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COMPARISON BETWEEN A VON NEUMANN-RICHTMYER HYDROCODE (AFWL'S PUFF) AND A LAX-WENDROFF HYDROCODE

Darrell Hicks

Robert Pelzl

TECHNICAL REPORT NO. AFWL-TR-68-112

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AIR FORCE WEAPONS LABORATORY
Air Force Systems Command
Kirtland Air Force Base
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FOREWORD

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This report has been reviewed and is approved.

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ABSTRACT

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A comparison between two one-dimensional Lagrangian hydrocodes has been made. The two hydrocodes are a von Neumann-Richtmyer hydrocode (AFWL's PUFF) and a Lax-Wendroff hydrocode (the two-step version with artificial viscosity). The comparison was made by applying the hydrocode test problems as described in HYDROCODE TEST PROBLEMS, AFWL-TR-67-127, February 1968. The most apparent difference between the von Neumann-Richtmyer hydrocode and the Lax-Wendroff is the greater tendency of the Lax-Wendroff scheme to oscillate. In those flows in which there are no strong shocks or strong rarefactions or vacuums, the Lax-Wendroff scheme is more accurate. However, in those flows in which there are strong shocks or strong rarefactions or vacuums the von Neumann-Richtmyer scheme is more accurate. The Lax-Wendroff scheme cannot handle vacuums because of the use of the specific volume instead of the density as a fluid variable. It appears that it might be possible to combine the better features of the von Neumann-Richtmyer and the Lax-Wendroff schemes to produce a better hydrocode.

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SECTION I

INTRODUCTION

This report is the description of a comparison between AFWL's PUFF hydrocode and LAX-WENDROFF two-step method with viscosity. The basis for comparison is the solutions these hydrocodes produce to a series of hydrocode test problems. The problems involve shocks and rarefactions and interactions. The hydrocode test problem solutions are known exactly. Brief descriptions of these test problems will be given here. For more details see "Hydrocode Test Problems" AFWL-TR-67-127.

1. THE PUFF HYDROCODE

Let the points of a rectangular network with spacings Δx and Δt be denoted by x_ℓ, t^n, ($\ell = 0, 1, 2, \ldots, L$: $n = 0, 1, 2, \ldots$). There will also be occasion to deal with intermediate points, having coordinates $x_{\ell+\frac{1}{2}} \equiv \frac{1}{2}(x_{\ell+1} + x_\ell)$, $t^{n+\frac{1}{2}} \equiv \frac{1}{2}(t^{n+1} + t^n)$. To facilitate the writing, introduce abbreviations such as $V^n_{\ell+\frac{1}{2}} \equiv V(x_{\ell+\frac{1}{2}}, t^n)$, etc.

$$\frac{U_\ell^{n+\frac{1}{2}} - U_\ell^{n-\frac{1}{2}}}{\Delta t} = - \frac{P^n_{\ell+\frac{1}{2}} + q^{n-\frac{1}{2}}_{\ell+\frac{1}{2}} - P^n_{\ell-\frac{1}{2}} - q^{n-\frac{1}{2}}_{\ell-\frac{1}{2}}}{(ZM_{\ell+\frac{1}{2}} + ZM_{\ell-\frac{1}{2}})/2} \qquad (1)$$

is PUFF's difference approximation to $\rho_0 \frac{\partial U}{\partial t} = - \frac{\partial (P+q)}{\partial x}$, where ZM is the zone mass, U is the fluid velocity, P is the fluid pressure and q is the artificial viscosity.

$$\frac{X_\ell^{n+1} - X_\ell^n}{\Delta t} = U_\ell^{n+\frac{1}{2}} \qquad (2)$$

is PUFF's difference approximation to $\frac{\partial X}{\partial t} = U$, where X is fluid position.

$$\rho^{n+1}_{\ell-\frac{1}{2}} = \frac{ZM_{\ell-\frac{1}{2}}}{X^{n+1}_{\ell} - X^{n+1}_{\ell-1}} \tag{3}$$

is PUFF's difference approximation to $\frac{\rho_0}{\rho} = \frac{\partial X}{\partial x}$, where ρ is the fluid density. Now let $\Delta U = U^{n+\frac{1}{2}}_{\ell} - U^{n+\frac{1}{2}}_{\ell-1}$.

Then PUFF's q is given by

$$q^{n+\frac{1}{2}}_{\ell-\frac{1}{2}} = \left(\Delta U \cdot C_0 - C_1 \cdot CS^{n-1}_{\ell-\frac{1}{2}}\right)\Delta U \cdot \frac{\left(\rho^{n+1}_{\ell-\frac{1}{2}} + \rho^{n}_{\ell-\frac{1}{2}}\right)}{2} \tag{4}$$

where

$$C_0 = 1.8$$

$$C_1 = .25$$

CS = isothermal sound speed

$$CS^2 \equiv \left.\frac{dP}{d\rho}\right|_{e\ \mathrm{const.}}$$

e is specific internal energy.

$$0 = \frac{e^{n+1}_{\ell-\frac{1}{2}} - e^{n}_{\ell-\frac{1}{2}}}{\Delta t} + \frac{\left(P^{n+1}_{\ell-\frac{1}{2}} + q^{n+\frac{1}{2}}_{\ell-\frac{1}{2}} + P^{n}_{\ell-\frac{1}{2}} + q^{n-\frac{1}{2}}_{\ell-\frac{1}{2}}\right)}{2} \cdot \frac{\Delta U}{ZM_{\ell-\frac{1}{2}}} \tag{5}$$

is PUFF's difference approximation to

$$0 = \frac{\partial e}{\partial t} + (P+q)\,\frac{1}{\rho_0}\,\frac{\partial U}{\partial x}$$

which results from

$$0 = \frac{\partial e}{\partial t} + (P+q)\frac{\partial V}{\partial t} \text{ and } \frac{\partial V}{\partial t} = \frac{1}{\rho_0}\frac{\partial U}{\partial x}$$

where $V = \frac{1}{\rho}$ is the specific volume.

Lastly, the equation of state:

$$P^{n+1}_{\ell-\frac{1}{2}} = P\left(e^{n+1}_{\ell-\frac{1}{2}}, \rho^{n+1}_{\ell-\frac{1}{2}}\right) \qquad (6)$$

PUFF's method of solution is this: Suppose all quantities are known for superscript n or $n-\frac{1}{2}$ (this is referred to as being at cycle n). Compute $U^{n+\frac{1}{2}}_{\ell}$ for each ℓ from (1), then compute X^{n+1}_{ℓ} for each ℓ from (2), then compute $\rho^{n+1}_{\ell-\frac{1}{2}}$ for each ℓ from (3), then compute $q^{n+\frac{1}{2}}_{\ell-\frac{1}{2}}$ for each ℓ from (4), then compute $e^{n+1}_{\ell-\frac{1}{2}}$ for each ℓ by simultaneously solving (13) and (14). At this point all variables have been advanced to cycle n+1. Next PUFF does its time-step computation.

$$\Delta t = .9 \min_{\ell} \frac{X^{n+1}_{\ell} - X^{n+1}_{\ell-1}}{CS^{n+1}_{\ell-\frac{1}{2}}(1+2\cdot C_1) - .4C_0^2\Delta U} \qquad (7)$$

where, as in the q calculation,

$$C_0 = 1.8, \; C_1 = .25$$

Remember

$$CS^2 \equiv \left.\frac{dP}{d\rho}\right|_{e \text{ const.}}$$

Therefore CS is the isothermal sound speed.

The isentropic sound speed is defined by

$$C^2 \equiv \left.\frac{dP}{d\rho}\right|_{S\ \text{const.}}$$

where S is the entropy.

For a γ - law gas $C^2 = \gamma CS^2$. Therefore for ΔU very small

$$\Delta t \approx \frac{.9\ \gamma}{(3/2)} \min_{\ell} \frac{X_{\ell}^{n+1} - X_{\ell-1}^{n+1}}{C_{\ell-\frac{1}{2}}^{n+1}}$$

$$\frac{.9\ \gamma}{(3/2)} \approx .7 \text{ for } \gamma = 1.4$$

If

$$\Delta t = \theta \min_{\ell} \frac{X_{\ell}^{n+1} - X_{\ell-1}^{n+1}}{C_{\ell-\frac{1}{2}}^{n+1}}$$

θ is called the effective CFL number.

For further details about PUFF see AFWL-TR-66-48 and AFWL-TR-67-127.

2. THE LAX-WENDROFF METHOD

The LAX-WENDROFF two-step method with viscosity uses

$$U_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \tfrac{1}{2}\left(U_{j+1}^{n} + U_{j}^{n}\right) - \tfrac{1}{2}\left(\frac{\Delta t}{\Delta z} + q\right)\left(F_{j+1}^{n} - F_{j}^{n}\right) \tag{8}$$

and

$$U_{j}^{n+1} = U_{j}^{n} - \frac{\Delta t}{\Delta z}\left(F_{j+\frac{1}{2}}^{n+\frac{1}{2}} - F_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) \tag{9}$$

as the difference approximation to

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial z} = 0$$

$$q = \frac{b\left|d^n_{j+1} - d^n_j\right|}{\left(d^n_{j+\frac{1}{2}}\right)^2} \tag{10}$$

where $d = V_0c/V$, c is isentropic sound speed, V is specific volume and V_0 is a constant with dimensions of specific volume defined by

$$z = V_0\int\rho_0(x)dx$$

In our case $V_0 = 1$ therefore, z is the Lagrangian mass variable. b is a dimensionless parameter which was chosen to be .5.

$$U = \begin{pmatrix} V \\ u \\ E \end{pmatrix}, \quad F(U) = V_0\begin{pmatrix} -u \\ P \\ Pu \end{pmatrix}$$

where $E = e + \frac{1}{2}u^2$ and e is the fluid's specific internal energy, u is the fluid velocity, and P is the fluid pressure.

The time step restriction is

$$\Delta t \leq \left(\left(1 + \frac{b^2}{4}\right)^{\frac{1}{2}} - \frac{b}{2}\right) \cdot \min \frac{X_j - X_{j-1}}{C_{j-\frac{1}{2}}}$$

where X is fluid position.

For

$$b = \tfrac{1}{2}, \left(1 + \frac{b^2}{4}\right)^{\frac{1}{2}} - \frac{b}{2} = .78$$

LAX-WENDROFF's method of solution:

Suppose all quantities are known for superscript n. Compute $U^{n+\frac{1}{2}}_{j+\frac{1}{2}}$ for each j from (8) then compute U^{n+1}_j for each j from (9). Now all variables are advanced to cycle n+1. Next LAX-WENDROFF does its time step computation

$$\Delta t = \theta \min_{\ell} \frac{X_{\ell}^{n+1} - X_{\ell-1}^{n+1}}{C_{\ell-\frac{1}{2}}^{n+1}}$$

where $\theta \leq .78$.

For more details see Richtmyer and Morton: Difference Methods for Initial Value Problems, Interscience Publishers, a division of John Wiley and Sons, 1967.

3. THE HYDROCODE TEST PROBLEMS

Since the geometry is one-dimensional slab, the problems may all be thought of as flows in a smooth pipe of constant cross section. There are seven problems. The first problem is the flow that results from a piston moving into the gas with a constant velocity. The second problem is the flow that results from pulling a piston away from the gas with a constant velocity. The third problem is the flow that results from a piston moving into the gas with a constant acceleration. The fourth problem is the flow that results from pulling the piston away from the gas with a constant acceleration. The fifth problem is the flow that results by removing a partition between two different states of the gas at rest. The sixth problem is the flow that results from the collision of two shock waves. The seventh problem is the flow that results when one shock overtakes another one. For more details see AFWL-TR-67-127.

SECTION II

COMPARISON OF THE HYDROCODES

For each test problem, the exact solution, the PUFF solution, and the LAX-WENDROFF solution will be described. In describing the PUFF and LAX-WENDROFF solutions an error table and graphs of their solutions will be used to compare with graphs of the exact solutions. In the error table the numbers presented are labeled Sum Abs. Error, Sum Sqr. Error, and Maximum Error. These numbers are now defined. Let $P_P(J)$ be the PUFF pressure in zone J and let $P_E(J)$ be the exact pressure in zone J. Let P_M be the maximum of the $P_E(J)$.

$$\text{Sum Abs. Error (for P)} = \sum_J \frac{|P_P(J) - P_E(J)|}{P_M}$$

$$\text{Sum Sqr. Error (for P)} = \frac{\sum_J \left(P_P(J) - P_E(J)\right)^2}{P_M^2}$$

$$\text{Maximum Error (for P)} = \frac{\max_J |P_P(J) - P_E(J)|}{P_M} \, \mathrm{SGN}\left(P_P(J_M) - P_E(J_M)\right)$$

J_M is the zone index of the maximum error and SGN is the sign function. The error functions are likewise defined for the velocity, density, and energy (specific internal).

Also in the error table are presented the sums of the internal energy, kinetic energy, and total energy of the exact solution, PUFF solution, and LAX-WENDROFF solution (the unit is ergs). In addition, the error table contains the problem time, computer time (CP time on the CDC 6600), cycle number, and the number of active zones.

The graphs are organized in this manner: pressure and density are plotted in the same graph as are velocity and energy (specific internal).

1. TEST PROBLEM SCTP-I

a. The Exact Solution

In this problem a piston moves to the right into the gas at a constant velocity. The solution has a steady profile. The solution profile is two constant states separated by the shock discontinuity. That is, each fluid parameter (pressure, density, fluid velocity, etc.) is a constant from the piston face to the shock and each is another constant to the right of the shock. The symbols used to describe the problem further are

C_ℓ sound speed to the left of the shock

C_r sound speed to the right of the shock

P_ℓ pressure to the left of the shock

P_r pressure to the right of the shock

ρ_ℓ density to the left of the shock

ρ_r density to the right of the shock

V_ℓ specific volume to the left of the shock

V_r specific volume to the right of the shock

v_ℓ fluid velocity to the left of the shock

v_P piston velocity

v_r fluid velocity to the right of the shock

v_S shock velocity

X_P piston position

X_Q quiet zone, a position far enough to the right so that the gas is still at rest; energy sums are taken out to X_Q

X_S shock position

There are two variations of SCTP-I and these are denoted SCTP-I-A and SCTP-I-B. For SCTP-I-A the piston is started at 0, and the shock is started at 50 meters, with the fluid parameters on the right at

$$P_r = 10^4 \text{ dynes/cm}^2$$

$$\rho_r = 10^{-6} \text{ gm/cm}^3$$

$$v_r = 0. \text{ cm/sec}$$

X_Q = 300 meters

This yields

$C_r = \sqrt{\gamma} \times 10^5$ cm/sec, which for γ=1.4 yields

$C_r = 1.18 \times 10^5$ cm/sec

Then one sets

$v_P = C_r = 1.18 \times 10^5$ cm/sec and this yields

$v_\ell = 1.18 \times 10^5$ cm/sec

$v_S = 2.09 \times 10^5$ cm/sec

$P_\ell = 3.47 \times 10^4$ dynes/cm^2

$V_\ell = 4.34 \times 10^5$ cm^3/gm

This problem is run for .1 second, with initial zones of 1 meter. For SCTP-I-B the shock is again started at 50 meters and with the fluid parameters on the right at

$P_r = 10^4$ dynes/cm^2

$\rho_r = 10^{-6}$ gm/cm^3

v_r = 0. cm/sec

X_Q = 300 meters

This yields

$C_r = 1.18 \times 10^5$ cm/sec

Then one sets

$v_P = 100\ C_r = 1.18 \times 10^7$ cm/sec and this yields

$v_\ell = 1.18 \times 10^7$ cm/sec

$P_\ell = 1.68 \times 10^8$ dynes/cm^2

$V_\ell = 1.67 \times 10^5$ cm^3/gm

$C_\ell = 6.26 \times 10^6$ cm/sec

$v_S = 1.42 \times 10^7$ cm/sec

This problem is run for 10^{-3} seconds with initial zones of 1 meter.

b. The PUFF Solution

The regions where the largest errors occurred were the regions where the shock was initially and where the shock is currently. See Tables I-A and I-B and Figures I-A and I-B.

c. The LAX-WENDROFF Solution

SCTP-I-A was run with a viscosity factor of .5 and a time factor of .78. SCTP-I-B was run with a viscosity factor of .5 and a time factor of .25. In order to get SCTP-I-B to run it was necessary to start off with 10 time steps with zero viscosity factor and .025 time factor.

The regions where the largest errors occurred were the regions where the shock was initially and where the shock is currently. The main difference between the PUFF and LAX-WENDROFF solutions is the oscillations behind the shock front. The oscillations are much more pronounced in the LAX-WENDROFF code. See Figures I-A and I-B and Tables I-A and I-B.

Table I-A

ERRORS ON SCTP-I-A

PUFF

Problem time = .1 sec
Computer time = 67 sec

Cycle = 1033
Number of Active Zones = 266

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.979	.407	+ .335	Current shock position
Velocity	1.42	.686	+ .578	Current shock position
Density	.996	.370	+ .289	Current shock position
Energy	.680	.271	+ .198	Current shock position
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	1.32369×10^9	2.26965×10^8	1.55066×10^9	
PUFF	1.32405×10^9	2.26327×10^8	1.55038×10^9	

LAX-WENDROFF

Problem time = .1 se.
Computer time = 186 sec

Cycle = 458
Number of Active Zones = 301

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.489	.313	- .292	Current shock position
Velocity	.551	.334	- .295	Current shock position
Density	.555	.256	- .218	Current shock position
Energy	.362	.150	+ .0988	Initial shock position
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	1.32369×10^9	2.26965×10^8	1.55066×10^9	
LAXWEN	1.32399×10^9	2.26790×10^8	1.55078×10^9	

Table I-B

ERRORS ON SCTP-I-B

PUFF

Problem time = 1. x 10^{-3} sec
Computer time = 74 sec

Cycle = 1463
Number of Active Zones = 197

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	1.06	.683	+ .644	Current shock position
Velocity	1.86	1.08	+ .850	Current shock position
Density	1.78	.799	+ .577	Current shock position
Energy	2.76	1.32	+ .865	Current shock position

	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	3.09498 x 10^{12}	3.09324 x 10^{12}	6.18823 x 10^{12}	
PUFF	3.09791 x 10^{12}	3.08713 x 10^{12}	6.18504 x 10^{12}	

LAX-WENDROFF

Problem time = 1. x 10^{-3} sec
Computer time = 661 sec

Cycle = 1762
Number of Active Zones = 302

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	1.33	.699	- .663	Current shock position
Velocity	.721	.396	- .372	Current shock position
Density	1.91	.636	- .492	Current shock position
Energy	1.67	.611	+ .449	Initial shock position

	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	3.09498 x 10^{12}	3.09324 x 10^{12}	6.18823 x 10^{12}	
LAXWEN	3.09016 x 10^{12}	3.08281 x 10^{12}	6.17297 x 10^{12}	

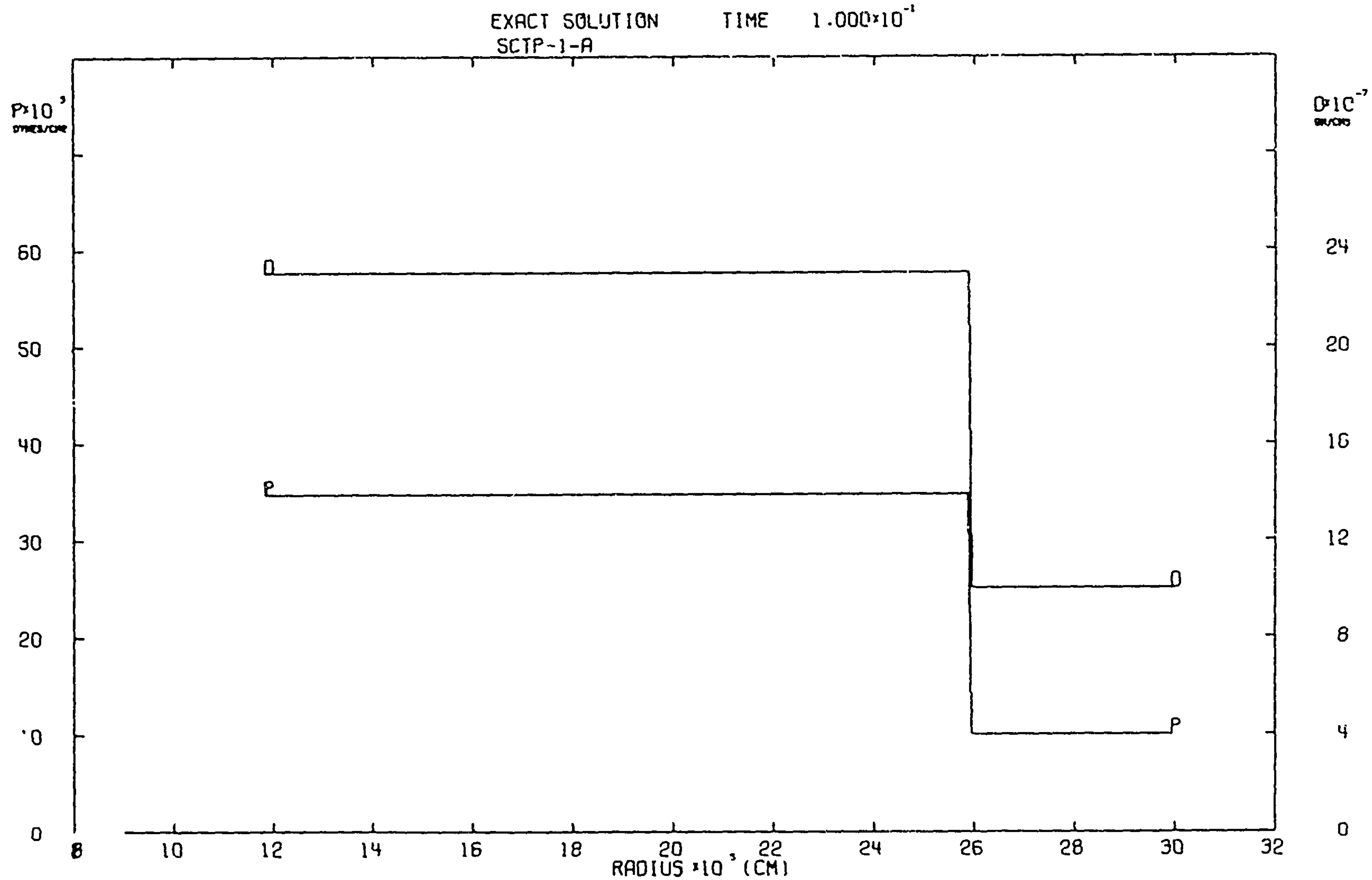

Figure I-A. PD-EXACT

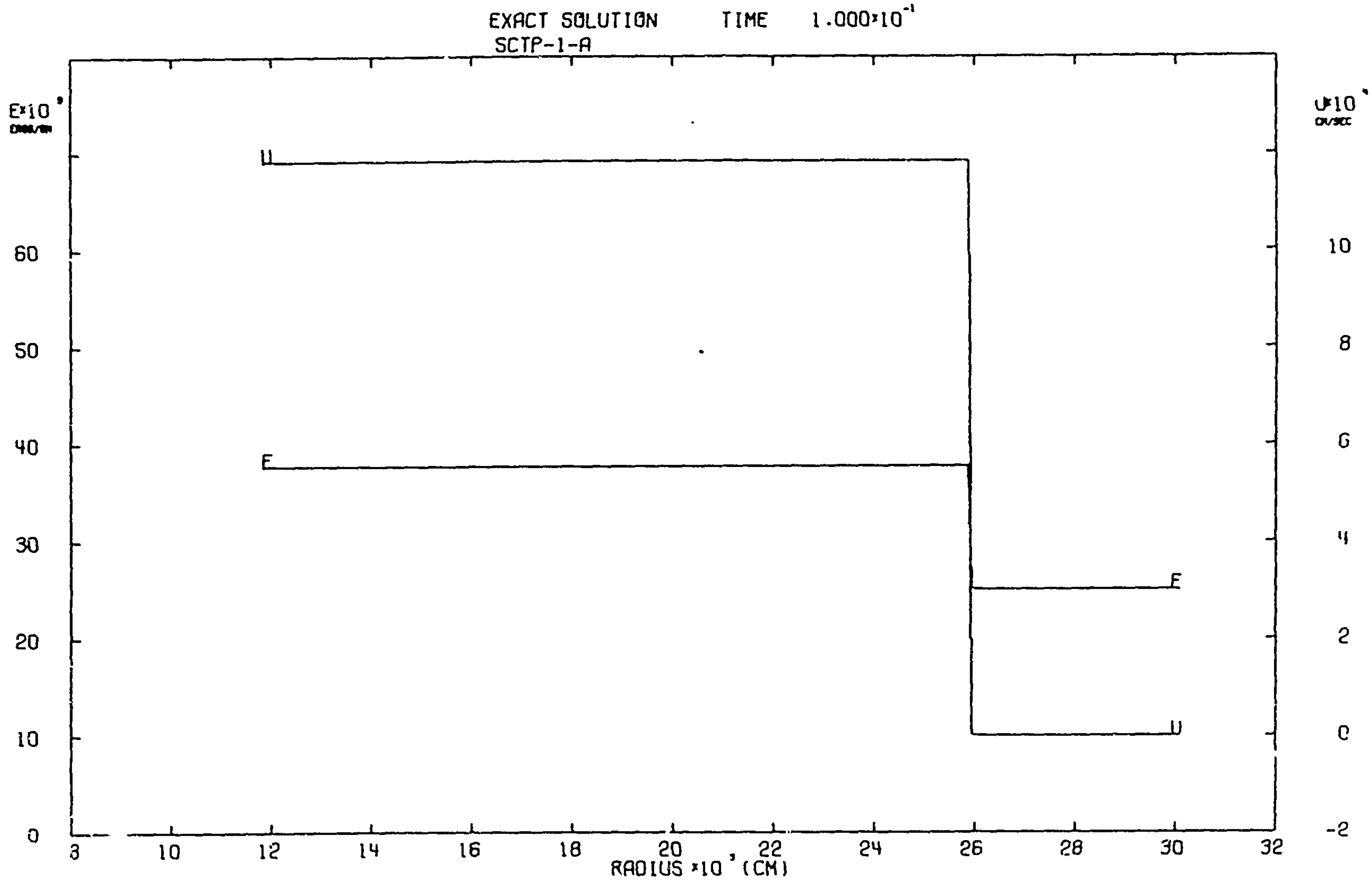

Figure 1-A. VE-EXACT

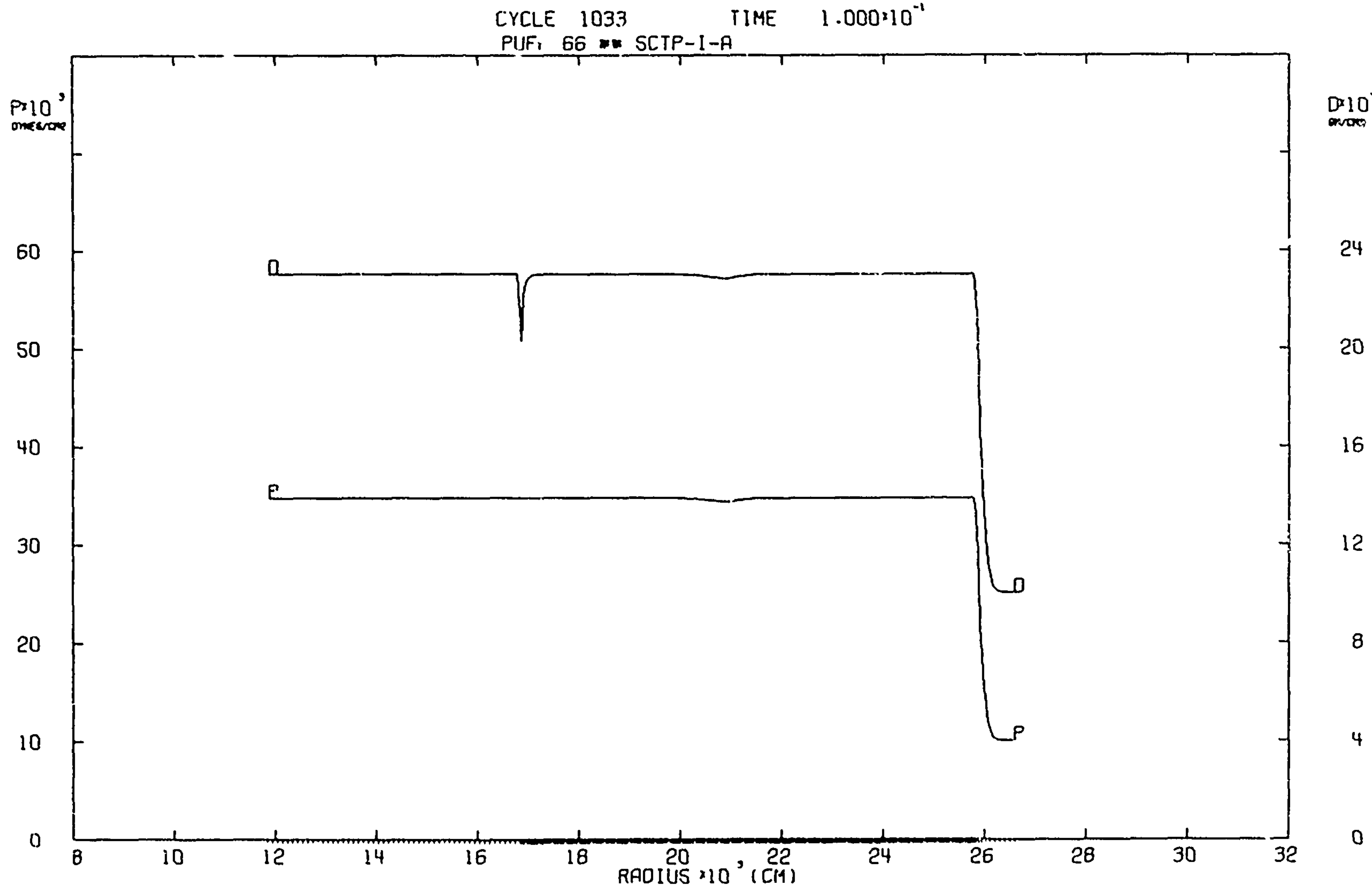

Figure I-A. PD-PUFF

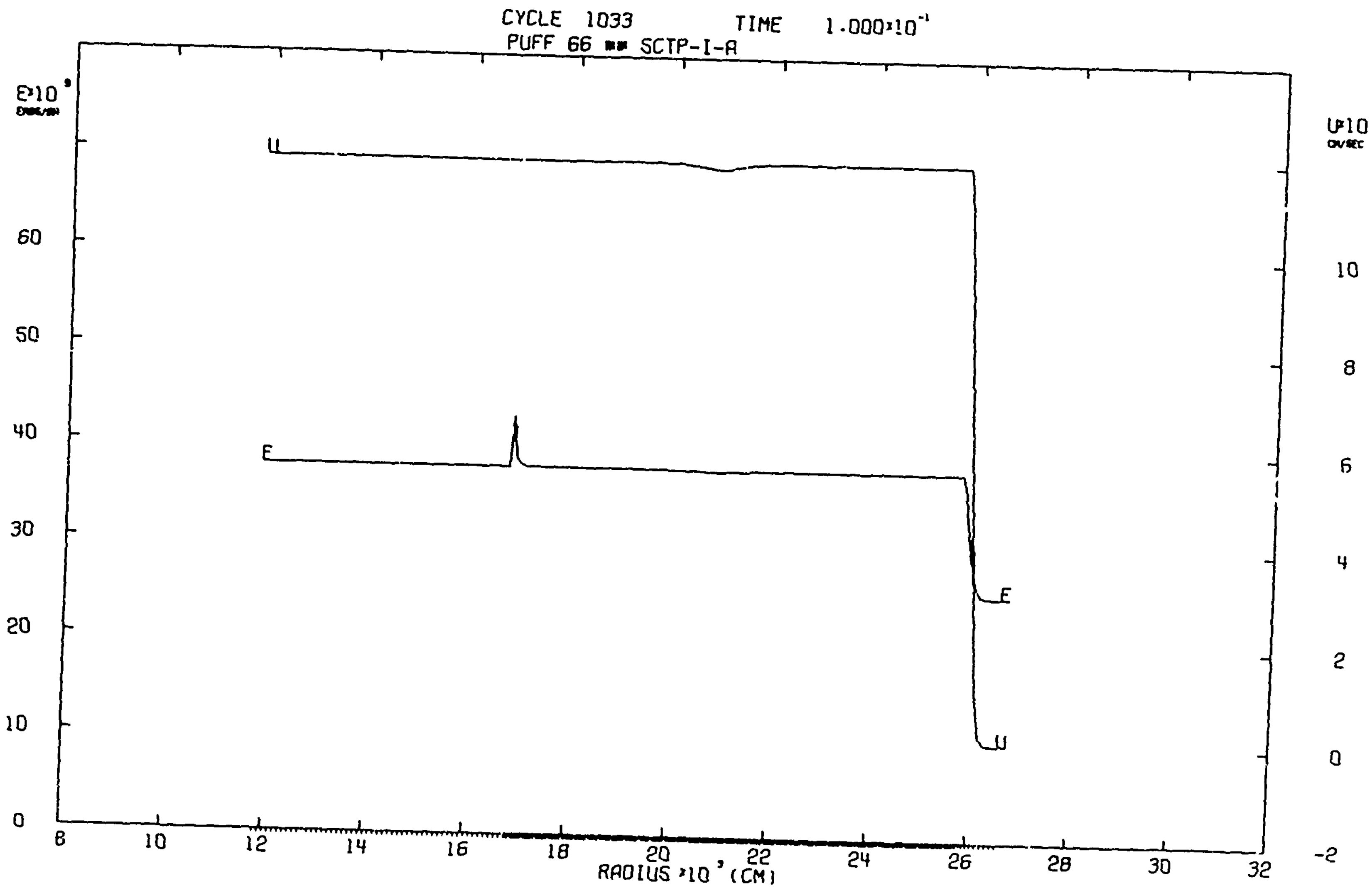

Figure I-A. VE-PUFF

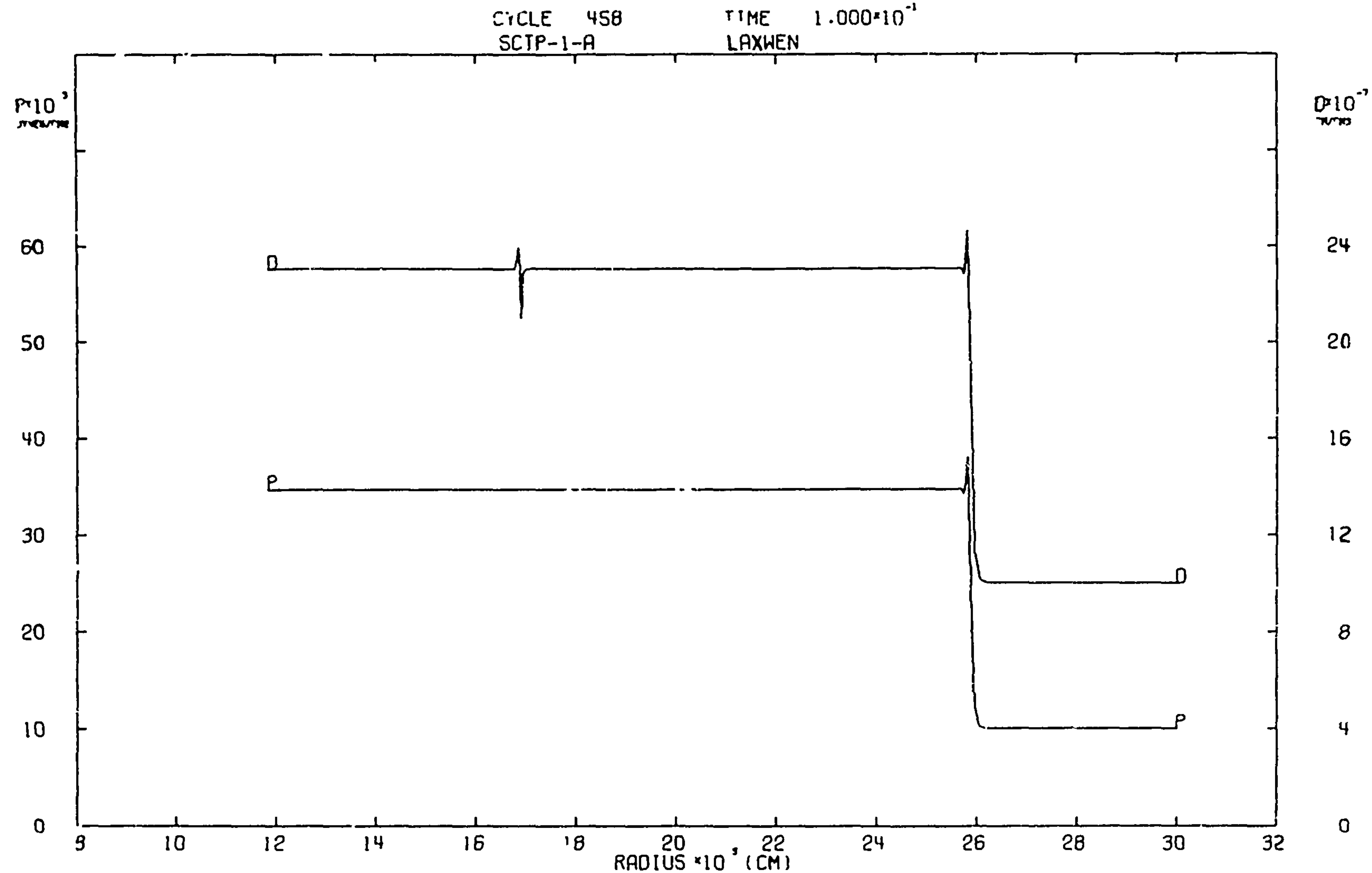

Figure 1-A. PD-LAX-WENDROFF

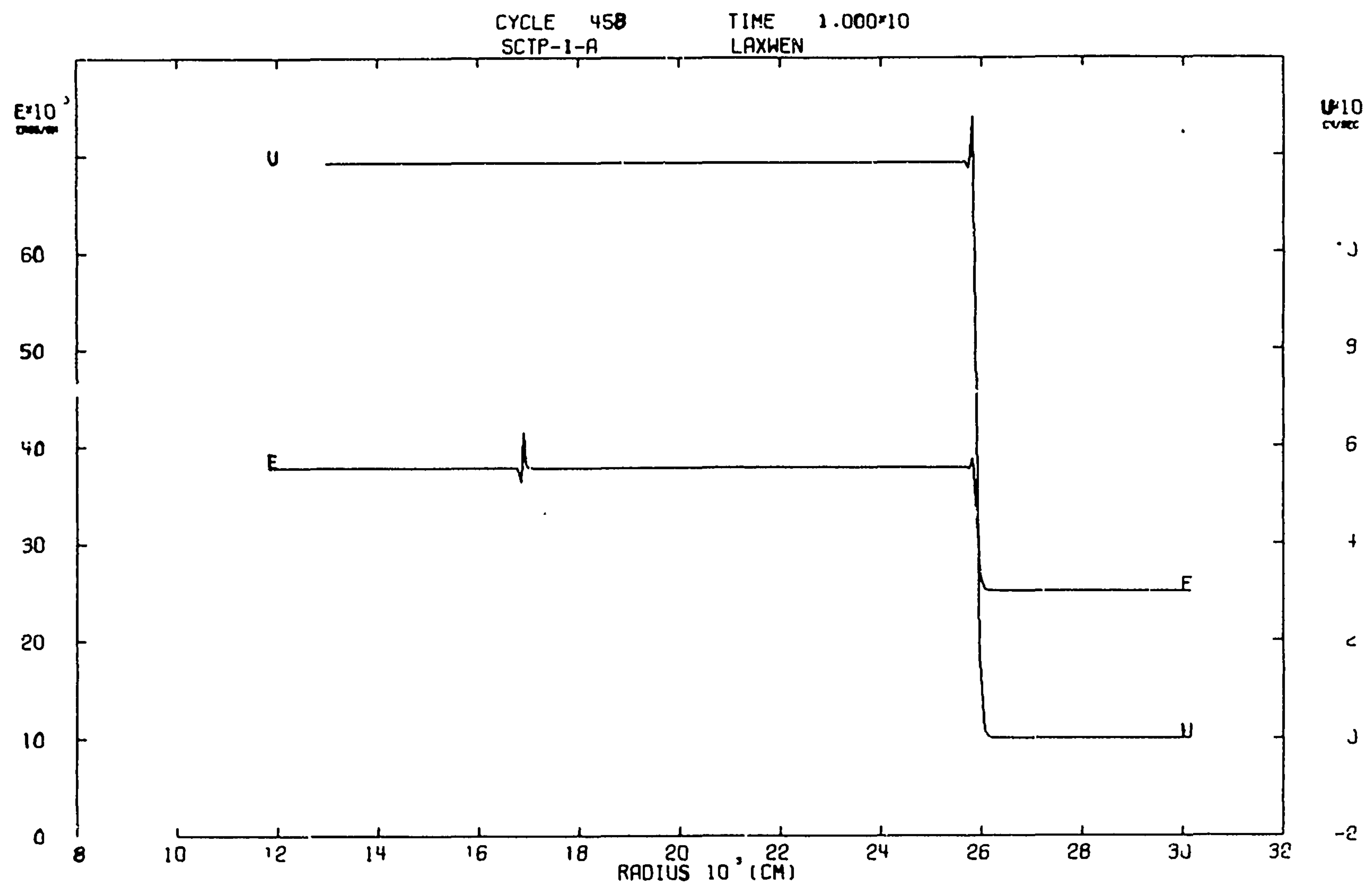

Figure I-A. VE-LAX-WENDROFF

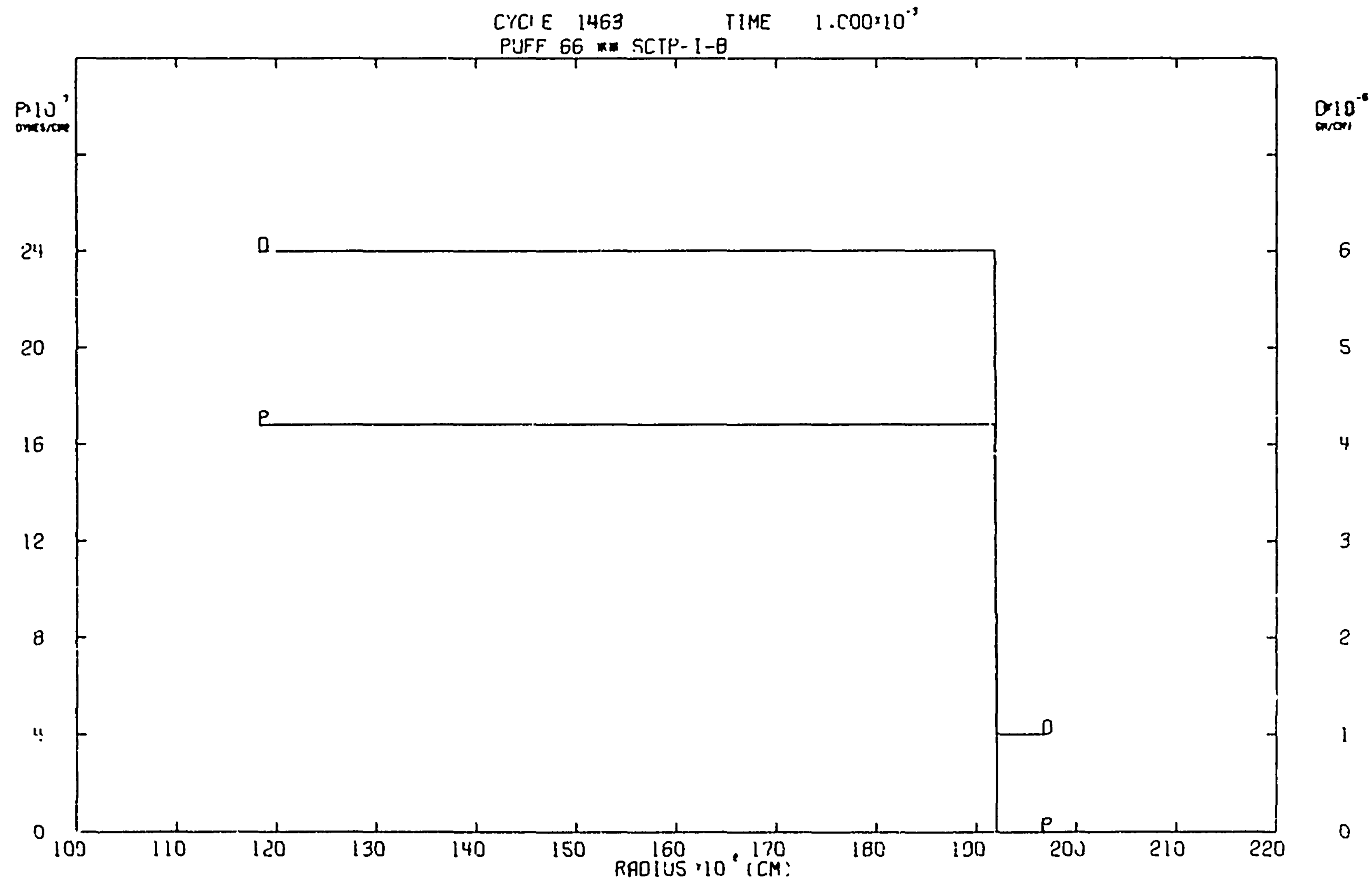

Figure I-B. PD-EXACT

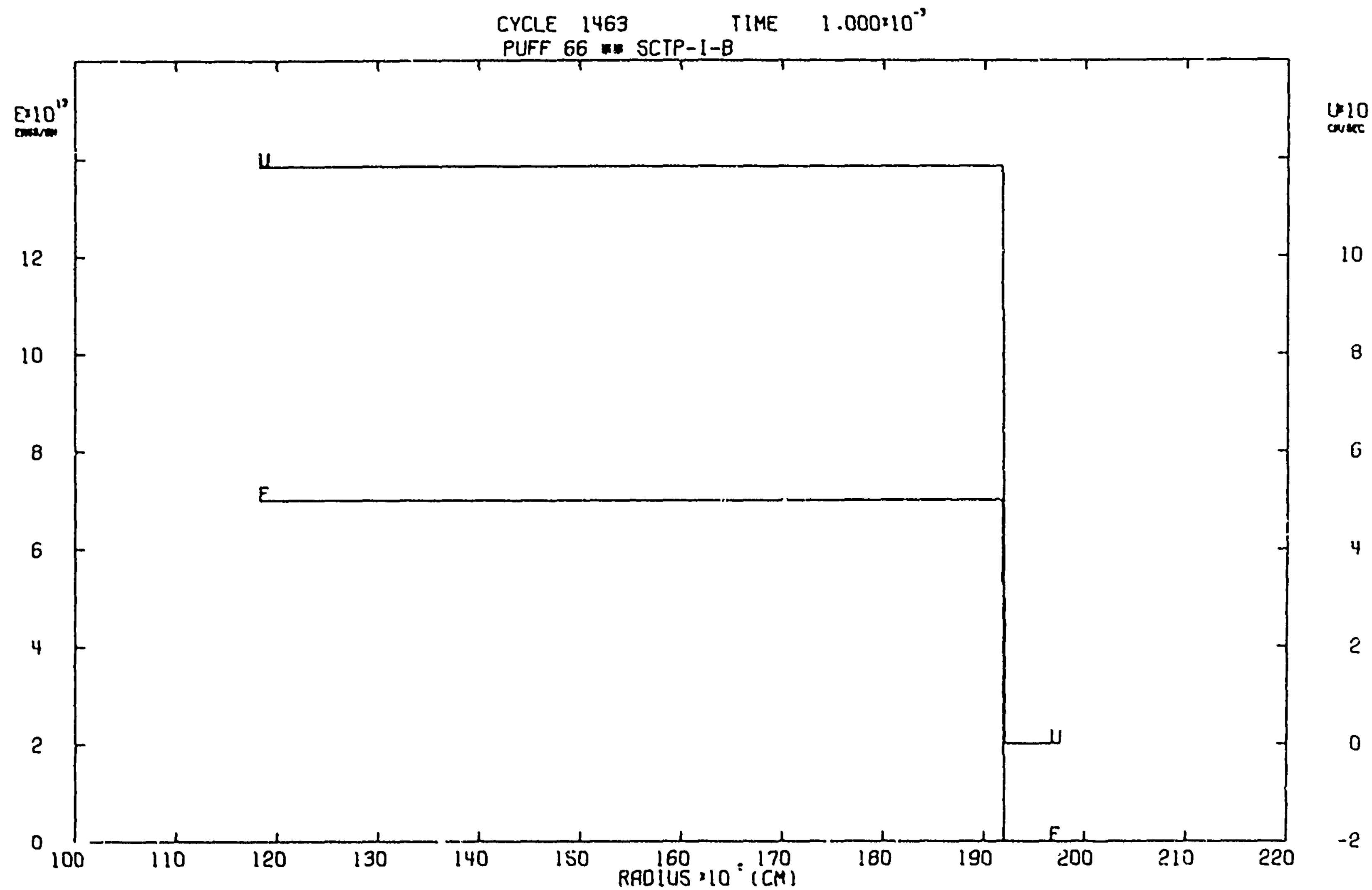

Figure I-B. VE-EXACT

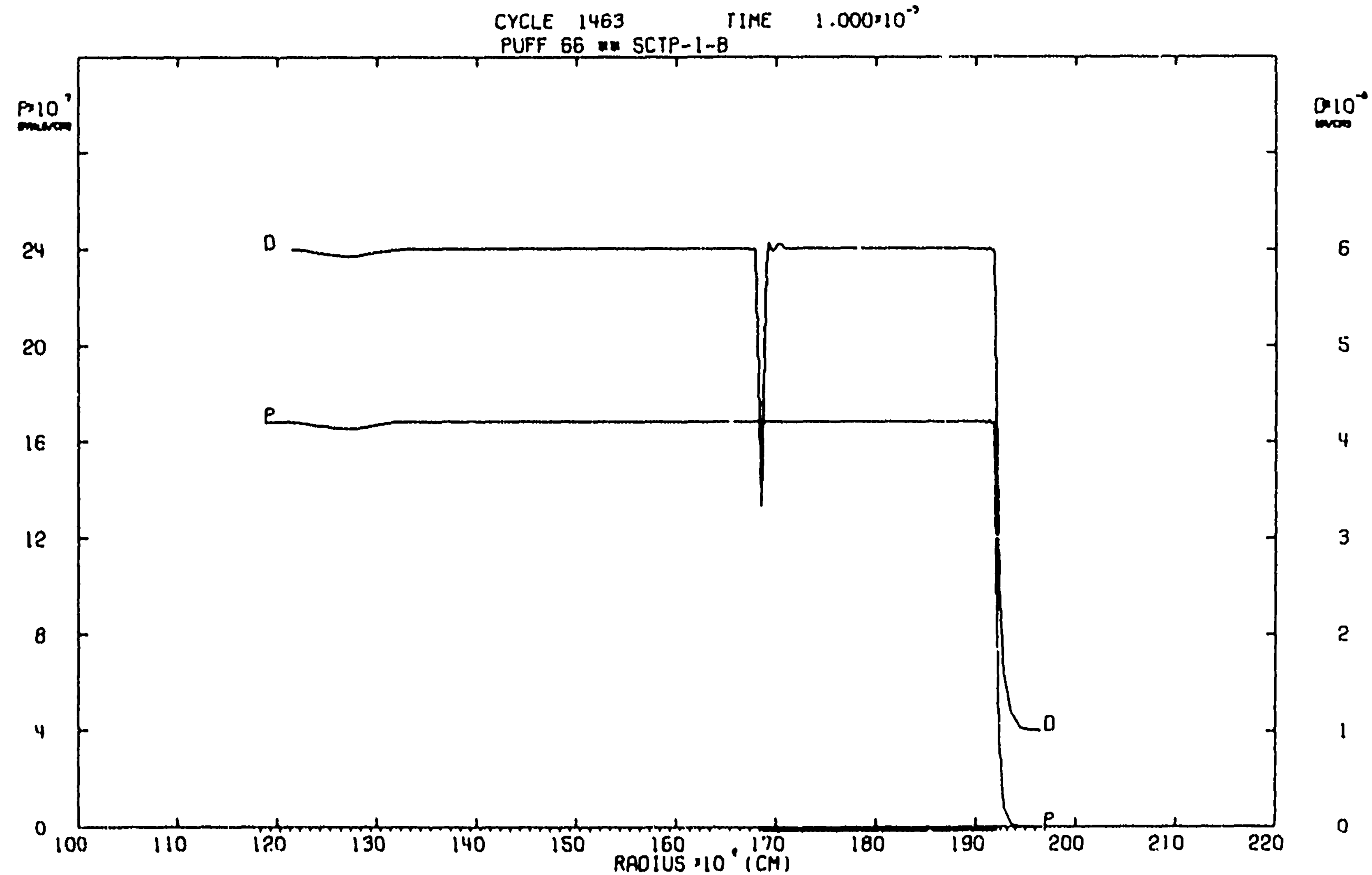

Figure I-B. PD-PUFF

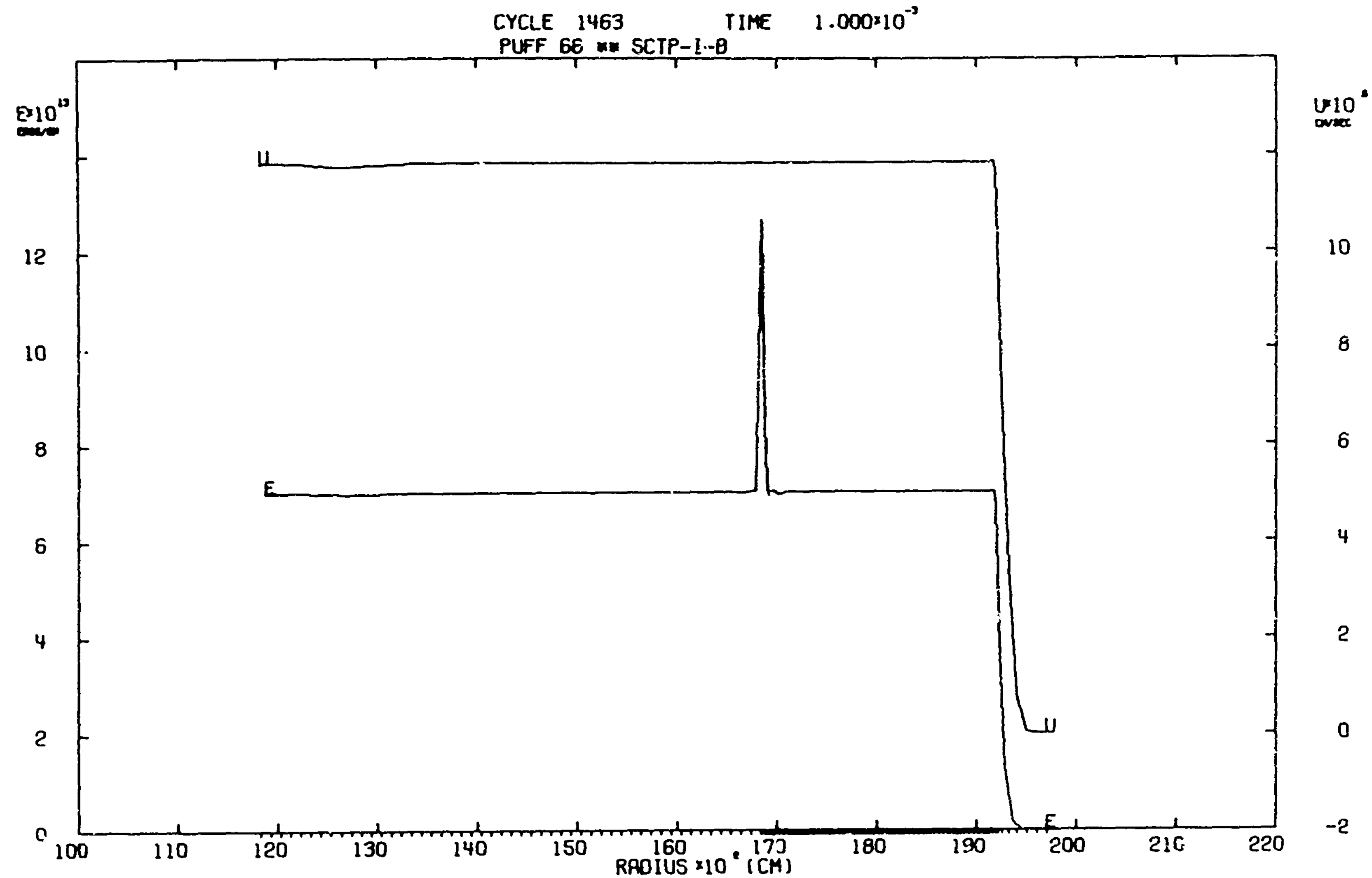

Figure I-B. VE-PUFF

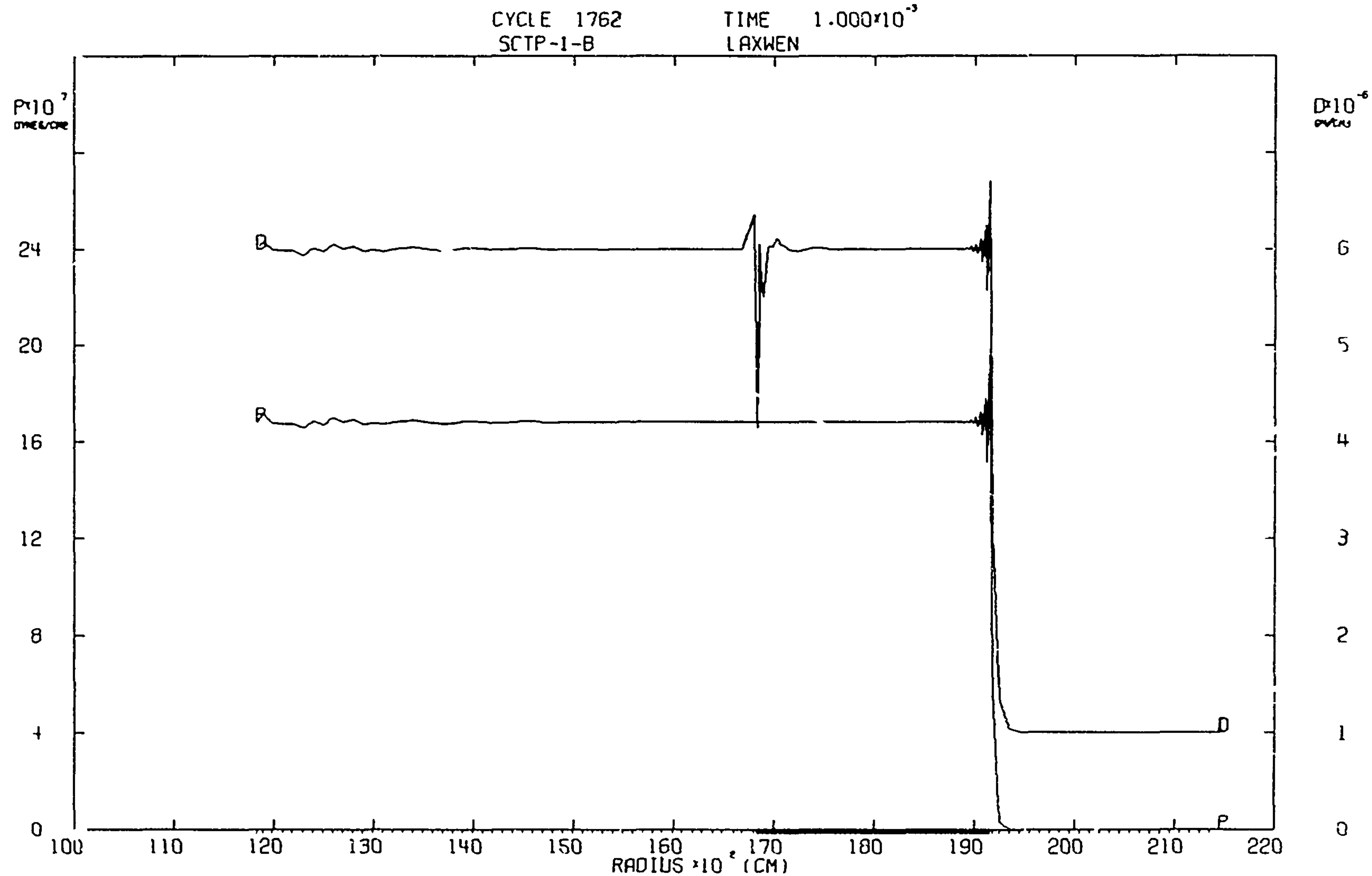

Figure I-B. PD-LAX-WENDROFF

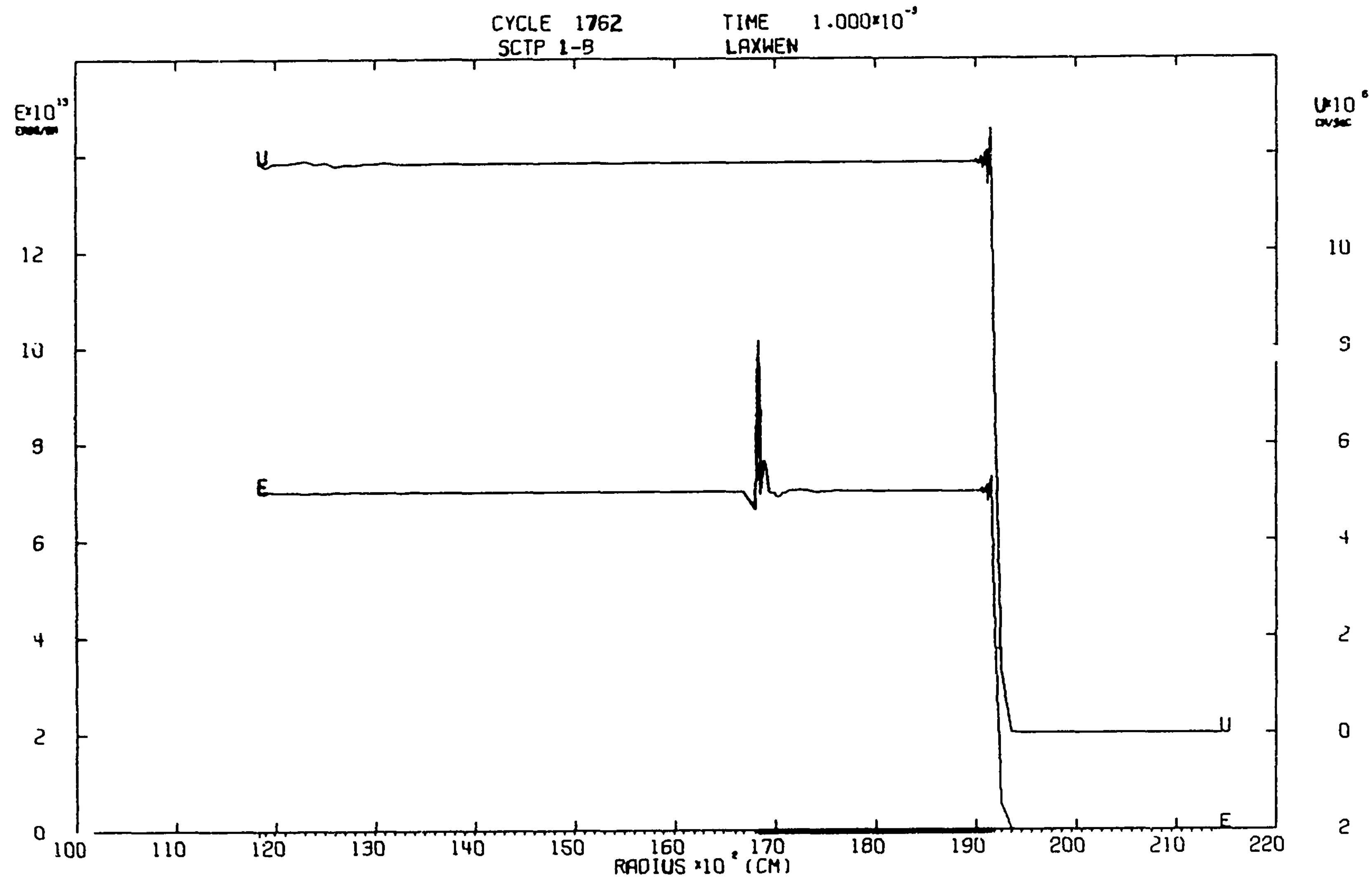

Figure I-B. VE-LAX-WENDROFF

2. TEST PROBLEM SCTP-II

a. The Exact Solution

In this problem a piston pulls away with constant velocity from the gas at rest in a pipe. The piston moves to the left with constant velocity $v_P < 0$ away from the gas on the right. This causes a rarefaction wave to move to the right. For a graphical description see the exact solution plots in Figures II. The exact solution for the velocity is piecewise linear as a function of X. Starting at the piston on the left at position $X_P(t)$ the velocity is the constant v_P from $X_P(t)$ to what is called the back of the rarefaction wave and denoted $X_R(t)$. From $X_R(t)$ rightwards to $X_C(t)$ the velocity rises from v_P linearly to zero. $X_C(t)$ is the front of the rarefaction wave. To the right of $X_C(t)$ the gas is at rest so the velocity is a constant zero.

$$X_P(t) = X_P(0) + v_P t$$

$$X_R(t) = X_P(0) + \left(C_r + \frac{\gamma+1}{2} v_P\right)t$$

$$X_C(t) = X_P(0) + C_r t$$

The rest of the variables are then determined by the simple wave formulas:

$$C(X,t) = C_r + \frac{\gamma-1}{2} v(X,t)$$

$$\rho(X,t) = \rho_r\left(\frac{C(X,t)}{C_r}\right)^{\frac{2}{\gamma-1}}$$

$$P(X,t) = P_r\left(\frac{C(X,t)}{C_r}\right)^{\frac{2\gamma}{\gamma-1}}$$

There are five variations on this problem. This much is common to all of them:

$$P_r = 10^4 \text{ dynes/cm}^2$$

$$\rho_r = 10^{-6} \text{ gm/cm}^3$$

$$C_r^2 = \gamma P_r/\rho_r = 1.4 \times 10^{10}\ cm^2/sec^2$$

$$\Delta X = 100\ cm$$

$$X_p(0) = 100\ meters$$

$$X_Q = 300\ meters$$

and all variations are run out to .1 second. The variations are in the piston velocity.

SCTP-II-A	$\|v_p\| = C_r/(\gamma+1)$
SCTP-II-B	$\|v_p\| = 2C_r/(\gamma+1)$
SCTP-II-C	$\|v_p\| = 2C_r/(\gamma-1)$
SCTP-II-D	$\|v_p\| = 4C_r/(\gamma-1)$
SCTP-II-E	Free boundary condition on the left in place of withdrawing piston condition. That is, it is as if at time zero one removes a separator to the left of which is a vacuum.

These variations were introduced to investigate the codes response to the following situations: in A, $X_R(t)$ moves to the right with velocity $C_r/2$; in B, $X_R(t)$ is stationary. In both A and B the piston is not pulled out too fast for the gas to follow; therefore the pressure and density are positive constants from $X_P(t)$ to $X_R(t)$. However, in C, D, E, $X_R(t)$ moves to the left with velocity $-2C_r/(\gamma-1)$ and between $X_p(t)$ and $X_R(t)$ there is a vacuum. In C the piston is pulled out with exactly the escape velocity of the gas, $-2C_r/(\gamma-1)$, therefore, $X_P(t) = X_R(t)$. In D the piston is pulled out faster than the gas can follow and so $X_P(t) < X_R(t)$. In E the code is allowed to compute its own escape velocity.

b. The PUFF solution

On A and B PUFF tended to underround at X_C then overround at X_R and undershoot just to the left of X_R. See Tables and Figures II-A and II-B. In

C, D, E PUFF again tended to underround at X_C. In E the gas front did not move as far to the left as it should. This is because of the finite mass in the left hand zone. If the zoning were made finer to the left so that the left hand zone had a smaller mass, then the left hand zone would move out more nearly at the rate at which the gas should escape. See Tables and Figures II-C, -D, -E.

c. The LAX-WENDROFF Solution

Because the LAX-WENDROFF scheme uses specific volume instead of density it is able only to run SCTP-II-A and B. This is because of the vacuums in C, D, and E. In the vacuum the density is zero and the specific volume is infinite. If this scheme is to be used for problems in which there are vacuums or near vacuums the specific volume must be changed over to density. The time step factor used was .78 and the artificial viscosity factor used was .5.

In A and B the LAX-WENDROFF scheme tended to underround at X_C, then overround at X_R, then undershoot to the left of X_R, and then dampingly oscillate toward the left.

Lastly, one zone to the right of the piston face there is a density dip. It is believed that the reason the LAX-WENDROFF scheme has larger errors on this problem is because it has a q-factor (artificial viscosity) in expansion. In the PUFF code q is not used in expansion--only in compression. It seems that for the LAX-WENDROFF scheme to compete with PUFF its q needs to be modified also for compression as noticed in SCTP-I. One other thing at this point: notice that the computer times for the LAX-WENDROFF scheme are longer. This is because PUFF is a production code and much time was spent making it run efficiently. On the other hand no time was spent trying to make the LAX-WENDROFF scheme coding efficient. The difference scheme was merely programmed to test its accuracy. So, if the LAX-WENDROFF scheme is to be used for production, then time should be spent in making the programming more efficient.

Table II-A

ERRORS ON SCTP-II-A

PUFF

Problem time = .1 sec
Computer time = 71 sec

Cycle = 170
Number of Active Zones = 132

	Sum Abs. Error	**Sum Sqr. Error**	**Maximum Error**	**Position of Maximum Error**
Pressure	.232	.037	- .013	X_C
Velocity	.539	.087	+ .025	X_R
Density	.184	.029	- .0095	X_C
Energy	.088	.014	+ .004	X_R

	Sum Int. Energy	**Sum Kin. Energy**	**Sum Tot. Energy**
EXACT	4.62916×10^8	1.02720×10^7	4.73188×10^8
PUFF	4.62885×10^8	1.02533×10^7	4.73138×10^8

LAX-WENDROFF

Problem time = .1 sec
Computer time ≈ 128 sec

Cycle = 160
Number of Active Zones = 200

	Sum Abs. Error	**Sum Sqr. Error**	**Maximum Error**	**Position of Maximum Error**
Pressure	.455	.066	+ .018	X_R
Velocity	1.09	.162	+ .052	X_R
Density	.382	.054	+ .015	X_R
Energy	.198	.028	+ .011	One zone to right of X_P

	Sum Int. Energy	**Sum Kin. Energy**	**Sum Tot. Energy**
EXACT	4.62916×10^8	1.02720×10^7	4.73188×10^8
LAXWEN	4.63009×10^8	1.01838×10^7	4.73192×10^8

Table II-B

ERRORS ON SCTP-II-B

PUFF

Problem time = .1 sec
Computer time = 79 sec

Cycle = 169
Number of Active Zones = 132

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.262	.037	- .014	X_C
Velocity	.389	.056	- .019	$\sim 2\frac{1}{2}$ zones left of X_R
Density	.226	.031	- .010	X_C
Energy	.127	.018	- .007	1 zone right of X_P

	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	4.43248×10^8	2.92345×10^7	4.72482×10^8	
PUFF	4.43133×10^8	2.91977×10^7	4.72331×10^8	

LAX-WENDROFF

Problem time = .1 sec
Computer time = 142 sec

Cycle = 160
Number of Active Zones = 200

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.509	.065	- .019	X_C
Velocity	.809	.111	+ .037	X_R
Density	.487	.063	- .027	1 zone right of X_P
Energy	.334	.062	+ .049	1 zone right of X_P

	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	4.43248×10^8	2.92345×10^7	4.72482×10^8	
LAXWEN	4.43411×10^8	2.89319×10^7	4.72343×10^8	

Table II-C

ERRORS ON SCTP-II-C

PUFF

Problem time = .1 sec
Computer time = 63 sec

Cycle = 169
Number of Active Zones = 132

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.314	.040	- .014	X_C
Velocity	.068	.007	- .002	X_C
Density	.255	.030	- .010	X_C
Energy	.125	.013	- .004	X_C
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	4.26049×10^8	7.39510×10^7	5.00000×10^8	
PUFF	4.24857×10^8	8.04407×10^7	5.05297×10^8	

LAX-WENDROFF

Problem time =
Computer time =

Cycle =
Number of Active Zones =

	Sum Abs. Error	Sum Sqr. Error	Maximum Err.	Position of Maximum Error
Pressure				
Velocity	The LAX-WENDROFF scheme cannot run this problem because it uses			
Density	specific volume instead of density.			
Energy				
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT				
LAXWEN				

Table II-D

ERRORS ON SCTP-II-D

PUFF

Problem time = .1 sec
Computer time = 61 sec

Cycle = 176
Number of Active Zones = 134

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.400	.056	- .019	X_C
Velocity	.078	.009	- .003	X_C
Density	.313	.041	- .013	X_C
Energy	.210	.063	+ .060	1 zone right of X_P

	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	3.64423×10^8	1.35577×10^8	5.00000×10^8	
PUFF	4.24743×10^8	9.88835×10^7	5.23626×10^8	

LAX-WENDROFF

Problem time =
Computer time =

Cycle =
Number of Active Zones =

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure				
Velocity	The LAX-WENDROFF scheme cannot run this problem because it uses			
Density	specific volume instead of density.			
Energy				

	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT				
LAXWEN				

Table II-E

ERRORS ON SCTP-II-E

PUFF

Problem time = .1 sec
Computer time = 66 sec

Cycle = 170
Number of Active Zones = 133

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.175	.029	- .014	X_C
Velocity	.050	.007	- .005	Free left boundary
Density	.145	.022	- .010	X_C
Energy	.082	.012	+ .007	Free left boundary
	Sum Int. Energy	**Sum Kin. Energy**	**Sum Tot. Energy**	
EXACT	3.64423×10^8	1.35577×10^8	5.00000×10^8	
PUFF	4.26052×10^8	7.37375×10^7	4.99790×10^8	

LAX-WENDROFF

Problem time =
Computer time =

Cycle =
Number of Active Zones

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure				
Velocity	The LAX-WENDROFF scheme cannot run this problem because it uses			
Density	specific volume instead of density.			
Energy				
	Sum Int. Energy	**Sum Kin. Energy**	**Sum Tot. Energy**	
EXACT				
LAXWEN				

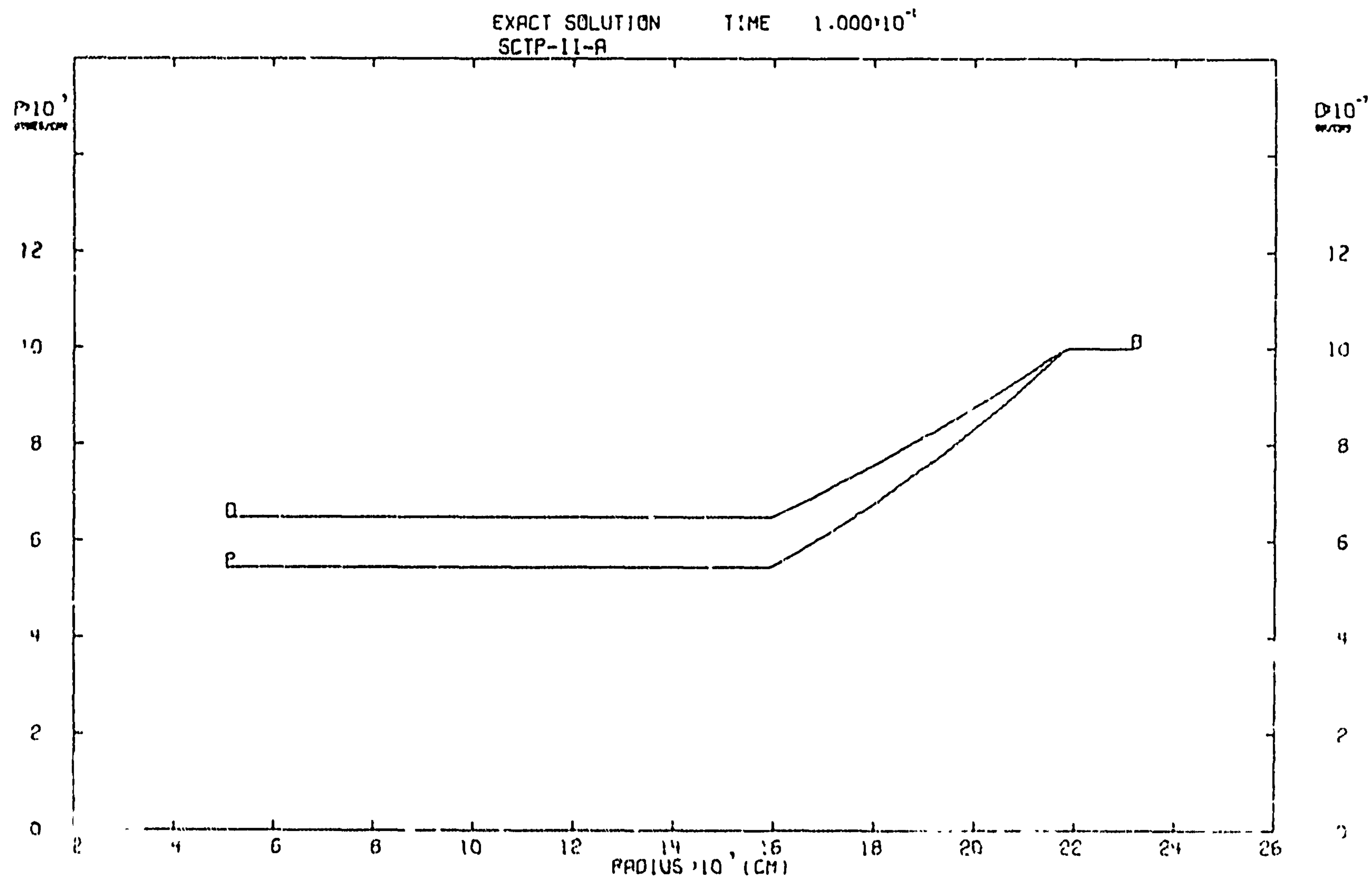

Figure II-A. PD-EXACT

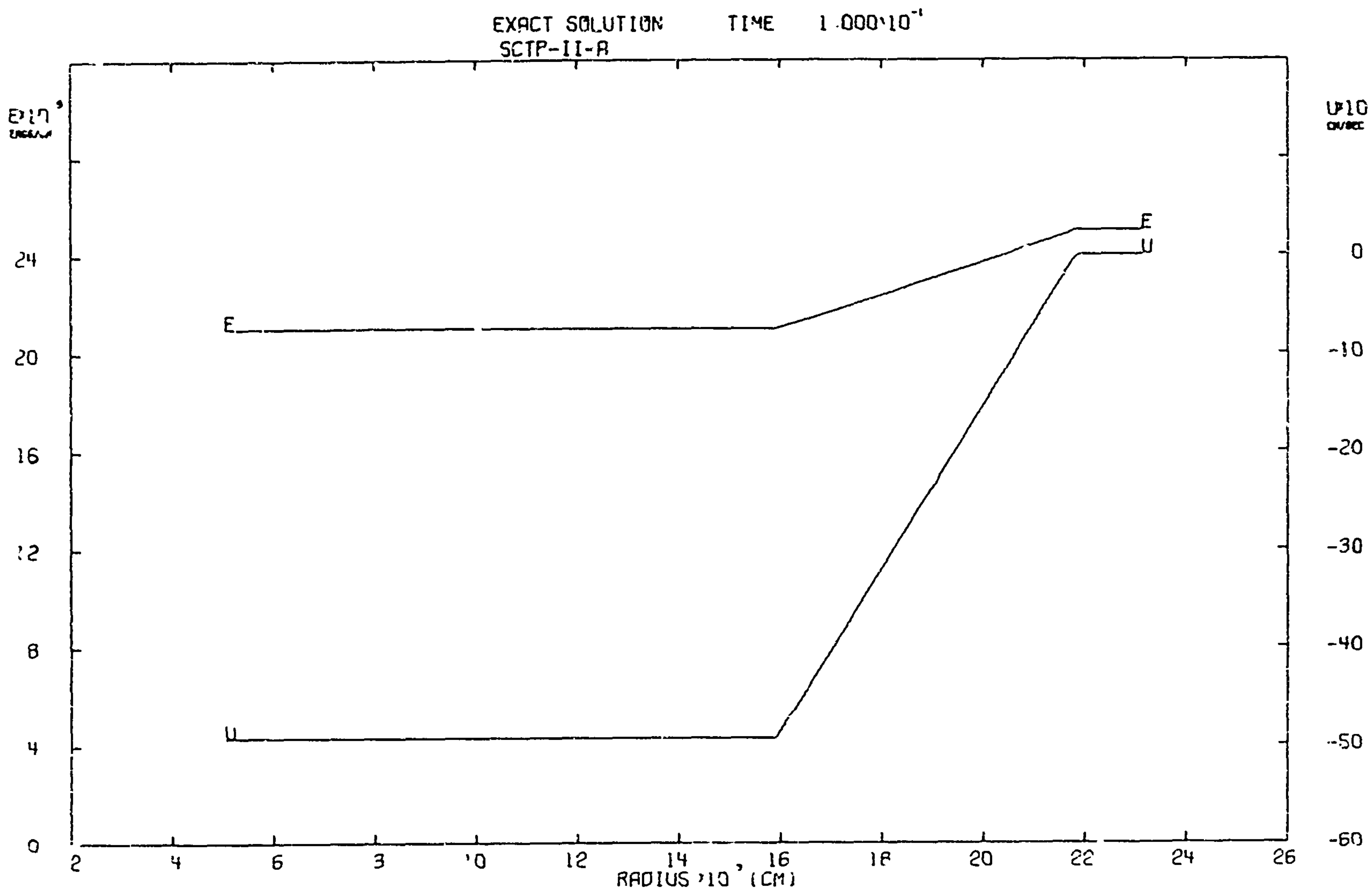

Figure II-A. VE-Exact

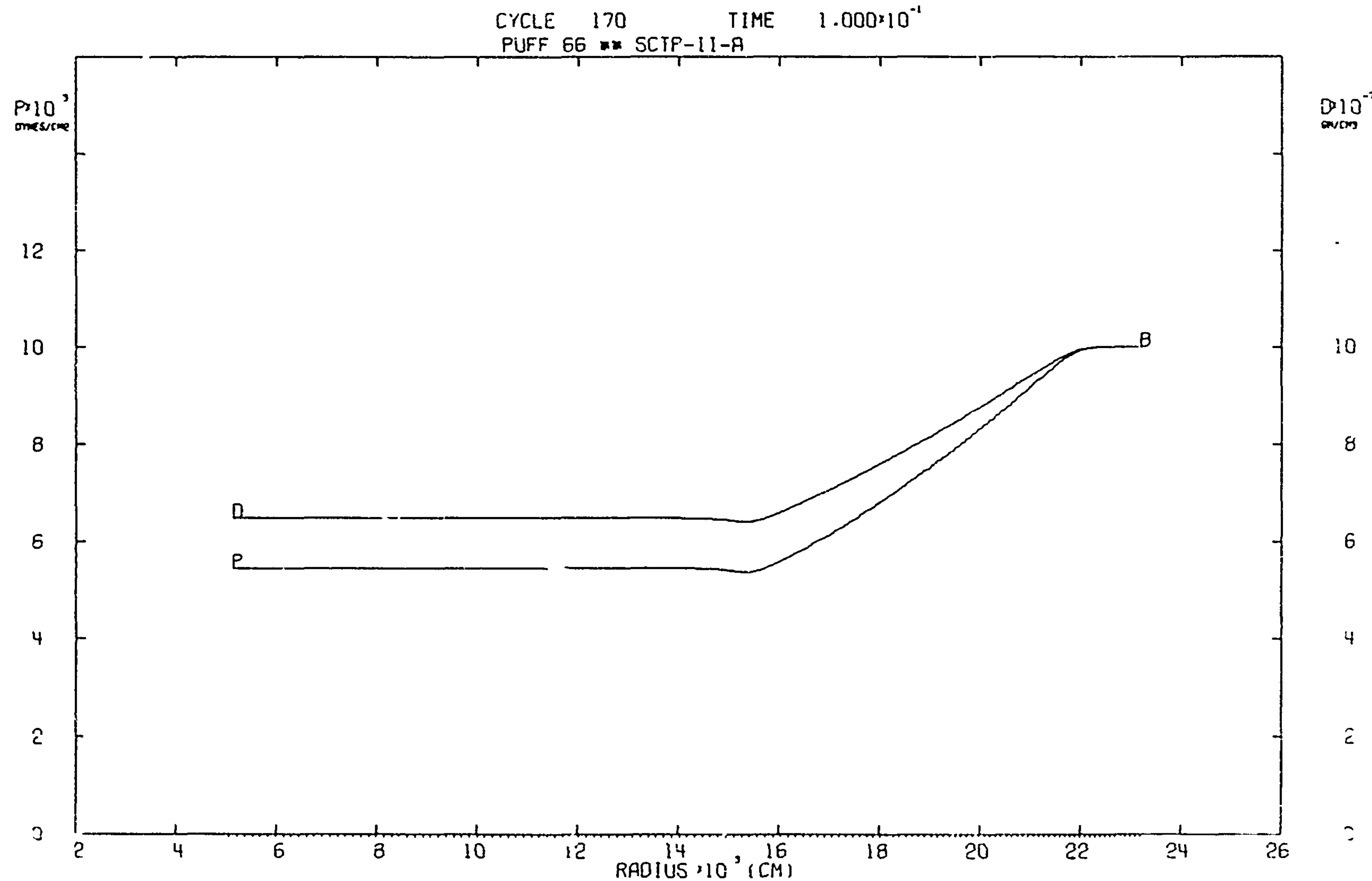

Figure II-A. PD-PUFF

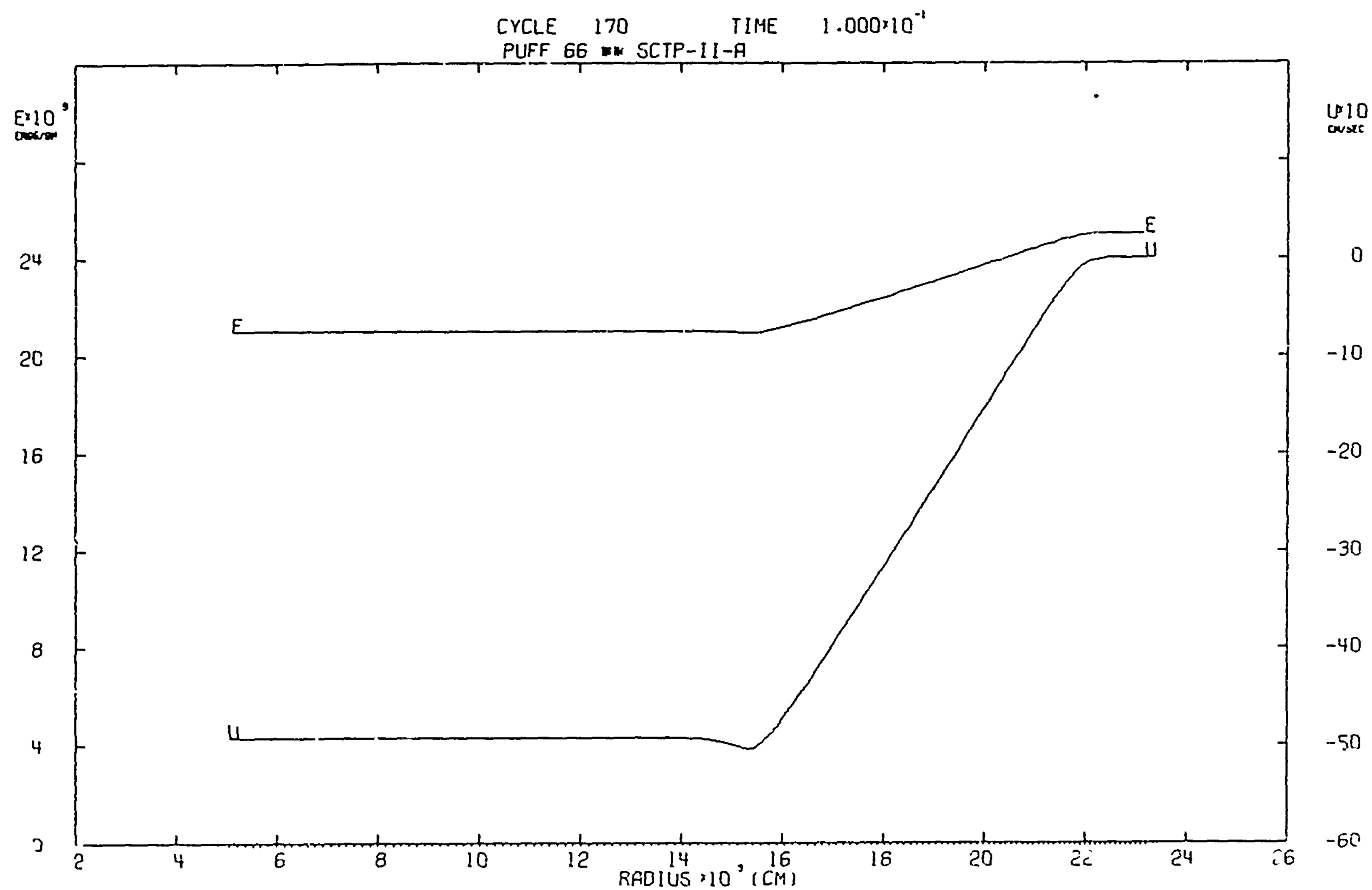

Figure II-A. VE-PUFF

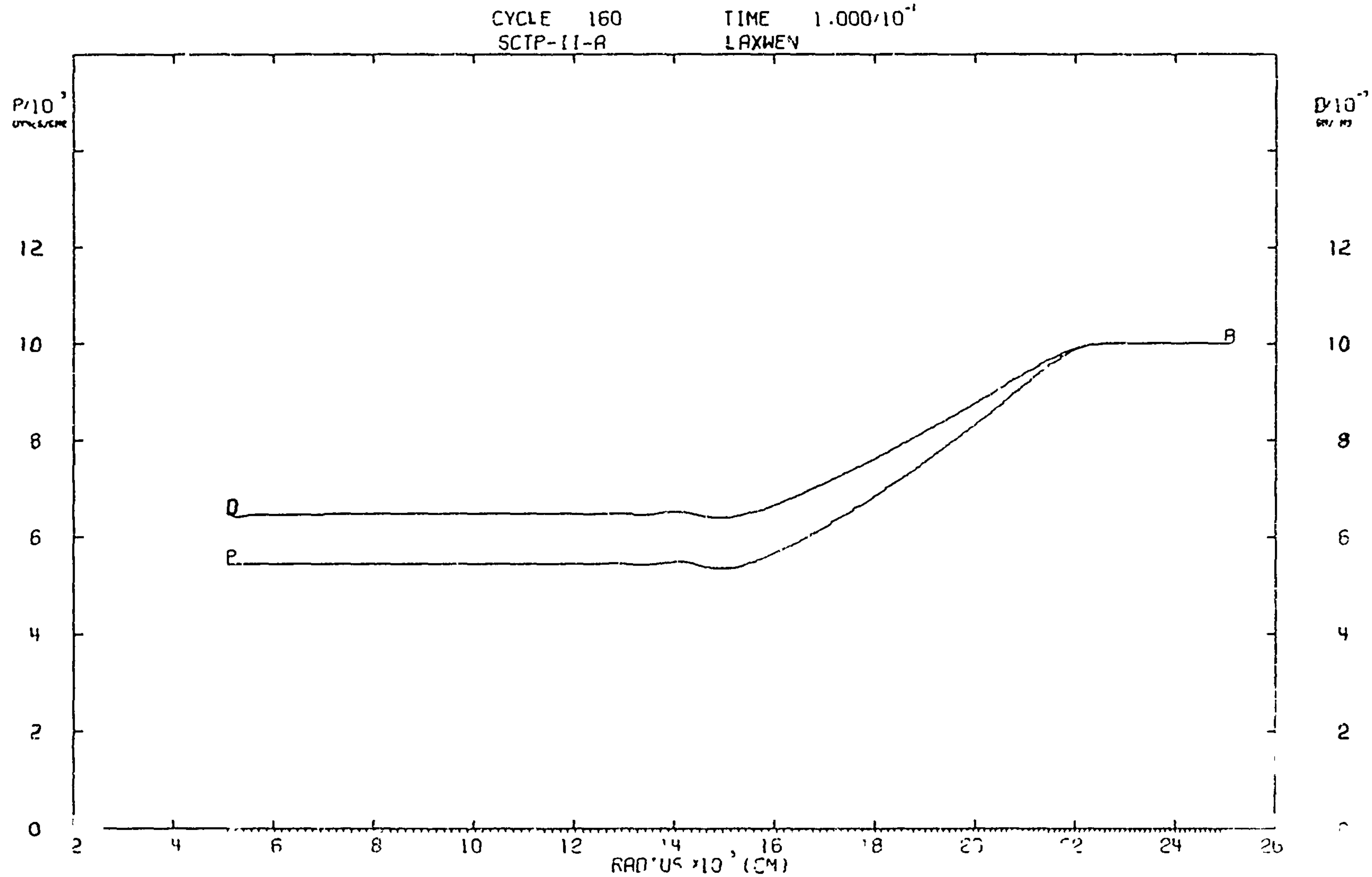

Figure II-A. PD-LAX-WENDROFF

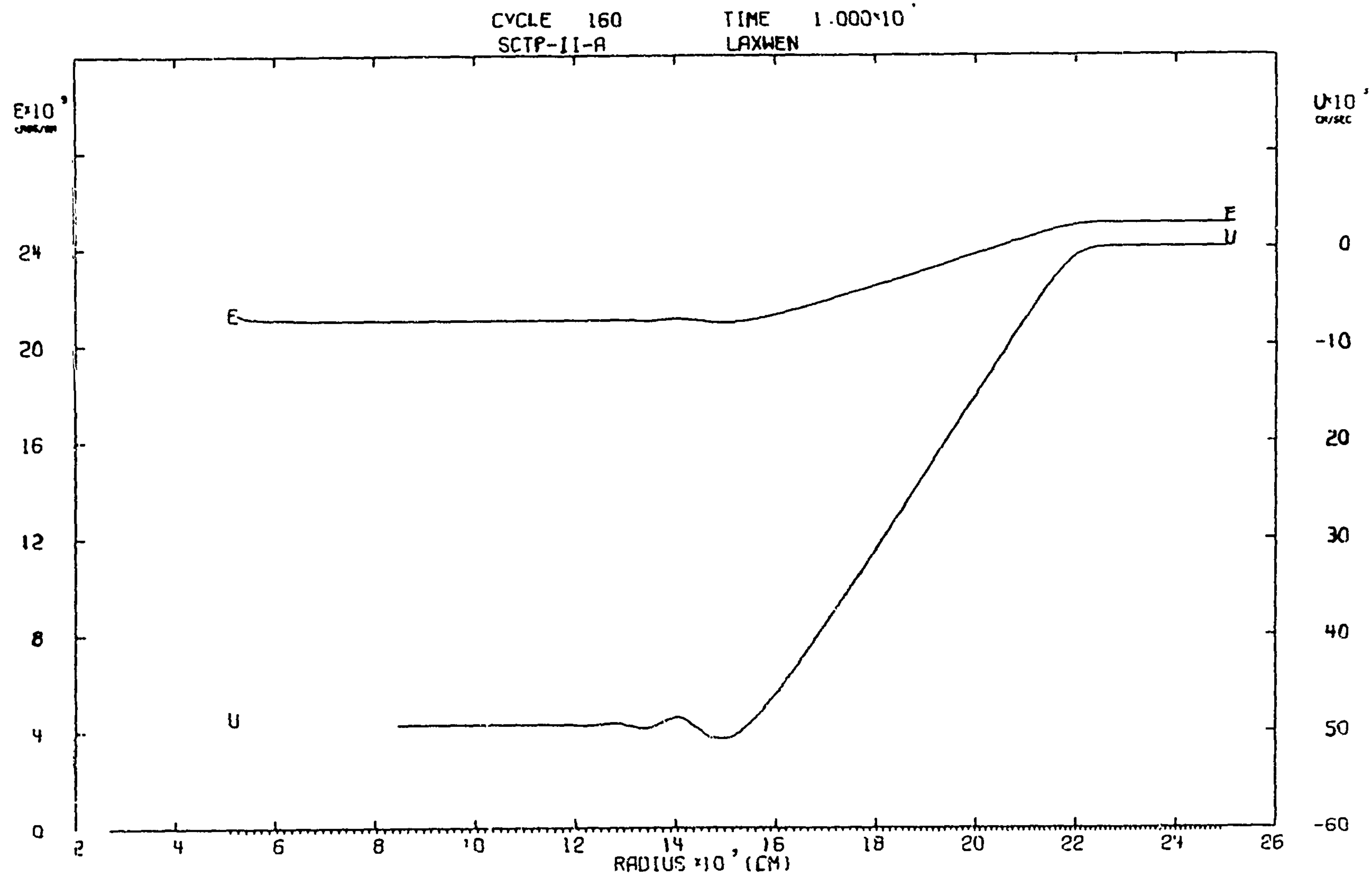

Figure 11-A. VE-LAX-WENDROFF

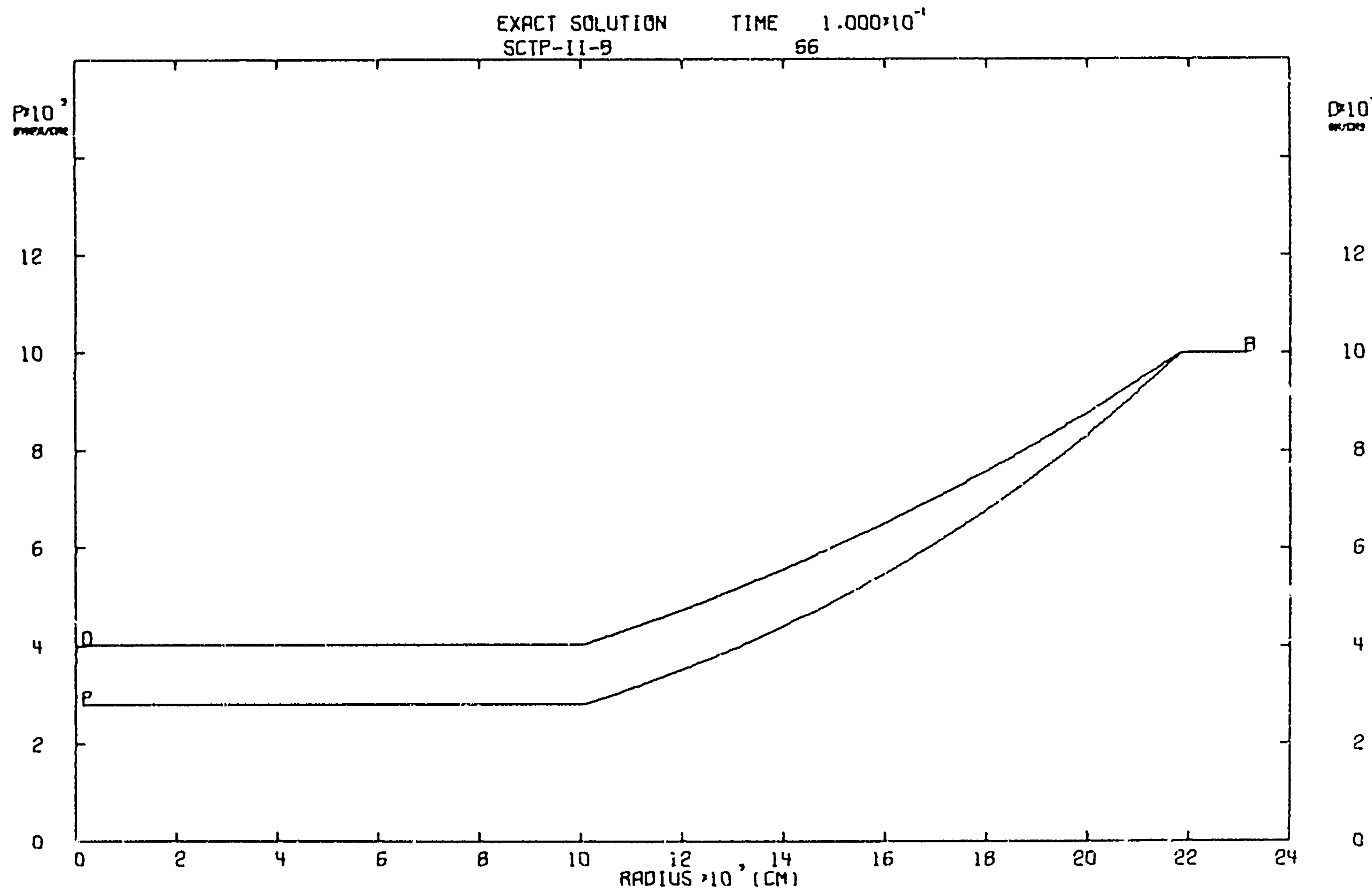

Figure II-B. PD-EXACT

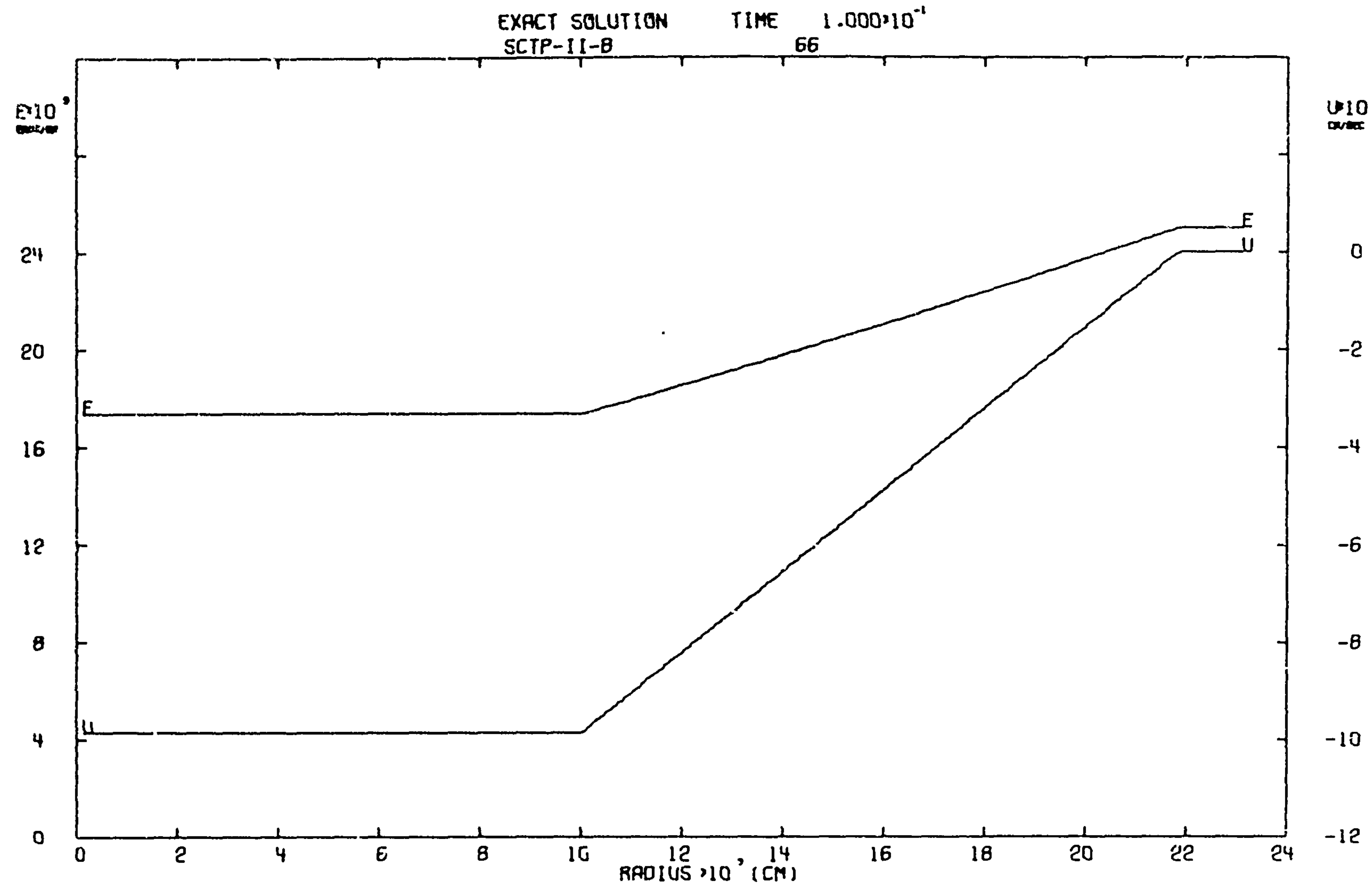

Figure II-B. VE-EXACT

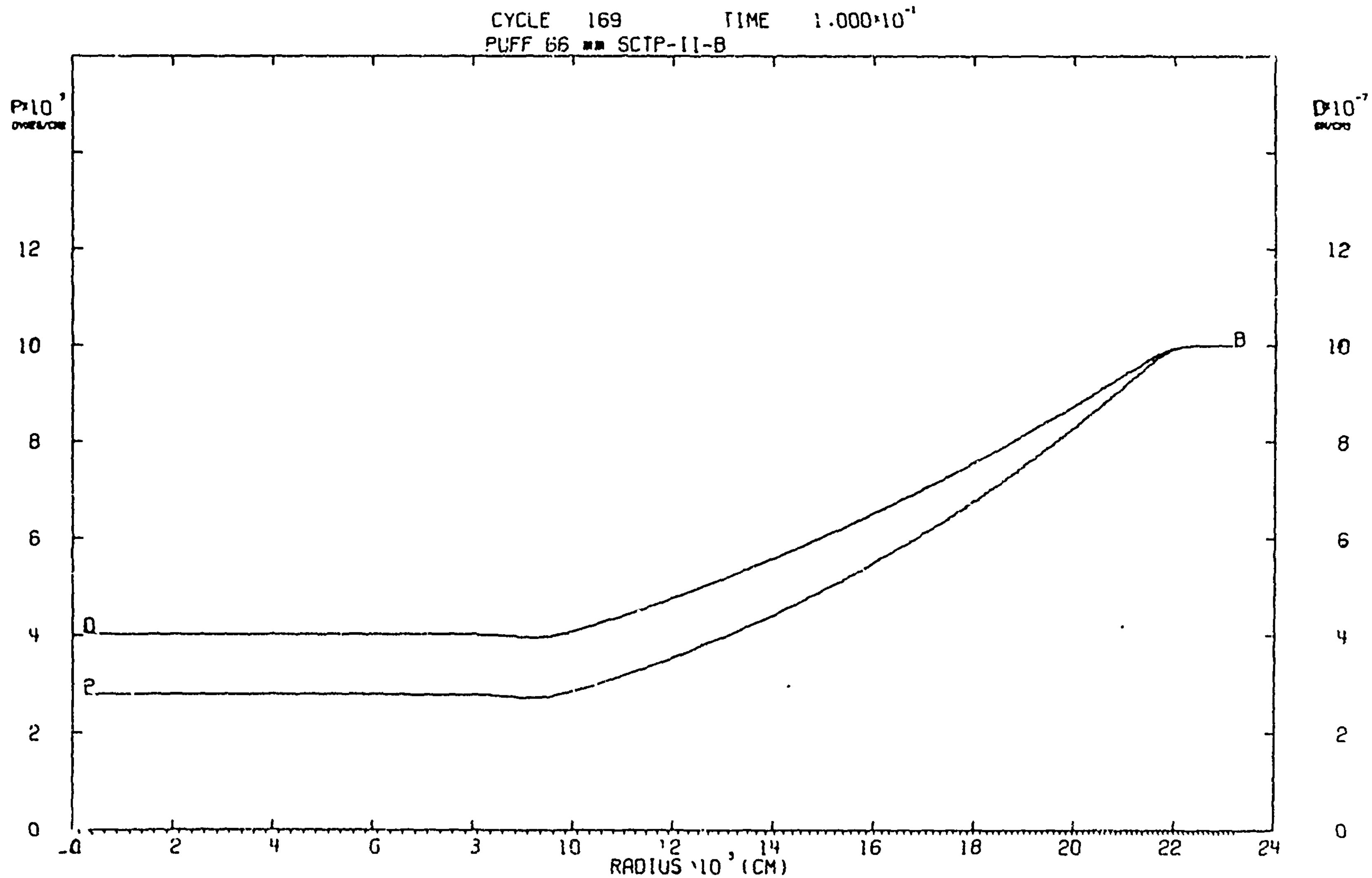

Figure II-B. PD-PUFF

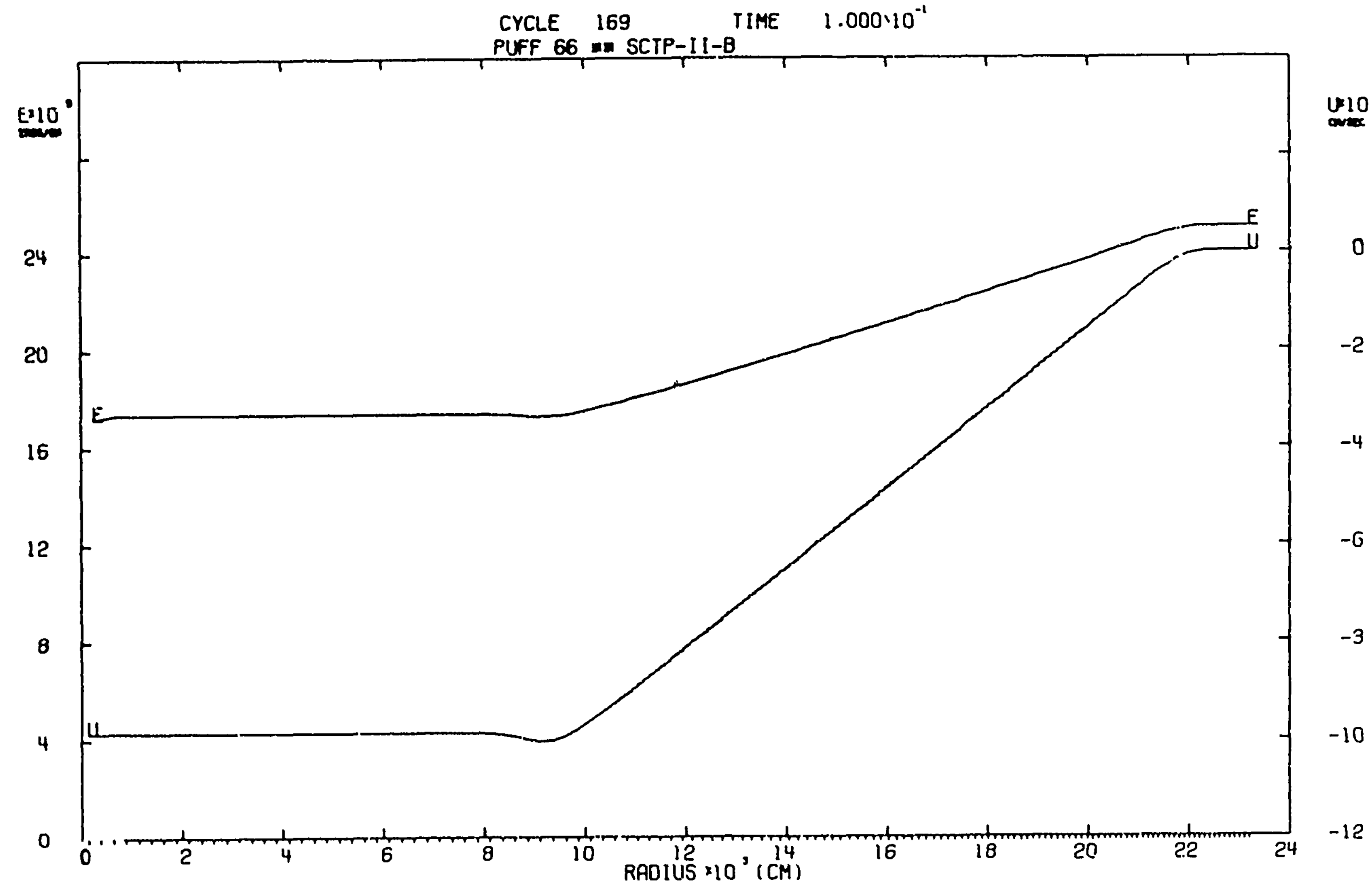

Figure II-B. VE-PUFF

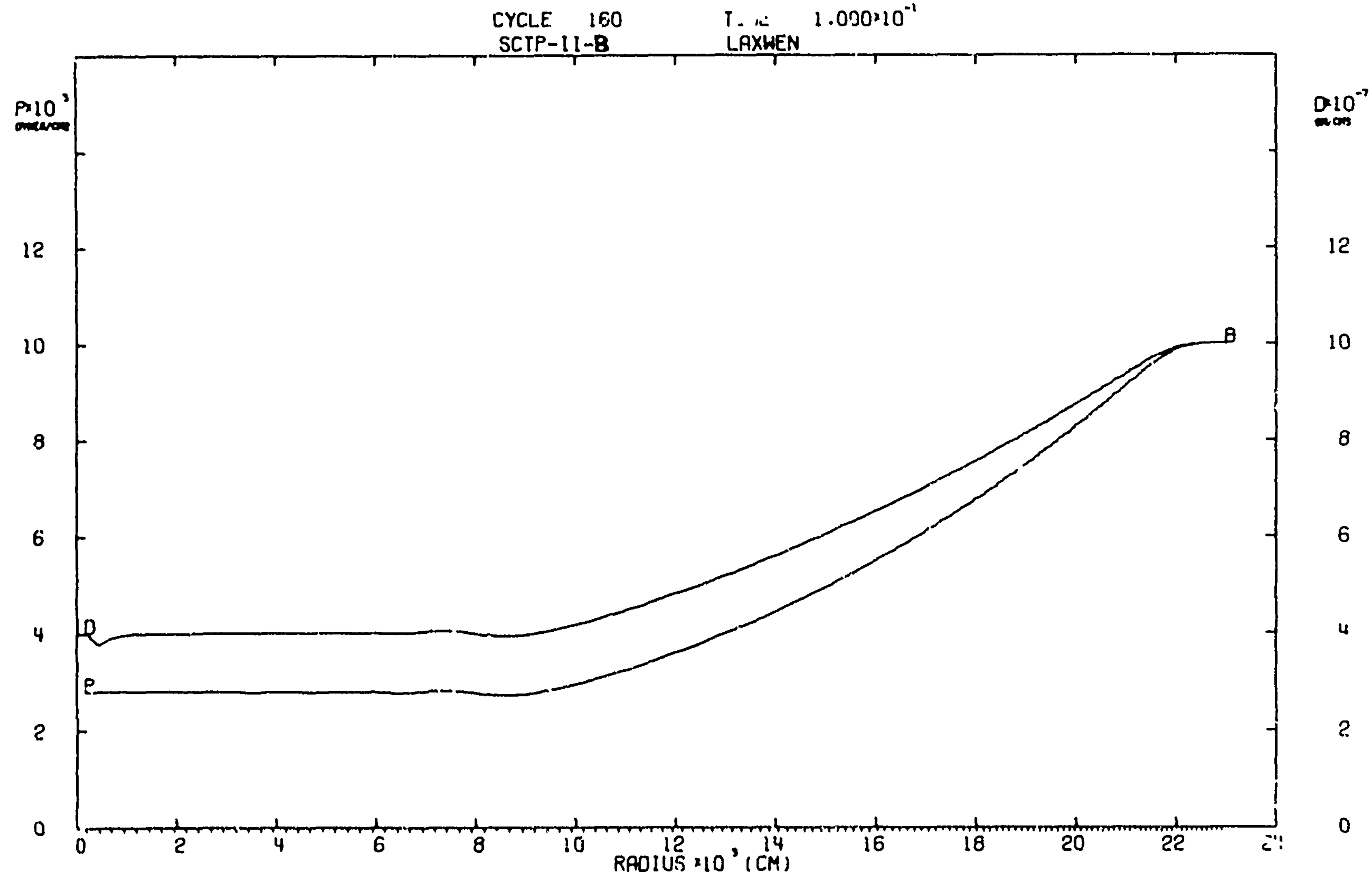

Figure II-B. PD-LAX-WENDROFF

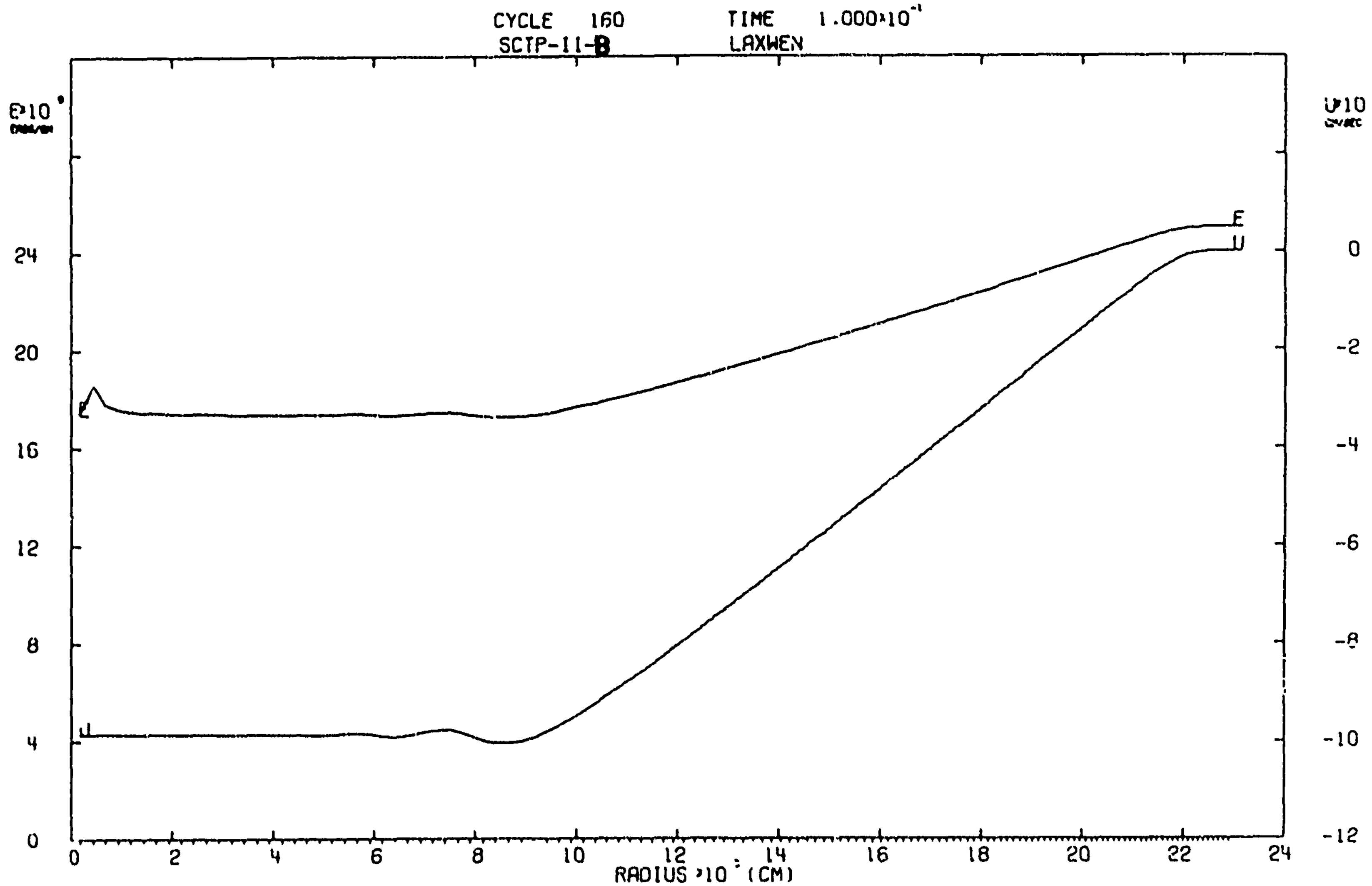

Figure 11-B. VE-LAX-WENDROFF

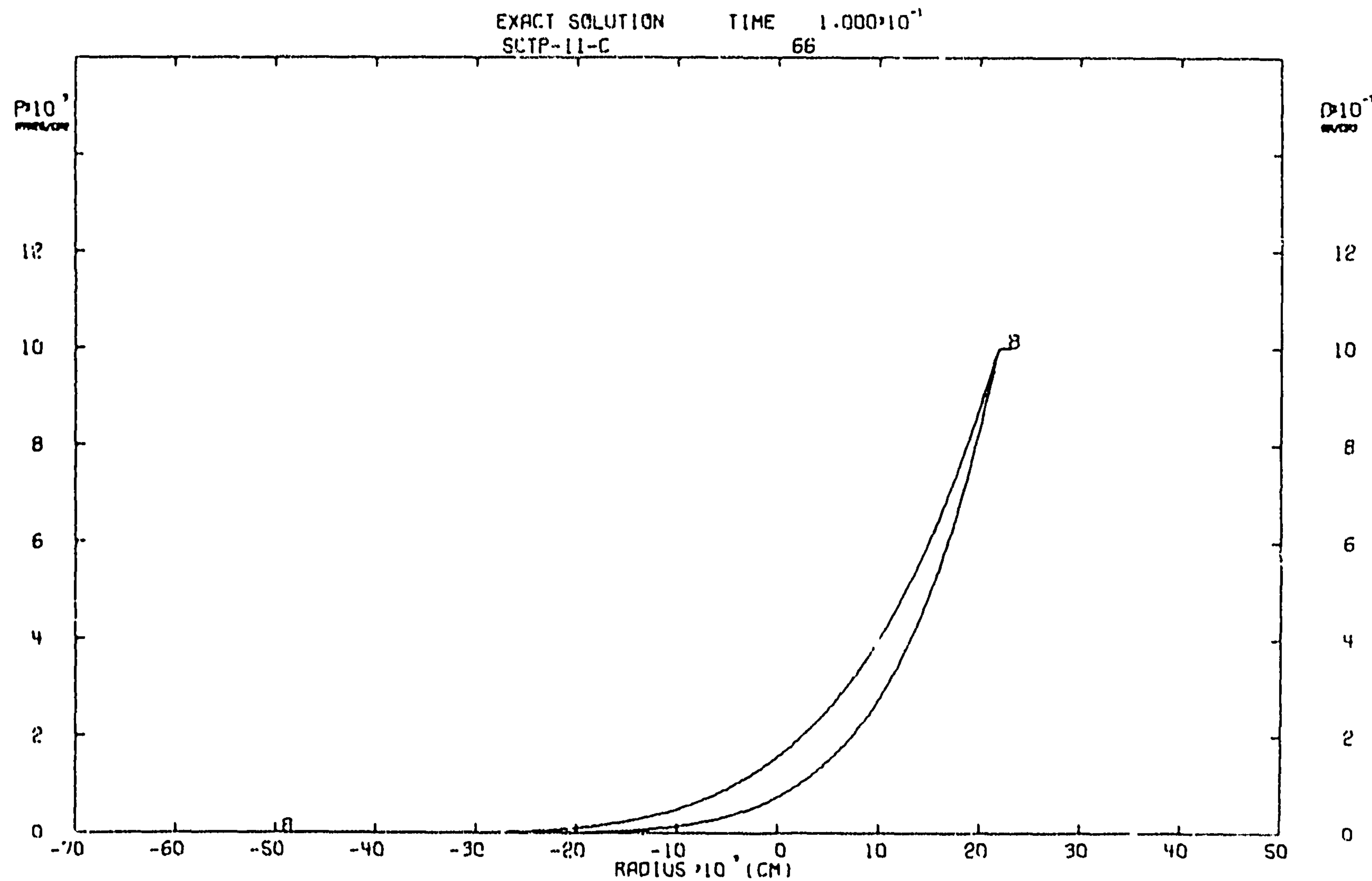

Figure II-C. PD-EXACT

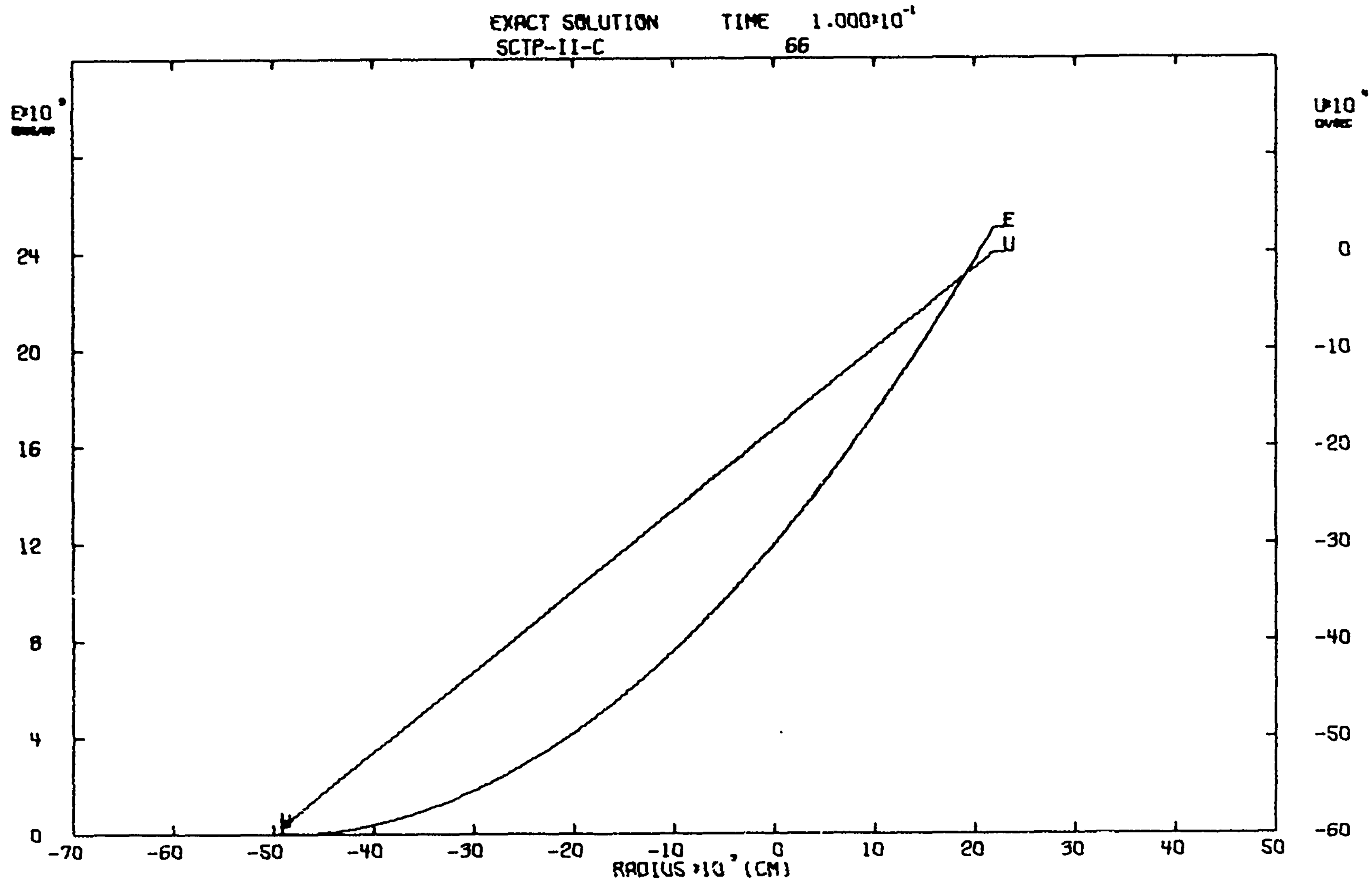

Figure II-C. VE-EXACT

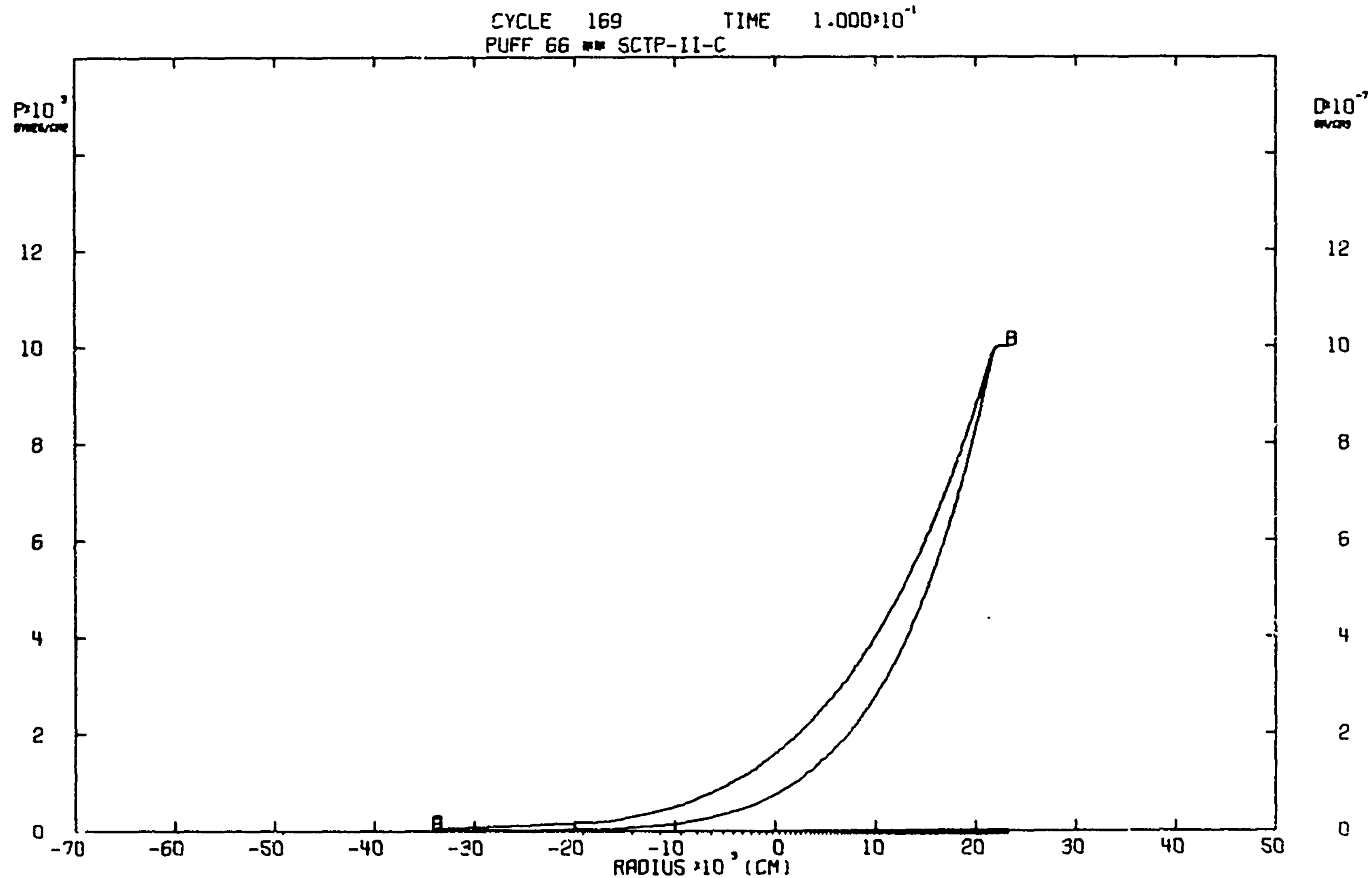

Figure II-C. PD-PUFF

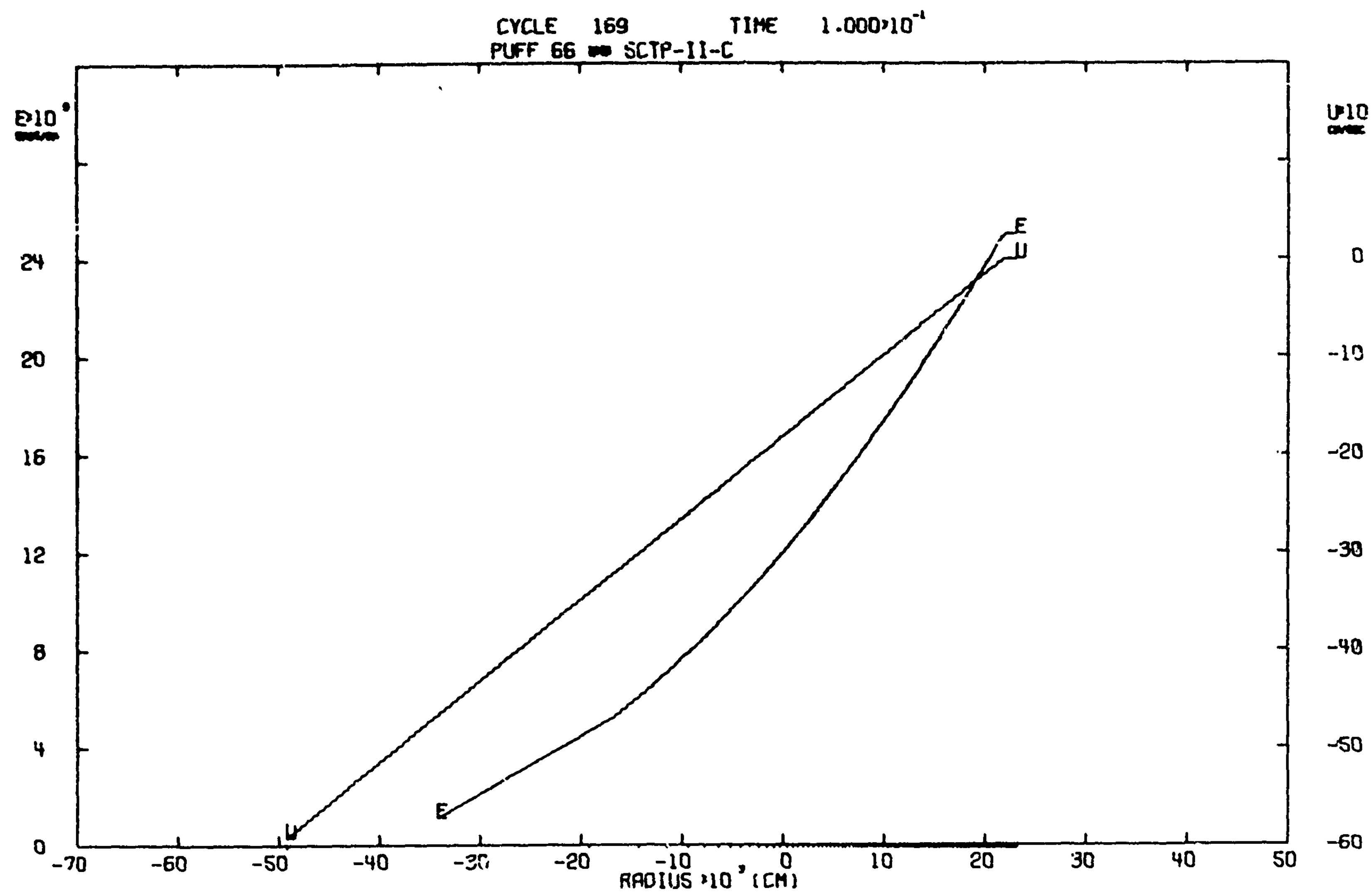

Figure II-C. VE-PUFF

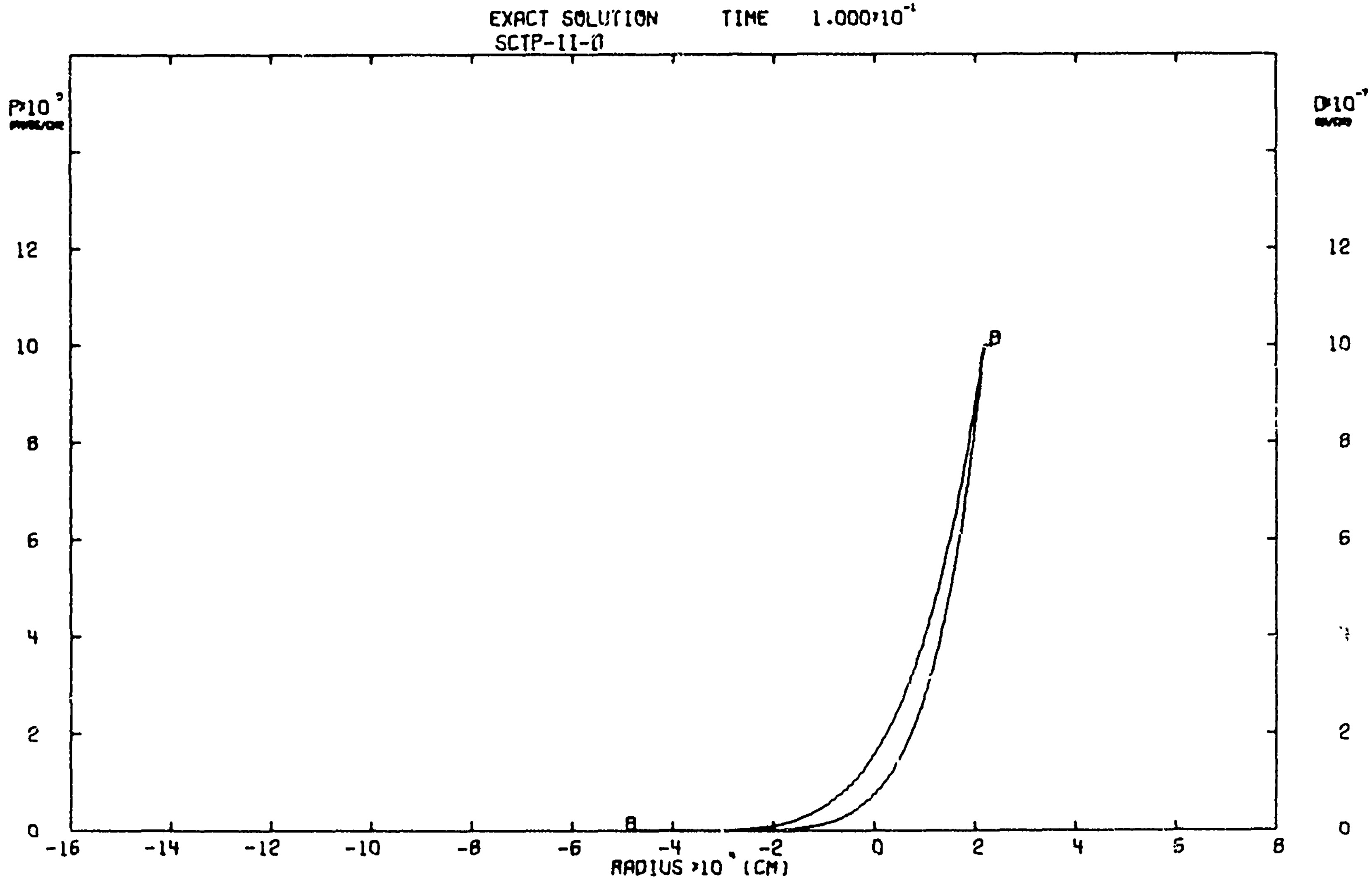

Figure II-D. PD-EXACT

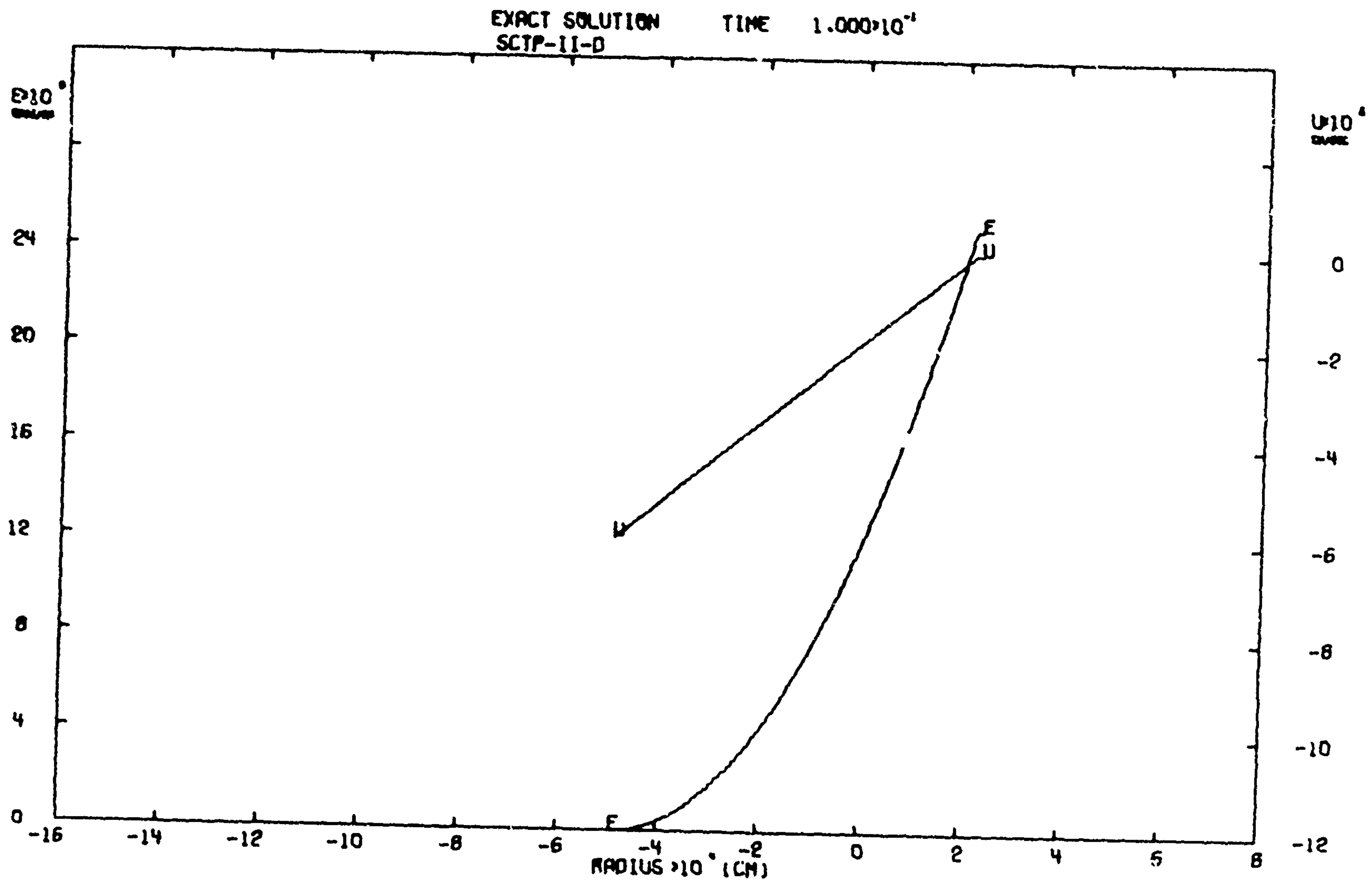

Figure II-D. VE-EXACT

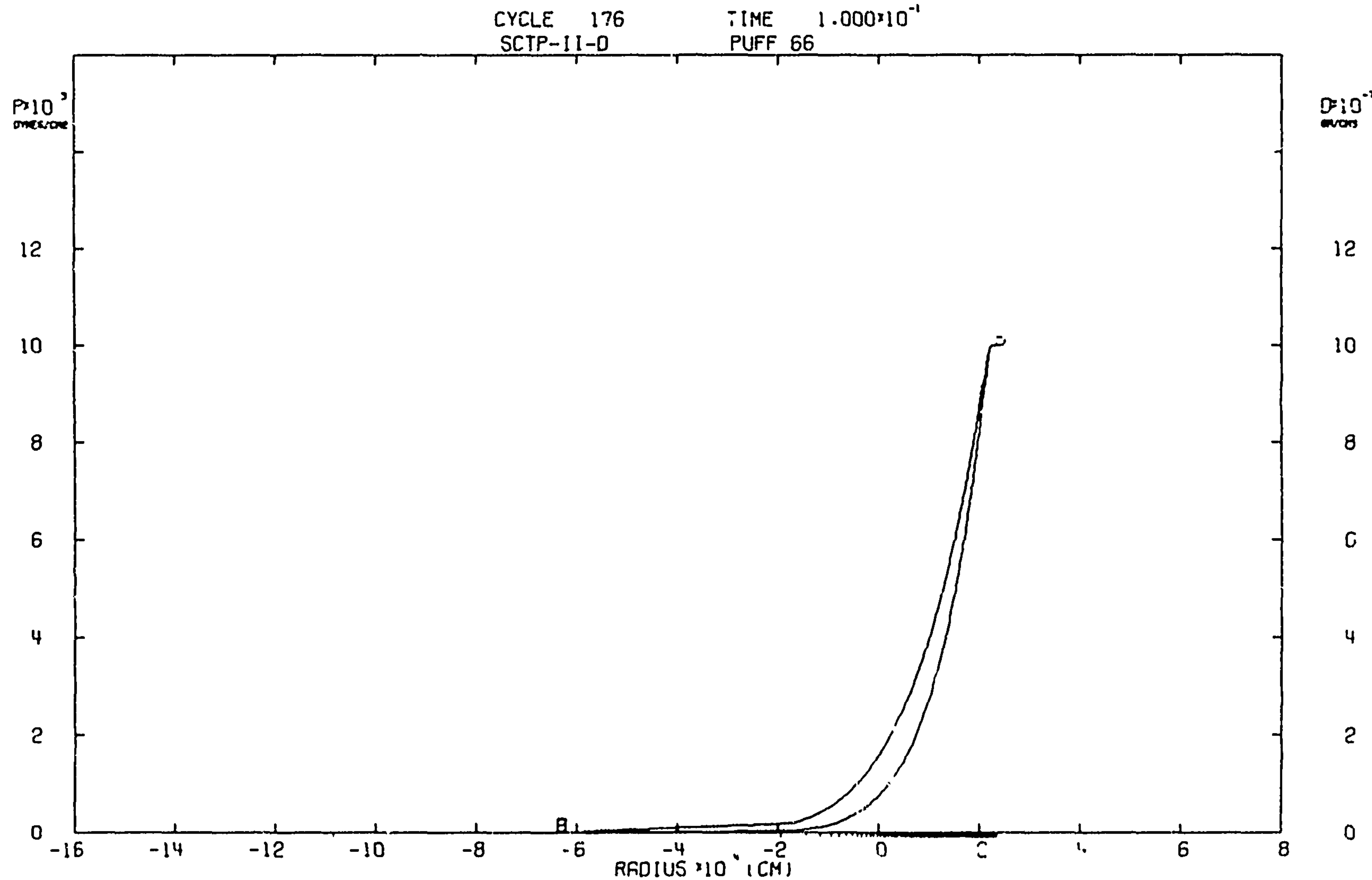

Figure II-D. PD-PUFF

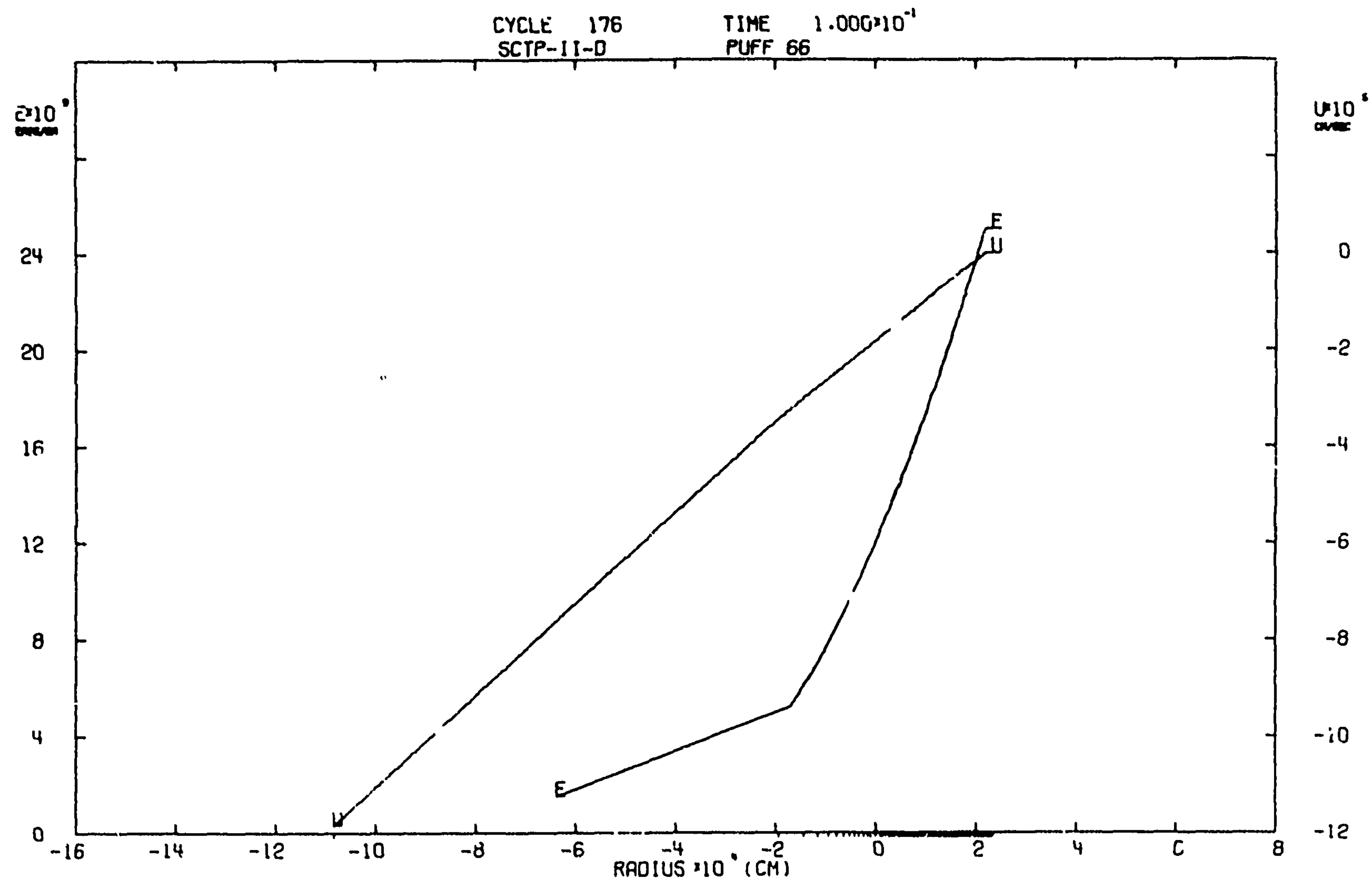

Figure II-D. VE-PUFF

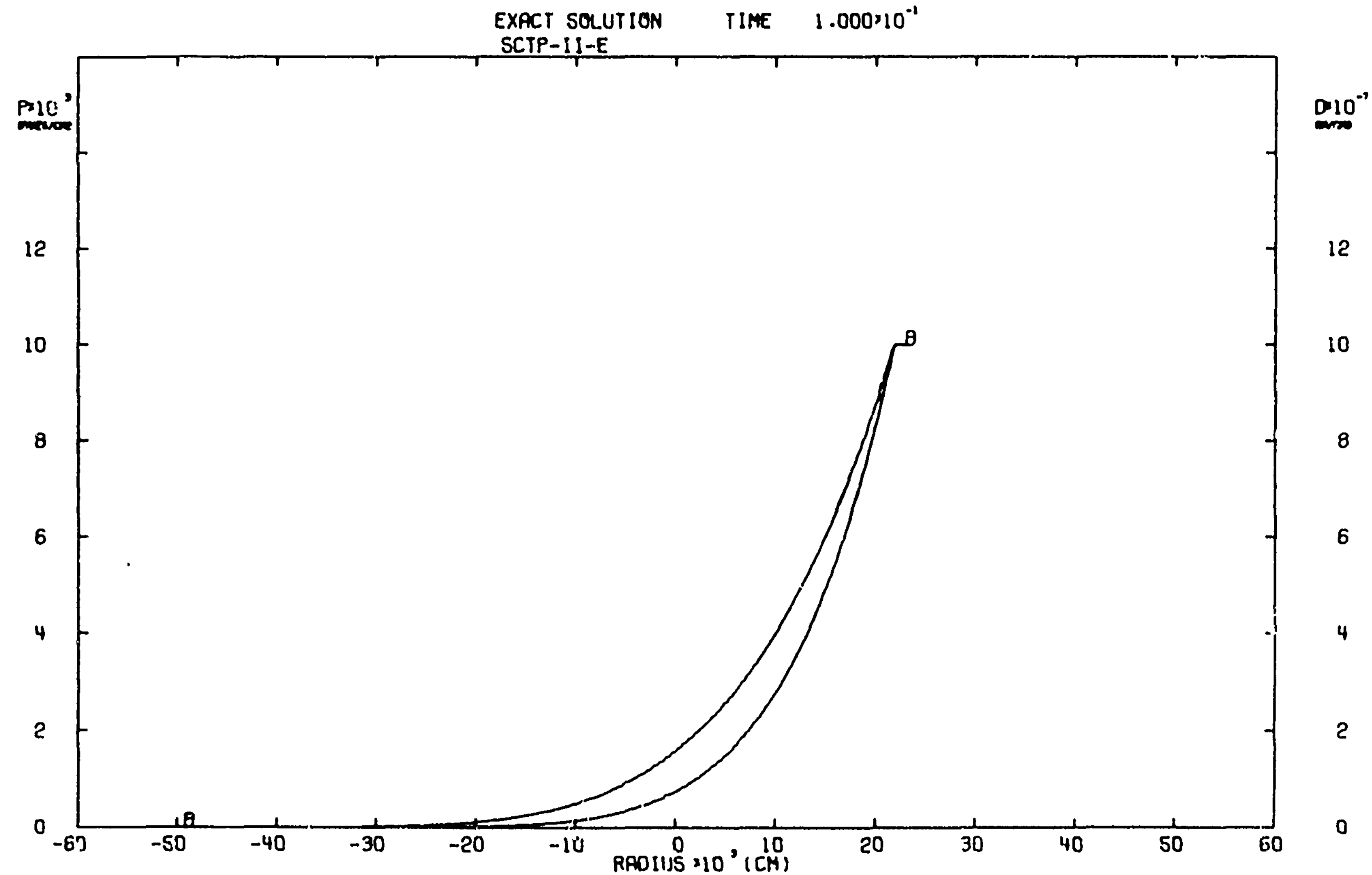

Figure II-E. PD-EXACT

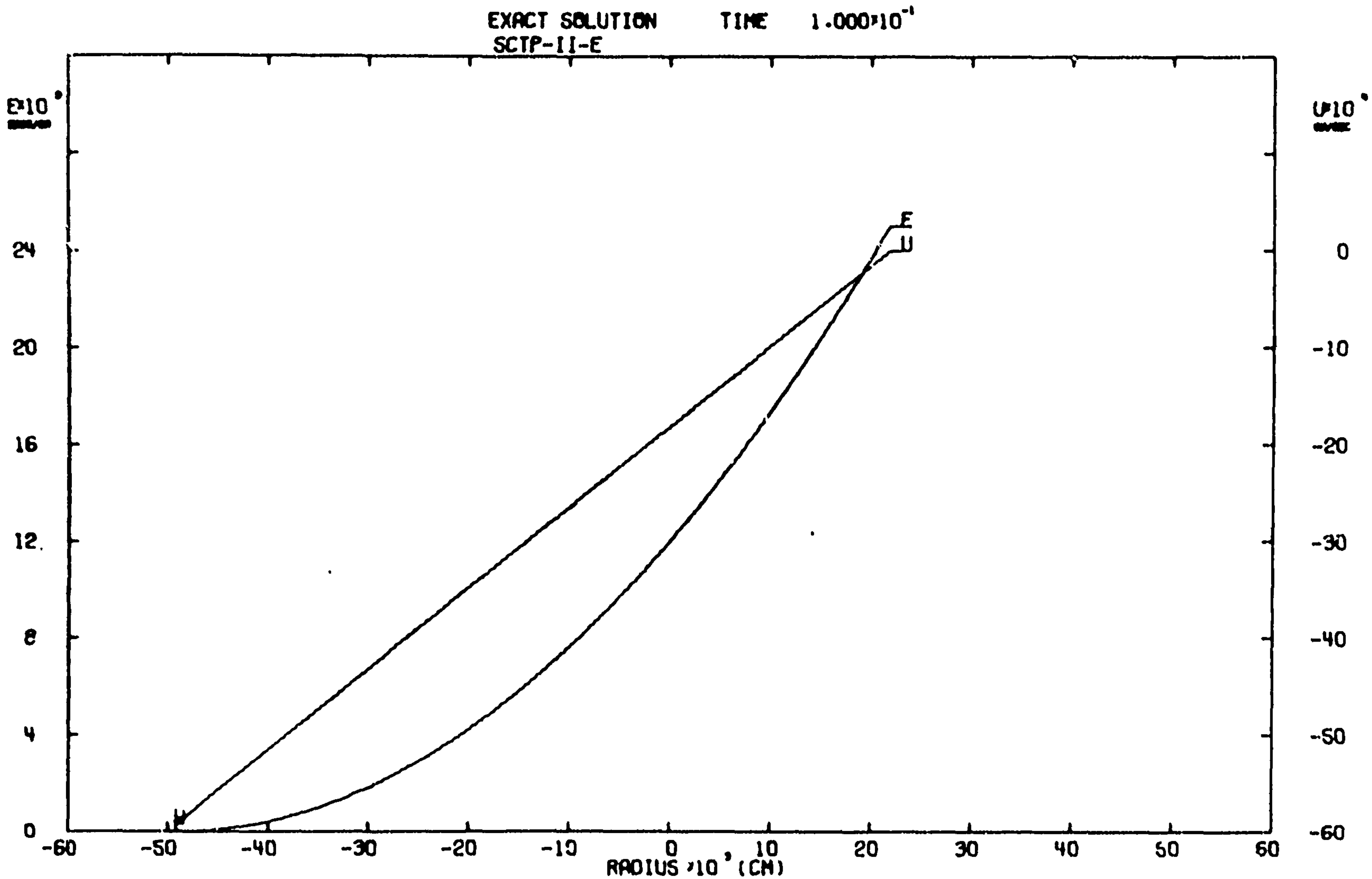

Figure II-E. VE-EXACT

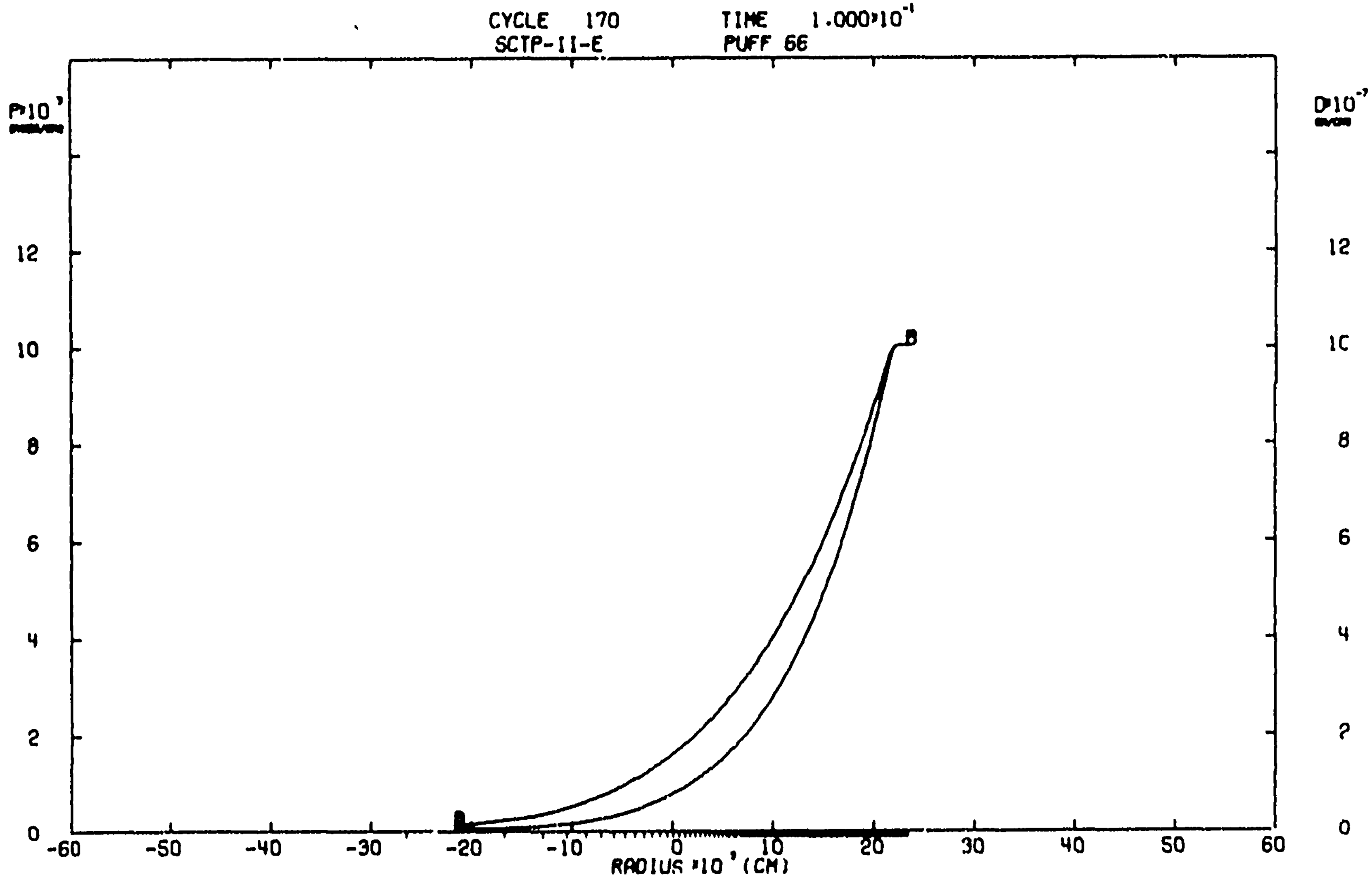

Figure II-E. PD-PUFF

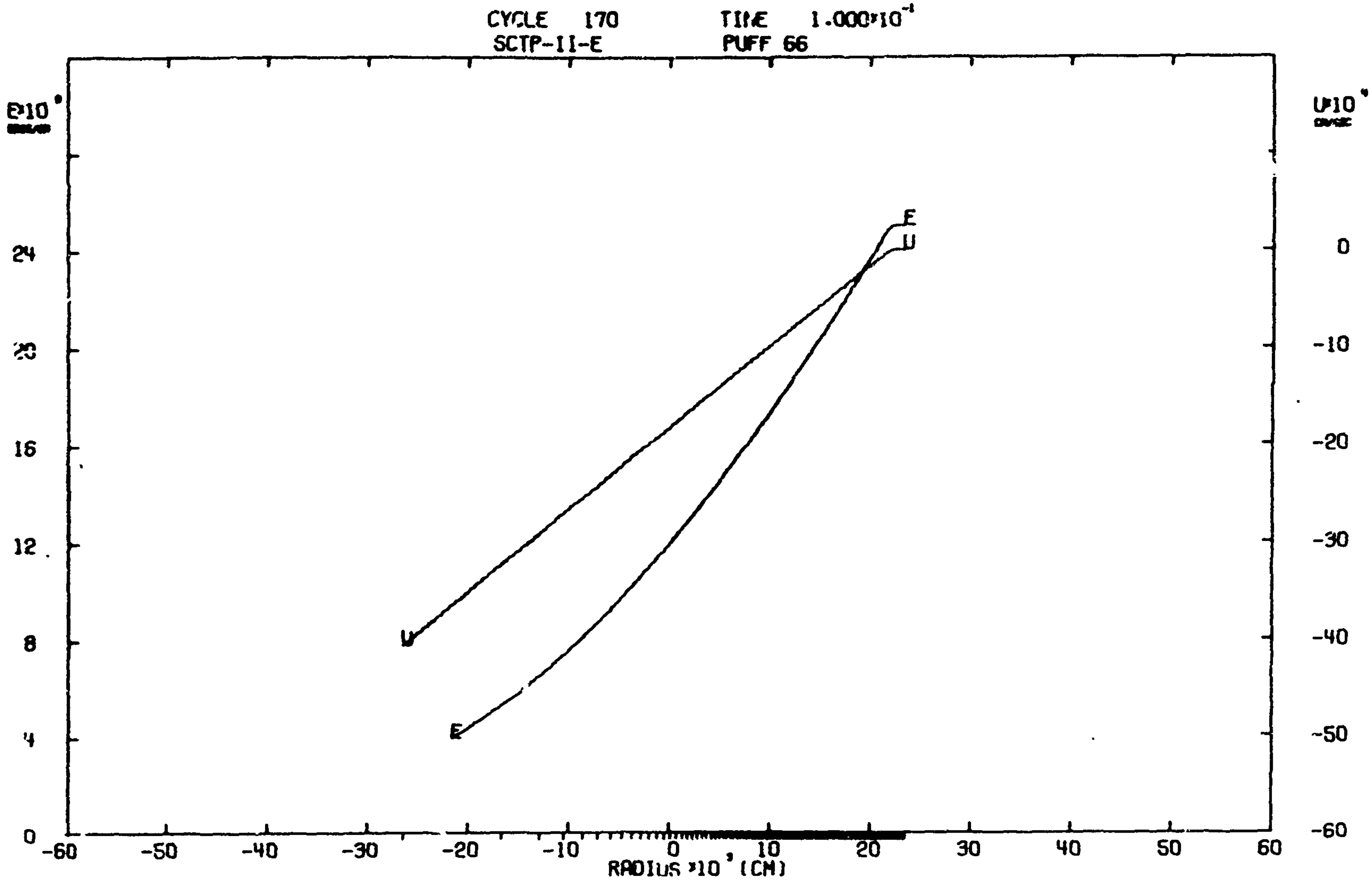

Figure II-E. VE-PUFF

3. TEST PROBLEM SCTP-III

a. The Exact Solution

In this problem, a piston proceeds with a constant acceleration into a gas initially at rest (by "gas initially at rest" is meant that the initial conditions are as follows: velocity is zero; density, pressure and all other fluid parameters are constant). This forms what is called a compression wave. At time $t_S = 2C_r/a(\gamma+1)$, a shock wave is formed (t_S = time of shock formation, C_r = sound speed of the gas at rest, a = acceleration of the piston). Until time t_S, the variables are continuous and the solution is easily found.* Moreover, except for one point (the front of the compression wave), the variables are smooth prior to t_S. The compression wave front up to time t_S is $X_C(t) = C_r t$. After that time, the compression wave front is a shock, i.e., there is a discontinuity in pressure, density, velocity, etc.

The solution for the velocity is

$$v(X,t) = \frac{-\left(C_r - \frac{\gamma+1}{2}\, a\, t\right) + \sqrt{\left(C_r - \frac{\gamma+1}{2}\, a\, t\right)^2 + 2a\gamma\left(C_r t - X\right)}}{\gamma}$$

for $X_P \leq X \leq X_C$, $0 \leq t \leq t_S$,

$$X_P = \frac{1}{2}\, a\, t^2, \quad X_C = C_r t$$

and

$$v(X,t) = 0 \text{ for } X > X_C$$

* See K. O. Friedrick's paper in 1948 Communications Pure and Applied Mathematics, page 211, for an investigation of the solution after shock formation.

Then the simple wave formulas yield

$$C = C_r\left(1 + \frac{\gamma-1}{2}\ \frac{v}{C_r}\right)$$

$$\rho = \rho\left(\frac{C}{C_r}\right)^{\frac{2}{\gamma-1}}$$

$$P = P_r\left(\frac{C}{C_r}\right)^{\frac{2\gamma}{\gamma-1}}$$

Notice that at time $t_S = 2C_r/a(\gamma+1)$ and position $X_S = C_r t_S$, the

$$\lim_{X \to X_S} v_X(X, t_S) = -\infty .$$

This indicates that a shock forms at (X_S, t_S). For further details see Hydrocode Test Problems, AFWL-TR-67-127.

The necessary data for this problem are:

Initial values: P_r, ρ_r, v_r.

Boundary values: At the piston position $X_P = \frac{1}{2}\,at^2$ the velocity is $v_P = at$.

There are two variations of this problem:

SCTP-III-A:

$P_r = 10^4$ dynes/cm^2

$\rho_r = 10^{-6}$ gm/cm^3

$v_r = 0$

$C_r^2 = \gamma P_r V_r = 1.4 \times 10^{10}$ cm^2/sec^2

$a = C_r/1$ sec

$\Delta X = 10$ meters

$X_Q = 1500$ meters

This problem is run to 1 second. The shock forms at .833... second.

SCTP-III-B

We vary this from A by setting

$a = 10\ C_r/1$ sec

$\Delta X = 1$ meter

$X_Q = 150$ meters

This problem is run to .1 second. The shock forms at .08333 second.

b. The PUFF Solution

The main error made by PUFF is a slight overround at X_C. See Tables and Figures III.

c. The LAX-WENDROFF Solution

The most noticeable error made by the LAX-WENDROFF scheme is the oscillation just left of X_C. See Tables and Figures III. The time step factor used was .78, the artificial viscosity factor used was .5.

Table III-A

ERRORS ON SCTP-III-A

PUFF

Problem time = .8333 sec
Computer time = 22 sec

Cycle = 244
Number of Active Zones = 118

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.235	.054	+ .038	X_C
Velocity	.324	.113	+ .081	X_C
Density	.246	.052	+ .036	X_C
Energy	.099	.031	+ .023	X_C
	Sum Int. Energy	**Sum Kin. Energy**	**Sum Tot. Energy**	
EXACT	4.36800×10^9	2.62864×10^8	4.63087×10^9	
PUFF	4.36884×10^9	2.62404×10^8	4.63124×10^9	

LAX-WENDROFF

Problem time = .8333 sec
Computer time = 48 sec

Cycle = 214
Number of Active Zones = 150

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.105	.035	- .021	2 zones left of X_C
Velocity	.230	.078	- .047	2 zones left of X_C
Density	.098	.033	- .020	2 zones left of X_C
Energy	.058	.019	- .012	2 zones left of X_C
	Sum Int. Energy	**Sum Kin. Energy**	**Sum Tot. Energy**	
EXACT	4.36794×10^9	2.62827×10^8	4.63077×10^9	
LAXWEN	4.36796×10^9	2.62788×10^8	4.63075×10^9	

Table III-B

ERRORS ON SCTP-III-B

PUFF

Problem time = .08333 sec
Computer time = 22 sec

Cycle = 244
Number of Active Zones = 118

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.235	.054	+ .038	X_C
Velocity	.224	.113	+ .081	X_C
Density	.246	.052	+ .036	X_C
Energy	.099	.031	+ .023	X_C
	Sum Int. Energy	**Sum Kin. Energy**	**Sum Tot. Energy**	
FXACT	4.36800×10^8	2.62864×10^7	4.63087×10^8	
PUFF	4.36884×10^8	2.62404×10^7	4.63124×10^8	

LAX-WENDROFF

Problem time = .08333
Computer time = 48 sec

Cycle = 214
Number of Active Zones = 150

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.105	.035	- .021	2 zones left of X_C
Velocity	.230	.078	- .047	2 zones left of X_C
Density	.098	.033	- .020	2 zones left of X_C
Energy	.057	.019	- .012	2 zones left of X_C
	Sum Int. Energy	**Sum Kin. Energy**	**Sum Tot. Energy**	
EXACT	4.36794×10^8	2.62827×10^7	4.63077×10^8	
LAXWEN	4.36796×10^8	2.62788×10^7	4.63075×10^8	

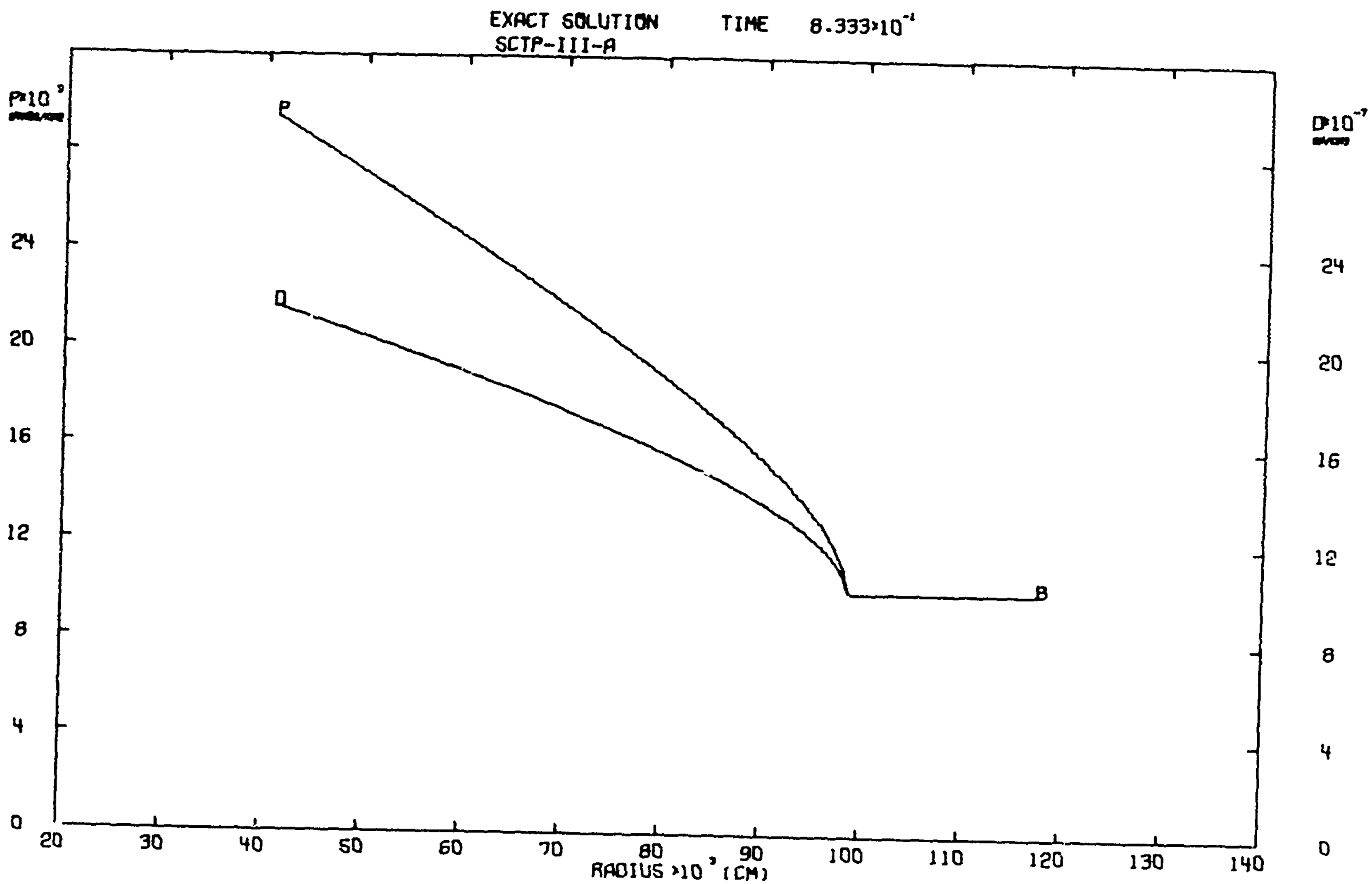

Figure III-A. PD-EXACT

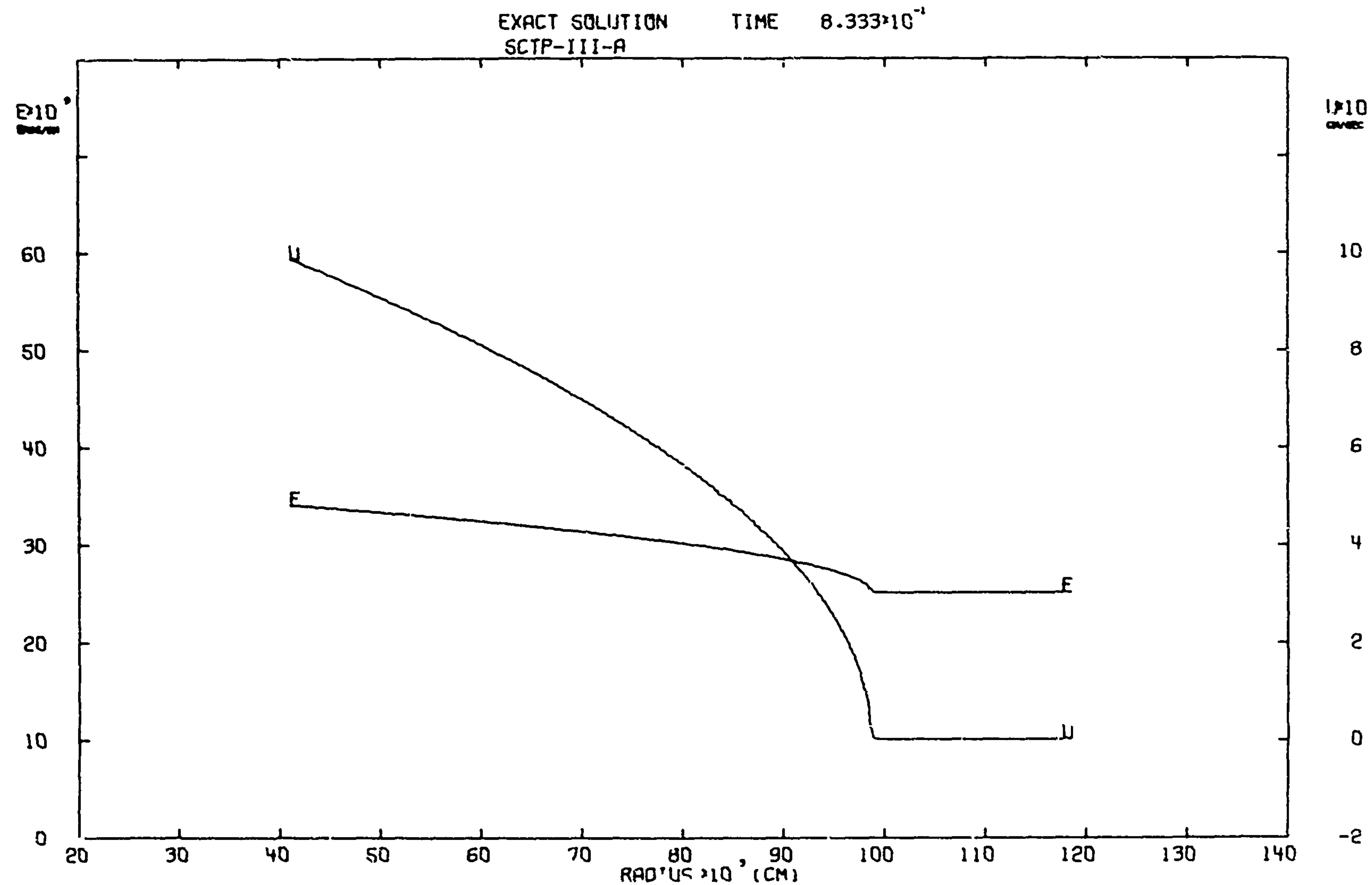

Figure III-A. VE-EXACT

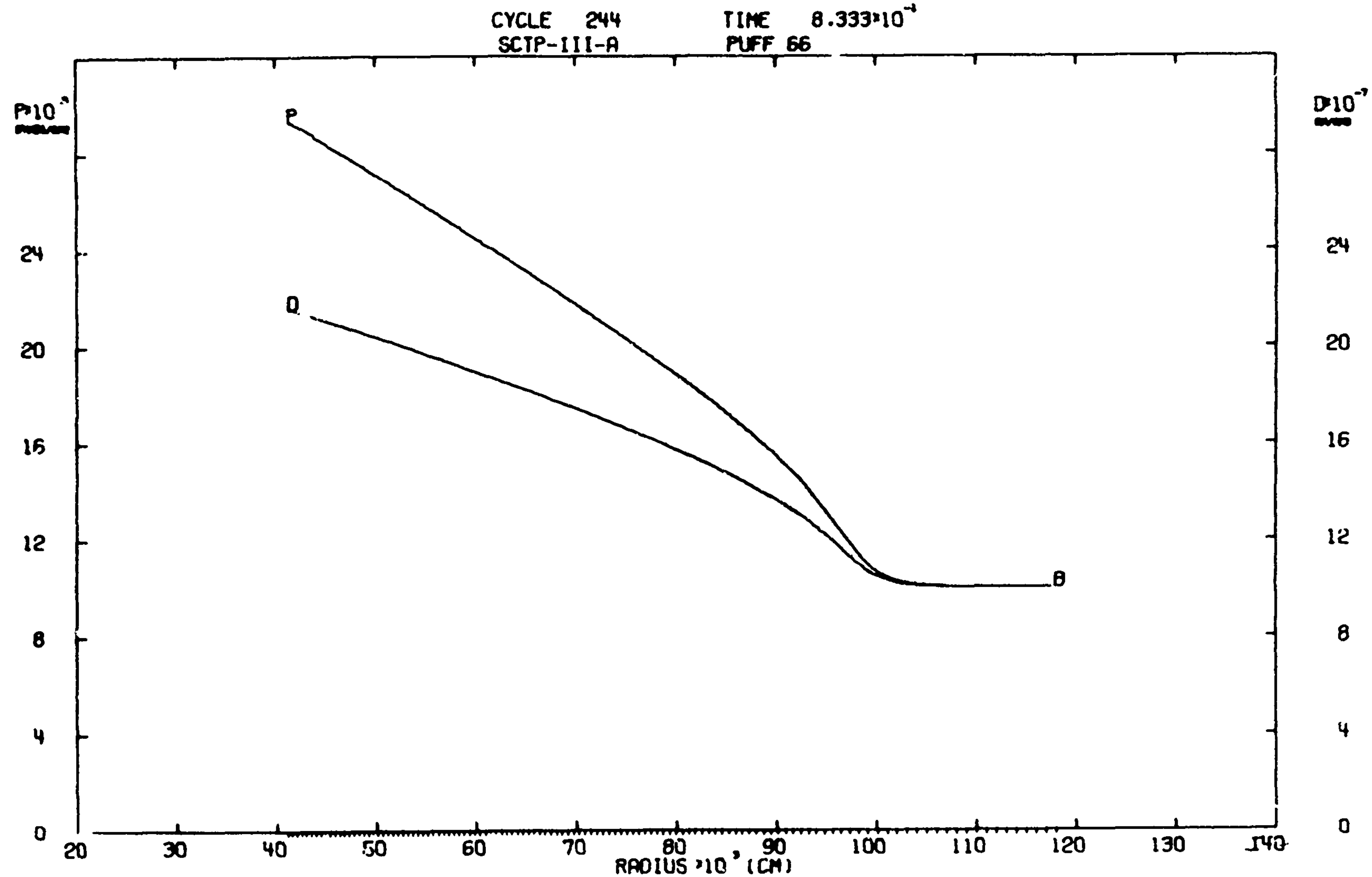

Figure III-A. PD-PUFF

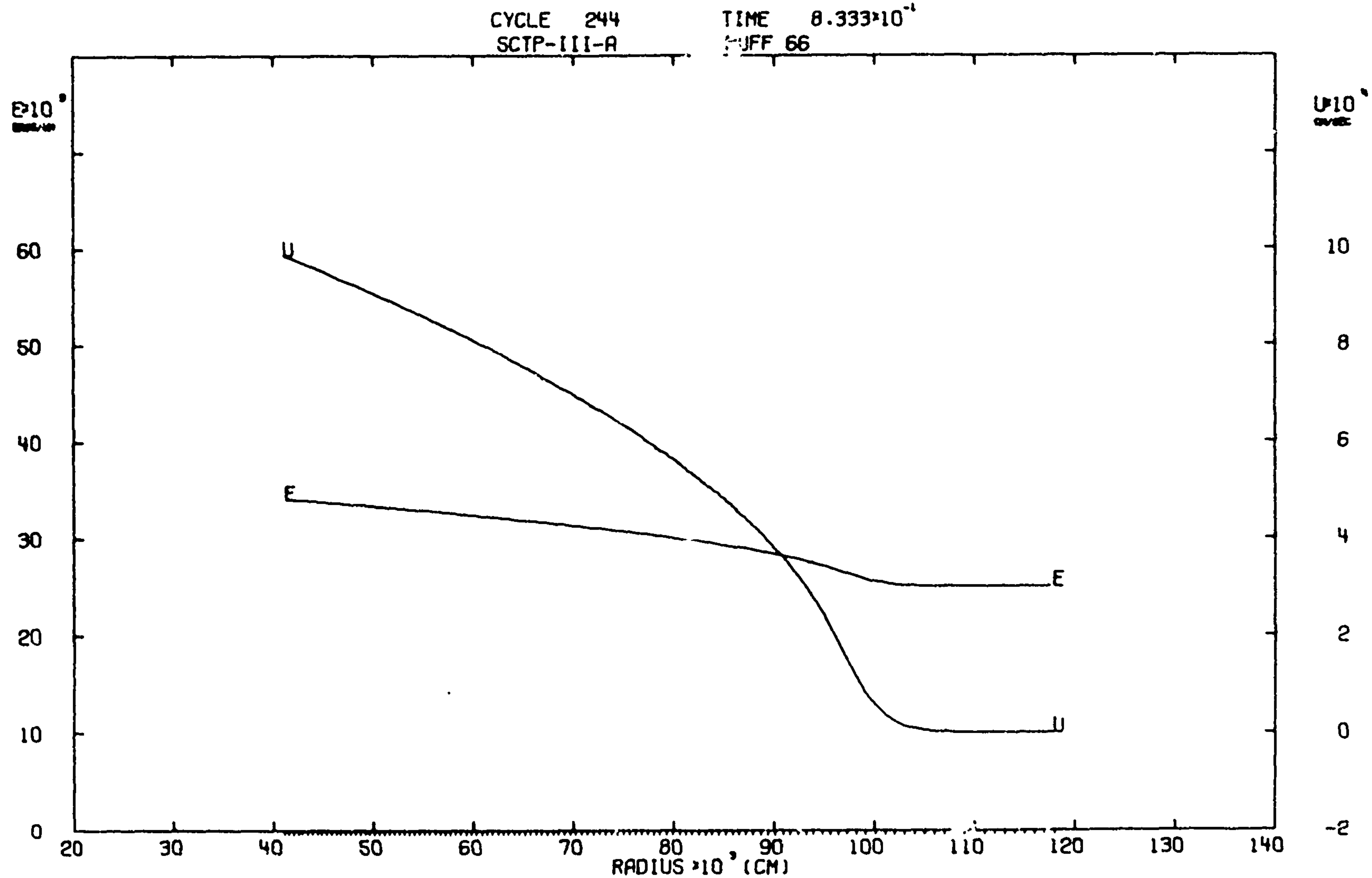

Figure III-A. VE-PUFF

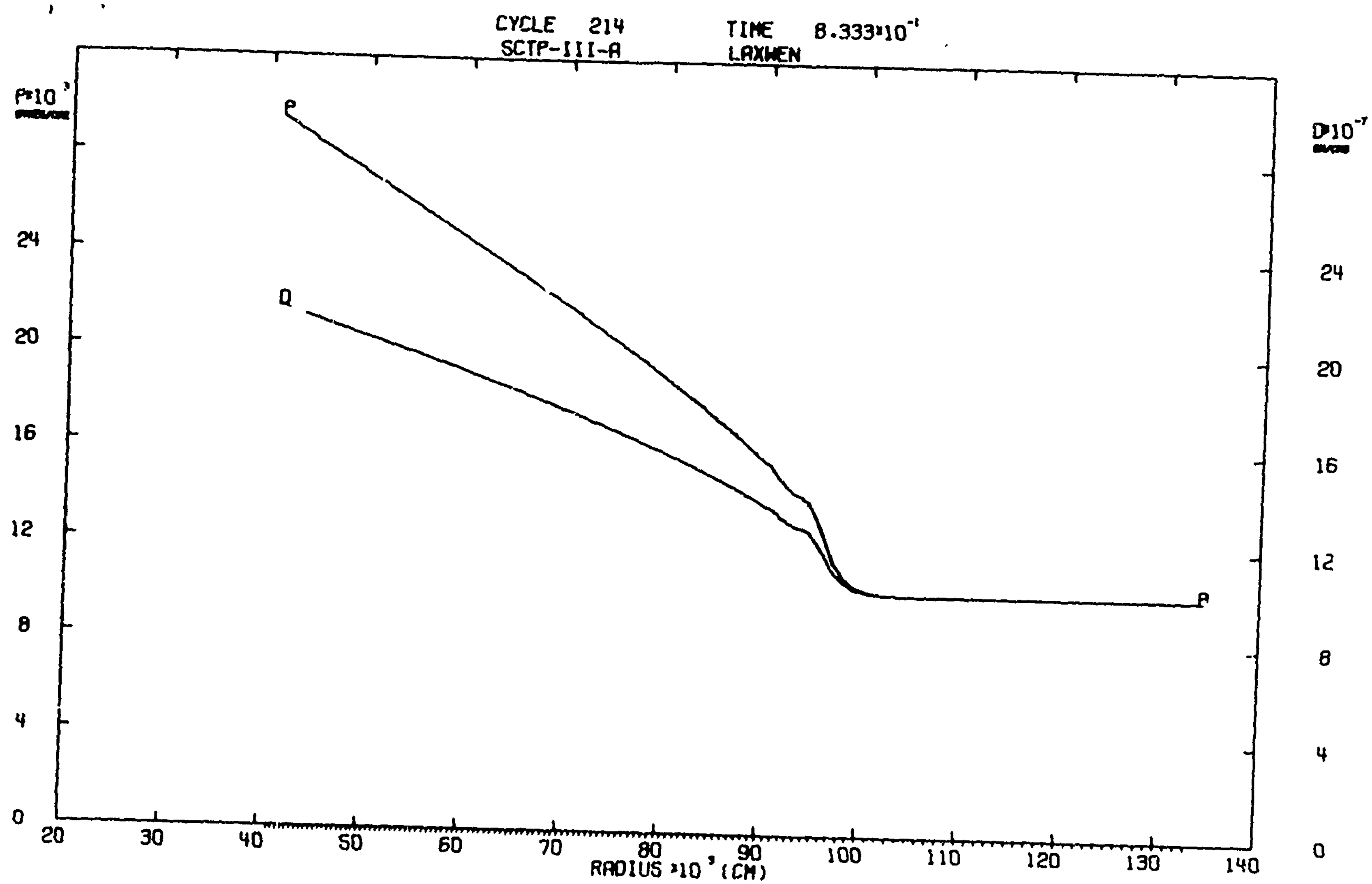

Figure III-A. PD-LAX-WENDROFF

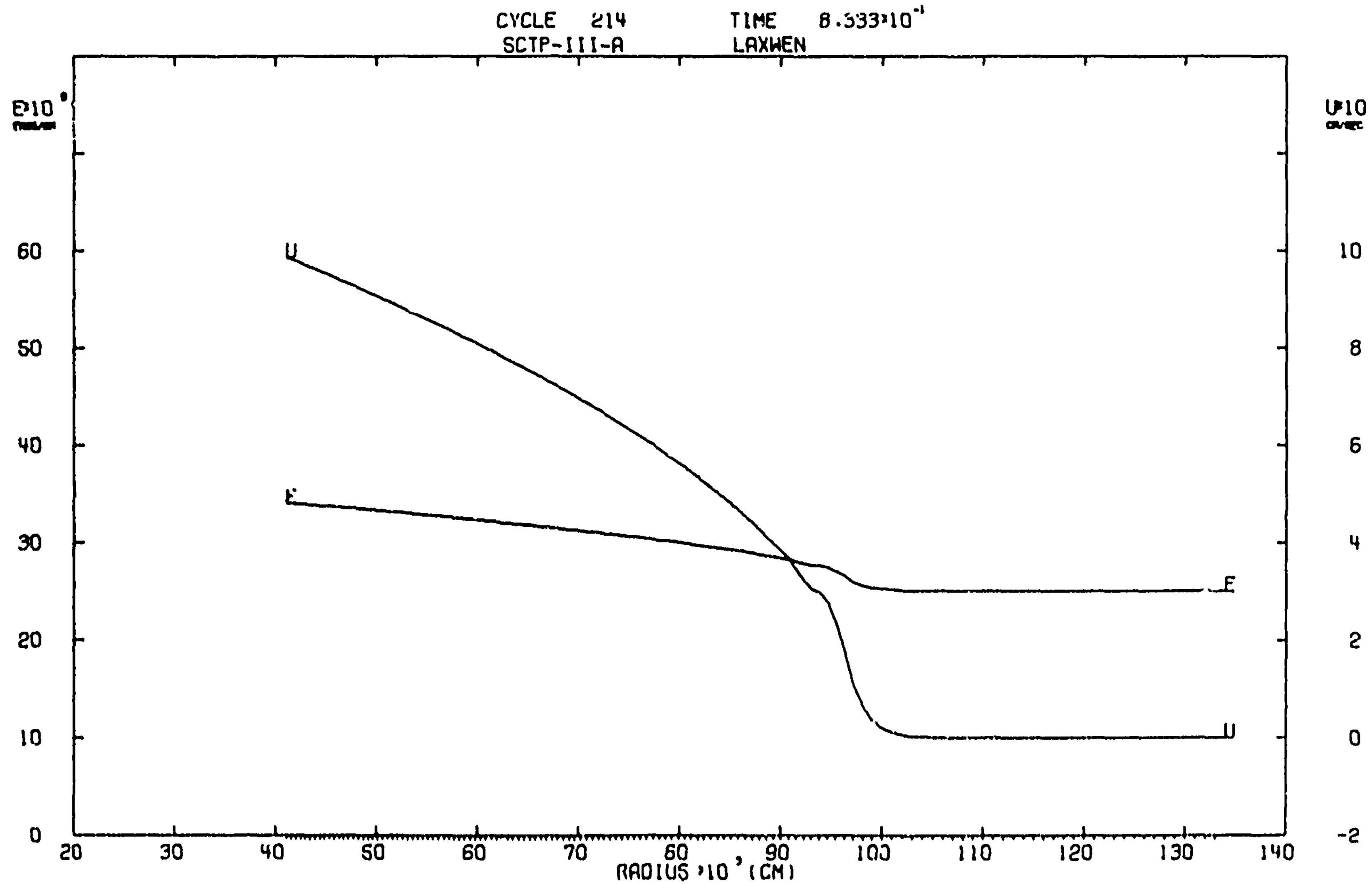

Figure III-A. VE-LAX-WENDROFF

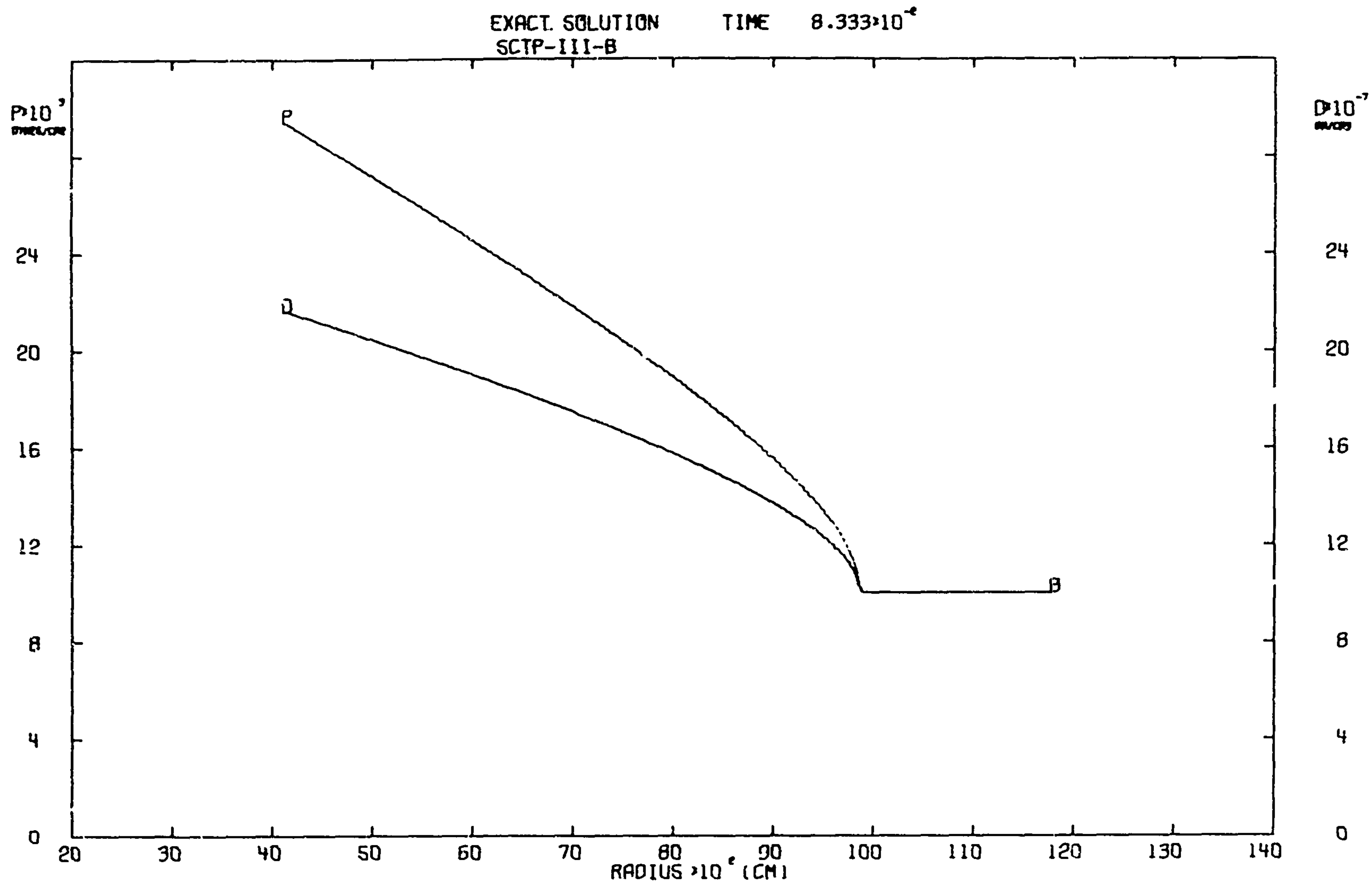

Figure III-B. PD-EXACT

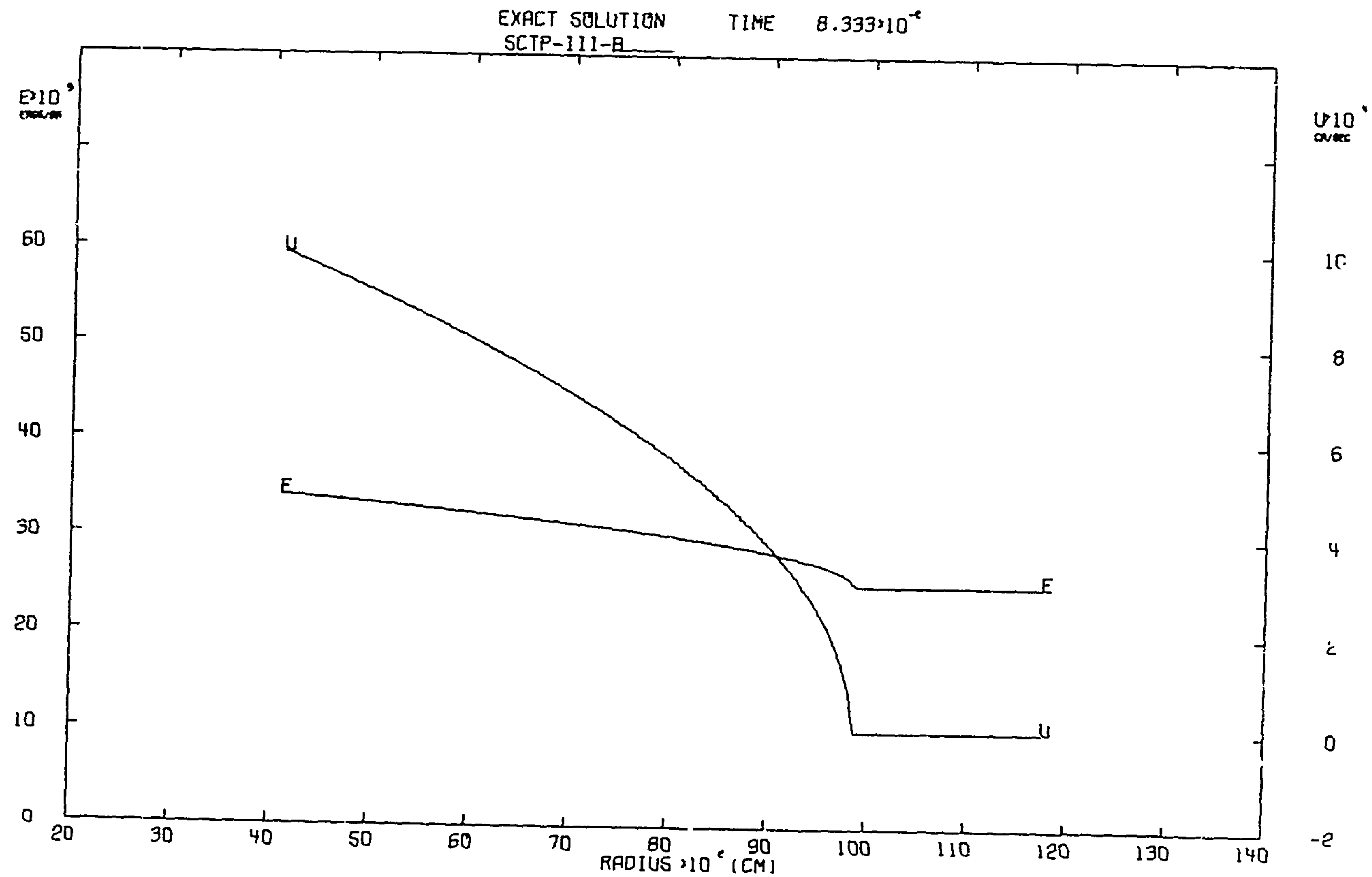

Figure III-B. VE-EXACT

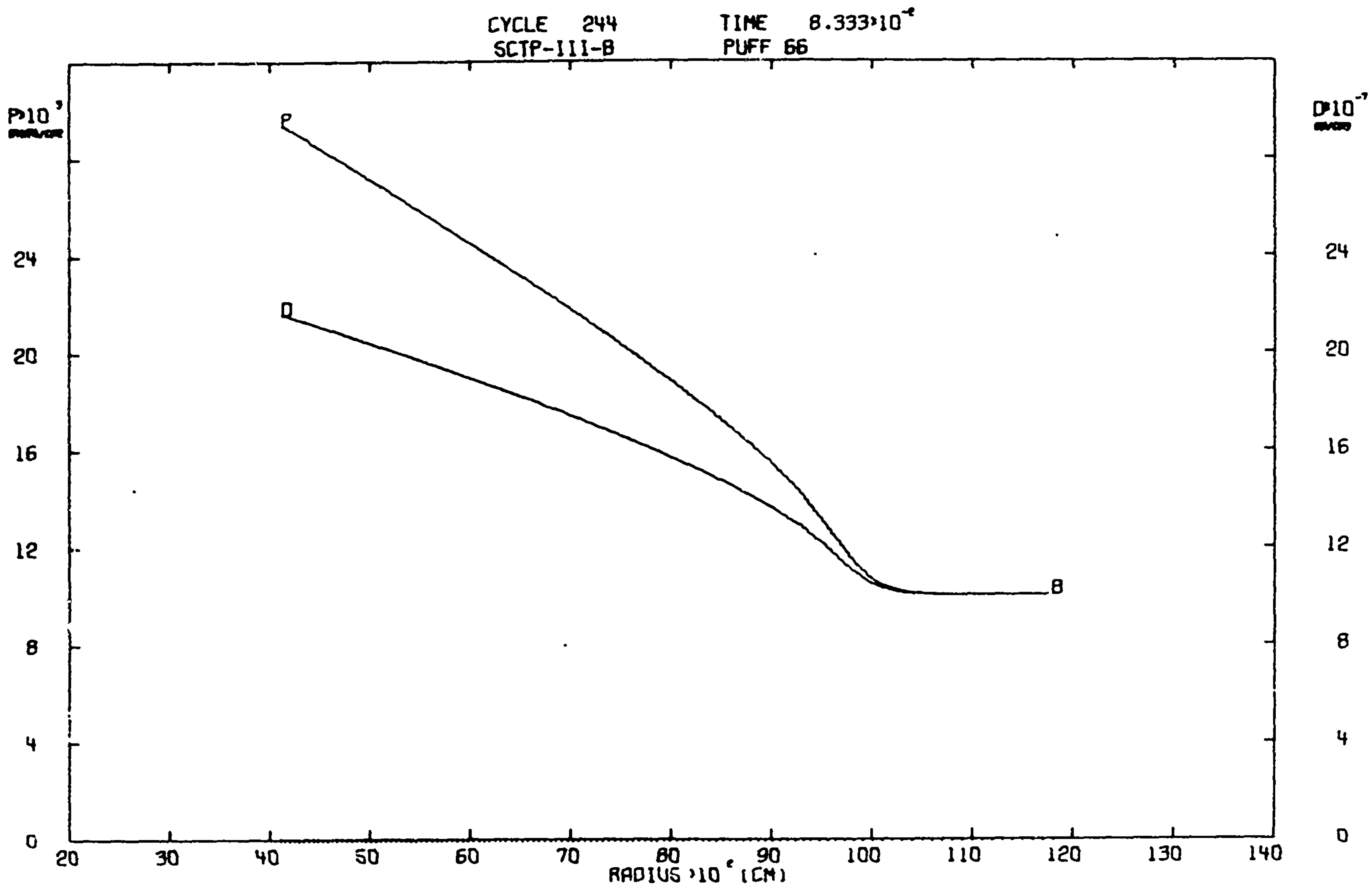

Figure III-B. PD-PUFF

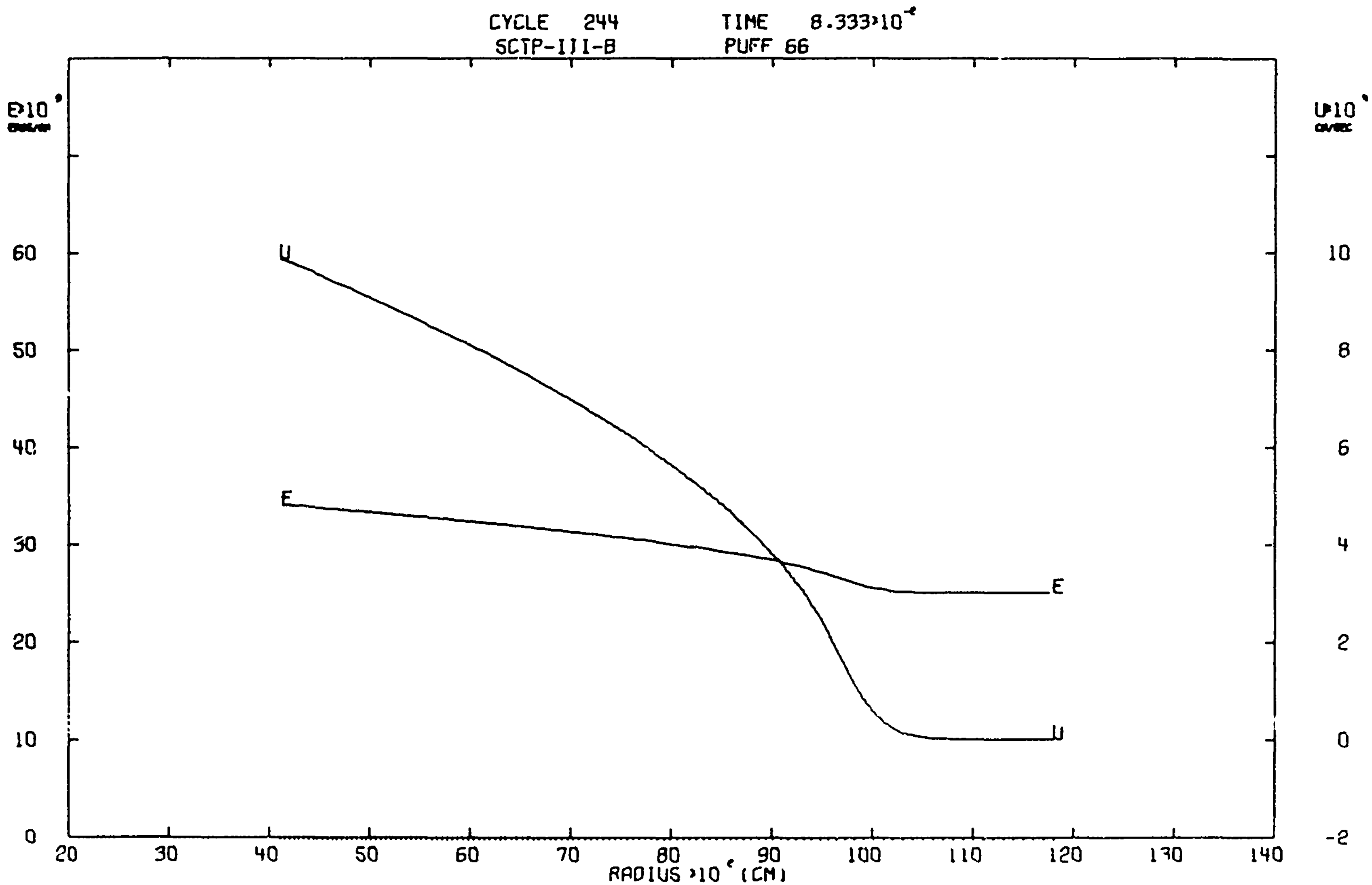

Figure III-B. VE-PUFF

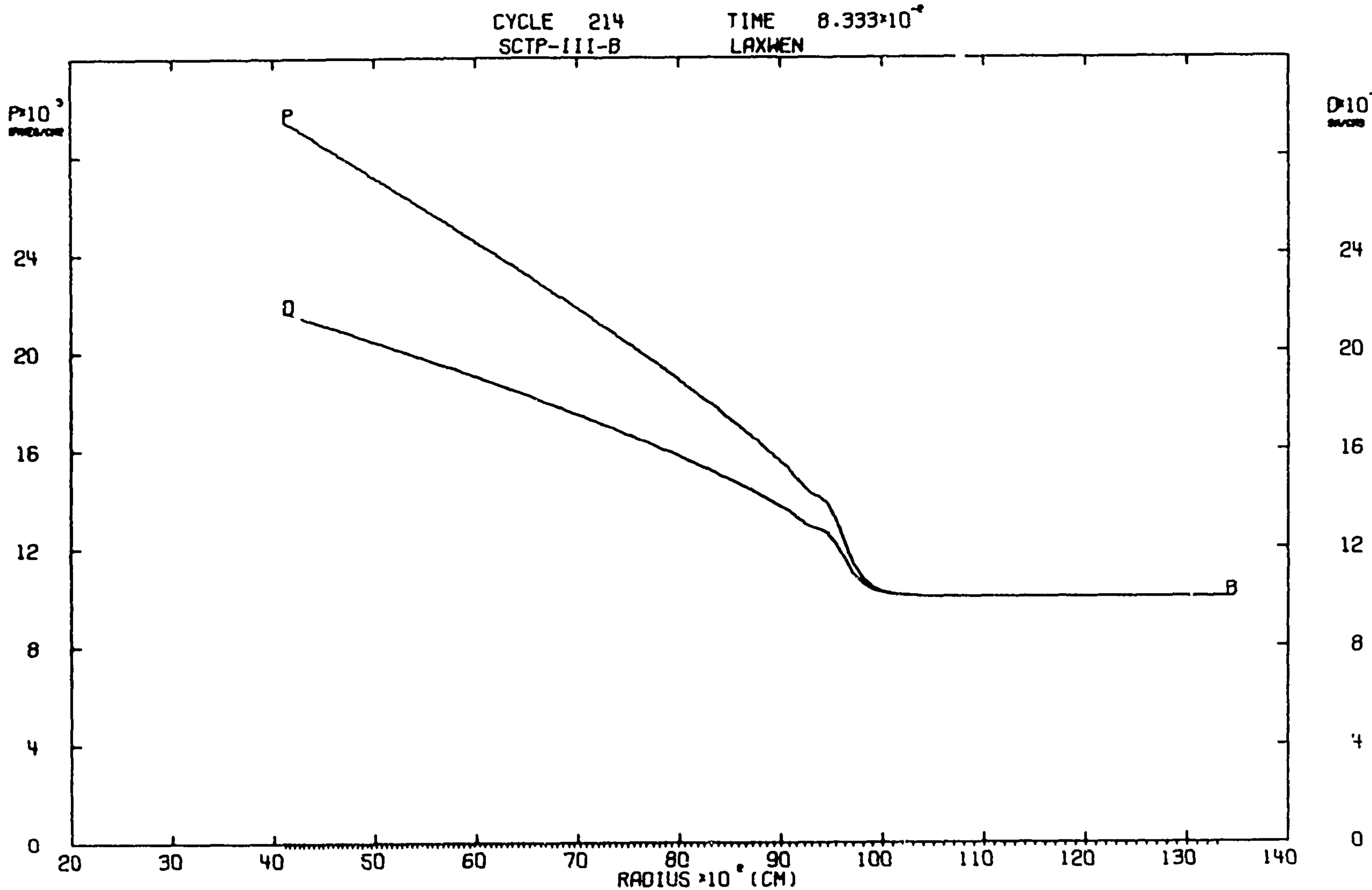

Figure III-B. PD-LAX-WENDROFF

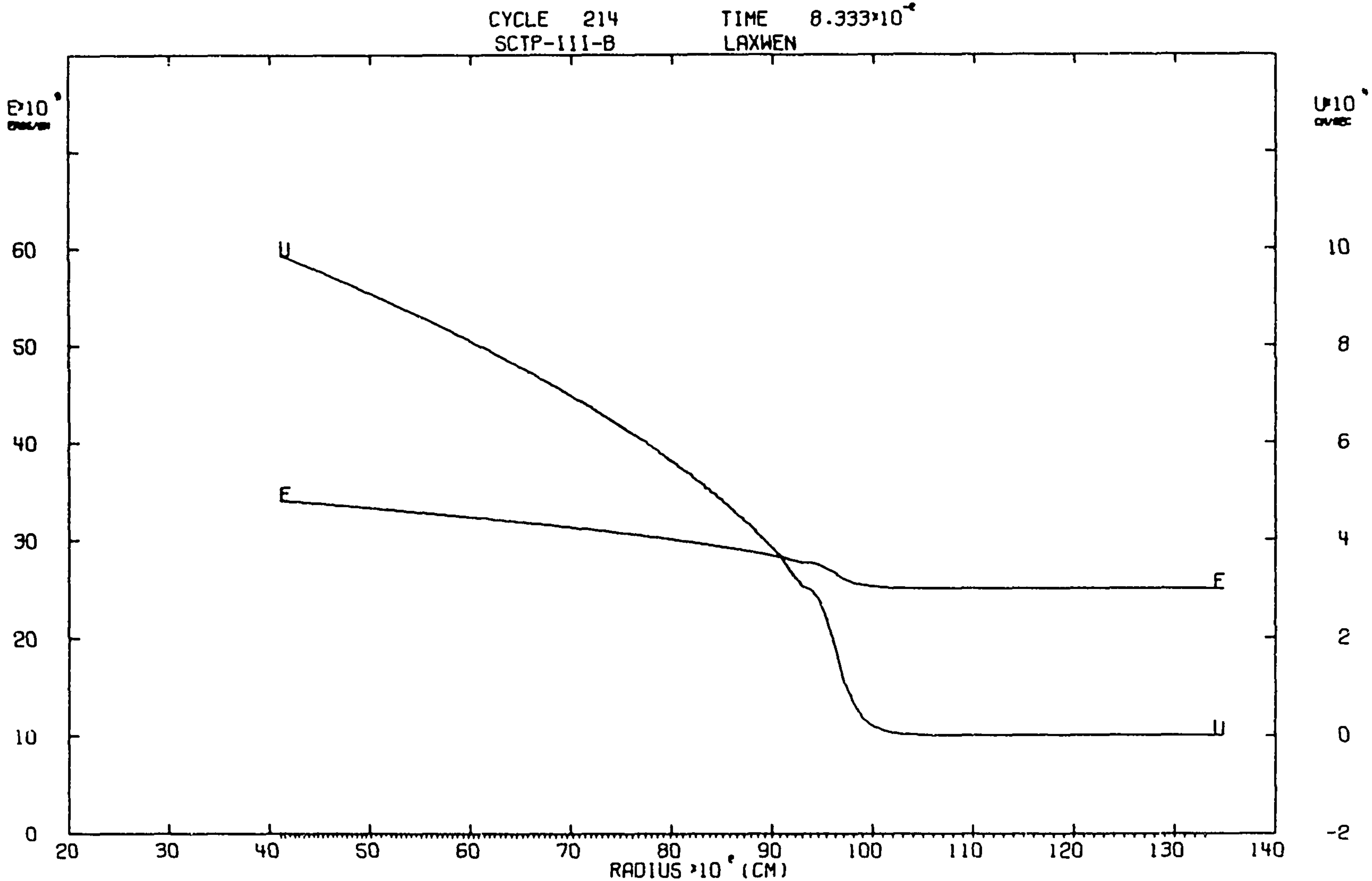

Figure III-B. VE-LAX-WENDROFF

4. TEST PROBLEM SCTP-IV

a. The Exact Solution

In this problem a piston has a constant acceleration, a, away from a gas at rest. The piston is traveling to the left. Eventually, the piston speed exceeds the speed with which the gas can follow. This speed is $2C_r/(\gamma-1)$, where C_r is the sound speed of the gas at rest. This speed is called the escape speed of the gas and when the piston exceeds this speed a vacuum occurs between the piston and the edge of the gas. The piston pulls away from the gas at time $t_v = 2C_r/|a|(\gamma-1)$ and position $X_p(t_v) = X_v = \frac{1}{2} at_v^2$. The solution for the velocity is

$$v(X,t) = \frac{-\left(C_r - \frac{\gamma+1}{2}\, at\right) + \sqrt{\left(C_r - \frac{\gamma+1}{2}\, at\right)^2 + 2a\gamma\left(C_r t - X\right)}}{\gamma}$$

for $X_p \le X \le X_C$, $0 \le t \le t_v$, $X_p = \frac{1}{2} at^2$, $X_C = C_r t$

$v(X,t) = 0$ for $X>X_C$ for all times.

For $t \ge t_v$ there is a vacuum from X_p to the gas front. The gas front is at $X_v - \frac{2C_r}{\gamma-1}(t-t_v)$. Therefore the velocity is really meaningless in this region. But the pressure, density, and sound speed are all zero in this region. From the gas front position on to X_C the above formula for the velocity holds. Once the velocity is known the simple wave formulas yield

$$C = C_r\left(1 + \frac{\gamma-1}{2}\,\frac{v}{C_r}\right)$$

$$\rho = \rho_r\left(\frac{C}{C_r}\right)^{\frac{2}{\gamma-1}}$$

$$P = P_r\left(\frac{C}{C_r}\right)^{\frac{2\gamma}{\gamma-1}}$$

The necessary data for this problem are:

Initial values: P_r, ρ_r, v_r

Boundary values: At the piston position $X_p = \frac{1}{2}at^2$ the velocity is $v_p = at$

There are two variations of this problem:

SCTP-IV-A:

$P_r = 10^4$ dynes/cm^2

$\rho_r = 10^{-6}$ gm/cm^3

$v_r = 0$

$C_r^2 = \gamma P_r V_r = 1.4 \times 10^{10}$ cm^2/sec^2

$a = - C_r/1$ sec

$\Delta X = 10$ meters

$X_Q = 1500$ meters

This problem is run to 10 seconds; the vacuum forms at 5 seconds.

SCTP-IV-B:

Same as A but $a = - 10C_r/1$ sec, run the problem to 1 second and the vacuum occurs at .5 second.

b. The PUFF solution

PUFF's main errors in this problem are at X_C and X_P. At X_C the error PUFF makes is an underround in pressure, density, velocity, and internal energy. The error PUFF makes just to the right of X_P is due to the fact that PUFF is a Lagrangian code and the mass that was originally in a zone remains in the zone and therefore the density and pressure can never go to zero. See Tables and Figures IV.

c. The LAX-WENDROFF Solution

Since a vacuum occurs in this problem and the LAX-WENDROFF code uses specific volume as a variable it cannot run this problem.

Table IV-A

ERRORS ON SCTP-IV-A

PUFF

Problem time = 10 sec
Computer time = 21 sec

Cycle = 169
Number of Active Zones = 132

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.067	.014	- .009	X_C
Velocity	.020	.006	- .005	X_P
Density	.061	.011	- .006	X_C
Energy	.132	.090	+ .089	X_P

	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	3.09266×10^{10}	6.98426×10^{9}	3.79108×10^{10}	
PUFF	3.04767×10^{10}	8.89789×10^{9}	3.93746×10^{10}	

LAX-WENDROFF

Problem time =
Computer time =

Cycle =
Number of Active Zones =

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure				
Velocity	LAX-WENDROFF scheme won't run on this one because of the vacuum			
Density				
Energy				

	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT				
LAXWEN				

Table IV-B

ERRORS ON SCTP-IV-B

PUFF

Problem time = 1 sec
Computer time = 15 sec

Cycle = 19
Number of Active Zones = 20

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.140	.054	- .040	X_C
Velocity	.023	.009	- .006	X_P
Density	.127	.045	- .029	X_C
Energy	.216	.161	+ .160	X_P
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	3.68427×10^{10}	6.98426×10^{8}	3.75411×10^{10}	
PUFF	3.67320×10^{10}	2.82060×10^{9}	3.95526×10^{10}	

LAX-WENDROFF

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure				
Velocity	LAX-WENDROFF scheme won't run on this one because of the vacuum			
Density				
Energy				
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT				
LAXWEN				

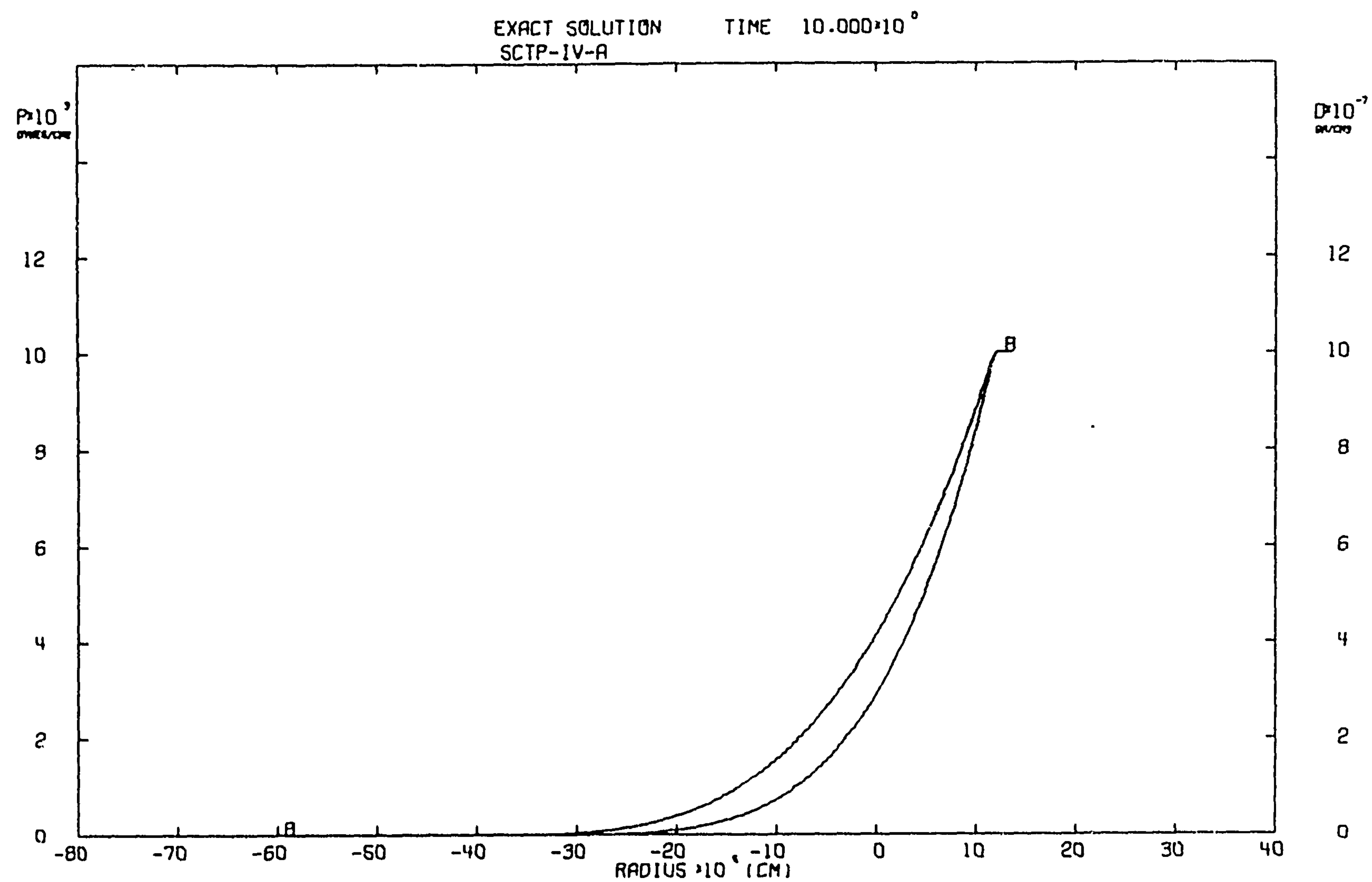

Figure IV-A. PD-EXACT

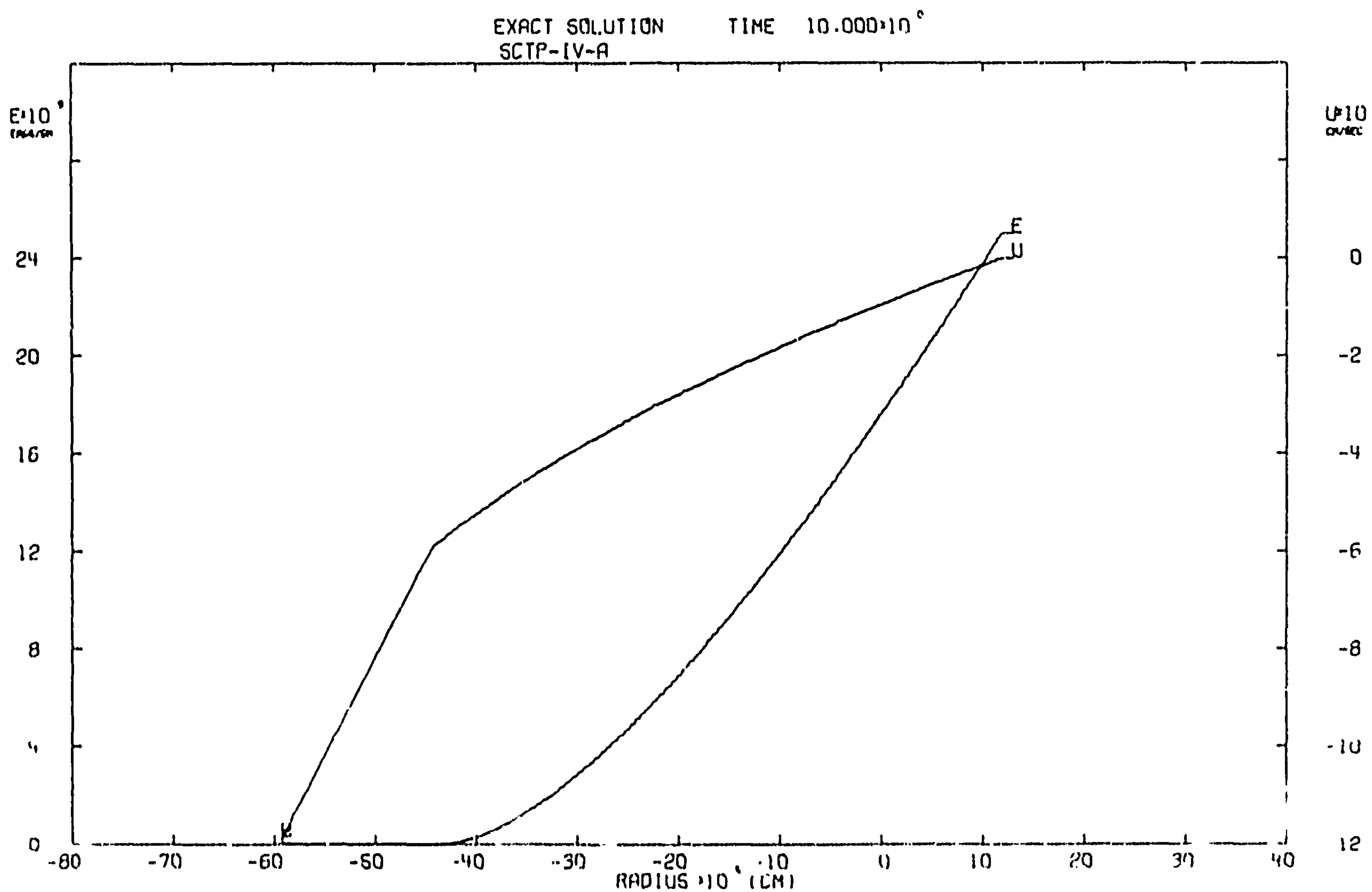

Figure IV-A. VE-EXACT

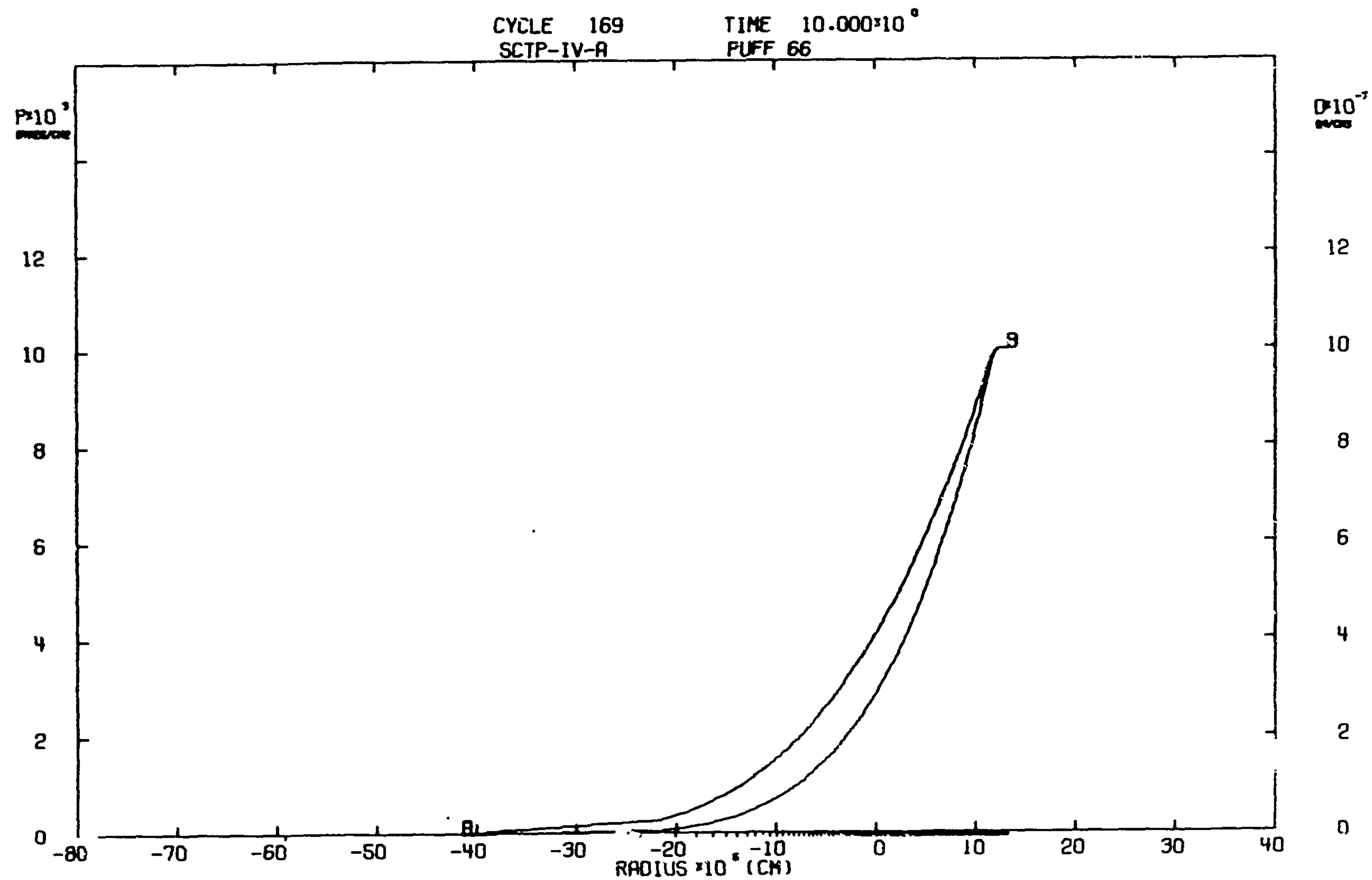

Figure IV-A. PD-PUFF

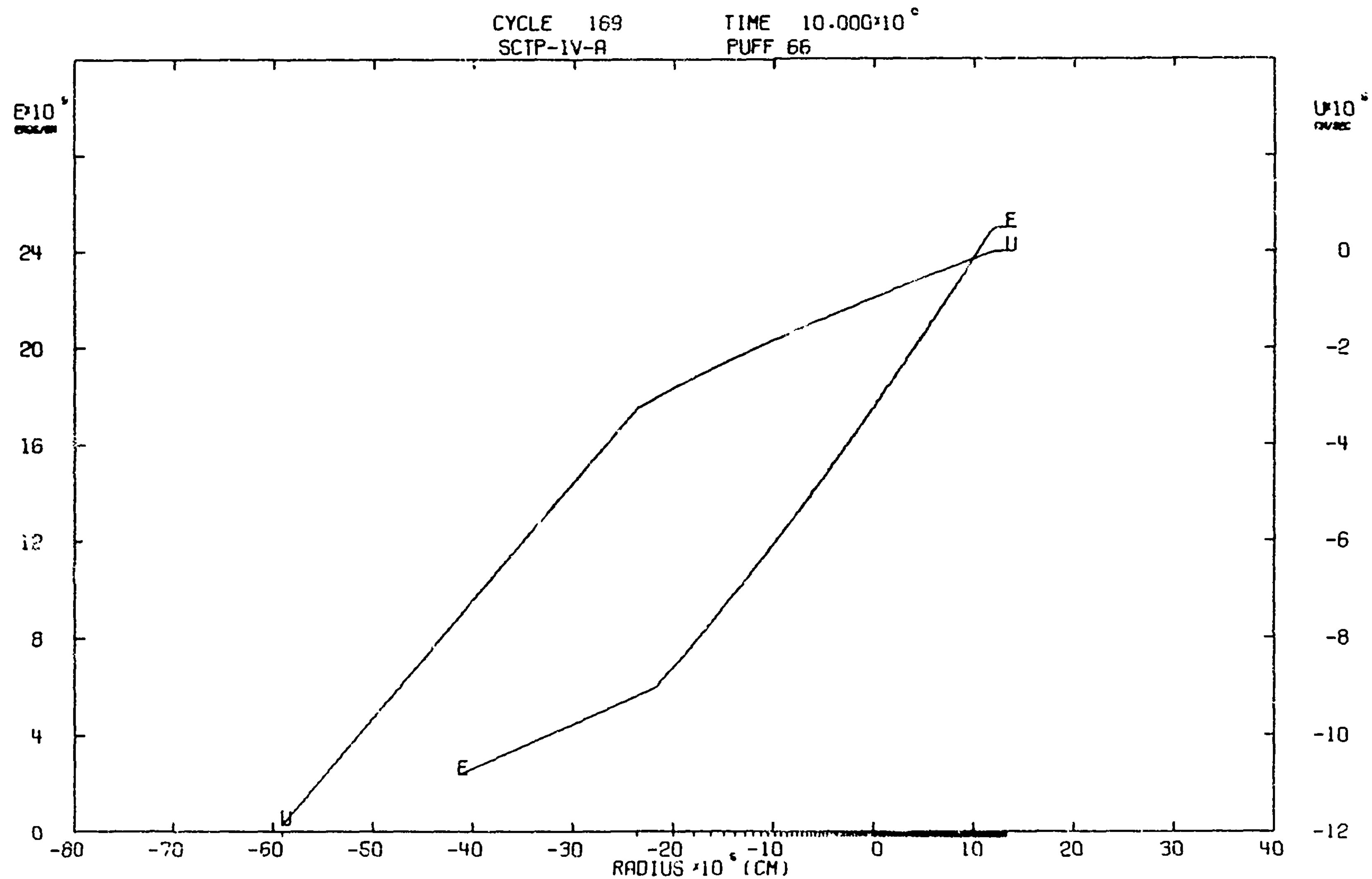

Figure IV-A. VE-PUFF

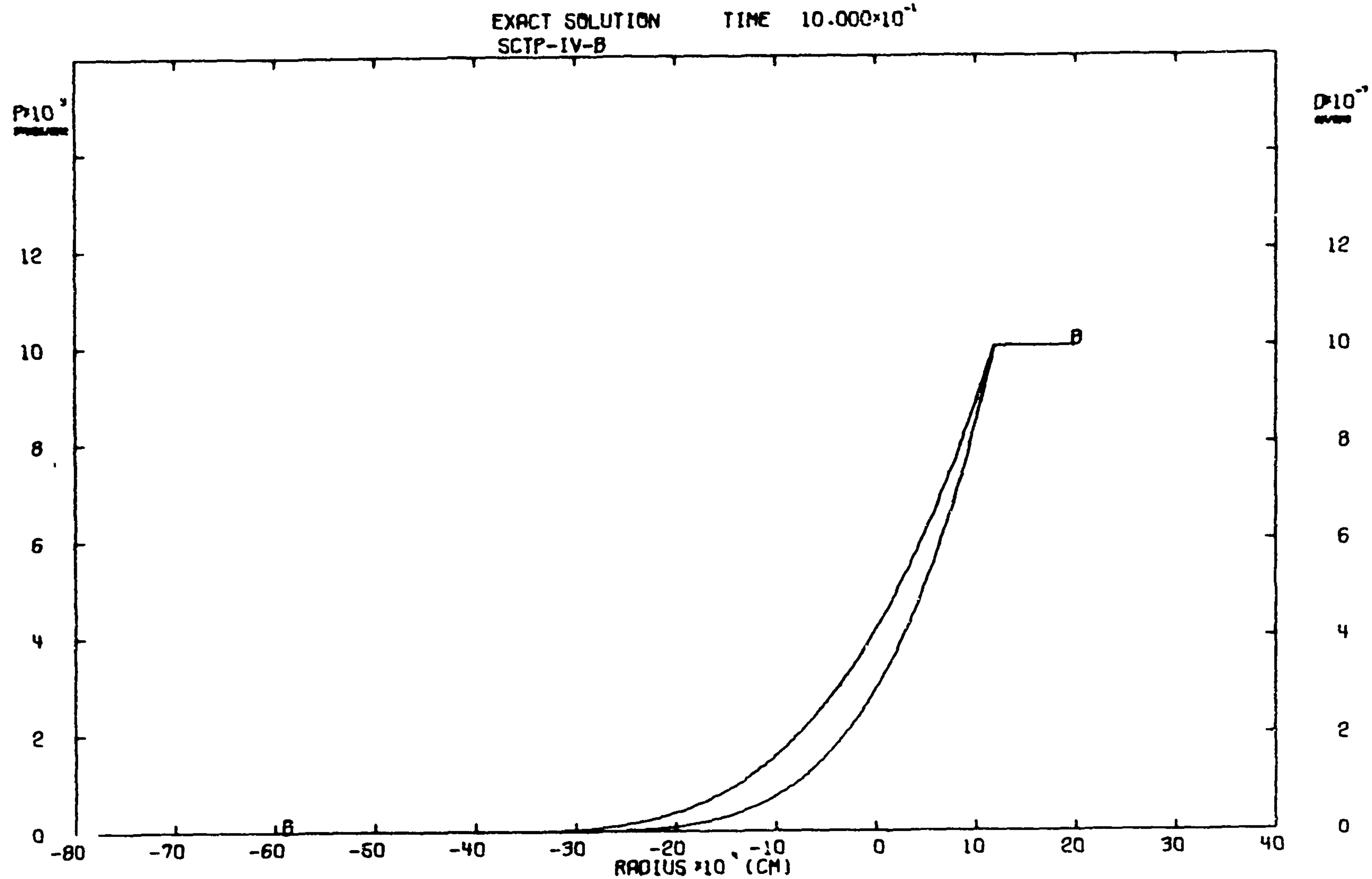

Figure IV-B. PD-EXACT

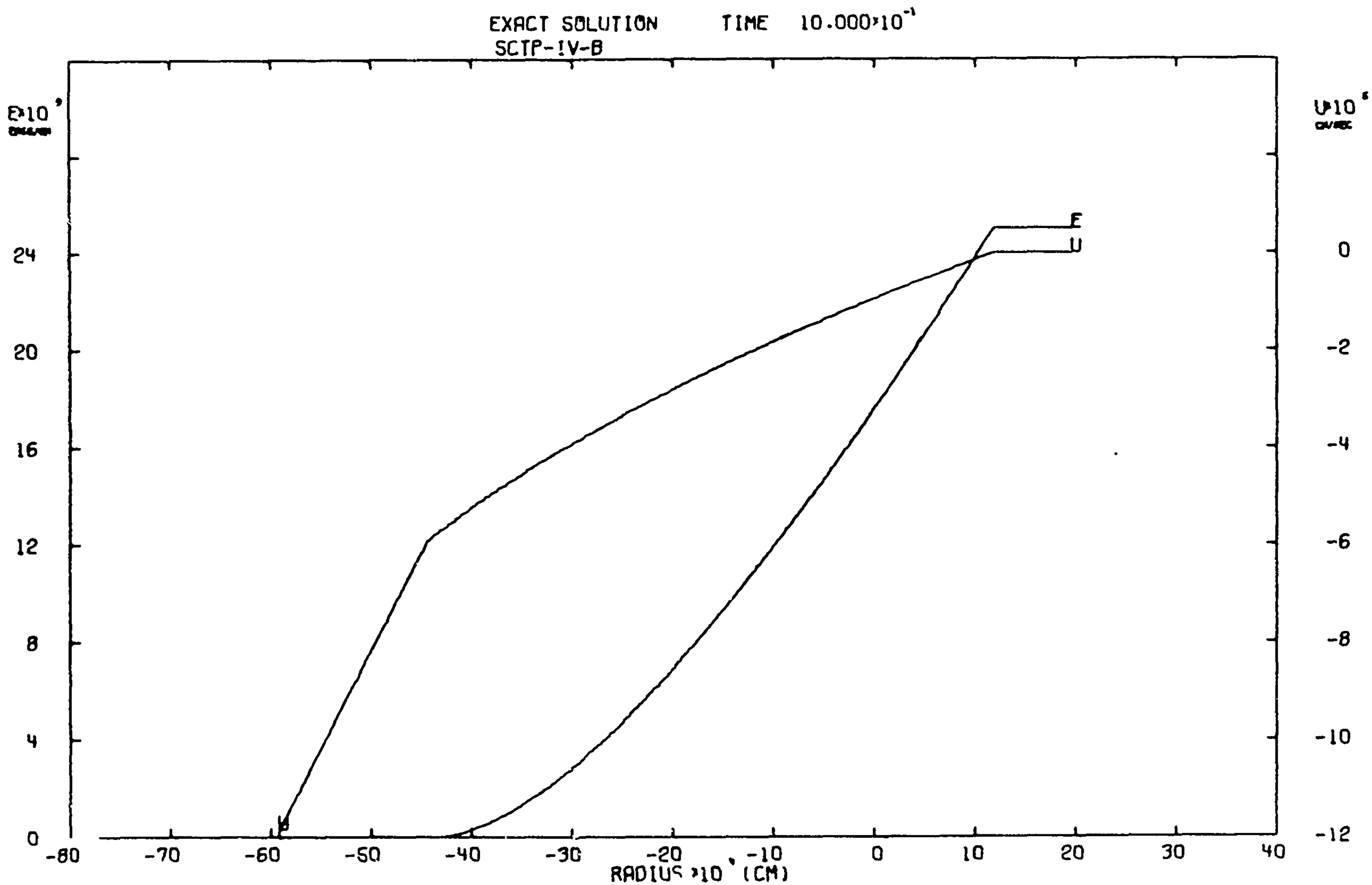

Figure IV-B. VE-EXACT

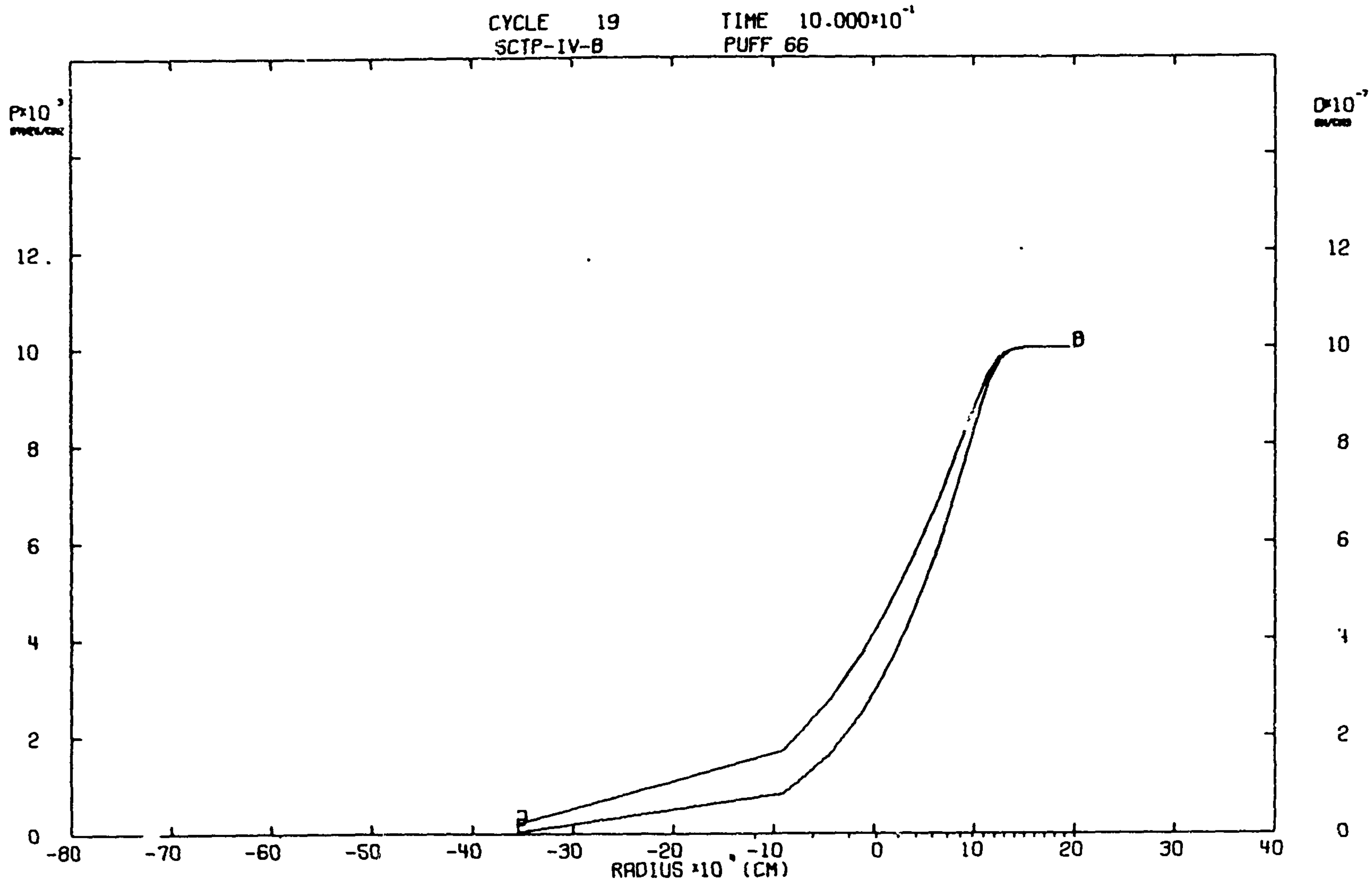

Figure IV-B. PD-PUFF

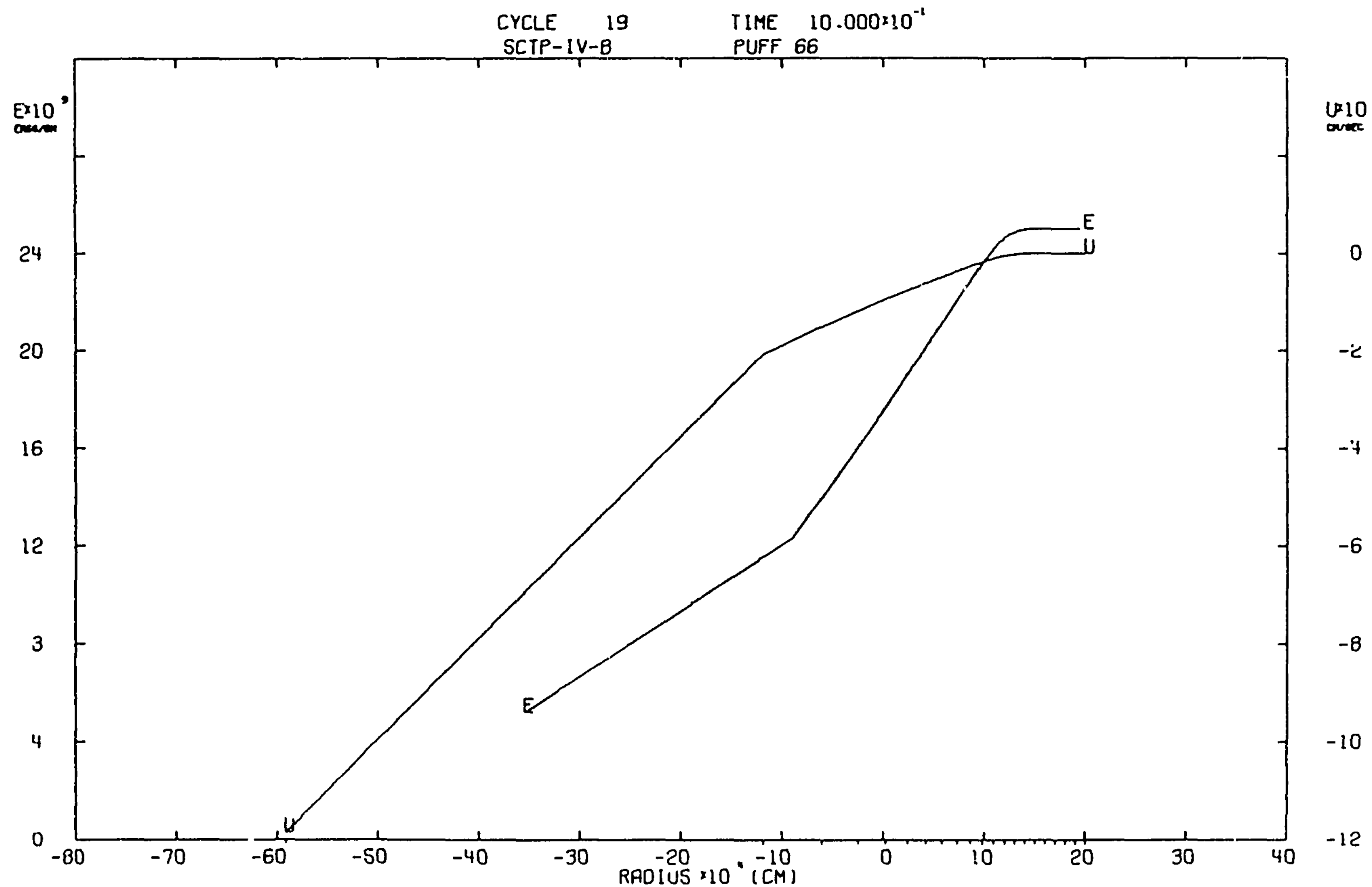

Figure IV-B. VE-PUFF

5. TEST PROBLEM SCTP-V

a. The Exact Solution

This is called the shock tube problem. It is an example of the more general Riemann problem. The Riemann problem is that of determining the flow after the conjunction of two states, left state and right state, with P_ℓ, ρ_ℓ, v_ℓ the constant values of the left state and P_r, ρ_r, v_r the constant values of the right state. In the shock tube problem, v_r and v_ℓ are no longer arbitrary but are set to zero. So the problem may be interpreted as the determination of the flow after removal of the membrane separating two constant states at rest. As a convention, take $P_\ell \geq P_r \geq 0$. Then in the code test problem the three possibilities, $\rho_\ell > \rho_r$, $\rho_\ell = \rho_r$, and $\rho_\ell < \rho_r$ will be explored.

At time zero, the membrane is removed. The resultant action is a rarefaction wave traveling into the left state and a shock traveling into the right state. The velocity is $v_\ell = 0$ to the left of the rarefaction wave. From the left of the rarefaction wave to the right, the velocity rises linearly from 0 to $v_m > 0$. The velocity is constant at v_m from the right of the rarefaction wave rightward toward the shock. At the shock, the velocity jumps from $v_m > 0$ down to $v_r = 0$. The pressure drops continuously across the rarefaction wave from P_ℓ to P_m. The pressure has the value P_m constantly from the right of the rarefaction wave to the shock. The pressure drops from P_m to P_r across the shock. The density drops continuously across the rarefaction wave from ρ_ℓ to a value $\rho_{m\ell}$, which it maintains from the right of the rarefaction wave to the point in the fluid where the initial discontinuity was and there the density jumps up to ρ_{mr}.

From the initial discontinuity point to the shock, the density jumps down from ρ_{mr} to ρ_r, which value is maintained all the way to the right.

The left side of the rarefaction wave is at $X_C(t) = X_S(0) - C_\ell t$, where $X_S(0)$ is the position of the shock at time zero which is also the position of the initial discontinuity.

The right side of the shock is at

$$X_R(t) = X_S(0) - \left(C - \frac{\gamma+1}{2} v_m\right)t$$

The shock wave is at

$$X_S(t) = X_S(0) + v_S t$$

where v_S is the velocity of the shock

$$v_S = v_r + \sqrt{V_r\left(\frac{\gamma+1}{2} P_m + \frac{\gamma-1}{2} P_r\right)}$$

The initial discontinuity point of the fluid is at $X_0(t) = X_S(0) + v_m t$. The middle values v_m and P_m are determined by simultaneously solving

$$v_m = v_r + (P_m - P_r)\sqrt{\frac{2V_r}{(\gamma+1) P_m + (\gamma-1) P_r}}$$

and

$$v_m = v_\ell + \frac{2\sqrt{\gamma}}{\gamma-1}\left(\frac{P_\ell}{\rho_\ell{}^{\gamma}}\right)^{\frac{1}{2\gamma}}\left[P_\ell{}^{\frac{\gamma-1}{2\gamma}} - P_m{}^{\frac{\gamma-1}{2\gamma}}\right]$$

Solution Summary:

LEFT REGION

For $X \le X_C(t)$, the values are P_ℓ, ρ_ℓ, v_ℓ

RAREFACTION REGION

For $X_C(t) \le X \le X_R(t)$, the values are

$$v(X,t) = \frac{X - X_C(t)}{X_R(t) - X_C(t)} v_m$$

$$C = C_\ell\left(1 - \frac{\gamma-1}{2} \frac{v}{C_\ell}\right)$$

$$P = P_\ell\left(\frac{C}{C_\ell}\right)^{\frac{2\gamma}{\gamma-1}}$$

$$\rho = \rho_\ell\left(\frac{C}{C_\ell}\right)^{\frac{2}{\gamma-1}}$$

MIDDLE REGION

For $X_R(t) \le X < X_S(t)$, the values are P_m, v_m

For $X_R(t) \le X < X_D(t)$, the density is $\rho_{m\ell}$

For $X_D(t) < X < X_S(t)$, the density is ρ_{mr}

RIGHT REGION

For $X > X_S(t)$, the values are P_r, ρ_r, v_r

The necessary data for this problem are

INITIAL VALUES: P_r, ρ_r, v_r, P_ℓ, ρ_ℓ, v_ℓ

BOUNDARY VALUES: At X = 0, hold the values at P_ℓ, ρ_ℓ, and v_ℓ at X_Q (the right boundary) hold the values at P_r, ρ_r, v_r.

There are three variations of this problem

SCTP-V-A:

$X_S(0) = 100$ meters

$\Delta X = 1$ meter

$P_\ell = 10^8$ dynes/cm^2

$\rho_\ell = 10^{-5}$ gm/cm^3

$v_\ell = 0$

$P_r = 10^4$ dynes/cm^2

$\rho_r = 10^{-6}$ gm/cm^3

$v_r = 0$

$X_Q = 250$ meters

These values imply the following values:

$P_m \doteq 1.888 \times 10^7$ dynes/cm^2

$v_m \doteq 3.964 \times 10^6$ cm/sec

$\rho_{m\ell} \doteq 3.040 \times 10^{-6}$ gm/cm^3

$\rho_{mr} \doteq 5.982 \times 10^{-6}$ gm/cm^3

$v_S \doteq 4.760 \times 10^6$ cm/sec

This problem was run to 2×10^{-3} seconds.

SCTP-V-B:

This problem is the same as SCTP-V-A except

$X_S(0) = 250$ meters

$\rho_\ell = 10^{-6}$ gm/cm^3

$X_Q = 500$ meters

These values imply the following values:

$P_m \doteq 4.610 \times 10^7$ dynes/cm^2

$v_m \doteq 6.196 \times 10^6$ cm/sec

$\rho_{m\ell} \doteq 5.751 \times 10^{-7}$ gm/cm^3

$\rho_{mr} \doteq 5.992 \times 10^{-6}$ gm/cm^3

$v_S \doteq 7.437 \times 10^6$ cm/sec

SCTP-V-C:

This problem is the same as A except

$X_S(0) = 250$ meters

$\rho_\ell = 10^{-6}$ gm/cm^3

$\rho_r = 10^{-5}$ gm/cm^3

$X_Q = 500$ meters

These values imply the following values:

$P_m \doteq 7.406 \times 10^7$ dynes/cm^2

$v_m \doteq 2.484 \times 10^6$ cm/sec

$\rho_{m\ell} \doteq 8.070 \times 10^{-7}$ gm/cm^3

$\rho_{mr} \doteq 5.995 \times 10^{-5}$ gm/cm^3

$v_S \doteq 2.981 \times 10^6$ cm/sec

b. The PUFF Solution

On SCTP-V-A, the most noticeable error was a smearing of the density discontinuity at X_D. The only errors were the typical underrounds and overrounds at corners.

On SCTP-V-B, there was a bit of oscillation in the density in the compressed region and a little overshoot in velocity and an undershoot in pressure at X_R.

On SCTP-V-C, the dominant error was a slight undershoot in the pressure at X_R. The other error was a slight undershoot in the pressure at X_R.

For more details see Tables and Figures V.

c. The LAX-WENDROFF Solution

In order to run this problem it was found necessary to cut the first time step and artificial viscosity factor down to one-twentieth the normal time step, cut the second time step and artificial viscosity down to two-twentieths of the normal time step, etc., until the twentieth step and thereafter allow the normal time step and artificial viscosity factor. The time factor used was .78 and the artificial viscosity factor used was .5.

The most noticeable difference between PUFF and LAX-WENDROFF in SCTP-V-A is the pronounced spikes at X_D and X_S in LAX-WENDROFF (see Figures V-A).

In SCTP-V-B the spikes are not so bad but there is quite an oscillation in the velocity just right of X_R (see Figures V-B).

In SCTP-V-C the overshoot in the velocity at X_R has grown more pronounced (see Figures V-C).

Table V-A

ERRORS ON SCTP-V-A

PUFF

Problem time = 2 x 10^{-3} seconds
Computer time = 61 seconds

Cycle = 976
Number of Active Zones = 201

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.401	.087	- .054	X_S
Velocity	2.32	1.13	+ .875	X_S
Density	1.74	.385	+ .172	X_S
Energy	1.13	.285	+ .180	X_S
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	2.17716 x 10^{12}	3.23215 x 10^{11}	2.50037 x 10^{12}	
PUFF	2.17807 x 10^{12}	3.21760 x 10^{11}	2.49983 x 10^{12}	

LAX-WENDROFF

Problem time = 2 x 10^{-3} seconds
Computer time = 122 seconds

Cycle = 363
Number of Active Zones = 251

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.743	.118	- .066	X_S
Velocity	1.39	.352	+ .276	X_S
Density	1.81	.327	- .150	X_S
Energy	1.11	.203	+ .087	X_D
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	2.17716 x 10^{12}	3.23215 x 10^{11}	2.50037 x 10^{12}	
LAXWEN	2.18008 x 10^{12}	3.19915 x 10^{11}	2.49999 x 10^{12}	

Table V-B

ERRORS ON SCTP-V-B

PUFF

Problem time = 2×10^{-3} sec
Computer time = 167 sec

Cycle = 1527
Number of Active Zones = 404

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.762	.276	- .263	X_S
Velocity	2.24	.900	+ .748	X_S
Density	1.52	.616	- .415	X_S
Energy	.285	.060	+ .039	X_D
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	5.66764×10^{12}	5.82988×10^{11}	6.25062×10^{12}	
PUFF	5.66880×10^{12}	5.81419×10^{11}	6.25022×10^{12}	

LAX-WENDROFF

Problem time = 2×10^{-3} sec
Computer time = 347 sec

Cycle = 535
Number of Active Zones = 501

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.692	.141	+ .122	X_S
Velocity	2.17	.605	+ .580	X_S
Density	1.11	.376	+ .284	X_S
Energy	3.63	.060	+ .045	X_S
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	5.66764×10^{12}	5.82988×10^{11}	6.25062×10^{12}	
LAXWEN	5.66694×10^{12}	5.81029×10^{11}	6.24797×10^{12}	

Table V-C

ERRORS ON SCTP-V-C

PUFF

Problem time = 2×10^{-3} sec
Computer time = 75 sec

Cycle = 611
Number of Active Zones = 315

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	1.22	.501	- .478	X_S
Velocity	3.49	.866	+ .701	X_S
Density	1.53	.639	- .471	X_S
Energy	.195	.023	+ .006	X_D
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	6.00501×10^{12}	2.45613×10^{11}	6.25062×10^{12}	
PUFF	6.00694×10^{12}	2.43446×10^{11}	6.25038×10^{12}	

LAX-WENDROFF

Problem time = 2×10^{-3} sec
Computer time = 221 sec

Cycle = 337
Number of Active Zones = 501

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.688	.194	+ .159	X_S
Velocity	2.90	.614	+ .517	X_S
Density	1.47	.846	- .792	X_D
Energy	.636	.432	- .428	X_D
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	6.00501×10^{12}	2.45613×10^{11}	6.25062×10^{12}	
LAXWEN	6.00238×10^{12}	2.42686×10^{11}	6.24506×10^{12}	

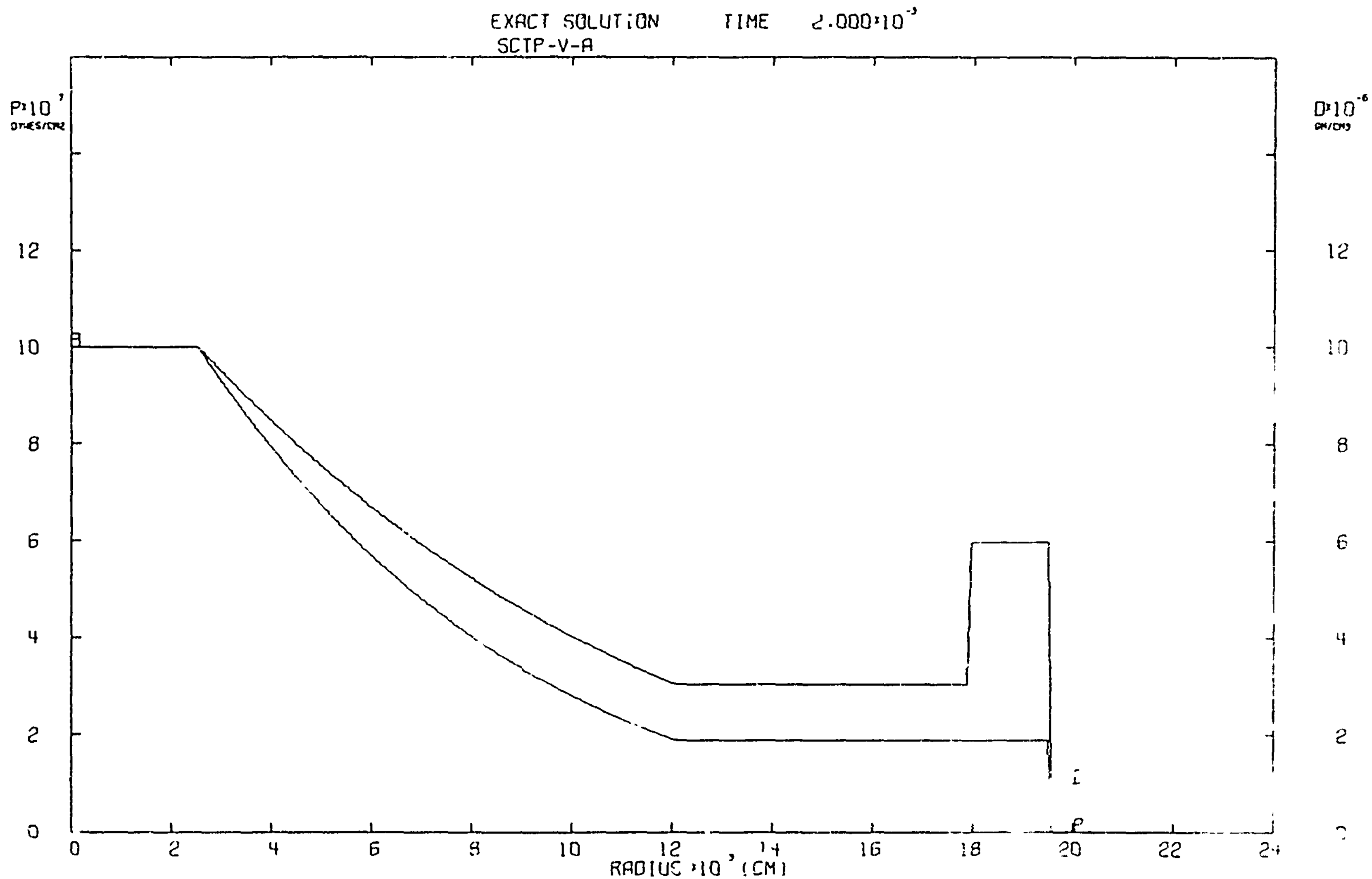

Figure V-A. PD-EXACT

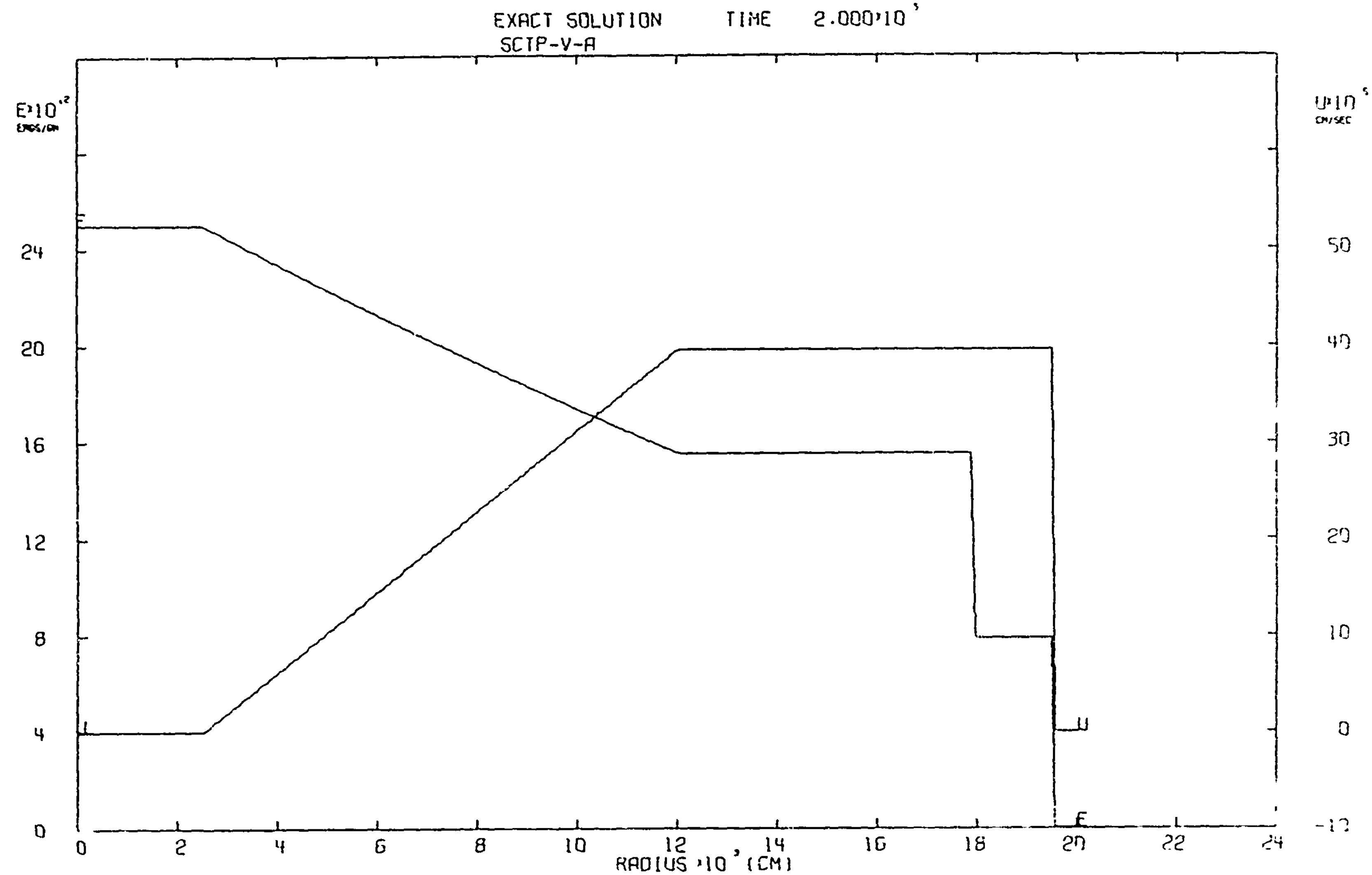

Figure V-A. VE-EXACT

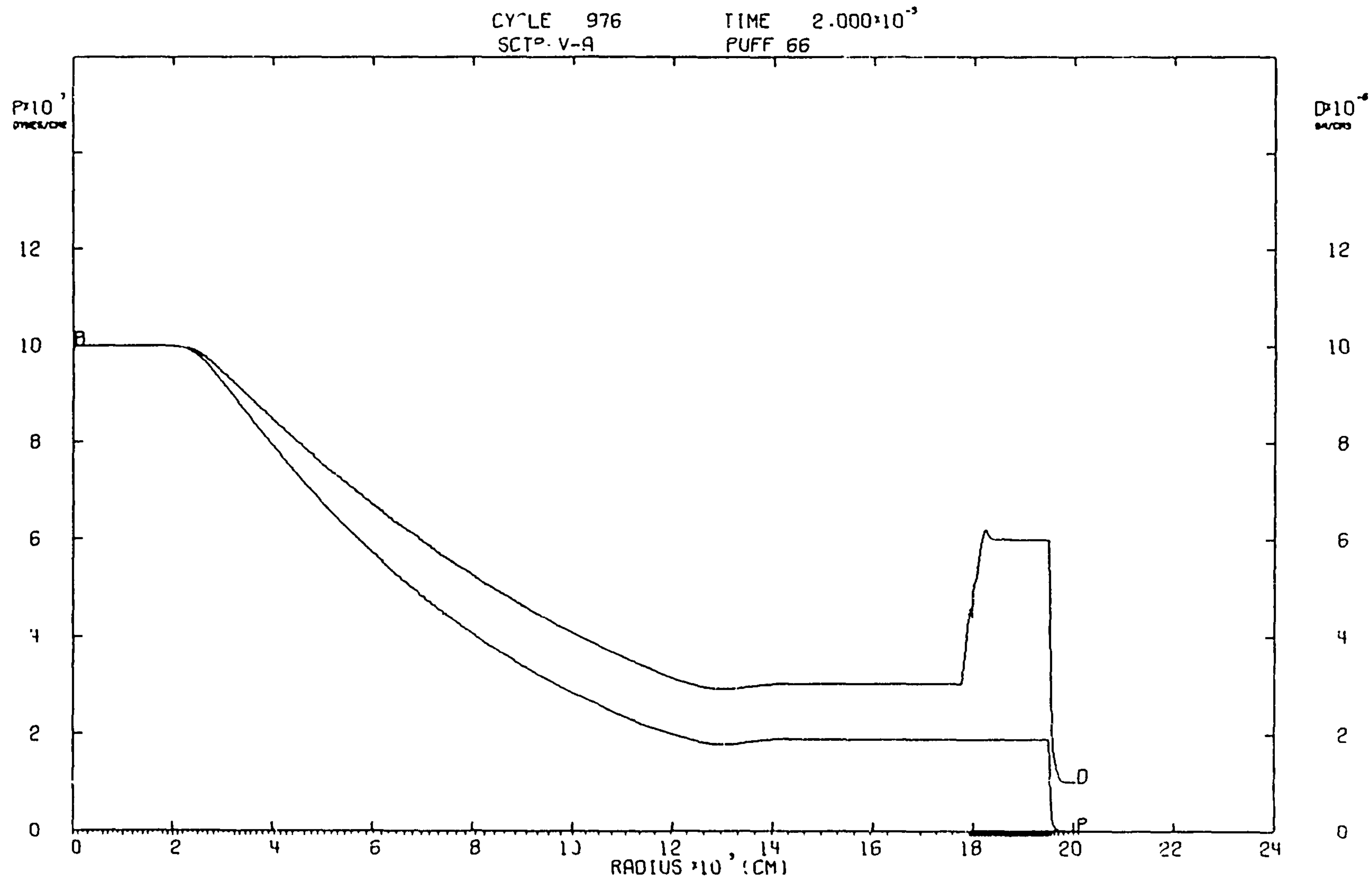

Figure V-A. PD-PUFF

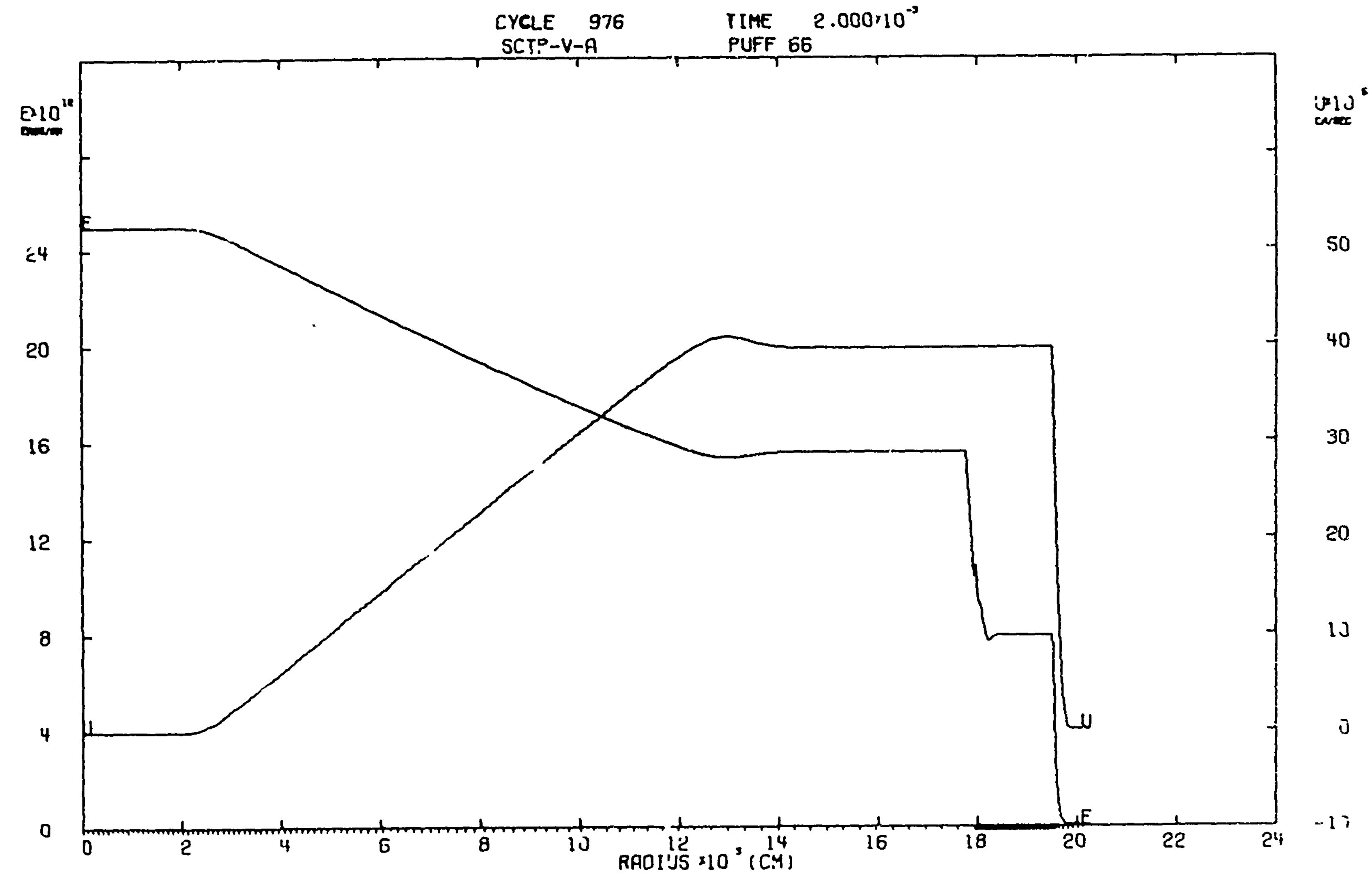

Figure V-A. VE-PUFF

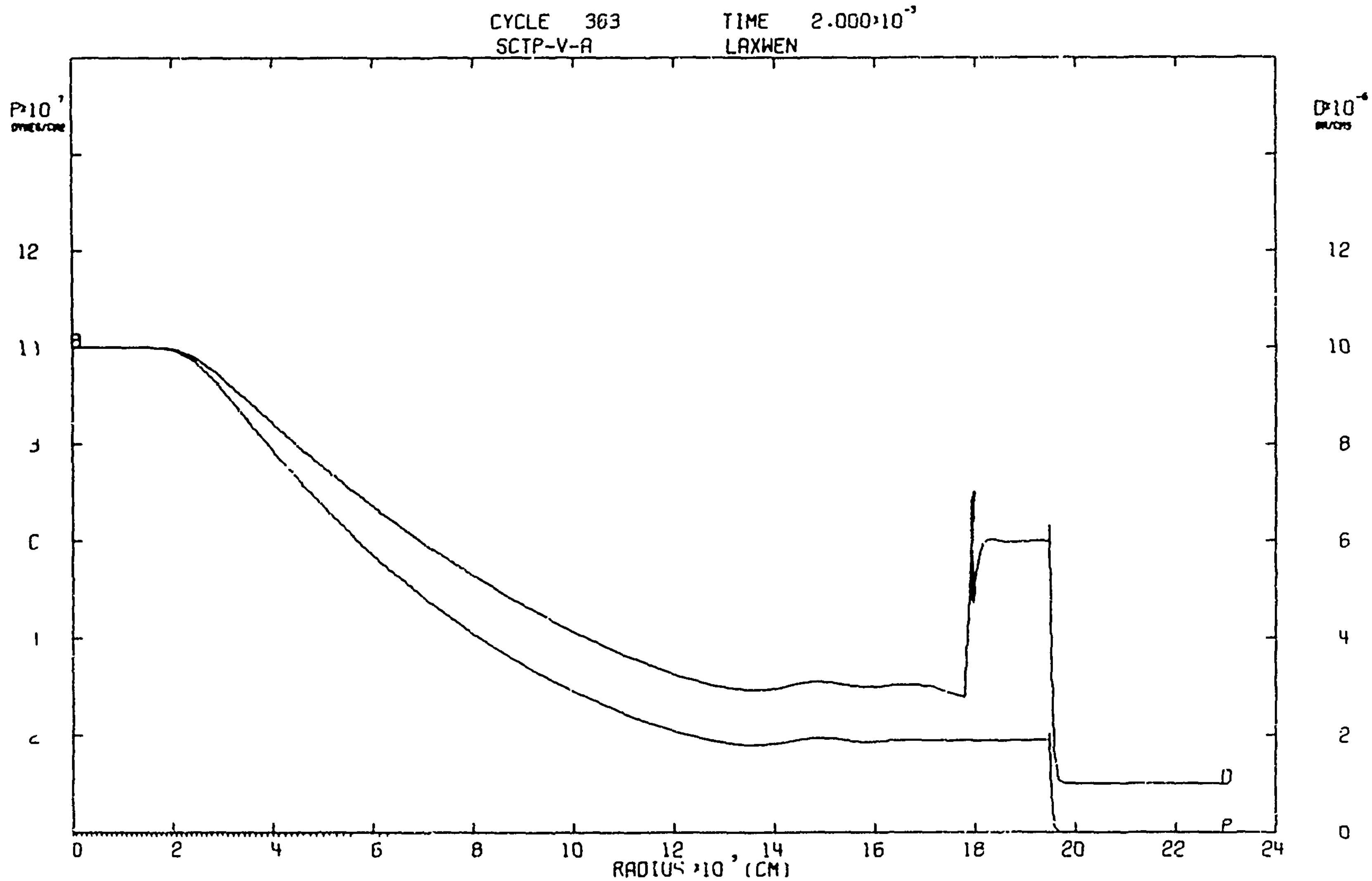

Figure V-A. PD-LAX-WENDROFF

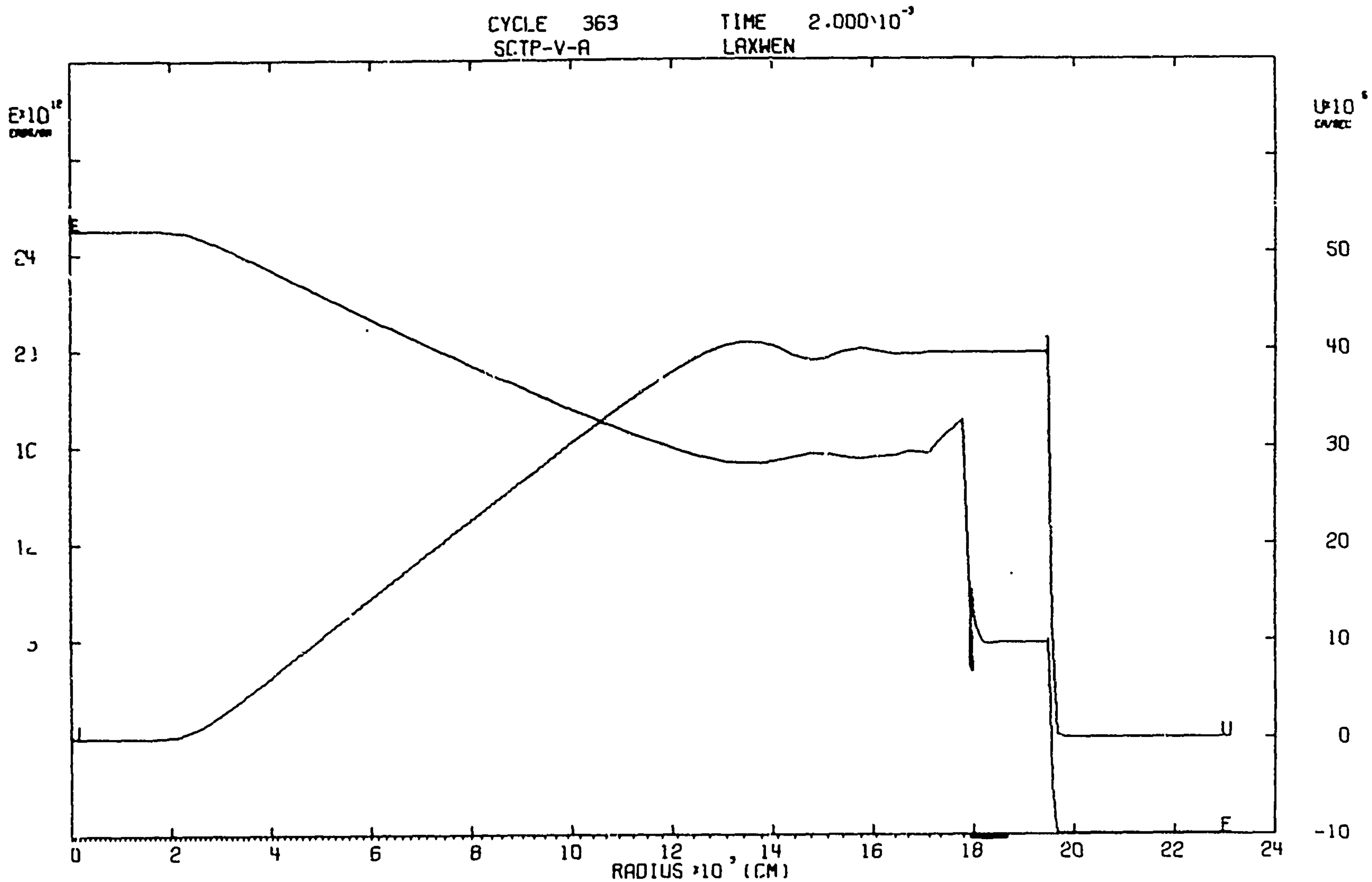

Figure V-A. VE-LAX-WENDROFF

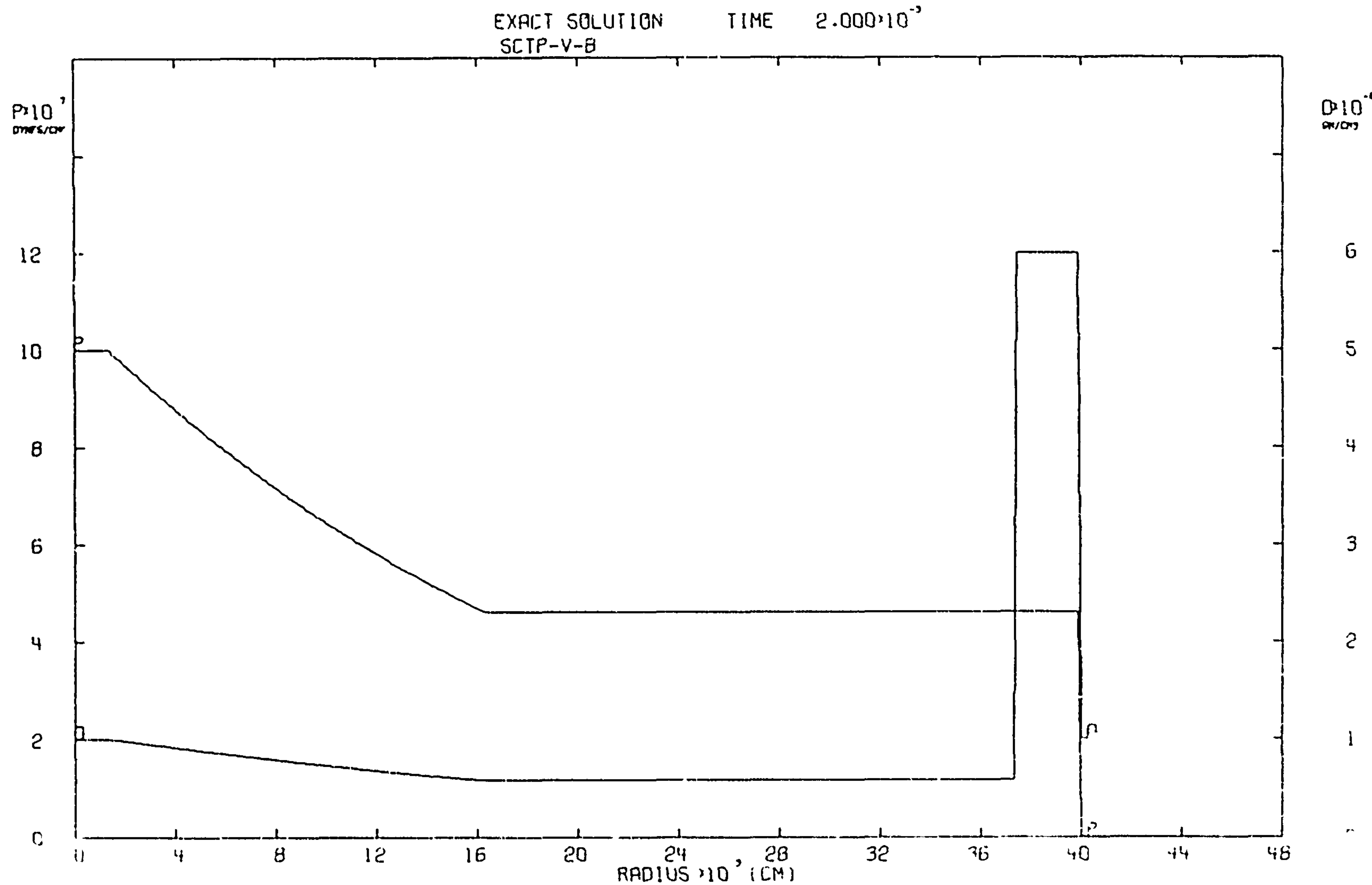

Figure V-B. PD-EXACT

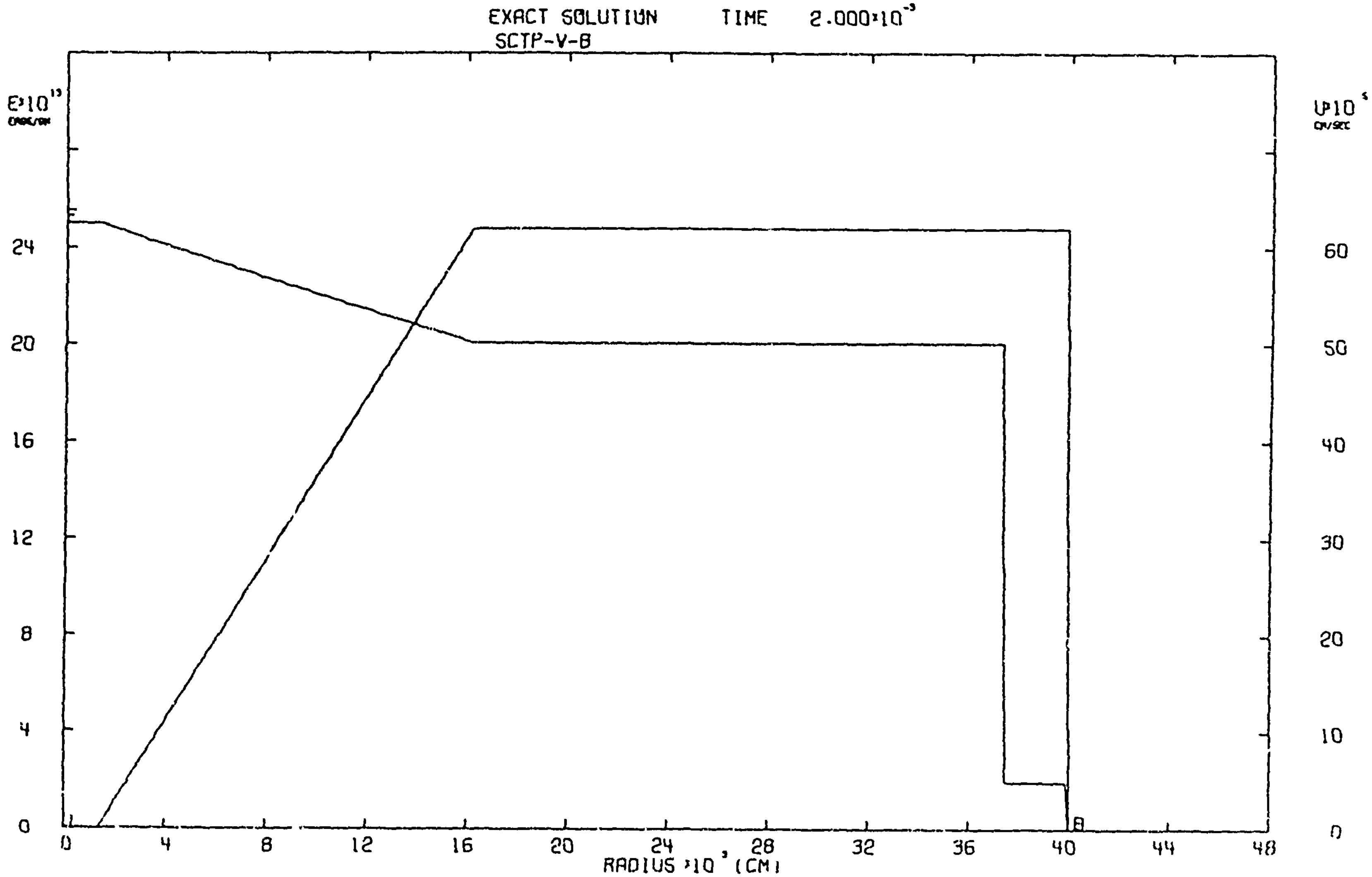

Figure V-B. VE-EXACT

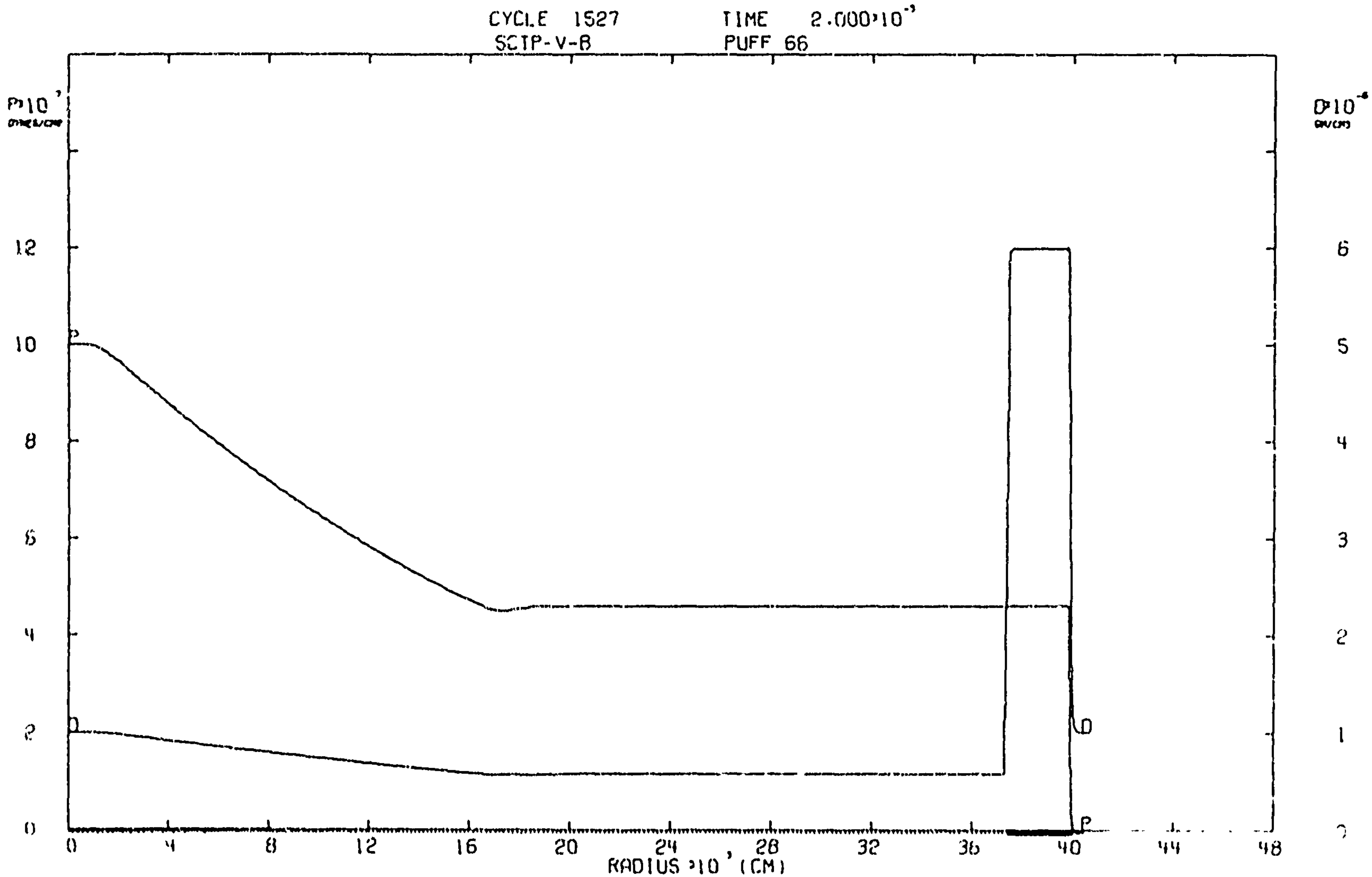

Figure V-8. PD-PUFF

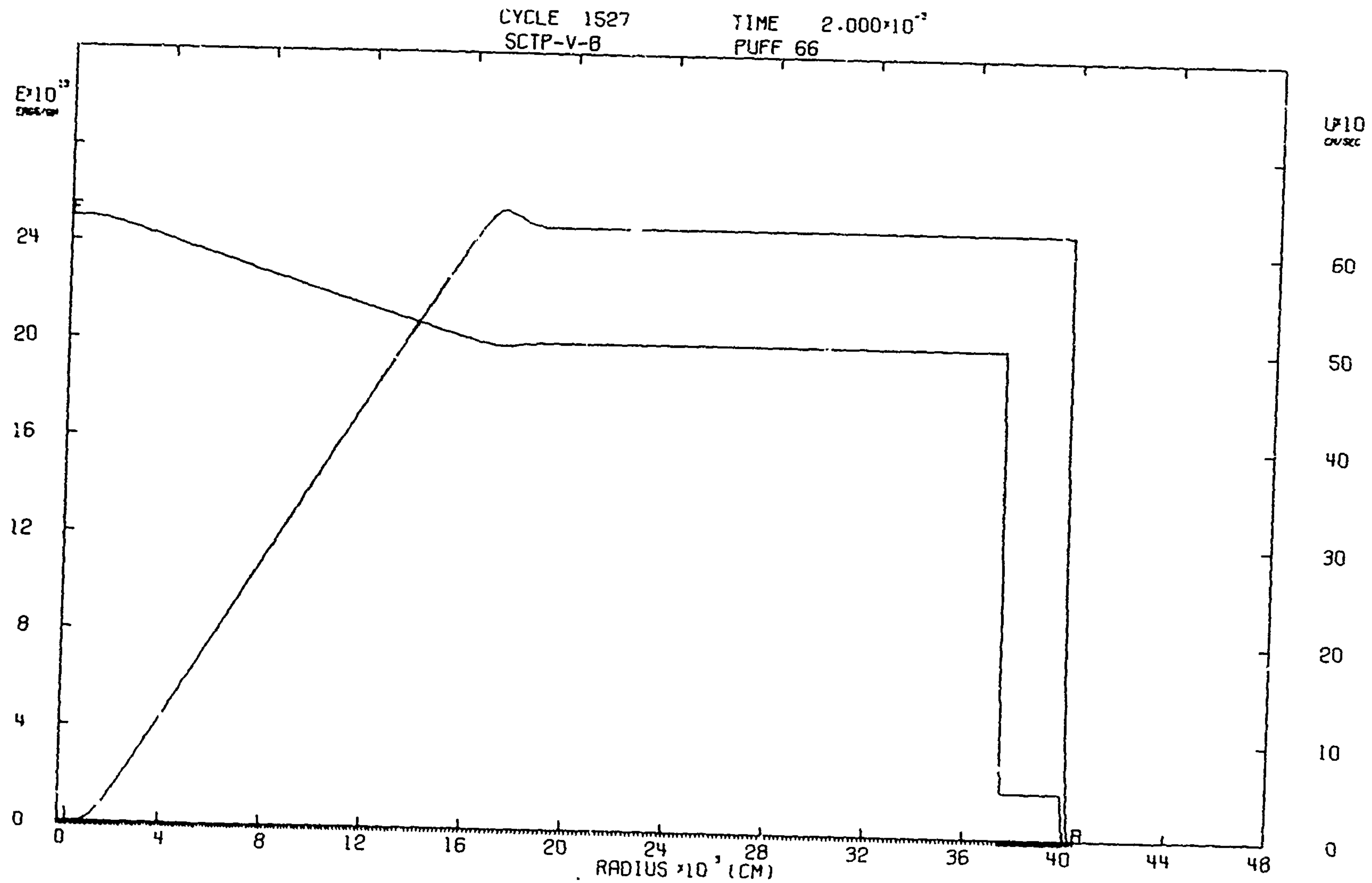

Figure V-B. VE-PUFF

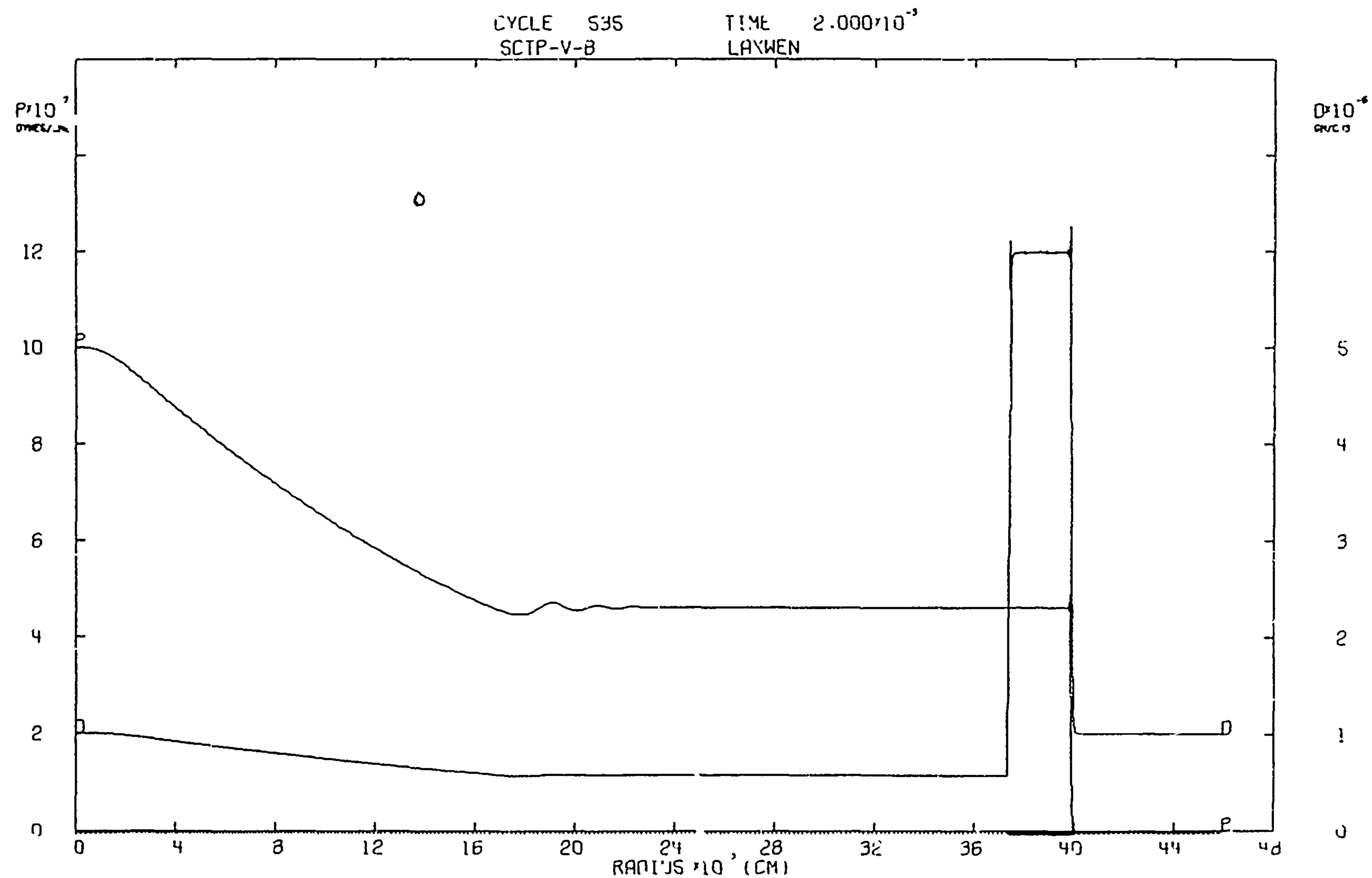

Figure V-B. PD-LAX-WENDROFF

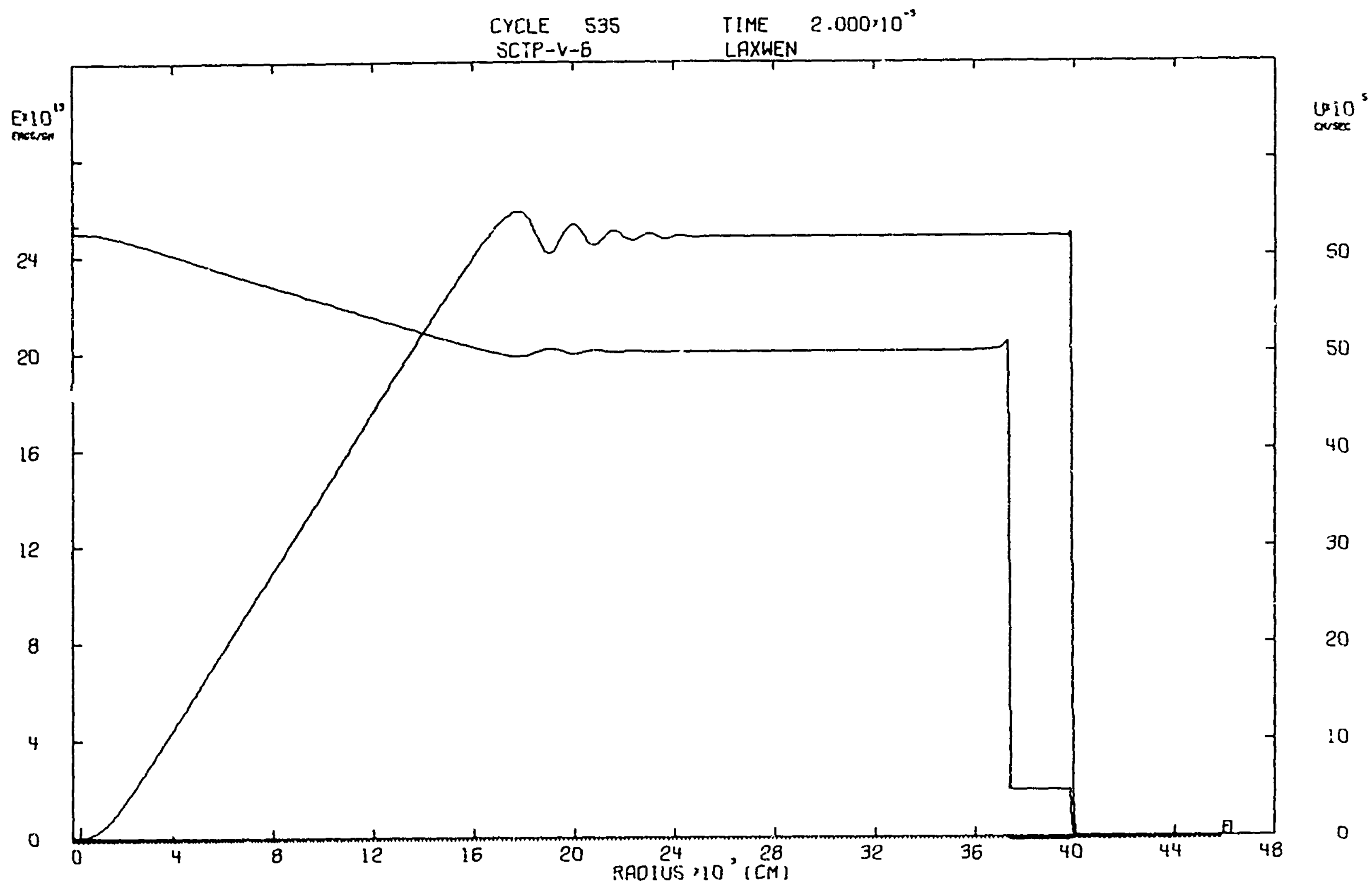

Figure V-B. VE-LAX-WENDROFF

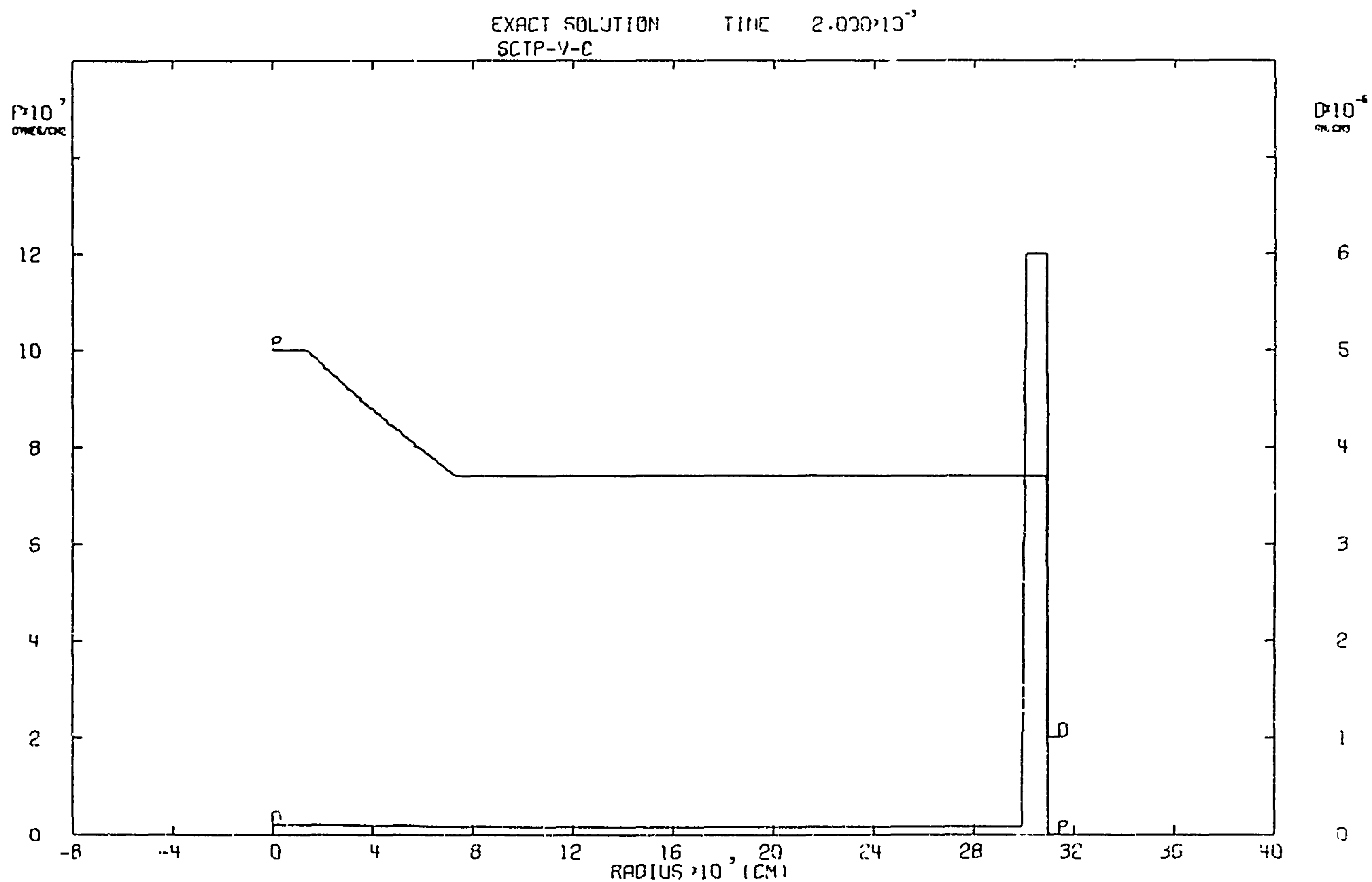

Figure V-C. PD-EXACT

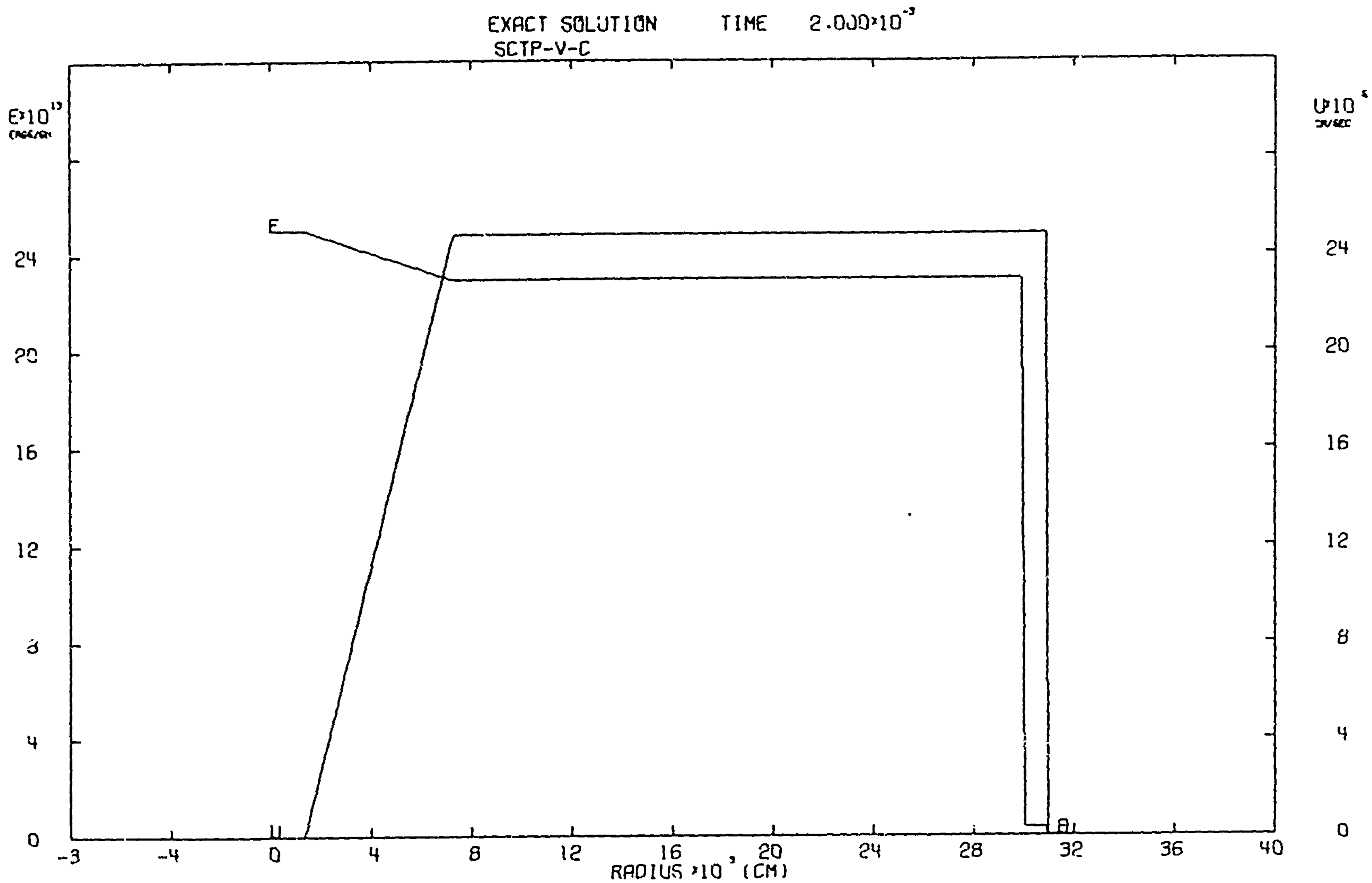

Figure V-C. VE-EXACT

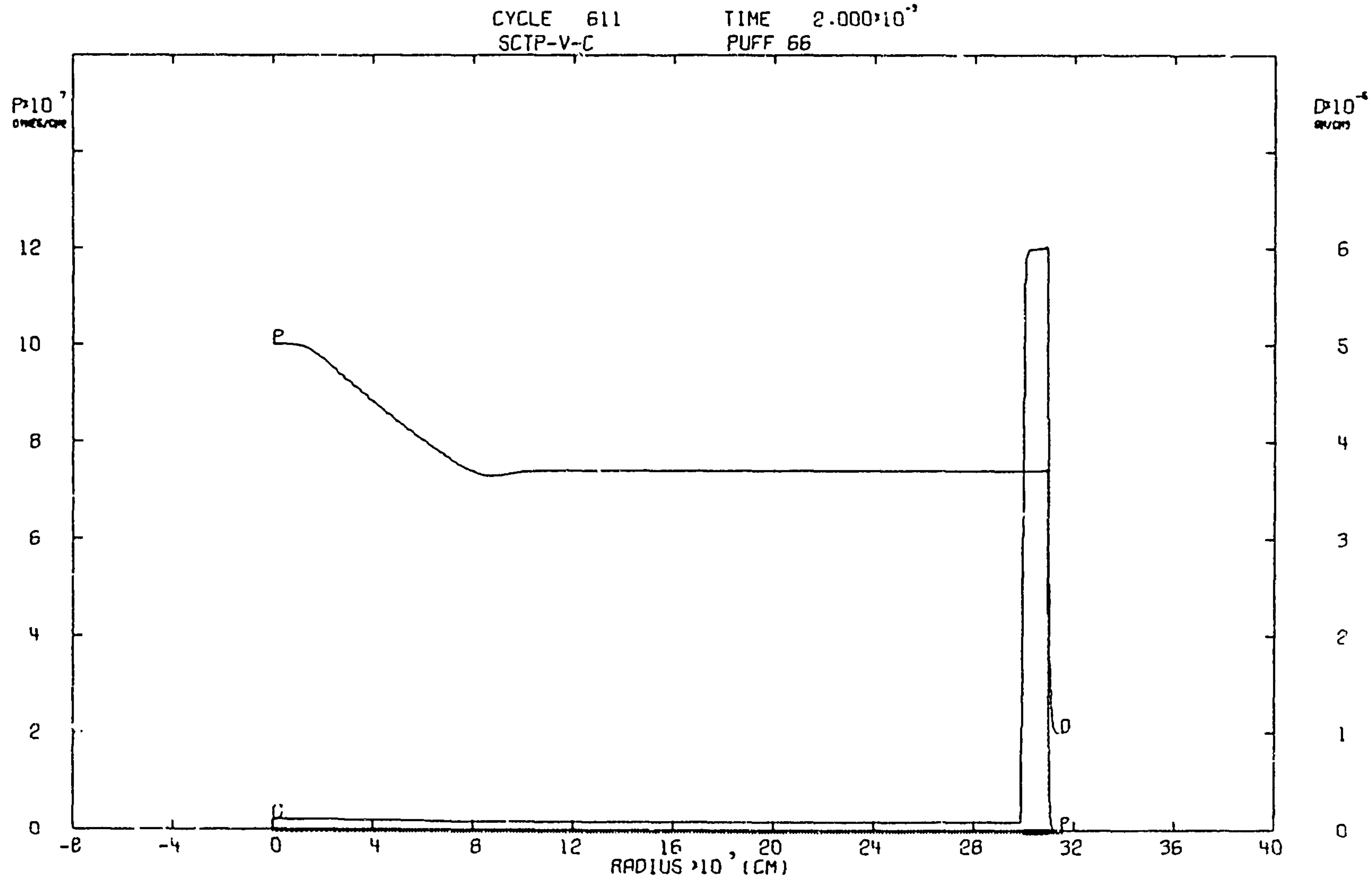

Figure V-C. PD-PUFF

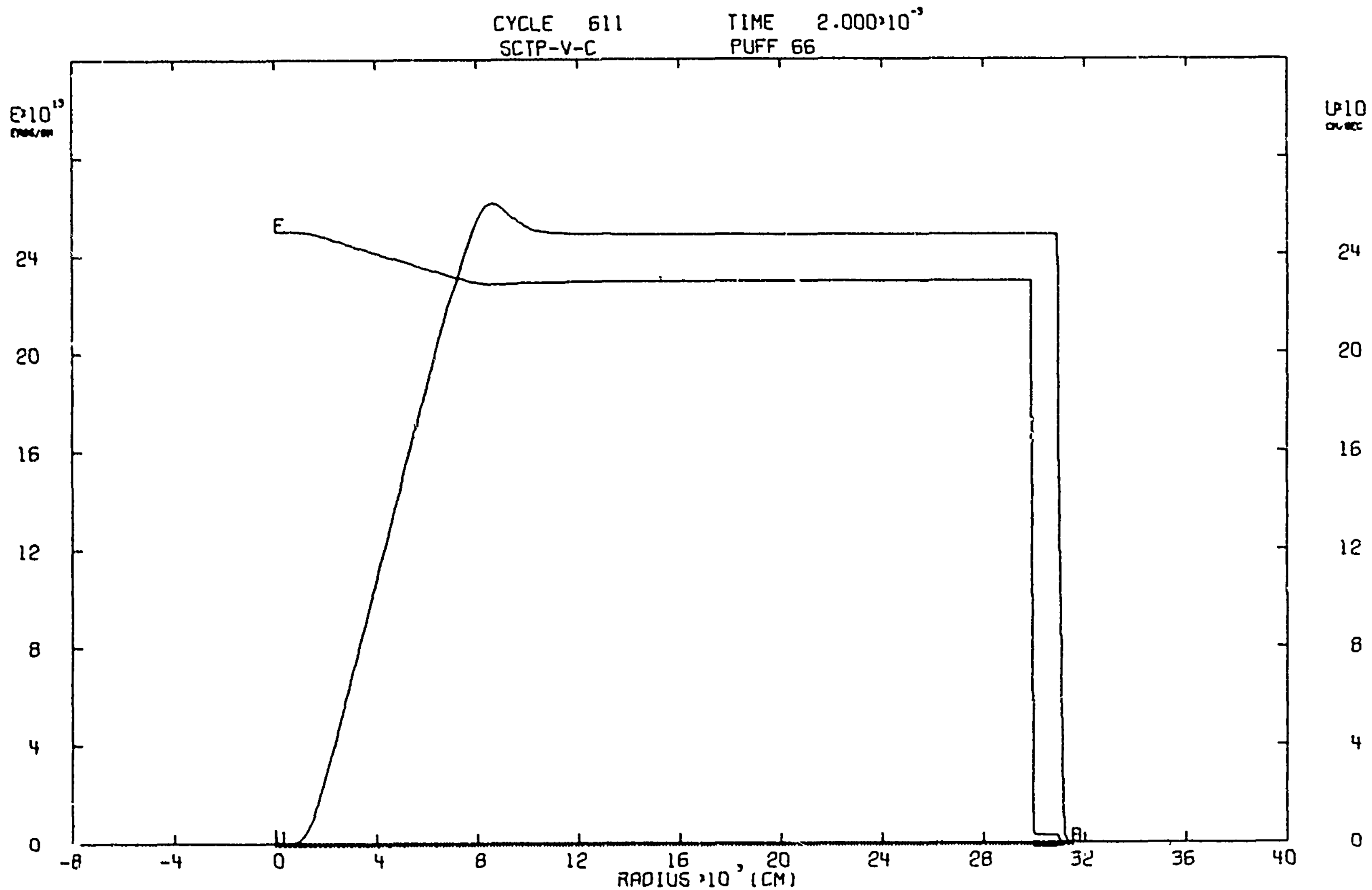

Figure V-C. VE-PUFF

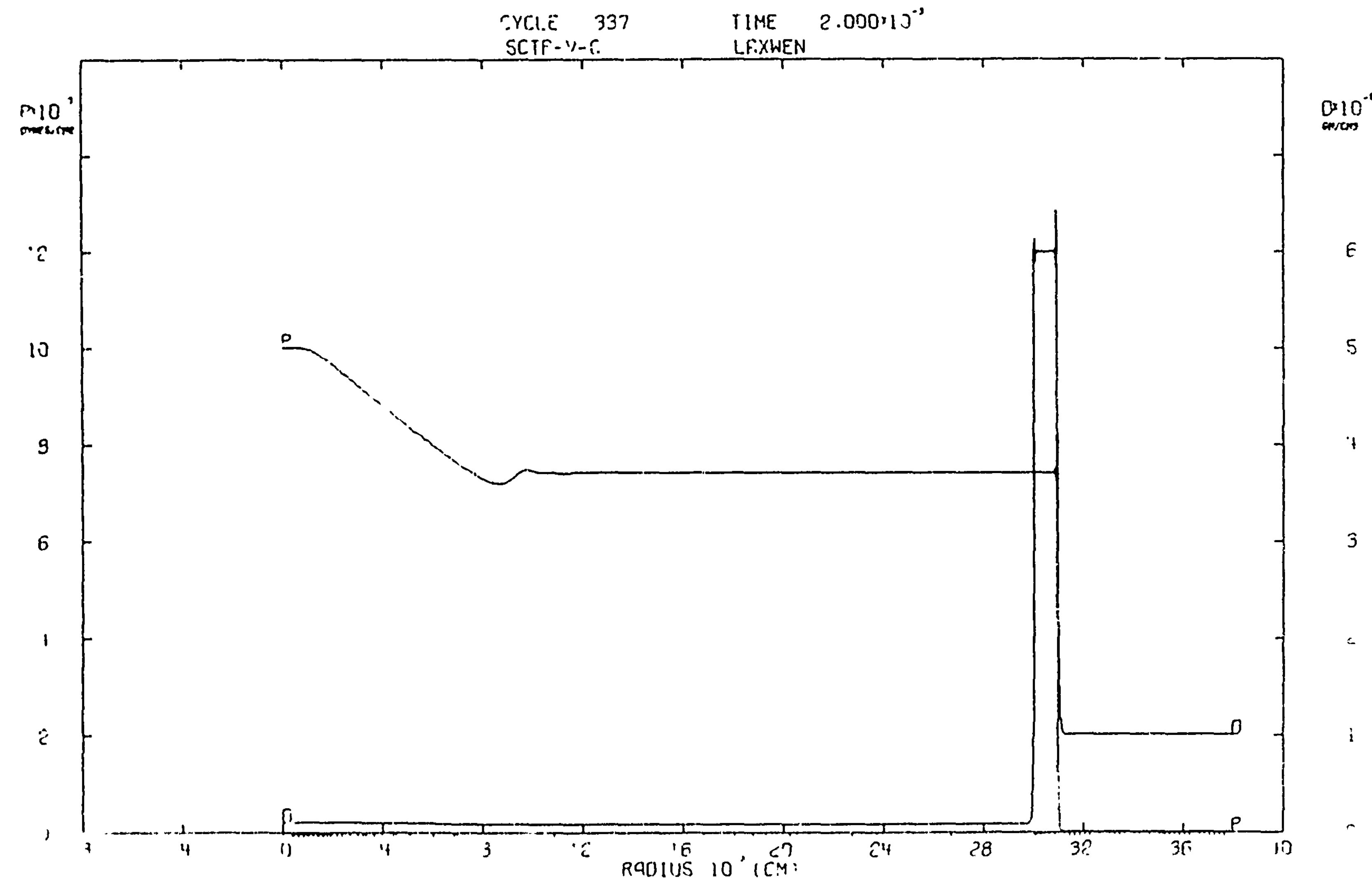

Figure V-C. PD-LAX-WENDROFF

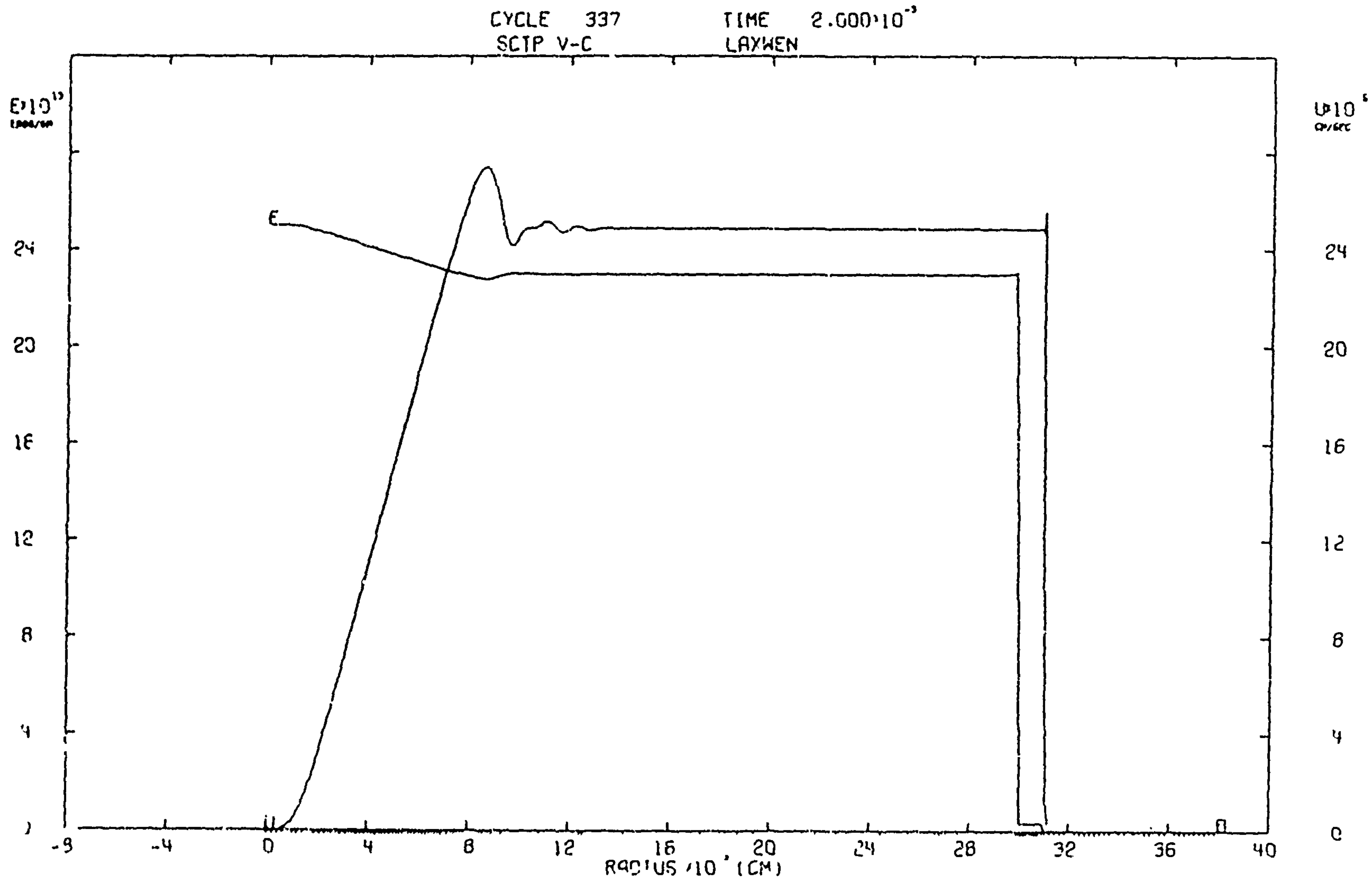

Figure V-C. VE-LAX-WENDROFF

6. TEST PROBLEM SCTP-VI

a. The Exact Solution

This problem is the collision of two shock waves. It is another special case of the Riemann problem. Proceeding from left to right, the initial values are P_ℓ, ρ_ℓ, v_ℓ connected by a right-facing shock to P_0, ρ_0, v_0, which in turn is connected by a left facing shock to P_r, ρ_r, v_r. As a convenient convention one takes $P_\ell \geq P_r \geq P_0$.

After collision a shock travels back to the left and a shock travels on to the right from a middle region in which the velocity and pressure are constants v_m and P_m.

For a shock facing to the right

$$v_m = v_r + \phi_r\left(P_m\right)$$

and for a shock facing to the left

$$v_m = v_\ell - \phi_\ell\left(P_m\right)$$

where

$$\phi_a(P) = \left(P - P_a\right)\frac{2V_a}{(\gamma+1)P + (\gamma-1)P_a}$$

From these two equations, P_m and v_m are determined.

The density profile proceeding from left to right is ρ_ℓ, then it jumps up to $\rho_{m\ell}$ at the left-facing shock, then at the point of collision (in the Lagrangian coordinates) the density jumps to ρ_{mr}, then at the right-facing shock the density jumps down to ρ_r. The Rankine-Hugoniot relation determines $\rho_{m\ell}$ and ρ_{mr}. That is,

$$0 = e_\ell - e_{m\ell} + \frac{P_\ell + P_m}{2}\left(V_\ell - V_{m\ell}\right)$$

and

$$0 = e_{mr} - e_r + \frac{P_m + P_r}{2}\left(V_{mr} - V_r\right)$$

where

$$e = \frac{PV}{\gamma-1}$$

and, of course,

$$V_{m\ell} = \rho_{m\ell}^{-1},\ V_{mr} = \rho_{mr}^{-1}$$

All of the shock velocities may be computed by

$$v_S = \frac{1+\gamma}{4} v_\ell + \frac{3-\gamma}{4} v_r \pm \sqrt{\left(\frac{1+\gamma}{4} v_\ell + \frac{3-\gamma}{4} v_r\right)^2 + C_r^{\,2}}$$

where + is taken for right-facing shocks and − is taken for left-facing shocks. For example, prior to collision (let $t_{co\ell}$ be the time of collision) the velocity of the right-facing shock (i.e., the shock on the left) is

$$v_{S\ell} = \frac{1+\gamma}{4} v_\ell + \frac{3-\gamma}{4} v_0 + \sqrt{\left(\frac{1+\gamma}{4} v_\ell + \frac{3-\gamma}{4} v_0\right)^2 + C_0^{\,2}}$$

and the velocity of the left-facing shock (i.e., the shock on the right) is

$$v_{Sr} = \frac{1+\gamma}{4} v_0 + \frac{3-\gamma}{4} v_r - \sqrt{\left(\frac{1+\gamma}{4} v_0 + \frac{3-\gamma}{4} v_r\right)^2 + C_r^{\,2}}$$

After collision (for $t > t_{co\ell}$) the velocity of the left-facing shock (i.e., the shock on the left) is

$$v_{Sr}^{\ *} = \frac{1+\gamma}{4} v_\ell + \frac{3-\gamma}{4} v_m - \sqrt{\left(\frac{1+\gamma}{4} v_\ell + \frac{3-\gamma}{4} v_m\right)^2 + C_m^{\,2}}$$

and the velocity of the right-facing shock (i.e., the shock on the right) is

$$v_{Sr}^* = \frac{1+\gamma}{4} v_m + \frac{3-\gamma}{4} v_r + \sqrt{\left(\frac{1+\gamma}{4} v_m + \frac{3-\gamma}{4} v_r\right)^2 + C_r^2}$$

where as before C stands for the isentropic sound speed.

Solution Summary:

Prior to collision ($t < t_{co\ell}$)

LEFT REGION: For $X < X_{S\ell}(t) = X_{S\ell}(0) + v_{S\ell}t$, the values are P_ℓ, ρ_ℓ, v_ℓ

MIDDLE REGION: For $X_{S\ell}(t) < X < X_{sr}(t) = X_{sr}(0) + v_{sr}t$, the values are P_0, ρ_0, v_0

RIGHT REGION: For $X > X_{sr}(t)$, the values are P_r, ρ_r, v_r

After collision ($t > t_{co\ell}$)

LEFT REGION: For $X < X^*_{S\ell}(t) = X_{co\ell} + v^*_{S\ell}(t-t_{co\ell})$, the values are P_ℓ, ρ_ℓ, v_ℓ

MIDDLE REGION: For $X^*_{S\ell}(t) < X < X^*_{Sr}(t) = X_{co\ell} + v^*_{sr}(t-t_{co\ell})$,

the values are P_m, v_m and $\rho_{m\ell}$ for $X < X_{co\ell} + v_m(t-t_{co\ell})$

and ρ_{mr} for $X > X_{co\ell} + v_m(t-t_{co\ell})$

RIGHT REGION: For $X > X^*_{sr}(t)$, the values are P_r, ρ_r, v_r

The necessary data for this problem are:

INITIAL VALUES: P_ℓ, P_0, ρ_0, v_0, P_r

From these all other initial values are determined.

BOUNDARY VALUES: At X = 0 (the left boundary) hold the values at P_ℓ, ρ_ℓ, v_ℓ. At X_Q (the right boundary) hold the values at P_r, ρ_r, v_r.

There are two variations of this problem:

SCTP-VI-A:

ΔX = 1 meter

$X_{S\ell}(0)$ = 75 meters

$X_{Sr}(0)$ = 125 meters

$X_Q(0)$ = 200 meters

$P_0 = 10^4$ dynes/cm^2

$\rho_0 = 10^{-6}$ gm/cm^3

$P_\ell = 10^8$ dynes/cm^2

$P_r = 10^7$ dynes/cm^2

$v_0 = 0$

These values then determine the following values:

$\rho_\ell \doteq 5.997 \times 10^{-6}$ gm/cm^3

$\rho_r \doteq 5.97 \times 10^{-6}$ gm/cm^3

$v_\ell \doteq 9.13 \times 10^6$ cm/sec

$v_r \doteq -2.88 \times 10^6$ cm/sec

$v_{S\ell} \doteq 1.095 \times 10^7$ cm/sec

$v_{Sr} \doteq -3.46 \times 10^6$ cm/sec

$t_{co\ell} \doteq 3.468 \times 10^{-4}$ sec

$X_{co\ell} \doteq 1.13 \times 10^4$ cm

$P_m \doteq 3.66 \times 10^8$ dynes/cm^2

$\rho_{m\ell} \doteq 1.43 \times 10^{-5}$ gm/cm^3

$\rho_{mr} \doteq 3.09 \times 10^{-5}$ gm/cm^3

$v_m \doteq 1.96 \times 10^6$ cm/sec

$v^*_{S\ell} \doteq 3.75 \times 10^5$ cm/sec

$v^*_{sr} \doteq 5.72 \times 10^6$ cm/sec

This problem was run to 7×10^{-4} sec.

SCTP-VI-B:

This problem is the same as A, except $P_\ell = P_r = 10^8$ dynes/cm^2.

This yields the following values:

$\rho_\ell = \rho_r \doteq 5.997 \times 10^{-6}$ gm/cm^3

$v_\ell = -v_r \doteq 9.13 \times 10^6$ cm/sec

$v_{S\ell} = -v_{Sr} \doteq 1.095 \times 10^7$ cm/sec

$t_{co\ell} \doteq 2.282 \times 10^{-4}$ sec

$X_{co\ell} = 1.00 \times 10^4$ cm

$P_m \doteq 7.995 \times 10^8$ dynes/cm^2

$v_m = 0$

$\rho_{m\ell} = \rho_{mr} \doteq 2.098 \times 10^{-5}$ gm/cm^3

$-v^*_{S\ell} = v^*_{Sr} \doteq 3.65 \times 10^6$ cm/sec

This problem was run to 7×10^{-4} sec.

b. The PUFF Solution

The major errors in evidence were the spikes in the density and internal energy. Hot-thin spikes resulted from the initial discontinuities and a cold-thick spike resulted from the shock collision. For more details see Tables and Figures VI.

c. The LAX-WENDROFF Solution

In addition to the spikes observed in the PUFF solution there is also a considerable amount of oscillation in evidence in the LAX-WENDROFF Solution. The time factor used was .39 and the artificial viscosity factor used was .25; as in SCTP-V the time and viscosity factors were multipled by one-twentieth on the first time step, two-twentieths on the second, etc., until the twentieth time step and thereafter when they were left at the values of .39 and .25 respectively. For more details see Tables and Figures VI.

Table VI-A

ERRORS ON SCTP-VI-A

PUFF

Problem time = 7×10^{-4} sec
Computer time = 89 sec

Cycle = 1283
Number of Active Zones = 200

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	1.92	.556	.352	X^*_{Sr}
Velocity	1.77	.687	.514	X^*_{Sr}
Density	6.40	1.46	.640	The collision point in the fluid
Energy	5.77	1.49	.981	The fluid position $x = X_{S\ell}(0)$
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	3.104×10^{12}	1.375×10^{12}	4.480×10^{12}	
PUFF	3.126×10^{12}	1.657×10^{12}	4.782×10^{12}	

LAX-WENDROFF

Problem time = 7×10^{-4} sec
Computer time = 433 sec

Cycle = 1719
Number of Active Zones = 200

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	5.39	1.13	- .563	$X^*_{S\ell}$
Velocity	2.41	.546	+ .367	$X^*_{S\ell}$
Density	5.45	.908	+ .370	$X_{co\ell}$
Energy	4.10	.722	+ .346	X^*_{Sr}
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	3.104×10^{12}	1.375×10^{12}	4.480×10^{12}	
LAXWEN	3.093×10^{12}	1.678×10^{12}	4.771×10^{12}	

Table VI-B

ERRORS ON SCTP-VI-B

PUFF

Problem time = 7×10^{-4} sec
Computer time = 101 sec

Cycle = 1500
Number of Active Zones = 200

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	1.55	.582	.332	X^*_{Sr}
Velocity	2.40	.993	.602	X^*_{Sr}
Density	7.38	1.29	- .516	The fluid position $x = X_{S\ell}(0)$
Energy	8.79	1.98	1.07	The fluid position $x = X_{S\ell}(0)$

	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy
EXACT	7.832×10^{12}	9.430×10^{11}	8.775×10^{12}
PUFF	7.882×10^{12}	8.940×10^{11}	8.776×10^{12}

LAX-WENDROFF

Problem time = 7×10^{-4} sec
Computer time = 623 sec

Cycle = 2474
Number of Active Zones = 200

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	6.12	1.30	.698	X^*_{Sr}
Velocity	3.68	.723	.306	X^*_{Sr}
Density	8.08	1.21	.504	X^*_{Sr}
Energy	5.37	.795	.316	The Langrangian position $x = X_{S\ell}(0)$

	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy
EXACT	7.83203×10^{12}	9.42979×10^{11}	8.77501×10^{12}
LAXWEN	7.81057×10^{12}	9.51231×10^{11}	8.76180×10^{12}

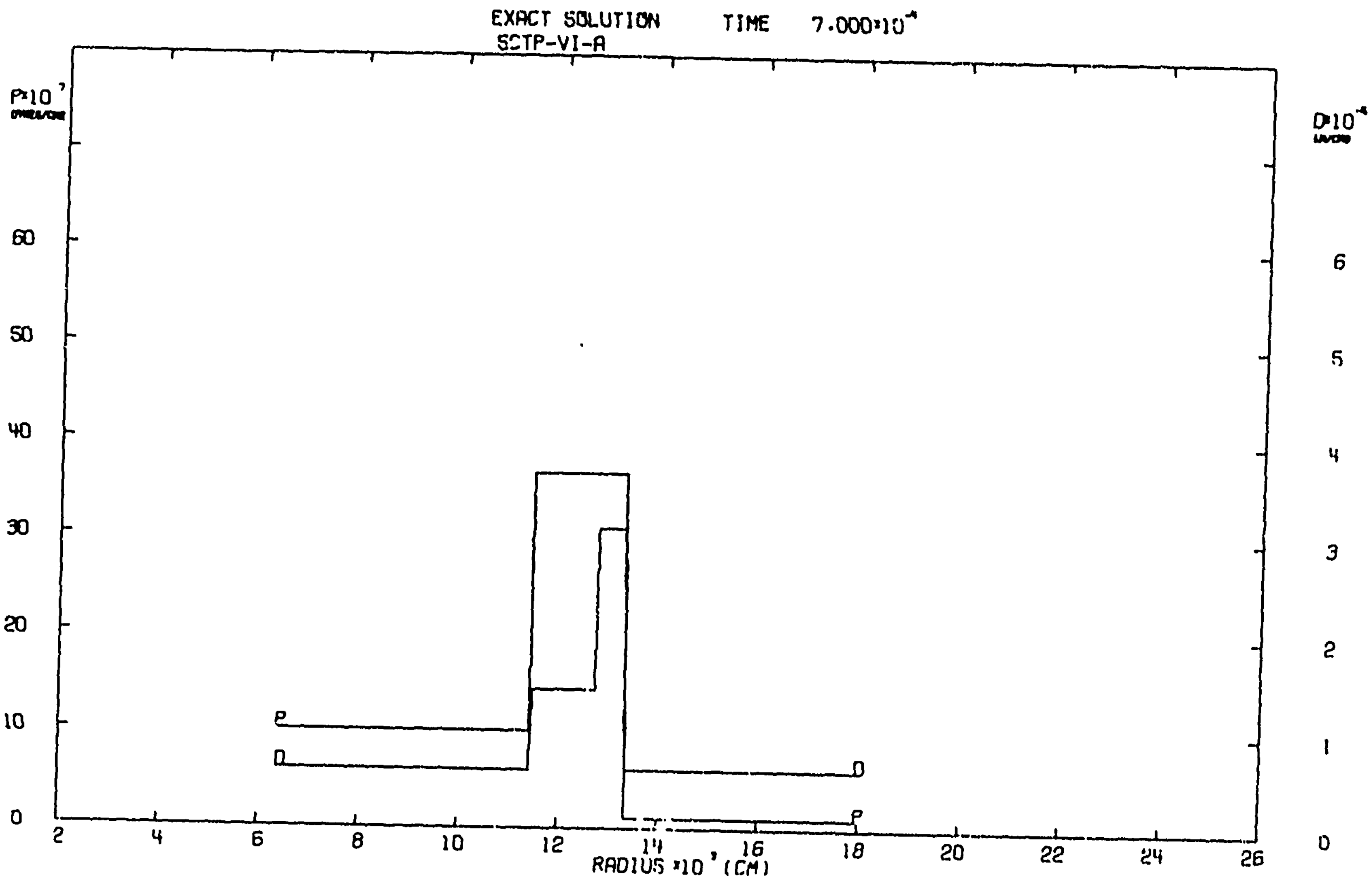

Figure VI-A. PD-EXACT

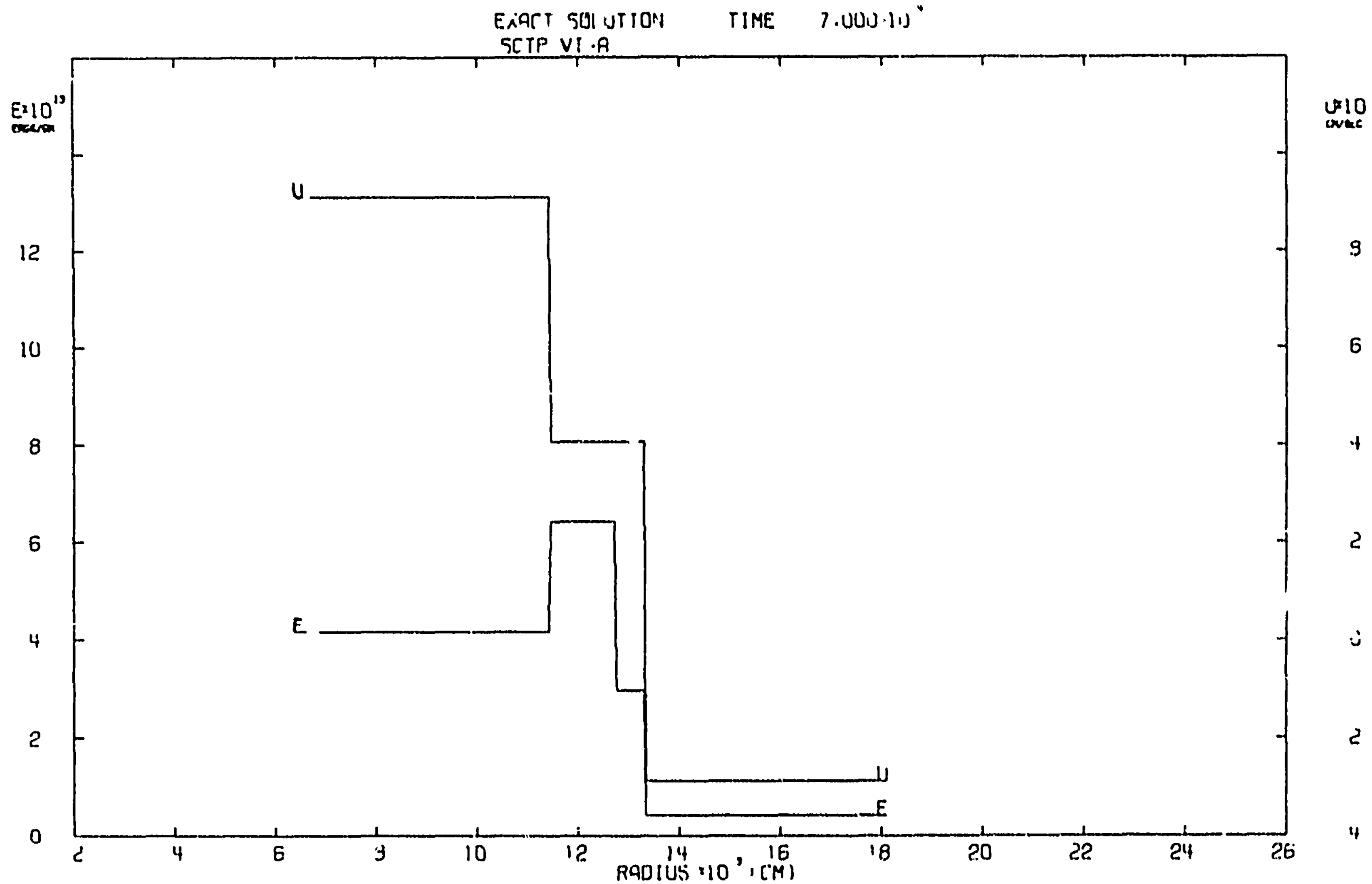

Figure VI-A. VE-EXACT

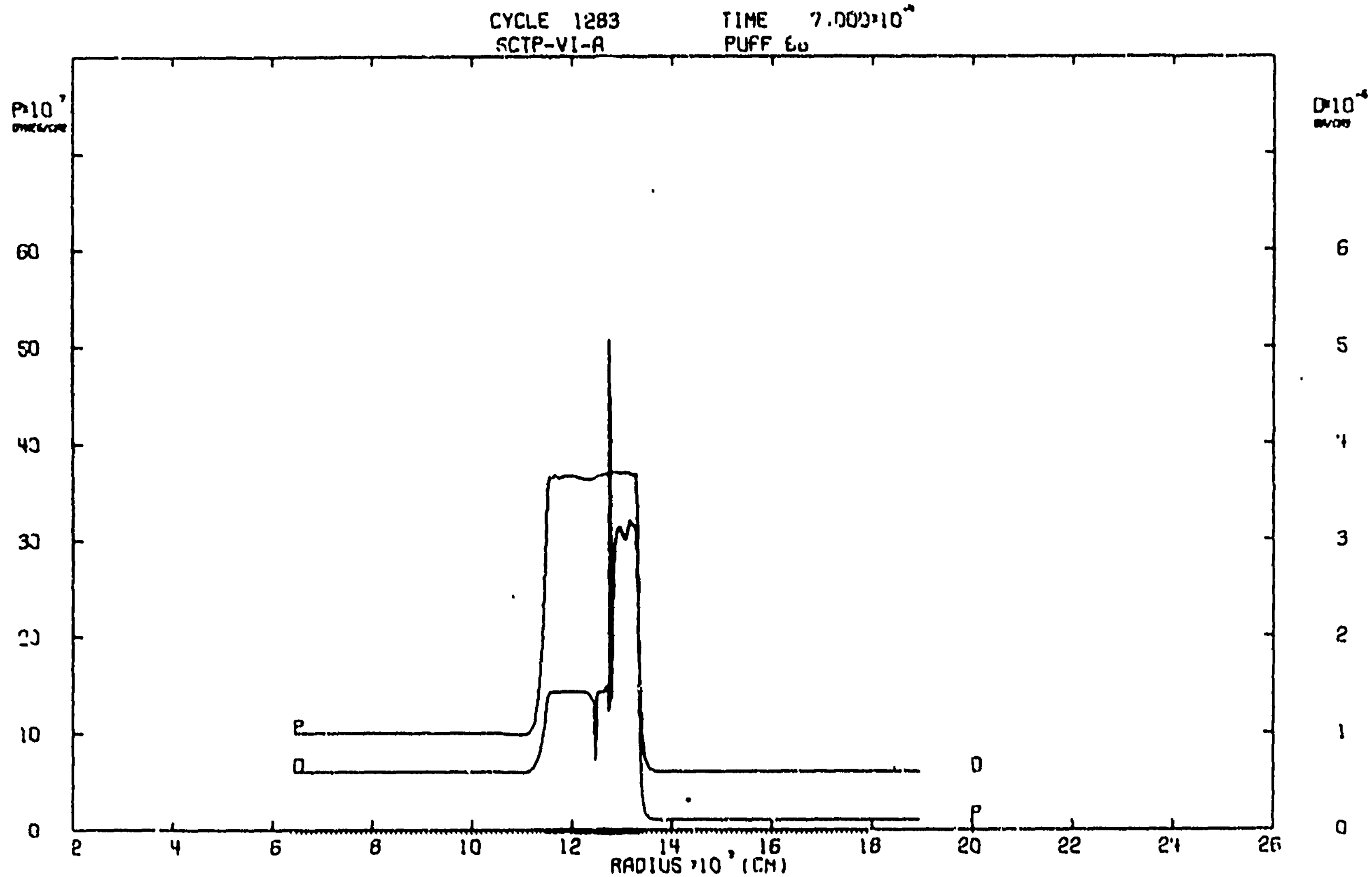

Figure VI-A. PD-PUFF

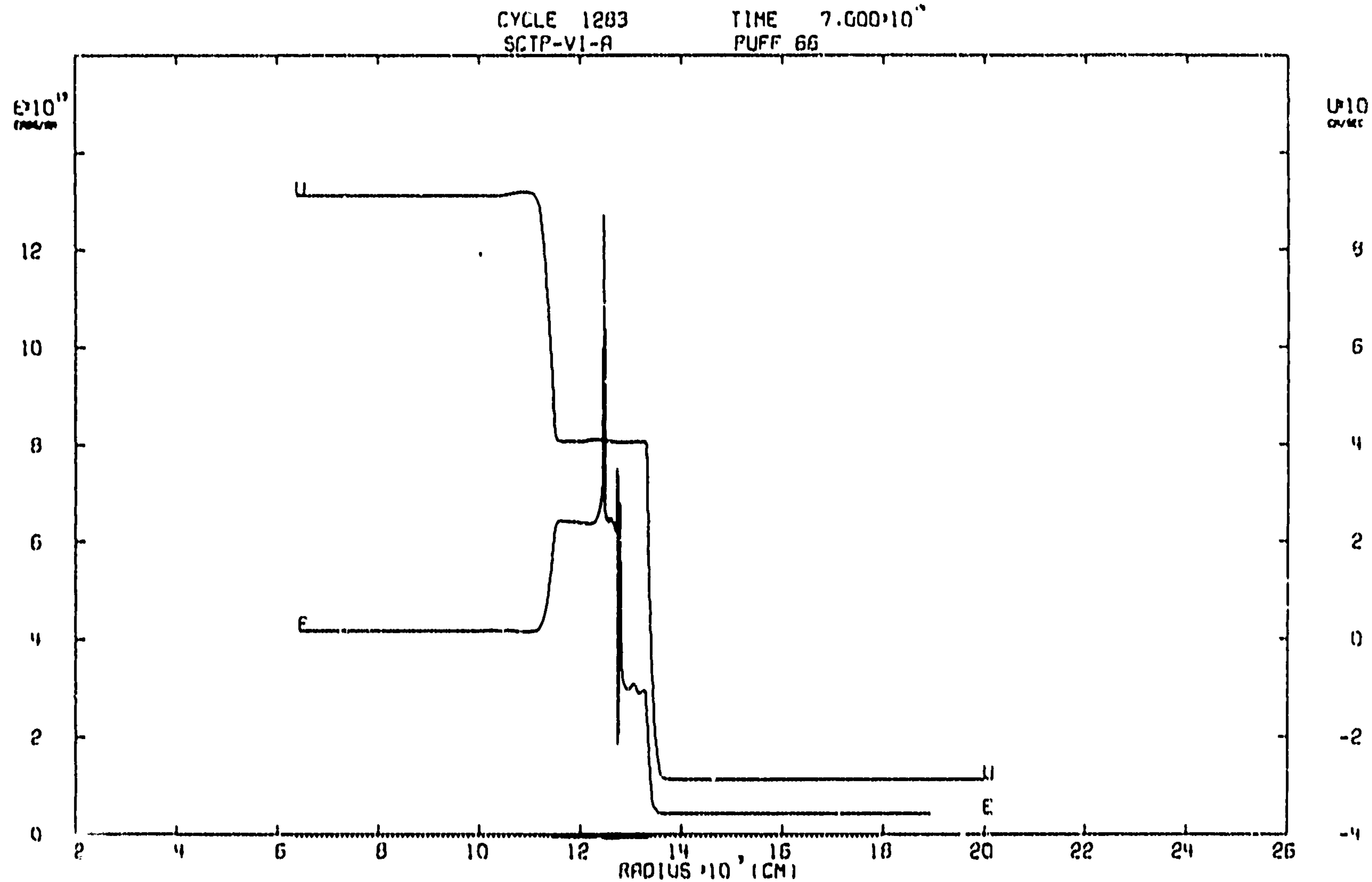

Figure VI-A. VE-PUFF

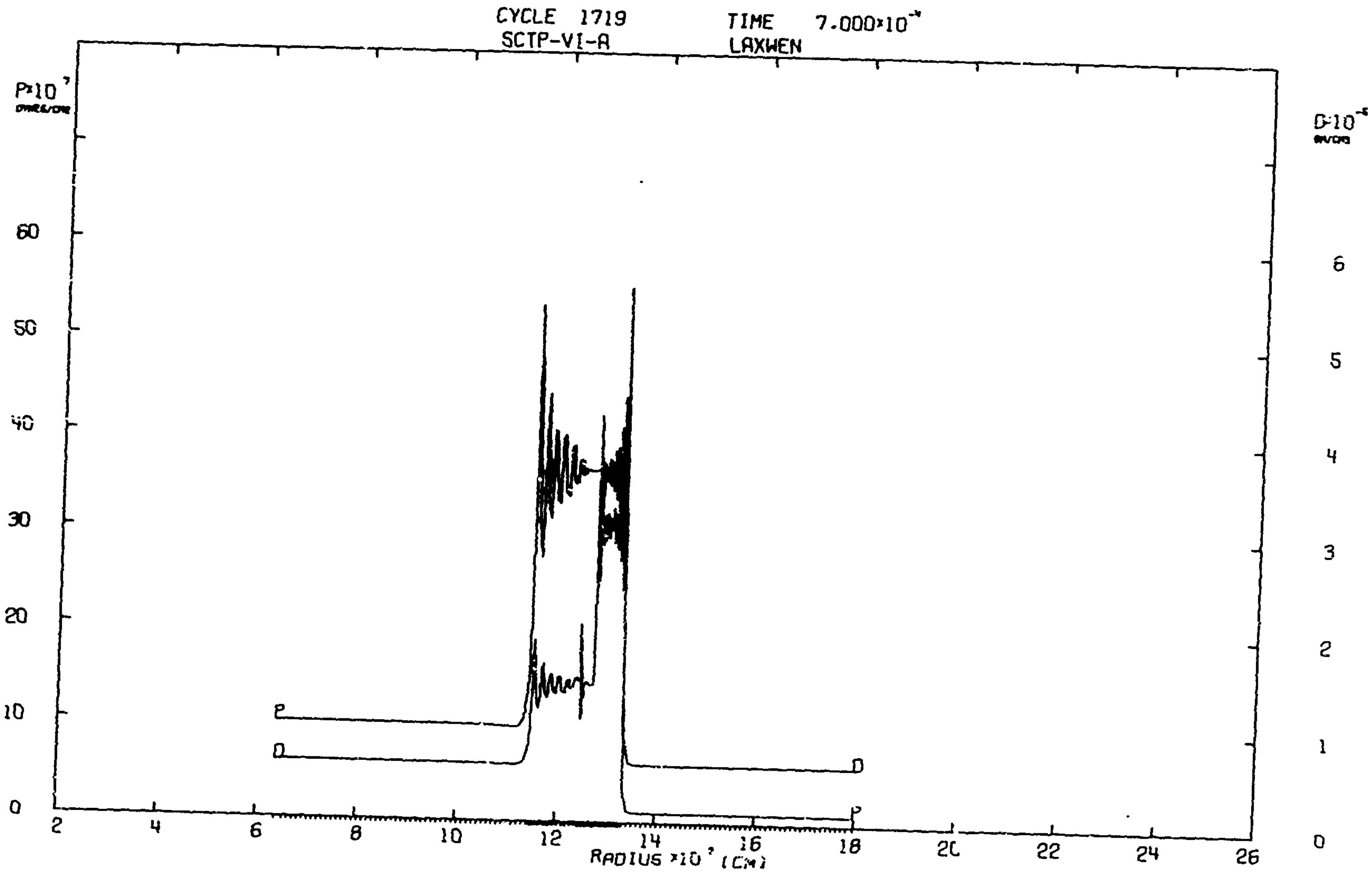

Figure VI-A. PD-LAX-WENDROFF

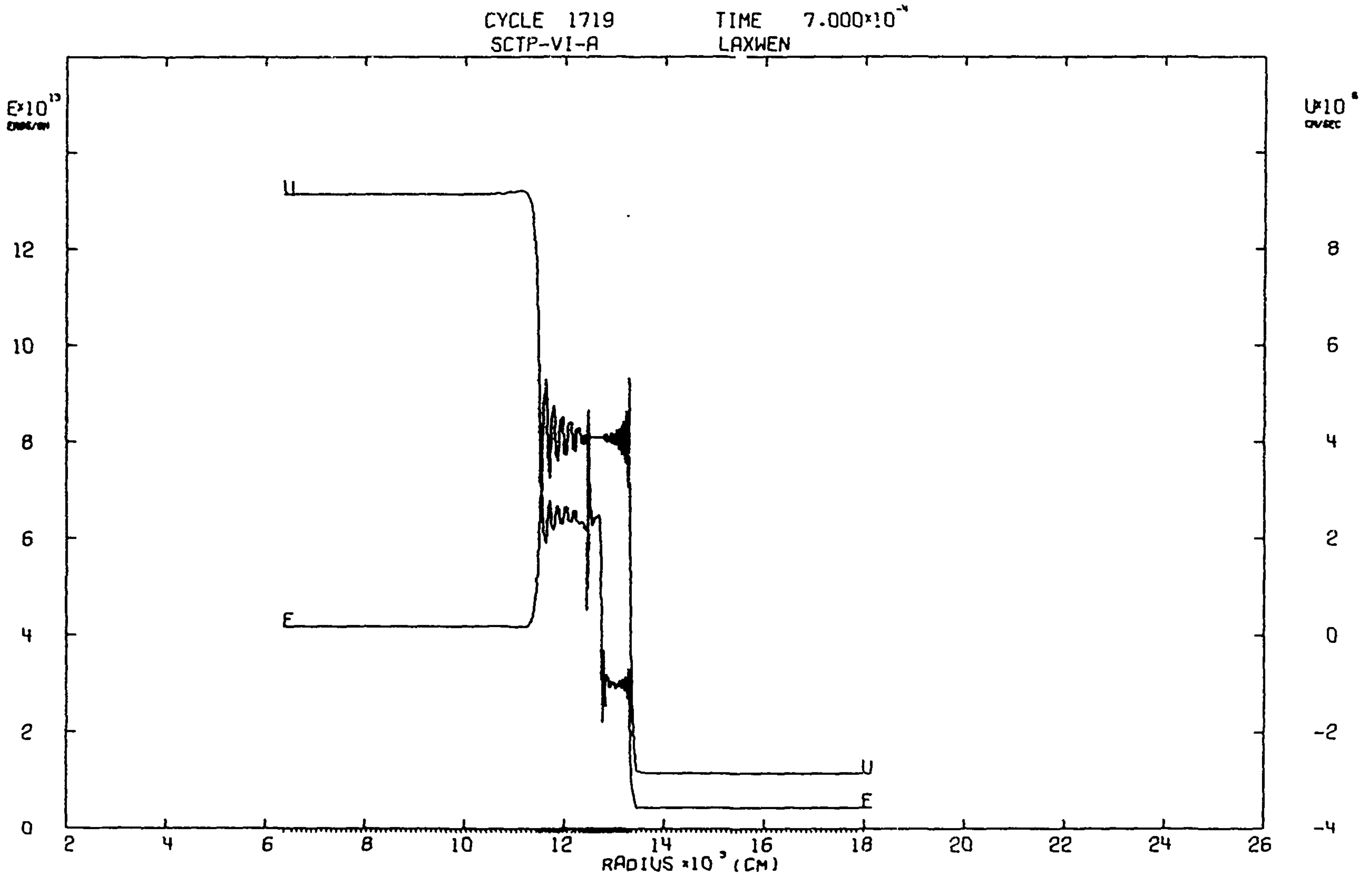

Figure VI-A. VE-LAX-WENDROFF

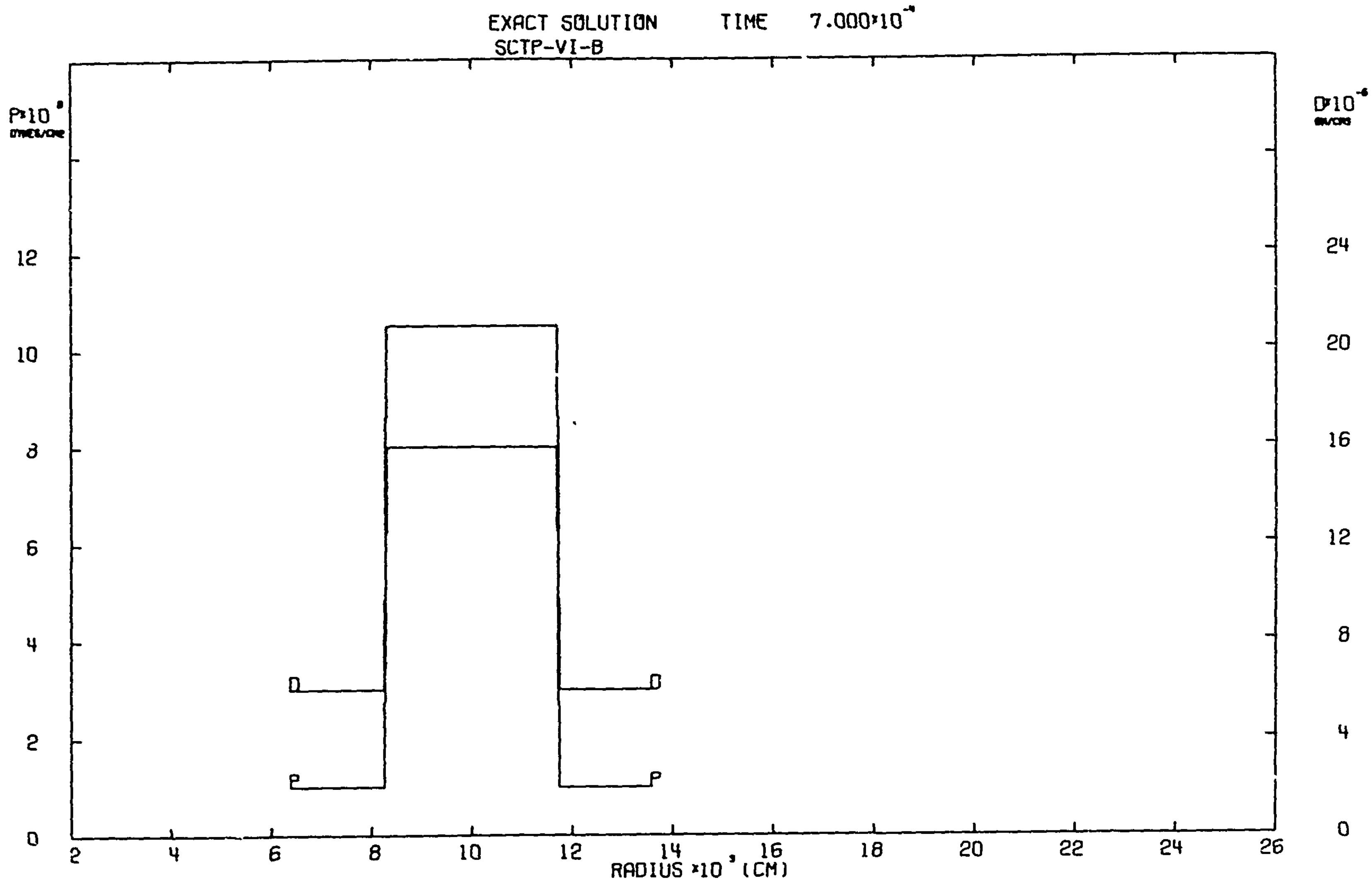

Figure VI-B. PD-EXACT

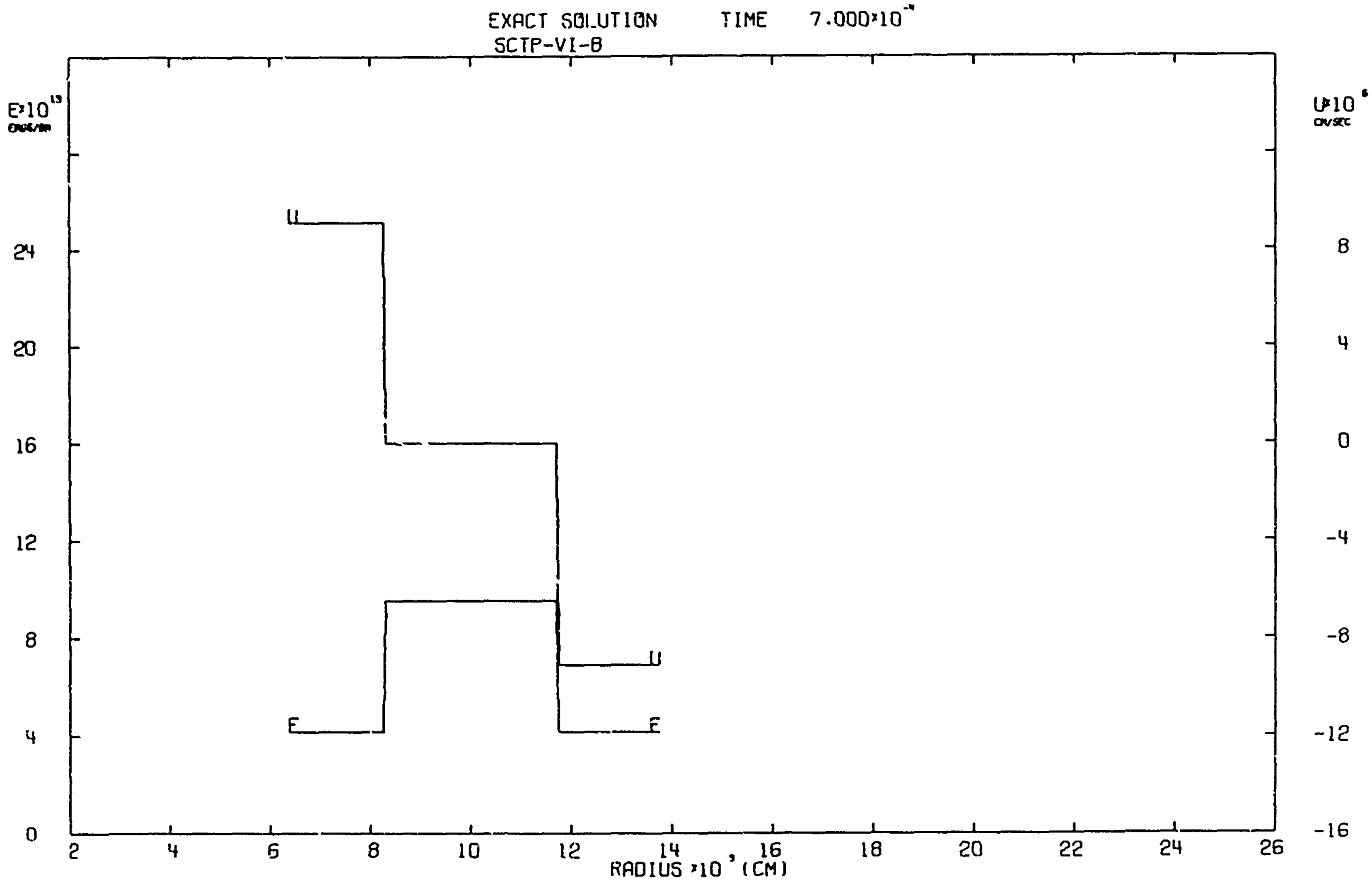

Figure VI-B. VE-EXACT

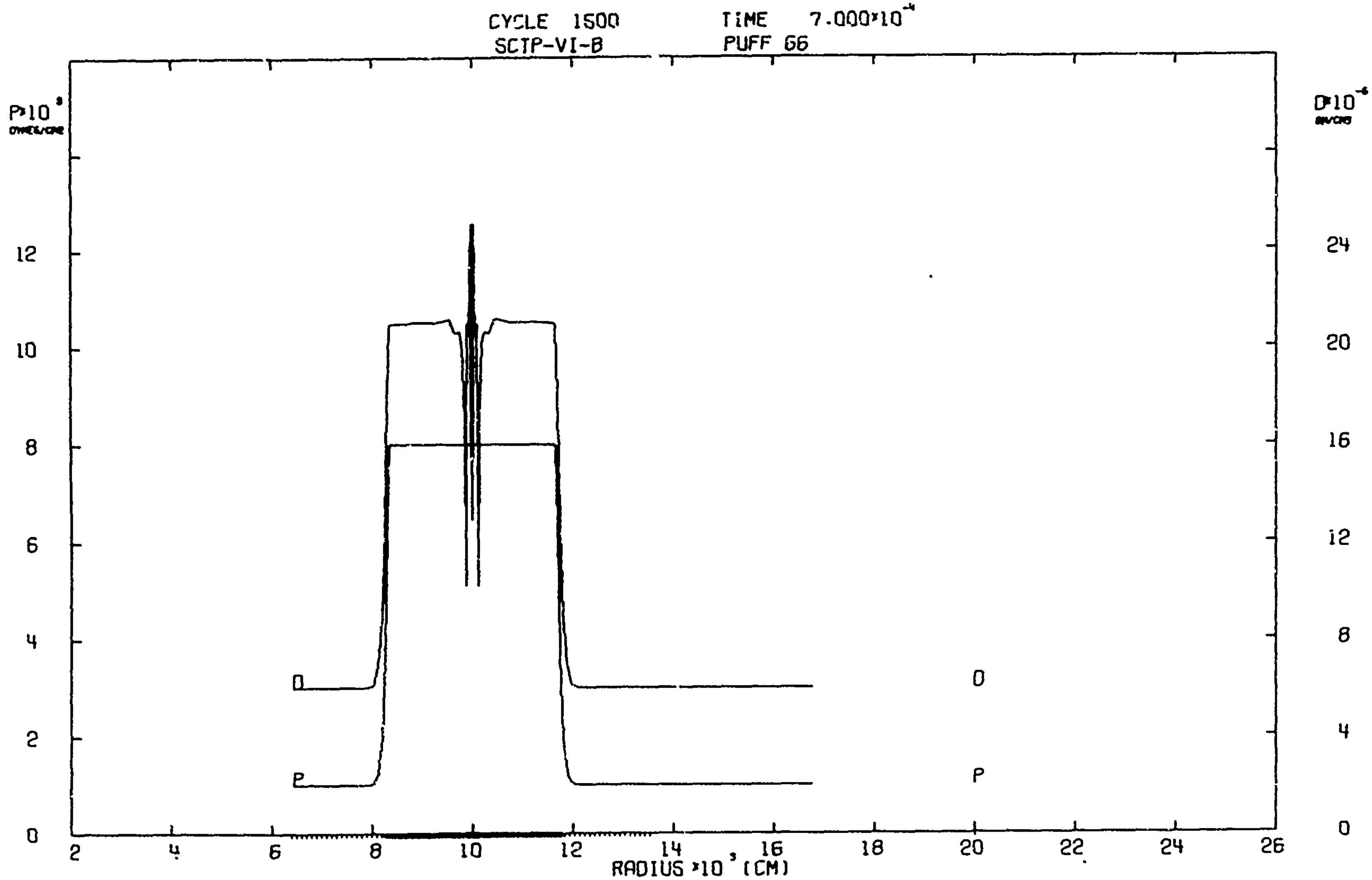

Figure VI-B. PD-PUFF

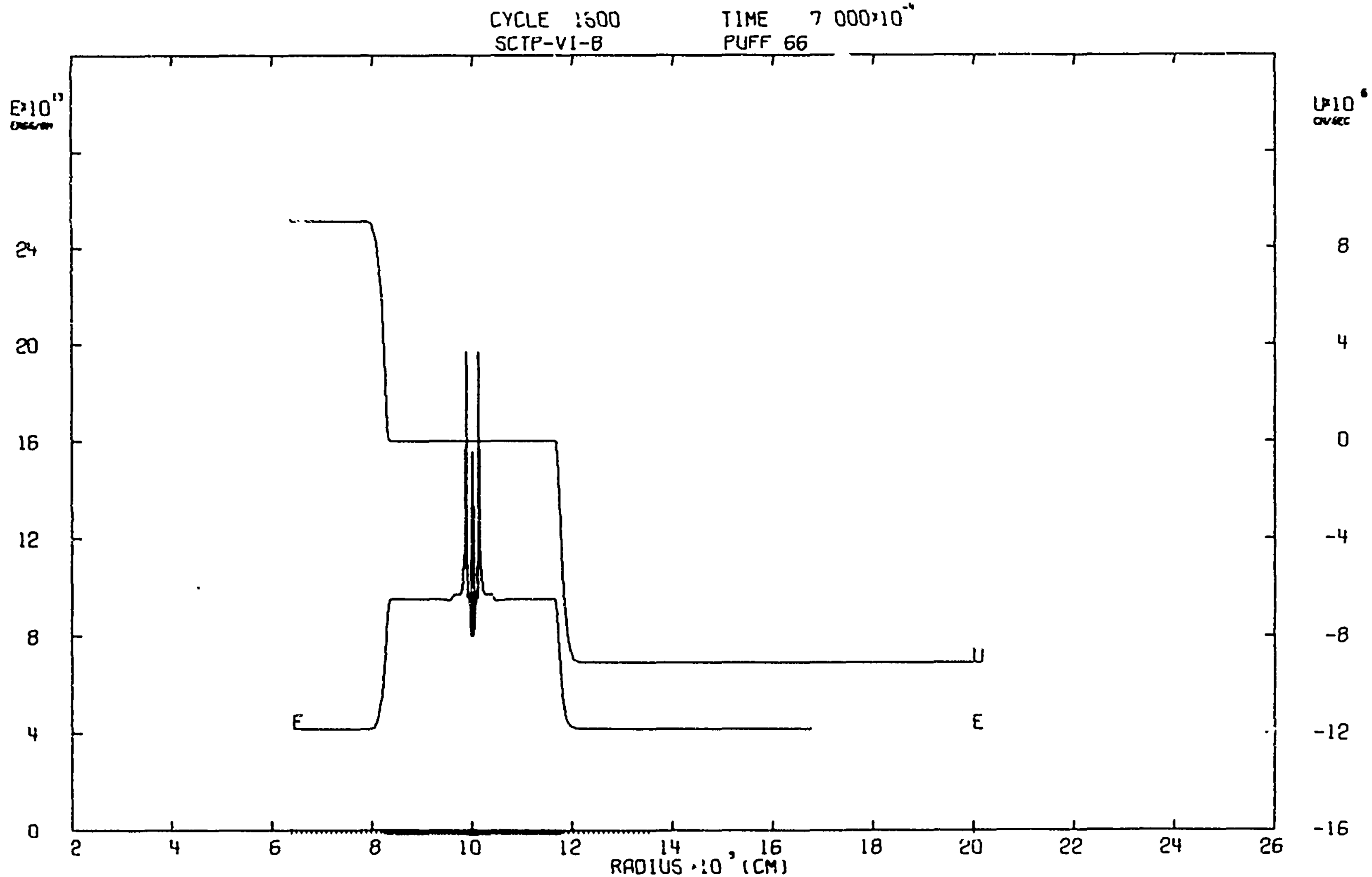

Figure VI-B. VE-PUFF

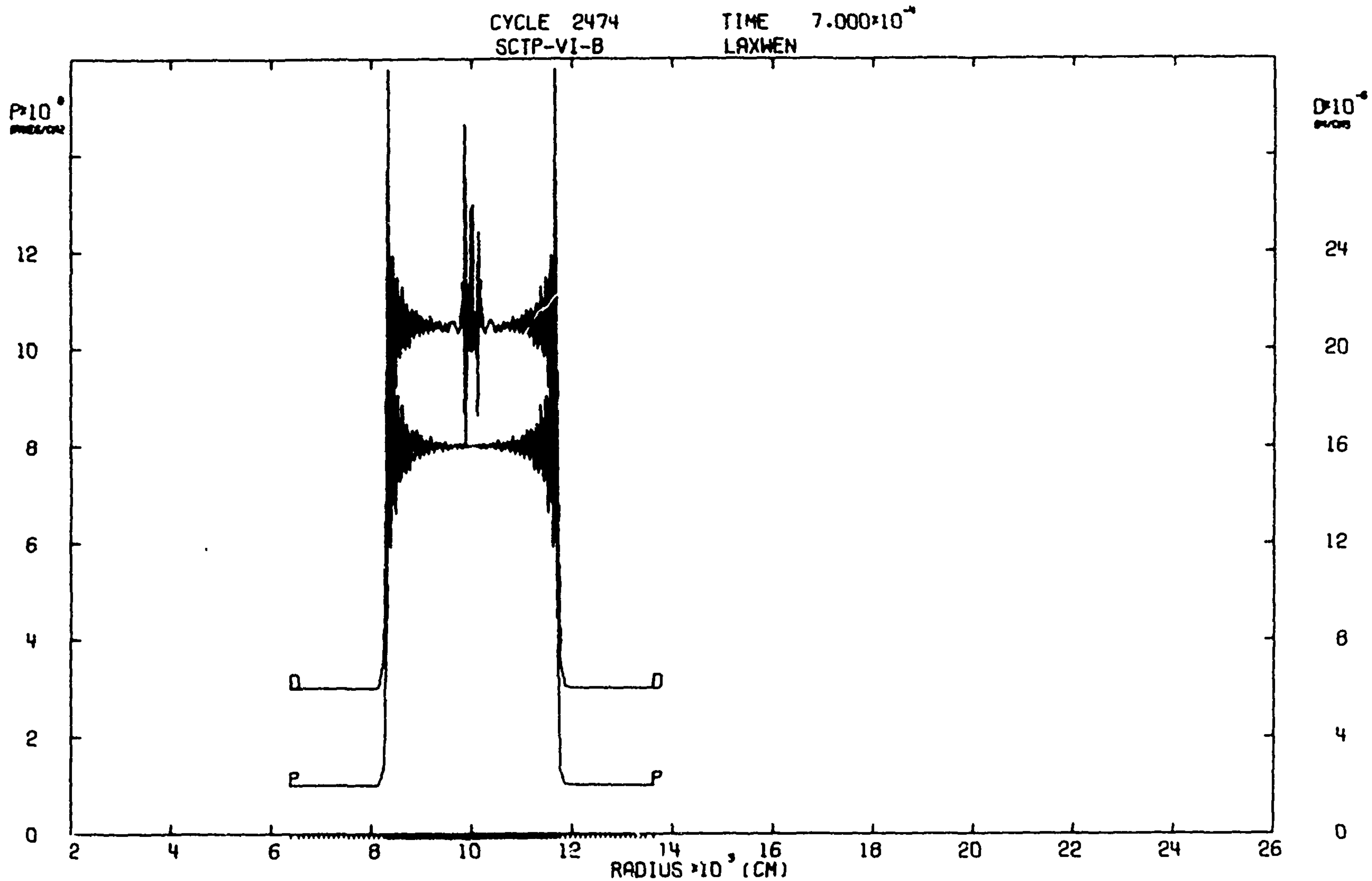

Figure VI-B. PD-LAX-WENDROFF

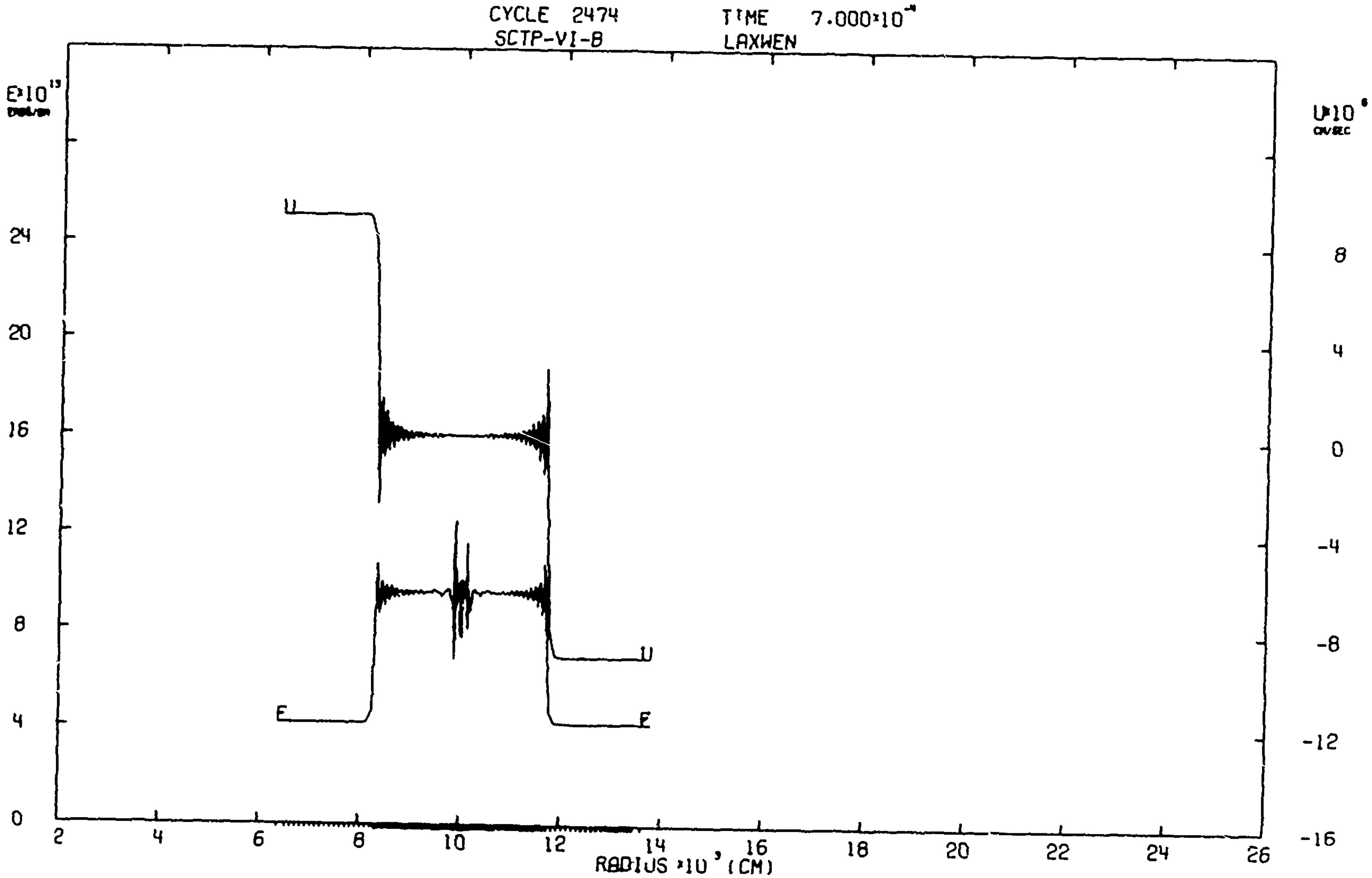

Figure VI-8. VE-LAX-WENDROFF

7. TEST PROBLEM SCTP-VII

a. The Exact Solution

In this problem one shock wave overtakes another. This problem is another special case of the Riemann problem. Two shock waves are traveling in the same direction, which is taken to the right. When two shock waves are traveling in the same direction, the one behind will always overtake the one in front. After overtake time, a rarefaction travels back to the left (for $\gamma \leq 5/3$) and a stronger shock travels on to the right and there is a middle region in which the velocity and pressure are constants v_m and P_m.

Proceeding from left to right, the initial values are P_ℓ, v_ℓ connected by a right-facing shock to $P_{\ell r}$, $\rho_{\ell r}$ which in turn is connected by a right-facing shock to P_r, ρ_r, v_r.

After overtake $v_m = v_r + \phi_r(P_m)$ for the shock traveling to the right and $v_m = v_\ell - \psi_\ell(P_m)$ for the rarefaction traveling to the left. Recall that

$$\phi_r\left(P_m\right) = \left(P_m - P_r\right) \frac{2V_r}{(\gamma-1)\ P_m + (\gamma-1)P_r}$$

and

$$\psi_\ell\left(P_m\right) = \frac{2\sqrt{\gamma}}{\gamma-1}\left(\frac{P_\ell}{\rho_\ell^{\ \gamma}}\right)^{\frac{1}{2\gamma}} \left[P_m^{\ \frac{\gamma-1}{2\gamma}} - P_\ell^{\ \frac{\gamma-1}{2\gamma}}\right]$$

From the above relations v_m and P_m are determined.

In the middle region there will be two values for the density; $\rho_{\ell m} = V_{\ell m}^{-1}$ to the left of the overtake point and $\rho_{mr} = V_{mr}^{-1}$ to the right of the overtake point. The Rankine-Hugoniot relation determines V_{mr} by

$$0 = \frac{1}{\gamma-1}\left(P_m\ V_{mr} - P_r\ V_r\right) + \frac{P_m + P_r}{2}\left(V_{mr} - V_r\right)$$

and $\rho_{\ell m}$ is determined from the fact that the entropy does not change through a rarefaction; therefore,

$$\frac{P_\ell}{P_m} = \left(\frac{\rho_\ell}{\rho_{\ell m}}\right)^\gamma$$

Let $v_{S\ell}$, $X_{S\ell}$ and v_{Sr}, X_{Sr} be the velocities and positions of the left and right shocks prior to overtake; (X_0, t_0) be the point where overtake occurs; v_S, X_S be the velocity and position of the shock after overtake; X_C be the left side of the rarefaction wave; X_r be the right side of the rarefaction wave and X_D be the position of the point in the fluid where overtake occurs.

Solution Summary:

Prior to overtake ($t < t_0$)

LEFT REGION — For $X < X_{S\ell}(t)$, the values are P_ℓ, v_ℓ, ρ_ℓ

MIDDLE REGION — For $X_{S\ell}(t) < X < X_{Sr}(t)$, the values are $P_{\ell r}$, $v_{\ell r}$, $\rho_{\ell r}$

RIGHT REGION — For $X > X_{Sr}(t)$, the values are P_r, v_r, ρ_r

After overtake ($t > t_0$)

LEFT REGION — For $X < X_0 + (v_\ell - C_\ell)(t-t_0) = X_C(t)$, the values are P_ℓ, v_ℓ, ρ_ℓ

MIDDLE REGION — For $X_C(t) < X < X_0 + \left(-C_\ell + \frac{\gamma+1}{2} v_m + \frac{\gamma+1}{2} v_\ell\right)(t-t_0) = X_R(t)$, the velocity goes linearly from v_ℓ at $X_C(t)$ up to v_m at $X_R(t)$ and

$$C = C_\ell - \frac{\gamma-1}{2}(v_\ell - v)$$

$$\rho = \rho_\ell \left(\frac{C}{C_\ell}\right)^{\frac{2}{\gamma-1}}$$

$$P_\ell = P_\ell \left(\frac{C}{C_\ell}\right)^{\frac{2\gamma}{\gamma-1}}$$

For the region $X_R(t) < X < X_0 + v_m(t-t_0) = X_D(t)$, the values are P_m, $\rho_{\ell m}$, and v_m. For the region $X_D(t) < X < X_0 + v_S(t-t_0) = X_S(t)$, the values are P_m, ρ_{mr}, v_m.

RIGHT REGION — For $X > X_s(t)$, the values are P_r, ρ_r, v_r.

The necessary data for this problem are:

INITIAL VALUES: P_ℓ, $P_{\ell r}$, P_r, ρ_r, v_r

From these values all other initial values are determined.

BOUNDARY VALUES: At $X = 0$ (the left boundary), hold the values at P_ℓ, ρ_ℓ, v_ℓ and at X_Q (the right boundary) hold the values at P_r, ρ_r, v_r.

Now the specific numerical values are presented for SCTP-VII:

$\Delta X = 1$ meter

$X_{S\ell}(0) = 25$ meters

$X_{Sr}(0) = 100$ meters

$P_r = 10^4$ dynes/cm^2

$\rho_r = 10^{-6}$ gm/cm^3

$v_r = 0$

$P_{\ell r} = 10^8$ dynes/cm^2

$P_\ell = 10^{12}$ dynes/cm^2

$X_Q = 200$ meters

These values yield

$C_\ell \doteq 1.97 \times 10^8$ cm/sec

$v_\ell \doteq 3.82 \times 10^8$ cm/sec

$\rho_\ell \doteq 3.60 \times 10^{-5}$ gm/cm^3

$v_{S\ell} \doteq 4.56 \times 10^8$ cm/sec

$v_{Sr} \doteq 1.095 \times 10^7$ cm/sec

$t_o \doteq 1.683 \times 10^{-5}$ sec

$X_0 \doteq 1.018 \times 10^4$ cm

$P_m \doteq 3.32 \times 10^{11}$ dynes/cm^2

$v_{\ell r} \doteq 9.13 \times 10^6$ cm/sec

$\rho_{\ell r} \doteq 5.997 \times 10^{-6}$ gm/cm^3

$v_m \doteq 5.26 \times 10^8$ cm/sec

$$v_S \doteq 6.31 \times 10^8 \text{ cm/sec}$$

$$\rho_{m\ell} \doteq 1.63 \times 10^{-5} \text{ gm/cm}^3$$

$$\rho_{mr} \doteq 6.000 \times 10^{-6} \text{ gm/cm}^3$$

This problem was run to 3×10^{-5} sec.

b. The PUFF Solution

As in SCTP-VI, the major errors in evidence were the spikes in density and internal energy. Hot-thin spikes resulted from the initial shock discontinuities and a cold-thick spike resulted from the shock overtake. For more details, see Table and Figures VII.

c. The LAX-WENDROFF Solution

In addition to the spikes observed in the PUFF solution, there is also more oscillation in the LAX-WENDROFF solution. The time factor used was .39, the artificial viscosity factor used was .25, and both factors were multiplied by one-twentieth on the first time step, two-twentieths on the second, etc., until the twentieth time step and thereafter when they were left at the values of .39 and .25 For more details, see Table and Figures VII.

Table VII

ERRORS ON SCTP-VII

PUFF

Problem time = 3×10^{-5} sec
Computer time = 106 sec

Cycle = 1642
Number of Active Zones = 200

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	.837	.214	- .184	X_S
Velocity	1.37	.565	+ .467	X_S
Density	1.60	.427	- .234	The fluid point $x = X_{S\ell}(0)$
Energy	2.64	.908	- .581	The fluid point $x = X_0$
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	9.374×10^{15}	1.489×10^{16}	2.426×10^{16}	
PUFF	9.370×10^{15}	1.488×10^{16}	2.425×10^{16}	

LAX-WENDROFF

Problem time = 3×10^{-5} sec
Computer time = 734 sec

Cycle = 2919
Number of Active Zones = 200

	Sum Abs. Error	Sum Sqr. Error	Maximum Error	Position of Maximum Error
Pressure	1.34	.380	- .255	X_S
Velocity	1.46	.570	- .508	X_S
Density	4.23	2.20	+ 2.14	The fluid point $x = X_0$
Energy	3.22	.961	+ .664	The fluid point $x = X_0$
	Sum Int. Energy	Sum Kin. Energy	Sum Tot. Energy	
EXACT	9.374×10^{15}	1.489×10^{16}	2.426×10^{16}	
LAXWEN	9.370×10^{15}	1.481×10^{16}	2.418×10^{16}	

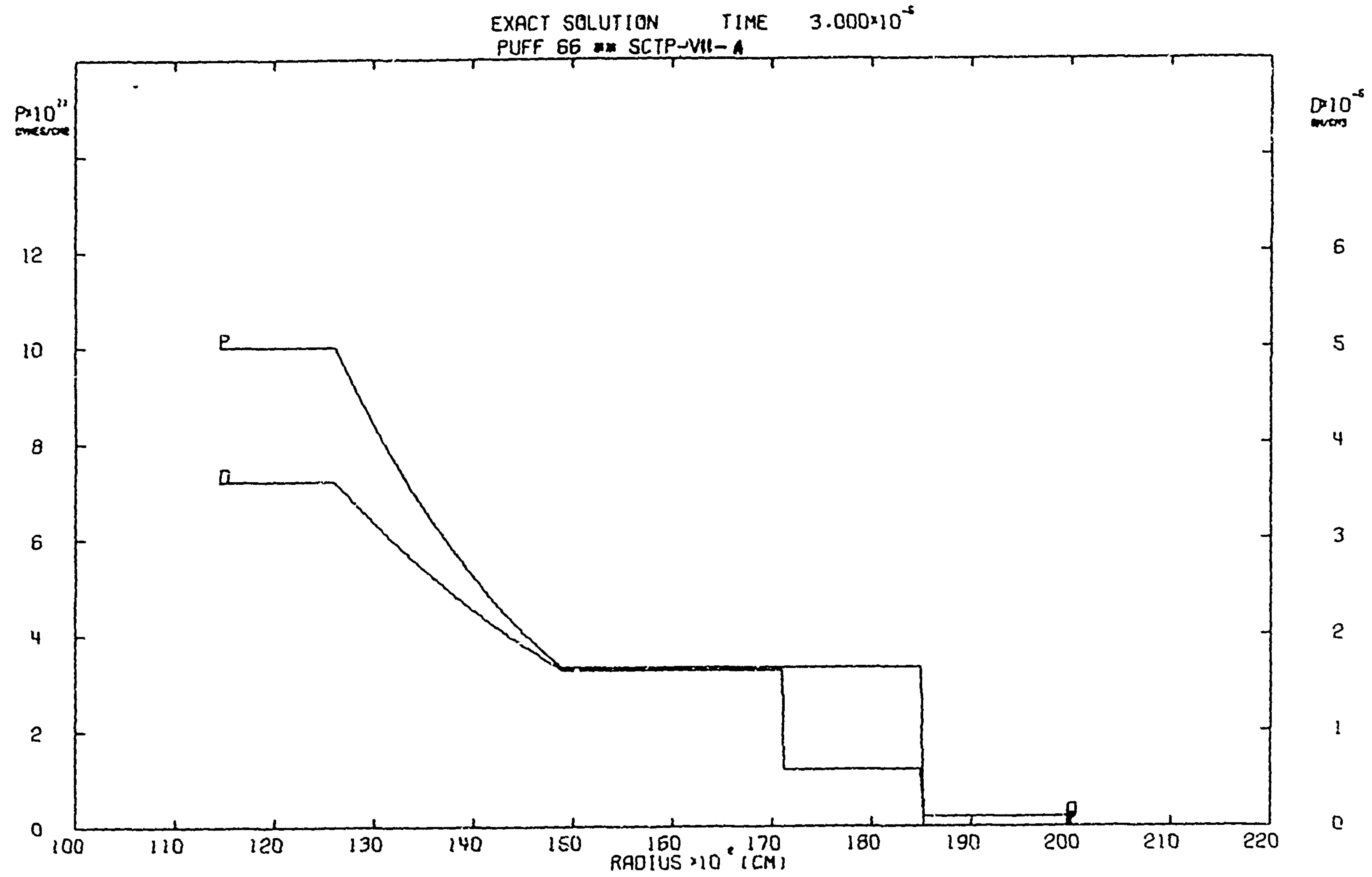

Figure VII. PD-EXACT

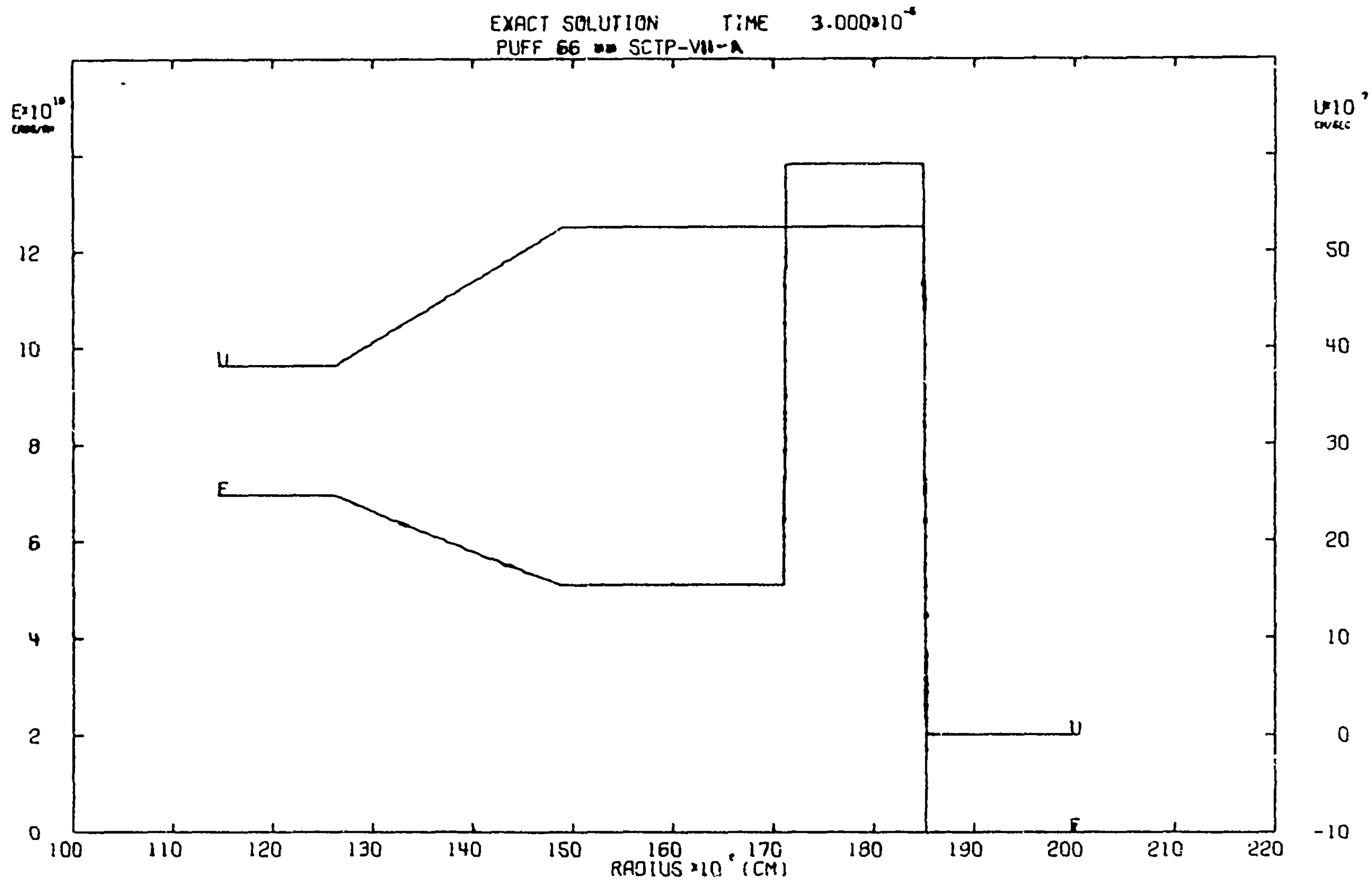

Figure VII. VE-EXACT

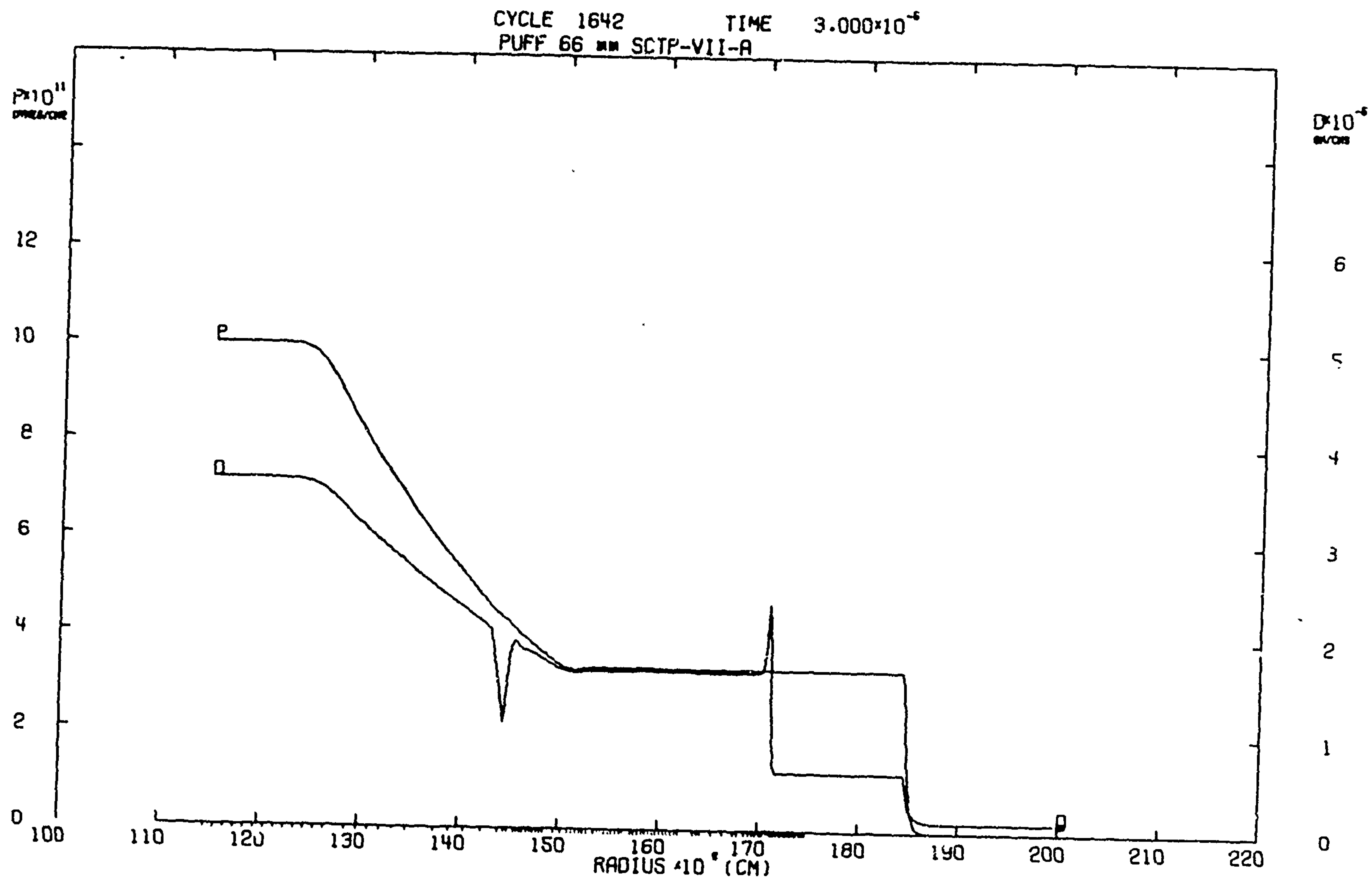

Figure VII. PD-PUFF

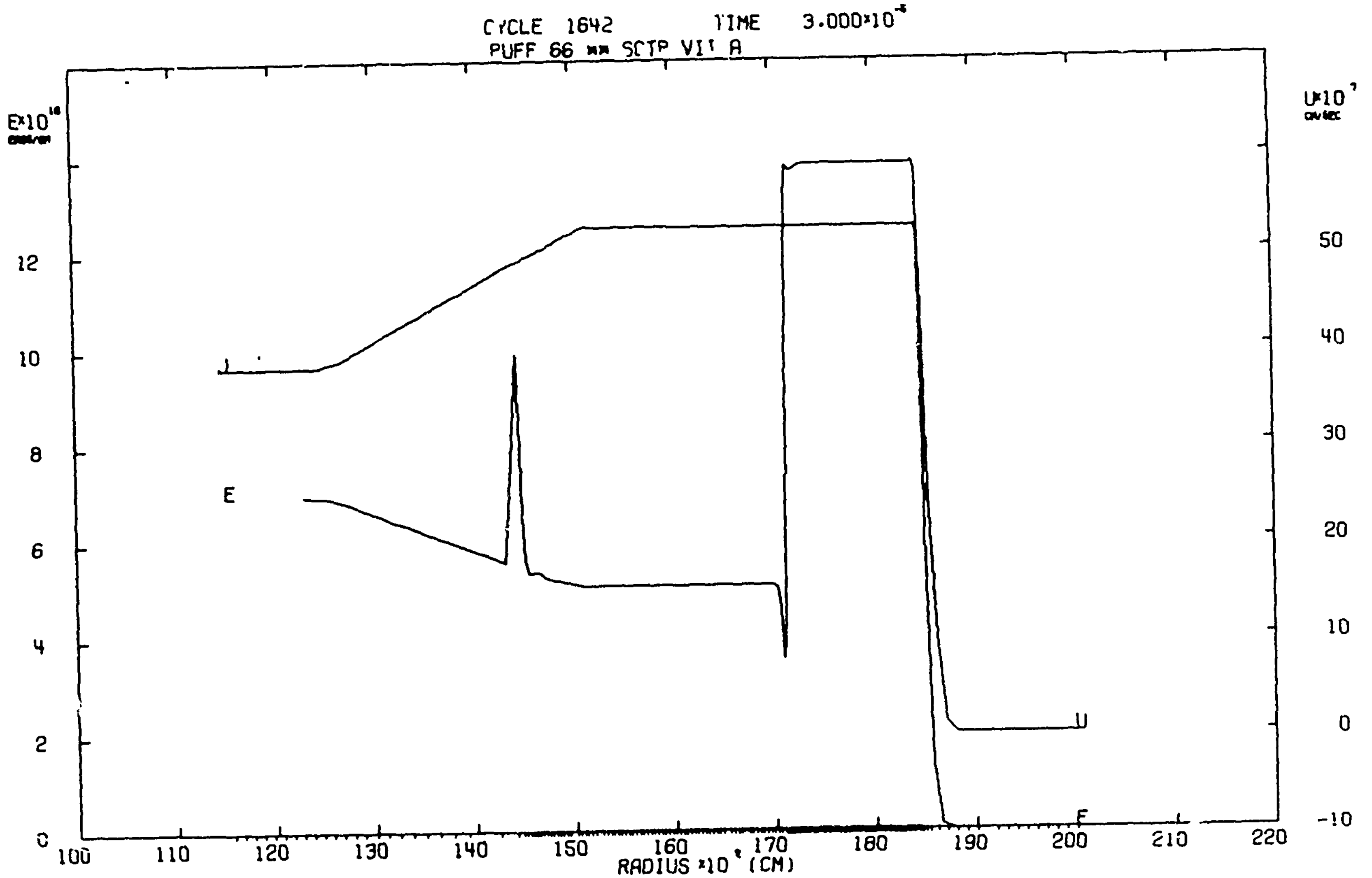

Figure VII. VE-PUFF

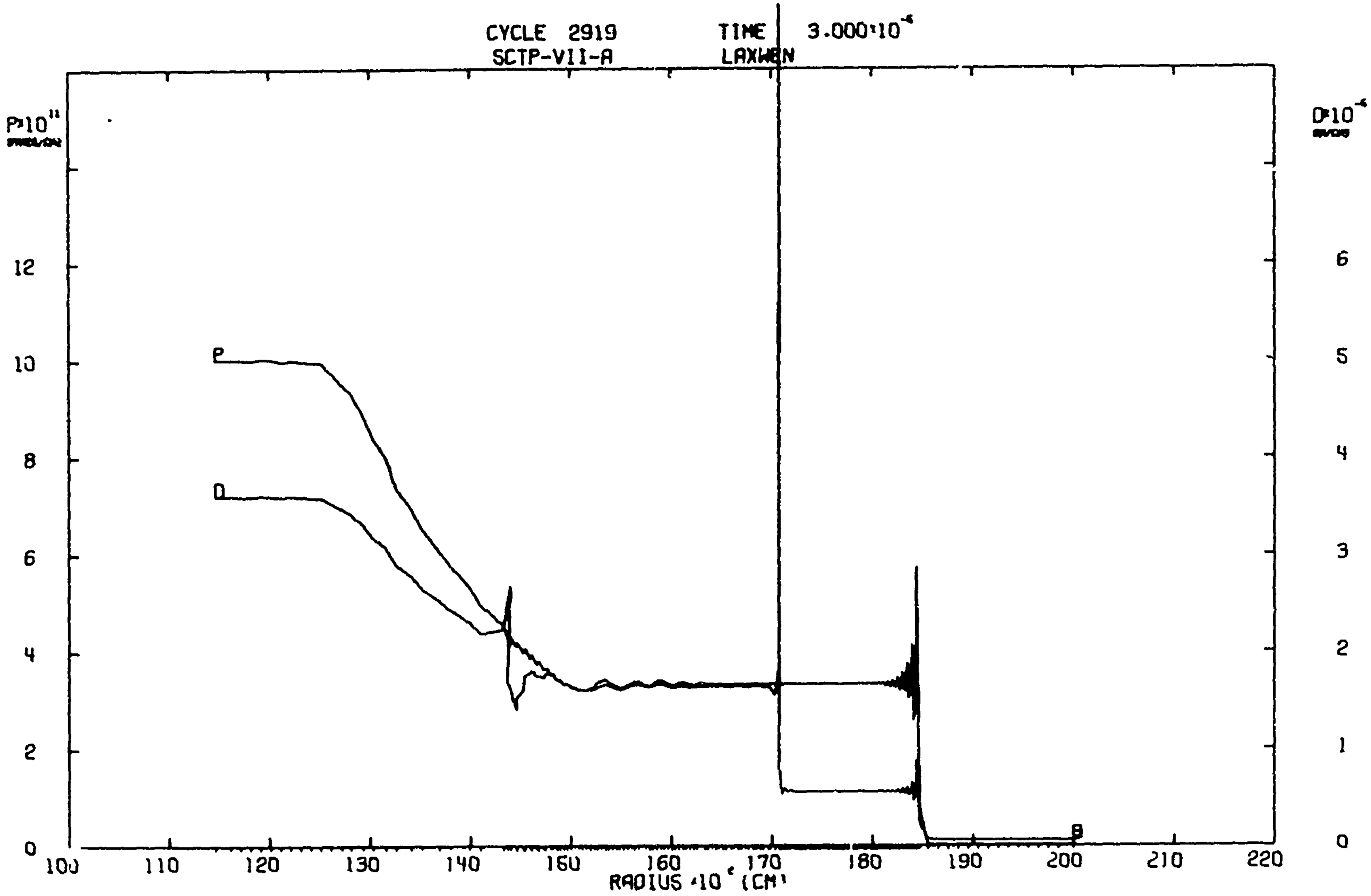

Figure VII. PD-LAX-WENDROFF

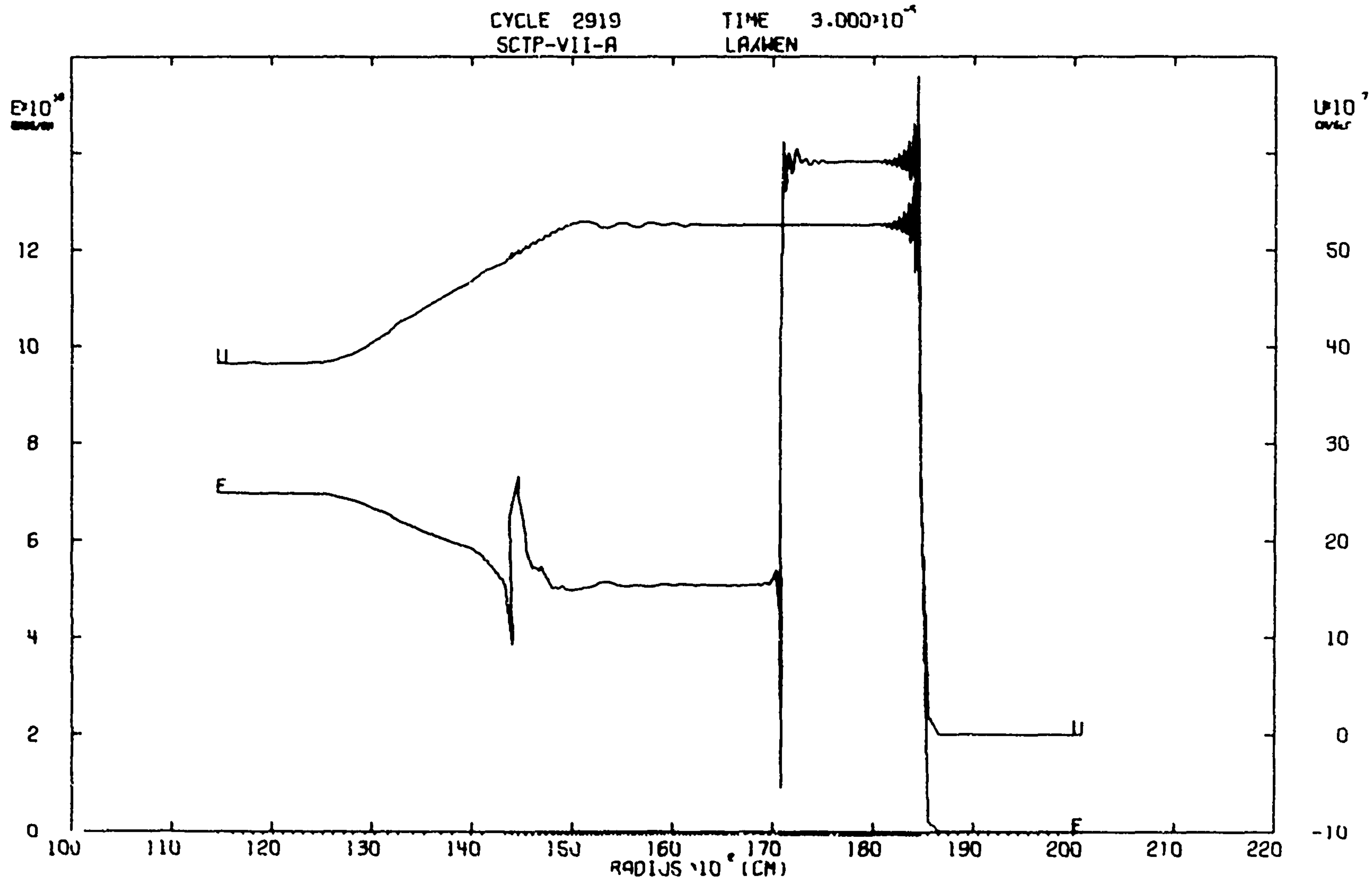

Figure VII. VE-LAX-WENDROFF

SECTION III

CONCLUSIONS

The most apparent difference between the PUFF and LAX-WENDROFF solutions is the greater tendency of the LAX-WENDROFF scheme to oscillate.

In those flows in which there are no strong shocks of strong rarefactions or vacuums, the LAX-WENDROFF scheme is more accurate than PUFF. However, in those flows in which there are strong shocks or strong rarefactions or vacuums the PUFF scheme is more accurate. The LAX-WENDROFF scheme cannot handle vacuums because of the use of the specific volume instead of the density as a fluid variable. It appears that the LAX-WENDROFF scheme could be improved by using an artificial viscosity of the type used in PUFF. And in general it appears that it might be possible to combine the better features of PUFF and LAX-WENDROFF to produce a superior hydrocode. This will be investigated and discussed in a forthcoming report.

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13. ABSTRACT

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A comparison between two one-dimensional Lagrangian hydrocodes has been made. The two hydrocodes are a von Neumann-Richtmyer hydrocode (AFWL's PUFF) and a Lax-Wendroff hydrocode (the two-step version with artificial viscosity). The comparison was made by applying the hydrocode test problems as described in HYDROCODE TEST PROBLEMS, AFWL-TR-67-127, February 1968. The most apparent difference between the von Neumann-Richtmyer hydrocode and the Lax-Wendroff is the greater tendency of the Lax-Wendroff scheme to oscillate. In those flows in which there are no strong shocks or strong rarefactions or vacuums, the Lax-Wendroff scheme is more accurate. However, in those flows in which there are strong shocks or strong rarefactions or vacuums the von Neumann-Richtmyer scheme is more accurate. The Lax-Wendroff scheme cannot handle vacuums because of the use of specific volume instead of the density as a fluid variable. It appears that it might be possible to combine the better features of the von Neumann-Richtmyer and the Lax-Wendroff schemes to produce a better hydrocode.

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PUFF Hydrocode						

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