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AN INTRODUCTION TO HIGH TEMPERATURE RADIATION GAS DYNAMICS

Philip M. Campbell

University of New Mexico
Department of Physics and Astronomy
Albuquerque, New Mexico

Contract No. F29601-68-C-0093

TECHNICAL REPORT NO. AFWL-TR-69-10

March 1969

AIR FORCE WEAPONS LABORATORY
Air Force Systems Command
Kirtland Air Force Base
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DESCRIPTORS: (*HYDRODYNAMICS, *THERMAL RADIATION),
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IDENTIFIERS: *AEROTHERMODYNAMICS, GAS
DYNAMICS (U)

THIS DOCUMENT IS INTENDED AS AN INTRODUCTION TO THE STUDY OF RADIATIVE TRANSFER AND HYDRODYNAMICS IN THE HIGH TEMPERATURE (>10,000K) REGIME. MOST OF THE BASIC REFERENCES ON RADIATIVE TRANSFER ARE ORIENTED TO ASTROPHYSICS. IT IS ALWAYS DIFFICULT FOR WORKERS IN OTHER AREAS OF RESEARCH TO CULL FROM THIS LITERATURE THOSE ASPECTS OF THE SUBJECT RELEVANT TO THEIR OWN INTERESTS. THE PURPOSE OF THIS DOCUMENT IS TO PRESENT THE BASIC PRINCIPLES OF RADIATION GAS DYNAMICS FROM THE POINT OF VIEW OF WEAPONS PHYSICS. THE FIRST CHAPTER IS CONCERNED WITH THE RADIATION FIELD ALONE. THE TRANSFER EQUATION IS DERIVED IN ITS VARIOUS FORMS, AND THE GENERAL FORMULATION OF RADIANT ENERGY TRANSFER IN MATERIALS IS DISCUSSED. THE SECOND CHAPTER DESCRIBES THE VARIOUS MECHANISMS BY WHICH THE RADIATION FIELD INTERACTS WITH A MATERIAL MEDIUM. THIS SECTION, THEREFORE, PROVIDES AN INTRODUCTION TO THE GENERAL PROBLEM OF OPACITIES. THE THIRD CHAPTER PRESENTS THE BASIC THEORY OF HYDRODYNAMICS AND THE INFLUENCE OF RADIATION FIELDS ON FLUID MOTION. THESE CHAPTERS COVER THE THREE FUNDAMENTAL ASPECTS OF THE SUBJECT: THE STATE OF THE RADIATION FIELD, THE RADIATION-MATERIAL INTERACTION, AND THE MOTION OF THE MATERIAL. (AUTHOR) (U)

AN INTRODUCTION TO HIGH TEMPERATURE RADIATION GAS DYNAMICS

Philip M. Campbell

University of New Mexico
Department of Physics and Astronomy
Albuquerque, New Mexico
Contract No. F29601-68-C-0093

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FOREWORD

This report was prepared by the University of New Mexico, Albuquerque, New Mexico, under Contract F29601-68-C-0093. The research was performed under Program Element 61102H, Project 5710, Subtask 08.0111, and was funded by the Defense Atomic Support Agency (DASA).

Inclusive dates of research were 20 June 1968 to 23 August 1968. The report was submitted 22 January 1969 by the AFWL Project Officer, Capt Chandler J. Kennedy (WLRT).

The author is grateful to Maj John Bode, Capt Marvin Alme, and Capt Kennedy for their interest and aid in this effort. Dr. G. Milton Wing and Dr. William J. Byatt served as general consultants in the collection of the material.

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This technical report has been reviewed and is approved.

CHANDLER J. KENNEDY
Capt, USAF
Project Officer

Truman L Franklin
TRUMAN L. FRANKLIN
Colonel, USAF
Chief, Theoretical Branch

CLAUDE K. STAMBAUGH
Colonel, USAF
Chief, Research Division

ABSTRACT

(~~Distribution Limitation Statement No. 2~~)

This document is intended as an introduction to the study of radiative transfer and hydrodynamics in the high temperature (> 10,000^{o}K) regime. Most of the basic references on radiative transfer are oriented to astrophysics. It is always difficult for workers in other areas of research to cull from this literature those aspects of the subject relevant to their own interests. The purpose of this document is to present the basic principles of radiation gas dynamics from the point of view of weapons physics.

The first chapter is concerned with the radiation field alone. The transfer equation is derived in its various forms, and the general formulation of radiant energy transfer in materials is discussed. The second chapter describes the various mechanisms by which the radiation field interacts with a material medium. This section, therefore, provides an introduction to the general problem of opacities. The third chapter presents the basic theory of hydrodynamics and the influence of radiation fields on fluid motion.

These chapters cover the three fundamental aspects of the subject: the state of the radiation field, the radiation-material interaction, and the motion of the material.

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Chapter I *The Radiation Field*

Chapter II *Interaction of Matter and Radiation*

Chapter I *The Radiation Field*

1. *Description of the Radiation Field*

Consider a radiation field that results from any arbitrary distribution of sources. The field is described (except for polarization) by the *monochromatic specific intensity*,

$$I^{\nu}(\vec{x},\vec{\mu},t), \tag{1-1}$$

which gives the distribution of radiant energy in position $\vec{x}$, direction $\vec{\mu}$, and frequency ν, at time t. The intensity is so defined that the radiant energy in the frequency interval $(\nu, d\nu)$ which passes through the surface element $d\vec{A}$ in time dt into the element of solid angle[1] $d\Omega$ about the direction $\vec{\mu}$ is given by

$$dW^{\nu} = I^{\nu}(\vec{x},\vec{\mu},t) \cos\alpha \, dA \, d\Omega \, d\nu \, dt. \tag{1-2}$$

The $\cos\alpha$ term enters because $dA' = dA \cos\alpha$ is the pro-

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[1] The solid angle subtended by an area $\vec{A}$ from a point P is defined as the projection of that area on a unit sphere centered at P: $\Omega = A \cos\alpha/R^2$, where $\vec{R}$ is the distance from P to A, and α is the angle between $\vec{A}$ and $\vec{R}$ (see Fig. 1-2). The unit of solid angle is the steradian, a dimensionless quantity, and Ω always lies in the range $-4\pi \le \Omega \le 4\pi$. Since $\vec{A}$ points outward from a curved surface, $\Omega < 0$ when P is outside the surface. The element of solid angle $d\Omega$ is given by $d\Omega = \sin\theta \, d\theta \, d\phi$, which is the element of area on a sphere of unit radius expressed in spherical coordinates.

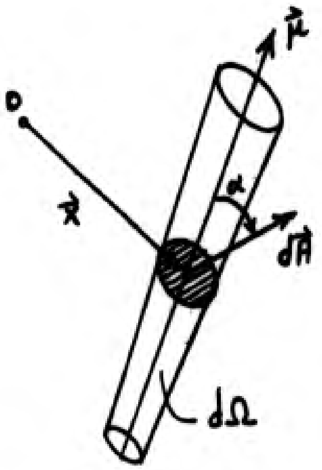

FIG. 1-1

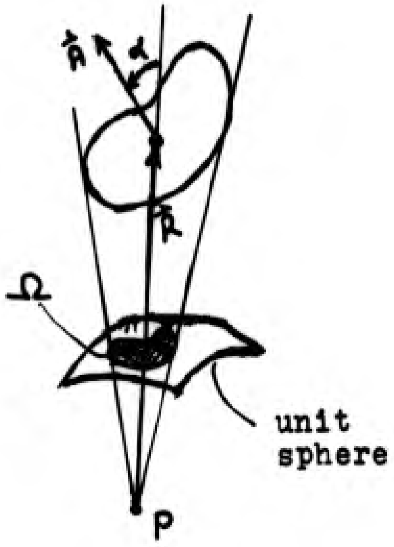

FIG. 1-2

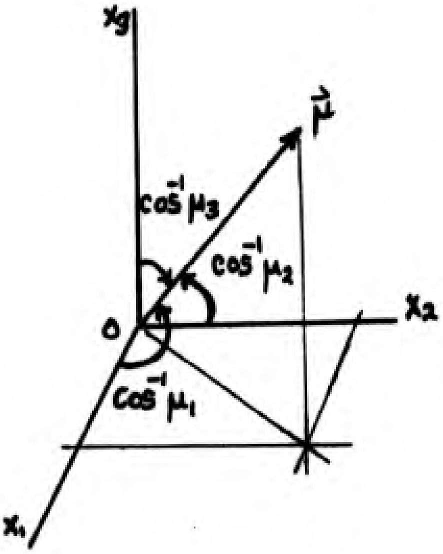

FIG. 1-3

jection of $d\vec{A}$ in the direction of the beam, $\vec{\mu}$ (see Fig. 1-1).

If the intensity is expressed in rectangular coordinates, it has the general form

$$I^{\nu}(\vec{x},\vec{\mu},t) = I^{\nu}(x_1x_2x_3;\mu_1\mu_2\mu_3;t),$$

where $(\mu_1\mu_2\mu_3)$ are the set of direction cosines of the unit vector $\vec{\mu}$ with respect to the $(x_1x_2x_3)$ axes as shown in Fig. 1-3. The radiation field is said to be *isotropic* if the intensity is independent of direction and *homogeneous* if the intensity is independent of position. The frequency integrated intensity,

$$I(\vec{x},\vec{\mu},t) = \int_0^{\infty} I^{\nu}(\vec{x},\vec{\mu},t)d\nu,$$

gives the total radiant energy of all frequencies that passes through $d\vec{A}$ into $d\Omega$ in time dt.

Three quantities of immediate physical interest are the energy density, the energy flux vector, and the radiation pressure tensor. These quantities are obtained directly from I^{ν} by virtue of the definition of intensity.[2]

The energy density $E^{\nu}(\vec{x},t)$ is the amount of radiant energy per unit volume in the frequency interval $(\nu,d\nu)$ that is in transit at time t in the neighborhood of the point $\vec{x}$. The energy density is given in terms of the

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[2] The derivation of these quantities in terms of I^{ν} is given by Chandrasekhar (1939), Chapter V, and by Pai (1966), Chapter II.

intensity by

$$E^{\nu}(\vec{x},t) = \frac{1}{c}\oint I^{\nu}(\vec{x},\vec{\mu},t)d\Omega. \qquad (1-3)$$

The frequency integrated energy density is $E(\vec{x},t) = \int_0^{\infty} E^{\nu} d\nu$ and has the units of ergs/cm^3.

The components of the energy flux vector $F_j^{\nu}(\vec{x},t)$ give the net amount of radiant energy in the frequency interval $(\nu, d\nu)$ that crosses a unit area perpendicular to each of the coordinate axes x_j per unit time. The flux is given in terms of the intensity by

$$F_j^{\nu}(\vec{x},t) = \oint I^{\nu}(\vec{x},\vec{\mu},t)\mu_j \, d\Omega. \qquad (1-4)$$

The frequency integrated flux is given by $F_j(\vec{x},t) = \int_0^{\infty} F_j^{\nu} d\nu$ with the units of ergs/cm^2/seccc.

It is known from the quantum theory of radiation that a photon of energy $h\nu$ carries momentum equal to $h\nu/c$. Any arbitrary radiation field, therefore, transports momentum, and since force is the time rate of change of momentum, the field will exert pressure on any absorbing surface emersed in it.

The flux vector gives the rate at which a *scalar* quantity (energy) is transported across a unit area. In the case of pressure, one calculates the rate at which a *vector* quantity (momentum) is transported across a unit area. Pressure must , therefore, be represented as a second rank tensor. The pressure tensor $P_{jk}^{\nu}(\vec{x},t)$ for radiation in the frequency range $(\nu, d\nu)$ is given in terms of the intensity by

$$P^{\nu}_{jk}(\vec{x},t) = \frac{1}{c}\oint I^{\nu}(\vec{x},\vec{\mu},t)\mu_j\mu_k \, d\Omega. \qquad (1\text{-}5)$$

The frequency integrated pressure tensor is $P_{jk}(\vec{x},t) = \int_0^{\infty} P^{\nu}_{jk} \, d\nu$ and has the units of energy density.[3] It is clear from (1-5) that the pressure tensor is symmetrical, $P_{jk} = P_{kj}$.

The mean pressure $\overline{P}$ is defined as

$$\overline{P}(\vec{x},t) = \frac{1}{3}(P_{11}+P_{22}+P_{33}).$$

For an isotropic radiation field,

$$P_{jk} = \begin{bmatrix} \overline{P} & 0 & 0 \\ 0 & \overline{P} & 0 \\ 0 & 0 & \overline{P} \end{bmatrix}, \quad \overline{P} = \frac{1}{3}E.$$

For radiation traveling as a plane wave along the x_1 axis,

$$P_{jk} = \begin{bmatrix} \overline{P} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \overline{P} = E.$$

The net force on any body in the radiation field is given in terms of P_{jk} and depends on the relative rates of absorption and emission of radiation by the body. The mechanical force exerted by radiation will be considered further in Chapter III.

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[3] P_{jk} is the Maxwell stress tensor for the radiation field as described by Panofsky and Phillips (1955), article 6-6.

Note that in constructing E^ν, F_j^ν, P_{jk}^ν,... we have formed a series of tensors of increasing rank from the mean values of products of the direction cosines. This procedure can be extended to quantities of higher rank, but these are of less significance physically.

§2. *Radiative Transfer in a Material Medium*

A radiation field in transit through a material medium interacts with the medium by means of three different processes: absorption, emission, and scattering.

Consider a small cylindrical volume of cross section dA and length ds in the direction $\vec{\mu}$, as shown in Fig. 2-1. The amount of radiation absorbed by the material within the cylindrical region is given by[4]

$$dI^\nu(\text{absorbed}) = -I^\nu \sigma_a{}^\nu ds, \tag{2-1}$$

where $\sigma_a{}^\nu$ is called the *absorption coefficient* and has the units of cm^{-1}. The amount of radiation emitted into the beam from the material in the cylinder is

$$dI^\nu(\text{emitted}) = J^\nu\, ds, \tag{2-2}$$

where J^ν is the *emission source function*. Both $\sigma_a{}^\nu$ and J^ν depend on the nature of the material and on its thermo-

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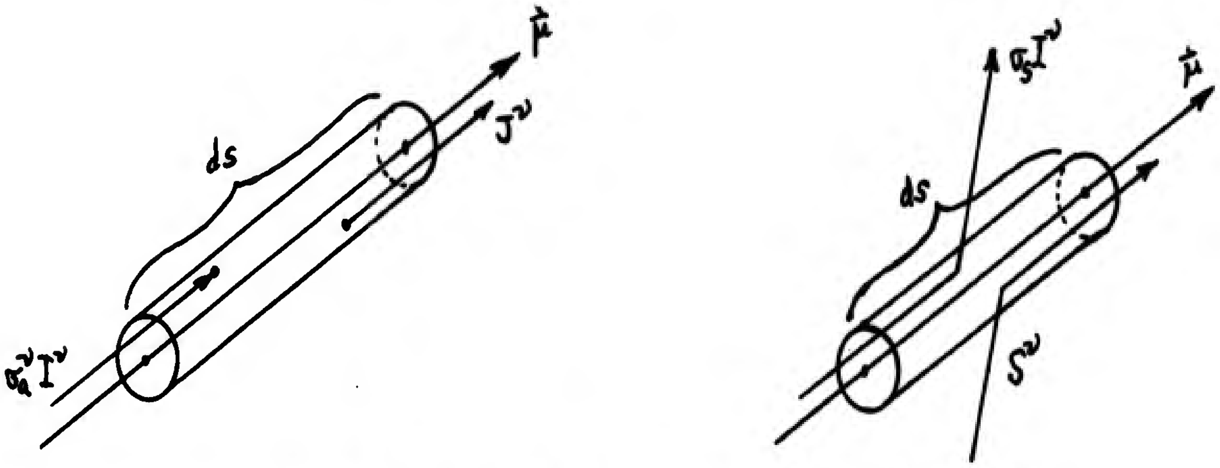
[4] The law of linear absorption is called Lambert's law and has been verified experimentally in most situations of practical interest. A mechanism contributing to non-linear absorption is the scattering of light by light.



(a) Emission and Absorption

(b) Scattering

FIG. 2-1

dynamic state. The nature of the emission source function is discussed in article 4.

As shown in Fig. (2-1), radiation in the cylindrical region will be affected both by scattering out of the cylinder and by scattering of radiation from other directions into the direction $\vec{\mu}$. The amount of radiation scattered out of the beam through interaction with material in the cylinder is

$$dI^{\nu}(\text{scattering out}) = -I^{\nu}\sigma_s{}^{\nu}ds\,, \qquad (2\text{-}3)$$

where $\sigma_s{}^{\nu}$ is the *scattering coefficient*. The radiation scattered into the beam is given by

$$dI^{\nu}(\text{scattering in}) = S^{\nu}ds\,, \qquad (2\text{-}4)$$

where S^{ν} is the *scattering source function.*

There are a number of different scattering mechanisms. For any mechanism, however, $\sigma_s{}^{\nu}$ depends on the state of the material, and S^{ν} depends on the distribution of the radiation field in direction and frequency. The nature of the scattering source function is discussed in §5.

The scattering and absorption coefficients are determined by the cross sections for the interaction of a photon with an individual atom or scattering center. An absorption coefficient can be expressed in two forms,

$$\sigma^{\nu}\ \text{cm}^{-1} = a^{\nu}\frac{\text{cm}^2}{\text{atom}} \times N\,\frac{\text{atoms}}{\text{cm}^3}$$

$$\kappa^{\nu}\ \frac{\text{cm}^2}{\text{gm}} = a^{\nu}\frac{\text{cm}^2}{\text{atom}} \times N'\,\frac{\text{atoms}}{\text{gm}},$$

where a^{ν} is the interaction cross section. The *mass absorption coefficient* κ^{ν} is related to σ^{ν} by

$$\sigma^{\nu} = \rho\kappa^{\nu},$$

where ρ is the mass density. The forms for the cross sections a^{ν} are discussed in Chapter II.

The absorption coefficient in the form σ has an important statistical interpretation. The probability that a photon will be absorbed (or scattered) in any infinitesimal element of its path ds is

$$P = \sigma ds.$$

The *mean free path* λ is the distance that a photon travels on the average before being absorbed (or scattered) and is given by

$$\lambda = \sigma^{-1} \text{ cm.} \qquad (2\text{-}6)$$

In a homogeneous medium, (2-1) leads to an exponential law of decay,

$$N = N_0 e^{-\sigma s}.$$

That is to say, a large number of photons N_0 will be reduced by absorption to N_0/e after traveling a distance equal to the mean free path λ.

The net transfer of radiation is obtained by adding the contributions of all the processes that enter into the interaction between the radiation field and the transport medium. The net change in the intensity in any direction $\vec{\mu}$ is

$$dI^\nu(\text{net}) = (J^\nu+S^\nu-\sigma_a{}^\nu I^\nu-\sigma_s{}^\nu I^\nu)ds,$$

or
$$\frac{dI^\nu}{ds} + (\sigma_a{}^\nu+\sigma_s{}^\nu)I^\nu = J^\nu + S^\nu. \qquad (2\text{-}7)$$

The equation (2-7) is the *equation of transfer* for the intensity $I^\nu(\vec{x},\vec{\mu},t)$, and when expressed in this form, the equation is valid in any coordinate system.

§3. *Differential and Integral Forms of the Transfer Equation*

In order to obtain the equation of transfer in a particular coordinate system, it is necessary to express dI/ds in terms of the coordinates chosen. For example, consider the case of rectangular coordinates as shown in Fig. 3-1. The change in the intensity along the path $\vec{\mu}$ is

$$\frac{d}{ds}I^\nu(x_1x_2x_3;\mu_1\mu_2\mu_3;t) = \frac{\partial I^\nu}{\partial t}\frac{dt}{ds} + \sum_j \frac{\partial I^\nu}{\partial x_j}\frac{dx_j}{ds} + \sum_j \frac{\partial I^\nu}{\partial \mu_j}\frac{d\mu_j}{ds}.$$

The velocity of light is ds/dt = c, and from the geometry, we see that

$$\frac{dx_j}{ds} = \mu_j,$$
$$\frac{d\mu_j}{ds} = 0.$$

In rectangular coordinates, the transfer equation takes the form

$$\frac{1}{c}\frac{\partial I^\nu}{\partial t} + \sum_j \mu_j \frac{\partial I^\nu}{\partial x_j} + (\sigma_a{}^\nu+\sigma_s{}^\nu)I^\nu = J^\nu + S^\nu. \qquad (3\text{-}1)$$

The transfer equation in other coordinates is obtained similarly. Note that $d\mu_j/ds \neq 0$ in general.

The intensity $I^{\nu}(\vec{x},\vec{\mu},t)$ is a function of 8 variables, and unless some simplification is possible, the equation of transfer will be intractable. It is usually possible to restrict the problem at hand to some geometrical configuration with one- or two-dimensional symmetry. The most commonly encountered case of two-dimensional symmetry is the finite cylindrical geometry of Fig. 3-2. Three important cases of one-dimensional symmetry are plane (or slab) geometry, spherical geometry, and cylindrical geometry, as shown in Fig. 3-3. The slab extends indefinitely in the y and z directions, and the cylinder is infinite along the z axis. In all cases, the transport medium must be uniform except along one direction (x or r.)

The transfer equation assumes its simplest form in slab geometry. From (3-1), we have

$$\frac{1}{c}\frac{\partial I^{\nu}}{\partial t} + \mu\frac{\partial I^{\nu}}{\partial x} + (\sigma_a{}^{\nu}+\sigma_s{}^{\nu})I^{\nu} = J^{\nu} + S^{\nu}. \qquad (3\text{-}2)$$

Spherical geometry is slightly more complicated. The derivative of I^{ν} along the path is

$$\frac{d}{ds}I^{\nu}(r,\mu,t) = \frac{\partial I^{\nu}}{\partial t}\frac{dt}{ds} + \frac{\partial I^{\nu}}{\partial r}\frac{dr}{ds} + \frac{\partial I^{\nu}}{\partial \mu}\frac{d\mu}{ds},$$

where $ds/dt = c$ as before. From Fig. 3-3, we see that $dr = \mu ds$, and since $L = r\sqrt{1-\mu^2} = \text{const.}$, differentiation gives

$$dr\sqrt{1-\mu^2} - \frac{r\mu d\mu}{\sqrt{1-\mu^2}} = 0,$$

or

$$\frac{d\mu}{ds} = \frac{(1-\mu^2)}{r}\ .$$

FIG. 3-1 Three-Dimensional Geometry

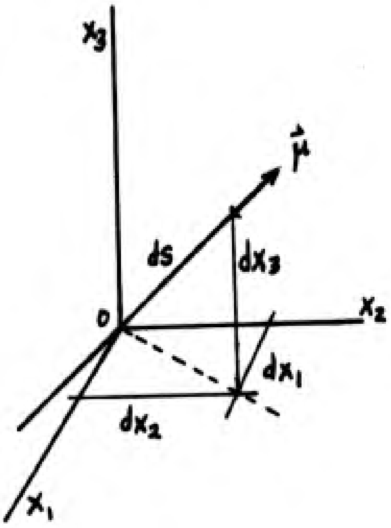


FIG. 3-2 Two-Dimensional Geometry

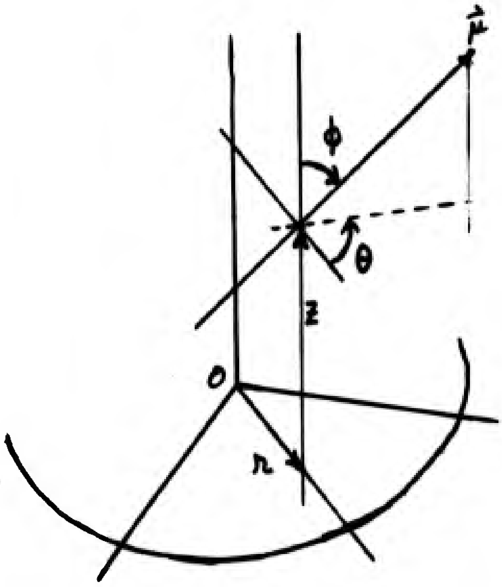
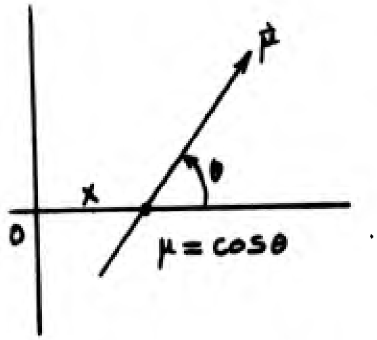
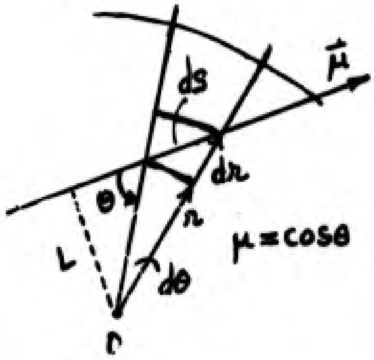


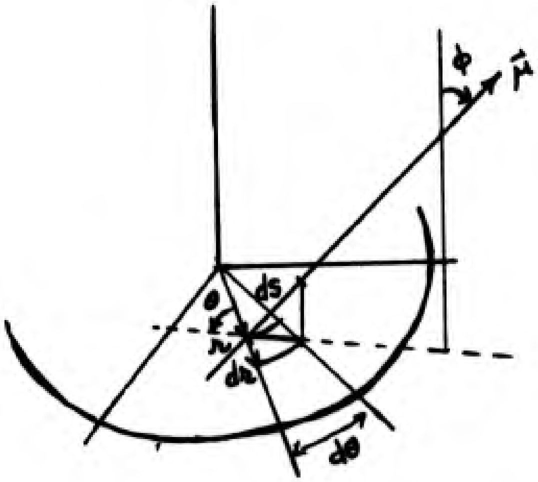
FIG. 3-3 One-Dimensional Geometries



(a) Plane geometry



(b) Spherical geometry



(c) Cylindrical geometry

The equation of transfer in spherical geometry takes the form

$$\frac{1}{c}\frac{\partial I^\nu}{\partial t} + \mu\frac{\partial I^\nu}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I^\nu}{\partial \mu} + (\sigma_a{}^\nu + \sigma_s{}^\nu)I^\nu = J^\nu + S^\nu. \tag{3-3}$$

The equation in cylindrical geometry is found similarly to be

$$\frac{1}{c}\frac{\partial I^\nu}{\partial t} + \sin\phi\left[\mu\frac{\partial I^\nu}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I^\nu}{\partial \mu}\right] + (\sigma_a{}^\nu + \sigma_s{}^\nu)I^\nu = J^\nu + S^\nu. \tag{3-4}$$

The finite cylindrical geometry of Fig. 3-2 is similar in all respects to the one-dimensional case, except that the variation in z must be included. The intensity is of the form $I^\nu(r,z,\mu,\phi,t)$ and the equation of transfer is

$$\frac{1}{c}\frac{\partial I^\nu}{\partial t} + \sin\phi\left[\mu\frac{\partial I^\nu}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I^\nu}{\partial \mu}\right] + \cos\phi\frac{\partial I^\nu}{\partial z} + (\sigma_a{}^\nu + \sigma_s{}^\nu)I^\nu = J^\nu + S^\nu. \tag{3-5}$$

The equations of transfer in various geometries are collected for reference in TABLE I.

The equation of transfer in the form (2-7) can be integrated directly. If we multiply by the integrating factor $\exp(\int_Q^{s'} \sigma_t^\nu ds'')$, (2-7) becomes

$$\frac{d}{ds'}\left(I^\nu e^{\int_Q^{s'} \sigma_t^\nu ds''}\right) = (J^\nu + S^\nu)e^{\int_Q^{s'} \sigma_t^\nu ds''}$$

where $\sigma_t{}^\nu = (\sigma_a{}^\nu + \sigma_s{}^\nu)$. This equation can be integrated from the point Q to any point s along a ray,

$$\int_Q^s d\left(I^\nu e^{\int_Q^{s'} \sigma_t^\nu ds''}\right) = \int_Q^{s'} (J^\nu+S^\nu) e^{\int_Q^{s'} \sigma_t^\nu ds''} ds',$$

giving

$$I^\nu(s)\, e^{\int_Q^s \sigma_t^\nu ds''} - I^\nu(Q) = \int_Q^s (J^\nu+S^\nu)\, e^{\int_Q^{s'} \sigma_t ds''} ds'.$$

Dividing by the exponential term, we get

$$I^\nu(s) = I^\nu(Q)\, e^{-\int_Q^s \sigma_t^\nu ds''} + \int_Q^s (J^\nu+S^\nu) e^{-\int_Q^s \sigma_t^\nu ds''} e^{\int_Q^{s'} \sigma_t^\nu ds''} ds',$$

which reduces to

$$I^\nu(s) = I^\nu(Q)\, e^{-\int_Q^s \sigma_t^\nu ds''} + \int_Q^s (J^\nu+S^\nu)\, e^{-\int_{s'}^s \sigma_t^\nu ds''} ds'. \tag{3-6}$$

This expression can be simplified by introducing the *optical depth* τ^ν defined by

$$\tau^\nu(Q,s) = \int_Q^s \sigma_t^\nu ds''. \tag{3-7}$$

In terms of the optical depth, the transfer equation in integral form becomes

$$I^\nu(s) = I^\nu(Q) e^{-\tau^\nu(Q,s)} + \int_Q^s (J^\nu+S^\nu) e^{-\tau^\nu(s',s)} ds'. \tag{3-8}$$

The equation (3-8) can be interpreted physically with the aid of Fig. 3-4. The intensity at the point s is given by the sum of two terms. The first term is the original intensity of the beam at Q attenuated by absorption over the distance s to Q. The second term gives the contri-

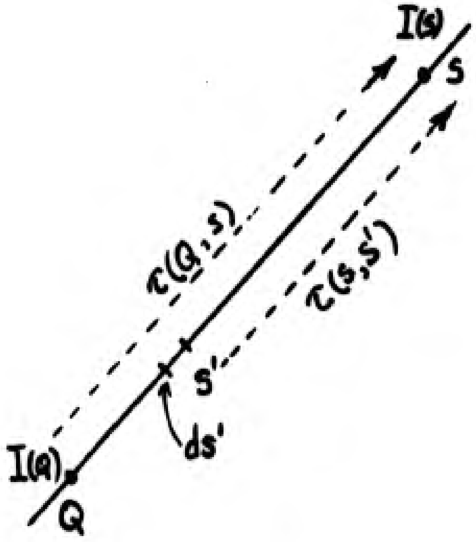


FIG. 3-4

TABLE I

Different Forms of the Equation of Transfer

1. In the plane geometry of Fig. 3-3 (a):

$$\frac{1}{c}\frac{\partial I^\nu}{\partial t} + \mu\frac{\partial I^\nu}{\partial x} + (\sigma_a^\nu+\sigma_s^\nu)I^\nu = J^\nu + S^\nu$$

2. In the spherical geometry of Fig. 3-3 (b):

$$\frac{1}{c}\frac{\partial I^\nu}{\partial t} + \mu\frac{\partial I^\nu}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I^\nu}{\partial \mu} + (\sigma_a^\nu+\sigma_s^\nu)I^\nu = J^\nu + S^\nu$$

3. In the cylindrical geometry of Fig. 3-3 (c):

$$\frac{1}{c}\frac{\partial I^\nu}{\partial t} + \sin\phi\left[\mu\frac{\partial I^\nu}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I^\nu}{\partial \mu}\right] + (\sigma_a^\nu+\sigma_s^\nu)I^\nu = J^\nu + S^\nu$$

4. In the finite cylindrical geometry of Fig. 3-2:

$$\frac{1}{c}\frac{\partial I^\nu}{\partial t} + \sin\phi\left[\mu\frac{\partial I^\nu}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I^\nu}{\partial \mu}\right] + \cos\phi\frac{\partial I}{\partial z} + (\sigma_a^\nu+\sigma_s^\nu)I^\nu = J^\nu + S^\nu$$

5. In the rectangular coordinates of Fig. 3-1:

$$\frac{1}{c}\frac{\partial I^\nu}{\partial t} + \sum_j \mu_j\frac{\partial I^\nu}{\partial x_j} + (\sigma_a^\nu+\sigma_s^\nu)I^\nu = J^\nu + S^\nu$$

6. In any coordinate system:

$$\frac{dI^\nu}{ds} + (\sigma_a^\nu+\sigma_s^\nu)I^\nu = J^\nu + S^\nu$$

7. In integral form:

$$I^\nu(s) = I^\nu(Q)e^{-\tau^\nu(Q,s)} + \int_Q^s (J^\nu+S^\nu)e^{-\tau^\nu(s',s)}ds'$$

bution to the intensity due to emission and scattering into the beam from each element ds' between Q and s. The radiation originating in each element ds' must be attenuated exponentially in traveling from s' to s.

Note that (3-8) is not a *solution* to the transfer problem, it is merely the transfer equation expressed in integral form. In general, the emission function and absorption coefficient depend on the past history of the radiation field. They are, therefore, unknown until (2-15) is solved for I^{ν}. Moreover, I^{ν} enters into the scattering term directly. Only when σ_a^{ν} and J^{ν} are independent of the radiation field, and there is no scattering, can (3-8) be considered as a solution to the transfer problem.

§4. *Local Thermodynamic Equilibrium*

Consider the radiation field that exists in the interior of an isothermal enclosure. Let the enclosure be filled with a uniform isotropic medium which interacts with the radiation through emission and absorption. We assume for the moment that scattering is negligible. Under these conditions, the radiation field is uniform in the cavity, and the equation of transfer (2-7) becomes

$$J^{\nu} = \sigma_a^{\nu} I^{\nu}. \tag{4-1}$$

The expression (4-1) is a statement of *Kirchhoff's law of radiation*: The ratio of the emission and absorption coefficients in an isothermal enclosure is equal to the specific intensity of the radiation. It can be shown also that in an isothermal enclosure, the specific intensity depends only on the temperature and not on the nature of the medium.[5] This particular radiation field is called *black body radiation* and is denoted by the symbol $B^{\nu}(T)$,

$$\frac{J^{\nu}}{\sigma_a^{\nu}} = B^{\nu}(T).$$

The determination of the form of $B^{\nu}(T)$ led to the development of the quantum theory. The history of the early investigations of the radiation law is given by Planck (1959). Quantum theory gives for $B^{\nu}(T)$,

$$B^{\nu}(T) = \frac{2}{h^2c^2}\,\frac{(h\nu)^3}{e^{h\nu/kT}-1}\,. \qquad (4\text{-}3)$$

The expression (4-3) is known as the *Planck function*, where $h = 6.6252 \cdot 10^{-27}$ erg-sec is Planck's constant, $k = 1.3804 \cdot 10^{-16}$ erg/^{o}K is Boltzmann's constant, and $c = 2.9979 \cdot 10^{10}$ cm/sec is the velocity of light. The Planck function is conveniently expressed in the form

$$B^{\nu}(T) = \frac{2}{h^2c^2}\,(kT)^3\, f(\xi), \qquad (4\text{-}4)$$

where $$f(\xi) = \frac{\xi^3}{e^{\xi}-1}.$$

The scaled distribution $f(\xi)$ is plotted in Fig. 4-1. The maximum of the curve is given by $df/d\xi = 0$, and occurs where

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[5] For the proof, see Chandrasekhar (1939), p. 201.

$$\xi_m = \frac{h\nu_m}{kT} = 2.8214. \qquad (4\text{-}5)$$

This expression gives the frequency ν_m for which the intensity is maximum at a given temperature. Wein's law (4-5) is usually expressed in the form

$$\lambda_m T = 0.28979 \text{ cm-}^{\circ}\text{K},$$

where $\lambda_m = c/\nu_m$ is the wavelength at maximum intensity.

The frequency integrated black body intensity is given by

$$B(T) = \frac{2}{h^2c^2}\int_0^{\infty} \frac{(h\nu)^3}{e^{h\nu/kT}-1}\, d\nu$$
$$= \frac{2k^4T^4}{h^3c^2}\int_0^{\infty} \frac{\xi^3}{e^{\xi}-1}\, d\xi.$$

The integration can be performed by expanding the denominator with the binomial theorem,

$$B(T) = \frac{2k^4T^4}{h^3c^2}\int_0^{\infty} \xi^3 e^{-\xi}(1+e^{-\xi}+e^{-2\xi}+\ldots)d\xi,$$

and integrating term by term,

$$B(T) = \frac{2k^4T^4}{h^3c^2}\, 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right)$$
$$= \frac{2\pi^4k^4}{15h^3c^2}\, T^4.$$

The energy density in the radiation field is

$$E = \frac{1}{c}\int B(T)d\Omega = \frac{4\pi}{c}B(T) \qquad (4\text{-}6)$$
$$= \left(\frac{8\pi^5k^4}{15h^3c^3}\right)T^4 = a\, T^4.$$

The coefficient, a, occuring in (4-6) is called the radiation constant. This constant is related to the Stefan-Boltzmann constant,[6] σ, by a = 4σ/c. The energy density can be expressed in either of the two forms

$$E = aT^4$$

$$= \frac{4}{c}\sigma T^4, \qquad (4\text{-}7)$$

where $a = 7.5634\cdot 10^{-15} \mathrm{erg/cm^3/{}^\circ K^4}$,

and $\sigma = 5.6686\cdot 10^{-5}\ \mathrm{erg/cm^2/sec/{}^\circ K^4}$.

In most problems of radiative transfer, the thermodynamic state of the material can be described by a temperature that varies from point to point. This will be the case when the relative population of the allowed energy states of free and bound electrons is given by Boltzmann's law[7]:

$$\frac{n(E_1)}{n(E_2)} = \frac{g_1}{g_2}\, e^{(E_2-E_1)/kT}. \qquad (4\text{-}8)$$

In this expression, n(E) is the number of electrons in a state of energy E, g is the statistical weight (degeneracy) of the state, and T is the temperature of the material in a small region about the point in question. The material

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[6] The symbol σ is standard notation for both the Stefan-Boltzmann constant and the absorption coefficient. In this work, σ will seldom appear as the Stefan-Boltzmann constant.

[7] For a discussion of thermal equilibrium in a quantum system, see Richtmyer, Kennard, and Lauritsen (1955), article 88.

can be described by a temperature according to (4-9) as long as the time scale for energy exchange between the different levels by collisions is small compared to the time scale for changes in the external radiation sources.

A material that can be described by a temperature at any point is said to be in *Local Thermodynamic Equilibrium* (LTE). The emission-source function for LTE is

$$J^{\nu} = \sigma_a^{\nu} B^{\nu}(T). \tag{4-10}$$

The conditions under which LTE breaks down are discussed by Sampson (1965).

It is sometimes convenient to define an *effective temperature* for a non-thermal source, when the emission function does not differ much from (4-10). If the non-thermal source emits energy at the rate R ergs/cm^2/sec, the effective temperature is defined by

$$R = \tfrac{1}{4} ac T_e{}^4. \tag{4-11}$$

Two nonthermal radiation sources of considerable importance are bremsstrahlung and synchrotron radiation. The emission function for bremsstrahlung is dominated by soft (low energy) photons, as shown qualitatively in Fig. 4-2. The figure shows the frequency spectrum that results as a particle of initial energy E loses energy through radiative collisions in a material[8]. Fig. 4-3

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[8] Heitler (1954), p.255.

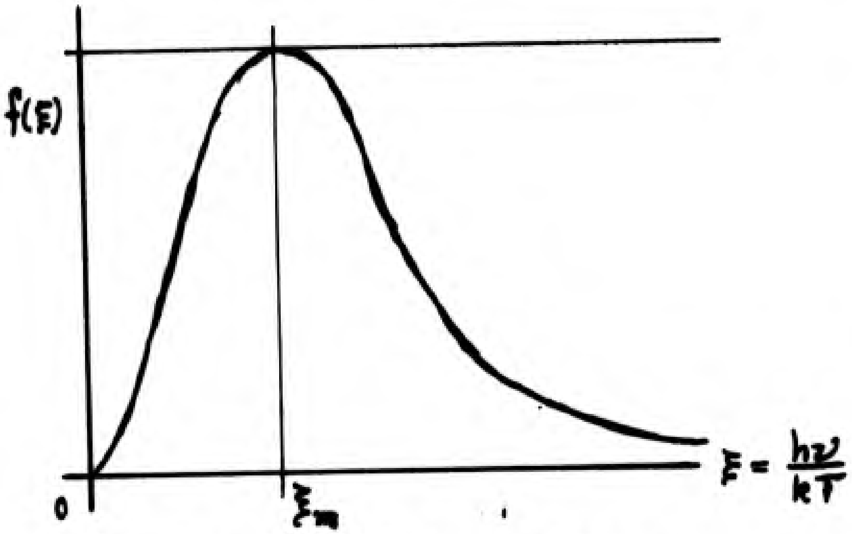

FIG. 4-1 Black Body Spectrum

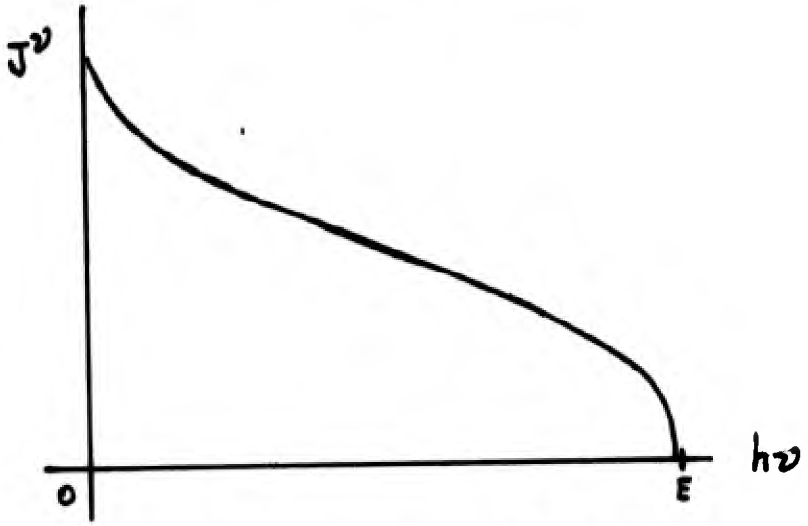

FIG. 4-2 Bremsstrahlung Spectrum

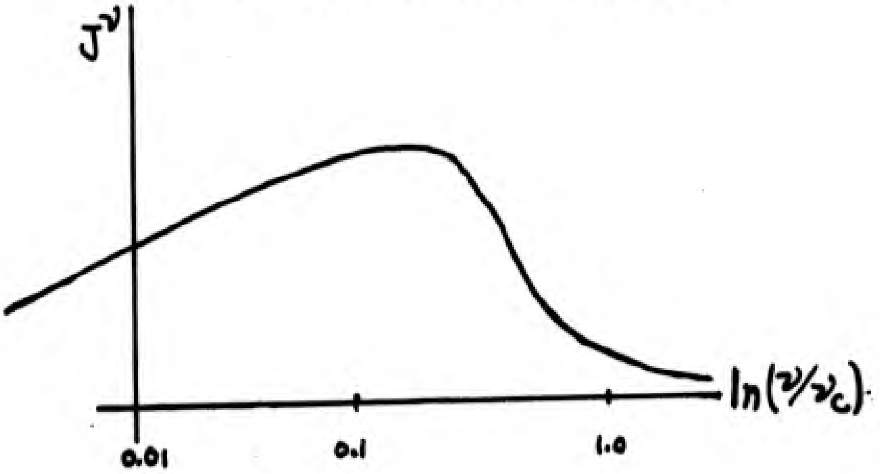

FIG. 4-3 Synchrotron Spectrum

shows the frequency spectrum for synchrotron radiation[9]. The unit of frequency is

$$\nu_c = 3\left(\frac{E}{mc^2}\right)^3 \frac{c}{2\pi\rho} \ .$$

§5. *Conservative Scattering*

In this section, the scattering source function S^ν is investigated. The two fundamental quantities that describe a scattering material are the scattering coefficient, σ_s, and the scattering kernel or *phase function*, $K(\cos\Theta)$. These quantities are defined as follows:

$\sigma_s ds$ = probability that a single photon will be scattered in traveling a distance ds,

$K(\cos\Theta)d\Omega$ = probability that a scattered photon will be found in the element of solid angle $d\Omega$ at an angle Θ with the incident direction.

If the phase function $K(\cos\Theta)$ is normalized to unity,

$$\oint K(\cos\Theta)d\Omega = 1,$$

the scattering is *conservative*; that is to say, no energy is lost in the scattering process. For nonconservative scattering,

$$\oint K(\cos\Theta)d\Omega = \varpi_0 < 1,$$

where ϖ_0 is called the *albedo for single scattering*.

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[9] Jackson (1962), p. 487.

In order to obtain the scattering source function, the scattering kernel must be expressed in terms of arbitrary incident and final directions in the particular coordinate system being used:

$$K(\cos\Theta) = K(\vec{\mu},\vec{\mu}'), \qquad (5\text{-}2)$$

where $\vec{\mu}\cdot\vec{\mu}' = \cos\Theta$. When the scattering kernel is expressed in this way, the scattering source function becomes

$$S^{\nu}(\vec{x},\vec{\mu},t) = \sigma_s \oint K(\vec{\mu},\vec{\mu}')I^{\nu}(\vec{x},\vec{\mu}',t)d\Omega'. \qquad (5\text{-}3)$$

Note that since $\cos\Theta = \vec{\mu}\cdot\vec{\mu}' = \vec{\mu}'\cdot\vec{\mu}$, the scattering kernel is symmetric in its arguments, $K(\vec{\mu},\vec{\mu}') = K(\vec{\mu}',\vec{\mu})$, and if $K(\cos\Theta)$ is normalized to unity, then

$$\oint K(\vec{\mu},\vec{\mu}')d\Omega' = \oint K(\vec{\mu}',\mu)d\Omega = 1. \qquad (5\text{-}4)$$

When the transfer equation is integrated over all solid angle , conservative scattering terms drop out. Consider the transfer equation (3-1) with the scattering term (5-3),

$$\frac{1}{c}\frac{\partial I^{\nu}}{\partial t} + \sum_j \mu_j \frac{\partial I^{\nu}}{\partial x_j} + (\sigma_a^{\nu}+\sigma_s^{\nu})I^{\nu} = J^{\nu} + \sigma_s\int K(\vec{\mu},\vec{\mu}')I^{\nu}(\vec{\mu}')d\Omega'.$$

If we multiply by $d\Omega$ and integrate over all solid angle, the scattering source function becomes

$$\sigma_s\iint K(\vec{\mu},\vec{\mu}')I^{\nu}(\mu')d\Omega' d\Omega = \sigma_s\int I^{\nu}(\vec{\mu}')d\Omega',$$

because of the second relation in (5-4). The scattering terms on both sides of the equation are the same and therefore cancel,

$$\sigma_s\int I^{\nu}(\vec{\mu})d\Omega = \sigma_s\int I^{\nu}(\vec{\mu}')d\Omega'.$$

The total radiation scattered out of the field is just equal to the total radiation scattered into the field from all directions.

The scattering coefficients and phase functions for several different scattering processes are shown in Table II. An excellent reference on the scattering of radiation by small particles is that of van de Hulst (1957). Electron scattering is of special interest in high temperature (x-ray) problems and is considered specifically in Chapter II.

The theory of the scattering process gives the phase function in the form $K(\cos\Theta)$. For use in the transfer equation, the phase function must be expressed in the form $K(\vec{\mu},\vec{\mu}')$. For example, consider Rayleigh's phase function for scattering in either plane or spherical geometry as shown in Fig. 5-1. The cosine of the scattering angle must be expressed in terms of the direction cosine μ and the azimuth angle ϕ. This can be done by means of the *addition theorem* for Legendre functions,[10]

$$P_{\ell}(\cos\Theta) = P_{\ell}(\mu)P_{\ell}(\mu') + 2\sum_{k=1}^{\ell} P_{\ell}^{k}(\mu)P_{\ell}^{k}(\mu')\frac{(\ell-k)!}{(\ell+k)!}\cos k(\phi-\phi'). \tag{5-5}$$

For $\ell = 1$ the addition theorem gives

$$\cos\Theta = \mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\,\cos(\phi-\phi'),$$

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[10] Margenau and Murphy (1956), p. 109.

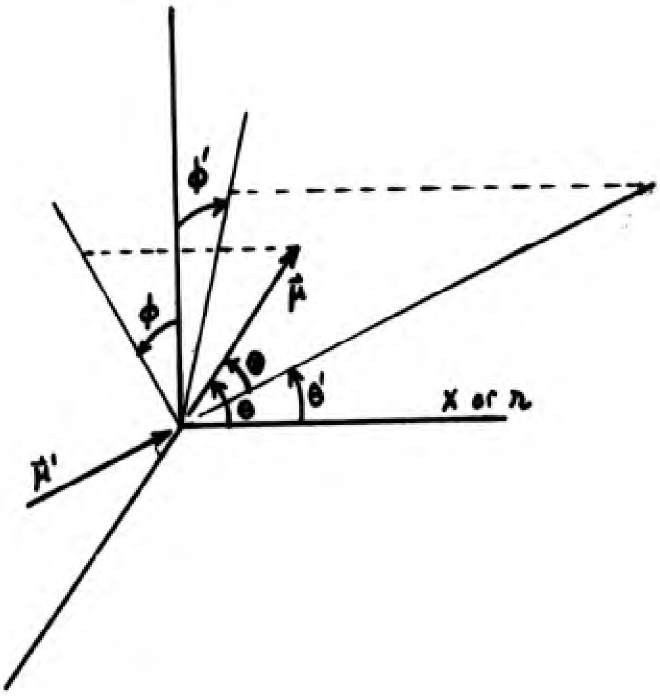

FIG. 5-1b $K(\cos\Theta) = \frac{3}{16\pi}(1+\cos^2\Theta)$

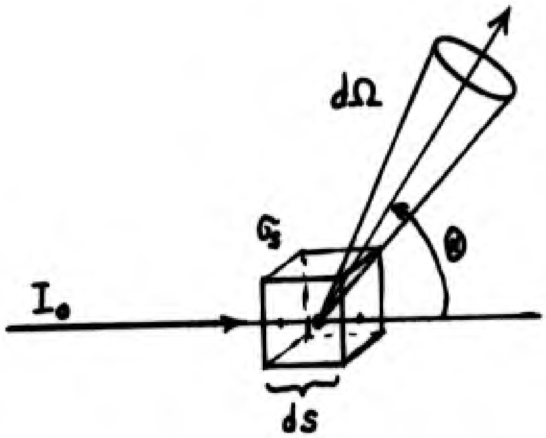

FIG. 5-1a Rayleigh's Phase Function

and Rayleigh's phase function becomes

$$K(\mu,\phi;\mu',\phi') = \frac{3}{16\pi}\left\{1 + \left[\mu\mu'+\sqrt{1-\mu^2}\ \sqrt{1-\mu'^2}\ \cos(\phi-\phi')\right]^2\right\}$$

$$= \frac{3}{16\pi}\left\{1 + \mu^2\mu'^2+(1-\mu^2)(1-\mu'^2)\cos^2(\phi-\phi') + 2\mu\mu'\sqrt{1-\mu^2}\ \sqrt{1-\mu'^2}\ \cos(\phi-\phi')\right\}. \qquad (5\text{-}6)$$

In both plane and cylindrical geometry, there is symmetry about the azimuth angle ϕ, and we may integrate over that part of the solid angle,

$$d\Omega = \sin\theta\ d\theta d\phi = -d\mu d\phi.$$

In either plane or spherical geometry, the scattering source function becomes

$$S^\nu(r,\mu,t) = \sigma_s\int_{-1}^{1} K(\mu,\mu')I^\nu(r,\mu',t)d\mu', \qquad (5\text{-}7)$$

where

$$K(\mu,\mu') = \frac{3}{8}\left[1+\mu'^2\mu^2+\tfrac{1}{2}(1-\mu^2)(1-\mu'^2)\right]$$

$$= \frac{3}{16}\left[(3-\mu^2)+(3\mu^2-1)\mu'^2\right]. \qquad (5\text{-}8)$$

Recall that the scattering kernel is symmetric, and therefore μ and μ' can be interchanged in (5-8).

§6. *Linear and Nonlinear Problems*

In the linear problem of radiative transfer, the emitting and absorbing properties of the material are given. A typical example of a linear problem is the transport of γ-rays through reactor shields. The radiation does not deposit enough energy in the shield

TABLE II

Properties of Typical Scattering Processes

Type of Scattering	Phase Function	Scattering Coefficient
Isotropic	$4\pi K(\cos\Theta) = \varpi_o$	$\sigma_s = \pi a^2$
Rayleigh Scattering by a Dielectric Sphere in Long Wavelength Limit[a]	$4\pi K(\cos\Theta) = \frac{3}{4}(1+\cos^2\Theta)$	$\sigma_s = \frac{128\pi^5}{3\lambda^4}\alpha^2 N$
Scattering by a Conducting Sphere in Long Wavelength Limit[b]	$4\pi K(\cos\Theta) =$ $\frac{3}{4}(1+\cos^2\Theta)-\frac{6}{5}\cos\Theta$	$\sigma_s =$ $\frac{10\pi}{3}a^6\,\frac{(2\pi)^4}{\lambda^4}N$
Colloidal Scattering[c]	$4\pi K(\cos\Theta) =$ $\varpi_o(1+x\cos\Theta)$ $\lvert x\rvert \leq 1$	$\sigma_s = \pi a^2 N$
Thompson Scattering by Free Electrons at Rest[d]	$4\pi K(\cos\Theta) =$ $\frac{3}{4}(1+\cos^2\Theta)$	$\sigma_s =$ $\frac{8\pi}{3}\,\frac{e^4}{m^2c^4}N$
Compton Scattering by Free Electrons at Rest[e]	$4\pi K(\cos\Theta) =$ $\frac{3}{4}(1+\cos^2\Theta)$ $[1+\alpha(1-\cos\Theta)]^{-2}$ $\left\{1+\frac{\alpha^2(1-\cos\Theta)^2}{(1+\cos^2\Theta)[1+\alpha(1-\cos\Theta)]}\right\}$	$\sigma_s =$ $\frac{8\pi}{3}\,\frac{e^4}{m^2c^4}N$

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[a] Lord Rayleigh (1871). (α = polarizability, N = particles/unit volume)

[b] Jackson (1962), p. 569. (a = radius of sphere, λ = wavelength)

[c] Van de Hulst (1957).

[d] Heitler (1954), p. 34.

[e] Heitler (1954), p. 219. ($\alpha = k_o/\mu$, $k_o = \hbar\nu_o$, $\mu = mc^2$)

to alter its thermodynamic state and hence its absorbing properties.

In the nonlinear problem, the state of the material depends on the temperature and must be obtained as part of the calculation. A problem of this type results when the radiation field carries so much energy that it controls the thermodynamic state of the material and drives the hydrodynamic motions.

In this section, the basic equations for the nonlinear problem are given, assuming local thermodynamic equilibrium and conservative scattering. The equation of transfer is

$$\frac{1}{c}\frac{\partial I^{\nu}}{\partial t} + \sum_{j} \mu_j \frac{\partial I^{\nu}}{\partial x_j} + (\sigma_a^{\nu}+\sigma_s^{\nu})I^{\nu} = \sigma_a^{\nu}B^{\nu}(T) + \sigma_s\int_{-1}^{1} K(\mu,\mu')I^{\nu}(\mu')d\mu'. \qquad (6\text{-}1)$$

The temperature at any point depends on the history of energy exchange between the material and the radiation field. When the transfer equation is integrated over all solid angle and frequencies, the *energy equation* is obtained,

$$-\left[\frac{\partial E}{\partial t} + \sum_{j} \frac{\partial F_j}{\partial x_j}\right] = \int_0^{\infty} c\sigma_a^{\nu}\left[E^{\nu}-\frac{4\pi}{c}B^{\nu}(T)\right]d\nu = \frac{d}{dt}E_m(T) + \frac{dW}{dt}. \qquad (6\text{-}2)$$

In this equation, E and $\vec{F}$ are the frequency integrated energy density and energy flux density. The middle term is the difference at any point between the rates of absorption and emission of radiant energy by the material.

This term is therefore equal to the net rate at which energy is being supplied to the material from the radiation field.

The energy taken up by the material goes into two forms, thermal energy and hydrodynamic motions. The thermal energy density in the material is given by the *equation of state*,

$$E_m(T) = \rho\, c_v\, T, \tag{6-3}$$

where ρ is the mass density and c_v the specific heat. In general, c_v will be a complicated function of temperature, depending on the relative number of bound and free electrons and the chemical composition of the material. Specific forms for c_v and the hydrodynamic work term dW/dt will be considered in Chapters II and III. In discussing the radiative transfer problem, it is often convenient to neglect the effects of mass motion for simplicity. In such cases, dW/dt is omitted from the energy equation (6-2).

In summary, the radiative transfer problem for LTE and conservative scattering is described by the following set of basic equations: (1) the equation of transfer (6-1) together with the appropriate boundary and initial conditions, (2) the energy equations (6-2), expressions for the cross sections and scattering kernel (as discussed in Chapter II), (3) the equation of state (6-3), and (4) the equations of mass motion (as discussed in Chapter III).

§7. *Approximate Formulations*

In this section, we consider some of the simplifying approximations that can sometimes be applied to the full transfer equation in nonlinear problems.

(a) *Gray Body Approximation.* When the interaction cross section of any material is independent of frequency, that material is called a *gray body*. Materials of practical interest seldom show this characteristic. However, when the conditions of a problem are such that the frequency spectrum is known approximately, a frequency averaged cross section can be used. This procedure is called a gray body approximation.

The transfer equation (6-1) averaged over frequency is

$$\frac{1}{c}\frac{\partial I}{\partial t} + \sum \mu_j \frac{\partial I}{\partial x_j} + \bar{\sigma}_a I + \bar{\sigma}_s I = \frac{c}{4\pi}\bar{\sigma}_p a T^4 + \sigma_s \int K(\vec{\mu},\vec{\mu}\,')I(\vec{\mu}\,')d\Omega'. \tag{7-1}$$

Two frequency averaged cross sections appear in this equation,

$$\bar{\sigma}_p(T) = \frac{\int_0^\infty B^\nu(T)\sigma_a^\nu d\nu}{\int_0^\infty B^\nu(T)d\nu}, \quad \bar{\sigma}_a(\vec{x},\vec{\mu},t) = \frac{\int_0^\infty I^\nu(\vec{x},\vec{\mu},t)\sigma_a^\nu d\nu}{\int_0^\infty I^\nu(\vec{x},\vec{\mu},t)d\nu}, \tag{7-2}$$

where $\bar{\sigma}_p(T)$ is the "Planck mean absorption coefficient," and $\bar{\sigma}_a(\vec{x},\vec{\mu},t)$ is the "transport mean absorption coefficient." The frequency averaged equation (7-1) cannot be solved unless the transport mean $\bar{\sigma}_a(\vec{x},\vec{\mu},t)$ is given, and this is generally impossible without a knowledge of the full frequency spectrum $I^\nu(\vec{x},\vec{\mu},t)$. Only in situations where the frequency spectrum is substantially determined by the conditions of the problem can the gray body approximation be used. Two situations of this type occur when the material medium is either optically very thick or very thin. These two cases are discussed below.

(b) *Moment Approximation*. When the transfer equation (6-1) is multiplied by various combinations of the direction cosines (1, μ_k, $\mu_k\mu_\ell$, etc.) and integrated over all solid angle, a series of equations for the moments is obtained

$$\frac{\partial E^\nu}{\partial t} + \sum_j \frac{\partial F_j^\nu}{\partial x_j} + c\sigma_a^\nu E^\nu = 4\pi\sigma_a^\nu B^\nu(T),$$

$$\frac{1}{c}\frac{\partial F_k^\nu}{\partial t} + c\sum_j \frac{\partial}{\partial x_j} P_{jk}^\nu + \sigma_T^\nu F_k^\nu = 0, \qquad (7\text{-}3)$$

etc.

The infinite set of moments is uniquely determined by the angular distribution of the radiation field, and conversley.[11] Thus, the moment equations (7-3) provide an alternate representation of the transfer problem in which the radiation field is described by the sequence of functions $E^\nu(\vec{x},t)$, $F_j^\nu(\vec{x},t)$, $P_{jk}^\nu(\vec{x},t)$,...

Inspection of (7-3) shows that the full set of moment equations is generally required, since the equation for

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[11] The moments determine the Taylor series expansion of the Fourier transform of the angular distribution. See, for example, Kittel (1958), article 25.

each moment contains the next higher one. However, since the angular distribution depends less strongly on the higher order moments, the sequence can be truncated by specifying arbitrarily one of the moments; for example, $P^{\nu}_{jk}(x,t)$. In problems where the angular distribution is known approximately, the truncation relation is essentially determined by the conditions of the problem.

(c) *Optically Thin Materials*. An optically thin material is one in which the interaction between the radiation and the material is very weak. The absorption and scattering coefficients are small, and the mean free paths are long. Most problems of transfer in optically thin materials are such that time changes in E^{ν} and F^{ν}_{j} are determined by the geometry and variations in the external source rather than by interaction with the material. The radiation field at any point is substantially *nonlocal*, and at points several mean free paths from the primary radiation source, the field is nearly all in one direction.

Let us then assume that the radiation is nearly unidirectional (the x direction) in an optically thin material. The pressure tensor has the form

$$P^{\nu}_{jk} = \begin{bmatrix} E^{\nu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the cross sections are small, we have

$$\frac{1}{c}\frac{\partial F_k}{\partial t} >> \sigma_T^\nu F_k^\nu; \qquad (7\text{-}4)$$

that is to say, the time changes in the flux due to absorption and scattering are very small compared to other effects. The second moment equation becomes

$$\frac{1}{c}\frac{\partial F_x^\nu}{\partial t} + c\,\frac{\partial E^\nu}{\partial x} = 0,$$
$$F_y^\nu \simeq F_z^\nu \simeq 0, \qquad (7\text{-}5)$$

and the first moment equation is

$$\frac{\partial E^\nu}{\partial t} + \frac{\partial F_x^\nu}{\partial x} + c\sigma_a^\nu E^\nu = 4\pi\sigma_a^\nu B^\nu(T). \qquad (7\text{-}6)$$

The moment approximation for an optically thin material is described by (7-5) and (7-6) together with the energy equation (6-2). Note that as $\sigma \to 0$, F_x^ν can be eliminated from these equations to give the wave equation. This represents the extreme limit of an optically thin material.

Since the interaction between the radiation and material is weak, the frequency spectrum at any point will not differ much from that of the external source. If it is necessary to use a gray body approximation in the case of an optically thin medium, the appropriate frequency average absorption coefficient in (7-6) is the Planck mean $\bar{\sigma}_P(T_o)$, assuming the source to be a black body at temperature T_o.

(d) *Optically Thick Materials.* In the case of an optically thick material, the radiation field interacts strongly with the transport medium. The absorption and scattering coefficients are large and the mean free paths are very small. The radiation field at any point is substantially *local* and therefore nearly isotropic. Radiative transfer in optically thick materials is a quasi-equilibrium process in which time changes in the radiation field occur slowly relative to the corresponding case in a thin material.

For an isotropic radiation field, the pressure tensor is

$$P_{jk} = \begin{bmatrix} \frac{1}{3}E^{\nu} & 0 & 0 \\ 0 & \frac{1}{3}E^{\nu} & 0 \\ 0 & 0 & \frac{1}{3}E^{\nu} \end{bmatrix} .$$

In this case, the cross sections are large, giving

$$\frac{1}{c}\frac{\partial F_k^{\nu}}{\partial t} << \sigma_T^{\nu} F_k^{\nu}. \qquad (7\text{-}7)$$

The second moment equation reduces to the relation for flux in a diffusing medium (Fick's law),

$$F_k^{\nu} = -\frac{c}{3}\lambda_T^{\nu}\frac{\partial E^{\nu}}{\partial x_k}, \qquad (7\text{-}8)$$

and the first moment equation is

$$\frac{\partial E^{\nu}}{\partial t} + \sum_k \frac{\partial F_k^{\nu}}{\partial x_k} + c\sigma_a^{\nu}E^{\nu} = 4\pi\sigma_a^{\nu}B^{\nu}(T). \qquad (7\text{-}9)$$

The moment approximation for radiative transfer in an optically thick material is described by (7-8) and (7-9) together with the energy equation (6-2). This formulation is called the *nonequilibrium diffusion formulation* and differs from equilibrium diffusion (described below) in that E^{ν} is calculated from (7-9) rather than set equal to $B^{\nu}(T)$.

If it is necessary to use a gray body approximation in the optically thick case, one can assume that the frequency dependence of E^ν is the same as $B^\nu(T)$. In this case, the flux equation becomes

$$F_k = -\frac{c}{3}\bar{\lambda}_R \frac{\partial E}{\partial x_k}, \tag{7-10}$$

where the "Rossland mean free path" is defined by

$$\bar{\lambda}_R = \frac{\displaystyle\int_0^\infty \frac{1}{\sigma_T^\nu}\frac{\partial B^\nu}{\partial T}d\nu}{\displaystyle\int_0^\infty \frac{\partial B^\nu}{\partial T}d\nu}. \tag{7-11}$$

The equation for E then becomes

$$\frac{\partial E}{\partial t} + \sum_k \frac{\partial F_k}{\partial x_k} + c\bar{\sigma}_P(T)E = \bar{\sigma}_P(T)acT^4. \tag{7-12}$$

The equations (7-10), (7-11), and (7-12) together with the energy equation (6-2) describe the gray body approximation for nonequilibrium diffusion.

(e) *Equilibrium Diffusion.* Consider the energy balance equation in the form

$$\frac{1}{c}\frac{\partial}{\partial t} E_m(T) = \int_0^\infty \sigma_a^\nu\left[E^\nu - \frac{4\pi}{c}B^\nu(T)\right] d\nu. \tag{7-13}$$

When σ_a^ν is large, the material heats up rapidly until $E^\nu \simeq \frac{4\pi}{c}B^\nu(T)$ at any point. In the nonequilibrium diffusion approximation (d), E^ν is calculated by (7-9) and therefore allowed to differ from $\frac{4\pi}{c}B^\nu(T)$.

When σ_a^ν is exceedingly large for all frequencies, the energy density can be set equal to $\frac{4\pi}{c}B^\nu(T)$. This is called *equilibrium diffusion*, since the radiation and material temperatures have come into equilibrium with each other. Under these conditions, the frequency spectrum of the radiation is strictly determined, making a frequency dependent calculation unnecessary.

The equations of the equilibrium diffusion formulation are

$$E = aT^4, \qquad F_j = -\frac{c}{3}\bar{\lambda}_R \frac{\partial E}{\partial x_j},$$

$$\frac{\partial E_m}{\partial t} + \frac{\partial E}{\partial t} + \sum_j \frac{\partial F_j}{\partial x_j} = 0. \tag{7-14}$$

Note that the differential form of the energy equation (6-2) must be used, since in this approximation the difference between E^ν and $\frac{4\pi}{c}B^\nu(T)$ which is required in the integral form (7-13) is not given by the calculation.

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Chapter II *Interaction of Matter and Radiation*

§8. *Basic Radiative Processes*

All of the forms of the transfer equation given in Table I on page 16 follow from geometry and the conservation of radiant energy. However, the laws of atomic physics are needed to determine the absorption and scattering cross sections and source functions. The calculation of these quantities requires a study of atomic and molecular processes together with the quantum theory of radiation (quantum electrodynamics).

In quantum electrodynamics, the atoms (or molecules) and the radiation field are considered as mutually interacting quantum systems. As is usual in quantum theory, the interaction between an atom and the radiation field is represented by an operator, $\hat{H}_{int}$. The interaction causes a perturbation that induces transitions between the possible states of the collective system of matter and radiation. The probability that a particular event (transition) will occur is proportional to the square of the matrix element of $\hat{H}_{int}$. The transition probability gives the cross section a^{ν}, and the cross section gives the inverse mean free path σ^{ν} for the interaction.

An excellent introduction to quantum electrodynamics is given by Leighton (1959), Chapters 2 and 6. The standard reference on the subject is Heitler (1954). A useful summary of radiation theory is given by Condon and Shortly (1957), Chapter IV. References on atomic and molecular physics are Leighton (1959), Eisberg (1961), and Richtmyer, Kennard, and Lauritsen (1955). More advanced treatments are given by Bates (1962) and Slater (1960).

a.) *Absorption.* The principal mechanisms of absorption of radiation by matter are bound-bound (line) absorption, bound-free (photoelectric) absorption, and free-free absorption. In line absorption, an electron in a bound state is excited to another bound state of higher energy by the absorption of a quantum of radiation (photon). The frequency of the absorption line is given by Bohr's relation, $h\nu_{nm} = E_n - E_m$, where E_n and E_m are the higher and lower energy states respectively. In photoelectric absorption, the electron is ejected from the atom and goes into one of the continuum of free energy states. Photoelectric absorption occurs whenever the energy of the incident photon is greater than the binding energy of the atom. In free-free absorption, an electron in a free state makes a transition to another free state of higher energy with the absorption of a photon. Fig. 8-1 shows the usual arrangement of energy levels and the possible transitions.

Similar processes occur when the material is composed of molecules. Molecular processes are somewhat more complicated because of the extra degrees of freedom (vibrational and rotational) that are present when atoms join together to form molecules. Two important references on molecular spectroscopy are Herzberg (1950) and Penner (1959).

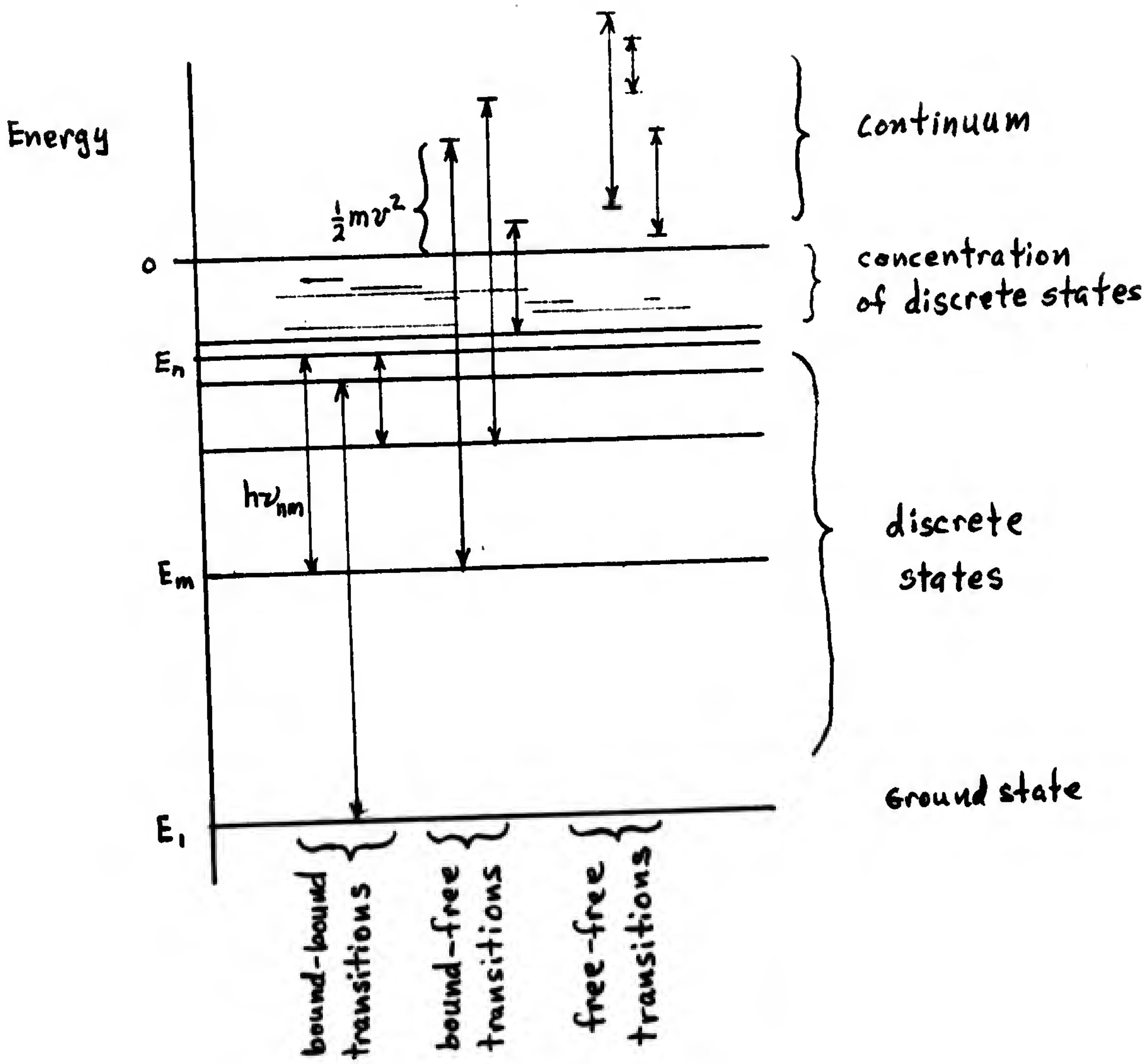

Fig. 8-1. Atomic Energy Levels and Transitions

Two absorption processes that occur at high (γ-ray) energies are electron-positron pair production and nuclear absorption. These processes are seldom of interest in the usual radiative transfer problems.

b.) *Emission.* Each absorption process has an inverse that results in the emission of radiation. The emission processes which correspond to bound-bound, bound-free, and free-free absorption are line emission, electron capture, and bremsstrahlung. The inverse of pair production is electron-positron annihilation. A nucleus in an excited state can become de-excited by the emission of a γ-ray.

When the allowed energy levels of any atomic system are maintained in an equilibrium population given by Boltzmann's law (4-8), the emission source function is given by Kirchhoff's law (4-10).

c.) *Scattering.* The scattering of electromagnetic radiation by an atom involves the absorption of a primary quantum and the nearly simultaneous emission of a secondary quantum in a different direction. If the frequency of the primary and secondary quanta are the same, the process is called *coherent scattering.* In some cases the scattered photon will not have the same frequency as the primary photon. During the scattering process, the atom may become excited or partially de-excited by collision or by spontaneous emission. The scattering process results in a change in frequency and is called *Raman scattering.*

Two special cases occur in coherent scattering. When the frequency of the primary photon is much different from the frequency of a line (i.e. a frequency of resonance), the scattering is called *Rayleigh scattering*. When the frequency of the incident photon is the same as that of a line, the process is called *resonance scattering* (or resonance flourescence).

Scattering by free electrons is called *Compton scattering*. In the nonrelativistic limit ($h\nu \ll mc^2$), Compton scattering reduces to classical Thompson scattering. The scattering cross section for bound electrons (coherent and Raman scattering) is small compared to the photoelectric cross section and of the same general form as the Compton cross section. It is usually sufficient, therefore, to assume that all electrons are free when evaluating the total effect of scattering by a material.

Photon-photon scattering has been shown to be negligible compared to Compton scattering at the usual temperatures and densities of interest.[12]

d.) *Energies and Temperatures.* The energy range where atomic radiative processes occur can be determined from the expression for the energy levels of a completely ionized (hydrogen-like) atom:

$$E_n = -\frac{2\pi^2 e^4 m}{h^2}\frac{z^2}{n^2} = -0.0136\frac{z^2}{n^2}\text{kev}, \qquad (8\text{-}1)$$

---

[12] D.H. Sampson, Astrophys.J.<u>129</u>, 734-751 (1959).

where Z is the atomic number, and n is an integer that labels the energy levels.[13] The energy levels run from about 0.01 kev for hydrogen to about 100 kev for fully ionized uranium. The wavelength of radiation in this energy range is obtained from

$$E = h\nu = \frac{hc}{\lambda}. \tag{8-2}$$

The wavelength λ ranges from 1300 Å to 0.13Å, where the angstrom unit Å is equal to 10^{-8}cm. This radiation lies in the ultraviolet and x-ray regions of the electromagnetic spectrum.

Molecular binding energies are typically several electron volts. The corresponding radiation falls in the visual and infared regions of the spectrum.

Pair production and nuclear processes occur for energies greater than 1 mev and wavelengths less than 0.01 Å. Radiation thus falls into the γ-ray region of the spectrum.

Material at temperature T radiates most of its energy in the neighborhood of the frequency at which the Planck function is maximum. According to (4-5) we have

$$h\nu_{max} = 2.8214kT.$$

---

[13] The unit of energy is the electron volt (ev). One ev is defined as the kinetic energy acquired by an electron in falling through a potential difference of one volt: 1 ev = $1.602 \cdot 10^{-12}$ erg, 1 kev = 10^3 ev, 1 mev = 10^6 ev.

If we change the energy units, the Boltzmann constant becomes

$$k^{-1} = 11{,}606\ {}^{\circ}\mathrm{K/ev} = 1.16\cdot 10^{7}\ {}^{\circ}\mathrm{K/kev}. \qquad (8\text{-}3)$$

Thus, photon energies in the range 0.01 kev to 100 kev correspond to emission temperatures in the range $4\cdot 10^{4}$ °K to $4\cdot 10^{8}$ °K. Molecular dissociation temperatures are in the neighborhood of 10^{4} °K.

§9. *Einstein's Transition Coefficients*

Consider an atom with energy levels E_n $(n=1,2,\ldots)$ in the presence of a radiation field. Transitions between any two states E_n and E_m, where $E_n > E_m$, are associated with the emission and absorption of radiation of frequency ν_{nm}, where

$$h\nu_{nm} = E_n - E_m. \qquad (9\text{-}1)$$

The Einstein transition coefficients A_{nm} and B_{nm} are defined as follows: the probability per unit time that the atom will make a transition from E_m to E_n with the absorption of a quantum of radiation is

$$P_{mn} = B_{mn} I^{\nu_{nm}}, \qquad (9\text{-}2a)$$

and the probability per unit time that the atom will make a downward transition from E_n to E_m with the emission of radiation is

$$P_{nm} = A_{nm} + B_{nm} I^{\nu_{nm}}. \qquad (9\text{-}2b)$$

The expression (9-2b) shows that the probability of transition to a lower energy is increased by the presence of an external radiation field. This is *induced* or *stimulated emission*. The coefficient A_{nm} describes the rate at which the atom undergoes *spontaneous emission* in the absence of a radiation field. The coefficient B_{nm} gives the rate at which the radiation field induces transitions, both upward and downward in energy, between the various states of the atom.

Although the spontaneous emission of photons is isotropic, induced emission has the same angular dependence as $I^{\nu_{nm}}(\vec{x},\vec{\mu},t)$. When the expressions (9-2) are properly normalized, they can be considered as the probability per unit time that a photon is emitted or absorbed in a unit solid angle. In terms of these probabilities, the energy emitted by one atom in time dt into the element of solid angle $d\Omega$ is

$$dW = h\nu_{nm}(A_{nm}+B_{nm}I^{\nu_{nm}})d\Omega\ dt,$$

and the corresponding energy absorbed is

$$dW = h\nu_{nm}\ B_{mn}\ I^{\nu_{nm}}\ d\Omega\ dt.$$

If N_n is the number of atoms per unit volume in the energy state E_n, the transfer equation for radiation of frequency ν_{nm} is

$$\frac{dI^{\nu_{nm}}}{ds} = N_n(A_{nm}+B_{nm}I^{\nu_{nm}})h\nu_{nm} - N_m\ B_{mn}\ I^{\nu_{nm}}\ h\nu_{nm}. \qquad (9\text{-}3a)$$

This equation can be rewritten to give

$$\frac{dI^{\nu_{nm}}}{ds} = N_n A_{nm} h\nu_{nm} - N_m B_{mn}\left(1-\frac{N_n B_{nm}}{N_m B_{mn}}\right) I^{\nu_{nm}} h\nu_{nm}. \qquad (9\text{-}3b)$$

Equations (9-3) give the equation of transfer when local thermodynamic equilibrium is not assumed. Let us now consider the radiation field in an isothermal enclosure. Since the system is in thermodynamic equilibrium, the rate of upward and downward transitions is the same, and from (9-3a) we have $dI^{\nu_{nm}}/ds = 0$. Under the condition of thermodynamic equilibrium, (9-3b) gives

$$N_n A_{nm} = N_m B_{mn}\left(1-\frac{N_n B_{nm}}{N_m B_{mn}}\right) I^{\nu_{nm}}.$$

This expression can be rearranged to yield

$$\frac{A_{nm}}{B_{nm}} = \frac{N_m}{N_n}\,\frac{B_{mn}}{B_{nm}}\left(1-\frac{N_n B_{nm}}{N_m B_{mn}}\right) I^{\nu_{nm}},$$

or, solving for $I^{\nu_{nm}}$, we obtain

$$I^{\nu_{nm}} = \frac{A_{nm}/B_{nm}}{\left(\frac{N_m B_{mn}}{N_n B_{nm}}-1\right)}. \qquad (9\text{-}4)$$

From §4 we know that the intensity of radiation in an isothermal enclosure at temperature T is given by the Planck function (4-3),

$$\frac{A_{nm}/B_{nm}}{\left(\frac{N_m B_{mn}}{N_n B_{nm}}-1\right)} = \frac{2}{(hc)^2}\,\frac{(h\nu_{nm})^3}{e^{h\nu_{nm}/kT}-1},$$

which gives the following relations for the transition coefficients:

$$\frac{A_{nm}}{B_{nm}} = \frac{2}{(hc)^2}(h\nu_{nm})^3, \quad e^{h\nu_{nm}/kT} = \frac{N_m B_{mn}}{N_n B_{nm}}. \tag{9-5}$$

It is possible to specify the transition coefficients further, since we know that in thermodynamic equilibrium the energy levels are populated by the Boltzmann distribution[14]

$$\frac{N_n}{N_m} = \frac{g_n}{g_m} e^{-(E_n-E_m)/kT} = \frac{g_n}{g_m} e^{-h\nu_{nm}/kT}. \tag{9-6}$$

When the exponential in (9-5) is evaluated by (9-6), we have[15]

$$A_{nm} = \frac{2}{(hc)^2}(h\nu_{nm})^3 B_{nm}, \tag{9-7}$$

$$g_m B_{mn} = g_n B_{nm}$$

Note that the transition coefficients A_{nm} and B_{nm} are fundamental measures of the interaction between an atom and a radiation field. They have the same values whether the material is in thermal equilibrium or not. We have used the case of thermal equilibrium as a simple way to derive the relations (9-7) between the different transition coefficients, but these relations do not depend on the equilibrium condition. We only need to calculate one set of coefficients, for example the A_{nm}, and use the relations (9-7) to determine the others.

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[14] See Richtmyer, Kennard, and Lauritsen (1955), p.168.

[15] The quantity g_n that appears in (9-6) and (9-7) is the statistical weight or degeneracy of the level n. The degeneracy is the number of different states with the same energy E_n. A degenerate level would be split into g_n different states if the degeneracy were removed by an external perturbation.

The Einstein A_{nm} coefficient has a useful physical interpretation. Consider N_n atoms in the state E_n in the absence of a radiation field. The probability that any one atom will decay in time dt is $A_{nm}dt$, and therefore the total number of decays in time dt is

$$dN_n = N_n \sum_m A_{nm}\, dt,$$

or

$$\frac{dN_n}{N_n} = -\sum_m A_{nm}\, dt,$$

where the summation is over all states $m<n$. If the original number of atoms is $(N_n)_o$, the number at any time is given by

$$N_n = (N_n)_o e^{-\sum_m A_{nm} t}.$$

We can define the mean life of the state E_n as the time required for a large number of excited atoms to decay to e^{-1} of their original number. The mean life τ is given in terms of the transition coefficients by

$$\tau_n = (\sum_m A_{nm})^{-1}. \tag{9-8}$$

The mean lifetimes for the hydrogen energy states are[16]

n	$\tau_n (10^{-8}$sec)
2	0.21
3	1.02
4	3.35
5	8.8
6	19.6

---

[16] Condon and Shortly (1957), p. 137.

The transfer equation (9-3) is valid for any population of the energy levels. If we assume that the Boltzmann population (9-6) is approximately valid in the neighborhood of every point in the transport medium, the transfer equation for LTE can be obtained. If we rearrange the two factors that appear in (9-3b),

$$N_n A_{nm} h\nu_{nm} = (N_m B_{mn} h\nu_{nm}) \frac{A_{nm}}{B_{nm}} \frac{N_n B_{nm}}{N_m B_{mn}}$$

$$= (N_m B_{mn} h\nu_{nm}) \frac{2}{(hc)^2} (h\nu_{nm})^3 e^{-h\nu_{nm}/kT},$$

and

$$N_m B_{mn} h\nu_{nm} \left(1 - \frac{N_n B_{nm}}{N_m B_{mn}}\right) = (N_m B_{mn} h\nu_{nm}) \left(1 - e^{-h\nu_{nm}/kT}\right),$$

we obtain the transfer equation in the form

$$\frac{dI^{\nu_{nm}}}{ds} = (N_m B_{mn} h\nu_{nm}) \left(1 - e^{-h\nu_{nm}/kT}\right) \left[B^{\nu_{nm}}(T) - I^{\nu_{nm}}\right]. \quad (9\text{-}9)$$

From this expression, the absorption coefficient is seen to be

$$\sigma^{\nu_{nm}} = (N_m B_{mn} h\nu_{nm}) \left(1 - e^{-h\nu_{nm}/kT}\right). \quad (9\text{-}10)$$

Thus, the absorption coefficient is a product of two factors, the *true absorption* coefficient

$$\sigma'^{\nu_{nm}} = N_m B_{mn} h\nu_{nm}, \quad (9\text{-}11)$$

and the correction for induced emission

$$1 - e^{-h\nu_{nm}/kT}.$$

When the absorption coefficient is expressed in this way, stimulated emission is considered as *negative absorption.* The effect of stimulated emission is to reduce the true absorption coefficient by the factor $(1-e^{-h\nu_{nm}/kT})$. The mass absorption coefficient corresponding to (9-11) is

$$\kappa'_{\nu nm} = \frac{N_m}{\rho} B_{mn} h\nu_{nm},$$

where N_n is the number of atoms per unit volume and ρ is the mass density.

From the results of this section, it is clear that the number of atoms per unit volume in a certain energy state is a quantity of fundamental importance in calculating the transfer of radiation. Although the Boltzmann distribution (9-6) gives the relative numbers, the absolute numbers are needed. The calculation of the state of excitation of the material is the subject of the next section. The results will be occupation numbers for the various energy levels in the atom and the number of free electrons per atom.

§10. *Energy Levels and Occupation Numbers*

The first steps in obtaining the absorption and scattering coefficients are to determine the energy levels E_j, the number of bound electrons for each level n_j (the occupation numbers), and the number of free electrons per atom n_f. The energy levels E_j depend on the state of ionization, since the potential energy function for the atom depends on the extent to which the nuclear charge Ze is screened by the cloud of bound electrons. For a fully ionized (hydrogen-like) atom, the energy levels are given by the usual Bohr formula. The energy levels for a partially ionized atom are given by

$$E_j = -\frac{2\pi^2 me^4}{h^2}\left(\frac{Z_j^*}{j}\right)^2, \qquad (10\text{-}1)$$

where Z_j^* is the *effective nuclear charge* seen by an electron in the state E_j. Z_j^* is given in terms of the *Slater screening constants*[17] σ_{jk} by

$$Z_j^* = Z - \sum_{k\neq j} n_k\sigma_{jk} + n_j\left(1-\frac{1}{g_j}\right)\sigma_{jj}, \qquad (10\text{-}2)$$

where $g_j = 2j^2$ is the degeneracy of the level E_j in a hydrogen-like atom. The screening constants are given in Table III.

The energy levels given by (10-1) must be corrected for the interaction of the bound electrons with free electrons in the neighborhood of the ion. The average ion has a net positive charge $Z'e$, where

$$Z' = Z - \sum_j n_j. \qquad (10\text{-}3)$$

Let $a_{Z'}$ be the radius of the spherical region surrounding the ion which contains zero net charge; that is to say, Z' free electrons. Any free electron outside this radius has no interaction with the ion. The region of interaction is called the ion sphere (see Fig. 10-1).

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[17] J. C. Slater, Phys. Rev. 36, 57 (1930).

The radius of the ion sphere is calculated as follows: The density of free electrons is

$$N_e = \frac{\rho}{AM} n_f \, \frac{\text{free electrons}}{\text{cm}^3} \tag{10-4}$$

where ρ is the mass density, A the atomic weight, M the mass of a nucleon, and n_f the number of free electrons per atom. The radius of the ion sphere is then given by

$$Z' = \frac{4}{3}\pi a_{Z'}^3 N_e,$$

or

$$a_{Z'} = \left(\frac{3}{4\pi}\frac{Z'}{N_e}\right)^{\frac{1}{3}}. \tag{10-5}$$

The electrostatic potential due to the free electrons within the ion sphere is

$$\phi_f(r) = -\frac{Z'e}{2a_{Z'}}\left[3-\frac{r^2}{a_{Z'}^2}\right].$$

In order to obtain the contribution to the energy levels, the potential ϕ_f must be averaged over the charge distribution $-e\psi_j^*\psi_j$ of the bound state described by the wave function ψ_j. With this correction, the energy levels become

$$E_j = -\frac{2\pi^2 me^4}{h^2}\left(\frac{Z_j^*}{j}\right)^2 + \frac{Z'e^2}{2a_{Z'}}\left[3 - \frac{\overline{(r^2)}_j}{a_{Z'}^2}\right]. \tag{10-6}$$

where $\overline{(r^2)}_j = \int \psi_j^*\psi_j r^2 d^3x$ is the mean square radius for the electron orbital ψ_j. The mean square radius for hydrogen-like atoms is given by Pauling and Wilson (1935) p.144:

$$\overline{(r^2)}_{j,\ell} = a_o^2\frac{j^4}{(Z_j^*)^2}\left\{1 + \frac{3}{2}\left[1-\frac{\ell(\ell+1)-1/3}{j^2}\right]\right\},$$

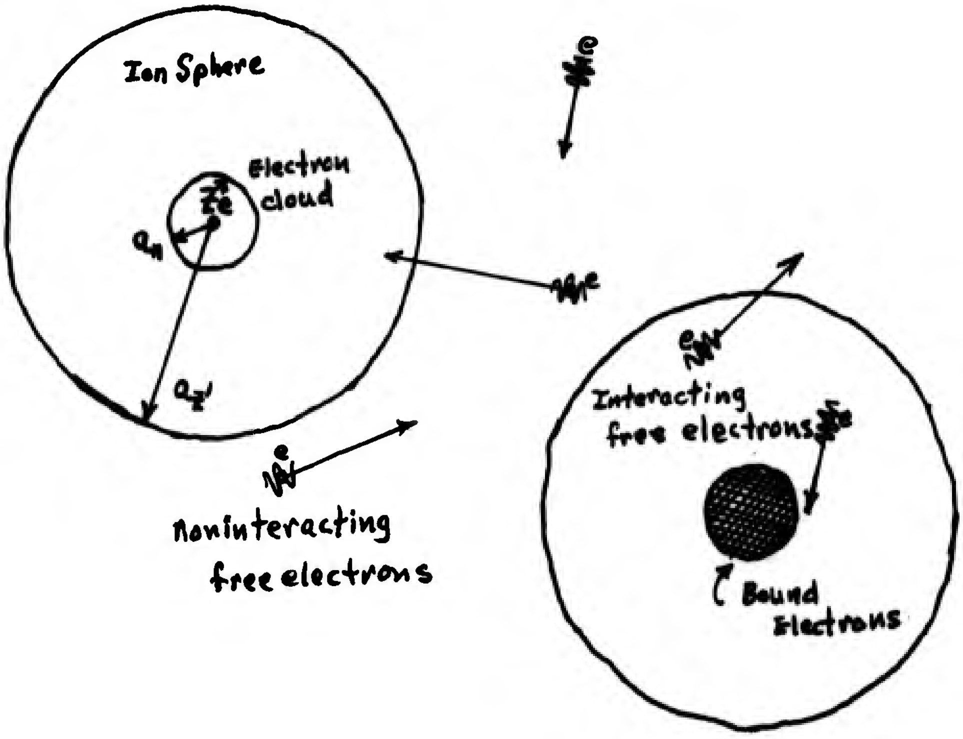

Fig. 10-1 The Ion Sphere

and if we average over the angular momentum states ℓ, we have

$$\overline{(r^2)}_j = a_o^2 \frac{2j^4}{(Z^*_j)^2} \left\{ 1 + \frac{1}{2j^2} \right\}. \qquad (10\text{-}7)$$

One further correction to the energy levels is needed. A free electron in the ion sphere has a certain interaction energy with the ion and with the other free electrons. For this reason, an electron in the ion sphere with zero kinetic energy will not have zero total energy according to (10-6). We therefore add a small correction to (10-6), so that any electron in the ion sphere has positive total energy and is therefore a free electron. With this correction,[18] the energy levels become

$$E_j = - \frac{2\pi^2 me^4}{h^2} \left(\frac{Z^*}{j} \right)^2 + \frac{Z' e^2}{2a_{Z'}} \left[3 - \frac{\overline{(r^2)}_j}{a_{Z'}^2} \right] + \frac{3}{5} \frac{Z'^2 e^2}{2a_{Z'}} \frac{1}{n_f}. \qquad (10\text{-}8)$$

The energy levels can be obtained from (10-8) if the occupation numbers n_j are known. The effective Zs are given by (10-2) and Table III; the radius of the ion sphere is given by (10-5) with (10-3) and (10-4); and the mean square radii $\overline{(r^2)}_j$ are given by (10-7). Note that with materials composed of a single element (A,Z), Z' is equal to the number of free electrons per atom n_f. In the case of a material composed of molecules, each molecule having N_Z atoms of atomic number Z, the number of free

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[18] Mayer (1947), Chapter IV.

electrons per molecule becomes $n_f = \sum_Z N_Z Z'$. For the case of a single element, the radius of the ion sphere is simply the mean distance between the atoms.

In order to get the occupation numbers n_j, we must use quantum statistical mechanics to find the probability that a given level is occupied. According to Fermi-Dirac statistics,[19] the probability that a state of energy E is occupied at temperature T is

$$f(E) = \frac{1}{\exp\left(\frac{E-\mu}{kT}\right) + 1},$$

where μ is called the chemical potential or Fermi energy. Since the degeneracy of hydrogen-like levels is $2j^2$, the occupation numbers can be written

$$n_j = \frac{2j^2}{\exp\left(\alpha + \frac{E_j}{kT}\right) + 1}, \qquad (10\text{-}9)$$

where $\mu = -\alpha kT$. The quantity α is not easy to calculate,[20] but it is given approximately by

$$e^{\alpha} \simeq \frac{2}{N_e}\left(\frac{2\pi mkT}{h^2}\right)^{3/2} = \frac{G(T)}{N_e}.$$

The expression for the occupation numbers then becomes

$$n_j = \frac{2j^2}{1 + \frac{G(T)}{N_e}\, e^{E_j/kT}}, \qquad (10\text{-}10)$$

where

$$G(T) = 2\left(\frac{2\pi mkT}{h^2}\right)^{3/2}. \qquad (10\text{-}11)$$

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[19] Kittel (1958), section 19.

[20] Mayer (1947), Chapter IV.

The complete set of equations giving the energy levels and occupation numbers are collected together in Table IV. Note that both N_e and E_j depend on n_j. The procedure is to guess at the n_j initially and iterate the set of equations until they converge to a consistent set of values for n_j, N_e, and E_j.

A more complete treatment of energy levels, occupation numbers, and the general problem of opacities is given by Mayer (1947) and Cox (1965). A number of important papers were presented at both of the Air Force Weapons Laboratory Conferences on Opacity.[21]

§11. *Continuous Absorption*

In this section we investigate the cross sections and absorption coefficients for bound-free and free-free absorption. These cross sections were first calculated by Kramers and Gaunt on the basis of semi-classical radiation theory.[22] An excellent discussion of continuous absorption is given by Chandrasekhar (1939), p.262.

a.) *Bound-free absorption.* The cross section for photoelectric absorption is given by

$$
\begin{aligned}
a_{bf}^{\nu}(j) &= \frac{64\pi^4 m e^{10}}{3\sqrt{3}\, c\, h^3} \frac{(Z^*)^4}{j^5} \frac{g_{bf}(j)}{(h\nu)^3}, && h\nu \geq |E_j| \\
a_{bf}^{\nu}(j) &= 0, && h\nu < |E_j|. \qquad (11\text{-}1)
\end{aligned}
$$

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[22] H. A. Kramers, Phil. Mag. 46, 836 (1923), and J.A. Gaunt, Phil. Trans. R. Soc. London, A229, 163 (1930).

[21] Papers of the first Conference, J. Quant. Spectrosc. Radiat. Transfer 4, 583-761 (1964); papers of the second Conference, J. Quant. Spectrosc. Radiat. Transfer 5, 3-271 (1965).

TABLE III

Slater Screening Constants σ_{ij}

j \ i	1s	2s	2p	3s	3p	3d
1s	.6250	.8395	.9712	.8954	.9795	.9992
2s	.2099	.6016	.6484	.7570	.8101	.9322
2p	.2428	.6484	.7266	.7808	.8455	.9570
3s	.0995	.3365	.3470	.5977	.6191	.6582
3p	.1088	.3600	.3758	.6191	.6464	.6924
3d	.1110	.4143	.4253	.6582	.6924	.7744

$\bar{j}$ \ $\bar{i}$	1	2	3	4	5	6	7	8	9
1	.6250	.9383	.9811	.987	.994	.997	.999	1.000	1.000
2	.2346	.6895	.8932	.94	.97	.984	.990	.993	.995
3	.1090	.3970	.7018	.85	.92	.955	.97	.98	.99
4	.06169	.2350	.4781	.705	.83	.90	.95	.97	.98
5	.03976	.1552	.3312	.531	.72	.83	.90	.95	.97
6	.02769	.1093	.2388	.400	.584	.735	.83	.90	.95
7	.02039	.08082	.1782	.3102	.459	.610	.745	.83	.90
8	.01563	.06250	.1378	.2425	.371	.506	.635	.750	.83
9	.01234	.04938	.1106	.1936	.299	.431	.544	.656	.760
10	.01000	.04000	.0900	.1584	.245	.353	.466	.576	.67

TABLE IV

Energy Levels and Occupation Numbers for a Single Element (A,Z)

1. Occupation numbers:

$$n_j = \frac{2j^2}{1+\frac{G(T)}{N_e}e^{E_j/kT}},$$

$$G(T) = \frac{2}{h^3}(2\pi mkT)^{3/2},$$

2. Number of free electrons:

$$n_f = Z' = Z - \Sigma n_j \quad \frac{\text{free electrons}}{\text{atom}},$$

$$N_e = \frac{\rho}{AM}n_f \quad \frac{\text{free electrons}}{\text{cm}^3},$$

M = atomic mass unit,

3. Energy levels:

$$E_j = -\frac{2\pi^2 me^4}{h^2}\left(\frac{Z_j^*}{j}\right)^2 + \frac{Z'e^2}{2a_{Z'}}\left[3-\frac{\overline{(r^2)}_j}{a_{Z'}^2}\right] + \frac{3}{5}\,\frac{Z'^2e^2}{2a_{Z'}}\,\frac{1}{n_f},$$

4. Effective nuclear charge:

$$Z_j^* = Z - \sum_{k\neq j} n_k\,\sigma_{jk} + n_j\left(1-\frac{1}{2j^2}\right)\sigma_{jj},$$

σ_{jk} = screening constants in Table III,

5. Radius of ion sphere:

$$a_{Z'} = \left(\frac{3}{4\pi}\,\frac{Z'}{N_e}\right)^{1/3},$$

6. Mean square radius of electron orbital:

$$\overline{(r^2)}_j = a_0^2\,\frac{j^4}{(Z_j^*)^2}\left\{1+\frac{1}{2j^2}\right\},$$

a_0 = first Bohr radius.

In this expression $g_{bf}(j)$ is the *Gaunt factor* for the level j. The Gaunt factor corrects the semi-classical result for quantum effects and is given approximately by[23]

$$g_{bf}(j) \cong 1 - 0.1728 \left[\frac{\nu}{R(Z^*_j)^2}\right]^{1/3} \left\{\frac{2}{j^2}\left[\frac{R(Z^*_j)^2}{\nu}\right] - 1\right\}, \qquad (11\text{-}2)$$

where R is the Rydberg constant, $R = 2\pi^2 me^4/ch^3 = 0.0136$ kev/h. The Gaunt factors are small corrections and can often be neglected. The cross section (11-1) is given in cm^2/electron. When the constants are evaluated, the result is

$$a^{\nu}_{bf}(j) = 8.19 \cdot 10^{-50} \frac{(Z^*_j)^4}{j^5} \frac{g_{bf}(j)}{(h\nu)^3} \frac{\text{cm}^2}{\text{electron}},$$

($h\nu$ in ergs)

or $$a^{\nu}_{bf}(j) = 1.99 \cdot 10^{-23} \frac{(Z^*_j)^4}{j^5} \frac{g_{bf}(j)}{(h\nu)^3} \frac{\text{cm}^2}{\text{electron}},$$

($h\nu$ in kev.)

Note that the cross section for an electron in the energy level E_j is zero unless $h\nu \geq |E_j|$. Only if the radiation has enough energy to remove an electron to the continuum, can that electron contribute to photoelectric absorption. The total photoelectric cross section will be given by

$$a^{\nu}_{bf} = \sum_j n_j \, a^{\nu}_{bf}(j), \qquad |E_j| \leq h\nu. \qquad (11\text{-}3)$$

In (11-3) n_j is the number of electrons in level j, and the sum extends only over those levels for which the

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[23] D. H. Menzel and C. H. Pekeris, Royal Astronom. Soc. Monthly Notices 96, 77 (1935).

ionization energy $|E_j|$ is less than the radiation energy $h\nu$. The cross sections for different levels are shown in Fig. 11-1.

The total cross section (11-3) is given in cm^2/atom, since we have summed over all bound electrons. The photoelectric absorption coefficient is then given by

$$\sigma_{bf}^{\nu} = \frac{\rho}{AM} \Sigma\, n_j\, a_{bf}^{\nu}(j) \qquad cm^{-1},$$

where ρ/AM gives the number of atoms/cm^3. The absorption coefficient is more conveniently expressed in the form

$$\sigma_{bf}^{\nu} = \rho\, D_k \;(h\nu)^{-3}, \qquad (11\text{-}4a)$$

where k is given by

$$|E_k| \leq h\nu < |E_{k+1}|,$$

and

$$D_k = \frac{1}{AM} \frac{64\pi^4 me^{10}}{3\sqrt{3}\; c\; h^3} \sum_{j=k}^{J} n_j \frac{(Z_j^*)^4}{j^5} g_{bf}(j). \qquad (11\text{-}4b)$$

The sum in (11-4b) starts at k, the first level for which $h\nu$ is greater than the ionization energy, and extends up to some maximum value J. Since the contribution of the higher levels falls off as $(h\nu)^{-3}$, it is usually sufficient to terminate the sum at $J = 5$ or 6.

It should be noticed that the absorption coefficient depends strongly on the temperature through n_j and Z_j^* because of ionization. The positions of the edges $|E_j|$ are temperature dependent as well. The form of the absorption coefficient is shown in Fig. 11-2. The K, L,... edges represent the ionization energies for the

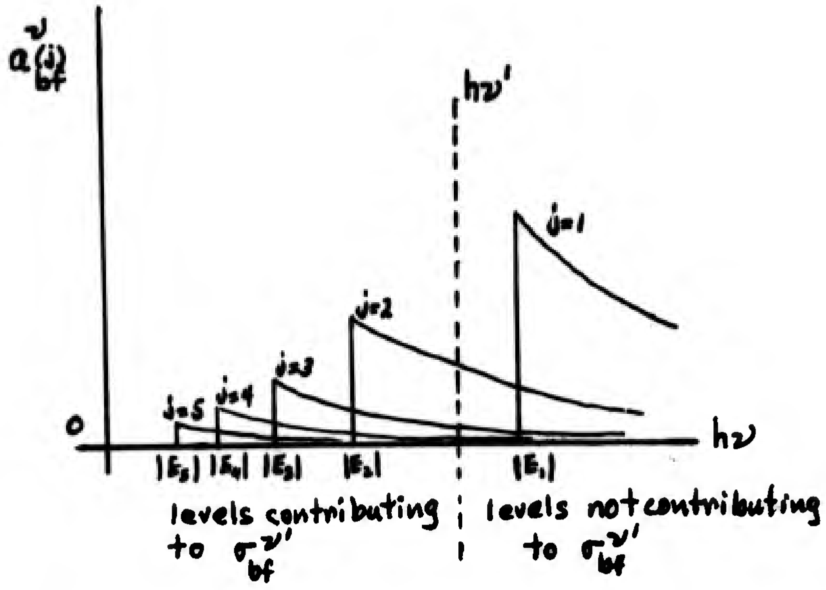

Fig. 11-1 Photoelectric Cross Sections for Bound Electrons

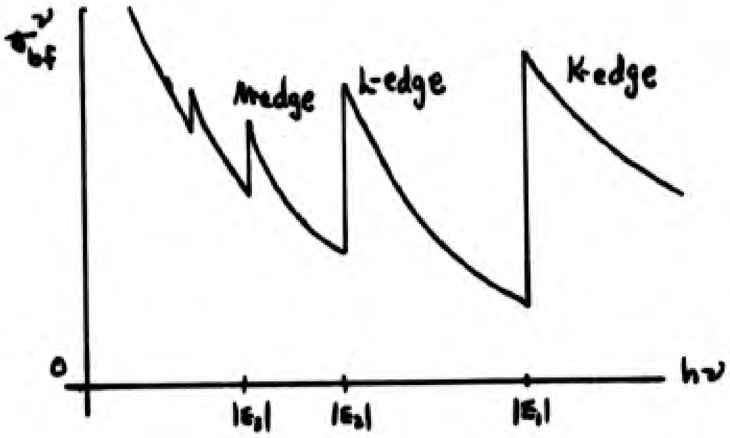

Fig. 11-2 Photoelectric Absorption Coefficient

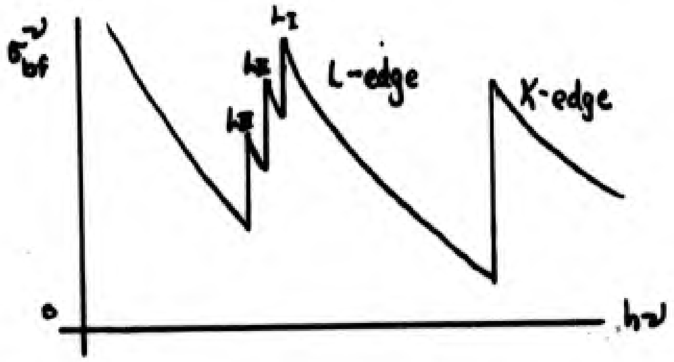

Fig. 11-3 Fine Structure of the L-edge

K, L,... shells. Actually the L, M,... edges have a fine structure, as shown in Fig. 11-3, due to the different binding energies of the angular momentum subshells. The corrections for the subshells are small and have been ignored in the relation for the energy levels (10-1). These corrections arise from relativity and spin-orbit coupling.

b.) *Free-free absorption.* The cross section for free-free absorption is given by

$$a_{ff}^{\nu} = \frac{16\pi^2 e^6 h^2}{3\sqrt{3}\, c\, (2\pi m)^{3/2}} \frac{N_e}{\sqrt{kT}} \frac{(Z')^2}{(h\nu)^3}\, g_{ff}, \qquad (11\text{-}5)$$

where N_e is the number of free electrons per unit volume, and $Z'e$ is the effective charge of the ion as seen by a free electron:

$$Z' = Z - \sum_j n_j,$$

$$N_e = \frac{\rho}{AM} Z'.$$

The free-free Gaunt factor is given by Menzel and Pekeris as

$$g_{ff} \cong 1 + 0.1728 \left[\frac{\nu}{R(Z')^2}\right]^{1/3} \left[1+2\frac{kT}{h\nu}\right] . \qquad (11\text{-}6)$$

The cross section (11-5) is given in cm^2/atom. When the constants are evaluated, we find

$$a_{ff}^{\nu} = 1.26\cdot10^{-78} \frac{N_e}{\sqrt{kT}} \frac{(Z')^2}{(h\nu)^3}\, g_{ff} \frac{cm^2}{atom},$$

(hν, kT in ergs)

or

$$a_{ff}^{\nu} = 7.68\cdot10^{-48} \frac{N_e}{\sqrt{kT}} \frac{(Z')^2}{(h\nu)^3}\, g_{ff} \frac{cm^2}{atom},$$

(hν, kT in kev)

where in both expressions N_e is the number of free electrons per cm^3.

Free electrons contribute to the absorption coefficient at all frequencies. The relative importance of bound-free and free-free absorption depends on the state of ionization. At low temperatures, there are very few free electrons, and bound-free absorption dominates. At very high temperatures, most electrons are free, and free-free absorption is dominant. However, for heavy elements free-free absorption is usually much smaller than photoelectric absorption for the usual range of temperatures.

The absorption coefficient is obtained by multiplying (11-5) by the number of atoms per unit volume:

$$\sigma^{\nu}_{ff} = \rho \, D_f \, (h\nu)^{-3}, \tag{11-7a}$$

where

$$D_f = \frac{1}{AM} \frac{16\pi^2 e^6 h^2}{3\sqrt{3}\, c(2\pi m)^{3/2}} \frac{N_e}{\sqrt{kT}} (Z')^2 \, g_{ff}. \tag{11-7b}$$

The total coefficient of continuous absorption is

$$\sigma'^{\nu} = \rho \, (D_k + D_f) \, (h\nu)^{-3}.$$

Recall that this is the *true absorption coefficient,* and it must be corrected for induced emission for use in the transfer equation (see §9):

$$\sigma^{\nu} = \rho(D_k + D_f) \, (1 - e^{-h\nu/kT}) \, (h\nu)^{-3}. \tag{11-8}$$

The coefficients of continuous absorption, corrected for induced emission, are summarized in Table V.

TABLE V

Coefficients of Continuous Absorption

1. Bound-Free Absorption:

$$\sigma_{bf}^{\nu} = \rho\ D_k\ (1-e^{-h\nu/kT})\ (h\nu)^{-3}$$

$$|E_k| \leq h\nu < |E_{k+1}|$$

$$D_k = \frac{1}{AM}\ \frac{64\pi^4 me^{10}}{3\sqrt{3}\ c\ h^3} \sum_{j=k}^{J} n_j\ \frac{(Z_j^*)^4}{j^5}\ g_{bf}(j),$$

2. Free-Free Absorption:

$$\sigma_{ff}^{\nu} = \rho\ D_f\ (1-e^{-h\nu/kT})\ (h\nu)^{-3}$$

$$D_f = \frac{1}{AM}\ \frac{16\pi^2 e^6 h^2}{3\sqrt{3}\ c(2\pi m)^{3/2}}\ \frac{N_e}{\sqrt{kT}}\ (Z')^2\ g_{ff},$$

3. Gaunt Factors:

$$g_{bf}(j) = 1 - 0.1728 \left[\frac{\nu}{R(Z_j^*)^2}\right]^{1/3} \left\{\frac{2}{j^2}\left[\frac{R(Z_j^*)^2}{\nu}\right] - 1\right\},$$

$$g_{ff} = 1 + 0.1728 \left[\frac{\nu}{R(Z')^2}\right]^{1/3} \left[1+2\frac{kT}{h\nu}\right],$$

4. State of Ionization:

n_j, N_e, Z_j^*, E_j given in Table IV.

§12. *Line Absorption*

In the semi-classical theory of line absorption, a bound electron in a periodic Bohr orbit behaves as a classical harmonic oscillator. The classical oscillator can emit and absorb radiation in a narrow spectral line. The classical theory of line absorption is described by Woolley and Stibbs (1953), Chapter VI, and Ambartsumyan (1956), Chapter 11.

The classical cross section for line absorption is

$$a_\ell^\nu = \left(\frac{\pi e^2}{mc}\right) \frac{(\gamma_0/4\pi)}{(\nu-\nu_0)^2+(\gamma_0/4\pi)^2} \quad \mathrm{cm}^2/\mathrm{atom}, \qquad (12\text{-}1)$$

where ν_0 is the frequency of the center of the line, and

$$\gamma_0 = \frac{8\pi^2 e^2 \nu_0^2}{3mc^3} \qquad (12\text{-}2)$$

is the *natural line width.* The natural line width is a result of radiation damping as shown by Heitler (1954), p. 32. In addition to the natural width, a line profile is broadened by collisional damping. A classical radiating oscillator is "quenched" when it exchanges energy by collisions. The effective width is then $\gamma = \gamma_0 + 2/\tau_c$, where

$$\frac{1}{\tau_c} = \sigma_0 \bar{v} N. \qquad (12\text{-}3)$$

In this expression, τ_c is the mean time between collisions, σ_0 is the cross section for collisions, $\bar{v}$ is the mean velocity of the atoms, and N is the number of atoms per unit volume. Lines are also broadened by the interatomic stark effect that results from the electric fields of neighboring ions.

So far we have discussed only the absorption cross section for a classical oscillator. The corresponding quantum mechanical treatment of an atom gives a similar result except for a correction factor f called the *oscillator strength.* The cross section for line absorption by an atom is

$$a_\ell^\nu = \left(\frac{\pi e^2}{mc}\right) f \, \frac{(\gamma/4\pi)}{(\nu-\nu_o)^2+(\gamma/4\pi)^2} \qquad \frac{\mathrm{cm}^2}{\mathrm{atom}} \tag{12-4}$$

where $\gamma = \gamma_o + 2/\tau_c$ with τ_c given by (12-3).

In order to determine the oscillator strength, we calculate the total energy of all frequencies absorbed in a cylinder of length ds that contains N classical oscillators per unit volume:

$$dW = N ds \left(\frac{\pi e^2}{mc}\right) \int_0^\infty \frac{(\gamma/4\pi) I^\nu d\Omega}{(\nu-\nu_o)^2+(\gamma/4\pi)^2} \, d\nu .$$

If we assume $I^\nu \simeq I^{\nu_o}$ over the line profile, the integral gives

$$dW = N ds \left(\frac{\pi e^2}{mc}\right) I^{\nu_o} d\Omega . \tag{12-5}$$

From the discussion in §9, we see that this result can also be written

$$dW = N_m ds \; B_{mn} I^{\nu_{nm}} d\Omega \; h\nu_{nm} , \tag{12-6}$$

where $\nu_{nm} = \nu_o$, the frequency of the line. The oscillator strength can now be obtained as the number of classical oscillators that produces the same absorption as N_m quantum atoms, $N_m = Nf$. From (12-5) and (12-6), we have

$$\frac{N}{N_m} = \frac{mc}{\pi e^2} B_{mn} h\nu_{nm} = f_{mn}.$$

From (9-7) and (12-2), the oscillator strength for absorption can be expressed in terms of the Einstein coefficient A_{nm} by

$$f_{mn} = \frac{4\pi}{3\gamma_o} \frac{g_n}{g_m} A_{nm}, \qquad (12\text{-}7a)$$

where $E_n > E_m$, and the oscillator strength for emission is

$$f_{nm} = \frac{4\pi}{3\gamma_o} A_{nm}. \qquad (12\text{-}7b)$$

Oscillator strengths for hydrogen-like atoms are calculated by Menzel and Pekeris, *op. cit.*

From the uncertainty relation $\Delta E \; \Delta t \sim \hbar$, we know that the breadth of a spectral line is associated with the level width ΔE and hence the lifetime of an excited state.

In addition to natural and collision broadening, a spectral line is smeared by Doppler shifts from the distribution of velocities of the emitting atoms. The cross section for line absorption assuming a Maxwellian distribution of velocities for the atoms is

$$a_\ell^\nu = \left(\frac{\pi e^2}{mc}\right) \frac{f}{\nu_o} \sqrt{\frac{Mc^2}{2\pi kT}}\, e^{-\frac{Mc^2}{2kT}\left(\frac{\nu-\nu_o}{\nu_o}\right)^2} \quad \frac{\text{cm}^2}{\text{atom}}, \qquad (12\text{-}8)$$

where M is the mass of the atom.

The atomic cross section for line absorption is conveniently expressed in the form

$$a_{mn}^\nu = \left(\frac{\pi e^2}{mc}\right) f_{mn}\; b(\nu) \quad \frac{\text{cm}^2}{\text{atom}}, \qquad (12\text{-}9)$$

where $b(\nu)$ is the "shape factor" or dispersion of the line. The shape factor is normalized to unity,

$$\int_0^\infty b(\nu)\, d\nu = 1,$$

so that the strength of the line is given by $N_m f_{mn}$ and does not depend on the shape. The inverse mean free path for absorption in the line corresponding to A_{nm} is

$$\sigma^{\nu_{nm}} = \left(\frac{\pi e^2}{m\,c}\right) N_m f_{mn} b(\nu), \qquad (12\text{-}10)$$

where N_m is the number of atoms per unit volume with bound electrons in the state E_m. The shape factor can be either the natural (or Lorentz) profile

$$b_L(\nu) = \frac{\gamma/4\pi}{(\nu-\nu_{nm})^2+(\gamma/4\pi)^2}, \qquad (12\text{-}11)$$

the Doppler profile,

$$b_D(\nu) = \frac{1}{\nu_{nm}} \sqrt{\frac{Mc^2}{2\pi kT}}\; e^{-\frac{Mc^2}{2kT}\left(\frac{\nu-\nu_{nm}}{\nu_{nm}}\right)^2}, \qquad (12\text{-}12)$$

or some combination.

Oscillator strengths for absorption in hydrogen are given in Table VI. Negative values indicate emission. From this table, the principal lines are seen to be at the heads of the series. The higher transitions become weaker and closer together and blend into the bound-free absorption edge as shown in Fig. 12-1. The effect of the large number

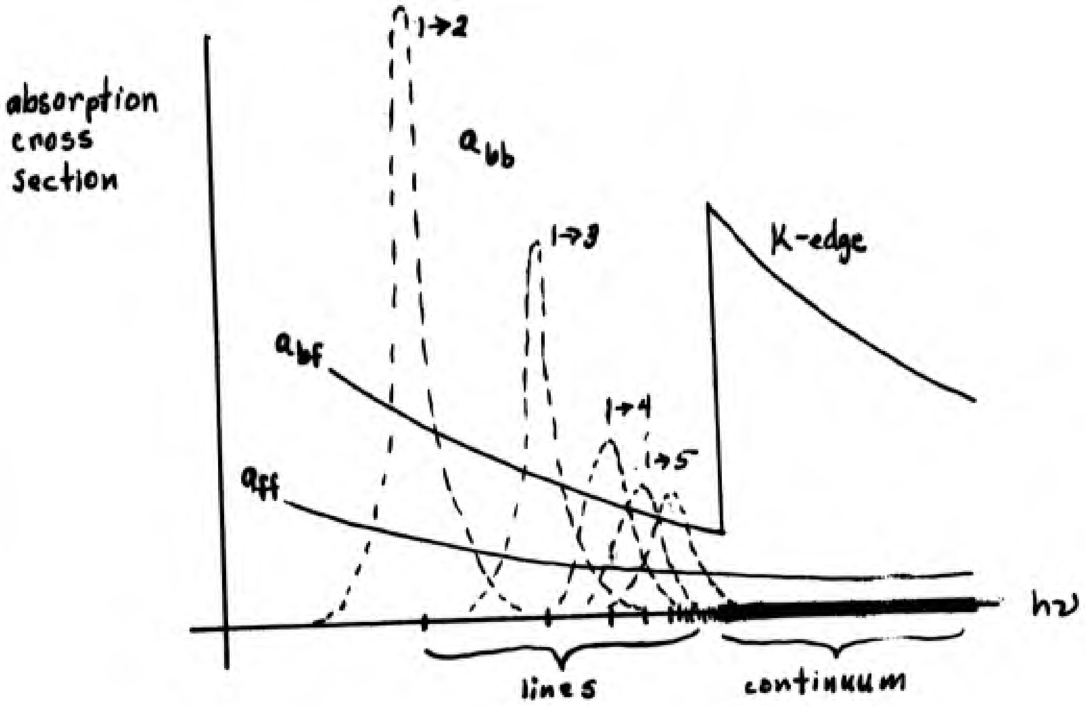

Fig. 12-1. Comparison of Absorption Processes

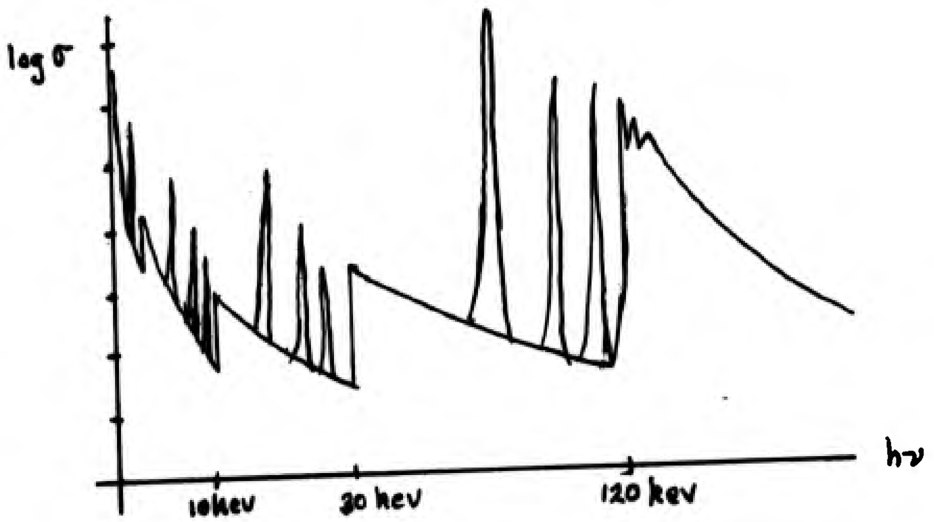

Fig. 12-2. Absorption Coefficient for Uranium at T = 10 kev

TABLE VI

Oscillator Strengths for Absorption in Hydrogen

Initial level *i* / Final level *j*	*i* = 1 *Lyman*	*i* = 2 *Balmer*	*i* = 3 *Paschen*
j = 1	..	-0·104	-0·0087
2	0·416	..	-0·284
3	0·0791	0·637	..
4	0·0290	0·119	0·841
5	0·0139	0·0443	0·150
6	0·0078	0·0212	0·0554
7	0·0048	0·0122	0·0269
8	0·0032	0·0080	0·0161
9 to ∞	0·0101	0·0237	0·0421

of weak, closely spaced lines is to extend the photoelectric ionization edge. This provides a means of approximating the effect of these lines.[24] The total absorption coefficient including bound-free, free-free, and bound-bound transitions is shown in Fig. 12-2. Note that the absorption coefficient varies over 5 orders of magnitude. Absorption effects at low frequency are depressed because the higher levels are depopulated at the temperature of 10 kev.

§13. *Electron Scattering*

We shall consider only the problem of Compton scattering from free electrons at rest. A photon of energy $h\nu$ collides with an electron and is scattered at an angle θ with the incident direction as shown in Fig. 13-1. The scattered photon has energy $h\nu'$, where

$$\begin{aligned} h\nu &= mc^2\,(\gamma-1) + h\nu', \\ \gamma &= (1-v^2/c^2)^{-\frac{1}{2}} \end{aligned} \tag{13-1}$$

and v is the recoil velocity of the electron. The relation (13-1) expresses the conservation of energy for the encounter. Conservation of both components of momentum is expressed by

$$\begin{aligned} \frac{h\nu}{c} &= \gamma m v \cos\phi + \frac{h\nu'}{c}\cos\theta, \\ 0 &= \gamma m v \sin\phi - \frac{h\nu'}{c}\sin\theta. \end{aligned} \tag{13-2}$$

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[24] B. Hunt and M. Sibulkin, J. Quant. Spectrosc. Radiat. Transfer 7, 951-964 (1967).

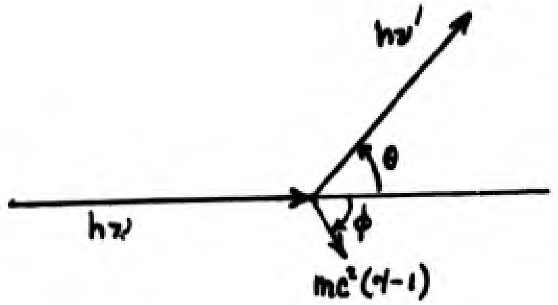

Fig. 13-1. Geometry of Compton Scattering from an Electron at Rest

The conservation equations (13-1) and (13-2) can be used to obtain the scattered photon energy, the electron recoil energy, and the scattering angles in terms of the initial photon energy:

$$h\nu' = \frac{h\nu}{1+\alpha(1-\cos\theta)} \qquad (13\text{-}3a)$$

$$T_e = h\nu\left[1 - \frac{1}{1+\alpha(1-\cos\theta)}\right] \qquad (13\text{-}3b)$$

$$\cos\theta = 1 - \frac{2}{(1+\alpha)^2\tan^2\phi+1}, \qquad (13\text{-}3c)$$

where $\alpha = h\nu/mc^2$.

a.) *Compton Scattering Cross Section.* The differential cross section for Compton scattering is given by the Klein-Nishina formula.[25] The cross section actually depends on polarization. However, if the incident radiation is unpolarized, we can average over the polarization angle to obtain

$$a_s^\nu(\cos\theta) =$$

$$= \frac{1}{2}r_o^2 \frac{1+\cos^2\theta}{\left[1+\alpha(1-\cos\theta)\right]^2}\left\{1 + \frac{\alpha^2(1-\cos\theta)^2}{(1+\cos^2\theta)\left[1+\alpha(1-\cos\theta)\right]}\right\} \frac{\text{cm}^2}{\text{electron}}, \qquad (13\text{-}4)$$

where $\alpha = h\nu/mc^2$ and $r_o = e^2/mc^2$. The expression (13-4) is derived in §22 of Heitler (1954). The differential scattering coefficient is given by

$$\sigma_s^\nu(\cos\theta) = N_e\, a_s^\nu(\cos\theta)\ \text{cm}^{-1},$$

where N_e is the number of free electrons per unit volume.

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[25] O. Klein and Y. Nishina, Z. Physik 52, 853 (1929).

The probability that a photon will be scattered through an angle θ in travelling a distance ds is $P = \sigma_s^\nu(\cos\theta)ds$.

For low energies, $\alpha \ll 1$, it is often simpler to expand (13-4) in powers of α. To second order, we have

$$a_s(\cos\theta) \cong \frac{1}{2}r_o^2(1+\cos^2\theta)\left[1 - 2\alpha(1-\cos\theta)+\alpha^2\,\frac{(1-\cos\theta)^2(2+\cos^2\theta)}{1+\cos^2\theta}\right],$$

$$h\nu' \cong h\nu\left[1-\alpha(1-\cos\theta)+\alpha^2(1-\cos\theta)^2\right], \quad \alpha \ll 1.$$

The classical expression for Thompson scattering is obtained when $\alpha = 0$:

$$a_s = \frac{1}{2}r_o^2(1+\cos^2\theta), \qquad h\nu' = h\nu. \tag{13-5}$$

The angular dependence of the differential cross section is shown in Fig. 13-2.

The total scattering cross section is obtained by integrating over all solid angle, $d\Omega = \sin\theta\, d\theta d\phi$. The Klein-Nishina formula gives

$$a_s^\nu = a_o\frac{3}{4}\left\{\frac{1+\alpha}{\alpha^3}\left[\frac{2\alpha(1+\alpha)}{1+2\alpha} - \ln(1+2\alpha)\right] + \frac{1}{2\alpha}\ln(1+2\alpha)-\frac{1+3\alpha}{(1+2\alpha)^2}\right\}, \tag{13-6}$$

where $a_o = 8\pi r_o^2/3$ and $\alpha = h\nu/mc^2$. The total cross section to second order in α is

$$a^\nu = a_o(1-2\alpha+\frac{26}{5}\alpha^2),$$

and the total cross section for Thompson scattering is

$$a_o = \frac{8\pi}{3}\,\frac{e^4}{m^2c^4} = 6.54\cdot10^{-25}\ \frac{\text{cm}^2}{\text{electron}}. \tag{13-7}$$

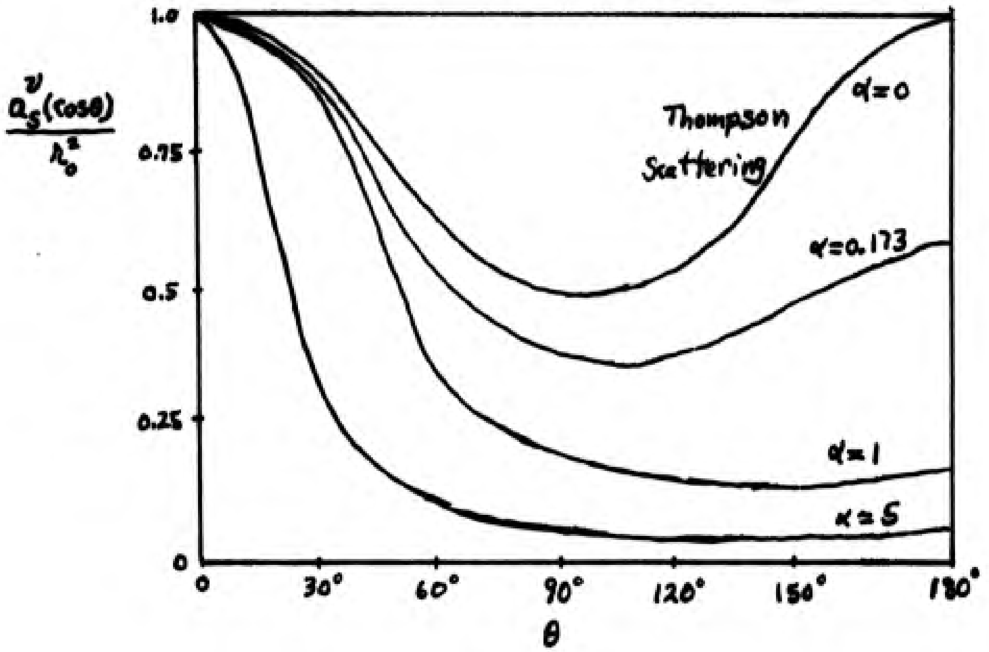

Fig. 13-2. Compton Differential Cross Section

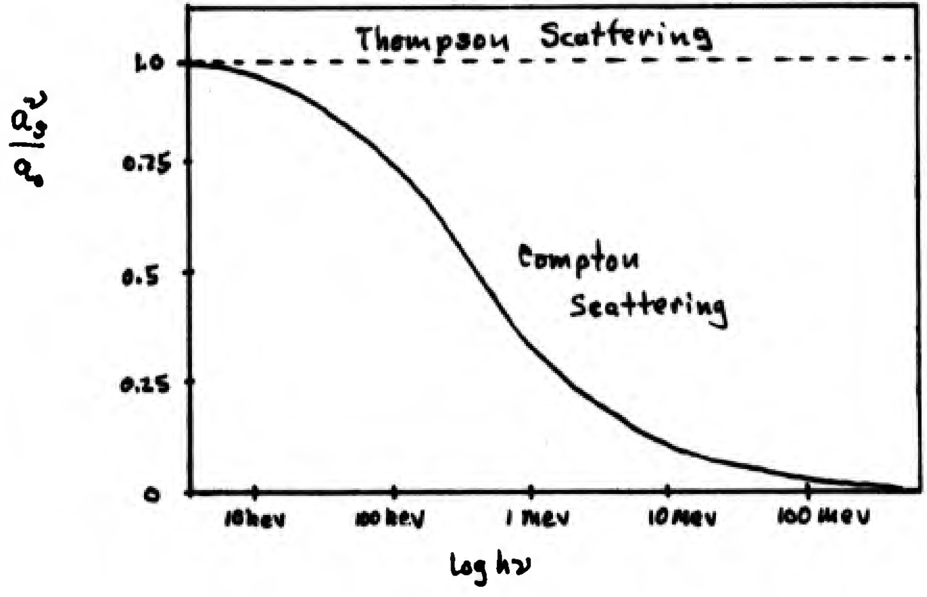

Fig. 13-3. Compton Total Cross Section

The frequency dependence of the total cross section is shown in Fig. 13-3.

b.) *Induced Scattering.* The scattering process can be considered from the point of view of quantum mechanics as an absorption followed immediately by emission. It was found in §9 that an emission process is enhanced by the presence of radiation in the final state. Therefore, we must correct the fundamental cross section (13-4) for induced scattering when radiation is present in the scattered beam.

On page 46 the energy emitted in time dt into the solid angle dΩ was found to be

$$dW = h\nu_{nm}A_{nm}\left(1+\frac{B_{nm}}{A_{nm}}I^{\nu_{nm}}\right)d\Omega\ dt,$$

where the second term in the parentheses is the induced emission. From (9-7) we have

$$dW = h\nu_{nm}A_{nm}d\Omega\ dt\left(1+\frac{c^2}{2h\nu_{nm}^3}I^{\nu_{nm}}\right). \qquad (13\text{-}8)$$

It can be shown that the correction for induced scattering is of the form $\left[1+n(\vec{\mu},\nu)\right]$, where $n(\vec{\mu},\nu)$ is the number of photons with energy hν travelling in the direction $\vec{\mu}$. Photons obey the Base-Einstein statistics with zero chemical potential.[26] Therefore, the number of photons in the energy state hν is

$$n = \frac{1}{e^{h\nu/kT}-1},$$

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[26] Kittel (1958), sections 21 and 22.

From (9-5) and (9-4) this expression can be written

$$n = \frac{1}{\frac{N_m B_{mn}}{N_n B_{nm}} - 1} = \frac{B_{nm}}{A_{nm}} I^{\nu_{nm}}(\vec{\mu}) = \frac{c^2}{2h\nu_{nm}^3} I^{\nu_{nm}}(\vec{\mu}),$$

which is the form that appears in (13-8). Thus the energy scattered out of the beam $I^{\nu}(\vec{\mu})$ into the direction $\vec{\mu}$ is enhanced by the factor

$$\left[1+n(\vec{\mu}',\nu')\right] = \left[1+\frac{c^2}{2h\nu'^3} I^{\nu'}(\vec{\mu}')\right] \qquad (13\text{-}9)$$

over that given by the cross section (13-4).

c.) *Transfer Equation for Scattering.* We can now obtain the transfer equation for Compton scattering by free electrons at rest when polarization is neglected. Consider a pencil of radiation confined to the solid angle $d\Omega$ about the direction $\vec{\mu}$. This radiation interacts with the free electrons in an elementary cylinder of length ds. The number of photons that pass through the cylinder per unit time is given by

$$\frac{I^{\nu}(\vec{r},\vec{\mu},t)}{h\nu}\, d\Omega\, d\nu .$$

The probability that a photon will be scattered into $d\Omega'$ at $\vec{\mu}'$ is given by

$$N_e ds a_s^{\nu}(\cos\theta) d\Omega' \left[1+\frac{c^2}{2h\nu'^3} I^{\nu'}(\vec{r},\vec{\mu}',t)\right],$$

where $\vec{\mu}\cdot\vec{\mu}' = \cos\theta$. Therefore, the total number of photons scattered out of the pencil per unit time in the length ds is

$$N_e I^{\nu}(\vec{r},\vec{\mu},t)d\Omega\frac{d\nu}{h\nu}ds\int\left[1+\frac{c^2}{2h\nu'^3}I^{\nu'}(\vec{r},\vec{\mu}',t)\right]a_s^{\nu}(\vec{\mu}\cdot\vec{\mu}')d\Omega'. \quad (13\text{-}10)$$

Similarly, the total number of photons scattered into the beam is

$$N_e\left[1+\frac{c^2}{2h\nu^3}I^{\nu}(\vec{r},\vec{\mu},t)\right]d\Omega ds\int\frac{I^{\nu'}(\vec{r},\vec{\mu}',t)d\nu'}{h\nu'}a_s^{\nu'}(\vec{\mu}\cdot\vec{\mu}')d\Omega'. \quad (13\text{-}11)$$

In this relation, we must express the frequency interval $d\nu'$ in terms of $d\nu$ in order to combine (13-11) with (13-10). From (13-3a), with $\alpha = h\nu/mc^2$, we find

$$\frac{d\nu'}{d\nu}=\left(\frac{\nu'}{\nu}\right)^2,$$

which gives for the number of photons scattered into the beam

$$N_e\left[1+\frac{c^2}{2h\nu^3}I^{\nu}(\vec{r},\vec{\mu},t)\right]d\Omega ds\,\frac{d\nu}{h\nu}\int\frac{\nu'}{\nu}I^{\nu'}(\vec{r},\vec{\mu}',t)a_s^{\nu'}(\cos\theta)d\Omega'.$$

The changes in the intensity due to scattering in and out of the beam are then given by

$dI^{\nu}(\text{scattering out}) =$

$$= -I^{\nu}(\vec{r},\vec{\mu},t)N_e ds\int\left[1+\frac{c^2}{2h\nu'^3}I^{\nu'}(\vec{r},\vec{\mu}',t)\right]a_s^{\nu}(\vec{\mu}\cdot\vec{\mu}')d\Omega',$$

$dI^{\nu}(\text{scattering in}) =$

$$= \left[1+\frac{c^2}{2h\nu^3}I^{\nu}(\vec{r},\vec{\mu},t)\right]N_e ds\int\frac{\nu'}{\nu}I^{\nu'}(\vec{r},\vec{\mu}',t)a_s^{\nu'}(\vec{\mu}\cdot\vec{\mu}')d\Omega'.$$

The transfer equation for emission, absorption and Compton scattering becomes

$$\frac{dI^\nu}{ds} + \sigma_a^\nu I^\nu + I^\nu \int \left[1+\frac{c^2}{2h\nu'^3} I^{\nu'}(\vec{\mu}')\right] N_e a_s^\nu(\vec{\mu}\cdot\vec{\mu}')d\Omega'$$

$$= \sigma_a^\nu B^\nu + \left[1+\frac{c^2}{2h\nu^3} I^\nu(\vec{\mu})\right] \int \frac{\nu'}{\nu} I^{\nu'}(\vec{\mu}') N_e a_s^{\nu'}(\vec{\mu}\cdot\vec{\mu}')d\Omega' ,$$

where $a_s^\nu(\vec{\mu}\cdot\vec{\mu}')$ is given by (13-4) and ν' is given in terms of ν and $\vec{\mu}'\cdot\vec{\mu}$ by (13-3a). As discussed in §5, $a_s^\nu(\cos\theta)$ must be expressed in the form $a_s^\nu(\vec{\mu},\vec{\mu}')$ before the integration can be performed. For example, see Fig. 5-1b.

The equations that describe Compton scattering by electrons at rest neglecting polarization are given for reference in Table VII. For scattering by hot electrons, one must first perform a Lorentz transformation to the electron restframe, calculate the scattering using the formulas described above, and then transform back to the restframe of the observer. Scattering by electrons in a Maxwellian distribution is discussed by Mayer (1947), and by Fraser.[27]

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[27] A. R. Fraser, "Radiative Transfer in Moving Media, Part II, Purely Scattering Media," AWRE (Aldermaston) Report TPN 78/58 (1958).

A. R. Fraser, "Fundamental Equations of Radiation Hydrodynamics," AWRE (Aldermaston) Report 0-82/65 (1965).

TABLE VII

Compton Scattering by Free Electrons at Rest

1. Equation of Transfer

$$\frac{1}{c}\frac{\partial I^{\nu}}{\partial t} + \vec{\mu}\cdot\nabla I^{\nu} + \sigma_a^{\nu} I^{\nu} + I^{\nu}\int\left[1+\frac{c^2}{2h\nu'^3}I^{\nu'}(\vec{\mu}')\right]\sigma_s^{\nu}(\vec{\mu},\vec{\mu}')\,d\Omega'$$

$$= \sigma_a^{\nu}B^{\nu}(T) + \left[1+\frac{c^2}{2h\nu^3}I^{\nu}(\vec{\mu})\right]\int\frac{\nu'}{\nu}I^{\nu'}(\vec{\mu}')\sigma_s^{\nu'}(\vec{\mu},\vec{\mu}')\,d\Omega'$$

2. Compton Cross Section

$$a_s^{\nu}(\cos\theta) =$$

$$= \frac{1}{2}r_o^2\frac{1+\cos^2\theta}{\left[1+\alpha(1-\cos\theta)\right]^2}\left\{1+\frac{\alpha^2(1-\cos\theta)^2}{(1+\cos^2\theta)\left[1+\alpha(1-\cos\theta)\right]}\right\}\frac{\mathrm{cm}^2}{\mathrm{electron}}$$

$$\sigma_s^{\nu} = N_s a_s^{\nu},$$

$$\cos\theta = \vec{\mu}\cdot\vec{\mu}',$$

$$r_o = e^2/mc^2,$$

3. Frequency Relation

$$h\nu' = \frac{h\nu}{1+\alpha(1-\cos\theta)}$$

$$\alpha = h\nu/mc^2.$$

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Chapter III *Motion of the Material*

§14. *Thermodynamic Properties of the Material*

At temperatures on the order of 1 kev, all materials are vaporized. The motion of the material, therefore, is that of gas dynamics. In this section, we first investigate the thermodynamic properties of ideal gases, and then consider various corrections that apply to real gases.

a.) *Ideal Gases.* An ideal gas is described by the equations[28]

$$\begin{aligned} pV &= nR'T, \\ U &= U(T), \end{aligned} \tag{14-1}$$

where $R' = 8.3166 \cdot 10^7$ erg/mole/°K is the gas constant, n is the number of moles in the volume V, and p is the pressure. The internal energy U of an ideal gas is a function of the temperature alone. The ideal gas law can also be written in the form

$$pV = NkT = \frac{M}{\mu m_H} kT = \frac{M}{\mu} RT, \tag{14-1a}$$

where $R = k/m_H = 8.2506 \cdot 10^7$ erg/gm/°K, $k = 1.3804 \cdot 10^{-16}$ erg/°K is the Boltzmann constant, μ is the mean molecular weight (gm/mole), m_H is the mass of the hydrogen atom,

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[28] The basic reference is Chandrasekhar (1939), Chapter II.

and N is the number of molecules of total mass M contained in the volume V.

By the First Law of Thermodynamics, the heat exchanged in a quasi-static change of state is

$$dQ = dU + pdV.$$

As heat is transferred between the gas and its surroundings, the temperature changes. The heat exchanged per unit change in temperature is

$$\frac{dQ}{dT} = \frac{dU}{dT} + p\frac{dV}{dT}. \qquad (14\text{-}2)$$

Let β be any physical variable. The *specific heat* per unit mass at constant β is defined by

$$c_\beta \equiv \frac{1}{M}\left(\frac{dQ}{dT}\right)_{\beta=\text{const}} \frac{\text{ergs}}{\text{gm.}^\circ\text{K}} .$$

For example, the specific heat at constant volume is found from (14-2) to be

$$c_V \equiv \frac{1}{M}\left(\frac{dQ}{dT}\right)_{V=\text{const}} = \frac{1}{M}\frac{dU}{dT}. \qquad (14\text{-}3)$$

If we differentiate (14-1a), we have

$$pdV + Vdp = M\frac{R}{\mu}dT.$$

If pdV in (14-2) is evaluated by this expression,

$$\frac{dQ}{dT} = \frac{dU}{dT} + M\frac{R}{\mu} - Vdp,$$

which gives for the specific heat at constant pressure,

$$c_p \equiv \frac{1}{M}\left(\frac{dQ}{dT}\right)_{p=\text{const}} = \frac{1}{M}\frac{dU}{dT} + \frac{R}{\mu}. \qquad (14\text{-}4)$$

This expression, together with (14-3), gives the important relation

$$c_p - c_V = \frac{R}{\mu}, \qquad (14\text{-}5)$$

or

$$\frac{R}{\mu} = c_V\left(\frac{c_p}{c_V} - 1\right) = c_V(\gamma - 1),$$

where $\gamma = c_p/c_V$, the ratio of specific heats. Thus, c_V can be expressed in terms of R and γ by

$$c_V = \frac{R}{\mu(\gamma-1)} = \frac{8.2506\cdot 10^7}{\mu(\gamma-1)}\ \frac{\text{ergs}}{\text{gm}^\circ\text{K}}. \qquad (14\text{-}6)$$

The equations (14-3), (14-4), (14-5), and (14-6) are fundamental expressions for all ideal gases.

If the gas is such that c_V is a constant for all changes of state, the gas is called *polytropic*. The internal energy for a polytropic gas is

$$U = Mc_VT \quad \frac{\text{ergs}}{\text{gm}}, \qquad (14\text{-}6)$$

$$c_V = \text{const.}$$

The polytropic equation of state is the simplest equation of state for an ideal gas. However, ideal gases are not in general polytropic, since the internal energy can be any function of the temperature alone, U(T).

We must now enquire into the problem of determining c_V [or γ through (14-6)] in terms of the properties of the gas. Let us assume that we have a mass M of gas at temperature T. The total internal energy of this gas is $U = Mc_VT$, and by Boltzmann's law we know that there are

$\frac{1}{2}kT$ ergs/degree of freedom. If there are $N = M/\mu m_H$ molecules in the system, and each molecule has α degrees of freedom, we can write

$$U = Mc_V T = \frac{\alpha}{2}NkT = \frac{M}{\mu}\frac{\alpha}{2}\left(\frac{k}{m_H}\right)T.$$

Since $R = k/m_H$ and $c_V = R/\mu(\gamma-1)$, we obtain c_V in terms of α, the number of degrees of freedom of the molecules in the gas:

$$c_V = \frac{\alpha}{2}\frac{R}{\mu} \text{ ergs/gm/}^\circ\text{K} \tag{14-7}$$

with

$$\gamma = 1 + 2/\alpha.$$

The molecules of a gas will have translational, vibrational and rotational degrees of freedom. We can therefore express the total number of degrees of freedom as $\alpha = 3 + r$, where r is the number of *internal* degrees of freedom. The ratio of specific heats then becomes

$$\gamma = \frac{5+r}{3+r}. \tag{14-8}$$

For a *monatomic gas* $(r=0, \alpha=3)$

$$\begin{aligned} \gamma &= 5/3 \\ c_V &= \frac{3}{2}\frac{R}{\mu} \\ c_p &= \frac{5}{2}\frac{R}{\mu}, \end{aligned} \tag{14-9}$$

and for a *diatomic gas* $(r=2, \alpha=5)$

$$\begin{aligned} \gamma &= 7/5 \\ c_V &= \frac{5}{2}\frac{R}{\mu} \\ c_p &= \frac{7}{2}\frac{R}{\mu}, \end{aligned} \tag{14-10}$$

Note that the gas will be polytropic as long as γ is constant; i.e. as long as the number of degrees of freedom does not change. If the gas begins to dissociate (or becomes ionized), α will change and the gas will no longer be polytropic.

It can be shown that thermal radiation enclosed in a cavity behaves as an ideal gas with $\gamma = 4/3$. First, it is known from §4 that the energy density of thermal radiation in a cavity is $E = aT^4$, so that the total internal energy in the cavity is

$$U = VaT^4,$$

that is to say, *a function of the temperature alone.* Since the radiation is isotropic,

$$p = \frac{1}{3}aT^4.$$

The "photon gas" then obeys the equation

$$pV = M\frac{R}{\mu}T = (\gamma-1)Mc_VT = (\gamma-1)U.$$

If we substitute the forms for p and U appropriate to the radiation field, we find

$$\frac{1}{3}aT^4V = (\gamma-1)VaT^4,$$

which gives $\gamma = 4/3$.

b). *Ionized Gases.* In an ideal gas, the molecules or atoms interact only by elastic collisions. In an ionized gas, the electrostatic interactions between the ions and electrons must be included.[29] Consider a gas composed of a single element (A,Z). The internal energy per unit volume is

$$I = PE_b + PE_f + KE_i + KE_e, \qquad (14\text{-}11)$$

where the potential energy of the bound electrons is

$$PE_b = -13.6\frac{\rho}{\mu m_H}\sum_j n_j\left(\frac{Z^*_j}{j}\right)^2 \frac{ev}{cm^3}, \qquad (14\text{-}11a)$$

the potential energy of the free electrons is

$$PE_f = -13.6\cdot\frac{9}{5}\cdot\frac{\rho}{\mu m_H}(Z')^2\left(\frac{a_o}{a_{Z'}}\right)\frac{ev}{cm^3}, \qquad (14\text{-}11b)$$

the kinetic energy of the ions is

$$KE_i = \frac{3}{2}\frac{\rho}{\mu m_H}kT, \quad (k^{-1} = 11{,}606\ {}^oK/ev) \qquad (14\text{-}11c)$$

and the kinetic energy of the free electrons is

$$KE_f = \frac{3}{2}\frac{\rho}{\mu m_H}n_f kT. \qquad (14\text{-}11d)$$

The calculation of these quantities is discussed in §10.

In order to calculate the gas law when electrostatic forces are present, we use the *virial theorem*.[30] As shown by Jeans (1925), section 164,

$$pV = \frac{2}{3}KE + \frac{1}{3}PE, \qquad (14\text{-}12)$$

which gives for the gas law

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[29] The basic reference is Mayer (1947), Appendix II.

[30] Goldstein (1950), section 3-4.

$$pV = \frac{\rho V}{\mu m_H} (1+n_f)kT - 14.4\cdot 10^{-12} \frac{3}{5} \frac{\rho V}{\mu m_H} (Z')^2 \left(\frac{a_o}{a_{Z'}}\right) .$$

The pressure is then given in terms of density and temperature by

$$p = \frac{\rho}{\mu m_H} \left[(1+n_f)kT - 14.4\cdot 10^{-12} \frac{3}{5} (Z')^2 \left(\frac{a_o}{a_{Z'}}\right)\right] \frac{\text{ergs}}{\text{cm}^3}. \tag{14-13}$$

Note that the internal energy now depends on ρ as well as T, since Z^*_j, n_j, Z', and $a_{Z'}$ all depend on $N_e = \rho n_f/\mu m_H$.

c). *Sound Speed.* The sound speed in a gas is given by[31]

$$c^2 = - v^2\left(\frac{\partial p}{\partial v}\right)_I + pv^2\left(\frac{\partial p}{\partial I}\right)_v , \tag{14-14}$$

where the specific volume $v = \rho^{-1}$, and I is the specific internal energy. For an ideal gas, $I = c_V T$ and

$$p = (\gamma-1)\rho c_V T = (\gamma-1)I/v$$

and the sound speed is

$$c^2 = (\gamma-1)I + (\gamma-1)^2 I$$

$$= \gamma(\gamma-1)I.$$

In an ideal gas, therefore, the sound speed is a function of temperature given by

$$c = \sqrt{\gamma(\gamma-1)c_V T}. \tag{14-15}$$

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[31] R. Lelevier, "Lectures on Hydrodynamics and Shock Waves," Lawrence Radiation Laboratory Report, UCRL-4333 (1954).

If the gas is shocked, the disturbance will propagate with the sound speed c.

In order to obtain the sound speed in an ionized gas, we must use (14-14) together with the gas law (14-13). This is a much more difficult problem owing to the subtle dependence of Z', $a_{Z'}$, and n_f on the density ρ.

§15. *Basic Equations in Eulerian Form.*

The state of the material is described by four variables: the pressure p, the density ρ, the velocity $\vec{u}$, and the temperature T. The equations that control these variables are the equation of state, the continuity equation, the momentum equation, and the energy equation. We shall first consider these equations in the absence of a radiation field. The derivation of the basic equations can be found in many books, for example, Knudsen and Katz (1958), and Harlow (1960).

a). *The Equation of State.* The equation of state for a gas is a functional relation between the pressure, density, and temperature. As discussed in §14, the equation of state for an ideal gas is

$$p = \frac{R}{\mu}\rho T, \tag{15-1}$$

and the equation of state for an ionized gas is given by (14-13).

b). *The Continuity Equation.* Except at energies where pair production and nuclear reactions occur, the total mass in any hydrodynamic system is a constant. The system, therefore, must obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0. \qquad (15\text{-}2)$$

In this equation, $\partial\rho/\partial t$ gives the rate at which mass in any elementary volume is increasing, and $-\nabla\cdot(\rho\vec{u})$ gives the net flux of mass into the elementary volume.

c). *The Momentum Equation.* The equation expressing the conservation of momentum is

$$\rho\frac{d\vec{u}}{dt} = -\nabla p + \nabla\cdot\overleftrightarrow{\tau} + \vec{f}, \qquad (15\text{-}3)$$

where $\vec{f}$ is any gravitational or electrodynamic force density, $\overleftrightarrow{\tau}$ is the viscous stress tensor, and p is the pressure. The expression (15-3) is Newton's law of motion for the flow of a fluid with viscous shearing stresses.

If $\vec{g}$ is the gravitational acceleration at any point, the force density is $\vec{f}_g = \rho\vec{g}$. If ρ_e is the net electric charge of an ionized fluid at any point, the electromagnetic force density is given by $\vec{f}_e = \rho_e(\vec{E}+\frac{1}{c}\vec{u}\times\vec{B})$.

Viscosity is the frictional effect that arises when two adjacent layers of fluid move relative to each other. The shearing force per unit area is proportional to the velocity gradient

$$\frac{F_V}{A} = \mu\frac{du}{dy},$$

where μ is called the *coefficient of viscosity*. The c.g.s. unit of μ is the poise, where 1 poise = 1 erg-sec/cm^3.

The components of the viscous stress tensor are

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) - \frac{2}{3}\,\mu(\nabla\cdot\vec{u})\delta_{ij},$$

where δ_{ij} is the Kronecker delta, and $(\nabla\cdot\overset{\leftrightarrow}{\tau})_i \equiv \sum_j \partial\tau_{ij}/\partial x_j$.

d). *The Energy Equation.* The equation of conservation energy is

$$\rho\frac{d}{dt}(I+\tfrac{1}{2}\vec{u}\cdot\vec{u}+\phi) = -\nabla\cdot(\vec{u}p) + \nabla\cdot(\vec{u}\cdot\overset{\leftrightarrow}{\tau}) + \nabla\cdot(k\nabla T) + Q. \quad (15\text{-}4)$$

In this equation, $I = c_V T$ is the specific internal energy;[32] ϕ is the potential energy per unit mass due to any external forces, k is the coefficient of heat conduction, and Q is any energy source such as chemical or nuclear reactions. The left hand side of (15-4) is the rate of change of the specific total energy of the gas. The first term on the right is the work done by the pressure gradients; the second term is the energy dissipated by viscous stresses; the third term is the energy transferred by heat conduction; and the last term gives the energy input due to external sources.

The equation (15-3) for the motion of a viscous, compressible fluid is called the *Navier-Stokes* equation. However, the viscosity for gases is usually so low that any terms in $\overset{\leftrightarrow}{\tau}$ can be neglected compared to terms in p. The momentum equation then becomes

$$\rho\frac{d\vec{u}}{dt} = -\nabla p+\vec{f}, \quad (15\text{-}5)$$

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[32] "Specific" quantities are quantities measured per unit mass.

which is called the *Euler equation*.

The total time derivative that appears in (15-3) and (15-4) can be expressed in terms of partial derivatives by means of the chain rule

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z},$$

or in more compact notation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla.$$

The hydrodynamic equations neglecting viscosity can then be written:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0, \qquad (15\text{-}6a)$$

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla(p+\rho\vec{u}\cdot\vec{u}) = \vec{f}, \qquad (15\text{-}6b)$$

$$\frac{\partial}{\partial t}(\rho I+\phi) + \nabla\cdot(\vec{u}p+\rho I\vec{u}) = \nabla\cdot(k\nabla T), \qquad (15\text{-}6c)$$

$$p = \frac{R}{\mu}\rho T, \qquad (15\text{-}6d)$$

$$I = c_V T. \qquad (15\text{-}6e)$$

The relations (15-6) are the equations of Eulerian hydrodynamics for an ideal gas in *conservative form*.[33]

In most hydrodynamic problems at high density and low temperature, the principal mechanism of heat transfer is conduction. At low densities and high temperature, heat transfer by radiation predominates, and the heat conduction

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[33] A *conservation equation* is any equation of the general form $\partial A/\partial t + \partial B/\partial x = 0$.

term in the energy equation must be replaced by the appropriate radiation terms. The equations of hydrodynamics in the presence of a radiation field are considered in §17.

§16. *Basic Equations in Lagrangian Form*

There are two alternate descriptions of hydrodynamics, both of which were conceived by Euler. In the *Eulerian representation,* the fluid is described as it appears to an observer fixed in an arbitrary external reference frame. In the *Lagrangian representation,* the fluid is described as it appears to an observer moving with the fluid at each point.

Imagine the ocean dotted with many islands, each with an observer. Imagine further that there are many rafts floating at rest with respect to the water and carried past the islands by the ocean currents. The men on the islands are Eulerian observers; they sample the temperature and density of the ocean at each fixed reference point as a function of time. The men on the rafts are Lagrangian observers; they sample the changing properties of the ocean at each reference point at rest with respect to the water.

The Lagrange coordinates are labels or indices attached to particles in the fluid. The most convenient labeling scheme is to take the Lagrange coordinates as the initial positions of the particles, $\vec{r}_0 \equiv \vec{r}(t=0)$.

a). *Lagrangian Equations in Two-Dimensions.* Let us consider the transformation to the Lagrangian description for the case of two-dimensional flow in rectangular coordinates.[34] The Eulerian coordinates are (x,y), and the Lagrangian coordinates are (x_o,y_o). The transformation between the two representations is

$$\begin{aligned} x &= x(x_o,y_o) \\ y &= y(x_o,y_o). \end{aligned} \tag{16-1}$$

In the Eulerian frame, the velocities are

$$u = \left(\frac{\partial x}{\partial t}\right)_{x_o,y_o=\text{const.}},$$
$$v = \left(\frac{\partial y}{\partial t}\right)_{x_o,y_o=\text{const.}}$$

The transformation (16-1) can be inverted to yield

$$\begin{aligned} x_o &= x_o(x,y) \\ y_o &= y_o(x,y). \end{aligned} \tag{16-2}$$

By the rules of implicit differentiation we have

$$\begin{aligned} \frac{\partial x_o}{\partial x} &= \frac{1}{J}\frac{\partial y}{\partial y_o}, \quad \frac{\partial y_o}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial x_o}, \\ \frac{\partial x_o}{\partial y} &= -\frac{1}{J}\frac{\partial x}{\partial y_o}, \quad \frac{\partial y_o}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial x_o}, \end{aligned} \tag{16-3}$$

where the Jacobian for the transformation is

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[34] The case of cylindrical coordinates is considered by D. Schulz, "Two Dimensional Lagrangian Hydrodynamic Difference Equations," Chapter I in Alder, *et.al.* (1964).

$$J = \begin{vmatrix} \frac{\partial x}{\partial x_o} & \frac{\partial y}{\partial x_o} \\ \frac{\partial x}{\partial y_o} & \frac{\partial y}{\partial y_o} \end{vmatrix} . \qquad (16\text{-}4)$$

Notice that the total time derivative of any quantity in the Eulerian frame,

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \vec{u}\cdot\nabla A,$$

is the sum of two terms. The first term gives the intrinsic time variation of $A(\vec{r},t)$, and the second arises from the nonhomogeniety of the field encountered by an observer moving with velocity $\vec{u}$. The total time derivative in the Eulerian frame is, therefore, the same as the partial time derivative in the Lagrangian frame,

$$\left(\frac{d}{dt}\right)_{\text{Eulerian}} = \left(\frac{\partial}{\partial t}\right)_{x_o y_o} . \qquad (16\text{-}5)$$

Consider now the continuity equation. In the Eulerian system we have

$$\frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0,$$

or

$$\frac{d\rho}{dt} + \rho\nabla\cdot\vec{u} = 0.$$

The first term can be transformed by (16-5). The second term can be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{J}\left(\frac{\partial u}{\partial x_o}\frac{\partial y}{\partial y_o} - \frac{\partial u}{\partial y_o}\frac{\partial y}{\partial x_o} - \frac{\partial v}{\partial y_o}\frac{\partial x}{\partial y_o} + \frac{\partial v}{\partial y_o}\frac{\partial x}{\partial x_o}\right)$$

$$= \frac{1}{J}\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_o}\frac{\partial y}{\partial y_o} - \frac{\partial x}{\partial y_o}\frac{\partial y}{\partial y_o} - \frac{\partial y}{\partial x_o}\frac{\partial x}{\partial y_o} + \frac{\partial y}{\partial y_o}\frac{\partial x}{\partial x_o}\right),$$

or $\nabla\cdot\vec{u} = \frac{1}{J}\frac{\partial J}{\partial t}$. (16-6)

The continuity equation in Lagrangian coordinates is

$$\frac{\partial \rho}{\partial t} + \frac{\rho}{J}\frac{\partial J}{\partial t} = 0. \qquad (16\text{-}7)$$

The expressions (16-5) and (16-6) are the two fundamental connections between the Eulerian and Lagrangian descriptions.

If we divide (16-7) by ρ and integrate, we obtain

$$\log(\rho J) = \text{const.},$$

$$\rho J = \text{const.}$$

Since $x_o = x(t=0)$ and $y_o = y(t=0)$, $J(t=0) = 1$. Therefore, the constant of integration must be

$$\rho J = \rho_o. \qquad (16\text{-}8)$$

Note that (16-8) is the solution of (16-7); the two relations are therefore equivalent.

Consider now the momentum equations in the absence of an external force. In the Eulerian frame, we have

$$\rho\frac{du}{dt} = -\frac{\partial p}{\partial x},$$

$$\rho\frac{dv}{dt} = -\frac{\partial p}{\partial y}.$$

The pressure gradient transforms by

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial x_o}\frac{\partial x_o}{\partial x} + \frac{\partial p}{\partial y_o}\frac{\partial y_o}{\partial x} = \frac{1}{J}\left(\frac{\partial p}{\partial x_o}\frac{\partial y}{\partial y_o} - \frac{\partial p}{\partial y_o}\frac{\partial y}{\partial x_o}\right)$$

$$= \frac{1}{J}\left[p,y\right],$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial x_o}\frac{\partial x_o}{\partial y} + \frac{\partial p}{\partial y_o}\frac{\partial y_o}{\partial y} = \frac{1}{J}\left(-\frac{\partial p}{\partial x_o}\frac{\partial x}{\partial y_o} + \frac{\partial p}{\partial y_o}\frac{\partial x}{\partial x_o}\right)$$

$$= \frac{1}{J}\left[x,p\right],$$

where $[p,y]$ and $[x,p]$ are Poisson brackets[35] with respect to x_o, y_o. The total time derivatives transform by (16-5) giving for the momentum equations in the Lagrange representation,

$$\frac{\partial u}{\partial t} + \frac{1}{\rho J}\left[p,y\right] = 0,$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho J}\left[x,p\right] = 0.$$

If we use the mass relation (16-8), the momentum equations become

$$\rho_o\frac{\partial u}{\partial t} + \left[p,y\right] = 0, \qquad (16\text{-}9a)$$

$$\rho_o\frac{\partial v}{\partial t} + \left[x,p\right] = 0,$$

where

$$\left[p,y\right] = \frac{\partial p}{\partial x_o}\frac{\partial y}{\partial y_o} - \frac{\partial p}{\partial y_o}\frac{\partial y}{\partial x_o}, \qquad (16\text{-}9b)$$

$$\left[x,p\right] = \frac{\partial x}{\partial x_o}\frac{\partial p}{\partial y_o} - \frac{\partial x}{\partial y_o}\frac{\partial p}{\partial x_o}$$

The Eulerian energy equation neglecting heat conduction is

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[35] Refer to Goldstein (1950), section 8-4.

$$\rho \frac{d}{dt}(I+\frac{1}{2}\vec{u}\cdot\vec{u}) + \nabla\cdot(\vec{u}p) = 0,$$

which can be put into the form

$$\rho \frac{dI}{dt} + p\nabla\cdot\vec{u} = 0$$

with the aid of the momentum equation (15-5). By means of the transformation equations (16-5) and (16-6), we obtain the energy equation in Lagrangian coordinates,

$$\rho_0 \frac{\partial I}{\partial t} + p\frac{\partial J}{\partial t} = 0. \qquad (16\text{-}10)$$

b). *Lagrangian Equations in One-Dimension.* In plane geometry, the Jacobian (16-4) is merely

$$J = \frac{dx}{dx_0}.$$

The mass equation (16-8) becomes

$$\rho dx = \rho_0 dx_0, \qquad (16\text{-}11a)$$

the momentum equations (16-9) reduce to

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x_0} = 0 \qquad (16\text{-}11b)$$

and the energy equation (16-10) is

$$\rho_0 \frac{\partial I}{\partial t} + p\frac{\partial u}{\partial x_0} = 0. \qquad (16\text{-}11c)$$

The corresponding equations in spherical coordinates are

$$r^2 \rho dr = r_0^2 \rho_0 dr_0, \qquad (16\text{-}12a)$$

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial r_0} = 0, \qquad (16\text{-}12b)$$

$$\rho_0 \frac{\partial I}{\partial t} + \frac{p}{r_0^2} \frac{\partial}{\partial r_0} (r_0^2 u) = 0. \qquad (16\text{-}12c)$$

In problems with one-dimensional symmetry, Lagrangian hydrodynamics is almost always used. Various finite difference schemes are discussed and compared by Fromm (1961). In two-dimensions, Lagrangian methods encounter difficulties that do not exist in one-dimensional problems. A discussion of the relative merits of two-dimensional hydrodynamic calculations can be found in Alder, *et al.* (1964).

§17. *Relativistic Equations of Radiation Hydrodynamics.*

In order to develop a consistent theory of radiation hydrodynamics, it is necessary to use relativistically correct equations of hydrodynamics rather than the Newtonian equations. Only in this way can approximations in u/c be made consistently. L. H. Thomas was the first to develop such a theory of radiation hydrodynamics.[36] However, his theory was restricted to radiation transport by diffusion. A. R. Fraser derived the relativistic equations of hydrodynamics in a general radiation field.[37] The application of these equations correct to terms in u/c is described by

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[36] L. H. Thomas, Quart. J. Math. 1, 239-251 (1930).

[37] A. R. Fraser, "Radiative Transfer in Moving Media, Part III, "Atomic Weapons Research Establishment (Aldermaston), TPN-95/59 (1959).

I. P. Grant and R. Thomas,[38] and the general problem is reviewed by B. Wendroff.[39]

Let ρ be the density in a local rest frame of the fluid when the internal energy, I, of the fluid is zero. A local rest frame is a coordinate system in which the fluid velocity is zero locally (a Lagrangian frame). The total relativistic energy density is $\rho_1 c^2$, where

$$\rho_1 = (\rho+\rho I/c^2+p_m/c^2)(1-u^2/c^2)^{-1}. \tag{17-1}$$

The exact equations of relativistic hydrodynamics in the presence of a radiation field expressed in an Eulerian coordinate system are as follows:

(1) Equation of continuity

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\sqrt{1-u^2/c^2}}\right) + \nabla\cdot\left(\frac{\rho\vec{u}}{\sqrt{1-u^2/c^2}}\right) = 0, \tag{17-2}$$

(2) Momentum equation

$$\rho_1 \frac{d\vec{u}}{dt} + \nabla p_m + \frac{\vec{u}}{c^2}\frac{\partial p_m}{\partial t} - \frac{\vec{u}}{c^2}\left[\frac{\partial E}{\partial t} + \nabla\cdot\vec{F}\right] + {} + \frac{1}{c^2}\frac{\partial \vec{F}}{\partial t} + \nabla\cdot\vec{P} = 0. \tag{17-3}$$

(3) Energy equation

$$\rho\frac{dI}{dt} - \frac{p_m}{\rho}\frac{d\rho}{dt} + \left[\frac{\partial E}{\partial t} + \nabla\cdot\vec{F}\right] - \vec{u}\cdot\left[\frac{1}{c^2}\frac{\partial \vec{F}}{\partial t}+\nabla\cdot\vec{P}\right] = 0. \tag{17-4}$$

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[38] I. P. Grant and R. Thomas, "GARTH-1," Atomic Weapons Research Establishment (Aldermaston), TPN-71/61 (1961), and "Energy Conservation in GARTH," TPN-15/62 (1962).

[39] B. Wendroff, "On the Computation of the Transport of Radiation," Los Alamos Scientific Laboratory Report, LAMS-2795 (1963).

In the formulas above, E, $\vec{F}$, and $\vec{P}$ are the frequency integrated radiation energy density, flux vector, and pressure tensor; $I = c_v T$, is the internal energy per unit mass; and p_m is the material pressure.

In these equations, not all terms are of the same importance. We are justified in neglecting all matter terms divided by c^2. This is the approximation used by Grant and Thomas in their gray body calculation. Their basic equations are

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0, \qquad (17\text{-}5a)$$

$$\rho\,\frac{d\vec{u}}{dt} + \nabla p_m + \nabla\cdot\vec{P} - \frac{\vec{u}}{c^2}\left(\frac{\partial E}{\partial t}+\nabla\cdot\vec{F}\right) + \frac{1}{c^2}\,\frac{\partial \vec{F}}{\partial t} = 0, \qquad (17\text{-}5b)$$

$$\rho\,\frac{dI}{dt} - \frac{p_m}{\rho}\,\frac{d\rho}{dt} + \left(\frac{\partial E}{\partial t}+\nabla\cdot\vec{F}\right) - \vec{u}\cdot\left(\frac{1}{c^2}\,\frac{\partial \vec{F}}{\partial t}+\nabla\cdot\vec{P}\right) = 0. \qquad (17\text{-}5c)$$

The equations (17-5) are correct to terms in (u/c) but neglect some terms in $(u/c)^2$. Since most hydrodynamic problems are such that $u/c \leq 1/300$, this approximation is sufficient.

Note that the equations (17-5) contain frequency integrated radiation terms. If one wishes to do a frequency dependent calculation to terms in (u/c), Lorentz transformations of frequency, cross sections, emission source functions, etc. will have to be made in transporting the radiation from one element of the fluid to another element with different velocity.

§18. *Radiation Hydrodynamics in Plane and Spherical Geometry.*

In plane and spherical geometry, the radiation pressure tensor is

$$P_{jk} = \begin{bmatrix} P-\frac{\alpha}{4}(3P-E) & 0 & 0 \\ 0 & P-\frac{\alpha}{4}(3P-E) & 0 \\ 0 & 0 & P \end{bmatrix}, \qquad (18\text{-}1)$$

where $\alpha = 0$ for plane geometry, and $\alpha = 2$ for spherical geometry.[40] The term $\nabla\cdot\vec{P} \equiv \sum_k \partial P_{jk}/\partial x_k$ that appears in the hydrodynamic equations becomes in this geometry

$$(\nabla\cdot\vec{P})_r = \frac{\partial P}{\partial r} + \frac{\alpha}{2}\,\frac{3P-E}{r},$$

where

$$E = \frac{2\pi}{c}\int I^{\nu}d\mu,$$

$$P = \frac{2\pi}{c}\int I^{\nu}\mu^2 d\mu.$$

The hydrodynamic equations (17-5) for plane and spherical geometry in the Eulerian form are

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho u) = 0, \qquad (18\text{-}2a)$$

$$\rho\frac{du}{dt} + \frac{\partial}{\partial r}(p_m+P) + \frac{\alpha}{2}\,\frac{(3P-E)}{r} - \frac{u}{c^2}\left[\frac{\partial E}{\partial t}+\frac{1}{r^{\alpha}}\,\frac{\partial}{\partial r}(r^{\alpha}F)\right] + \frac{1}{c^2}\,\frac{\partial F}{\partial t} = 0, \qquad (18\text{-}2b)$$

$$\rho\frac{dI}{dt} - \frac{p_m}{\rho}\,\frac{d\rho}{dt} + \left[\frac{\partial E}{\partial t}+\frac{1}{r^{\alpha}}\,\frac{\partial}{\partial r}(r^{\alpha}F)\right] - \frac{u}{c^2}\,\frac{\partial F}{\partial t} - u\left[\frac{\partial P}{\partial r}+\frac{\alpha}{2}\,\frac{(3P-E)}{r}\right] = 0, \qquad (18\text{-}2c)$$

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[40] Refer to Wendroff, *op.cit.*, Chapter I, section 3.

where $u = dr/dt$. These equations can be transformed to the Lagrangian coordinate frame, i.e. the frame in which the fluid velocity is locally zero. The hydrodynamic equations in Lagrangian form are

$$dh = \rho r^{\alpha} dr, \tag{18-3a}$$

$$\frac{\partial u}{\partial t} + r^{\alpha} \frac{\partial}{\partial h} (p_m + P) + \frac{\alpha}{2} \frac{(3P-E)}{\rho r} = 0, \tag{18-3b}$$

$$\frac{\partial I}{\partial t} - \frac{p_m}{\rho^2} \frac{\partial \rho}{\partial t} + \frac{1}{\rho} \left[\frac{\partial E}{\partial t} + \frac{\partial (r^{\alpha} F)}{\partial h} \right] = 0. \tag{18-3c}$$

Finite difference approximations for these equations are discussed by Wendroff, *op.cit.*, Chapter III and by Grant and Thomas, *op.cit.*

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13. ABSTRACT (Distribution Limitation Statement No. 2)

This document is intended as an introduction to the study of radiative transfer and hydrodynamics in the high temperature (>10,000°K) regime. Most of the basic references on radiative transfer are oriented to astrophysics. It is always difficult for workers in other areas of research to cull from this literature those aspects of the subject relevant to their own interests. The purpose of this document is to present the basic principles of radiation gas dynamics from the point of view of weapons physics. The first chapter is concerned with the radiation field alone. The transfer equation is derived in its various forms, and the general formulation of radiant energy transfer in materials is discussed. The second chapter describes the various mechanisms by which the radiation field interacts with a material medium. This section, therefore, provides an introduction to the general problem of opacities. The third chapter presents the basic theory of hydrodynamics and the influence of radiation fields on fluid motion. These chapters cover the three fundamental aspects of the subject: the state of the radiation field, the radiation-material interaction, and the motion of the material.

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