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BLACKBODY RADIATION, PHOTON EMISSION, AND THE CALCULATION OF DEBYE FUNCTIONS

Donald C. Todd

ARO, Inc.

December 1968

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NOTICES

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Donald C. Todd
ARO, Inc.

FOREWORD

The work reported herein was sponsored by Air Force Avionics Laboratory (AVRO) Wright-Patterson AFB, Ohio, under Program Element 62403F, Project 4163, Task 06.

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This technical report has been reviewed and is approved.

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Colonel, USAF
Director of Test

ABSTRACT

This report aids in the calculation of blackbody radiation. The equations of blackbody radiation and photon emission are summarized and a computer program is described that is useful for calculations concerning blackbodies. The input to the program consists of the units desired, the temperature, and wavelengths. The output includes the physical constants in the desired units and the evaluation of the equations for the temperature and wavelengths specified. The equations for photon and power emission are written in terms of Debye functions. The computer program uses a subroutine that calculates to five significant figures the Debye functions of orders from one through six. The method and error analysis of this calculation are included.

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NOMENCLATURE

a	Conversion factor (Eq. (10))
a_{nk}^{ℓ}	Term of infinite series (Eq. (49))
b_2	Ratio of ν_{Mf} over T_a
b_3	Ratio of ν_{Me} over T_a
b_4	Product of ν_{Mf} and T_a
b_5	Product of ν_{Me} and T_a
B_{2k}	Bernoulli numbers
c	Speed of light
c_1	First radiation constant
c_2	Second radiation constant
$D_n(x)$	Debye function
$D_n^{\ell}(x)$	Debye function or its complement, depending on ℓ
$e_{b\lambda}$	Monochromatic emissive power in terms of wavelength
$e_{b\nu}$	Monochromatic emissive power in terms of frequency
E_b	Total emissive power
$E_{b\lambda}$	Emissive power in terms of wavelength
$E_{b\nu}$	Emissive power in terms of frequency
E_{nm}^{ℓ}	Relative error
$f_{b\lambda}$	Monochromatic photon emittance in terms of wavelength
$f_{b\nu}$	Monochromatic photon emittance in terms of frequency
F_b	Total photon emittance
$F_{b\lambda}$	Photon emittance in terms of wavelength
$F_{b\nu}$	Photon emittance in terms of frequency
h	Planck constant
k	Boltzmann constant
k	Subscript indexing terms in infinite series

ℓ	Superscript distinguishing different series
m	Number of terms of infinite series retained
n	Order of Debye function
r	Defined in Eq. (57)
R_{nm}^{ℓ}	Remainder
S_{n}^{ℓ}	Infinite series
S_{nm}^{ℓ}	Partial sum
T	Temperature
T_a	Absolute temperature
x	Defined in Eq. (20) or independent variable of Debye function
ζ	Zeta function
λ	Wavelength
λ_{Me}	Wavelength of maximum $e_{b\lambda}$
λ_{Mf}	Wavelength of maximum $f_{b\lambda}$
ν	Frequency
ν_{Me}	Frequency of maximum $e_{b\nu}$
ν_{Mf}	Frequency of maximum $f_{b\nu}$
σ	Stefan-Boltzmann constant
τ	Ratio of F_b to T_a^3

SECTION I
INTRODUCTION

The blackbody concept is important in theoretical and applied physics and in radiation heat transfer. Some of its properties, such as the Stefan-Boltzmann law and Weins displacement law can be derived from the second law of thermodynamics. The search for a theoretical derivation of the spectral distribution function of blackbody radiation, which fits experimental data, led to Planck's Radiation law, which was the birth of quantum mechanics (Ref. 1). The complete knowledge of the spectral distribution of blackbody radiation makes it a valuable standard for checking the performance of wavelength-dependent devices such as optical filters, spectrometers, and radiometers. Many calculations of radiation heat transfer are based on the assumption that the surfaces are black or gray.

In certain areas, calculations involving blackbody radiation are frequently required. These calculations necessitate looking up or remembering the equations and physical constants, performing unit conversions, using tables, and evaluating the equations. There are several useful aids in the calculations, such as the photon slide rule, the radiation slide rule, and tables of Refs. 2, 3, and 4, respectively. This report and the described computer program should also be useful. A summary of the equations of blackbody radiation and photon emission is included. The input to the computer program includes the units desired, the temperature, and the wavelengths; the output consists of the physical constants in the desired units and the evaluation of the equations for the temperature and wavelengths specified.

The photon and power emissions between wavelengths are written in terms of Debye functions. The computer program uses a subroutine that calculates to five significant figures the Debye functions of orders from one through six. The method and error analysis of this calculation are good examples of numerical application of infinite series; this analysis is given in Section IV, independent of the other material.

SECTION II
EQUATIONS

The energy emitted by a blackbody per unit time and per unit area in a frequency range $d\nu$ is $e_{b\nu}\, d\nu$ where $e_{b\nu}$ is the monochromatic emissive power in terms of frequency and is given by

$$e_{b\nu} = \frac{2\pi h \nu^3}{c^2\left[\exp\left(\frac{h\nu}{kT_a}\right) - 1\right]} \tag{1}$$

The energy emitted per unit time and per unit area in a frequency range from 0 to ν is the emissive power in terms of frequency and is

$$E_{b\nu} = \int_0^{\nu} e_{b\nu}\, d\nu \tag{2}$$

The total energy emitted per unit time and per unit area from all frequencies is the total emissive power

$$E_b = \int_0^{\infty} e_{b\nu}\, d\nu \tag{3}$$

The maximum value of $e_{b\nu}$ occurs at the frequency ν_{Me} which is the nonzero root of

$$\frac{de_{b\nu}}{d\nu} = 0 \tag{4}$$

A photon is a packet of energy, $h\nu$. Thus, the number of photons emitted by a blackbody per unit time and per unit area in a frequency range $d\nu$ is

$$f_{b\nu}\, d\nu = \frac{e_{b\nu}}{h\nu}\, d\nu \tag{5}$$

and the monochromatic photon emittance in terms of frequency, $f_{b\nu}$, is

$$f_{b\nu} = \frac{e_{b\nu}}{h\nu} \tag{6a}$$

or

$$f_{b\nu} = \frac{2\pi \nu^2}{c^2\left[\exp\left(\frac{h\nu}{kT_a}\right) - 1\right]} \tag{6b}$$

The number of photons emitted per unit time and per unit area in a frequency range from 0 to ν is the photon emittance in terms of frequency and is

$$F_{b\nu} = \int_0^{\nu} f_{b\nu}\, d\nu \tag{7}$$

The total number of photons emitted per unit time and per unit area from all frequencies is the total photon emittance and is

$$F_b = \int_0^{\infty} f_{b\nu}\, d\nu \tag{8}$$

The maximum value of $f_{b\nu}$ occurs at the frequency ν_{Mf} which is the nonzero root of

$$\frac{df_{b\nu}}{d\nu} = 0 \tag{9}$$

Wavelength and frequency are related by

$$\nu = \frac{c}{a\lambda} \tag{10}$$

where a is a conversion factor, such as cm per μ, since sometimes different length units are used for c and λ. The energy emitted by a blackbody per unit time and per unit area in a wavelength range $d\lambda$ is $e_{b\lambda}\, d\lambda$, where $e_{b\lambda}$ is the monochromatic emissive power in terms of wavelength and is given by

$$e_{b\lambda} = -e_{b\nu} \frac{d\nu}{d\lambda} \tag{11a}$$

or

$$e_{b\lambda} = \frac{2\pi hc^2}{a^4\lambda^5\left[\exp\left(\frac{hc}{ak\lambda T_a}\right) - 1\right]} \tag{11b}$$

which is commonly written as

$$e_{b\lambda} = \frac{c_1}{\lambda^5\left[\exp\left(\frac{c_2}{\lambda T_a}\right) - 1\right]} \tag{11c}$$

where

$$c_1 = \frac{2\pi hc^2}{a^4} \tag{12}$$

and

$$c_2 = \frac{hc}{ak} \tag{13}$$

The energy emitted per unit time and per unit area in a wavelength range from 0 to λ is the emissive power in terms of wavelength and is

$$E_{b\lambda} = \int_0^{\lambda} e_{b\lambda}\, d\lambda \tag{14}$$

The maximum value of $e_{b\lambda}$ occurs at the wavelength λ_{Me} which is the nonzero root of

$$\frac{de_{b\lambda}}{d\lambda} = 0 \tag{15}$$

Note that λ_{Me} is not the wavelength corresponding to ν_{Me}.

The number of photons emitted per unit time and per unit area in a wavelength range $d\lambda$ is $f_{b\lambda} d\lambda$, where $f_{b\lambda}$ is the monochromatic photon emittance in terms of wavelength and is

$$f_{b\lambda} = -f_{b\nu} \frac{d\nu}{d\lambda} \tag{16a}$$

which may be written as

$$f_{b\lambda} = \frac{2\pi c}{a^3 \lambda^4 \left[\exp\left(\frac{c_2}{\lambda T_a}\right) - 1\right]} \tag{16b}$$

The number of photons emitted per unit time and per unit area in a wavelength range from 0 to λ is the photon emittance in terms of wavelength and is

$$F_{b\lambda} = \int_0^\lambda f_{b\lambda}\, d\lambda \tag{17}$$

The maximum value of $f_{b\lambda}$ occurs at the wavelength λ_{Mf} which is the nonzero root of

$$\frac{df_{b\lambda}}{d\lambda} = 0 \tag{18}$$

Note that λ_{Mf} is not the wavelength corresponding to ν_{Mf}.

The emissive power and photon emittance can be written in terms of Debye functions (Ref. 5). A table of Debye functions is given in Ref. 5, and a method to calculate them is given in Section IV of this report. The nth order Debye function is defined as

$$D_n(x) = \int_0^x \frac{t^n}{\exp(t) - 1}\, dt \tag{19}$$

making the substitution

$$x = \frac{h\nu}{kT_a} \tag{20}$$

Combination of Eqs. (1), (2), and (19) results in

$$E_{b\nu} = \frac{2\pi k^4}{c^2 h^3} T_a^4 D_3(x) \tag{21}$$

The substitution into Eq. (7) results in

$$F_{b\nu} = \frac{2\pi k^3}{c^2 h^3} T_a^3 D_2(x) \tag{22}$$

The value of the Debye functions at infinity (Ref. 4) is

$$D_n(\infty) = n!\,\zeta(n+1) \tag{23}$$

where the zeta function is defined as

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \tag{24}$$

Using this notation, and Eqs. (3) and (21), one obtains

$$E_b = \frac{12\pi k^4}{c^2 h^3}\,\zeta(4)\,T_a^4 \tag{25a}$$

which is commonly written as

$$E_b = \sigma T_a^4 \tag{25b}$$

The value of $\zeta(4)$ is (Ref. 5)

$$\zeta(4) = \frac{\pi^4}{90} = 1.08232\,32337\,11138\,19152 \tag{26}$$

thus, σ is

$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3} \tag{27}$$

From Eqs. (8) and (22) one obtains

$$F_b = \frac{4\pi k^3}{c^2 h^3}\,\zeta(3)\,T_a^3 \tag{28}$$

which can be written as

$$F_b = \tau T_a^3 \tag{29}$$

where

$$\tau = \frac{4\pi k^3}{c^2 h^3}\,\zeta(3) \tag{30}$$

There is no exact expression for $\zeta(3)$ as there was for $\zeta(4)$; however, its value is (Ref. 5)

$$\zeta(3) = 1.20205\,69031\,59594\,28540 \tag{31}$$

The substitution of Eq. (20) into Eq. (14) results in

$$E_{b\lambda} = E_b - E_{b\nu} \tag{32}$$

and similarly

$$F_{b\lambda} = F_b - F_{b\nu} \tag{33}$$

Performing the differentiation of Eqs. (4), (9), (15), and (18), the result in each case is of the form

$$x = n(1 - e^{-x}) \tag{34}$$

The nonzero roots, x_n, of Eq. (34) are given in Table I for the values of n from 2 through 10. Thus is obtained

$$\nu_{Me} = b_3 T_a \tag{35}$$

$$\nu_{Mf} = b_2 T_a \tag{36}$$

$$\lambda_{Me} T_a = b_5 \tag{37}$$

$$\lambda_{Mf} T_a = b_4 \tag{38}$$

where

$$b_3 = \frac{k x_3}{h} \tag{39}$$

$$b_2 = \frac{k x_2}{h} \tag{40}$$

$$b_5 = \frac{hc}{akx_5} \tag{41}$$

$$b_4 = \frac{hc}{akx_4} \tag{42}$$

The equations of this section summarize the expression for the quantities of blackbody radiation and photon emission which are most frequently to be calculated. The next section describes a computer program that relieves the calculator of the labor of finding physical constants and conversion factors, performing unit conversions, using tables, and evaluating the equations.

TABLE I
ROOTS OF $x = n(1 - e^{-x})$

n	x_n
2	1.59362 42601
3	2.82143 93721
4	3.92069 03949
5	4.96511 42317
6	5.98490 12264
7	6.99357 56867
8	7.99730 90676
9	8.99888 80761
10	9.99954 57944

SECTION III
COMPUTER PROGRAM

The computer program was written in IBM System/360 FORTRAN IV language. Its calculations can be divided into three groups:

1. Unit conversions and calculation of the physical constants.
2. Calculation of quantities dependent on temperature, but not wavelength.
3. Calculation of wavelength-dependent quantities.

The flow chart of the program is given in Fig. 1, and a description of the logic is given below followed by an explanation of the input and output.

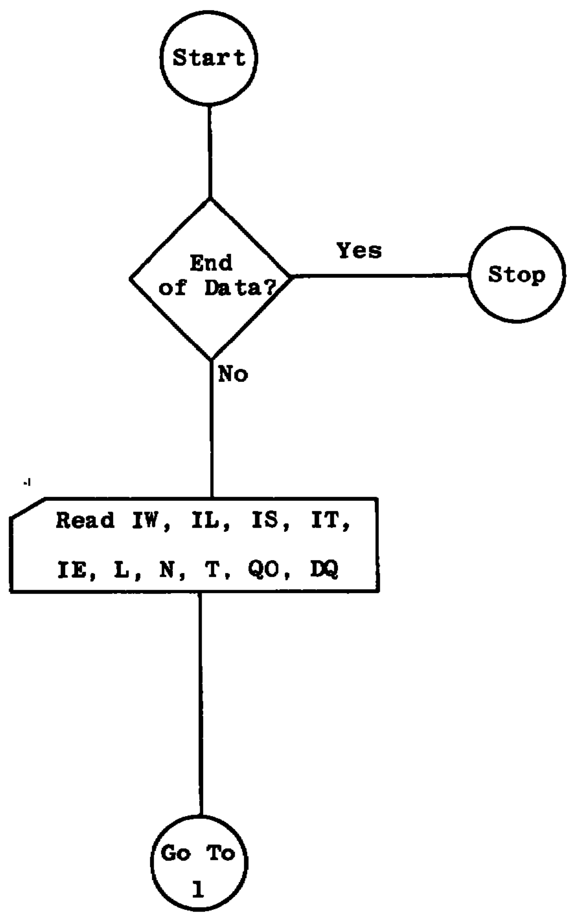

Fig. 1 Flow Chart of the Computer Program

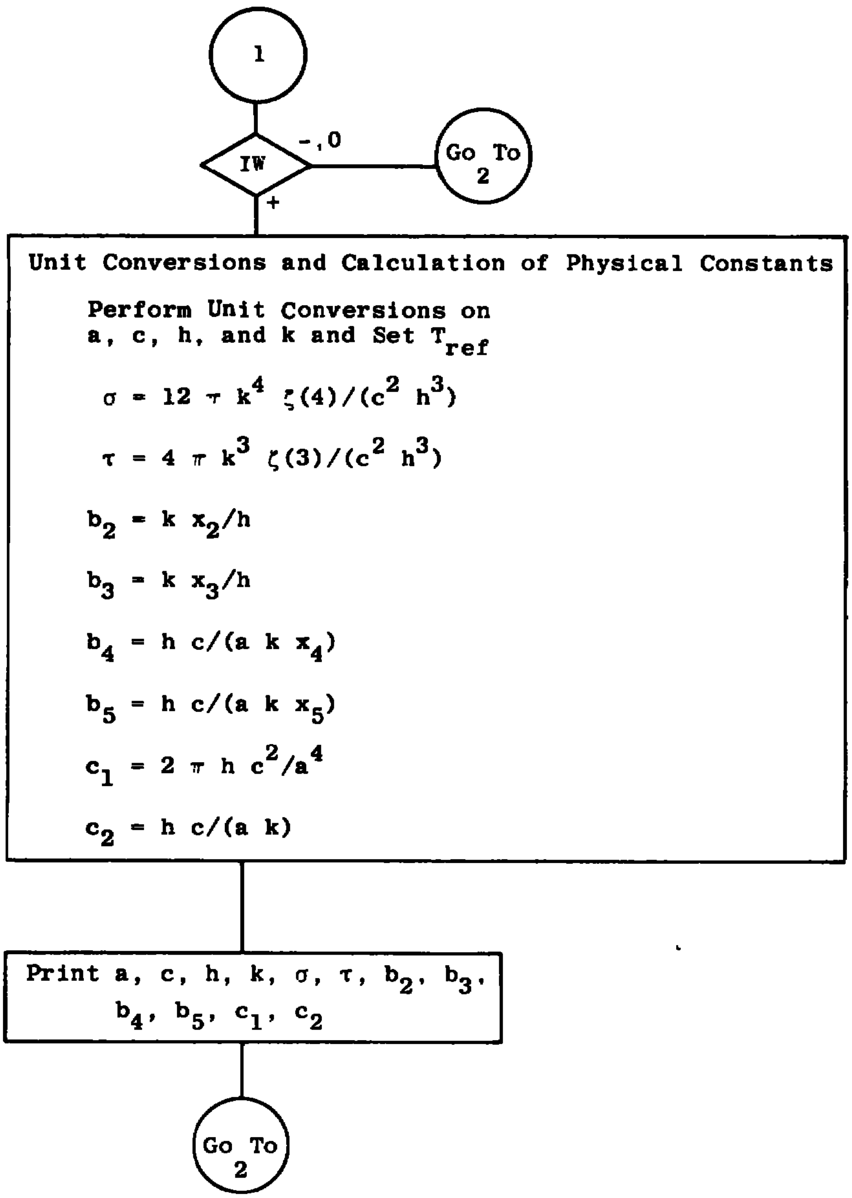

Fig. 1 Continued

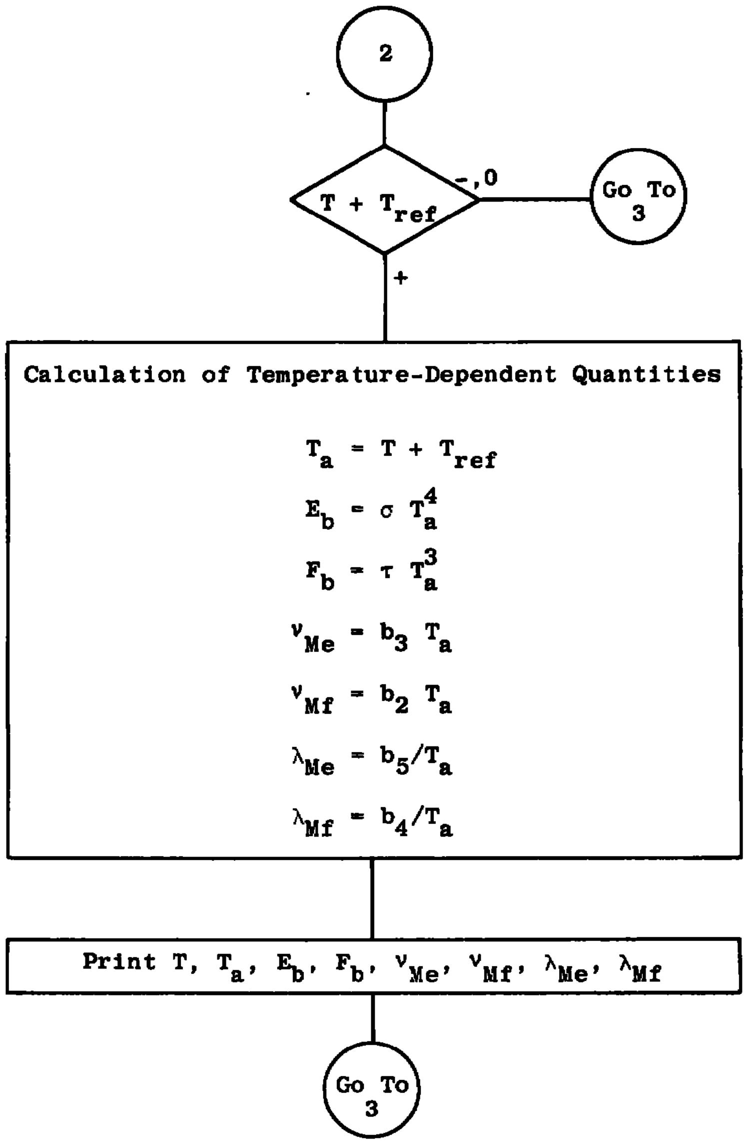

Fig. 1 Continued

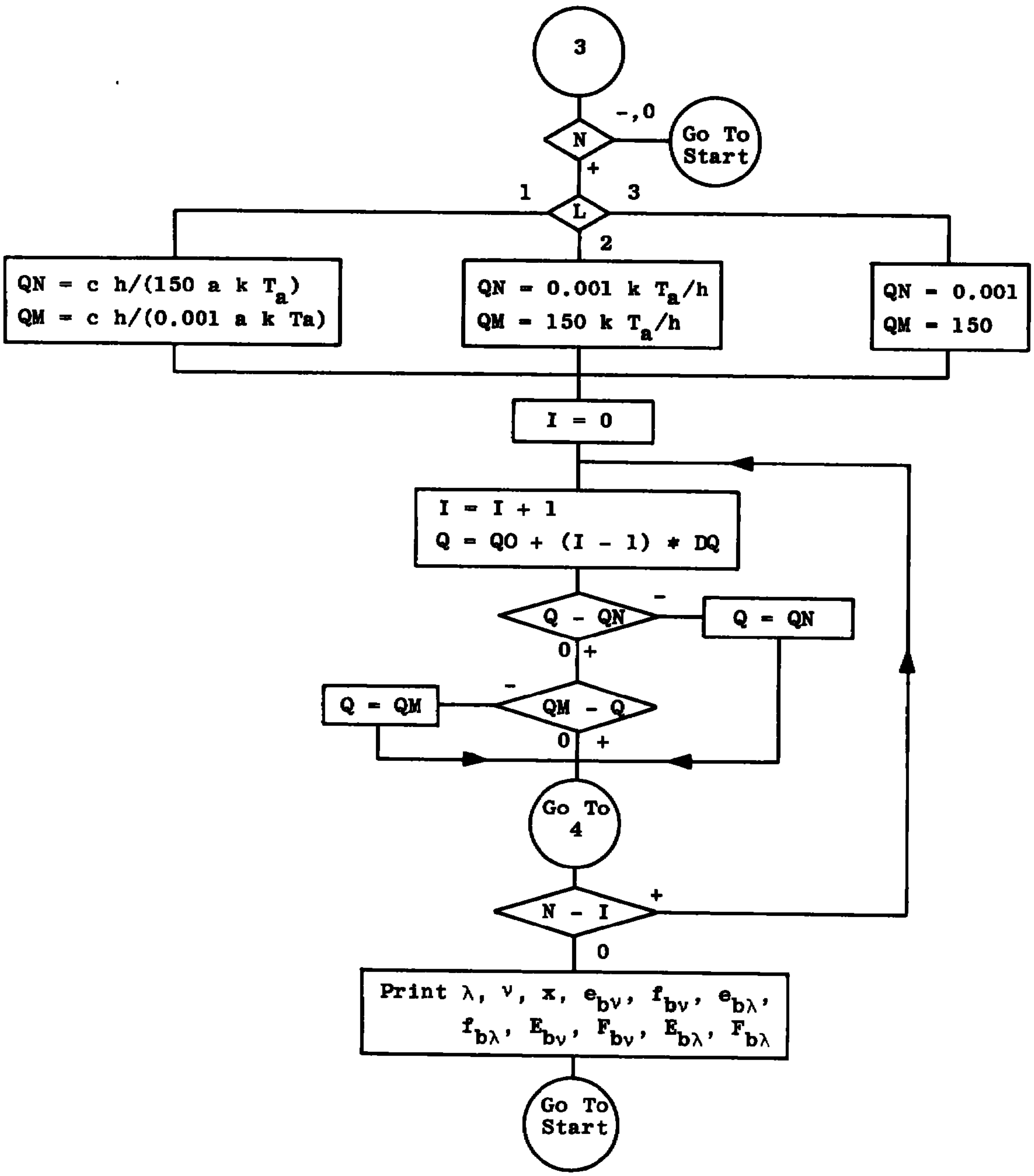

Fig. 1 Continued

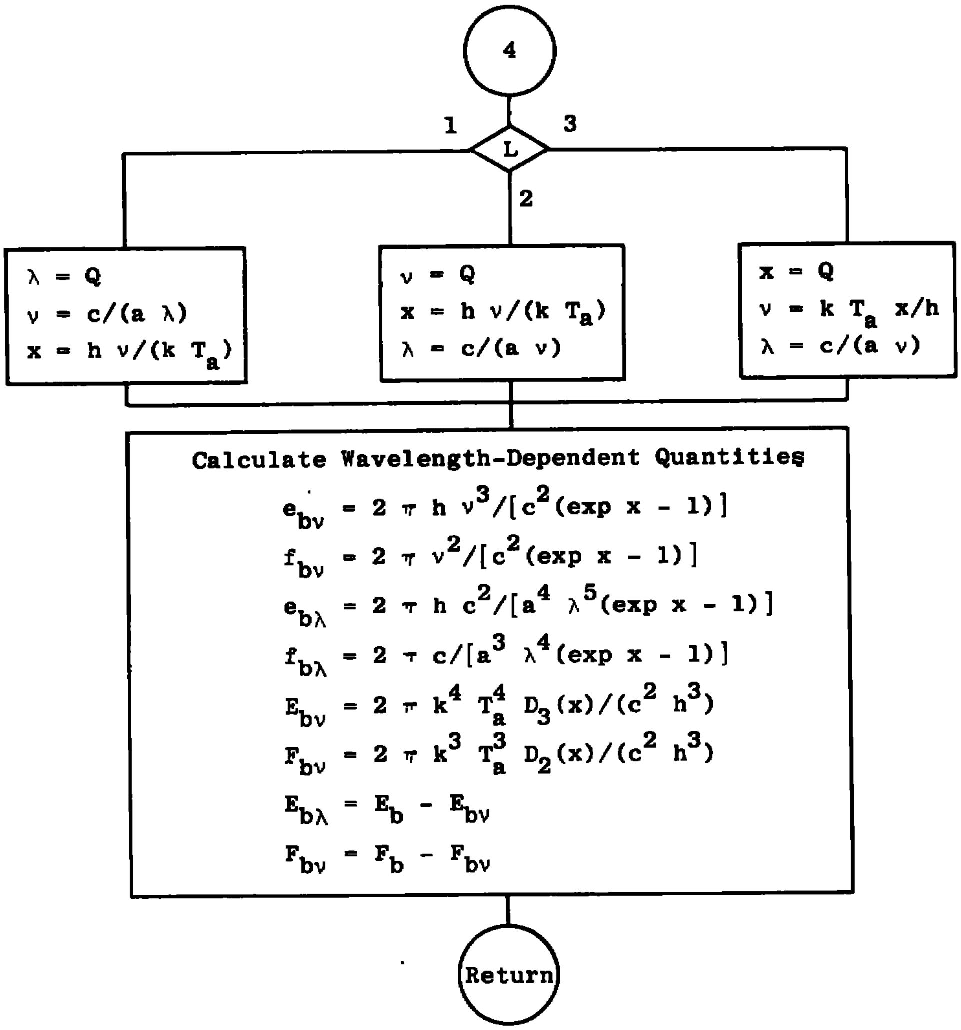

Fig. 1 Concluded

One good feature of the program is the simplicity of its logic. Input is via one READ statement:

READ IW, IL, IS, IT, IE, L, N, T, QO, DQ

According to the format:

6I6, I6, 3E12.0

The input variables are:

IW	Indicator of the length unit of λ
IL	Indicator of the length unit of c
IS	Indicator of the time unit
IT	Indicator of the temperature unit
IE	Indicator of the energy unit
L	Indicator of whether λ, ν, or x is the independent variable
N	Number of values of the independent variable for which the wavelength dependent quantities are to be calculated
T	Temperature
QO	Starting value of the independent variable
DQ	Increment of the independent variable

The program reads a card and performs the indicated computations explained below. That is one complete cycle of the program. The program then branches to the start and begins another cycle by reading another card. Execution is terminated when the end of the data is reached.

The first calculations are the unit conversions and the calculation of the physical constants. The basic physical constants are a, c, h, k, and T_{ref}. All the other constants, σ, τ, b_2, b_3, b_4, b_5, c_1, and c_2, are expressed in terms of the basic constants. The program contains the values of the basic constants in the International System of Units (SI) (Ref. 6) and the values of various conversion factors (Table II). Unit conversions indicated by the first five input variables are performed on the four basic constants, then the other constants are calculated by their defining equations. The valid values for the first five input variables and the units which the values indicate are given in Table II. This table should be referred to when preparing input data for the program. If IW is left blank on an input card, then this first group of calculations is omitted. The values of the physical constants used in the previous cycle are retained for the second and third group of calculations.

TABLE II
VALUES OF BASIC PHYSICAL CONSTANTS AND CONVERSION FACTORS

Values of Basic Physical Constants in SI Units (Ref. 5)
a = 1 m/m
c = 2.997925 x 10^{8} m/s
h = 6.6256 x 10^{-34} Js
k = 1.38054 x 10^{-23} J/°K

Length Units (Ref. 5)		
IW and IL	Units Indicated	Conversion Factor
1	μ	1 x 10^{-6} m/μ
2	cm	0.01 m/cm
3	m	1 m/m
4	in.	0.0254 m/in.
5	ft	0.3048 m/ft

Time Units (Ref. 5)		
IT	Units Indicated	Conversion Factor
1	s	1 s/s
2	min	60 s/min
3	hr	3600 s/hr

Temperature Units (Ref. 5)		
IT	Units Indicated	Conversion Factor
1	°K	1 °K/°K, T_{ref} = 0
2	°R	0.5555556 °K/°R, T_{ref} = 0
3	°C	1 °K/°C, T_{ref} = 273.15
4	°F	0.5555556 °K/°F, T_{ref} = 459.67

Energy Units (Ref. 5)		
IE	Units Indicated	Conversion Factor
1	erg	1 x 10^{-7} J/erg
2	J	1 J/J
3	cal (mean)	4.19002 J/cal
4	ft-lb	1.35582 J/ft-lb
5	Btu (mean)	1055.87 J/Btu

Note: s = seconds
J = joules

The quantities T_a, E_b, F_b, ν_{Me}, ν_{Mf}, λ_{Me}, and λ_{Mf} depend on the temperature, but not upon the wavelength. These are the quantities calculated by the second group of calculations. If T is less than or equal to absolute zero, then the second group of calculations is omitted. The values of the temperature-dependent quantities used in the previous cycle are retained for use in the third group of calculations.

The wavelength-dependent parameters are λ, ν, x, $e_{b\nu}$, $f_{b\nu}$, $e_{b\lambda}$, $f_{b\lambda}$, $E_{b\nu}$, $F_{b\nu}$, $E_{b\lambda}$, and $F_{b\lambda}$. These are the quantities calculated by the third group of calculations. The variables λ, ν, and x are interdependent and any one can be chosen as the independent variable. This choice is indicated by L, with values of 1, 2, and 3 indicating λ, ν, and x, respectively. The use of x as the independent variable has the advantage that the principal part of the spectrum is in the range $0.1 \leq x \leq 10$, whereas the corresponding range for λ and ν depends on the temperature. The wavelength-dependent variables are calculated for N values of the independent variable, starting with QO and incrementing by DQ each time. The value of Q is limited to values corresponding to x greater than 0.001 and less than 150. If Q is outside this range, it is set equal to the nearest limiting value. If N is left blank on an input card, then the third group of calculations is omitted.

Figure 2 illustrates an input form designed for this program and data for a sample run. The corresponding output is shown in Fig. 3. The FORTRAN listing of the computer program is given in Appendix I. A list of FORTRAN variables is given in Table III.

Columns	1	2	3	4	5	6	7	8	9	10	11	12	13 - 24	25 - 36	37 - 48
Input Variable / Card No.	IW	IL	IS	IT	IE	L	N						T	QO	DQ
1	1	5	3	4	5								-500.		
2	1	4	2	2	3										
3	3	3	1	1	2										
4	2	2	1	3	1								-500.		
5	1	2	1	1	2								20.		
6													77.		
7													300.		
8						3					1	1	6000.	0.	1.
9						3						8		20.	20.

Fig. 2 Input Data for Sample Run

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IA	IL	IS	IT	IE	A	C	F	K	TREF
SIGMA	TAU	B2	B3	B4	B5	C1	C2		
MICR	FT	HR	FDEG	BTU	3.2808E-06	3.5409E 12	1.7431E-40	7.2638E-27	4.5967E 02
1.7108E-09	8.7192E 16	6.6411E 13	1.1758E 14	6.6055E 03	5.2160E 03	1.1851E 08	2.5898E 04		
MICR	IN	MIN	RDEG	CAL	3.9370E-05	7.0817E 11	2.6355E-36	1.8205E-24	0.0
4.9397E-11	1.0092E 13	1.1068E 12	1.9596E 12	6.6055E 03	5.2160E 03	3.4566E 06	2.5898E 04		
M	M	S	KDEG	J	1.0000E 00	2.9979E 08	6.6256E-34	1.3805E-23	0.0
5.6698E-08	1.5204E 15	3.3205E 10	5.8789E 10	3.6697E-03	2.8978E-03	3.7415E-16	1.4388E-02		
CM	CM	S	CDEG	ERG	1.0000E 00	2.9979E 10	6.6256E-27	1.3805E-16	2.7315E 02
5.6698E-05	1.5204E 11	3.3205E 10	5.8789E 10	3.6697E-01	2.8978E-01	3.7415E-05	1.4388E 00		
MICR	CM	S	KDEG	J	1.0000E-04	2.9979E 10	6.6256E-34	1.3805E-23	0.0
5.6698E-12	1.5204E 11	3.3205E 10	5.8789E 10	3.6697E 03	2.8978E 03	3.7415E 04	1.4388E 04		

T	TA	EB	FB	NME	NMF	LME	LMF
2.0000E 01	2.0000E 01	9.0716E-07	1.2163E 15	1.1758E 12	6.6411E 11	1.4489E 02	1.8349E 02
7.7000E 01	7.7000E 01	1.9931E-04	6.9412E 16	4.5267E 12	2.5568E 12	3.7634E 01	4.7659E 01
3.0000E 02	3.0000E 02	4.5975E-02	4.1051E 18	1.7637E 13	9.9616E 12	9.6593E 00	1.2232E 01
6.0000E 03	6.0000E 03	7.3480E 03	3.2841E 22	3.5273E 14	1.9923E 14	4.8297E-01	6.1162E-01

LAMBDA	NL	X	EN	FN	EL	FL	EBN	FBN	EBL	FBL
2.3980E 03	1.2502E 11	1.0000E-03	9.0472E-19	1.0922E 05	4.7167E-10	5.6943E 12	2.7703E-07	6.8279E 15	7.3480E 03	3.2841E 22
2.3980E 00	1.2502E 14	1.0000E 00	3.2674E-12	6.3591E 07	2.7461E 02	3.3153E 21	2.5437E 02	4.8349E 21	7.0936E 03	2.8006E 22
1.1990E 00	2.5004E 14	2.0000E 00	1.1333E-11	6.8409E 07	2.3634E 03	1.4266E 22	1.2311E 03	1.3471E 22	6.0170E 03	1.9370E 22
7.9933E-01	3.7506E 14	3.0000E 00	1.2804E-11	5.1526E 07	6.0078E 03	2.4177E 22	2.8879E 03	2.1061E 22	4.4601E 03	1.1780E 22
5.9950E-01	5.0007E 14	4.0000E 00	1.0807E-11	3.2618E 07	9.0150E 03	2.7209E 22	4.3870E 03	2.6288E 22	2.9611E 03	6.5527E 21
4.7960E-01	6.2509E 14	5.0000E 00	7.6747E-12	1.8531E 07	1.0003E 04	2.4152E 22	5.5443E 03	2.9426E 22	1.8037E 03	3.4151E 21
3.9966E-01	7.5011E 14	6.0000E 00	4.3579E-12	9.7747E 06	9.1177E 03	1.8346E 22	6.3205E 03	3.1146E 22	1.0275E 03	1.6948E 21
3.4257E-01	8.7513E 14	7.0000E 00	2.8335E-12	4.8867E 06	7.2384E 03	1.2484E 22	6.7927E 03	3.2031E 22	5.5532E 02	8.1000E 20
2.9975E-01	1.0001E 15	8.0000E 00	1.5551E-12	2.3467E 06	5.1887E 03	7.8301E 21	7.0603E 03	3.2465E 22	2.8776E 02	3.7562E 20
2.6644E-01	1.1252E 15	9.0000E 00	8.1436E-13	1.0924E 06	3.4390E 03	4.6131E 21	7.2039E 03	3.2671E 22	1.4412E 02	1.7028E 20
2.3980E-01	1.2502E 15	1.0000E 01	4.1093E-13	4.9609E 05	2.1424E 03	2.5864E 21	7.2778E 03	3.2765E 22	7.0174E 01	7.5664E 19
1.1990E-01	2.5004E 15	2.0000E 01	1.4924E-16	9.0086E 01	3.1123E 00	1.8787E 18	7.3480E 03	3.2841E 22	2.1750E-02	1.2445E 16
5.9950E-02	5.0007E 15	4.0000E 01	2.4609E-24	7.4273E-07	2.0528E-07	6.1955E 10	7.3490E 03	3.2841E 22	3.3191E-10	9.7613E 07
3.9966E-02	7.5011E 15	6.0000E 01	1.7119E-32	3.4445E-15	3.2129E-15	6.4648E 02	7.3490E 03	3.2841E 22	2.2508E-18	4.4522E-01
2.9975E-02	1.0001E 16	8.0000E 01	8.3637E-41	1.2621E-23	2.7907E-23	4.2113E-06	7.3480E 03	3.2841E 22	1.0858E-26	1.6179E-09
2.3980E-02	1.2502E 16	1.0000E 02	3.3670E-49	4.0648E-32	1.7554E-31	2.1192E-14	7.3480E 03	3.2841E 22	4.3382E-35	5.1844E-18
1.9983E-02	1.5002E 16	1.2000E 02	1.1992E-57	1.2065E-40	9.0030E-40	9.0574E-23	7.3480E 03	3.2841E 22	1.5373E-43	1.5336E-26
1.7128E-02	1.7503E 16	1.4000E 02	3.9250E-66	3.3847E-49	4.0108E-48	3.4586E-31	7.3480E 03	3.2841E 22	5.0137E-52	4.2923E-35
1.5987E-02	1.8753E 16	1.5000E 02	2.1917E-70	1.7640E-53	2.5710E-52	2.0692E-35	7.3480E 03	3.2841E 22	2.7956E-56	2.2349E-39

THE END

Fig. 3 Output of Sample Run

TABLE III
FORTRAN VARIABLES

FORTRAN Variable	Algebraic Variable
A	a
B2	b_2
B3	b_3
B4	b_4
B5	b_5
C	c
CR	c in m/s
C1	c_1
C2	c_2
EB	E_b
EBL	$E_{b\lambda}$
EBN	$E_{b\nu}$
EL	$e_{b\lambda}$
EN	$e_{b\nu}$
FB	F_b
FBL	$F_{b\lambda}$
FBN	$F_{b\nu}$
FL	$f_{b\lambda}$
FN	$f_{b\nu}$
H	h
HR	h in Js
K	k
KØH	k/h
KR	k in J/°K
LAM	λ
LME	λ_{Me}
LMF	λ_{Mf}
NME	ν_{Me}
NMF	ν_{Mf}
NU	ν
PI	π
SIG	σ
T	T
TA	T_a
TAU	τ
X	x
X2	x_2
X3	x_3
X4	x_4
X5	x_5
ZETA3	$\zeta(3)$
ZETA4	$\zeta(4)$

SECTION IV
DEBYE FUNCTION CALCULATION

The Debye function can be calculated by truncated infinite series. Theorems of infinite series were used to perform an error analysis on this calculation. Of special significance, a criterion was obtained for the error that was independent of the order of the function. This criterion was used to deduce conditions for the calculation of the Debye function to five significant figures.

4.1 DEFINITIONS AND NOTATION

Adding superscripts, 1 and 2, to distinguish, respectively, the Debye function and its complement, the Debye function is

$$D_n^1(x) = \int_0^x \frac{t^n}{e^t - 1}\, dt \tag{43}$$

and its complement is

$$D_n^2(x) = \int_x^\infty \frac{t^n}{e^t - 1}\, dt \tag{44}$$

The sum of the Debye function and its complement is

$$D_n^1(x) + D_n^2(x) = \int_0^\infty \frac{t^n}{e^t - 1}\, dt = n!\ \zeta(n + 1) \tag{45}$$

where the zeta function is defined by Eq. (24).

An infinite series expansion for the Debye function is

$$D_n^1(x) = x^n \left[\frac{1}{n} - \frac{x}{2(n + 1)} + \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k}}{(2k + n)\,(2k)!}\right] \tag{46}$$

for $|x| \leq 2\pi$ and for $n \geq 1$.

An infinite series expansion for the complement of the Debye function is

$$D_n^2(x) = \sum_{k=1}^{\infty} \sum_{j=0}^{n} \frac{n!}{(n - j)!} \frac{x^{n-j} e^{-kx}}{k^{j+1}} \tag{47}$$

for $x \geq 0$ and for $n \geq 1$.

The B_{2k} of Eq. (46) are Bernoulli numbers and are discussed in Ref. 5. Of basic importance to this analysis is the inequality

$$\frac{2(2k)!}{(2\pi)^{2k}} < (-1)^{k-1} B_{2k} < \frac{2(2k)!}{(2\pi)^{2k}} \left[\frac{1}{1 - 2^{1-2k}}\right] \tag{48}$$

The Debye function can be calculated by using a partial sum of one or the other of the two series, depending on the value of x. The remaining function can be calculated from Eq. (45).

In the following analysis, the theorems used are proved and discussed in Ref. 7. The notation used for infinite series will be

$$S_n^{\ell} = \sum_{k=1}^{\infty} a_{nk}^{\ell} \tag{49}$$

where the n subscript denotes the n-dependence, the k subscript denotes the term and the ℓ superscript will be used to distinguish different series. The mth partial sum will be denoted by

$$S_{nm}^{\ell} = \sum_{k=1}^{m} a_{nk}^{\ell} \tag{50}$$

and the remainder after m terms as

$$R_{nm}^{\ell} = S_n^{\ell} - S_{nm}^{\ell} \tag{51}$$

4.2 FIRST SERIES

4.2.1 Ratio Test

Define

$$a_{nk}^{1} = \frac{B_{2k}\, x^{2k}}{(2k + n)\,(2k)!} \tag{52}$$

and for purposes of comparison define

$$a_{nk}^{3} = \frac{2x^{2k}}{(2\pi)^{2k}\,(2k + n)\,(1 - 2^{1-2k})} \tag{53}$$

Performing the ratio test on S_n^3 one obtains

$$\frac{a_{n,k+1}^{3}}{a_{n,k}^{3}} = \left[\frac{2k + n}{2k + n + 2}\right]\left[\frac{1 - 2^{1-2k}}{1 - 2^{-1-2k}}\right]\left[\frac{x}{2\pi}\right]^2 \tag{54}$$

and

$$\lim_{k\to\infty} \left[\frac{a_{n,k+1}^{3}}{a_{n,k}^{3}}\right] = \left[\frac{x}{2\pi}\right]^2 \tag{55}$$

Thus, S_n^3 converges for $|x| < 2\pi$. But by Eq. (48)

$$|a_{nk}^2| < a_{nk}^3 \qquad (56)$$

and thus, by the comparison test, S_n^1 converges absolutely for $|x| < 2\pi$. This result can be used for obtaining a value for an upper limit for $|R_{nm}^1|$. Defining

$$r = \left(\frac{x}{2\pi}\right)^2 \qquad (57)$$

the upper limit obtained from the ratio test is

$$R_{nm}^1 < \frac{2r^{m+1}}{(2m+n+2)(1-2^{-1-2m})(1-r)} \qquad (58)$$

However, a better upper limit can be obtained from the alternating series test.

4.2.2 Alternating Series Test

It is noted from Eq. (48) that S_n^1 is an alternating series. The difference between the absolute values of sequential terms is

$$|a_{nk}^1| - |a_{n,k+1}^1| = \frac{|B_{2k}|\, x^{2k}}{(2k+n)(2k)!} - \frac{|B_{2k+2}|\, x^{2k+2}}{(2k+n+2)(2k+2)!} \qquad (59)$$

Replacing the first term by a smaller value and the second term by a larger value obtained from Eq. (48) results in

$$|a_{nk}^1| - |a_{n,k+1}^1| > \frac{2r^k}{2k+n} - \frac{2r^{k+1}}{(2k+n+2)}\left(\frac{1}{(1-2^{-1-2k})}\right) \qquad (60)$$

which becomes, after factorization of the right side,

$$|a_{nk}^1| - |a_{n,k+1}^1| > \left[1 - \left(\frac{2k+n}{2k+n+2}\right)\left(\frac{2^{2k+1}}{2^{2k+1}-1}\right) r\right]\left[\frac{2r^k}{2k+n}\right] \qquad (61)$$

Noting that

$$\frac{2k+n}{2k+n+2} < 1$$

and that

$$\frac{2^{2k+1}}{2^{2k+1}-1} \leq \frac{8}{7}$$

for all k, the difference of Eq. (61) will be positive if

$$r < \frac{7}{8}$$

or certainly if

$$|x| < 5$$

Thus, from the alternating series test, an upper bound for $\left| R^1_{nm} \right|$ for this range of x is

$$|R^1_{nm}| < \frac{|B_{2m+2}|\, x^{2m+2}}{(2m+n+2)\ (2m+2)!} \tag{62}$$

4.2.3 Upper Bound for the Absolute Value of the Relative Error

The absolute value of the relative error resulting from using a partial sum instead of the series in Eq. (46) is

$$|E^1_{nm}| = \frac{x^n R^1_{nm}}{D^1_n\ (x)} \tag{63}$$

Any expression obtained from Eq. (63) by replacing the numerator by something greater and the denominator by something smaller will be an upper bound for the absolute value of the relative error. One obvious replacement is to replace $\left| R^1_{nm} \right|$ by its upper bound. Another replacement would be to replace $|B_{2m+2}|$ by its upper bound obtained from Eq. (48). These replacements result in

$$x^n\,|R^1_{nm}| < \frac{2r^{m+1}\, x^n}{(2m+n+2)\ (1-2^{-1-2m})} \tag{64}$$

Since S^1_n is convergent for $|\ x\ | < 2\pi$, it can be grouped in any manner. Thus, grouping S^1_n as

$$S^1_n = (a^1_{n1} + a^1_{n2}) + (a^1_{n3} + a^1_{n4}) + \ldots . \tag{65}$$

and noting that the first term of each group is positive and the second term is negative, from Eq. (61) it is deduced that each group is positive for x < 5. Thus, S^1_n is positive and one obtains

$$D^1_n(x) > x^n \left[\frac{1}{n} - \frac{x}{2(n+1)}\right] \tag{66}$$

Using Eqs. (64) and (65) to replace the numerator and denominator, respectively, of Eq. (63) one obtains

$$E^1_{nm} < \frac{2^{2m+2}\ r^{m+1}}{(2^{2m+1}-1)\ (2m+n+2)\ \left[\frac{1}{n} - \frac{x}{2(n+1)}\right]} \tag{67}$$

But

$$(2m+n+2)\left[\frac{1}{n} - \frac{x}{2(n+1)}\right] = \frac{(2m+n+2)\ (2n-nx+2)}{2n(n+1)}$$

and

$$\frac{2n - nx + 2}{n} = 2 - x + \frac{2}{n} > 2 - x$$

and

$$\frac{2m + n + 2}{n + 1} > 1$$

so

$$|E^{1}_{mn}| < \frac{2^{2m+3}}{2^{2m+1} - 1} \frac{r^{m+1}}{2 - x} \tag{68}$$

This upper limit approaches zero as m approaches infinity and is easily calculated. Note that it is also independent of n and thus holds for all orders.

4.3 SECOND SERIES

4.3.1 Integral Test

The terms of the second series are

$$a^{2}_{nk} = \sum_{j=0}^{n} \frac{n!}{(n-j)!} \frac{x^{n-j} e^{-kx}}{k^{j+1}} \tag{69}$$

It is shown that a^2_{nk} satisfies all of the conditions for the integral test. Thus an upper limit for $\left| R^2_{nm} \right|$ can be obtained as

$$|R^{2}_{nm}| < \sum_{j=0}^{n} \frac{n!}{(n-j)!} x^{n-j} \int_{m}^{\infty} e^{-xt} t^{-j-1} dt \tag{70}$$

Integrating successively by parts, one obtains for $j > 0$

$$\int_{m}^{\infty} e^{-xt} t^{-j-1} dt = \left[e^{-xt}\right] \left[(-1) \frac{t^{-j}}{j}\right]\Bigg|_{m}^{\infty} - \left[(-x) e^{-xt}\right]\left[(-1)^2 \frac{t^{-j+1}}{j(j-1)}\right]\Bigg|_{m}^{\infty}$$

$$+ \left[(-x)^2 e^{-xt}\right] \left[(-1)^3 \frac{t^{2-j}}{j(j-1)(j-2)}\right]\Bigg|_{m}^{\infty} - \dots$$

$$+ (-1)^{j-1} \left[(-x)^{j-1} e^{-xt}\right]\left[(-1)^j \frac{t^{-1}}{j!}\right]\Bigg|_{m}^{\infty}$$

$$+ \frac{(-1)^j (-x)^j (-1)^j}{j!} \int_{m}^{\infty} \frac{e^{-xt}}{t} dt$$

or after simplification

$$\int_{m}^{\infty} e^{-xt}\, t^{-j+1}\, dt = \sum_{i=1}^{j} (-1)^{i-1} \frac{(j-i)!}{j!} \frac{(mx)^{i-1}}{m^{j}} e^{-mx} + \frac{(-x)^{j}}{j!} E(mx) \qquad (71)$$

where E(mx) is the exponential integral discussed in Ref. 5. Substituting Eq. (71) into Eq. (70) results in

$$\left|R^{2}_{nm}\right| < \sum_{j=1}^{n} \sum_{i=1}^{j} \left[(-1)^{i-1} \frac{n!(j-i)!}{(n-j)!j!} x^{n} (mx)^{i-j-1} e^{-mx}\right]$$

$$+ \sum_{j=1}^{n} (-1)^{j} \frac{n!}{j!(n-j)!} x^{n} E(mx) + \frac{n!}{n!} x^{n} E(mx) \qquad (72)$$

Note that the last term can be included in the last summation by letting the range of the index j start at zero. Making this change one finds

$$\sum_{j=0}^{n} (-1)^{j} \frac{n!}{j!(n-j)!} x^{n}E(mx) = x^{n}E(mx) \sum_{j=0}^{n} (-1)^{j} \binom{n}{j}$$

The resulting summation is zero as can be found in Ref. 5. Thus, the right-hand side of Eq. (72) reduces to the double summation. This can be simplified to

$$|R^{2}_{nm}| < \sum_{i=1}^{n} \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} (i-1)! \frac{x^{n-i}}{m^{i}} e^{-mx} \qquad (73)$$

It can be proved that the double summation of Eq. (73) is just a rearrangement of the terms of the double summation of Eq. (72). Simplifying further

$$|R^{2}_{nm}| < e^{-mx} \sum_{i=1}^{n} (-1)^{i} (i-1)! \frac{x^{n-i}}{m^{i}} \sum_{j=i}^{n} (-1)^{j} \binom{n}{j} \qquad (74)$$

But, from Ref. 5

$$\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j},\quad 0 < j < n$$

so

$$\sum_{j=i}^{n} (-1)^{j} \binom{n}{j} = \sum_{j=i}^{n-1} (-1)^{j} \binom{n-1}{j-i} + \sum_{j=i}^{n-1} (-1)^{j} \binom{n-1}{j} + (-1)^{n}$$

$$= \sum_{j=i-1}^{n-2} (-1)^{j+1} \binom{n-1}{j} + \sum_{j=i}^{n-1} (-1)^{j} \binom{n-1}{j} + (-1)^{n}$$

$$= \sum_{j=1}^{n-2} \left[(-1)^{j+1} + (-1)^{j}\right] \binom{n-1}{j} + (-1)^{i}\binom{n-1}{i-1} + (-1)^{n-1} \binom{n-1}{n-1} + (-1)^{n}$$

It is readily shown that the summation is zero and that the last two terms cancel leaving only

$$\sum_{j=1}^{n} (-1)^j \binom{n}{j} = (-1)^i \binom{n-1}{i-1} \tag{75}$$

Substituting Eq. (75) into Eq. (74) one obtains

$$|R_{nm}^2| < e^{-mx} \sum_{i=1}^{n} (i - 1)! \binom{n-1}{i-1} \frac{x^{n-i}}{m^i}$$

or finally

$$|R_{nm}^2| < (n - 1)!\, e^{-mx} \sum_{i=0}^{n-1} \frac{x^i}{i!\, m^{n-i}} \tag{76}$$

4.3.2 Upper Bound to the Absolute Value of the Relative Error

The absolute value of the relative error caused by using a partial sum instead of a series in Eq. (47) is

$$|E_{nm}^2| = \frac{|R_{nm}^2|}{D_n^2(x)} \tag{77}$$

Any expression obtained from Eq. (77) by replacing the numerator by something greater and the denominator by something smaller will be an upper bound for the absolute value of the relative error. The obvious replacement for the numerator is the upper bound just obtained for $|R_{nm}^2|$. This bound will be further increased in value by replacing each term in the summation of Eq. (76) by a greater value, specifically

$$|R_{nm}^2| < (n - 1)!\, e^{-mx} \sum_{i=0}^{n-1} \frac{x^i}{i!} \tag{78}$$

Since all of the terms of the double summation of Eq. (47) are positive, then $D_n^2(x)$ is greater than the first n terms. Thus after reindexing it is found that

$$D_n^2(x) > \sum_{j=0}^{n-1} \frac{n!}{j!} e^{-x} x^j$$

thus, obviously

$$D_n^2(x) > (n - 1)!\, e^{-x} \sum_{j=0}^{n-1} \frac{x^j}{j!} \tag{79}$$

Using Eqs. (78) and (79) to replace the numerator and denominator, respectively, of Eq. (77), one obtains

$$|E^2_{nm}| < e^{-(m-1)x} \tag{80}$$

This upper limit is easily calculated and approaches zero as m approaches infinity. Again the bound is independent of n and thus holds for all orders.

4.4 RANGES OF CALCULATION

The upper bound for the absolute value of the relative error is related to the number of significant figures by a theorem proved in Ref. 8:

If the absolute value of the relative error of any number is less than $5 \times 10^{-(s+1)}$ then the number is certainly correct to s significant figures.

Using this relation the accuracy of each approximation was plotted and is shown in Fig. 4. Given a value of x and the number of significant figures desired, from Fig. 4 one can find which approximation and what order is required. Thus one can construct a table dividing the calculation into ranges of x and the best approximation to use. Table IV, for example, gives such a division for five significant figures.

It is shown that the approximation for each range falls below the line s = 5 in Fig. 4. Since the small-x approximation of order 5 and the large-x approximation of order 8 intersect below this line, one never needs to go higher than these orders to obtain five significant figures. Ranges 13 and 14 were determined by the limitation of the IBM System/360 computer that the order of the exponential function be less than 174.

A subroutine was written to calculate the Debye function to five significant figures using the ranges of Table IV. The subroutine is called by the statement:

CALL DEBYE (N, X, F, G)

The arguments N and X input n and x, respectively, where the order of the Debye function must be in the range $0 \leq n \leq 6$. The Debye function and its complement are returned by the arguments F and G, respectively. The listing of this subroutine is given in Appendix II.

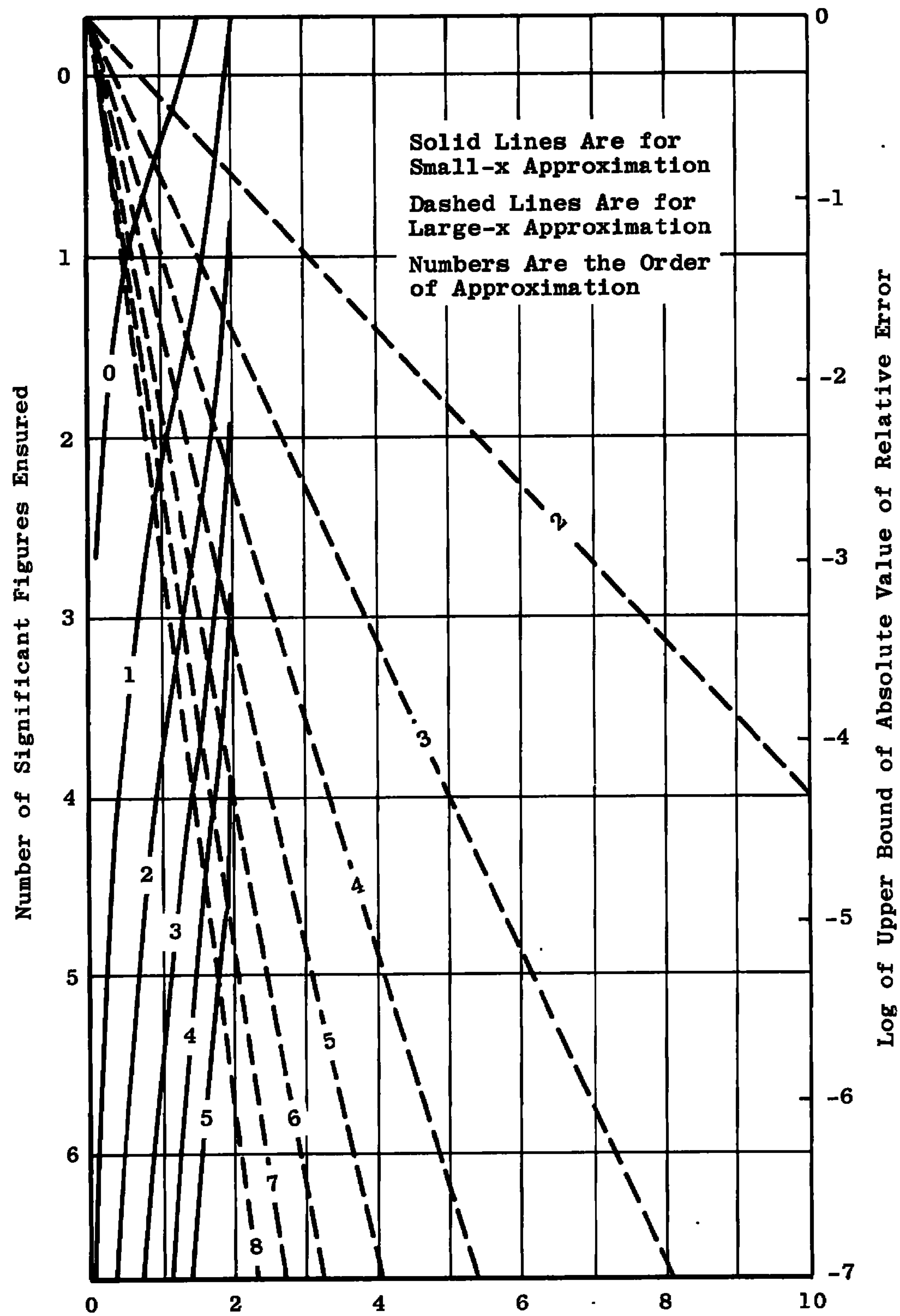

Fig. 4 Accuracy of Approximations

TABLE IV
RANGES OF CALCULATION

Range No.	Range	ℓ	m
1	$0 \leq x \leq 0.007$	1	0
2	$0.007 < x \leq 0.22$	1	1
3	$0.22 < x \leq 0.68$	1	2
4	$0.68 < x \leq 1.12$	1	3
5	$1.12 < x \leq 1.52$	1	4
6	$1.52 < x \leq 1.80$	1	5
7	$1.80 < x < 2.03$	2	8
8	$2.03 \leq x < 2.44$	2	7
9	$2.44 \leq x < 3.05$	2	6
10	$3.05 \leq x < 4.07$	2	5
11	$4.07 \leq x < 6.10$	2	4
12	$6.10 \leq x < 12.21$	2	3
13	$12.21 \leq x < 87$	2	2
14	$87 \leq x < 174$	2	1

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APPENDIXES

I. FORTRAN LISTING OF MAIN PROGRAM
II. FORTRAN LISTING OF SUBROUTINE DEBYE

APPENDIX I
FORTRAN LISTING OF MAIN PROGRAM

```
C AEA00052 BLACKBODY EMISSION  D.C. TODD  2-12-68  SY80C2-Y00
      REAL UNIT(5,4)/'MICR',' CM ','  M ',' IN ',' FT ','  S ',' MIN',
     1    ' HR ',2*0.,'KDEG','RDEG','CDEG','FDEG',0.,' ERG','  J ',
     2    ' CAL','FTLB',' BTU'/,DIST(5)/1.E-6,.01,1.,.0254,.3048/,
     3    TIME(3)/1.,60.,3600./,TEMP(4)/1.,.5555556,1.,.5555556/,
     4    TURE(4)/0.,0.,273.15,459.67/,ERGY(5)/1.E-7,1.,4.19002,1.35582,
     5    1055.87/,CR/2.997925E8/,HR/6.6256E-34/,KR/1.38054E-23/,
     6    PI/3.141593/,ZETA3/1.202057/,ZETA4/1.082323/,X2/1.593624/,
     7    X3/2.821439/,X4/3.920697/,X5/4.965114/,
     8    K,LAM,NU,NME,NMF,LME,LMF,KOH
      IP=0
      WRITE(6,1000)
    1 READ(5,1001,END=900)IW,IL,IS,IT,IE,L,N,T,Q0,DQ
C UNIT CONVERSIONS AND CALCULATION OF PHYSICAL CONSTANTS
      IF(IW.LE.0)GO TO 41
      A=DIST(IW)/DIST(IL)
      C=CR*TIME(IS)/DIST(IL)
      H=HR/(ERGY(IE)*TIME(IS))
      K=KR*TEMP(IT)/ERGY(IE)
      TREF=TURE(IT)
      KOH=K/H
      TAU=4.*PI*(KOH/C)**2*KOH
      SIG=3.*ZETA4*K*TAU
      TAU=ZETA3*TAU
      B2=KOH*X2
      B3=KOH*X3
      B4=C/(A*KOH*X4)
      B5=C/(A*KOH*X5)
      C1=2.*PI*H*C*C/A**4
      C2=C/(A*KOH)
      IF(IP.EQ.1)GO TO 21
      IP=1
      WRITE(6,1005)
   21 WRITE(6,1002)UNIT(IW,1),UNIT(IL,1),UNIT(IS,2),UNIT(IT,3),
     1    UNIT(IE,4),A,C,H,K,TREF
      WRITE(6,1003)SIG,TAU,B2,B3,B4,B5,C1,C2
C CALCULATION OF TEMPERATURE DEPENDENT QUANTITIES
   41 TCK=T+TREF
      IF(TCK.LE.0.)GO TO 61
      TA=TCK
      EB=SIG*TA*TA*TA*TA
      FB=TAU*TA*TA*TA
      NME=B3*TA
      NMF=B2*TA
      LME=B5/TA
      LMF=B4/TA
      IF(IP.EQ.2)GO TO 42
      IP=2
      WRITE(6,1006)
   42 WRITE(6,1003)T,TA,EB,FB,NME,NMF,LME,LMF
C CALCULATION OF WAVELENGTH DEPENDENT QUANTITIES
   61 IF(N.LE.0)GO TO 1
      GO TO(72,73,74),L
   72 QN=C/(150.*A*KOH*TA)
      QM=C/(.001*A*KOH*TA)
      GO TO 75
```

APPENDIX I (Concluded)

```
   73 QN=.001*KOH*TA
      QM=150.*KOH*TA
      GO TO 75
   74 QN=.001
      QM=150.
   75 CONTINUE
      DO 66 I=1,N
      Q=Q0+(I-1)*DQ
      IF(Q.LT.QN)Q=QN
      IF(Q.GT.QM)Q=QM
      GO TO(62,63,64),L
   62 LAM=Q
      NU=C/(A*LAM)
      X=NU/(KOH*TA)
      GO TO 65
   63 NU=Q
      X=NU/(KOH*TA)
      LAM=C/(A*NU)
      GO TO 65
   64 X=Q
      NU=KOH*TA*X
      LAM=C/(A*NU)
   65 FN=2.*PI*(NU/C)**2/(EXP(X)-1.)
      EN=H*NU*FN
      EL=NU*EN/LAM
      FL=NU*FN/LAM
      CALL CEBYE(3,X,EBN,EBL)
      EBN=EB*EBN/(6.*ZETA4)
      EBL=EB*EBL/(6.*ZETA4)
      CALL CEBYE(2,X,FBN,FBL)
      FBN=FB*FBN/(2.*ZETA3)
      FBL=FB*FBL/(2.*ZETA3)
      IF(IP.EQ.3)GO TO 66
      IP=3
      WRITE(6,1007)
   66 WRITE(6,1003)LAM,NU,X,EN,FN,EL,FL,EBN,FBN,EBL,FBL
      GO TO 1
  900 WRITE(6,1004)
      STOP
 1000 FORMAT('1AFAC0C52  D.C. TODD  BLACKBODY EMISSION')
 1001 FORMAT(6I1,I6,3E12.0)
 1002 FORMAT(1H03XA4,4(8XA4),4X1P5E12.4)
 1003 FORMAT(1P11E12.4)
 1004 FORMAT('0THE END'/1H1)
 1005 FORMAT(1H04X2HI 10X2HIL10X2HIS10X2HIT10X2HIE11X1HA11X1HC11X1HH11X
     1      1HK9X4HTREF/4X5HSIGMA3X3HTAU9X2HB210X2HB310X2HB410X2HB5
     2      10X2HC110X2HC2)
 1006 FORMAT(1H05X1HT10X2HTA10X2HEB10X2HFB10X3HNME9X3HNMF9X3HLME9X3HLMF
     1      /)
 1007 FORMAT(1H02X6HLAMEDA8X2HNU11X1HX10X2HEN10X2HFN10X2HEL10X2HFL10X
     1      3HEBN9X3HFBN9X3HEBL9X3HFBL/)
      END
```

APPENDIX II
FORTRAN LISTING OF SUBROUTINE DEBYE

```
FORTRAN IV G LEVEL 1, MOD 2              DEBYE              DATE = 68313              19/32/19

      C AEA00048  DEBYE FUNCTION  D.C. TODD  1-18-68  ST80C2-Y00
            SUBROUTINE CEBYE(N,X,F,G)
            REAL B2(5)/.1666667,-.03333333,.C2380952,-.03333333,.07575758/,
          1 ZETA1(6)/1.644934,1.202057,1.C82323,1.036928,1.017343,1.008349/,
          2 FAC(10)/1.,2.,6.,24.,12C.,720.,5040.,40320.,36288C.,3628800./,
          3 A(5)/.007,.22,.68,1.12,1.52/,
          4 B(8)/174.,87.,12.21,6.1,4.C7,3.C5,2.44,2.C3/
            IF(X.GT.1.8)GO TO 20
      C F CALCULATED
            M=0
            DO 11 J=1,5
            IF(X.LE.A(J))GC TC 12
         11 M=M+1
         12 XS=X*X
            SUM=1./N-.5*X/(N+1)
            IF(M.EQ.0)GC TO 14
            POW=1.
            K2=0
            DO 13 K=1,M
            K2=K2+2
            POW=POW*XS
         13 SUM=SUM+B2(K)*POW/((K2+N)*FAC(K2))
         14 F=SUM*X**N
            G=FAC(N)*ZETA1(N)-F
            RETURN
      C G CALCULATED
         20 M=0
            DO 21 J=1,8
            IF(X.GE.B(J))GO TO 22
         21 M=M+1
         22 IF(M.GT.0)GO TC 23
            G=0.
            GO TO 26
         23 SUM=0.
            POW=0.
            DO 25 K=1,M
            POW=POW+X
            SM=1./K**(N+1)
            PW=SM
            DO 24 J=1,N
            PW=PW*X*K
         24 SM=SM+PW/FAC(J)
         25 SUM=SUM+SM*EXP(-POW)
            G=FAC(N)*SUM
         26 F=FAC(N)*ZETA1(N)-G
            RETURN
            END
```

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13. ABSTRACT

This report aids in the calculation of blackbody radiation. The equations of blackbody radiation and photon emission are summarized and a computer program is described that is useful for calculations concerning blackbodies. The input to the program consists of the units desired, the temperature, and wavelengths. The output includes the physical constants in the desired units and the evaluation of the equations for the temperature and wavelengths specified. The equations for photon and power emission are written in terms of Debye functions. The computer program uses a subroutine that calculates to five significant figures the Debye functions of orders from one through six. The method and error analysis of this calculation are included.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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programs, computers						
2 Debye functions						
3 photons						
4 emissivity						

5. Computer program -- Blackbody radiation

1 - 2.

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