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# A BRIEF SURVEY OF TRANSFER MATRIX TECHNIQUES WITH SPECIAL REFERENCE TO THE ANALYSIS OF AIRCRAFT PANELS


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Based on Lectures by

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# A BRIEF SURVEY OF TRANSFER MATRIX TECHNIQUES WITH SPECIAL REFERENCE TO THE ANALYSIS OF AIRCRAFT PANELS

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## FOREWORD

This report was prepared by the Strength and Dynamics Branch, Metals and Ceramics Division, under Project No. 7351, "Metallic Materials", Task No. 735106, "Behavior of Metals". This research work was conducted in the Air Force Materials Laboratory, Directorate of Laboratories, Wright-Patterson Air Force Base, Ohio by B. K. Donaldson based on lectures by Y. K. Lin.

This work began in June 1966 and was completed in September 1966. Manuscript released by the author in February 1967 for publication as an AFML Technical Report.

This technical report has been reviewed and is approved.

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ABSTRACT

This report offers an introduction to the transfer matrix method of analyzing the dynamic behavior of common engineering structures, followed by an explanation of the application of the transfer matrix method to an array of aircraft panels which are continuous over supporting stringers. The skin-stringer problem, important to the prediction of fatigue failures, is discussed for rather general conditions. The rectangular panels may vary in thickness and length while the stringers may vary in cross-sectional shape and size. The panels may be flat or curved, and the curved panels may vary in radius of curvature.

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## ILLUSTRATIONS

## SYMBOLS

| | |
|---|---|
| $a$ | radius of curvature of a cylindrical shell segment. in. |
| $b$ | panel system width, ie. distance between frames. in. |
| $c$ | viscous or equivalent viscous damping factor. $lb\text{-}sec/in^2$. |
| $c_y$, $c_z$ | distances defined in Fig. 10. in. |
| $\hat{c}, \hat{d}, \hat{e}$ | functions of $\omega$ defined by Eqs. (39) |
| $f_i$ | independent solution to a structural equation of motion. |
| $\hat{f}, \hat{g}, \hat{h}$ | functions of $\omega$ defined by Eqs. (69) |
| $g_i$ | functions defined by Eqs. (59) |
| $h$ | plate thickness. in. |
| $h_{pq}(t)$ | impulse response function |
| $k$ | $h^2/12a^2$ |
| $k_j$ | linear spring constant at the $j^{th}$ Myklestad beam station. in/lb. |
| $\tilde{k}_j$ | torsional spring constant at the $j^{th}$ Myklestad beam station. rad/in-lb. |
| $\ell_j$ | length of the $j^{th}$ beam segment (also used without subscript). in. |
| $m$ | mass of a single degree of freedom system. $lb\text{-}sec^2/in.$ |
| $n$ | index number for panel normal modes in x direction. |
| $p$ | $[\,12a^2(1-\nu^2)\,/\,h^2]^{1/4}$ |
| $\bar{p}, \bar{q}, \bar{r}$ | distributed load in the vertical direction, in the horizontal direction, and torque, respectively, acting at the stringer shear center. |
| $q_n$ | $n\pi a/b$ |

| | |
|---|---|
| $s_z$ | distance defined in Fig. 10. in. |
| $t$ | time, an independent variable. sec. |
| $t_{jk}$ | (j,k) element of a transfer matrix. |
| $u$ | deflection of a panel segment parallel to the stringers; positive with the x-axis. in. |
| $v$ | deflection of a panel (segment) or a stringer at the point of attachment, parallel to the frames; positive, with the corresponding coordinate axis. in. |
| $v_c, v_0$ | deflections parallel to the frames of a stringer cross-section at the centroid and shear center, respectively; positive with the corresponding coordinate axis. in. |
| $w$ | vertical deflection of a panel (segment), or of a stringer at the point of attachment ; positive down. in. |
| $w_c$, $w_o$ | vertical deflection of a stringer cross-section at the centroid and shear center respectively; positive down. in. |
| $w_j$ | vertical deflection of the $j^{th}$ beam segment, positive down. in. |
| $x$ | panel cartesian coordinate parallel to stringers. |
| $x_j$ | independent cartesian space coordinate for the $j^{th}$ beam segment, positive to the right. (also used without subscript). in. |
| $y$ | panel cartesian coordinate parallel to frames. |
| $A$ | beam or stringer cross-sectional area. $in.^2$ |
| $[B]$ | conversion matrix defined by Eq. (32) for the flat panel and Eq. (66) for curved panels. |
| $C$ | Saint- Venant constant of uniform torsion. $in.^4$ |
| $C_w$ | warping constant of stringer cross-section. $in.^6$ |

| | |
|---|---|
| $C_{ws}$ | $C_w + I_\zeta s_z^2$ in.$^6$ |
| D | $\frac{Eh^3}{12(1-\nu^2)}$ a plate stiffness factor. lb-in. |
| $[\mathfrak{D}]$ | differential operator defined by Eqs. (61). rad. |
| $E_j$ | Young's modulus for $j^{th}$ structural segment (also used without subscript).lb/in.$^2$. |
| $[F]_j$ | field transfer matrix defined by Eqs. (6a) and (7) for the $j^{th}$ beam segment; by Eq. (33) for a flat panel. |
| G | shear elastic modulus. lb/in.$^2$ |
| $[G]_j$ | beam point transfer matrix for station j defined by Eq. (8) or stringer transfer matrix defined by Eq. (40). |
| $H_{pq}(\omega)$ | frequency response function. |
| $I_j$ | area moment of inertia of the $j^{th}$ beam segment. in.$^4$ |
| $I_\eta, I_\zeta, I_{\eta\zeta}$ | stringer area moments of inertia and product of inertia about the stringer centroid. in.$^4$ |
| $J_c$, $J_s$ | stringer cross-section area polar moment of inertia about the centroid and point of attachment, respectively. |
| $J_j$ | discrete rotary inertia at station j of Myklestad beam (lb-in-sec$^2$) or rotary inertia per unit mass line of panel model (lb-sec$^2$). |
| [K] | matrix defined in Eq. (62). |
| [L] | matrix defined in Eq. (62). |
| $M(x_j)$ | moment amplitude factor for $j^{th}$ segment of Mykelstad beam, positive tension top side. in.-lb. |
| $M_n$ | component of $M_y$ correspondint to the $n^{th}$ panel mode in the x direction. in.-lb./in. |
| $M_y$ | moment in a flat panel about an x-axis per unit length of x-axis. in-lb/in. |

$M_\phi$ moment in a curved panel about an x-axis per unit length of x-axis. in. -lb. /in.

N number of Myklestad massless beam segments (or one less than the total number of beam stations.

$N_\phi$ tensile force in a curved panel parallel to the frames per unit length in the x direction. lb/in.

$N_{\phi x}$ shear force in the x-plane parallel to the frames, per unit length in the $a\phi$ direction. lb/in.

$P_q$ an arbitrary force or moment at station q.

$[R]_j$ field tramsfer matrix defined by equation (29).

$[T]$ transfer matrix from position indicated by subscript and superscript on right side, to position indicated on left side.

$V(x_j)$ shear amplitude factor for $j^{th}$ segment of Myklestad beam, positive up on right hand end. lb.

$V_n$ component of $V_y$ corresponding to the $n^{th}$ panel mode in the x direction. lb. /in.

$V_y$ shear in a flat panel in the z direction on a unit length in the x direction, positive down. lb. /in.

$V_\phi$ shear in a curved panel in the z direction on a unit length in the x direction, positive down. lb/in.

$X(x_j)$ vertical deflection amplitude factor for $j^{th}$ segment of a beam, positive down. in.

$Y_n$ panel normal modal amplitude function of y for $n^{th}$ mode. in.

$\{Z\}$ beam state vector defined by Eqs. (6) and (7); or a panel state vector defined by Eq. (32) or before Eq. (54).

$\alpha_{ij}$ coefficients to be determined for the functions $f_i$. See Eqs. (60).

| | |
|---|---|
| $\beta_{ij}$ | coefficients to be determined for the functions $g_i$. See Eqs. (60). |
| $\gamma, \delta, \varepsilon$ | structural damping factors. non-dimensional. |
| $\gamma_i$, $\delta_i$ | real and imaginary parts respectively of the characteristic roots of the shell segment equation. See Eqs. (50), i=1,2. rad. |
| $\delta(t)$ | Dirac delta function - See footnote, page 20. |
| $\zeta$ | damping factor for single degree of freedom system = $c/2\sqrt{km}$. rad. |
| $\eta_i$ | quantities defined in Eq. (56). (i=1,2,3). |
| $\theta_j$ | angular width of curved panel segment (See Fig. 11). rad. |
| $\lambda_j$ | panel span width between stringers. |
| $\mu_j$ | discrete mass at station j of Mykelstad beam. lb-sec$^2$/in. |
| $\nu$ | Poisson's ratio. rad. |
| $\{\xi\}$ | column matrix of biharmonic functions defined by Eqs. (60). rad. |
| $\rho$ | mass density lb-sec$^2$/in.$^4$ |
| $\sigma$ | a root of the characteristic equation of the equation of motion -- defined by Eq. (19a) for the distributed mass beam, by Eqs. (26) for the flat panel system. rad./in. |
| $\phi$ | angular coordinate of curved panel segment whose corresponding arc is parallel to the frames. rad. |
| $\omega$ | circular frequency of vibration. A subscript n indicates a natural frequency. |
| $\Delta(\omega)$ | frequency determinant value as a function of $\omega$. |
| $\Theta(x_j)$ | bending slope amplitude factor for $j^{th}$ segment of Mykelstad beam = $X'(x_j)$ rad. |

| | |
|---|---|
| $\Lambda_i$ | constant of integration of a structural equation of motion. |
| $\{\Xi\}$ | state vector defined by Eq. (65). |
| $T_n, \Phi_n, \Psi_n$ | amplitude functions of $\phi$ for the deflections u, w, and v respectively. See Eqs. (48). |
| $\Omega_p$ | a deflection or bending slope response. |

## I. INTRODUCTION

The following presentation is an attempt to clearly outline the transfer matrix method of analyzing the dynamic behavior of an elastic system and, at the same time, to explain a recent extension of this theory to aircraft type panel-stringer construction[1]. For additional introductory information the reader is referred to Reference [2].

The technique of transfer matrices is related to the methods of dynamic analysis developed by Holzer[3] and Myklestad[4]. Like these previous methods, the transfer matrix method is a calculation of the deflections and internal forces at successive values of a single space coordinate (stations) by utilizing a knowledge of the system intertia, damping, and stiffness properties between stations. Thus, the similarity of these procedures extends to the computation of system natural frequencies by iterative procedures, to the determination of normal modes, and to the calculation of deflection and force and moment type responses in the case of forced vibrations. In other words, the transfer matrix method accomplishes the same types of objectives in much the same manner as the Holzer and Myklestad methods. The difference between these older techniques and the transfer matrix technique lies in the advantage of conciseness that results from the use of matrix algebra by the latter method. This advantage has made practicable the analysis in detail of complex structures such as single rows of curved panels supported at varying intervals by not necessarily symmetrical stringers. The use of matrix algebra does not compromise the original advantages of the Holzer-Myklestad style of analysis. For example, the transfer matrix method allows the introduction of appropriate and separate damping descriptions at each component of the structure. Considering the advances that are currently being made in damping technology[5], this is no small advantage. On the other hand, the transfer matrix method leads to certain difficulties in numerical computation. These problems and their remedies that have proven to be effective will be briefly discussed.

To explain the principles of the transfer matrix method a series of examples of increasing complexity will be employed. These examples also illustrate the scope of this procedure which is presently limited to a structure undergoing sinusoidal motion (or being in static equilibrium--a special case), and to a structure being one dimensional in space or whose mathematical description is reducible to that where the unknown amplitude factor is a function of just one variable. An example of the latter alternative is a thin plate of rectangular shape which is simply supported at two opposite edges. In the case of a single frequency vibration, we could write for the vertical deflection

$$w(x,y,t) = e^{i\omega t} Y_n(y) \sin \frac{n\pi x}{\ell}$$

where y is the space coordinate parallel to the simply supported edges, and $Y_n(y)$ describes the variation of w in the y direction. In other words, we have reduced a plate problem to a one dimensional problem susceptible of solution by transfer matrix methods by (justifiably) assuming the form of the deflection with respect to one of the space coordinates.

The examples in the order of their discussion are (1) an undamped beam with discrete masses, (2) an undamped and damped beam with distributed mass properties, (3) the calculation of frequency response functions for a beam, (4) a flat panel-stringer row, and (5) a curved panel-stringer row. The first three examples are for explanatory purposes. The last two are summaries of recent extensions of transfer matrix methods.

## II. THE MYKLESTAD BEAM PROBLEM

A convenient place to begin an explanation of the principles of the transfer matrix method of analysis is with an undamped straight beam constructed of discrete masses connected by massless beam segments--the Myklestad "lumped mass" beam model. Figure 1 is a drawing of a general Myklestad model beam, where $\mu_j$ and $J_j$ are the discrete mass and rotary inertia at station j, respectively. We could begin our analysis by examining either of the two essential features of this beam, the elastic supports and the concentrated masses located at the stations, or the beam segments between stations. Let us begin by taking a detailed look at typical massless beam segment (j), Figure 2, where for generality we will include all possible combinations of stiffness and inertia at the stations that mark the segment endpoints. If in an actual application any of these properties are absent, their descriptive constant is, of course, zero. The combination of a linear and a torsional spring could, for example, be supplied by another beam of small mass running perpendicular to the beam under consideration and attached to it at the given station.

To locate a position along this massless beam segment we will make use of the local coordinate $x_j$. If $w_j(x_j, t)$ is the vertical deflection of the segment, positive down, the equation of motion is easily found from the basic beam equation to be

$$(EI)_j \partial^4 w_j / \partial x_j^4 = 0 \tag{1}$$

where for the sake of simplicity we have chosen the beam segment to have uniform stiffness properties. (If we could not reasonably describe the stiffness properties between our desired mass stations as being uniform, we could of course approximate the non-uniform stiffness by piecewise uniform segments separated by stations without inertia properties, or even if deemed necessary at the cost of complicating the equation of motion, we could describe the non-uniform stiffness as an analytic function of $x_j$.) If we are concerned with a single frequency vibration of the beam, we know that every point of the beam will move in a sinusoidal fashion; i.e., we may describe the motion in complex notation as being the real part of

$$w_j(x_j, t) = X(x_j)\, e^{i\omega t} \tag{2}$$

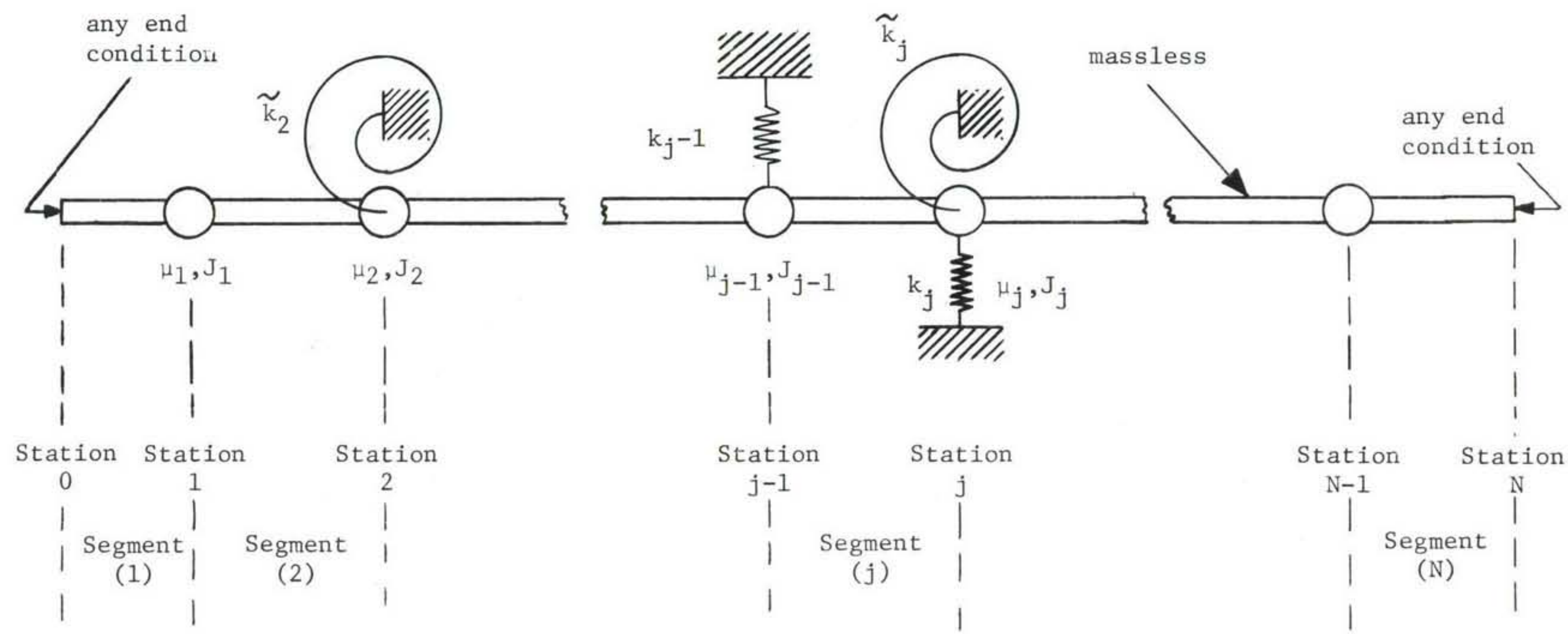



Figure 1. Myklestad Model Beam

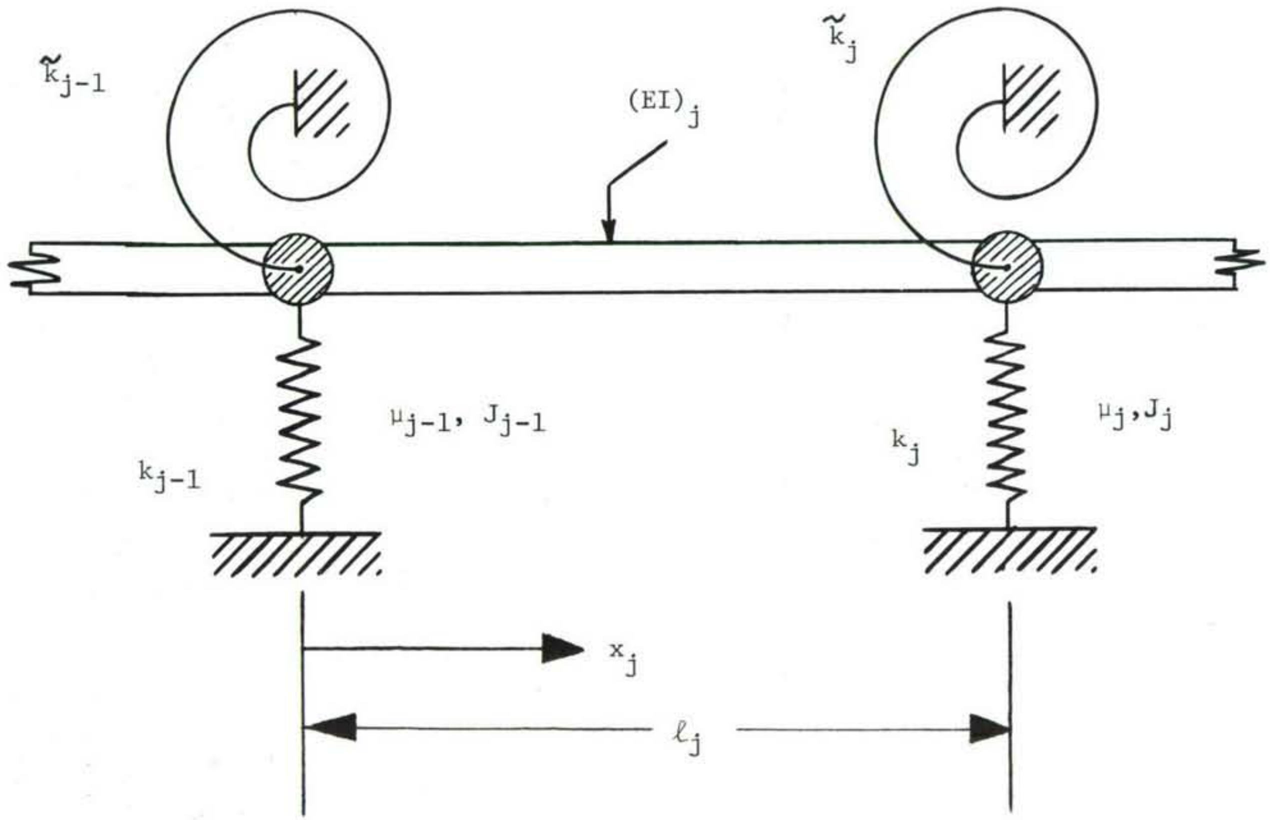



Figure 2. Massless Beam Segment

Substituting into Equation (1), we have

$$(EI)_j \frac{d^4 X(x_j)}{d x_j^4} \quad e^{i\omega t} = 0$$

or $$X^{(4)}(x_j) = 0 \tag{3}$$

Equation (3) obviously has the solution

$$X(x_j) = Ax_j^3 + Bx_j^2 + Cx_j + D \tag{4}$$

The physical meaning of the arbitrary constants A, B, C, and D is easily seen to be

$$\begin{aligned} D &= X(0) \\ C &= X'(0) \\ B &= \frac{1}{2} X''(0) = \frac{1}{2} \cdot \frac{M(0)}{(EI)_j} \\ A &= \frac{1}{6} X'''(0) = \frac{1}{6} \frac{V(0)}{(EI)_j} \end{aligned} \tag{5}$$

Except for the oscillatory factor $e^{i\omega t}$, D is the deflection at $x_j = 0$, C is the slope at $x_j = 0$, B is $1/2(EI)_j$ times the moment at $x_j = 0$, and A is $1/6(EI)_j$ times the shear at $x_j = 0$. We have now essentially accomplished a first step in the transfer matrix procedure. We have related the deflections and internal force quantities which describe the condition or state of the structural segment at a boundary point of that segment to the state at any point in that segment. In the strict sense, the mechanical state of the straight beam segment is completely specified by the continuous deflection function. However, in this and later examples, we will when we refer to the state of an elastic element mean not only the necessary, basic deflection components, but also the internal force and other deflection components, (or equivalently, the derivatives of the deflection components) necessary to calculate the basic deflections at one point when they are known at another. We complete this first step by substituting Equations (5) into Equation (4) and differentiating thrice to obtain

$$X(x_j) = \frac{V(0)}{6(EI)_j} x_j^3 + \frac{M(0)}{2(EI)_j} x_j^2 + \theta(0)\, x_j + X(0)$$

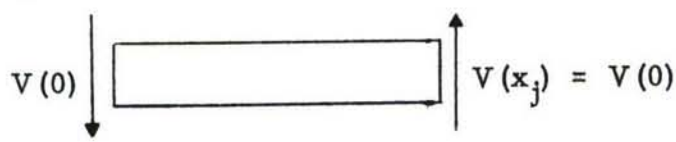
$$\theta(x_j) = \frac{V(0)}{2(EI)_j} x_j^2 + \frac{M(0)}{(EI)_j} x_j + \theta(0) \qquad (6)$$

$$M(x_j) = V(0)\, x_j + M(0)$$

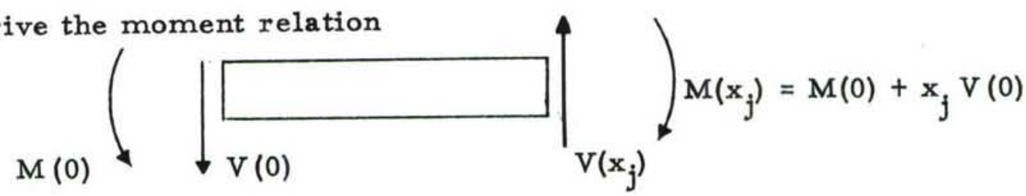
$$V(x_j) = V(0)$$

Equations (6) will be the basis of what we will call the field transfer matrix.

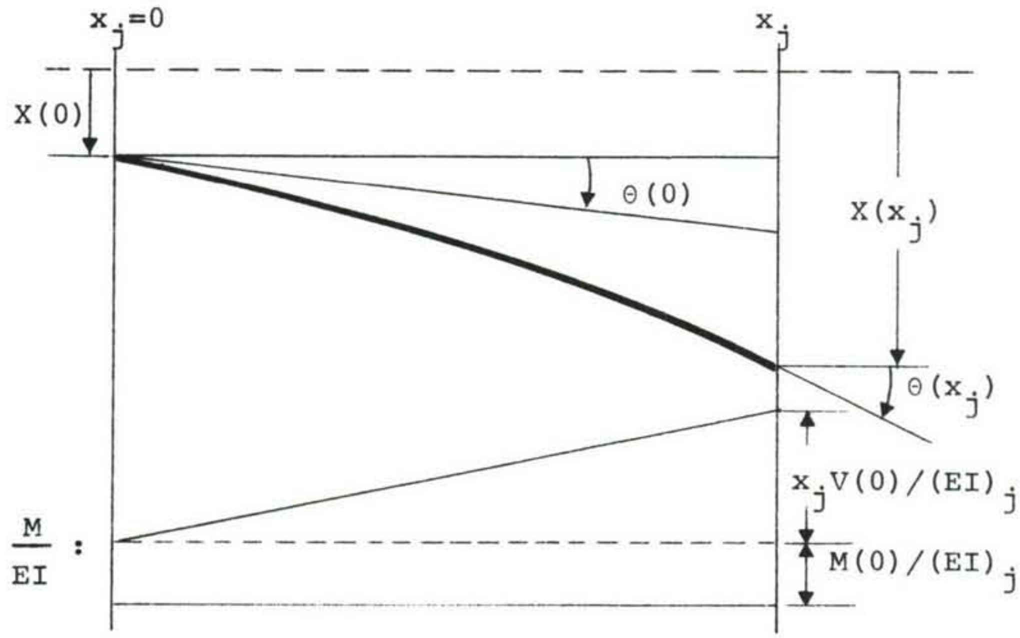
To show by example that other sufficient analyses of the structure will lead to these same relations, we will now note that Equations (6) could have been derived from the elementary theory of strength of materials. By so doing, by means of the following diagrams, we will explicitly state our sign convention. First, to derive the shear relation




To derive the moment relation




If the moment-area method is used to derive the expressions for $\theta(x_j)$ and $X(x_j)$, we have




Once again

$$\Theta(x_j) = \Theta(0) + \frac{M(0)}{(EI)_j} x_j + \frac{V(0)}{2(EI)_j} x_j^2$$

$$X(x_j) = X(0) + \Theta(0)\, x_j + \frac{M(0)}{2(EI)_j} x_j^2 + \frac{V(0)}{6(EI)_j} x_j^3$$

If we now specialize Eqs. (6) by letting $x_j = \ell_j$, we may view the result as a means of carrying over or transferring the mechanical state of the beam at the right hand side of station j-1 to the left hand side of station j. We may easily arrange this series of transfer equations in matrix form as follows

$$\begin{Bmatrix} X \\ \Theta \\ M \\ V \end{Bmatrix}_j^{\ell} = \begin{bmatrix} 1 & \ell & \frac{\ell^2}{2EI} & \frac{\ell^3}{6EI} \\ 0 & 1 & \frac{\ell}{EI} & \frac{\ell^2}{2EI} \\ 0 & 0 & 1 & \ell \\ 0 & 0 & 0 & 1 \end{bmatrix}_j \begin{Bmatrix} X \\ \Theta \\ M \\ V \end{Bmatrix}_{j-1}^{r} \qquad (6a)$$

where both the superscripts ($\ell$ = left of, r = right of) and subscripts on the column matrices (called state vectors) refer to a station, and the subscript on the square matrix indicates that the quantities that compose the elements of this matrix are those of the indicated beam segment. As mentioned previously, the amplitudes X, $\theta$, M and V are not the only choice as the four elements of our state vector. (For example, we could have selected X and its first three derivatives.) This choice, however, is much more convenient since all of these elements are of interest in themselves, and, more importantly, they will facilitate the introduction of the boundary conditions into the final equation. For the sake of being more concise, we rewrite Eqs. (6) as

$$\{Z\}_j^{\ell} = [F]_j \{Z\}_{j-1}^{r} \tag{7}$$

where the respective definitions of terms are obvious. The matrix [ F ] is called a field transfer matrix. We remark that if the displacement and force elements of the state vector are separated and the corresponding deflection and force elements are arranged in mirror image position, and if a suitable sign convention is chosen, then the associated transfer matrices are usually symmetrical about the cross-diagonal. This is the case here and in the following two examples. Another interesting feature of field transfer matrices is that their inverses can always be calculated by simply writing the field transfer matrix for the opposite direction.

To have complete freedom of description, another basic transfer matrix, called a point transfer matrix, is needed. It transfers the state vector across a station. Again let us consider the most general case; i. e., let there be a mass, a translational and a torsional spring at station j. The free-body diagram of station j is as shown in Figure 3.

Utilizing d'Alembert's principle, we may write:

$$M_j^r = M_j^{\ell} + (-\omega^2 J_j)\,\Theta_j + k_j\,\Theta_j$$

$$V_j^r = V_j^{\ell} - (-\omega^2 \mu\, X_j) - k_j X_j$$

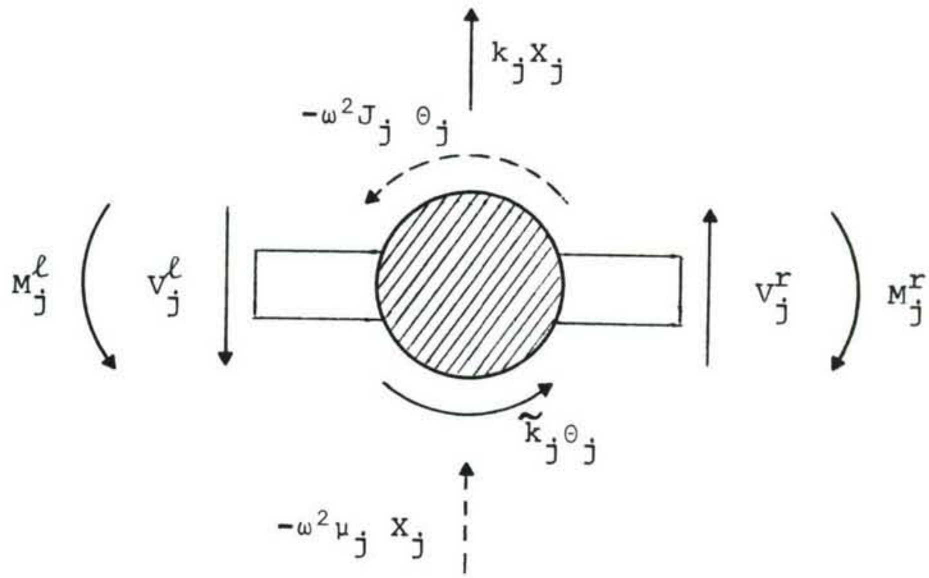



Figure 3. Forces and Moments Acting as a Discrete Mass

We also may write, because of the continuity of the deformation across station j:

$$X_j^r = X_j^\ell$$

$$\theta_j^r = \theta_j^\ell$$

Again arranging our result in matrix form, we have

$$\begin{Bmatrix} X \\ \theta \\ M \\ V \end{Bmatrix}_j^r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \tilde{k}-\omega^2 J & 1 & 0 \\ -k_j+\omega^2\mu & 0 & 0 & 1 \end{bmatrix}_j \begin{Bmatrix} X \\ \theta \\ M \\ V \end{Bmatrix}_j^\ell \qquad (8)$$

or

$$\{Z\}_j^r = [G]_j \{Z\}_j^\ell \tag{8a}$$

We are now ready to undertake the analysis of any undamped Myklestad beam possessing any end conditions. To make clear the general case, we choose the specific beam shown in Figure 4 where any combination of elastic restraints may be present at the three intermediate stations.

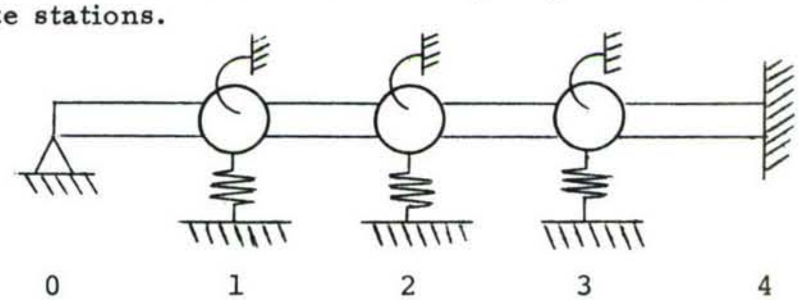



Figure 4. Example Myklestad Beam

Proceeding from left to right, we may by successively applying Eqs. (7) and (8) write

$$\{Z\}_1^\ell = [F]_1 \{Z\}_0^r$$

$$\{Z\}_1^r = [G]_1 [F]_1 \{Z\}_0^r$$

$$\{Z\}_4^\ell = [F]_4 [G]_3 \cdots [G]_1 [F]_1 \{Z\}_0^r$$

or more concisely

$$\{Z\}_4^\ell = {}_4^\ell[T]_0^r \{Z\}_0^r \tag{9}$$

We call $[T]$ a transfer matrix; i.e., no adjective. Its superscripts and subscripts indicate the extent of the transfer of the state vector. The boundary conditions of our example are:

at $(\ell, 4)$ $X = \Theta = 0$

at $(r, 0)$ $X = M = 0$

(A similar result would ensue for other boundary conditions.) Inserting these boundary conditions in the matrix equation (9) we have

$$\left\{\begin{matrix} 0 \\ 0 \\ M \\ V \end{matrix}\right\}_4^{\ell} = {}^{\ell}_{4}\begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix}_0^r \left\{\begin{matrix} 0 \\ \Theta \\ 0 \\ V \end{matrix}\right\}_0^r$$

from which we note that we may extract two linear homogeneous equations, i. e., the submatrix equation

$$\left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\} = {}^{\ell}_{4}\begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \end{bmatrix}_0^r \left\{\begin{matrix} \theta \\ V \end{matrix}\right\}_0^r \tag{10}$$

The existence of a nontrivial solution to Eq. (10) requires that the determinant of the coefficients be equal to zero. Since these coefficients are all functions of the free vibration frequency, our equation for the natural frequencies of the system is then

$${}^{\ell}_{4}\begin{vmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \end{vmatrix}_0^r = \Delta(\omega) = 0 \tag{11}$$

Since $\Delta(\omega)$ will for this model, be a polynomial in $\omega^2$ of order equal to the number of intertia parameters, both rotary and translational, a numerical rather than an exact solution for $w$ is to be obtained in almost all cases. Of course, all the roots for $\omega^2$ are real and positive. Once the natural frequencies are known from equation (11), it is a simple matter to compute the force and deflection mode shapes for this beam. We start with Eq. (10) which tells us

$$\Theta = -\frac{t_{14}}{t_{12}} V = -\frac{t_{24}}{t_{22}} V \tag{12}$$

If, for example, we normalize (i. e. assign a unit value to) V, we know all four components of the modal state vector at (r, 0). It is now only necessary to use the transfer matrices $[G(\omega_n)]$ and $[F(x_j)]$ to compute the modal state vector at any point of continuity on the beam.

The above steps are the basic elements of the transfer matrix approach. In review, we start at one point of the structure and proceed to an adjacent point using whatever state vector is necessary to describe plus transfer the mechanical state of the structure. The transfer matrices that carry the state vector from one point to another can be calculated by solving the overall equation(s) of motion of each structural segment. When the proper field and point transfer matrices are determined, a transfer matrix from one boundary to another can be constructed, and then partitioned to solve for the natural frequencies and mode shapes.

The present analysis could be expanded, for instance, to consideration of the vertical vibrations of a cantilevered beam with a horizontal bend. We could further slightly complicate the matter by placing the concentrated masses and rotary inertias on rigid extensions from the beam axis. This model might be a suitable model for a cantilevered high aspect ratio wing. See Figure 5.

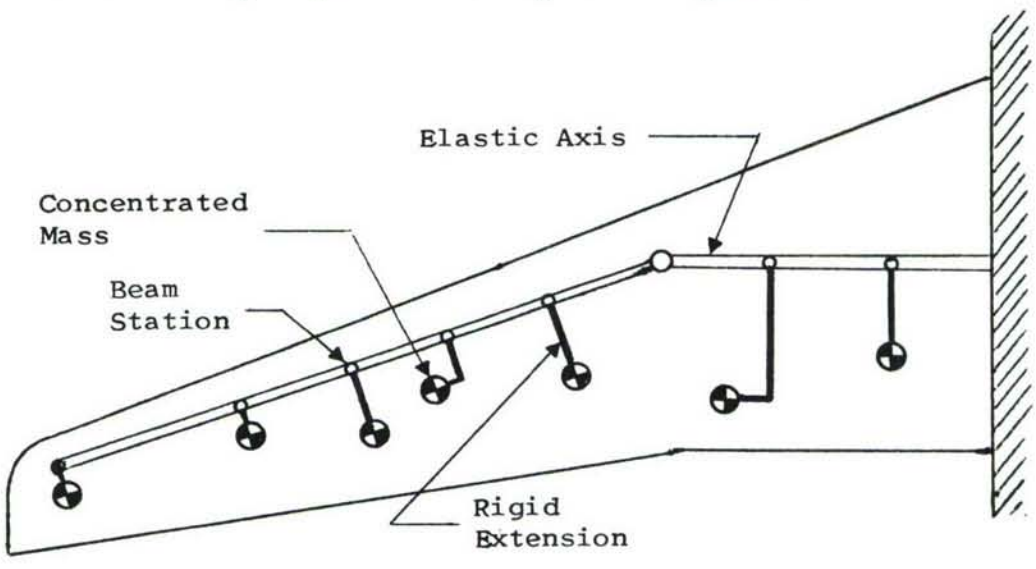



Figure 5. Bent Beam

We won't stop to analyze this model, but only comment on those of its features that are not found in the previous example. First, simply because of the horizontal bend, a vertical force or bending moment on the outboard section of the beam will produce a torsional moment (and deflection) as well as a bending moment on the inboard section. Therefore we would have to include the local beam twist and torsional moment in an analysis of this structure. Secondly, even without the horizontal bend, we would be compelled to consider torsional twist and torsional moment because the vertical accelerations of the offset concentrated masses would produce inertial torques about the beam axis. Therefore, our model would properly include not only the rotary inertias associated with bending slope deflections, but also the rotary inertias corresponding to torsional deflections, and cross products of inertia.

We could start our analysis at the beam tip and work our way toward the bend just as in the previous example. (The additional equation of motion for a massless beam segment is: the effective torsional stiffness times the second derivative of the twisting angle equals zero. The equations of motion of beam stations can again be derived using D'Alembert's principle.) At the bend we simply connect the locally oriented state vectors on each side of the bend by a matrix that resolves the outboard set of deflections and internal forces into components referenced to the inboard coordinate system. (Matrices are a rather efficient way of handling coordinate rotations.)

We are not limited to models where the inertia properties are made discrete. Let us now investigate how we may apply the transfer matrix method to a beam with distributed inertia properties.

## III. THE CONTINUOUS BEAM PROBLEM

Consider the case of a uniform beam segment of symmetric cross-section and of uniform density between two stations, say j-1 and j. See Figure 6.

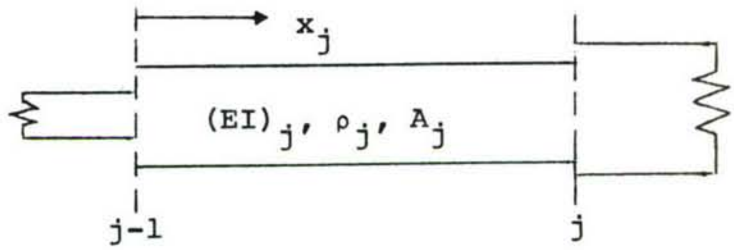



Figure 6. Continuous Mass Beam

The equation of motion of this segment can be derived from the basic beam bending equation. It is, for vibrating motion

$$EI \frac{\partial^4 w}{\partial x^4} = - \rho A \ddot{w} \tag{13}$$

where $\rho$ is the mass density, A the cross-sectional area, and each dot above a symbol indicates one partial differentiation with respect to time. We have dropped the subscript j from our various quantities for the sake of simplifying the writing of our equations. If we wish to examine a single frequency sinusoidally varying motion, we may again describe the vertical deflection as

$$w(x, t) = X(x) e^{i\omega t}$$

Thus our equation of motion reduces to

$$EI \frac{d^4 X}{dx^4} = \rho A \omega^2 X$$

This equation may be rewritten as

$$\frac{d^4 X}{dx^4} - \sigma^4 X = 0 \tag{14}$$

where $\sigma^4 = \frac{\rho A \omega^2}{EI}$ (14a)

Four independent solutions to the differential equation (14) are $\exp(\sigma x)$, $\exp(-\sigma x)$, $\exp(i\sigma x)$, and $\exp(-i\sigma x)$. Note that these solutions are independent regardless of which of the four roots of $\sigma^4$ is used, and in fact, the same four functions are obtained regardless of which of the four roots is used. For later convenience we will specify the use of the positive real root. Now any four independent linear combinations of the above four solutions are also a complete set of solutions. The most advantageous set of combinations is

$$
\begin{aligned}
f_0(x) &= \frac{1}{2}(\cosh \sigma x + \cos \sigma x) \\
f_2(x) &= \frac{1}{2}(\cosh \sigma x - \cos \sigma x) \\
f_1(x) &= \frac{1}{2}(\sinh \sigma x + \sin \sigma x) \\
f_3(x) &= \frac{1}{2}(\sinh \sigma x - \sin \sigma x)
\end{aligned}
\tag{15}
$$

The advantages of this set of combinations, the reasons for their selection, are

(1) $f_0$ and $f_2$ are real even functions of $x$

$f_1$ and $f_3$ are real odd functions of $x$

(2) The index order rotates under differentiation or integration; e. g.

$$
\begin{aligned}
f_0'(x) &= \sigma f_3(x) \\
f_1'(x) &= \sigma f_0(x) \\
f_2'(x) &= \sigma f_1(x) \\
f_3'(x) &= \sigma f_2(x)
\end{aligned}
$$

and (3) $f_0(0) = 1$; $f_j(0) = 0$ for $j \neq 0$.

Then, when we write our solution in the standard form

$$X(x) = \Lambda_0 f_0(x) + \Lambda_1 f_1(x) + \Lambda_2 f_2(x) + \Lambda_3 f_3(x) \tag{16}$$

we can see by virtue of items (2) and (3) above that we obtain the happy result

$$\Lambda_0 = X(0)$$

$$\sigma \Lambda_1 = X'(0) \qquad \text{or} \qquad \Lambda_1 = \frac{\Theta(0)}{\sigma}$$

$$\sigma^2 \Lambda_2 = X''(0) \qquad \text{or} \qquad \Lambda_2 = \frac{M(0)}{\sigma^2 EI} \tag{17}$$

$$\sigma^3 \Lambda_3 = X'''(0) \qquad \text{or} \qquad \Lambda_3 = \frac{V(0)}{\sigma^3 EI}$$

Therefore

$$X(x) = X(0)\, f_0(x) + \frac{\Theta(0)}{\sigma} f_1(x) + \frac{M(0)}{\sigma^2 EI} f_2(x) + \frac{V(0)}{\sigma^3 EI} f_3(x) \tag{16a}$$

So, once again by obtaining a suitable solution to the equation of motion of an elastic element, we have obtained the basic relation for the field transfer matrix of that element. The remaining relations are, of course, obtained by differentiating Eq. (16a)

$$X'(x) = \sigma X(0)\, f_3(x) + \Theta(0)\, f_0(x) + \frac{M(0)}{\sigma EI} f_1(x) + \frac{V(0)}{\sigma^2 EI} f_2(x)$$

$$X''(x) = \sigma^2 X(0)\, f_2(x) + \sigma\, \Theta(0)\, f_3(x) + \frac{M(0)}{EI} f_0(x) + \frac{V(0)}{\sigma EI} f_1(x)$$

$$X'''(x) = \sigma^3 X(0)\, f_1(x) + \sigma^2\, \Theta(0)\, f_2(x) + \sigma \frac{M(0)}{EI} f_3(x) + \frac{V(0)}{EI} f_0(x)$$

Combining these results for $x_j = \ell_j$ in matrix form, we have

$$\begin{Bmatrix} X \\ \Theta \\ M \\ V \end{Bmatrix}_j^{\ell} = \begin{bmatrix} f_0(\ell) & \frac{f_1(\ell)}{\sigma} & \frac{f_2(\ell)}{\sigma^2 EI} & \frac{f_3(\ell)}{\sigma^3 EI} \\ \sigma f_3(\ell) & f_0(\ell) & \frac{f_1(\ell)}{\sigma} & \frac{f_2(\ell)}{\sigma EI} \\ \sigma^2 EI f_2(\ell) & \sigma EI f_3(\ell) & f_0(\ell) & \frac{f_1(\ell)}{\sigma} \\ \sigma^3 EI f_1(\ell) & \sigma^2 EI f_2(\ell) & \sigma f_3(\ell) & f_0(\ell) \end{bmatrix}_j \begin{Bmatrix} X \\ \Theta \\ M \\ V \end{Bmatrix}_{j-1}^{r} \tag{18}$$

Note that this field transfer matrix is also cross-symmetrical.

At this point, let us include damping effects as part of our analysis. We do so in the realization that for a damped system sinusoidal motion exists only when the system is subjected to a sinusoidal excitation. Structural damping at joints and in elastic segments can easily be included in the conventional manner in our equations of motion. At joints, that is at the location of elastic springs, we introduce structural damping by simply replacing $k$ by $k(1 + i\delta)$ and $\tilde{k}$ by $\tilde{k}(1 + i\varepsilon)$ where $\delta$ and $\varepsilon$ are structural damping factors. Similarly, structural damping in an elastic element is described by replacing EI by $EI(1 + i\gamma)$, where $\gamma$ is another structural damping factor. Of course, these changes in the equations of motion are carried through in identical form to the point and field transfer matrices. We may also take into account an equivalent viscous damping by adding another term to our equations of motion. For example, the equation for an elastic segment becomes

$$EI(1 + i\gamma) \frac{\partial^4 w}{\partial x^4} + c\dot{w} + \rho A \ddot{w} = 0$$

with no external excitation on this particular segment. To solve this equation we again let $w = X(x) \exp(i\omega t)$, and substitute to obtain

$$\frac{d^4 X}{dx^4} - \frac{\rho w^2 A - icw}{EI(1+i\gamma)} X = 0 \tag{19}$$

We can return to the form of equation (14) by simply redefining $\sigma$ as

$$\sigma^4 = \frac{\rho\omega^2 A - ic\omega}{EI(1+i\gamma)} \tag{19a}$$

Thus, again, the field transfer matrix, Eq. (18) retains the same form. In this case all the quartic roots are complex. As before it is permissible to use any one of the four roots as $\sigma$ in the computation since we only require that $\sinh\sigma x$, $\cosh\sigma x$, $\sin\sigma x$, and $\cos\sigma x$ be independent of each other.

Thus, since the form of the transfer matrices remains unaltered by the presence of damping effects, in our subsequent examples we will take for granted the presence of the various damping parameters in their appropriate locations.

As a final comment, before we pass to the topics of forced vibration and skin-stringer construction, the method of transfer matrices can also be applied to beam grids and frames[2]. Very briefly, a single path through the structure is chosen and by means of the continuity and equilibrium equations of the joints and the boundary conditions at beam ends other than the two that mark the beginning and end of the chosen path, the structure that lies off the chosen path is represented by point transfer matrices on the chosen path.

## IV. RESPONSE TO FORCED VIBRATION

Before we see how to use transfer matrices to calculate the response of a structure to forced vibration, let us pause to review a bit of the basic theory of mechanical vibrations.

The response to a structure under forced vibration, either deterministic or random, can be described by either of the following two basic functions [6]:

the impulse response function $h_{pq}(t)$

the frequency response function $H_{pq}(\omega)$

The respective meanings of these two functions are as follows. Let the structure be at rest at $t = 0$, and let the excitation be a unit impulse* applied at station q at time $t = 0$. Then $h_{pq}(t)$ is the response at station p. See Figure 7 below. Let the structure be stable (say positively damped), and let the excitation be a unit sinusoidally varying load ($e^{i\omega t}$) applied at station q. Then the <u>steady state</u> response at station p is $H_{pq}(\omega)\, e^{i\omega t}$.

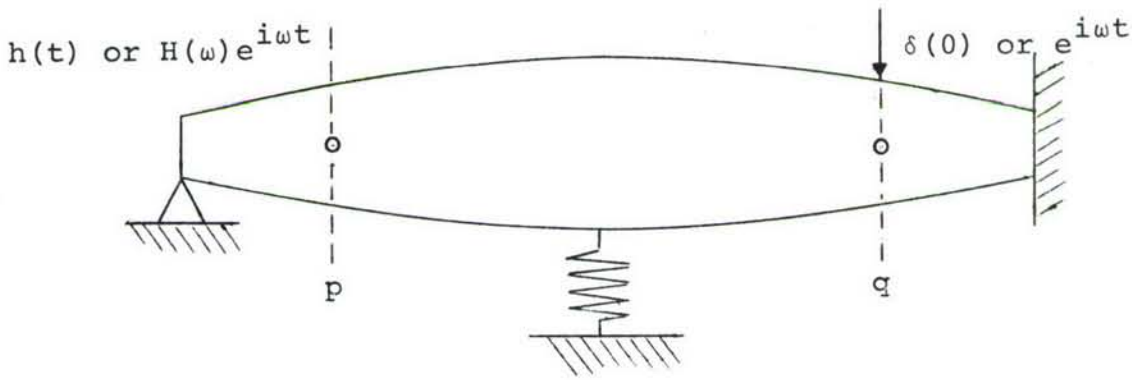



Figure 7. Beam of General Configuration

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* A unit impulse at time $\tau = t$ is described mathematically by the Dirac delta function, $\delta(t-\tau)$. Its definition is, for an arbitrary function $\Gamma(t)$

$$\Gamma(t) = \int_{-\infty}^{\infty} \Gamma(\tau) \cdot \delta(t-\tau)d\tau = \int_{t-\varepsilon}^{t+\varepsilon} \Gamma(\tau) \cdot \delta(t-\tau)d\tau$$

where $\varepsilon$ is arbitrarily small. It is sometimes described as a functional because it is not a function in the ordinary sense, i.e. while $\delta(t) = 0$ for almost all t, it is not specifically defined for t within the interval $(t-\varepsilon, t+\varepsilon)$, $\varepsilon$ arbitrarily small.

An arbitrary forcing function can be considered as a continuous sequence of impulses, i.e. we may write

$$P_q(t) = \int_{-\infty}^{+\infty} P_q(\tau)\,\delta(t-\tau)\,d\tau$$

A response at station p to the forcing function can be described with the aid of the appropriate impulse response function for the two stations, i.e.

$$\Omega_p(t) = \int_{-\infty}^{+\infty} P_q(\tau)\,h_{pq}(t-\tau)\,d\tau$$

On the other hand, if a spectral analysis of the forcing function is performed so that

$$P_q(t) = \int_{-\infty}^{+\infty} \overline{P}_q(\omega)\,e^{i\omega t}\,d\omega$$

where

$$\overline{P}_q(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} P_q(t)\,e^{-i\omega t}\,dt$$

then the response to the forcing function $P_q(t)$ can be written

$$\Omega_p(t) = \int_{-\infty}^{\infty} \overline{P}_q(\omega)\,H_{pq}(\omega)\,e^{i\omega t}\,d\omega$$

Using a few simple steps we can show that

$$H_{pq}(\omega) = \int_{-\infty}^{\infty} h(t)\,e^{-i\omega t}\,dt$$

$$h_{pq}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{pq}(\omega)\,e^{i\omega t}\,d\omega$$

That is,outside of the adjustment of a constant factor, these functions are Fourier transform pairs.

A simple example of these functions is to be found in the case of a single degree of freedom system. Here p = q = 1 (omitted), and

$$h(t) = \begin{cases} \dfrac{-i}{m\omega_n \sqrt{1-\zeta^2}} \exp\left(-\zeta\omega_n t + i\omega_n \sqrt{1-\zeta^2}\; t\right), & \text{for } t > 0 \\ 0, & \text{for } t \leq 0 \end{cases}$$

and

$$H(\omega) = \frac{1}{m(\omega_n^2 - \omega^2 + 2 i \zeta \omega \omega_n)}$$

The technique of transfer matrices can be used to compute frequency response functions. For illustration, consider the non-uniform beam with end conditions sketched below.

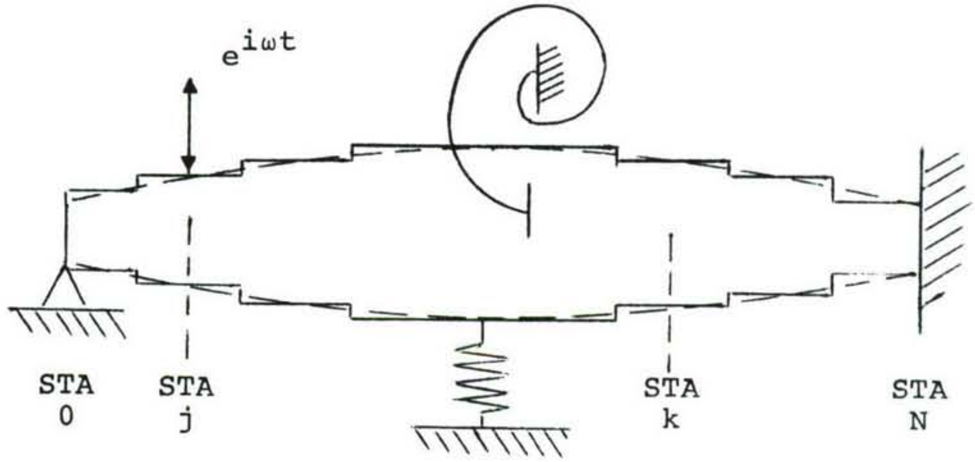


Figure 8. Segmented Beam of General Configuration.

The beam is approximated by a finite sequence of uniform sections between stations. Any arrangement of interior elastic supports may be included in the problem. We are interested in obtaining the response at station k due to a unit sinusoidal force at station j. We may need to be more specific about the response, that is, decide whether the response on the left hand side or the right hand side of station k is the response to be calculated. For the sake of definiteness, let us decide we want to know the response on the right hand side.

To develop the equations in a natural way, let us again start at the left end of the beam and proceed to the right. By our previous work, we can immediately work our way to the left hand side of the impressed force.

$$\{Z\}_j^{\ell} = {}_j^{\ell}[\,T\,]_o^r \; \{Z\}_o^r$$

The impressed force is simply introduced by writing

$$\{Z\}_j^{r} = {}_j^{r}[\,T\,]_o^r \, \{Z\}_o^r + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

(Recall the factor $e^{i\omega t}$ is common to all terms of these equations.) Proceeding from this point, we have

$$\{Z\}_k^{r} = {}_k^{r}[\,T\,]_j^r \; \{Z\}_j^r$$

so that

$$\{Z\}_k^{r} = {}_k^{r}[\,T\,]_o^r \; \{Z\}_o^r + {}_k^{r}[\,T\,]_j^r \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{20}$$

This is our first of two basic equations. In expanded form it is

$$\begin{Bmatrix} X \\ \Theta \\ M \\ V \end{Bmatrix}_k^r = {}_k^{r}[\,T\,]_o^r \begin{Bmatrix} 0 \\ \Theta \\ 0 \\ V \end{Bmatrix}_o^r + {}_k^{r}[\,T\,]_j^r \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{20a}$$

where we have inserted the boundary conditions at station 0. In addition to the elements of the state vector at (r, k), the slope and shear at (r, o) are unknown. These elements can be eliminated by use of the boundary conditions at station N in our second basic equation, which is merely a continuation of the first equation. It is

$$\begin{Bmatrix} 0 \\ 0 \\ M \\ V \end{Bmatrix}_N^{\ell} = {}_N^{\ell}[\,T\,]_0^r \begin{Bmatrix} 0 \\ \Theta \\ 0 \\ V \end{Bmatrix}_0^r + {}_N^{\ell}[T]_j^r \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{21}$$

Now, from the above two (4 x 1) matrix equations, (20a) and (21), we can extract the following two equations, respectively.

$$
{}^{r}_{k}\begin{Bmatrix} X \\ \Theta \\ M \\ V \end{Bmatrix}^{r}_{k} = {}^{r}_{k}\begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \\ t_{32} & t_{34} \\ t_{42} & t_{44} \end{bmatrix}^{r}_{0} \begin{Bmatrix} \Theta \\ V \end{Bmatrix}^{r}_{0} + {}^{r}_{k}\begin{Bmatrix} t_{14} \\ t_{24} \\ t_{34} \\ t_{44} \end{Bmatrix}^{r}_{j}
$$

$$
\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = {}^{\ell}_{N}\begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \end{bmatrix}^{r}_{0} \begin{Bmatrix} \Theta \\ V \end{Bmatrix}^{r}_{0} + {}^{\ell}_{N}\begin{Bmatrix} t_{14} \\ t_{24} \end{Bmatrix}^{r}_{j}
$$

Then solving the second of these for

$$
\begin{Bmatrix} \Theta \\ V \end{Bmatrix}^{r}_{0} = -\left( {}^{\ell}_{N}\begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \end{bmatrix}^{r}_{0} \right)^{-1} {}^{\ell}_{N}\begin{Bmatrix} t_{14} \\ t_{24} \end{Bmatrix}^{r}_{j}
$$

and substituting this quantity into the first, we obtain

$$
\begin{Bmatrix} X \\ \Theta \\ M \\ V \end{Bmatrix}^{r}_{k} = - {}^{r}_{k}\begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \\ t_{32} & t_{34} \\ t_{42} & t_{44} \end{bmatrix}^{r}_{0} \left( {}^{\ell}_{N}\begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \end{bmatrix}^{r}_{0} \right)^{-1} {}^{\ell}_{N}\begin{Bmatrix} t_{14} \\ t_{24} \end{Bmatrix}^{r}_{j} + {}^{r}_{k}\begin{Bmatrix} t_{14} \\ t_{24} \\ t_{34} \\ t_{44} \end{Bmatrix}^{r}_{j} \tag{22}
$$

The left hand side is the set of amplitudes of the response as functions of the frequency $\omega$ due to a unit sinusoidal force excitation, which by definition are the frequency response functions for these circumstances. Of course, the frequency response functions for a unit sinusoidal moment can be obtained in an exactly analogous way. The result is

$$
\begin{Bmatrix} X \\ \Theta \\ M \\ V \end{Bmatrix}^{r}_{k} = - {}^{r}_{k}\begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \\ t_{32} & t_{34} \\ t_{42} & t_{44} \end{bmatrix}^{r}_{0} \left( {}^{\ell}_{N}\begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \end{bmatrix}^{r}_{0} \right)^{-1} {}^{\ell}_{N}\begin{Bmatrix} t_{13} \\ t_{23} \end{Bmatrix}^{r}_{j} + {}^{r}_{k}\begin{Bmatrix} t_{13} \\ t_{23} \\ t_{33} \\ t_{43} \end{Bmatrix}^{r}_{j} \tag{22a}
$$

Of course, other boundary conditions lead to the selection of other elements of the transfer matrices.

## V. FLAT STRINGER-PANEL SYSTEMS

We will discuss two types of stringer-panel systems. We will begin with flat stringer-panel systems, and when we have determined the method of solution for these, we will proceed to discuss the more complex case of curved panel systems. A sketch of a typical flat panel array is shown in Figure 9.

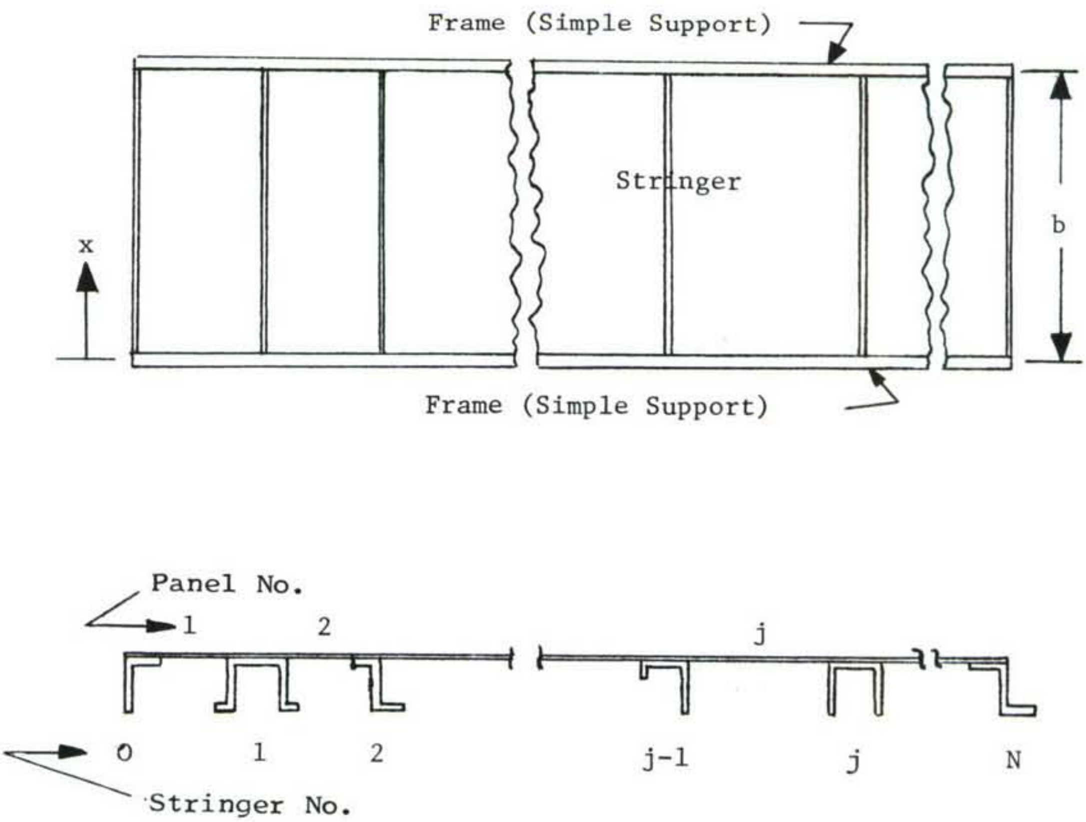


Figure 9. Flat stringer-panel model

To simplify the presentation of both the flat and curved panel analyses, we will limit our attention to free, undamped vibrations. The treatment of the cases of damped and forced vibrations, as has been explained, is little different. A simple forced vibration example will be given in Section XII.

Note that, while in the case of beams we made no restrictions concerning the boundary conditions, in our flat panel mathematical model we specify simple support boundary conditions at the frames. As will be seen, this particular arrangement allows us to reduce this essentially two dimensional problem to a one dimensional problem. As pointed out before, once we have a one dimensional problem, we may employ the method of transfer matrices. Most actual construction is not such that the conditions of simple support exist at the frames. However, because the distance between stringers is usually one half or less than one half the distance between frames, and since fatigue failure can be expected to begin in the panel skin adjacent to the center of a stringer, this model is quite practical for the study of the important problem of fatigue stresses in actual panel construction. More generally, this same argument can be extended to claim validity for applying the results of this model to construction with different boundary conditions at the frames if the response of interest is located near the mid-distance line between frames.

The standard plate bending equation of motion for the panel skin is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\rho\frac{h}{D}\ddot{w} \tag{23}$$

For a discussion of the underlying assumptions and derivation of Eq. (23) see Reference [7], pages 39-1 through 39-3. In this problem, with simple support at the frames, we may obtain a solution in the form

$$w(x,\, y,\, t) = e^{i\omega t}\, Y_n\,(y)\,\sin\frac{n\pi x}{b} \tag{24}$$

where we take advantage of the known modal form in the x-direction. For each n, the function $Y_n$ is the amplitude function in the y-direction It corresponds to X(x) in the beam problem, and this unknown is, of course, only a function of one space coordinate.

Substituting Eq. (24) into Eq. (23), after cancelling $\exp(i\omega t)$ and $\sin(m\pi x/b)$, we obtain for each stringerwise modal number n

$$Y_n^{(4)} - 2\left(\frac{n\pi}{b}\right)^2 Y_n'' + \left(\frac{n\pi}{b}\right)^4 Y_n = \frac{\rho\omega^2 h}{D} Y_n \qquad (25)$$

We have two unknowns in the one equation, $\omega$ and $Y_n$. We need to determine those special values of $\omega$ (in mathematical terms the eigenvalues, or in dynamical terms the natural frequencies) for which we can also determine solutions for $Y_n$ up to an arbitrary multiplicative constant, (the eigenfunctions, or normal modes of an undamped system.)

We can write the characteristic roots of Eq. (25) as $+\sigma_1$, $-\sigma_1$, $+i\sigma_2$, and $-i\sigma_2$ where

$$\sigma_1 = \left[\omega\sqrt{\frac{\rho h}{D}} + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

$$\sigma_2 = \left[\omega\sqrt{\frac{\rho h}{D}} - \left(\frac{n\pi}{b}\right)^2\right]^{1/2} \qquad (26)$$

Since for each n, $(D^{1/4} n\pi / h^{1/4} \rho^{1/4} b)^2$ is the smallest natural frequency of an infinitely long panel row without stringers[8], and because the addition of stringers and boundary conditions at a finite distance would increase the stiffness and hence the natural frequencies of such a built-up, finite system, we can conclude that $\sigma_2$ is a real quantity.

Let us now turn to our transfer matrix techniques. We are here interested in determining the deflection, slope, moment, and shear in the panel skin along any plane perpendicular to the x-axis. The deflection and slope, of course, depend upon $Y_n$ and $Y'_n$ respectively. Since[7]

$$M_y = +D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$V_y = +D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\right)$$

we see that we can describe these quantities of interest by employing $Y_n$ and its first three derivatives. Therefore, as a stepping-stone to our final state vector $\lfloor Y_n \; Y_n' \; M_n \; V_n \rfloor$, let us consider the state vector $\lfloor Y_n \; Y_n' \; Y_n'' \; Y_n''' \rfloor$ and discover how it may be transferred across a field, i.e. across a panel.

We already have the first part of the solution to the panel problem in terms of the solutions for the chatacteristic roots of our reduced equation of motion, Eq. (26). These roots, of course, allow us to write a solution for $Y_n$ in terms of (e.g. expodential functions of) $\omega$ for each panel. Then, analogous to the continuous beam problem, our next step is to express the panel solution in a form appropriate to our state vector $\lfloor Y_n \; Y_n' \; Y_n'' \; Y_n''' \rfloor$ . Again we start by expressing our solution in the form of the sum of four independent functions, i.e.

$$Y_n(y) = \Lambda_o \; f_o(y) + \Lambda_1 \; f_1(y) + \Lambda_2 \; f_2(y) + \Lambda_3 \; f_3(y) \qquad (27)$$

so that among other conveniences, $\Lambda_o = Y_n(0)$, $\Lambda_1 = Y_n'(0)$, $\Lambda_2 = Y_n''(0)$, and $\Lambda_3 = Y_n'''(0)$. As in the case of the beam, we by-pass solutions in the form of exponential functions, and concentrate on combinations of the hyperbolic and circular functions. Specifically, these independent solutions are sinh $\sigma_1 y$, cosh $\sigma_1 y$, sin $\sigma_2 y$, and cos $\sigma_2 y$. The combination of these that we seek is

$$\begin{aligned}
f_o(y) &= \frac{1}{\sigma_1^2 + \sigma_2^2} (\sigma_2^2 \cosh \sigma_1 y + \sigma_1^2 \cos \sigma_2 y) \\
f_1(y) &= \frac{1}{\sigma_1^2 + \sigma_2^2} (\frac{\sigma_2^2}{\sigma_1} \sinh \sigma_1 y + \frac{\sigma_1^2}{\sigma_2} \sin \sigma_2 y) \\
f_2(y) &= \frac{1}{\sigma_1^2 + \sigma_2^2} (\cosh \sigma_1 y - \cos \sigma_2 y) \\
f_3(y) &= \frac{1}{\sigma_1^2 + \sigma_2^2} (\frac{1}{\sigma_1} \sinh \sigma_1 y - \frac{1}{\sigma_2} \sin \sigma_2 y)
\end{aligned} \qquad (28)$$

(A general procedure by which we can calculate the desired functional forms of our solution will be illustrated for the more complex case of a curved panel segment.) The functions $f_o$ and $f_2$ are even while $f_1$ and $f_3$ are odd, $f_s^{(r)}(0) = 0$ when $r \neq s$, and $f_s^{(r)} = 1$ when $r = s$, for r, s = 0, 1, 2, 3. However, because $\sigma_1 \neq \sigma_2$, $f_s'(y) \neq f_{s+1}(y)$; i.e. the nice index rotating property that was present in the case of the beam problem does not exist here. As a result of this, we will require more than four functions to construct the transfer matrix for $\lfloor Y_n \; Y_n' \; Y_n'' \; Y_n''' \rfloor$.

Now that we have as our solution Eqs. (28), we can write

$$Y_n(y) = Y_n(0)\, f_0(y) + Y'_n(0)\, f_1(y) + Y''_n(0)\, f_2(y) + Y'''_n(0)\, f_3(y) \tag{27a}$$

Thus, again, by differentiating and then setting $y = \lambda_j$, we can establish the relations between $Y_n$ and its derivatives at $y = \lambda_j$ on one hand, and $Y_n(0)$, $Y'_n(0)$, $Y''_n(0)$, and $Y'''_n(0)$ on the other hand. These relations in matrix form are

$$\begin{Bmatrix} Y_n \\ Y'_n \\ Y''_n \\ Y'''_n \end{Bmatrix}_j^{\ell} = [\,R\,]_j \begin{Bmatrix} Y_n \\ Y'_n \\ Y''_n \\ Y'''_n \end{Bmatrix}_{j-1}^{r} \tag{29}$$

where $y = \lambda_j$ corresponds to $(\ell, j)$ and $y = 0$ corresponds to $(r, j-1)$, and

$$[\,R\,]_j = \begin{bmatrix} C_0 & S_{-1} & C_{-2} & S_{-3} \\ S_1 & C_0 & \tilde{S}_{-1} & C_{-2} \\ C_2 & S_1 & \tilde{C}_0 & \tilde{S}_{-1} \\ S_3 & C_2 & \tilde{S}_1 & \tilde{C}_0 \end{bmatrix}_j$$

where
$$C_0 = f_0(\lambda_j)$$
$$S_{-1} = f_1(\lambda_j)$$
$$C_{-2} = f_2(\lambda_j)$$
$$S_{-3} = f_3(\lambda_j)$$

$$\tilde{C}_0 = \frac{1}{\sigma_1^2 + \sigma_2^2} (\sigma_1^2 \cosh \sigma_1 \lambda_j + \sigma_2^2 \cos \sigma_2 \lambda_j)$$

$$C_2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} (\cosh \sigma_1 \lambda_j - \cos \sigma_2 \lambda_j)$$

$$\tilde{S}_{-1} = \frac{1}{\sigma_1^2 + \sigma_2^2} (\sigma_1 \sinh \sigma_1 \lambda_j + \sigma_2 \sin \sigma_2 \lambda_j)$$

$$S_1 = \frac{\sigma_1 \sigma_2}{\sigma_1^2 + \sigma_2^2} (\sigma_2 \sinh \sigma_1 \lambda_j - \sigma_1 \sin \sigma_2 \lambda_j)$$

$$\tilde{S}_1 = \frac{1}{\sigma_1^2 + \sigma_2^2} (\sigma_1^3 \sinh \sigma_1 \lambda_j - \sigma_2^3 \sin \sigma_2 \lambda_j)$$

$$S_3 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} (\sigma_1 \sinh \sigma_1 \lambda_j + \sigma_2 \sin \sigma_2 \lambda_j)$$

At this point we will convert the state vector $\lfloor Y_n \; Y_n' \; Y_n'' \; Y_n''' \rfloor$ to one which will represent the variations in moment and shear in the y direction. We do this by expressing the moment and shear in the same form as the vertical deflection, i.e.

$$M_y = e^{i\omega t} M_n(y) \sin \frac{n\pi x}{b}$$

$$V_y = e^{i\omega t} V_n(y) \sin \frac{n\pi x}{b} \qquad (30)$$

From Eqs. (27) and (30), by means of the orthogonality properties of the sine function, we can obtain

$$M_n(y) = D \left[ Y_n'' - \nu \left(\frac{n\pi}{b}\right)^2 Y_n \right]$$

$$V_n(y) = D \left[ Y_n''' - (2 - \nu) \left(\frac{n\pi}{b}\right)^2 Y_n' \right] \qquad (31)$$

This immediately leads to the matrix form

$$\{Z\} = \begin{Bmatrix} Y_n \\ Y'_n \\ M_n \\ V_n \end{Bmatrix} = [B]_j \begin{Bmatrix} Y_n \\ Y'_n \\ Y''_n \\ Y'''_n \end{Bmatrix} \tag{32}$$

where the conversion matrix and its inverse in detail are

$$[B]_j = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -D\nu\left(\frac{n\pi}{b}\right)^2 & 0 & D & 0 \\ 0 & -(2-\nu)\left(\frac{n\pi}{b}\right)^2 D & 0 & D \end{bmatrix}_j$$

$$[B]_j^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \nu\left(\frac{n\pi}{b}\right)^2 & 0 & \frac{1}{D} & 0 \\ 0 & (2-\nu)\left(\frac{n\pi}{b}\right)^2 & 0 & \frac{1}{D} \end{bmatrix}_j$$

Now we are in a position to write

$$\begin{Bmatrix} Y_n \\ Y'_n \\ M_n \\ V_n \end{Bmatrix}_j^{\ell} = [F]_j \begin{Bmatrix} Y_n \\ Y'_n \\ M_n \\ V_n \end{Bmatrix}_{j-1}^{r} \tag{33}$$

where $[F]_j = [B]_j [R]_j [B]_j^{-1}$ is, of course, the field matrix which transfers the state vector $\lfloor Y_n \; Y'_n \; M_n \; V_n \rfloor$ across a panel. The matrix $[F]$ is symmetrical about its cross-diagonal, and $[F(b_n)]^{-1} = [F(-b_n)]$.

Now that we can leap panels as we wish, we turn to our last hurdle, the crossing of stringers. To find the point transfer matrix which transfers a state vector across a stringer, we must take a closer look at the stringer. The stringer does not interfere with the continuity of the deflections and slopes in the skin on either side of the line of attachment between the stringer and skin. Thus, very simply,

$$Y_n(r, j) = Y_n(\ell, j)$$

$$Y'_n(r, j) = Y'_n(\ell, j)$$

However, the stringer, because of its elastic and inertial properties, does produce what we will consider to be an abrupt change in the moment and shear in the skin at the line of attachment. To determine the form of the change in the shear and moment components of the state vector we will make use of an existing theory which deals with the bending and torsion of a general thin-walled member of open cross-section. This theory is the result of a number of distinguished engineers, a few of which are: Timoshenko (in 1908); Wagner (in 1929); Goodier (in 1941); and Argyris (in the early 1950's).

The basic deflection equations of the stringer are

$$EI_\eta w_o^{(4)} + EI_{\eta\zeta} v_o^{(4)} = \overline{p}(x)$$

$$EI_\zeta v_o^{(4)} + EI_{\eta\zeta} w_o^{(4)} = \overline{q}(x) \qquad (34)$$

$$EC_w \psi^{(4)} - GC\psi'' = \overline{r}(x)$$

where $I_\eta$, $I_\zeta$, and $I_{\eta\zeta}$ are the centroidal moments of inertia and product moment of inertia; G is the shear modulus of elasticity; the subscript o refers the deflections w and v to the shear center ; $C_w$ is the warping constant of the stringer cross section with respect to the shear center ; C is the Saint-Venant constant of uniform torsion; $\overline{p}$ and $\overline{q}$

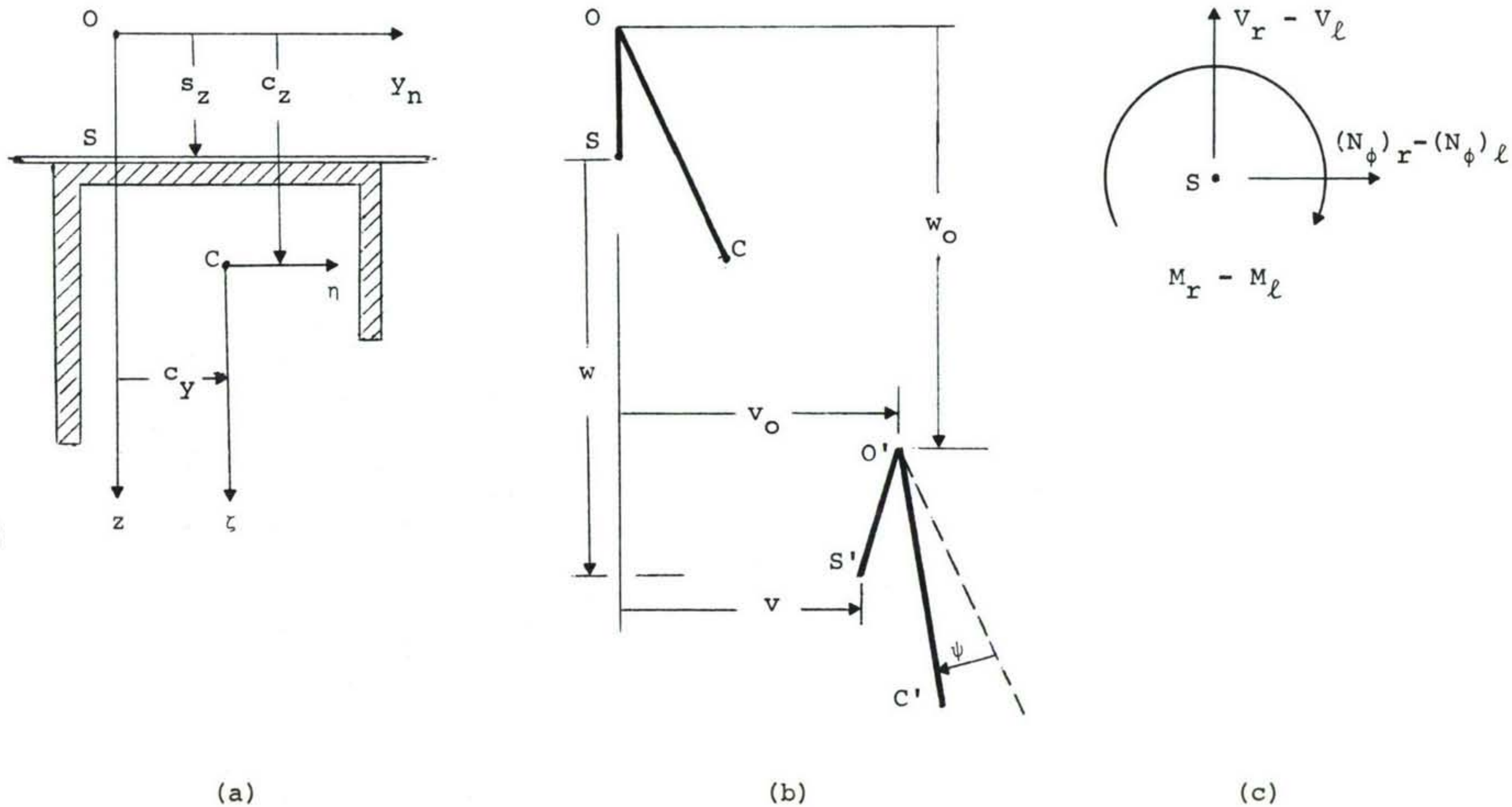


Figure 10. A Typical Stiffening Stringer of Open Thin-walled Type (a) cross section; (b) displacement; (c) forces and moment transferred to stringer from skin.

are the distributed loads at the shear center along the stringer in the z and y directions respectively; $\overline{r}$ is the distributed torque about the shear center; and primes indicate differentiation with respect to x. The first two of equations (34) can be derived by simply (a) superimposing the strains of a stringer cross-section due to curvatures in the two perpendicular planes (plane sections assumed to remain plane); (b) integrating the corresponding stresses to obtain the values of the moments in terms of the deflections; and (c) differentiating to obtain the distributed loads. The derivation of the third of Eqs. (34) can be found in Reference [7], section 36. 8.

The distributed forces and the distributed torque about the shear center are a result of the acceleration of the stringer, and (reversing our original point of view) the differences between the forces and moments in the skin on opposite sides of the point of attachment S. Specifically

$$\overline{p}\,(x) = (-V^{r} + V^{\ell}) - \rho A \ddot{w}_c$$

$$\overline{q}\,(x) = (N^{r}_{\phi} - N^{\ell}_{\phi}) - \rho A \ddot{v}_c \tag{35}$$

$$\overline{r}\,(x) = (M^{r} - M^{\ell}) - (N^{r}_{\phi} - N^{\ell}_{\phi})\, s_z - \rho A \ddot{v}_c\, c_z + \rho A \ddot{w}_c\, c_y - \rho J_c \ddot{\psi}$$

where $N_{\phi}$ is the amplitude of circumferential tension per unit length; $\rho$ is the stringer mass density; A is the stringer cross sectional area; $J_C$ is the stringer area polar moment of inertia about the centroid C; $c_z$, $c_y$, and $s_z$ are distances illustrated in Figure 10, and the subscript c on the deflections refers them to the centroid C.

Our basic equations are expressed in terms of the deflections at the stringer shear center 0 and the stringer centroid C. Our purpose is, however, to express ourselves in terms of the y-coordinate continuous deflections of the panel skin. This can be accomplished by noting that the deflections of the stringer and those of the skin are the same at the point of attachment S. As can be seen from Figure 10, the deflections of the stringer at point S (no subscript) are geometrically related to those at the centroid and shear center as follows

$$
\begin{aligned}
w &= w_o - s_z(1 - \cos\psi) \approx w_o \\
v &= v_o - s_z \psi \\
w_c &= w_o + c_y \psi \\
v_c &= v_o - c_z \psi
\end{aligned}
\tag{36}
$$

Substituting these relations into our equations of motion (34) and (35), and combining and rearranging them, we obtain

$$
\begin{aligned}
M^r - M^\ell &= E C_{ws} \psi^{(4)} - G C \psi'' + EI_\zeta s_z v^{(4)} + EI_{\eta\zeta} s_z w^{(4)} \\
&\quad + \rho J_s \ddot{\psi} - \rho A (c_z - s_z) \ddot{v} + \rho A c_y \ddot{w} \\
V^r - V^\ell &= - EI_\eta w^{(4)} - EI_{\eta\zeta} v^{(4)} - EI_{\eta\zeta} s_z \psi^{(4)} \\
&\quad - \rho A \ddot{w} - \rho A c_y \ddot{\psi} \\
N_\phi^r - N_\phi^\ell &= EI_\zeta v^{(4)} + EI_{\eta\zeta} w^{(4)} + EI_\zeta s_z \psi^{(4)} \\
&\quad + \rho A \ddot{v} + \rho A (c_z - s_z) \ddot{\psi}
\end{aligned}
\tag{37}
$$

where $C_{ws} = C_w + I_\zeta s_z^2$ is the warping constant of the stringer cross-section with respect to S as the center of twist, and $J_s = J_c + A c_y^2 + A (c_z - s_z)^2$ is the area polar moment of inertia of the stringer cross-section with respect to S.

We now recognize that the flat panel system undergoes negligible lateral motion, i. e. mathematically $v = 0$. This is in keeping with the underlying assumptions of our equation of skin motion, Eq. (23). That is, Eq. (23) is a bending theory equation in which bending of the plate produces no horizontal motion of points in the plate middle plane. (On the other hand a bending theory of curved panels will obviously have to include lateral deflections.) So, substituting $v = 0$ and $\frac{\partial w}{\partial y} = \psi$ into Eqs. (37) we obtain

$$M^r - M^\ell = EC_{ws} \frac{\partial^5 w}{\partial x^4 \partial y} - GC \frac{\partial^3 w}{\partial x^2 \partial y}$$

$$+ EI_{\eta\zeta} s_z \frac{\partial^4 w}{\partial x^4} + \rho I_s \frac{\partial^3 w}{\partial y \partial t^2} + \rho A c_y \frac{\partial^2 w}{\partial t^2}$$

(38)

$$V^r - V^\ell = - EI_\eta \frac{\partial^4 w}{\partial x^4} - EI_{\eta\zeta} s_z \frac{\partial^5 w}{\partial x^4 \partial y}$$

$$- \rho A \frac{\partial^2 w}{\partial t^2} - \rho A c_y \frac{\partial^3 w}{\partial y \partial^2 t}$$

We conclude the derivation of the point transfer matrix by substituting the expressions for w, M, and V, Eqs. (24) and (31), into Eqs. (38) to arrive at

$$(M_n)^r - (M_n)^\ell = [EC_{ws} (\frac{n\pi}{\ell})^4 + GC(\frac{n\pi}{\ell})^2 - \omega^2 \rho I_s] Y'_n$$

$$+ [EI_{\eta\zeta} s_z (\frac{n\pi}{\ell})^4 - \omega^2 \rho A c_y] Y_n$$

$$= \hat{d} Y_n + \hat{c} Y'_n \tag{39}$$

$$(V_n)^r - (V_n)^\ell = -[EI_\eta (\frac{n\pi}{\ell})^4 - \omega^2 \rho A] Y_n$$

$$- [EI_{\eta\zeta} s_z - \omega^2 \rho A c_y] Y_n'$$

$$= -\hat{e}Y_n - \hat{d}Y'_n$$

We note that the quantities $\hat{c}$, $\hat{d}$, and $\hat{e}$ are functions of the natural frequency. Thus, in conclusion, we can write

$$\{Z\}_j^r = [G]_j \{Z\}_j^\ell \tag{40}$$

where

$$[G]_j = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hat{d} & \hat{c} & 1 & 0 \\ -\hat{e} & -\hat{d} & 0 & 1 \end{bmatrix}_j$$

and the subscript j associated with $[G]$ indicates that the elements of the point transfer matrix are computed from the physical data of the $j^{th}$ stringer.

The only exceptions to the cross-symmetry of the stringer transfer matrix are the elements $\hat{d}$ and $-\hat{d}$. These exceptions are a result of the stringer cross-section being unsymmetrical; ie. d=0 if the section is symmetrical, and when the direction of transfer of the state vector is reversed causing a far-side eccentricity to become a near-side eccentricity, d becomes -d.

We now can proceed to calculate the natural frequencies and modes of the entire panel array. Let $[T]$, with proper subscripts and superscripts, denote the general transfer matrix relating the state vectors at any two stations on a panel system. For example

$${}_k^\ell[T]_j^r = [F]_k \, [G]_{k-1} \, [F]_{k-1} \cdots [G]_{j+1} \, [F]_{j+1}$$

and

$${}_k^r[T]_j^\ell = [G]_k \, [F]_k \, [G]_{k-1} \cdots [F]_{j+1} \, [G]_j$$

For the purposes of discussion, let us assume that the two extreme ends of the panel system are supported by stringers. In this case we write

$$\{Z\}^{r}_{N} = {}^{r}_{N}[T]^{\ell}_{0}\ \{Z\}^{\ell}_{0} \tag{41}$$

where the state vectors are for the free edges just outside the lines of attachment of the end stringers. Hence in both state vectors the moment and shear are zero; ie.

$$\{Z\}^{r}_{N} = \begin{Bmatrix} Y_n \\ Y'_n \\ 0 \\ 0 \end{Bmatrix}^{r}_{N} \qquad \{Z\}^{\ell}_{0} = \begin{Bmatrix} Y_n \\ Y'_n \\ 0 \\ 0 \end{Bmatrix}^{\ell}_{0}$$

Thus the following two by one matrix equation can be extracted from equation (41).

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = {}^{r}_{N}\begin{bmatrix} t_{31} & t_{32} \\ t_{41} & t_{42} \end{bmatrix}^{\ell}_{0} \begin{Bmatrix} Y_n \\ Y'_n \end{Bmatrix}^{\ell}_{0} \tag{42}$$

For this equation to have a non-trivial solution, we must have the frequency determinant equation

$${}^{r}_{N}\begin{vmatrix} t_{31} & t_{32} \\ t_{41} & t_{42} \end{vmatrix}^{\ell}_{0} = \Delta(\omega) = 0 \tag{43}$$

which can be solved for the natural frequencies of the skin-stringer system. The simplist method of obtaining a numerical solution to this complicated transcendental equation is that of graphing $\Delta(\omega)$. Its zeros are, of course, the natural frequencies.

For a given root $\omega$ of the frequency determinant equation, the associated normal mode can be obtained as follows. Let $Y_n$ at $(\ell, 0)$ be arbitrary. Then the magnitude of $Y_n'$ at $(\ell, 0)$ is determined from equation (42) as

$$Y_n' = -\frac{t_{31}}{t_{32}} Y_n = -\frac{t_{41}}{t_{42}} Y_n$$

Let us say we are interested in the modal deflections of the $j^{th}$ panel. By use of transfer matrices we can obtain

$$\{Z\}_{j-1}^{r} = {}_{j-1}^{\;\;r}[T]_0^{\ell} \begin{Bmatrix} Y_n \\ Y_n' \\ 0 \\ 0 \end{Bmatrix}_0^{\ell} \qquad (44)$$

From this we have $\{Y\}_{j-1}^{r} =$

$$\begin{Bmatrix} Y_n(0) \\ Y_n'(0) \\ Y_n''(0) \\ Y_n'''(0) \end{Bmatrix}_{j-1}^{r} = [B]_j^{-1} \{Z\}_{j-1}^{r} \qquad (45)$$

The elements of $\{Y\}_{j-1}^{r}$ are the constants $\Lambda_0$, $\Lambda_1$, $\Lambda_2$, and $\Lambda_3$ appearing in equation (27a) from which we can calculate $Y_n(y)$.

## VI. CURVED STRINGER-PANEL SYSTEMS

The analysis for curved panels is more involved that that for flat panels. It is possible to approach the problem by means of a distributed mass mathematical model, but to reduce the complexity of the problem we will lump the distributed panel mass along discrete "mass lines" running parallel to the supporting stringers. The mass lines will be numbered ..., j-1, j, j+1, ... . The corresponding masses per unit length of mass line are designated ..., $\mu_{j-1}$, $\mu_j$, $\mu_{j+1}$ ..., and the rotational moments of inertia per unit length of mass line are ..., $J_{j-1}$, $J_j$, $J_{j+1}$, ... . The line masses are connected by massless segments of thin cylindrical shells. This discrete approximation of a curved panel is depicted in Figure 11.

The usefulness of this model is as before dependent upon being able to consider the panels as simply supported at the frames, i.e. along the curved edges. When this ability exists the problem is reducible to one of one dimension, and the same general procedure as carried out in our previous problems will yield a solution.

We will first concern ourselves with developing the field transfer matrix, the matrix which will carry us across the massless shell segments between mass lines. We will base our development of the field transfer matrix on Donnell's shell theory, an approximate theory which is obtained by eliminating those terms of the more exact theory which are of minor importance to the case at hand. For an unloaded and massless cylindrical shell element, the Donnell equations [7] are

$$a^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{2}a(1+\nu)\frac{\partial^2 v}{\partial x \partial \phi} - \nu a \frac{\partial w}{\partial x} = 0$$

$$\frac{1}{2}a(1+\nu)\frac{\partial^2 u}{\partial x \partial \phi} + \frac{\partial^2 v}{\partial \phi^2} + \frac{1}{2}a^2(1-\nu)\frac{\partial^2 v}{\partial x^2} - \frac{\partial w}{\partial \phi} = 0 \qquad (46)$$

$$\nu a \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \phi} - w - k\left(a^4 \frac{\partial^4 w}{\partial x^4} + 2a^2 \frac{\partial^4 w}{\partial x^2 \gamma \phi^2} + \frac{\partial^4 w}{\partial \phi^4}\right) = 0$$

where

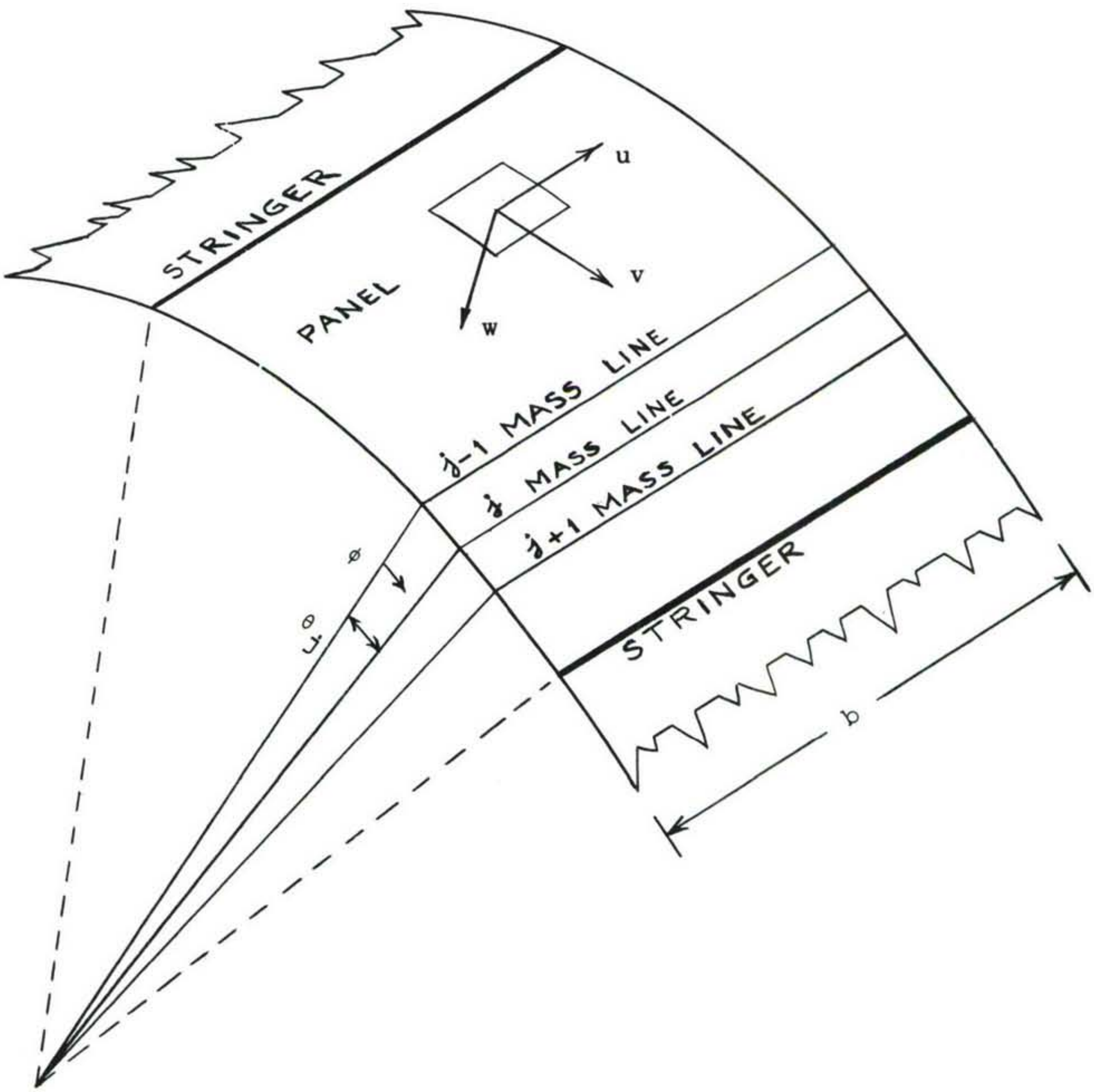


Figure 11. A Typical Curved Panel

$a$ = the radius of curvature of the segment (note that this quantity may change from segment to segment)

$k$ = $h^2/12a^2$

$\nu$ = Poisson's ratio

$\phi$ = angular coordinate of the shell segment ($a\phi$ corresponds to $y$ of the flat panel)

These are two second order and one fourth order partial differential equations in three unknowns, the displacements u, v, and w. They can be combined into the following single equation of the eighth order in one unknown.

$$a^8 k\nabla^8 w + a^4(1-\nu^2) \frac{\partial^4 w}{\partial x^4} = 0 \qquad (47)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2}$ and $\nabla^8 = (\nabla^2)^4$

With simple support conditions at the frames, the solutions for the case of a single frequency oscillation can be written as

$$u = e^{i\omega t} T_n(\phi) \cos \frac{n\pi x}{b}$$

$$v = e^{i\omega t} \Psi_n(\phi) \sin \frac{n\pi x}{b} \qquad (48)$$

$$w = e^{i\omega t} \Phi_n(\phi) \sin \frac{n\pi x}{b}$$

Substituting the above expression for w into equation (47), we find that $\Phi_n(\phi)$ must satisfy the following ordinary differential equation.

$$\Phi_n^{(8)} - 4q_n^2 \Phi_n^{(6)} + 6q_n^4 \Phi_n^{(4)} - 4q_n^6 \Phi_n''$$

$$+ q_n^4 (q_n^4 + p^4)\Phi_n = 0 \qquad (49)$$

where $q_n = \frac{n\pi a}{b}$ and $p = \sqrt[4]{\frac{12a^2(1-\nu^2)}{h^2}}$

The eight characteristic roots of this equation can be written as

$$\pm \sqrt{q_n + \frac{pq_n}{\sqrt{2}} (\pm 1 \pm i)}$$

or, to better suit our purposes, in real and imaginary component form as

$$\pm (\gamma_1 \pm i\delta_1) \qquad \text{and}$$

$$\pm (\gamma_2 \pm i\delta_2)$$

The values of these components work out to be

$$\gamma_1 = \sqrt{q_n}(q_n^2 + \sqrt{2}\, q_n p + p^2)^{1/4} \cos(\tfrac{1}{2} \arctan \frac{p}{\sqrt{2}\, q_n + p})$$

$$\delta_1 = \sqrt{q_n}(q_n^2 + \sqrt{2}\, q_n p + p^2)^{1/4} \sin(\tfrac{1}{2} \arctan \frac{p}{\sqrt{2}\, q_n + p}) \qquad (50)$$

$$\gamma_2 \quad \sqrt{q_n}(q_n^2 - \sqrt{2}\, q_n p + p^2)^{1/4} \cos[\tfrac{1}{2} \arctan\ p/(\sqrt{2}\, q_n - p)]$$

$$\delta_2 = \sqrt{q_n}(q_n^2 - \sqrt{2}\, q_n p + p^2)^{1/4} \sin[\tfrac{1}{2} \arctan\ p/(\sqrt{2}\, q_n - p)]$$

and, in all cases the principal value of the arctangents are to be used. When the characteristic roots are expressed in terms of their real and imaginary components, we can say that Eq. (49) is satisfied by any of the following eight beharmonic functions

$\cosh\gamma_1\phi \cdot \cos\delta_1\phi,$ $\sinh\gamma_1\phi \cdot \sin\delta_1\phi,$

$\cosh\gamma_2\phi \cdot \cos\delta_2\phi,$ $\sinh\gamma_2\phi \cdot \sin\delta_2\phi,$

$\sinh\gamma_1\phi \cdot \cos\delta_1\phi,$ $\cosh\gamma_1\phi \cdot \sin\delta_1\phi,$

$\sinh\gamma_2\phi \cdot \cos\delta_2\phi,$ $\cosh\gamma_2\phi \cdot \sin\delta_2\phi.$

The eight solutions we will use will be linear combinations of the above solutions so as to facilitate the construction of the field transfer matrix. We will write our desired solution in the form

$$\Phi_n(\phi) = \sum_{i=0}^{7} \Lambda_i \, f_i(\phi) \tag{51}$$

and, so as to have even subscripts of f imply even functions of $\phi$, we further write for i=0, 2, 4, and 6

$$\begin{aligned} f_i = {} & \alpha_{i1}\cosh\gamma_1\phi\cos\delta_1\phi + \alpha_{i2}\sinh\gamma_1\phi\sin\delta_1\phi \\ & + \alpha_{i3}\cosh\gamma_2\phi\cos\delta_2\phi + \alpha_{i4}\sinh\gamma_2\phi\sin\delta_2\phi \end{aligned} \tag{52}$$

and to have odd subscripts of f imply odd functions of $\phi$, for i=1, 3, 5 and 7

$$\begin{aligned} f_i = {} & \alpha_{i1}\sinh\gamma_1\phi\cos\delta_1\phi + \alpha_{i2}\cosh\gamma_1\phi\sin\delta_1\phi + \\ & \alpha_{i3}\sinh\gamma_2\phi\cos\delta_2\phi + \alpha_{i4}\cosh\gamma_2\phi\sin\delta_2\phi \end{aligned} \tag{53}$$

Before we can construct a field transfer matrix, we must agree on a suitable state vector. The eight quantities necessary to specify the state of the cylindrical shell segment are

$T_n$, $\Psi_n$, $\Phi_n$, $\frac{1}{a}\Phi'_n$, $(M_\phi)_n$, $(V_\phi)_n$, $(N_\phi)_n$, and $(N_{\phi x})_n$.

These eight quantities in the order given will constitute our state vector. The meaning of the last four quantities is to be seen from the following relations[7]. The second part of these relations are part of Donnell's theory of cylindrical shells.

$$M_\phi = e^{i\omega t}(M_\phi)_n \sin\frac{n\pi x}{\ell} = D\left(\frac{1}{a^2}\cdot\frac{\partial^2 w}{\partial\phi^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$V_\phi = e^{i\omega t} (V_\phi)_n \sin \frac{n\pi x}{\ell} = D \left[ \frac{1}{a^3} \cdot \frac{\partial^3 w}{\partial \phi^3} + (2-\nu) \frac{1}{a} \cdot \frac{\partial^3 w}{\partial \phi \partial x^2} \right]$$

$$N_\phi = e^{i\omega t} (N_\phi)_n \sin \frac{n\pi x}{\ell} = \frac{Eh}{1-\nu^2} \left( \frac{1}{a} \frac{\partial v}{\partial \phi} - \frac{w}{a} + \nu \frac{\partial u}{\partial x} \right)$$

and

$$N_{\phi x} = e^{i\omega t} (N_{\phi x})_n \sin \frac{n\pi x}{\ell} \tag{54}$$

$$= \frac{Eh}{2(1+\nu)} \left( \frac{1}{a} \cdot \frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x} \right)$$

See Figure 12 for visualization of $M_\phi$, $V_\phi$, $N_\phi$, and $N_{\phi x}$. From these relations, Eqs. (54), and their counterparts for the deflections, Eqs. (48), we can see that

$(M_\phi)_n$ depends upon $\Phi''_n$ and $\Phi_n$

$(V_\phi)_n$ depends upon $\Phi'''_n$ and $\Phi'_n$

$(N_\phi)_n$ depends upon $\Psi'_n$, $\Phi_n$, and $T_n$

$(N_{\phi x})_n$ depends upon $T'_n$ and $\Psi_n$

Therefore our initially chosen state vector can be replaced by a state vector composed of the eight quantities $\Phi_n$, $\Phi'_n$, $\Phi''_n$, $\Phi'''_n$, $\Psi_n$, $\Psi'_n$ $T_n$ and $T'_n$. Moreover, from the second and third expressions of Equations (46), we find that $T_n$ can be expressed in terms of $\Psi'_n$, $\Phi_n$, $\Phi''_n$, and $\Phi_n^{(4)}$, $T'_n$, in terms of $\Phi'_n$ $\Psi_n$, and $\Psi''_n$. Thus $\lfloor \Phi_n \; \Phi'_n \; \Phi''_n \; \Phi'''_n \; \Phi_n^{(4)} \; \Psi_n \; \Psi'_n \; \Psi''_n \rfloor$ is still another possible state vector. It is this latter form of the state vector that we shall use as a convenient temporary replacement for our initially chosen, more descriptive state vector in order to determine the essential features of the field transfer matrix. Once this is done we will use a conversion matrix to return to our initial state vector.

At this point we return to equation (51) and resume our search for the field transfer matrix. Another attribute that we will require of our solution is that the $f_i(\phi)$ are chosen so that

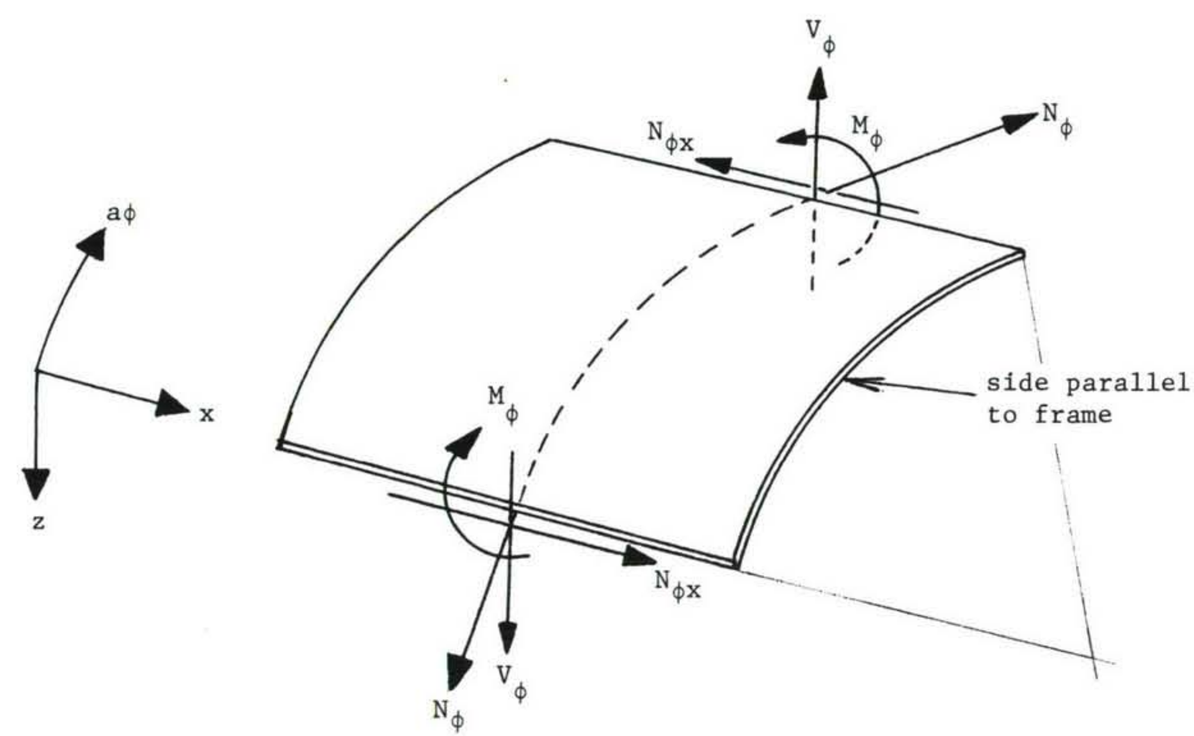


Figure 12. Force and Moment Components of the State Vector Shown in their Positive Directions

$$\Lambda_0 = \Phi_n(0) \qquad \Lambda_1 = \Phi'_n(0)$$
$$\Lambda_2 = \Phi''_n(0) \qquad \Lambda_3 = \Phi'''_n(0)$$
$$\Lambda_4 = \Phi_n^{(4)}(0) \qquad \Lambda_5 = \Psi_n(0) \tag{55}$$
$$\Lambda_6 = \Psi'_n(0) \qquad \Lambda_7 = \Psi''_n(0)$$

**One further consideration before we fully develop our solution: the solution for $\Psi_n$ is not independent of $\Phi_n$. From the second and third equations of (46) we may obtain**

$$\Psi''_n + \nu q_n^2 \Psi_n = \eta_1 \Phi'_n + \eta_2 \Phi'''_n + \eta_3 \Phi_n^{(5)} \tag{56}$$

where $\eta_1 = 1 + \frac{1+\nu}{1-\nu} k\, q_n^4$

$$\eta_2 = -2 \left( \frac{1+\nu}{1-\nu} \right) k\, q_n^2$$

$$\eta_3 = \frac{1+\nu}{1-\nu} k$$

**Now let us define**

$$\Psi_n(\phi) = \sum_{i=0}^{7} \Lambda_i\, g_i(\phi) \tag{57}$$

**Then**

$$\sum_{i=0}^{7} \Lambda_i (g''_i + \nu q_n^2 g_i) = \sum_{i=0}^{7} \Lambda_i (\eta_1 f'_i + \eta_2 f'''_i + \eta_3 f_i^{(5)})$$

**Since this is true for any set of $\Lambda_i$, Eq. (56) becomes a relation between the functions $g_i(\phi)$ and $f_i(\phi)$ for any particular $i$; i.e.**

$$g''_i + \nu q_n^2 g_i = \eta_1 f'_i + \eta_2 f'''_i + \eta_3 f_i^{(5)} \tag{58}$$

**We note that $g_i(\phi)$ is an odd function if $f_i(\phi)$ is an even function, and vice versa. A property of the $f_i$ functions, which we will use again later, is their essential permanence of form under differentiation ex-**

cept that the odd functions become even, and vice versa. Thus a solution for $g_i(\phi)$ may be written in the form

$$g_i(\phi) = \beta_{i1} \sinh \gamma_1\phi \cos \delta_1\phi + \beta_{i2} \cosh \gamma_1\phi \sin \delta_1\phi + \beta_{i3} \sinh \gamma_2\phi \cos \delta_2\phi + \beta_{i4} \cosh \gamma_2\phi \sin \delta_2\phi \qquad (59a)$$

for i=0, 2, 4, 6;and

$$g_i(\phi) = \beta_{i1} \cosh \gamma_1\phi \cos \delta_1\phi + \beta_{i2} \sinh \gamma_1\phi \sin \delta_1\phi + \beta_{i3} \cosh \gamma_2\phi \cos \delta_2\phi + \beta_{i4} \sinh \gamma_2\phi \sin \delta_2\phi$$

for i=1, 3, 5, and 7.

It will be convenient to collect these various definitions of $f_i$ and $g_i$ into matrix form as:

for i=0, 2, 4, or 6.

$$f_i(\phi) = \lfloor \alpha_{i1} \quad \alpha_{i2} \quad \alpha_{i3} \quad \alpha_{i4} \rfloor \{\xi_e\}$$

$$g_i(\phi) = \lfloor \beta_{i1} \quad \beta_{i2} \quad \beta_{i3} \quad \beta_{i4} \rfloor \{\xi_o\} \qquad (60a)$$

for i=1, 3, 5, or 7

$$f_i(\phi) = \lfloor \alpha_{i1} \quad \alpha_{i2} \quad \alpha_{i3} \quad \alpha_{i4} \rfloor \{\xi_o\}$$

$$g_i(\phi) = \lfloor \beta_{i1} \quad \beta_{i2} \quad \beta_{i3} \quad \beta_{i4} \rfloor \{\xi_e\} \qquad (60b)$$

where

$$\{\xi_e\} = \begin{Bmatrix} \cosh \gamma_1\phi & \cos \delta_1\phi \\ \sinh \gamma_1\phi & \sin \delta_1\phi \\ \cosh \gamma_2\phi & \cos \delta_2\phi \\ \sinh \gamma_2\phi & \sin \delta_2\phi \end{Bmatrix}$$

$$\{\xi_0\} = \begin{Bmatrix} \sinh \gamma_1\phi \ \cos \delta_1\phi \\ \cosh \gamma_1\phi \ \sin \delta_1\phi \\ \sinh \gamma_2\phi \ \cos \delta_2\phi \\ \cosh \gamma_2\phi \ \sin \delta_2\phi \end{Bmatrix}$$

To obtain our desired solutions for $\Phi_n$ and $\Psi_n$ we have to evaluate four constants for each of eight $f_i$ functions and each of eight $g_i$ functions or a total of sixty-four constants. We begin by returning to our differential relation between $\Phi_n$ and $\Psi_n$ in order to relate the $\beta$ coefficients to the $\alpha$ coefficients, and thereby reduce immediately the problem to that of evaluating thirty-two coefficients.

As a step preliminary to shaping the relation between the $\alpha$ and $\beta$ coefficients, we want an economical means of describing differentiation When properly organized, straight forward differentiation yields

$$\frac{d}{d\phi} f_i(\phi) = \lfloor \alpha_{i1} \quad \alpha_{i2} \quad \alpha_{i3} \quad \alpha_{i4} \rfloor [D] \{\xi_0\} \qquad (61a)$$

for i=<u>even</u>, and

$$\frac{d}{d\phi} f_i(\phi) = \lfloor \alpha_{i1} \quad \alpha_{i2} \quad \alpha_{i3} \quad \alpha_{i4} \rfloor [D] \{\xi_e\} \qquad (61b)$$

for i= <u>odd</u>, where

$$[D] = \begin{bmatrix} \gamma_1 & -\delta_1 & 0 & 0 \\ \delta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & \gamma_2 & -\delta_2 \\ 0 & 0 & \delta_2 & \gamma_2 \end{bmatrix}$$

Of course, the similar situation that exists for $g_i(\phi)$ can easily be deduced from the above. Therefore differentiation of either $f_i(\phi)$ or $g_i(\phi)$ is equivalent to the application of the matrix of transformation $[D]$ to the opposite column matrix of biharmonic functions; that is, for example, differentiation of $f_i$ with i even, means replacing $\{\xi_e\}$ with $[D]\{\xi_0\}$.

Now we are ready to relate the $\beta$ coefficients to the $\alpha$ coefficients by means of Eq. (58). Direct application of the differential operator yields

$$\lfloor \beta_i \rfloor \left([D]^2 + \nu q_n^2 [I]\right)$$

$$= \lfloor \alpha_i \rfloor \left(\eta_1 [D] + \eta_2 [D]^3 + \eta_3 [D]^5\right)$$

or

$$\lfloor \beta_i \rfloor [K] = \lfloor \alpha_i \rfloor [L]$$

$$\lfloor \beta_i \rfloor = \alpha_i [L][K]^{-1} \qquad (62)$$

where [I] is the identity matrix, and the definitions of [K] and [L] are clear. Let the coefficients for $g_i'(\phi)$ be $\kappa_{i1}$, $\kappa_{i2}$, $\kappa_{i3}$, and $\kappa_{i4}$, and those for $g_i''(\phi)$ be $\tau_{i1}$, $\tau_{i2}$, $\tau_{i3}$, and $\tau_{i4}$. Then

$$\lfloor \kappa_i \rfloor = \lfloor \alpha_i \rfloor [D][L][K]^{-1}$$

$$\lfloor \tau_i \rfloor = \lfloor \alpha_i \rfloor [D][L][K]^{-1}$$

The task that remains is that of specifying the thirty-two $\alpha$ coefficients. We do this by returning our attention to Eq. (51), Eqs.(52) and (53), and Eq. (55). It is our purpose to guarantee that Eq. (55) is satisfied. Since (1) we segregated the functions $f_i(\phi)$ and $g_i(\phi)$ into odd and even functions, since (2) we are interested in the values of these functions for zero arguments, since (3) and even number of derivatives of odd functions and an odd number of derivatives of even functions produce odd functions, and since (4) the value of any odd function of zero argument is zero, we may conveniently treat the odd and even functions separately as follows. The sixteen conditions the even functions must satisfy in order to fulfill Eqs. (55) are concisely stated in the matrix form

$$\begin{bmatrix} f_0(0) & f_0''(0) & f_0^{(4)}(0) & g'_0(0) \\ f_2(0) & f_2''(0) & f_2^{(4)}(0) & g'_2(0) \\ f_4(0) & f_4''(0) & f_4^{(4)}(0) & g'_4(0) \\ f_6(0) & f_6''(0) & f_6^{(4)}(0) & g'_6(0) \end{bmatrix} = [I] \qquad (63a)$$

The sixteen conditions for the odd functions are

$$\begin{bmatrix} f'_1(0) & f'''_1(0) & g_1(0) & g''_1(0) \\ f'_3(0) & f'''_3(0) & g_3(0) & g''_3(0) \\ f'_5(0) & f'''_5(0) & g_5(0) & g''_5(0) \\ f'_7(0) & f'''_7(0) & g_7(0) & g''_7(0) \end{bmatrix} = [I] \qquad (63b)$$

One advantage of treating the even and odd functions separately is that it reduces the size of the above matrix statements. This is a total of thirty-two conditions for the thirty-two $\alpha$ coefficients. Let us look at how these statements in terms of the $f_i(\phi)$ and $g_i(\phi)$ functions may be conveniently rephrased in terms of the $\alpha$ coefficients. Again considering the even functions as examples, $f_o(0) = 1$ means

$$\lfloor \alpha_{01} \quad \alpha_{02} \quad \alpha_{03} \quad \alpha_{04} \rfloor \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} = 1$$

$f_0''(0) = 0$ means

$$\lfloor \alpha_0 \rfloor \; [\mathfrak{D}]^2 \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} = 0$$

$f_0^{(4)}(0) = 0$ means

$$\lfloor \alpha_0 \rfloor \; [\mathfrak{D}]^4 \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} = 0$$

$g'_0(0) = 0$ means

$$\lfloor \alpha_0 \rfloor \; [\mathfrak{D}][L][K]^{-1} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} = 0$$

If we repeat this analysis for the remaining twelve expressions, and combine our results, they are

$$\begin{bmatrix} \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \alpha_{61} & \alpha_{62} & \alpha_{63} & \alpha_{64} \end{bmatrix} [Q] = [I] \qquad (64)$$

where [Q] has as its columns

$$\begin{Bmatrix}1\\0\\1\\0\end{Bmatrix}, \quad [\mathcal{D}]^2\begin{Bmatrix}1\\0\\1\\0\end{Bmatrix}, \quad [\mathcal{D}]^4\begin{Bmatrix}1\\0\\1\\0\end{Bmatrix}, \quad \text{and} \quad [\mathcal{D}][L][K]^{-1}\begin{Bmatrix}1\\0\\1\\0\end{Bmatrix}.$$

Hence, for the even functions

$$[A] = [Q]^{-1} \tag{64a}$$

and for the odd functions

$$[A] = [U]^{-1} \tag{64b}$$

where [U] has as its columns

$$[\mathcal{D}]\begin{Bmatrix}1\\0\\1\\0\end{Bmatrix}, \quad [\mathcal{D}]^3\begin{Bmatrix}1\\0\\1\\0\end{Bmatrix}, \quad [L][K]^{-1}\begin{Bmatrix}1\\0\\1\\0\end{Bmatrix}, \quad \text{and} \quad [\mathcal{D}]^2[L][K]^{-1}\begin{Bmatrix}1\\0\\1\\0\end{Bmatrix}.$$

At this point we have determined all the functions $f_i(\phi)$ and $g_i(\phi)$ in

$$\Phi_n(\phi) = \sum_{i=0}^{7} \Lambda_i \, f_i\,(\phi) \tag{51}$$

$$\Psi_n(\phi) = \sum_{i=0}^{7} \Lambda_i \, g_i\,(\phi) \tag{57}$$

and we did this so that Eqs. (55) are satisfied. Hence the state vector

$$\lfloor \Xi \rfloor = \lfloor \Phi_n \;\; \Phi'_n \;\; \Phi''_n \;\; \Phi'''_n \;\; \Phi_n^{(4)} \;\; \Psi_n \;\; \Psi'_n \;\; \Psi''_n \rfloor \tag{65}$$

can be determined at any value of the argument $\phi$ from its value at $\phi = 0$. That is, in particular we now have the transfer matrix by which to obtain $\{\Xi\}$ at $\phi = \theta_j$.

$$\{\Xi\}_j^{\ell} = [R]_j \{\Xi\}_{j-1}^{r}$$

where the elements of $[R]_j$ are given by

$$r_{k\ell} = \lfloor \alpha_{i1} \quad \alpha_{i2} \quad \alpha_{i3} \quad \alpha_{i4} \rfloor [D]^{k-1} \{\xi(\theta_j)\}$$

for $\ell = i+1$, $k = 1, 2, 3, 4$, and 5

and

$$r_{k\ell} = \lfloor \alpha_{i1} \quad \alpha_{i2} \quad \alpha_{i3} \quad \alpha_{i4} \rfloor [D]^{k-6} [L] [K]^{-1} \{\xi(\theta_j)\}$$

for $\ell = i+1$, $k = 6, 7$, and 8

and where

$$\{\xi(\theta_j)\} = \{\xi_e(\theta_j)\} \quad \text{for } \ell+k = \text{even}$$

$$\{\xi(\theta_j)\} = \{\xi_o(\theta_j)\} \quad \text{for } \ell+k = \text{odd}$$

The $\{\Xi\}$ vector is not suitable for the incorporation of boundry conditions, nor does it conveniently mesh with our analysis of stringers. Therefore we will now convert $\lfloor \Xi \rfloor$ to $\lfloor Z \rfloor = \lfloor T_n \; \Psi_n \; \Phi_n \; \frac{1}{a}\Phi'_n \; (M_\phi)_n (V_\phi)_n (N_\phi)_n (N_{\phi x})_n \rfloor$ The first component of our mixed displacement and force type state vector can be expressed in terms of the components of $\lfloor \Xi \rfloor$ by referring to the third of Eqs. (46). From this we find that

$$T_n(\phi) = \frac{1}{\nu q_n} [\Psi'_n - (1+kq_n^4)\, \Phi_n + 2kq_n^2\, \Phi''_n - k\Phi_n^{(4)}]$$

The second, third, and fourth elements of $\lfloor Z \rfloor$ present no difficulty. Referring to Eqs. (54), we have

$$M_\phi = D\left(\frac{1}{a^2} \frac{\partial^2 w}{\partial \phi^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

So,

$$(M_\phi)_n = \frac{D}{a^2}(\Phi''_n - \nu q_n^2 \Phi)$$

Similarly

$$V_\phi = \frac{D}{a} \frac{\partial}{\partial\phi} \left[ \frac{1}{a^2} \frac{\partial^2 w}{\partial\phi^2} + (2-\nu) \frac{\partial^2 w}{\partial x^2} \right]$$

and

$$(V_\phi)_n = \frac{D}{a^3} [\Phi'''_n - (2-\nu) q_n^2 \Phi'_n]$$

Similarly

$$N_\phi = \frac{Eh}{1-\nu^2} \left( \frac{1}{a} \frac{\partial v}{\partial\phi} - \frac{w}{a} + \nu \frac{\partial u}{\partial x} \right)$$

so

$$(N_\phi)_n = \frac{Eh}{a(1-\nu^2)} (\Psi'_n - \Phi_n - \nu q_n T_n)$$

$$= \frac{D}{a^3} (q_n^4 \Phi_n - 2q_n^2 \Phi''_n + \Phi_n^{(4)})$$

Similarly

$$N_{\phi x} = \frac{Eh}{2(1+\nu)} \left( \frac{1}{a} \frac{\partial u}{\partial\phi} + \frac{\partial v}{\partial x} \right)$$

so

$$(N_{\phi x})_n = \frac{Eh}{2a(1+\nu)} (T'_n + q_n \Psi_n)$$

It follows from the second of Donnell's equations, Eqs. (46) that

$$T'_n = \frac{2}{(1+\nu) q_n} \Psi''_n - \frac{1-\nu}{1+\nu} q_n \Psi_n - \frac{2}{(1+\nu) q_n} \Phi'_n$$

Therefore

$$(N_{\phi x}) = \frac{Eh}{a(1+\nu)^2} \left( \frac{1}{q_n} \Psi''_n + \nu q_n \Psi_n - \frac{1}{q_n} \Phi' \right)$$

Now we can set up our conversion matrix so that

$$\{Z\} = [B]_j \{\Xi\} \tag{66}$$

where the subscript j is associated with [B] indicates that this matrix is to be computed from the physical constants of the $j^{th}$ shell segment. The details of the matrix $[B]_j$ can be found on the following two pages.

The $\{Z\}$ vectors at the two opposite edges of the $j^{th}$ shell segment can now be related by writing

$$\begin{aligned} \{Z\}_j^{\ell} &= [B]_j \, [R]_j \, [B]_j^{-1} \, \{Z\}_{j-1}^{r} \\ &= [F]_j \, \{Z\}_{j-1}^{r} \end{aligned} \tag{67}$$

This completes our development of the necessary field transfer matrix, and we now turn our attention to the point transfer matrix mecessary to account for the effects produced by a stringer and a mass line. Eqs. (37) are our starting point for the case of a stringer. If we insert the series expansions for the deflections, monent, shear, and tensile force into these equations, we can write the result for a stringer as

$$\begin{aligned} (M_\phi)_n^r - (M_\phi)_n^\ell &= \hat{c} \; \frac{1}{a} \Phi'_n + \hat{d} \, \Phi_n + \hat{f} \, \Psi_n \\ (V_\phi)_n^r - (V_\phi)_n^\ell &= \hat{e} \, \Phi_n - \hat{d} \frac{1}{a} \, \Phi'_n - \hat{g} \, \Psi_n \\ (N_\phi)_n^r - (N_\phi)_v^\ell &= \hat{h} \, \Psi_n + \hat{f} \frac{1}{a} \, \Phi'_n + \hat{g} \, \Phi_n \end{aligned} \tag{68}$$

where the quantities $\hat{c}$ , $\hat{d}$, and $\hat{e}$ are detailed in Eqs. (39), and

$$
[B]_j = \begin{bmatrix}
-\frac{1+kq_m}{\nu q_m} & 0 & \frac{2kq_m}{\nu} & 0 & -\frac{k}{\nu q_m} & 0 & \frac{1}{\nu q_m} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{a} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\nu D q_m^2}{a^2} & 0 & \frac{D}{a^2} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{(2-\nu) D q_m^2}{a^3} & 0 & \frac{D}{a^3} & 0 & 0 & 0 & 0 \\
+\frac{D q_m^4}{a^3} & 0 & -\frac{2 D q_m^2}{a^3} & 0 & +\frac{D}{a^3} & 0 & 0 & 0 \\
0 & \frac{-Eh}{a q_m (1+\nu)^2} & 0 & 0 & 0 & \frac{\nu E h q_m}{a(1+\nu)^2} & 0 & \frac{Eh}{a q_m (1+\nu)^2}
\end{bmatrix}_j
$$

$$[B]_j^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu q_m^2 & 0 & \frac{a^2}{D} & 0 & 0 & 0 \\ 0 & 0 & 0 & (2-\nu) a q_m^2 & 0 & \frac{a^3}{D} & 0 & 0 \\ 0 & 0 & -(1-2\nu) q_m^4 & 0 & \frac{2a^2 q_m^2}{D} & 0 & \frac{a^3}{D} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu q_m & 0 & 1 & 0 & 0 & 0 & \frac{a(1-\nu^2)}{Eh} & 0 \\ 0 & -\nu q_m^2 & 0 & a & 0 & 0 & 0 & \frac{a(1+\nu)^2 q_m}{Eh} \end{bmatrix}_j$$

$$\hat{f} = EI_{\zeta}\, s_z \left(\frac{n\pi}{\ell}\right)^4 + \rho A(c_z - s_z)\omega^2$$

$$\hat{g} = EI_{\eta\zeta}\left(\frac{n\pi}{\ell}\right)^4 \qquad (69)$$

$$\hat{h} = EI_{\zeta}\left(\frac{n\pi}{\ell}\right)^4 - \rho A\omega^2$$

The corresponding contributions of a panel skin mass line located at the same station are $-J\omega^2$, $+\mu\omega^2$, and $-\mu\omega^2$ respectively. Thus we may compose a general point transfer matrix as

$$[G]_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \hat{f} & \hat{d} & \hat{c}-J\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\hat{g} & -\hat{e}+\mu\omega^2 & -\hat{d} & 0 & 1 & 0 & 0 \\ 0 & \hat{h}-\mu\omega^2 & \hat{g} & \hat{f} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_j$$

Of course, if no stringer exists at station j, we merely set $\hat{c} = \hat{d} = \hat{e} = \hat{f} = \hat{g} = \hat{h} = 0$. On the other hand, if there is no panel mass line, $\mu_j = J_j = 0$.

Now that we have described the basic transfer matrices [F] and [G], we can relate the state vectors at any two locations in the panel system. We determine the natural frequencies as before by relating the state vectors at the two ends of the panel row. For example, if both ends of the panel row are simply supported (i.e. $\Psi_n = \Phi_n = (M_\phi)_n = (N_\phi)n = 0$), these two vectors and the connection between them are, for any value of n

$$\begin{Bmatrix} T \\ 0 \\ 0 \\ \Phi'/a \\ 0 \\ v \\ 0 \\ N_{\Phi x} \end{Bmatrix}_N^{\ell} = {}_N^{\ell}[T]_0^r \begin{Bmatrix} T \\ 0 \\ 0 \\ \Phi'/a \\ 0 \\ V \\ 0 \\ N_{\Phi x} \end{Bmatrix}_0^r$$

where ${}_N^{\ell}[T]_0^r = [F]_N \, [G]_{N-1} \cdots [G]_1 \, [F]_1.$

Then proceding as before, the determinant equation for the natural frequencies is, for any n,

$$\begin{vmatrix} t_{21} & t_{24} & t_{26} & t_{28} \\ t_{31} & t_{34} & t_{36} & t_{38} \\ t_{51} & t_{54} & t_{56} & t_{58} \\ t_{71} & t_{74} & t_{76} & t_{78} \end{vmatrix}_{N\;\;0}^{\ell\;\;r} = \Delta(\omega) = 0 \qquad (70)$$

The determination of the normal mode associated with a natural frequency can in theory be approached in a manner analogous to the case of flat panels. However, in both the flat and curved panel cases, difficulties may arise when actually carrying out these frequency and mode shape computations.

## VII. NUMERICAL COMPUTATIONS

Two types of difficulties can arise when carrying out the numerical solution of any of the frequency determinant equations. Reference [2], section 7.1 is a good and sufficient explanation of these difficulties. The first of these difficulties is the loss of significant figures when the frequency determinant is reduced in an ordinary manner. This case is exemplified by Eq. 7-3 of Reference [2]. The second difficulty arises when the elastic supports are very stiff in comparison with the remainder of the structure. The error introduced is explained by Eq. 7-4. In sections 7-2 and 7-3 there are presented two methods of avoiding these obstacles. The first method, the Delta Matrix method, is limited in the size of the frequency determinant to which it can be applied. It is not difficult to apply successfully to a four by four frequency determinant, such as that of a flat panel system, and by dint of much programming, it can also be applied to an eight by eight frequency determinant. A larger determinant would probably be beyond the capabilities of most present day digital computors. In section 7-4 a trial and error scheme is explained. A short summary of this method is available in Reference [1], section IV.

Even when a direct graphical solution of the frequency determinant equation is no problem, there is a possibility that the values of $\Delta(\omega)$ may pass through zero very abruptly[9]. Thus, while it is easy to make a very good estimate of $\omega_n$ per se, it is very difficult to obtain near zero values of $\Delta(\omega)$ for the purpose of computing the mode shapes. As a result, if many transfer matrices are necessary to describe the structure, the error in the initial state vector will grow to the point where the final state vector is not even approximately representative of the given boundry condition at that end. A means of circumventing this error accumulation is that of adopting a forced vibration point of view. To begin, sinusoidal shear forces (or moments) of frequencies just on either side of the natural frequency of interest are applied in turn to the structure, and the respective states of the structure are calculated. If it is found, as would be expected, that the computed states corresponding to the two such frequencies are very nearly the same, then it may be concluded that both states are dominated by the same mode--the mode corresponding to the bracketed natural frequency. Therefore it is reasonable to call the averages of the deflection components of these states the mode shapes of that nearby natural frequency.

The following example will illustrate the features of the forced vibration procedure of calculating modal states. Let us apply a sinusoidal shear loading of a unit amplitude at station s of a curved panel with simple support end conditions. By combining our successive transfer matrix equations we arrive at

$$\{Z\}^{\ell}_{N} = {}^{\ell}_{N}[T]^{r}_{0}\,\{Z\}^{r}_{0} + {}^{\ell}_{N}[T]^{r}_{s} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

where the transfer matrices are evaluated at one of the two selected frequencies. Again we make use of the boundry conditions and reduce the above eight by one matrix equation to the following four by one equation

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}^{\ell}_{N} = {}^{\ell}_{N}\begin{bmatrix} t_{21} & t_{24} & t_{26} & t_{28} \\ t_{31} & t_{34} & t_{36} & t_{38} \\ t_{51} & t_{54} & t_{56} & t_{58} \\ t_{71} & t_{74} & t_{76} & t_{78} \end{bmatrix}^{\ell}_{0} \begin{Bmatrix} T \\ \phi'/a \\ V \\ N_{\phi} \end{Bmatrix}^{\ell}_{0} + {}^{\ell}_{N}\begin{Bmatrix} t_{26} \\ t_{36} \\ t_{56} \\ t_{76} \end{Bmatrix}^{r}_{s}$$

This matrix equation can be solved for $\lfloor T \quad \phi'/a \quad V \quad N_{\phi} \rfloor^{\ell}_{0}$ by inverting the four by four reduced transfer matrix. This inversion is always possible even in the case of an undamped model analysis because the determinant of this matrix is identical to $\Delta(\omega)$ which, of course, is non-zero by our choice of a frequency other than a natural frequency. Furthermore, the more abrupt the change in the value of $\Delta(\omega)$, i.e. the more acute the difficulty, the closer the values of the selected frequencies can be to the natural frequency. (Differences of 0.1 radians have been satisfactory in all respects in the cases known to the author.) It is of course necessary to avoid nodal points when applying the driving force or moment. This can be assured by altering the point of application of the driving load and then comparing the results.

## References


1. Y. K. Lin, et al,
"Free Vibration of Continuous Skin-Stringer Panels with Non-Uniform Stringer Spacing and Panel Thickness", AFML-TR-64-347, Part I, February 1965.

2. E. C. Pestel and F. A. Leckie,
"Matrix Methods in Elastomechanics", McGraw-Hill, 1963.

3. H. Holzer,
"Die Berechnung Der Drehschwingungen", Springer-Verlag, Berlin, 1921.

4. N. O. Mykelstad,
"A New Method of Calculating Natural Modes of Uncoupled Bending Vibration of Airplane Wings and Other Types of Beams," J. Aero. Sci., April, 1944, p. 153.

5. D. I. G. Jones, J. P. Henderson and G. H. Bruns,
"Use of Tuned Viscoelastic Dampers for Reduction of Vibration in Aerospace Structures" Proceedings of the Thirteenth Air Force Science and Engineering Symposium, 1967.

6. Y. K. Lin,
"Probabalistic Theory of Structural Dynamics", McGraw-Hill, 1967.

7. W. Flugge,
"Handbook of Engineering Mechanics", McGraw-Hill, 1962.

8. Y. K. Lin,
"Free Vibration of Continuous Skin-Stringer Panels", Journal of Applied Mechanics, Dec. 1960.

9. T. J. McDaniel and B. K. Donaldson,
"Free Vibration of Continuous Skin-Stringer Panels with Non-Uniform Stringer Spacing and Panel Thickness". AFML-TR-64-347, Part II.

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13. ABSTRACT

This report offers an introduction to the transfer matrix method of analyzing the dynamic behavior of common engineering structures, followed by an explanation of the application of the transfer matrix method to an array of aircraft panels which are continuous over supporting stringers. The skin-stringer problem, important to the prediction of fatigue failures, is discussed for rather general conditions. The rectangular panels may vary in thickness and length while the stringers may vary in cross-sectional shape and size. The panels may be flat or curved, and the curved panels may vary in radius of curvature.

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