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ON THE THEORY OF OPTIMUM AERODYNAMIC SHAPES

by

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On the Theory of Optimum Aerodynamic Shapes[1,2]

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Abstract. The determination of optimum aerodynamic shapes has interested the scientific community for centuries. Historically speaking, the first problem of this kind was the study by Newton of the body of revolution having minimum drag for a given length and diameter. In the early part of this century, the use of advanced mathematical techniques in the analysis of subsonic and supersonic flows stimulated a renewed interest in optimization problems, exemplified by the work of Munk and Van Kármán. In more recent times, the advent of jet and rocket engines as aircraft propulsion systems and the parallel increase in flight velocities and altitudes have made it necessary to extend the optimization of aerodynamic shapes to a wider range of Mach and Reynolds numbers, thereby including the hypersonic and free-molecular flow regimes.

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Since the distributions of pressure and friction coefficients depend on the flow regime, a single optimum body does not exist; rather, a succession of optimum configurations exist, that is, one for each flow regime and each set of free-stream conditions. In addition, the optimum geometry depends on the quantity being extremized (drag, lift, lift-to-drag ratio, heat-transfer rate, sonic boom, thrust) as well as on the constraints employed in the optimization process, whether geometric quantities (length, thickness, volume, wetted area, planform area, frontal area) or aerodynamic quantities (lift, pitching moment, position of the center of pressure).

In this paper, the physical models of interest in the theory of optimum aerodynamic shapes are reviewed. The corresponding mathematical models are illustrated for both problems involving one independent variable and problems involving two independent variables. Then, the solution process is considered. Finally, new trends in the theory of optimum aerodynamic shapes as well as problems of interest in the immediate future are outlined.

1. Introduction

The determination of optimum aerodynamic shapes has interested the scientific community for centuries. Historically speaking, the first problem of this kind was the study by Newton of the body of revolution having minimum drag for a given length ℓ and thickness t (Fig. 1). Not only did Newton employ analytical techniques analogous to the modern calculus of variations, but he also formulated pressure laws which are good approximations for certain physical flows. In studying the impact of a gas molecule with the body, Newton postulated two possible models: (a) the normal velocity component is reversed, while the tangential velocity component is conserved; and (b) the normal velocity component is annihilated, while the tangential velocity component is conserved. In modern aerospace terminology, we recognize that Model (a) is the specular-reflection model of free-molecular flow and Model (b) is an approximation to that of a hypersonic, inviscid flow.

In the early part of this century, the use of advanced mathematical techniques in the analysis of subsonic and supersonic flows stimulated a renewed interest in optimization problems. In particular, Munk determined the lift distribution which minimizes the induced drag of a subsonic wing having given span and lift; furthermore, Von Kármán determined the shape of the slender forebody of revolution of given length and thickness which minimizes the pressure drag in linearized supersonic flow. In more recent times, the advent of jet and rocket engines as aircraft propulsion systems and the parallel increase in flight velocities and altitudes have made it necessary to extend the optimization of aerodynamic shapes to a wider range of Mach and Reynolds numbers, thereby including the hypersonic and free-molecular flow regimes.

Since the distributions of pressure coefficients and friction coefficients depend on the flow regime, it is clear that a single optimum body does not exist; rather a succession of optimum configurations exist, that is, one for each flow regime and set of free-stream conditions (Ref. 1). In addition, the optimum geometry depends on the quantity being extremized as well as the constraints employed in the optimization process, whether aerodynamic constraints or geometric constraints. A summary of the variety of problems which may be encountered in the study of optimum aerodynamic shapes is shown in Table 1.

Table 1. Study of optimum aerodynamic shapes

Flow regimes	Subsonic, transonic, supersonic, hypersonic, free-molecular
Criteria of optimization	Pressure drag, total drag, lift, lift-to-drag ratio, heat-transfer rate, sonic boom, thrust
Aerodynamic constraints	Lift, pitching moment, center of pressure
Geometric constraints	Length, thickness, wetted area, planform area, frontal area, volume

Ideally, one would like to optimize an aircraft or a missile as a whole. Since this approach is extremeley difficult, optimization studies have been concerned only with the main components of a configuration. In this connection, the categories of shapes most frequently investigated are shown in Table 2.

Table 2. Categories of shapes

Wings	Two-dimensional, three-dimensional
Bodies	Axisymmetric, three-dimensional
Nozzles, diffusers	Axisymmetric, two-dimensional, three-dimensional

In this paper, the physical models of interest in the theory of optimum aerodynamic shapes are reviewed in Section 2. Then, the corresponding mathematical models are illustrated in Section 3. Concerning variational problems, those involving one independent variable are reviewed in Section 4 and those involving two independent variables are reviewed in Section 5. The solution process is considered in Section 6. Finally, new trends in the theory and certain physical problems of interest in the immediate future are outlined in Section 7. Here, only considerations of a general nature are presented;

for detailed results, the reader should consult the specialized literature on the subject (see, for example, Refs. 1 and 2).

2. Physical Models

In this section, some of the physical models of current interest in the theory of optimum aerodynamic shapes are reviewed.

2.1. Linearized Supersonic Flow. For relatively slender shapes in flight at Mach numbers not too close to unity and yet not too large with respect to unity, the small-perturbation theory can be employed when estimating the aerodynamic forces acting on a body. In other words, the set of nonlinear equations governing the motion can be replaced by one which is linear: this is equivalent to assuming that the Mach lines originating from the surface of the body are parallel (Fig. 2).

Because of the linearity, the method of superposition can be employed, and general analytical solutions can be derived for the aerodynamic forces acting on either a two-dimensional shape or an axisymmetric shape whose contour is arbitrarily prescribed. For a two-dimensional shape (Fig. 2-a), the pressure coefficient at a point P has the form

$$C_p = C_1\dot{y} \tag{1}$$

where C_1 is a constant which depends on the free-stream Mach number and $\dot{y}$ is the inclination of the tangent to a surface element with respect to the free-stream direction[4]. On the other hand, for an axisymmetric shape (Fig. 2-b), the pressure coefficient no longer depends on the local slope of a surface element, but it is governed by the geometry of the entire body portion preceding that element. Symbolically, this can

[4] The symbol x denotes a coordinate in the undisturbed flow direction, y a coordinate perpendicular to x, and $\dot{y}$ the derivative dy/dx.

be written as

$$C_p = C_p \text{ (shape)} \tag{2}$$

2.2. Nonlinearized Supersonic Flow. Whenever the combination of thickness ratio and Mach number is such that the linearization process is not permissible, a more precise approach to the determination of the fluid properties is necessary. In this connection, one can employ a pressure coefficient derived from second or higher order approximations to the equations of motion or, where possible, one can use the complete set of equations.

Neglecting the interaction between the shock wave originating at the leading edge of the body I and the Mach lines originating at points downstream (Fig. 3), one can employ a pressure coefficient derived from a second-order approximation to the compression processes and expansion processes. Thus, for a two-dimensional shape, the pressure coefficient at a point P has the form

$$C_p = C_1\dot{y} + C_2\dot{y}^2 \tag{3}$$

where C_1 and C_2 are constants which depend on the free-stream Mach number.

If more precision is desired, one can employ shock-expansion theory. That is, the compression through the shock wave originating at the leading edge I is calculated using the exact equations of a shock wave; the subsequent expansion from point I to point P is calculated using the equations of a Prandtl-Meyer expansion. For a two-dimensional shape, this leads to the following functional expression:

$$C_p = C_p(\dot{y}, \dot{y}_1) \tag{4}$$

If further precision is desired, the interaction between the shock-wave and the Mach lines must be accounted for (Fig. 4). That is, one must study the fluid region (R) limited by the body surface IF, the shock wave IC originating at the leading edge, and the right-going characteristic line CF passing through the final point. Along the line IF, the fluid velocity must be tangent to the body; along the line IC, the equations of a shock wave are valid; and, along the line CF, the direction and compatibility conditions hold. Finally, within the region (R), the partial differential equations governing the gas flow must be satisfied. This type of study is called method of characteristics and leads to a pressure law of the form

$$C_p = C_p \text{ (shape)} \tag{5}$$

This means that the pressure coefficient at a point P depends on the geometry of the entire body portion preceding this point.

2.3. Newtonian Hypersonic Flow. Whenever the free-stream Mach number is sufficiently large with respect to unity, the shock wave generated at the leading edge of the body lies so close to the body that it can be regarded to be identical with it (Fig. 5). Consequently, the pressure distribution can be determined with the assumption that the tangential velocity component of the particles striking the body is conserved, while the normal velocity component is annihilated. This is precisely Model (b) of the introduction. For slender, two-dimensional shapes and axisymmetric shapes, the pressure coefficient at a point P is given by

$$C_p = 2\dot{y}^2 \tag{6}$$

2.4. Newton-Busemann Hypersonic Flow. A basic hypothesis of the Newtonian flow model is that the pressure at a point immediately behind the shock wave is identical with the pressure at the corresponding point of the body. Even if one admits that the layer of gas between the shock wave and the body is infinitely thin, the equality of the pressures is justified only if the gas particles, after crossing the shock wave, move along rectilinear paths; this is precisely the case for a wedge or a cone. On the other hand, if the body surface is either convex or concave, the gas particles in the thin layer between the shock wave and the body move along curvilinear paths, that is, they are subjected to centripetal accelerations. Therefore, the actual pressure on the body is lower than that predicted with the Newtonian theory for convex bodies but higher for concave bodies.

The resulting pressure correction was first calculated by Busemann; hence, this flow model is called the Newton-Busemann model and, while more complicated than the Newtonian model, it is still relatively simple for analytical purposes. The reason is that, if the slender body approximation is made, the pressure coefficient at a point P is given by

$$C_p = 2\dot{y}^2 + ky\ddot{y} \tag{7}$$

where $k = 2$ for two-dimensional flow and $k = 1$ for axisymmetric flow. Therefore, C_p depends only on the geometric properties of a surface element and is independent of the geometry of the body portion preceding that element.

2.5. Free-Molecular Flow. In the previous sections, it was tacitly assumed that the gas is a continuum, that is, the mean free path is small with respect to a characteristic dimension of the body. Whenever the mean free path is large with respect to a characteristic dimension of the body, the nature of the flow is free-molecular. The incident molecules are undisturbed by the presence of the vehicle, that is, the incoming and reflected flows are transparent to each other. For analytical purposes, two idealized models have been employed thus far and are now illustrated.

In the specular reflection model (Fig. 6-a), the molecules hitting the surface are reflected optically, which means that the tangential velocity component is unchanged while the normal velocity component is reversed. This is Model (a) of the introduction. Under convenient approximations, the pressure coefficient at a point P of a slender body is given by

$$C_p = 4\dot{y}^2 \tag{8}$$

that is, it is twice that of Newtonian hypersonic flow.

In the diffuse reflection model (Fig. 6-b), the molecules hitting the surface are first absorbed and then reemitted with a Maxwellian velocity distribution corresponding to an equilibrium temperature intermediate between that of the incoming flow and that of the solid surface. Under convenient approximations, the pressure coefficient at a point P of a slender body is given by

$$C_p = 2\dot{y}^2 + 2k\dot{y} \tag{9}$$

where the constant k depends on the surface temperature and the free-stream conditions. Clearly, C_p depends only on the orientation of a surface element with respect to the free-stream direction and is independent of the geometry of the body portion preceding that element.

3. Mathematical Models

In the previous section, a discussion of the principal flow regimes was given. After the physical model has been established, and after the criterion of optimization, the aerodynamic constraints, and the geometric constraints have been decided upon, a well-defined optimization problem arises. In this connection, two mathematical models can be identified: (a) problems in which the optimum is sought with respect to a finite number of parameters and (b) problems in which the optimum is sought with respect to a finite number of functions. Problems of type (a) belong to the theory of maxima and minima, also called mathematical programming; problems of type (b) belong to the calculus of variations, also called optimal control theory. For the sake of brevity, only problems of type (b) are reviewed in the following sections. Specifically, the case of one independent variable is considered in Section 4 and the case of two independent variables is considered in Section 5.

4. Variational Problems in One Independent Variable

In the theory of optimum aerodynamic shapes, certain functional forms involving one independent variable and one or several dependent variables are of frequent interest.

4.1. Simplest Problem. The simplest problem of the calculus of variations consists of extremizing the line integral

$$J = \int_{x_i}^{x_f} f(x, y, \dot{y})dx \tag{10}$$

with respect to the class of continuous functions y(x) which satisfy certain prescribed boundary conditions. In this relation, x denotes the independent variable, y the dependent variable, and $\dot{y}$ the derivative dy/dx; the subscripts i, f stand for the initial and final points respectively.

Variational problems of this type arise whenever two requirements are met. First, the configuration must have special geometric properties so that the body is described by a single curve; this is precisely the case with a two-dimensional wing, a body of revolution, and a conical body. Next, the flow regime must be such that the pressure and friction coefficients are functions of, at most, the local coordinates and the slope of the contour; this situation occurs in linearized supersonic flow, Newtonian hypersonic flow, and free-molecular flow.

Examples of functionals of type (10) are the following:

$$D_p/2C_1 q_\infty = \int_0^{\ell} \dot{y}^2 dx \tag{11}$$

and

$$D_p/4\pi q_\infty = \int_0^{\ell} y\dot{y}^3 dx \tag{12}$$

where D_p is the pressure drag, q_∞ the free-stream dynamic pressure, x a coordinate in the flow direction, and y a coordinate perpendicular to x. Equation (11) pertains to a two-dimensional wing, symmetric with respect to the chord, in linearized supersonic flow; Eq. (12) pertains to a body of revolution in Newtonian hypersonic flow.

4.2. Isoperimetric Problem. A modification of the previous problem arises whenever the following integrals are considered:

$$J = \int_{x_i}^{x_f} f(x, y, \dot{y})\, dx, \qquad K = \int_{x_i}^{x_f} \varphi(x, y, \dot{y})\, dx \tag{13}$$

where K is a given constant. The extremization of (13-1) is sought with respect to the class of continuous functions y(x) which satisfy certain prescribed boundary conditions and the isoperimetric constraint (13-2).

The following are examples of problems of this type:

$$D_p/2C_1 q_\infty = \int_0^{\ell} \dot{y}^2 dx, \qquad A/2 = \int_0^{\ell} y dx \tag{14}$$

and

$$D_p/4\pi q_\infty = \int_0^{\ell} y\dot{y}^3 dx, \qquad V/\pi = \int_0^{\ell} y^2 dx \tag{15}$$

Problem (14) pertains to a two-dimensional wing, symmetric with respect to the chord, in linearized supersonic flow: the pressure drag D_p must be minimized for a given enclosed area A. Problem (15) pertains to a body of revolution in Newtonian hypersonic flow: the pressure drag D_p must be minimized for a given volume V.

4.3. Ratio of Integrals. A modification of the isoperimetric problem arises whenever the following integrals are considered:

$$J_1 = \int_{x_i}^{x_f} f_1(x, y, \dot{y})dx, \qquad J_2 = \int_{x_i}^{x_f} f_2(x, y, \dot{y})dx \tag{16}$$

and the extremization of the ratio

$$J = J_1/J_2 \tag{17}$$

is sought with respect to the class of continuous functions y(x) which satisfy certain prescribed boundary conditions.

The following example illustrates the above situation:

$$L/2q_\infty = \int_0^\ell \dot{y}^2 dx, \qquad D/2q_\infty = \int_0^\ell (\dot{y}^3 + C_f)dx \tag{18}$$

$$E = L/D \tag{19}$$

where L is the lift, D the total drag, E the lift-to-drag ratio, and C_f the surface-averaged friction coefficient. Problem (18)-(19) pertains to a two-dimensional, flat-top wing in Newtonian hypersonic flow: the lift-to-drag ratio E is to be maximized for given length ℓ and thickness t.

4.4. Bolza Problem. In the previous sections, several particular problems were considered. Here, we formulate a very general problem, which includes all of the previous problems as particular cases. We consider the set of derivated variables

$$y_k = y_k(x), \qquad k = 1, \ldots, n \tag{20}$$

and nonderivated variables

$$u_k = u_k(x), \qquad k = 1, \ldots, m \tag{21}$$

which satisfy the isoperimetric constraints

$$K_j = \int_{x_i}^{x_f} \varphi_j(x, y_k, \dot{y}_k, u_k)dx + [\gamma_j(x, y_k)]_i^f , \qquad j = 1, \ldots, p \tag{22}$$

the differential constraints

$$\psi_j(x, y_k, \dot{y}_k, u_k) = 0 , \qquad j = 1, \ldots, q \tag{23}$$

and certain prescribed boundary conditions. It is required to find the combination (20)-(21) which extremizes the functional

$$J = \int_{x_i}^{x_f} f(x, y_k, \dot{y}_k, u_k)dx + [g(x, y_k)]_i^f \tag{24}$$

This problem, called the Bolza problem, is the most general problem of the calculus of variations in one independent variable. It reduces to the Lagrange problem for

$$g = 0, \qquad \gamma_j = 0, \qquad j = 1, \ldots, p \tag{25}$$

and to the Mayer problem for

$$f = 0, \qquad \varphi_j = 0, \qquad j = 1, \ldots, p \tag{26}$$

In turn, the Mayer problem reduces to the Pontryagin problem when the differential constraints (23) have the form

$$\dot{y}_j - \omega_j(x, y_k, u_k) = 0, \qquad j = 1, \ldots, q \tag{27}$$

with

$$q = n \tag{28}$$

Problems of the Bolza type arise in the study of two-dimensional or axisymmetric bodies in nonlinearized supersonic flow, providing the aerodynamic forces and the geometric constraints can be expressed as one-dimensional integrals to be evaluated along the same reference line (e.g., the contour of the body or a characteristic line of the flow field). As an example, consider the shock-free, supersonic expansion of a gas in a two-dimensional or axisymmetric nozzle of given length (Fig. 7). In this problem, the thrust, the mass flow, and the length can be expressed as integrals of quantities evaluated along the left-going characteristic line DIF joining the axis of symmetry with the final point. The minimal problem is a Bolza problem, with this understanding: the quantity J is the thrust; the constants K_j are the mass flow and the length; and the constraints $\Psi_j = 0$ are the differential equations to be satisfied along a characteristic line, namely, the direction and compatibility conditions.

Problems of the Bolza type also arise in the study of two-dimensional or axisymmetric bodies in Newtonian hypersonic flow, Newton-Busemann hypersonic flow, and free-molecular flow whenever an inequality constraint is imposed on the configuration and/or derivatives of second and higher order are present. At first glance, these problems do not seem to be covered by the Bolza formulation: in Eqs. (22)-(24), inequality constraints are not mentioned and only first-order derivatives are present. However, by the judicious use of auxiliary variables, each problem can be converted into a Bolza problem. As an example, the slope of a configuration in Newtonian hypersonic flow may be required to be nonnegative everywhere, that is, the inequality constraint

$$\dot{y} \geq 0 \tag{29}$$

is to be accounted for. This inequality constraint can be converted into a differential constraint if the real auxiliary variable u defined by the relationship

$$\dot{y} - u^2 = 0 \tag{30}$$

is introduced.

5. Variational Problems in Two Independent Variables

In the theory of optimum aerodynamic shapes, certain functional forms involving two independent variables and one or several dependent variables are of frequent interest.

5.1. Simplest Problem. The simplest problem of the calculus of variations consists of extremizing the surface integral

$$J = \iint_S f(x, y, z, z_x, z_y)\,dxdy \tag{31}$$

with respect to the class of continuous functions z(x, y) which satisfy certain prescribed boundary conditions. In this relation, x and y denote the independent variables, z the dependent variable, z_x the derivative $\partial z/\partial x$, and z_y the derivative $\partial z/\partial y$; the symbol S denotes the domain of integration in the xy-plane.

Variational problems of this type arise whenever the flow regime is such that the pressure and friction coefficients are functions of, at most, the local coordinates and the slopes of the surface defining the body. This situation occurs in certain problems of linearized supersonic flow, Newtonian hypersonic flow, and free-molecular flow.

Examples of functions of type (31) are the following:

$$D_p/2q_\infty = \iint_S z_x^3\,dxdy \tag{32}$$

and

$$D/2q_\infty = \iint_S (z_x^3 + C_f)\,dxdy \tag{33}$$

where x and y are planform coordinates and z is a coordinate perpendicular to the xy-plane. Equation (32) represents the pressure drag of a three-dimensional, flat-top wing in Newtonian hypersonic flow; Eq. (33) is the total drag of the same wing.

5.2. Isoperimetric Problem. A modification of the previous problem arises whenever the following integrals are considered:

$$J = \iint_S f(x, y, z, z_x, z_y)\, dxdy, \qquad K = \iint_S \varphi(x, y, z, z_x, z_y)\, dxdy \tag{34}$$

where K is a given constant. The extremization of (34-1) is sought with respect to the class of continuous functions z(x, y) which satisfy certain prescribed boundary conditions and the isoperimetric constraint (34-2).

The following is an example of a problem of this type:

$$D/2q_\infty = \iint_S (z_x^3 + C_f)\, dxdy, \qquad V = \iint_S z\, dxdy \tag{35}$$

Problem (35) pertains to a three-dimensional, flat-top wing in Newtonian hypersonic flow: the total drag D must be minimized for a given volume V.

5.3. Ratio of Integrals. A modification of the isoperimetric problem arises whenever the following integrals are considered:

$$J_1 = \iint_S f_1(x, y, z, z_x, z_y)\, dxdy, \qquad J_2 = \iint_S f_2(x, y, z, z_x, z_y)\, dxdy \tag{36}$$

and the extremization of the ratio

$$J = J_1/J_2 \tag{37}$$

is sought with respect to the class of continuous functions z(x, y) which satisfy certain prescribed boundary conditions.

The following example illustrates the above situation:

$$L/2q_\infty = \iint_S z_x^2 dxdy, \qquad D/2q_\infty = \iint_S (z_x^3 + C_f) dxdy \tag{38}$$

$$E = L/D \tag{39}$$

Problem (38)-(39) pertains to a three-dimensional, flat-top wing in Newtonian hypersonic flow: the lift-to-drag ratio E is to be maximized for a given planform.

5.4. Bolza Problem. In the previous sections, several particular problems were considered. Here, we formulate a very general problem, which includes all of the previous problems as particular cases. We consider the set of derivated variables

$$z_k = z_k(x, y), \qquad k = 1, \ldots, n \tag{40}$$

and nonderivated variables

$$u_k = u_k(x, y), \qquad k = 1, \ldots, m \tag{41}$$

which satisfy the isoperimetric constraints

$$K_j = \iint_S \varphi_j(x, y, z_k, z_{kx}, z_{ky}, u_k) dxdy + \oint_B \gamma_j(x, \dot{x}, y, \dot{y}, z_k, \dot{z}_k) ds, \quad j = 1, \ldots, p \tag{42}$$

the differential constraints

$$\psi_j(x, y, z_k, z_{kx}, z_{ky}, u_k) = 0, \qquad j = 1, \ldots, q \tag{43}$$

and certain prescribed boundary conditions. Here, S is the domain of integration, B the boundary of this domain, and s a curvilinear abscissa along B; the dot sign denotes total derivative with respect to s. It is required to find the combination (40)-(41) which extremizes the functional

$$J = \iint_S f(x, y, z_k, z_{kx}, z_{ky}, u_k)\, dxdy + \oint_B g(x, \dot{x}, y, \dot{y}, z_k, \dot{z}_k)\, ds \qquad (44)$$

This problem, called the Bolza problem, is the most general problem of the calculus of variations in two independent variables. It reduces to the Lagrange problem for

$$g = 0, \qquad \gamma_j = 0, \qquad j = 1, \ldots, p \qquad (45)$$

and to the Mayer problem for

$$f = 0, \qquad \varphi_j = 0, \qquad j = 1, \ldots, p \qquad (46)$$

Problems of the Bolza type arise in the study of axisymmetric bodies in linearized or nonlinearized supersonic flow, whenever constraints are imposed not only on the length and the diameter but also on integrated quantities such as the wetted area or the volume. As an example, consider the problem of finding the axisymmetric closed body which minimizes the drag in linearized supersonic flow for given constraints imposed on the length and the volume (Fig. 8). One deals with the flow properties in a region (R) limited by a boundary B formed by the body contour, the left-going characteristic through the initial point I, and the right-going characteristic through the final point F. After the drag and the volume are expressed as integrals of quantities evaluated along the body contour IF, the minimal problem can be treated as a Bolza problem, with this understanding: the quantity J is the drag; the constant K is the value prescribed for the volume; and the constraints $\Psi_j = 0$ are the irrotationality condition and the continuity equation which hold at every point of the region (R). Of course, the tangency conditions must be satisfied along the body contour IF and the direction and compatibility conditions must be satisfied along the remainder of the contour B, that is, the characteristic lines IC and CF.

6. Solution Process

In the previous sections, a discussion of the principal mathematical models was given. After an optimization problem has been formulated, the classical tools of the calculus of variations must be employed: they involve first-order conditions as well as second and higher order conditions. While a detailed analysis is beyond the scope of this paper, a summary of these conditions is presented in Table 3. We note that the basic equations, the Euler equations, are ordinary differential equations for the problems of Section 4 and partial differential equations for the problems of Sections 5. Therefore, the problems of Section 5 are considerably more difficult than those of Section 4.

With the aid of the above variational tools, a wide variety of problems has been solved in recent years. Wings, bodies, and wing-body combinations have been optimized in supersonic, hypersonic, and free-molecular flow. The discussion of all the results obtained goes beyond the scope of this paper. Therefore, the reader is referred to the specialized literature on the subject (see, for example, Refs. 1 and 2).

Table 3. Optimum conditions

First-order conditions	Euler equations, transversality condition, corner conditions
Higher-order conditions	Legendre condition, Weierstrass condition, Jacobi condition

7. Engineering Trends and Unsolved Problems

Despite the variety of the results already obtained, the theory of optimum aerodynamic shapes is only at its beginning. There are interesting and useful variational problems in one independent variable yet to be solved in every flow regime. An analogous remark is even more appropriate for variational problems involving two, three, or four independent variables, since these problems have been treated in the literature only occasionally. Among the engineering problems which deserve to be investigated in the near future, the following deserve to be mentioned:

(a) Supersonic flow: Determination of the axisymmetric closed body, forebody, or ducted forebody which minimizes the total drag, the sum of the pressure drag and the friction drag, for a given volume.

(b) Supersonic flow: Determination of three-dimensional wings, fuselages, and wing-fuselage combinations which minimize the total drag under the condition that the lift is given, the volume is given, and the boom intensity on the ground does not exceed a prescribed limit.

(c) Hypersonic flow: Determination of the axisymmetric body which minimizes the surface-integrated heat transfer rate.

(d) Hypersonic flow: Determination of three-dimensional wings, fuselages, and wing-fuselage combinations which minimize the total drag or maximize the lift-to-drag ratio for given conditions imposed on the lift and the volume.

(e) Free-molecular flow: Determination of three-dimensional shapes having minimum drag for a given volume.

Mathematically speaking, these problems are problems of the Bolza type in one or several independent variables, not generally amenable to analytical solutions. This being the case, numerical techniques must be developed, more specifically, first-variation methods (steepest-descent methods) and second-variation methods (quasi-linearization methods).

While the theory of optimum aerodynamic shapes is only at its beginning, the vista is expanding rapidly on its promising applications. Therefore, it is not difficult to predict that, providing sufficient research effort is expended in this area and providing the present rate of progress is maintained in the design of digital computing machines, the calculus of variations approach will become a fundamental instrument in the design of optimum aerodynamic configurations.

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List of Captions

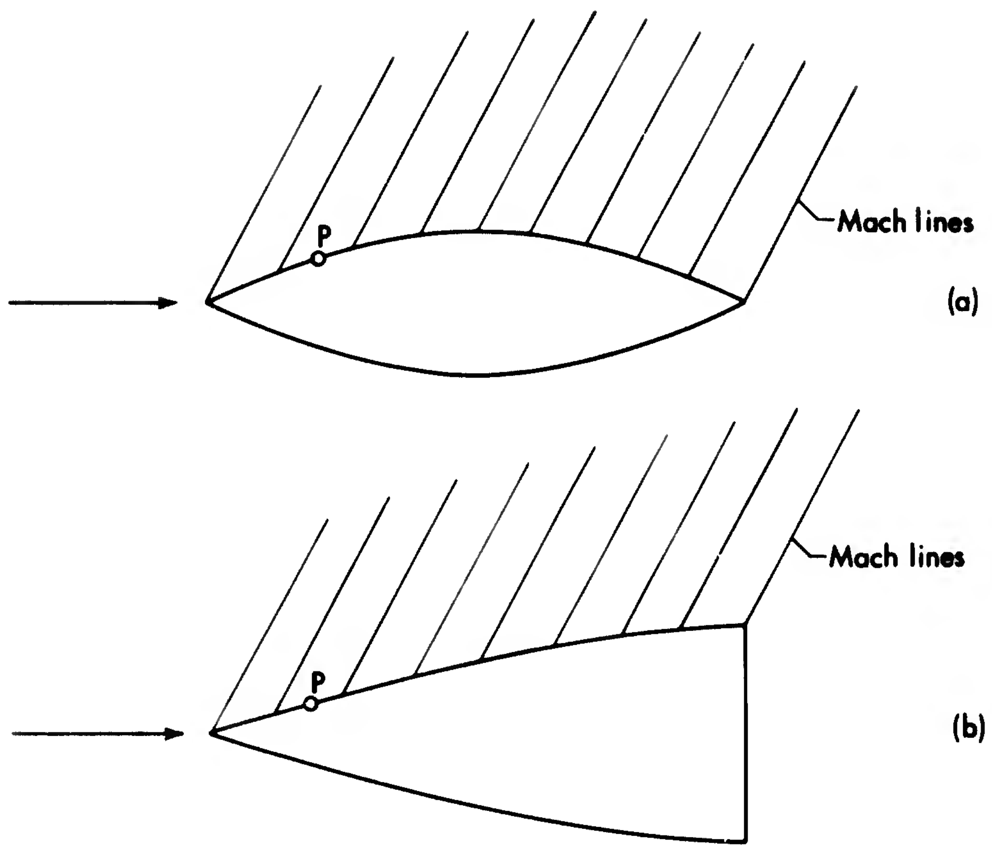

Fig. 1 Newton's problem.

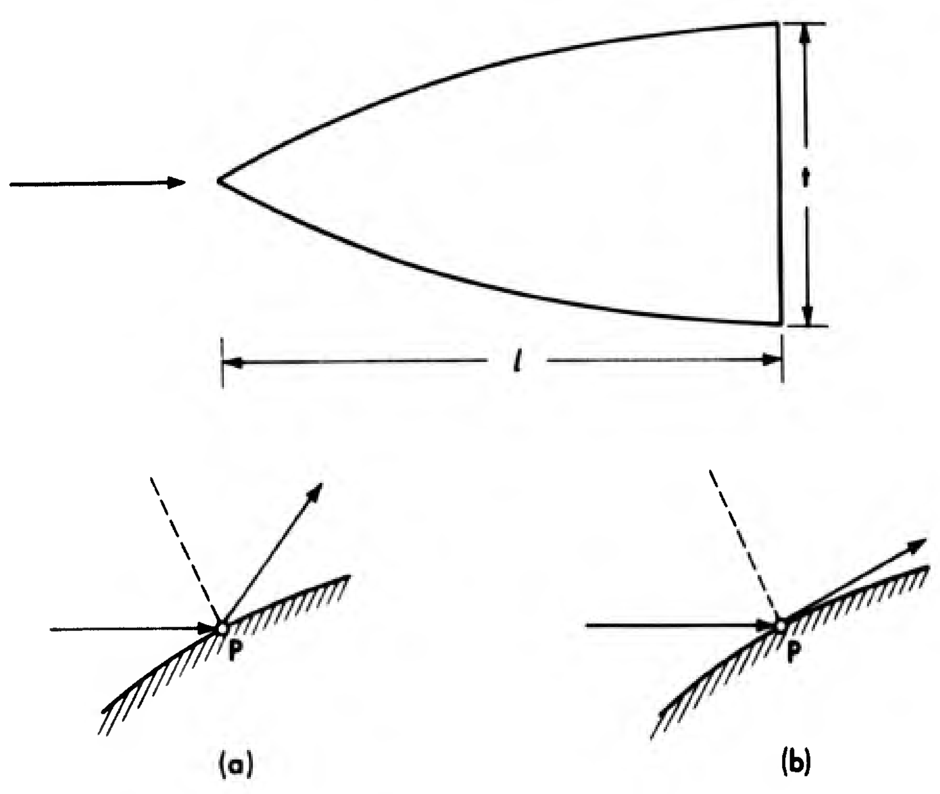

Fig. 2 Linearized supersonic flow.

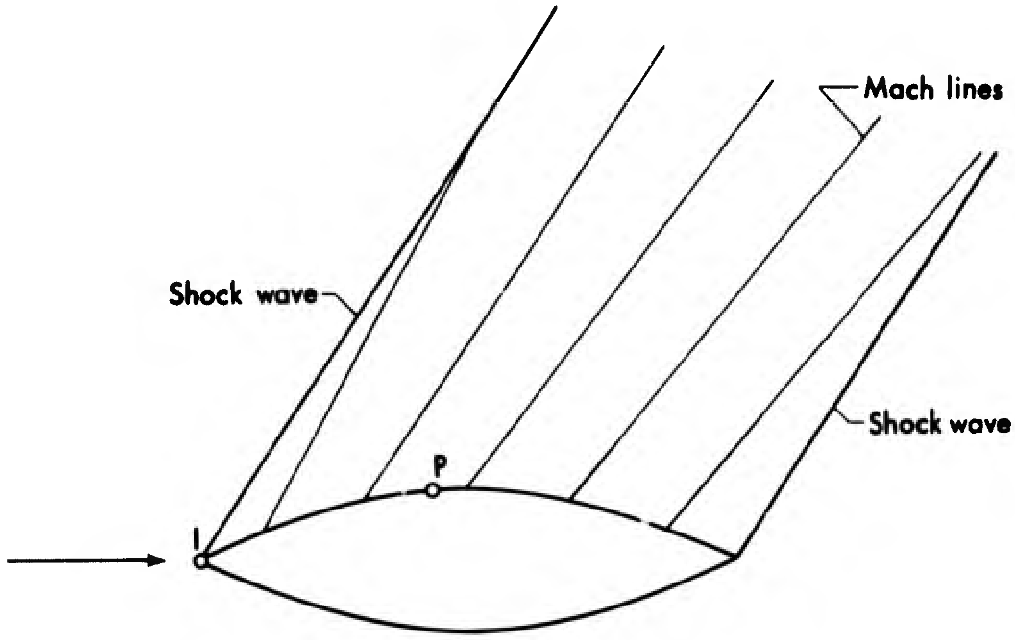

Fig. 3 Higher order approximations.

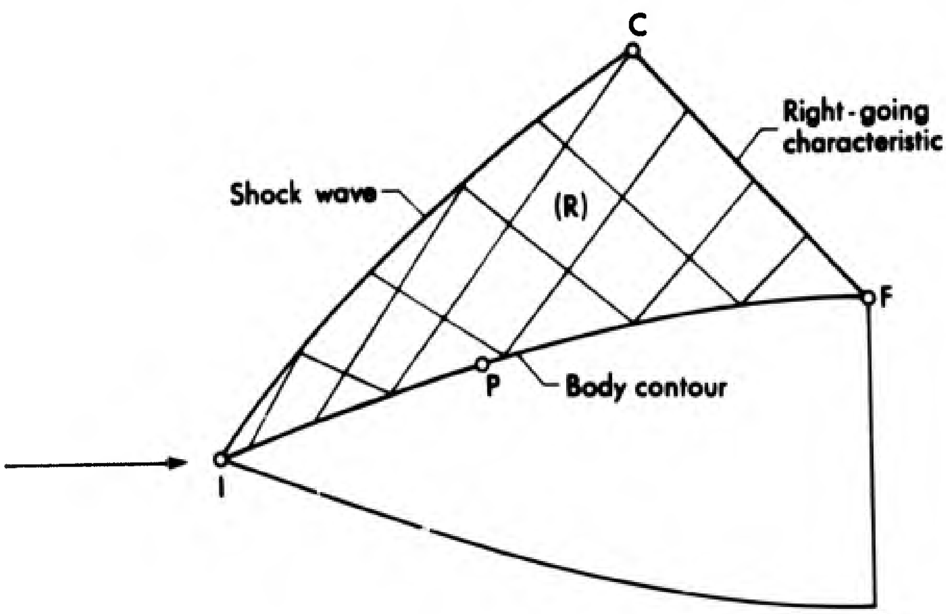

Fig. 4 Nonlinearized supersonic flow.

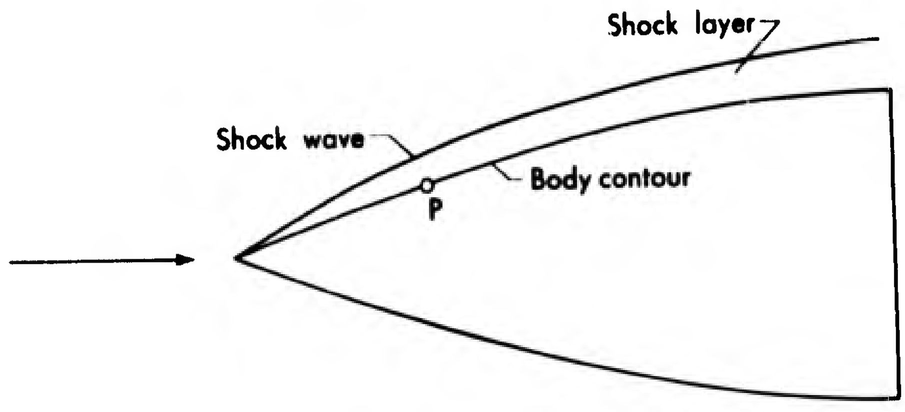

Fig. 5 Newtonian hypersonic flow.

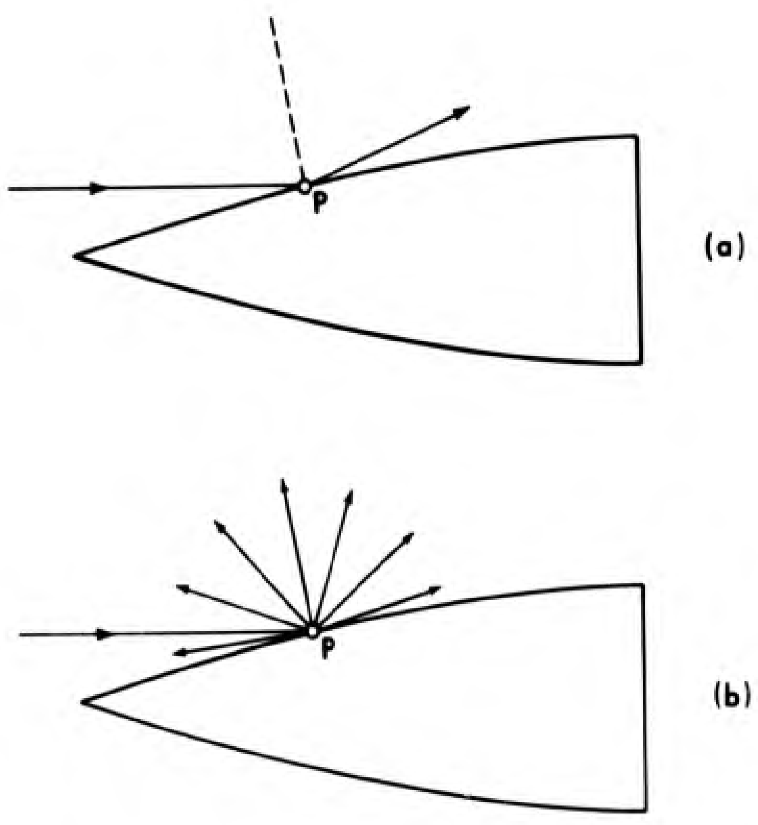

Fig. 6 Free-molecular flow.

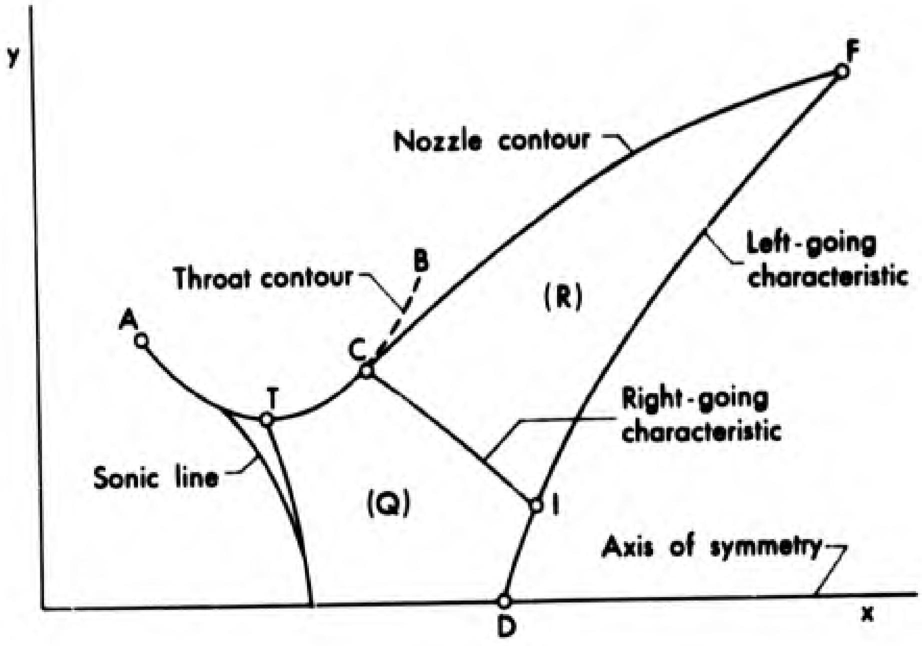

Fig. 7 Rocket nozzle.

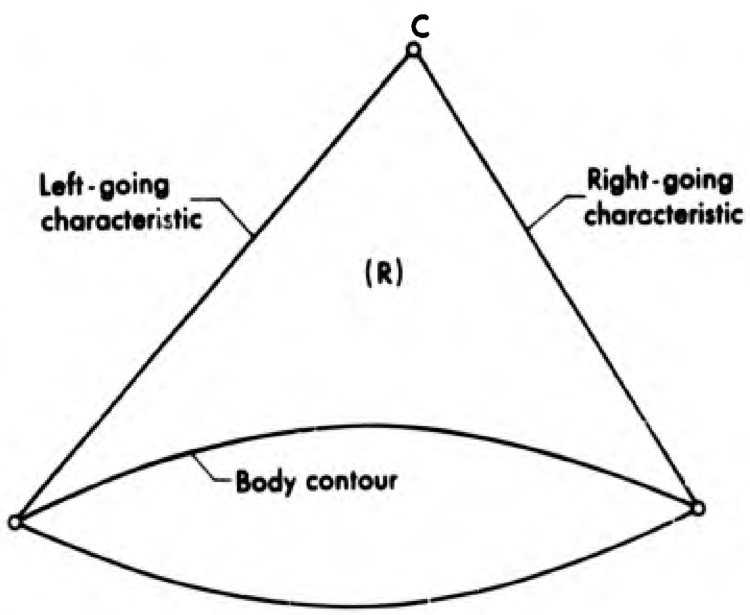

Fig. 8 Closed body of revolution.