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ANALYTICAL STUDIES OF TECHNIQUES FOR THE COMPUTATION OF HIGH-RESOLUTION WAVENUMBER SPECTRA

ADVANCED ARRAY RESEARCH
Special Report No. 9

Prepared by

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TEXAS INSTRUMENTS INCORPORATED
Science Services Division
P.O. Box 5621
Dallas, Texas 75222

Contract:	F33657-68-C-0867
Contract Date:	15 February 1968
Amount of Contract:	$341,000
Contract Expiration Date:	14 March 1969

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ADVANCED RESEARCH PROJECTS AGENCY
Nuclear Test Detection Office
ARPA Order No. 624
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Prepared for

AIR FORCE TECHNICAL APPLICATIONS CENTER
Washington, D. C. 20333

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ABSTRACT

A theoretical study of several computational techniques for estimating wavenumber spectra is presented. Relationships among the techniques used for computing high-resolution wavenumber spectra are investigated. Stability and resolution of their estimates are compared. Methods for computing 2-dimensional maximum-entropy spectra are found to be impractical, and it is concluded that the maximum-likelihood technique provides the best estimate for most purposes.

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SECTION I

INTRODUCTION AND SUMMARY

This report discusses the multiplicity of techniques available for computing frequency-wavenumber power-density spectra. Four techniques are investigated to illustrate typical properties of the high-resolution spectral methods in comparison with the conventional beamsteer output spectrum, with emphasis on the mathematical properties of the various spectra. When selection of a most appropriate spectral technique is discussed, however, practical considerations and the author's aesthetic preferences are introduced into the report.

A. TYPES OF SPECTRA

Four t pes of frequency-wavenumber power-density spectra are discussed.

The first of these,

$$P = V^H \Phi V$$

where V is the column vector

$$\begin{bmatrix} e^{i2\pi\vec{k}\cdot\vec{x}_1} \\ \vdots \\ e^{i2\pi\vec{k}\cdot\vec{x}_N} \end{bmatrix}$$

and Φ is the crosspower spectrum matrix for a sample of measured data, is the output of a beamsteer filter set which time-aligns the arrival of wavefronts coming from the direction specified by the wavenumber $\vec{k}$. By varying the wavenumber of the probe vector V, spectral values can be obtained for any set of wavenumber values.

The second type of spectrum is

$$Q = \frac{1}{V^H \Phi^{-1} V}$$

which is called the maximum-likelihood spectrum. It is the output of an optimum filter designed to pass a unit-amplitude plane-wave signal without distortion; i.e., if A is the optimum filter set to be designed, filter A minimizes the mean-square error $1 - V^H A - A^H V + A^H V V^H A + A^H \Phi A$, subject to the constraint $A^H V V^H A = 1$. This filter (and any scalar multiple of it) also maximizes the signal-to-noise ratio $(A^H V V^H A) / (A^H \Phi A)$.

The third type of spectrum is

$$R = \frac{V^H \Phi^{-1} V}{V^H \Phi^{-2} V}$$

which is called the constant-norm frequency-wavenumber power-density spectrum. It is the output of an optimum-gain multichannel filter designed to pass a unit-amplitude plane-wave signal under the constraint that the filter is of unit amplitude; i.e., it maximizes the output signal-to-noise ratio $(A^H V V^H A) / (A^H \Phi A)$, subject to the constraint $A^H A = 1$.

The fourtn type of spectrum,

$$S = \frac{1}{V^H \Lambda V}$$

where

$$\iint_R \frac{V V^H}{V^H \Lambda V} \, d\vec{k} = \Phi$$

is called the maximum-entropy spectrum. It maximizes the integral

$$\iint_R \log P(\vec{k})\, d\vec{k}$$

over the region R in wavenumber space where power is confined, subject to the constraint that the power spectrum's inverse Fourier transform

$$\iint_R P(\vec{k})\, VV^H\, d\vec{k}$$

agrees with the crosspower spectrum matrix Φ.

B. COMPARISON BETWEEN MAXIMUM-LIKELIHOOD AND BEAMSTEER SPECTRA

To ascertain the qualitative differences between the maximum-likelihood spectrum and the conventional beamsteer spectrum, three simple noise fields are studied:

- Random noise only
- Random noise plus a single plane wave
- Random noise plus two plane waves

The principal difference between the maximum-likelihood spectrum and the conventional spectrum in these cases is that spectral-window effects disappear in the maximum-likelihood spectrum as the random noise component vanishes. This feature, of course, does not apply to the conventional spectrum.

C. MAXIMUM-ENTROPY SPECTRUM

The analytic solution for the maximum-entropy spectrum is known only for an equally spaced line array. For this case, the maximum-entropy spectrum is

$$S(k) = \frac{1}{U} \frac{P}{V^H \Gamma \Gamma^H V}$$

where

P and $\Gamma^H = \{1, \gamma_2^*, \gamma_3^*, \ldots, \gamma_N^*\}$ are solutions of the matrix equation

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1N} \\ \Phi_{21} & \Phi_{22} & \cdots & \Phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{N1} & \Phi_{N2} & \cdots & \Phi_{NN} \end{bmatrix} \begin{bmatrix} 1 \\ \gamma_2 \\ \vdots \\ \gamma_N \end{bmatrix} = \begin{bmatrix} P \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$U = 1/\Delta x$, and

V is the column vector

$$\begin{bmatrix} 1 \\ e^{i2\pi k\Delta x} \\ \vdots \\ e^{i2\pi(N-1)k\Delta x} \end{bmatrix}$$

corresponding to a plane wave specified by the wavenumber k.

Studied in an attempt to extend the maximum-entropy spectrum to multidimensional arrays is an equilateral-triangle array with the associated crosspower spectrum matrix

$$\begin{bmatrix} \phi_0 & \phi_1 & \phi_1 \\ \phi_1 & \phi_0 & \phi_1 \\ \phi_1 & \phi_1 & \phi_0 \end{bmatrix}$$

A direct integration of the constraint equations

$$\iint_R \frac{VV^H}{V^H \Lambda V} \, d\vec{k} = \Phi$$

will lead to extremely complicated expressions for ϕ_0 and ϕ_1 in terms of elliptic integrals of the first and third kind. The arguments of these elliptic integral functions contain expressions involving square roots of square roots of the elements of the matrix Λ. A power series expansion will be found for the ratio of the off-diagonal elements to the main-diagonal elements of the matrix Λ in terms of the ratio ϕ_1/ϕ_0, but this expansion does not provide any recognizable clue regarding the nature of the general analytic solution of the multidimensional maximum-entropy spectrum. However, a similar technique might be used to express the matrix Λ in terms of a matrix power series involving the crosspower spectrum matrix.

D. COMPARISON OF TECHNIQUES

A hierarchy of resolution is determined for the constant-norm spectrum (R), the maximum-likelihood spectrum (Q), and the beamsteer spectrum (P). From empirical results, the maximum-entropy spectrum (S) is known to have resolution comparable to the maximum-likelihood spectrum.

An interesting property of the maximum-likelihood and constant-norm spectra is that they are equal to zero whenever the unit-amplitude probe vector U in the formulas

$$Q = \frac{1}{U^H \Phi^{-1} U} \quad \text{and} \quad R = \frac{U^H \Phi^{-1} U}{U^H \Phi^{-2} U}$$

lies outside the subspace spanned by the nonzero eigenvectors of Φ. The conventional beamsteer spectrum (P) is not identically zero under the same conditions; sidelobe effects make it nonzero except under the much more restrictive condition that the probe vector U is perpendicular to every nonzero eigenvector of Φ.

Of all four types of spectra studied in this report, only the maximum-entropy spectrum (S) is consistent with the original crosspower spectrum matrix upon inverse Fourier transformation from wavenumber space.

In selecting a computational technique from the four studied during this contract, several salient points must be considered: (1) the conventional beamsteer spectrum (P) suffers from undesirable sidelobe effects and poor directional resolution; (2) the constant-norm spectrum (R) is the most sensitive to measurement errors and is based on a constraint which has no clear physical significance; (3) the maximum-likelihood spectrum (Q) appears to be a suitable compromise which avoids the undesirable sidelobe characteristics and poor resolution of the beamsteer spectrum and the extreme sensitivity of the constant-norm spectrum but, like the first two techniques, is not consistent with the original crosspower spectrum matrix; (4) the maximum-entropy spectrum(S) eliminates this defect while preserving all of its advantages. Unfortunately, the maximum-entropy algorithm is known only for an equally-spaced line array.

It appears that the best course of action is to use the maximum-entropy spectrum in those cases where the algorithm is now known or can be found sometime in the future. Otherwise, the maximum-likelihood spectrum is the most advantageous alternative available.

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SECTION II

METHODS OF FREQUENCY-WAVENUMBER SPECTRAL COMPUTATION

This section describes several methods of computing frequency-wavenumber power-density spectra and explains how they originated. More specific discussions of the properties of these techniques follow in later sections.

A. BEAMSTEER OUTPUT

The simplest form of a frequency-wavenumber power-density spectrum is obtained by designing sets of beamsteer filters which time-align the arrival of wavefronts associated with the directions or wavenumber vectors at which spectral values are desired.

Consider a plane wave of wavelength λ propagating in the direction specified by the direction cosines $(\gamma_1, \ldots, \gamma_n)$ in an n-dimensional space. Figure II-1 illustrates this situation in three dimensions. Let its time waveform at the origin be given by $g(t) = \cos 2\pi(f_0 t + c)$ and let $\vec{k}_0$ be the vector wavenumber $(\gamma_1, \ldots, \gamma_n) \cdot \frac{1}{\lambda}$, which points in the direction of propagation of the plane wave and has the magnitude $\frac{1}{\lambda}$.

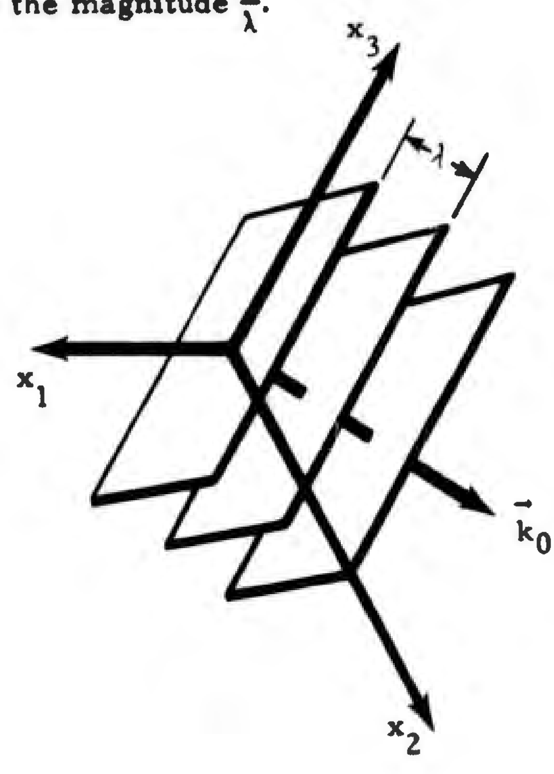

Figure II-1. Plane Wave Specified by Vector Wavenumber $\vec{k}_0 = (\gamma_1, \ldots, \gamma_n)/\lambda$

Inasmuch as the magnitude of the velocity of propagation $\vec{V}_p$ = frequency times wavelength λ and since the perpendicular distance to any point $\vec{x}$ from a plane through the origin with a normal specified by the direction cosines $(\gamma_1, \ldots, \gamma_n)$ is given by the formula $(\gamma_1, \ldots, \gamma_n) \cdot \vec{x}$, the plane wave will arrive at the position of the point $\vec{x}$ after a time lag

$$\tau = \frac{(\gamma_1, \ldots, \gamma_n) \cdot \vec{x}}{f_0 \lambda} = \frac{\vec{k}_0 \cdot \vec{x}}{f_0} .$$

Therefore, the equation of this plane wave at position $\vec{x}$ is given by

$$g(\vec{x}, t) = \cos 2\pi (f_0 t + c - \vec{k}_0 \cdot \vec{x})$$

To steer the array so that it enhances plane waves from a particular direction, time delays can be applied to the sensor outputs to time-align the arrival of wavefronts associated with that direction. As just shown, the time difference $\tau = t_{\vec{x}} - t_{\vec{0}}$ between the time of arrival $t_{\vec{x}}$ at $\vec{x}$ and the time of arrival $t_{\vec{0}}$ at the origin for a plane wave $g(\vec{x}, t) = \cos 2\pi (ft + c - \vec{k}_0 \cdot \vec{x})$ is given by the formula

$$\tau = \frac{\vec{k}_0 \cdot \vec{x}}{f}$$

The appropriate processing time delay to cancel the effect of this propagation delay at the sensor location $\vec{x}_j$ is

$$\tau_j = - \frac{\vec{k}_0 \cdot \vec{x}_j}{f} = - \frac{\vec{V}_p \cdot \vec{x}_j}{\vec{V}_p \cdot \vec{V}_p}$$

where $\vec{V}_p$ is the velocity of propagation of the plane wave $g(\vec{x}, \tau) = \cos 2\pi (ft + c - \vec{k}_0 \cdot \vec{x})$. The application of a time delay τ_j to the j^{th} sensor is mathematically equivalent to the application of a convolution filter $v_j(\tau) = \delta(\tau - \tau_j)$ whose Fourier transform is

$$V_j(f) = \int_{-\infty}^{\infty} \delta(\tau - \tau_j)\; e^{-i2\pi f\tau} dt$$

$$= e^{-i2\pi f\tau_j}$$

$$= e^{i2\pi \vec{k}_0 \cdot \vec{x}_j}$$

If such a steering filter is applied to each sensor of an N-sensor array, the associated beamsteer filter set for the wavenumber $\vec{k}$ is specified by the column vector

$$V = \begin{bmatrix} e^{i2\pi\vec{k}\cdot\vec{x}_1} \\ e^{i2\pi\vec{k}\cdot\vec{x}_2} \\ \vdots \\ e^{i2\pi\vec{k}\cdot\vec{x}_N} \end{bmatrix}$$

and the power output of the filter set is given by

$$P = V^H \Phi V$$

where Φ is the crosspower spectrum matrix

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1N} \\ \Phi_{21} & \Phi_{22} & \cdots & \Phi_{2N} \\ \vdots & \vdots & & \vdots \\ \Phi_{N1} & \Phi_{N2} & \cdots & \Phi_{NN} \end{bmatrix}$$

corresponding to some data sample measured by the array.

For any desired direction specified by some wavenumber $\vec{k}$, the beamsteer filter output P can be computed by premultiplying and postmultiplying the crosspower spectrum matrix Φ by the appropriate column vector V. Thus the column vector V serves the function of a probe vector which permits determination of a frequency-wavenumber power-density spectrum at any wavenumber $\vec{k}$ by the matrix operations just described.

The beamsteer output method of frequency-wavenumber spectral computation suffers from the disadvantage that the true frequency-wavenumber spectrum is convolved with the spectral window $V^H V$ corresponding to the array used. The effect is to distort the true spectrum with a smudge function in wavenumber space. This smudge function is the spectral window. A delta function spectrum in wavenumber space, for example, will appear as a wavenumber-translated spectral window.* When the logarithm of such a wavenumber power spectrum is plotted, a broad convex lobe occurs at the wavenumber corresponding to the true spectral peak and is echoed at those wavenumbers which correspond to sidelobes in the spectral window. Devised in an attempt to reduce or eliminate this disadvantage is a multiplicity of computational techniques which are generally known as high-resolution frequency-wavenumber spectra; this designation is used to contrast them with the beamsteer technique, which is known as the conventional frequency-wavenumber spectrum. The remainder of this section discusses three such techniques.

B. MAXIMUM-LIKELIHOOD FILTER OUTPUT

The maximum-likelihood frequency-wavenumber power-density spectrum is obtained by designing sets of maximum-likelihood filters which minimize the mean-square-error for unit-amplitude plane-wave signals arriving at the directions or wavenumbers for which spectral values are desired. These maximum-likelihood filter sets are subject to the constraint that the unit-amplitude plane-wave signals must have unit-amplitude output upon application of the associated filter sets.

*Texas Instruments Incorporated, 1966: Array Configuration Selection Report, Contract NONR-4045 (00) (FBM) (X), 1 Dec.

For a unit-amplitude signal model specified by the column vector

$$V = \begin{bmatrix} e^{i2\pi\vec{k}\cdot\vec{x}_1} \\ e^{i2\pi\vec{k}\cdot\vec{x}_2} \\ \vdots \\ e^{i2\pi\vec{k}\cdot\vec{x}_N} \end{bmatrix}$$

the mean-square-error for the filter set

$$V = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix}$$

is equal to

$$1 - V^H A - A^H V + A^H V V^H A + A^H \Phi A$$

where matrix Φ is the noise crosspower spectrum matrix corresponding to some data sample recorded by an N-sensor array. This quantity must be minimized subject to the constraint that

$$A^H V V^H A = 1$$

Therefore, the partial derivatives of the quantity

$$\mathcal{E} = 1 - V^H A - A^H V + A^H V V^H A + A^H \Phi A + \lambda\,(A^H V V^H A - 1)$$

$$= (1 - V^H A)\,(1 - A^H V) + A^H \Phi A + \lambda\,(A^H V V^H A - 1)$$

with respect to the real part Re A_k and the imaginary part Im A_k must equal zero.

The Lagrangian multiplier λ is to be determined from the constraint equation just described:

$$\frac{\partial \mathcal{E}}{\partial \mathrm{Re} A_k} = - V_k^*(1 - A^H V) - V_k(1 - V^H A) + (\Phi A)_k$$

$$+ (A^H \Phi)_k + \lambda \left[V_k^* (A^H V) + V_k (V^H A) \right]$$

$$\frac{\partial \mathcal{E}}{\partial \mathrm{Im} A_k} = - iV_k^* (1 - A^H V) + iV_k (1 - V^H A) - i(\Phi A)_k$$

$$+ i(A^H \Phi)_k + \lambda \left[iV_k^* (A^H V) - iV_k (V^H A) \right]$$

$$\frac{\partial \mathcal{E}}{\partial \mathrm{Re} A_k} + i \frac{\partial \mathcal{E}}{\partial \mathrm{Im} A_k} = 0 = - 2V_k (1 - V^H A) + 2(\Phi A_k) + 2\lambda V_k (V^H A)$$

so that

$$\Phi A = V(1 - V^H A - \lambda V^H A)$$

$$A = [1 - (\lambda + 1) V^H A] \Phi^{-1} V$$

$$A^H V V^H A = [1 - (\lambda + 1) V^H A] (A^H V)(V^H \Phi^{-1} V)$$

$$= [A^H V - (\lambda + 1)(A^H V V^H A)] (V^H \Phi^{-1} V)$$

Under the constraint $A^H V V^H A = 1$, the last equation reduces to

$$\frac{1}{V^H \Phi^{-1} V} = A^H V - (\lambda + 1)$$

or

$$\lambda + 1 = A^H V - \frac{1}{V^H \Phi^{-1} V}$$

Substitution for $\lambda + 1$ yields

$$\Phi A = \left[1 - \left(A^H V - \frac{1}{V^H \Phi^{-1} V}\right) V^H A\right] V$$

$$= \frac{V V^H A}{V^H \Phi^{-1} V}$$

or

$$A^H \Phi A = \frac{A^H V V^H A}{V^H \Phi^{-1} V} = \frac{1}{V^H \Phi^{-1} V}$$

The mean-square-error $1 - V^H A - A^H V + A^H V V^H A + A^H \Phi A$ then equals

$$\left|1 - V^H A\right|^2 + \frac{1}{V^H \Phi^{-1} V}$$

It is minimized when $V^H A = 1 = A^H V$, a condition consistent with the constraint equation $A^H V V^H A = 1$. Therefore, the filter set which minimizes the mean-square-error is

$$A = \left(\frac{V^H A}{V^H \Phi^{-1} V}\right) \Phi^{-1} V = \frac{\Phi^{-1} V}{V^H \Phi^{-1} V}$$

The mean-square-error can be written $\left|1 - V^H A\right|^2 + A^H \Phi A$. The term $\left|1 - V^H A\right|^2$ under the constraint $1 = A^H V V^H A = \left|V^H A\right|^2$ or $\left|V^H A\right| = 1$ can be affected only by the phase of A. On the other hand, the term $A^H \Phi A$ is unaffected by the phase of A, for $e^{-i\phi} A^H \Phi A e^{i\phi} = A^H \Phi A$; it is affected solely by the magnitudes of the components of A and by their relative phases. The minimization of the mean-square-error subject to the constraint $A^H V V^H A = 1$ is accomplished by setting $A = e^{i\phi} \Phi^{-1} V / (V^H \Phi^{-1} V)$ to minimize the term $A^H \Phi A$ and by setting $e^{i\phi} = 1$ to zero the term $|1 - V^H A|^2$. Thus, any filter set $A = e^{i\phi}\, \Phi^{-1} V / (V^H \Phi^{-1} V)$ minimizes the output noise power $A^H \Phi A$ subject to the constraint $A^H V V^H A = 1$.

Under the constraint $A^H V V^H A = 1$, the output signal-to-noise ratio

$$\frac{A^H V V^H A}{A^H \Phi A}$$

is maximized by minimizing the output noise power $A^H \Phi A$. However, multiplying A by any positive real-valued scale factor ρ in no way affects the output signal-to-noise ratio

$$\frac{\rho A^H V V^H A \rho}{\rho A^H \Phi A \rho} = \frac{A^H V V^H A}{A^H \Phi A}$$

The constraint $A^H V V^H A = 1$ can therefore be removed without affecting the output signal-to-noise ratio, and any filter set $A = \rho\, e^{i\phi}\, \Phi^{-1} V / (V^H \Phi^{-1} V)$ maximizes the output signal-to-noise ratio.

It has been shown that:

- Any complex-valued scalar multiple of $\Phi^{-1} V / (V^H \Phi^{-1} V)$ maximizes the output signal-to-noise ratio $(A^H V V^H A)/(A^H \Phi A)$
- Any unit-amplitude complex-valued scalar multiple of $\Phi^{-1} V / (V^H \Phi^{-1} V)$ minimizes the output noise power $A^H \Phi A$ subject to the constraint $A^H V V^H A = 1$
- The filter set $A = \Phi^{-1} V / (V^H \Phi^{-1} V)$ minimizes the mean-square-error $|1 - V^H A|^2 + A^H \Phi A$, subject to the same constraint

Under the constraint condition $A^H V V^H A = 1$, the noise output power $A^H \Phi A$ of the filter set which minimizes the mean-square-error $|1 - V^H A|^2 + A^H \Phi A$, minimizes the output noise power $A^H \Phi A$, and maximizes the output signal-to-noise ratio $(A^H V V^H A)/(A^H \Phi A)$, is equal to

$$A^H \Phi A = \left(\frac{e^{i\phi} V^H \Phi^{-1}}{V^H \Phi^{-1} V}\right) \Phi \left(\frac{\Phi^{-1} V e^{i\phi}}{V^H \Phi^{-1} V}\right) = \frac{V^H \Phi^{-1} \Phi \Phi^{-1} V}{(V^H \Phi^{-1} V)^2} = \frac{1}{V^H \Phi^{-1} V}$$

C. CONSTANT-NORM FILTER OUTPUT

The constant-norm frequency-wavenumber power-density spectrum is obtained by designing sets of optimum-gain filters for unit-amplitude plane-wave signals at the directions or wavenumbers for which spectral values are desired. These filter sets are then gained so that they are of unit amplitude.

An optimum-gain filter set was previously shown to be achieved when the filter set

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix}$$

is equal to any scalar multiple of $\Phi^{-1} V$. If A is set equal to $(\Phi^{-1} V)/\sqrt{V^H \Phi^{-2} V}$, the constraint $A^H A = 1$ is satisfied:

$$A^H A = \frac{V^H \Phi^{-1} \Phi^{-1} V}{V^H \Phi^{-2} V} = 1$$

The power output for such a filter set is

$$A^H \Phi A = \frac{V^H \Phi^{-1} \Phi \Phi^{-1} V}{V^H \Phi^{-2} V} = \frac{V^H \Phi^{-1} V}{V^H \Phi^{-2} V} .$$

Thus, another method of computing frequency-wavenumber spectra can be implemented by varying the wavenumber probe vector $V(\vec{k})$ over the set of wavenumber values for which a spectrum is desired.

D. MAXIMUM-ENTROPY SPECTRUM

This method of spectral estimation uses the criterion that the spectral estimate must be the most random or have the maximum entropy of any spectrum which is consistent with the measured crosspower spectrum matrix.

The entropy of a frequency-wavenumber power-density spectrum $P(\vec{k})$ can be defined to be the integral of the logarithm of the spectrum over the region in wavenumber space in which the power is confined. Stated as a formula, the entropy is

$$\iint_R \log P(\vec{k})\, d\vec{k}$$

where R denotes the region in wavenumber space in which the power is confined. For a uniformly spaced array, the power can be considered to be confined to the unit cell in wavenumber space which corresponds to the geometry of the array elements, so that R can be chosen to be the unit cell in wavenumber space.

A frequency-wavenumber spectrum, to be consistent over the unit cell with the measured crosspower spectrum matrix, must satisfy the equations

$$\iint_R P(\vec{k}) \cos 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell)\, d\vec{k} = \mathrm{Re}\ \Phi_{j\ell}$$

$$\iint_R P(\vec{k}) \sin 2\pi\vec{k} \cdot (x_j - x_\ell)\, d\vec{k} = \mathrm{Im}\ \Phi_{j\ell}$$

corresponding to each crosspower spectrum value $\Phi_{j\ell}$ of the crosspower spectrum matrix. In these equations, the indices j and ℓ correspond to the sensor pair consisting of the j^{th} and ℓ^{th} sensors. The vectors $\vec{x}_j$ and $\vec{x}_\ell$ denote the position of the j^{th} and ℓ^{th} sensors, respectively. The vector $\vec{k}$ denotes any of the set of wavenumber vectors in the unit cell over which the integration is performed. For an array having N sensors, there is a total of N^2 entries $\Phi_{j\ell}$ in the cross-power spectrum matrix, so a total of $2N^2$ equations must be satisfied. These equations are not independent of each other, however. When $j \neq \ell$,

$$\cos 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell) = \cos 2\pi\vec{k} \cdot (\vec{x}_\ell - \vec{x}_j)$$

and

$$\sin 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell) = -\sin 2\pi\vec{k} \cdot (\vec{x}_\ell - \vec{x}_j)$$

so that

$$\text{Re}\ \Phi_{j\ell} = \text{Re}\ \Phi_{\ell j}$$

and

$$\text{Im}\ \Phi_{j\ell} = -\text{Im}\ \Phi_{\ell j}$$

Furthermore, when $j = \ell$

$$\cos 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell) = 1$$

and

$$\sin 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell) = 0$$

so that

$$\text{Re}\ \Phi_{jj} = \iint_R P(\vec{k})\, d\vec{k}$$

and

$$\text{Im}\ \Phi_{jj} = 0$$

These properties are consistent with the fact that the crosspower spectrum matrix must be Hermitian. Also implicit in this formulation is the equality of all autopower spectra along the main diagonal of the matrix. This equality is a consequence of the assumption of space-stationarity: all cross-power spectra between sensor pairs at the same vector displacement $\vec{x}_j - \vec{x}_\ell$ must be equal. When the assumption of space stationarity is not satisfied, the concept of a wavenumber power-density spectrum does not make sense.

By eliminating redundant constraint conditions, the problem of determining the maximum-entropy frequency-wavenumber power-density spectrum reduces to the maximization of the integral

$$\iint_R \log P(\vec{k})\, d\vec{k}$$

subject to the constraints

$$\iint_R P(\vec{k})\, d\vec{k} = \Phi_{jj} \qquad \text{for all } j$$

$$\iint_R P(\vec{k}) \cos 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell)\, d\vec{k} = \mathrm{Re}\, \Phi_{j\ell} \qquad \text{for all } j < \ell$$

$$\iint_R P(\vec{k}) \sin 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell)\, d\vec{k} = \mathrm{Im}\, \Phi_{j\ell} \qquad \text{for all } j < \ell$$

This is an isoperimetric problem in the calculus of variations. Let $\mathcal{P}(\vec{k})$ denote the function

$$\log P(\vec{k}) + P(\vec{k})\left\{\rho + \sum_{j=1}^{N-1} \sum_{\ell=j+1}^{N} \left[\mu_{j\ell} \cos 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell) + \nu_{j\ell} \sin 2\pi\vec{k} \cdot (x_j - x_\ell)\right]\right\}$$

In order that the entropy be maximized subject to the specified constraints,

$$0 = \frac{\partial \mathcal{P}(\vec{k})}{\partial P(\vec{k})} = \frac{1}{P(\vec{k})} + \rho + \sum_{j=1}^{N-1} \sum_{\ell=j+1}^{N} \left[\mu_{j\ell} \cos 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell) + \nu_{j\ell} \sin 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell)\right]$$

For all $j < \ell$, let $\lambda_{j\ell} = -\frac{1}{2}(\mu_j + i\nu_j)$ and $\lambda_{\ell j} = -\frac{1}{2}(\mu_j - i\nu_j)$. Furthermore, let $\lambda_{jj} = -\frac{\rho}{N}$. Then,

$$0 = \frac{1}{P(\vec{k})} - \sum_{j=1}^{N} \sum_{\ell=1}^{N} \lambda_{j\ell}\, e^{-i 2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell)}$$

and

$$P(\vec{k}) = \frac{1}{V^H \Lambda V}$$

where V is the column vector

$$\begin{bmatrix} e^{i2\pi\vec{k} \cdot \vec{x}_1} \\ e^{i2\pi\vec{k} \cdot \vec{x}_2} \\ \vdots \\ e^{i2\pi\vec{k} \cdot \vec{x}_N} \end{bmatrix}$$

and Λ is the Hermitian matrix

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} \cdots & \lambda_{1N} \\ \lambda_{21} & \lambda_{22} \cdots & \lambda_{2N} \\ \vdots & \vdots & \vdots \\ \lambda_{N1} & \lambda_{N2} \cdots & \lambda_{NN} \end{bmatrix}$$

and the superscript H denotes conjugate transpose. The matrix Λ is determined by the set of constraint equations

$$\iint_R \frac{e^{i2\pi \vec{k} \cdot (\vec{x}_j - \vec{x}_\ell)} \, d\vec{k}}{V^H \Lambda V} = \Phi_{j\ell} \qquad \left\{ j = 1, 2, \ldots, N; \ell = 1, 2, \ldots, N \right\}$$

The analytic solution for matrix Λ is known only for the case of an equally spaced line array. For other types of arrays, iterative or approximating techniques appear necessary for specifying the matrix Λ .

SECTION III

MATHEMATICAL PROPERTIES OF MAXIMUM-LIKELIHOOD FILTER OUTPUT

A. DETERMINANT EXPRESSION FOR FILTER OUTPUT POWER

Any crosspower spectrum matrix can be written in the form

$$\Phi = \rho I + V_1 V_1^H + \ldots + V_n V_n^H$$

In some cases, it is easy to associate the rank 1 matrices $V_j V_j^H$ with the arrival of specific plane waves; in other cases where data sensed by an array are not produced by a small number of plane waves, this particular form of the crosspower spectrum matrix does not have much physical significance. In either event, this way of expressing the crosspower spectrum matrix allows the maximum-likelihood filter output power $1/(U^H \Phi^{-1} U)$, corresponding to a unit-amplitude plane-wave signal model U, to be expressed as the ratio of two determinants. Appendix A shows that the filter output power is equal to

$$\rho \frac{\begin{vmatrix} \rho + |V_1|^2 & V_1^H V_2 & \ldots & V_1^H V_n \\ V_2^H V_1 & \rho + |V_2|^2 & \ldots & V_2^H V_n \\ \vdots & \vdots & & \vdots \\ V_n^H V_1 & V_n^H V_2 & \ldots & \rho + |V_n|^2 \end{vmatrix}}{\begin{vmatrix} |U|^2 & U^H V_1 & \ldots & U^H V_n \\ V_1^H U & \rho + |V_1|^2 & \ldots & V_1^H V_n \\ \vdots & \vdots & & \vdots \\ V_n^H U & V_n^H V_1 & \ldots & \rho + |V_n|^2 \end{vmatrix}}$$

This determinant form of the filter output power is used in this section to investigate the way in which the maximum-likelihood frequency-wavenumber power-density spectrum portrays noise fields composed of a small number of plane waves.

B. SPECTRUM FOR NOISE FIELD WITH RANDOM NOISE AND SINGLE PLANE WAVE

When a noise field consists of spatially uncorrelated noise and a single plane wave, the crosspower spectrum matrix can be written as

$$\Phi = \rho I + VV^H$$

where column vector V is equal to

$$a_V \begin{bmatrix} e^{i2\pi \vec{k} \cdot \vec{x}_1} \\ e^{i2\pi \vec{k} \cdot \vec{x}_2} \\ \vdots \\ e^{i2\pi \vec{k} \cdot \vec{x}_N} \end{bmatrix}$$

The constant a_V is the amplitude of the plane wave, which propagates toward the direction specified by the wavenumber $\vec{k}$. The maximum-likelihood wavenumber spectrum for this noise field is equal to

$$\frac{\rho \left| \rho + |V|^2 \right|}{\begin{vmatrix} |U|^2 & U^H V \\ V^H U & \rho + |V|^2 \end{vmatrix}}$$

or

$$\frac{\rho\left(\rho + |V|^2\right)}{|U|^2\left(\rho + |V|^2\right) - |U^H V|^2}$$

To illustrate the qualitative difference between the maximum-likelihood wavenumber spectrum and the conventional spectrum, let the random noise component approach zero in the limit. If U is a complex-valued scalar multiple of V, $|U^H V|^2 = |U|^2 |V|^2$, so the expression $|U|^2 (\rho + |V|^2) - |U^H V|^2 = \rho |U|^2$. The maximum-likelihood spectrum is then given by

$$\lim_{\rho \to 0} \frac{1}{U^H \Phi^{-1} U} = \lim_{\rho \to 0} \frac{\rho(\rho + |V|^2)}{\rho |V|^2} = \frac{|U|^2}{|V|^2}$$

In particular, if $U = V/|V|$,

$$\lim_{\rho \to 0} \frac{1}{U^H \Phi^{-1} U} = |V|^2$$

Under the same conditions, the conventional spectrum yields the same spectral value

$$\lim_{\rho \to 0} U^H \Phi V = \lim_{\rho \to 0} \rho |U|^2 + |U^H V|^2 = |V|^2$$

if $U = V/|V|$.

On the other hand, if U is not a scalar multiple of V, the expression $|U|^2 (\rho + |V|^2) - |U^H V|^2$ approaches the finite value $|U|^2 |V|^2 - |U^H V|^2$ as ρ approaches zero. However, the expression $\rho(\rho + |V|^2)$ approaches zero as ρ approaches zero. Thus, the maximum-likelihood spectrum is

$$\lim_{\rho \to 0} \frac{1}{U^H \Phi^{-1} U} = \lim_{\rho \to 0} \frac{\rho(\rho + |V|^2)}{|U|^2 |V|^2 - |U^H V|^2} = 0$$

if $U \neq \alpha V$.

When U is not a scalar multiple of V, the conventional spectrum consists of a spectral window term

$$\lim_{\rho \to 0} U^H \Phi U = \lim_{\rho \to 0} \rho |U|^2 + |U^H V|^2 = |U^H V|^2$$

if $U \neq \alpha V$. For this particular case, the spectral window term disappears in the maximum-likelihood spectrum as the random noise component vanishes.

C. SPECTRUM FOR NOISE FIELD WITH RANDOM NOISE AND TWO PLANE WAVES

For a noise field consisting of spatially uncorrelated noise and two plane waves, the crosspower spectrum matrix can be written as

$$\Phi = \rho I + VV^H + WW^H$$

where column vectors V and W are equal to

$$a_V \begin{bmatrix} e^{i2\pi \vec{k}_V \cdot \vec{x}_1} \\ e^{i2\pi \vec{k}_V \cdot \vec{x}_2} \\ \vdots \\ e^{i2\pi \vec{k}_V \cdot \vec{x}_N} \end{bmatrix} \quad \text{and} \quad a_W \begin{bmatrix} e^{i2\pi \vec{k}_W \cdot \vec{x}_1} \\ e^{i2\pi \vec{k}_W \cdot \vec{x}_2} \\ \vdots \\ e^{i2\pi \vec{k}_W \cdot \vec{x}_N} \end{bmatrix}$$

The constants a_V and a_W represent the respective amplitudes of the plane waves, which propagate toward the directions specified by wavenumbers $\vec{k}_V$ and $\vec{k}_W$. The corresponding maximum-likelihood wavenumber spectrum is

$$\frac{\rho \begin{vmatrix} \rho + |V|^2 & V^H W \\ W^H V & \rho + |W|^2 \end{vmatrix}}{\begin{vmatrix} |U|^2 & U^H V & U^H W \\ V^H U & \rho + |V|^2 & V^H W \\ W^H U & W^H V & \rho + |W|^2 \end{vmatrix}} = \frac{\rho \left[\rho^2 + \rho |V|^2 + \rho |W|^2 + \begin{vmatrix} |V|^2 & V^H W \\ W^H V & |W|^2 \end{vmatrix} \right]}{\rho^2 |U|^2 + \rho \begin{vmatrix} |U|^2 & U^H V \\ V^H U & |V|^2 \end{vmatrix} + \rho \begin{vmatrix} |U|^2 & U^H W \\ W^H U & |W|^2 \end{vmatrix} + \begin{vmatrix} |U|^2 & U^H V & U^H W \\ V^H U & |V|^2 & V^H W \\ W^H U & W^H V & |W|^2 \end{vmatrix}}$$

In the event that U is not a linear combination of V and W, i. e., cannot be written as $\alpha V + \beta W$ (where α and β are complex-valued scalars), either

$$\begin{vmatrix} |U|^2 & U^H V & U^H W \\ V^H U & |V|^2 & V^H W \\ W^H U & W^H V & |W|^2 \end{vmatrix} \neq 0 \quad \text{or} \quad \begin{vmatrix} |V|^2 & V^H W \\ W^H V & |W|^2 \end{vmatrix} = 0$$

so that

$$\lim_{\rho \to 0} \frac{1}{U^H \mathbb{S}^{-1} U} = 0$$

if $U \neq \alpha V + \beta W$. Under the same conditions, the conventional spectrum is

$$\lim_{\rho \to 0} U^H \mathbb{S} U = \lim_{\rho \to 0} \rho |U|^2 + |U^H V^2| + |U^H W|^2 = |U^H V|^2 + |U^H W|^2$$

The conventional spectrum contains the spectral-window terms $|U^H V|^2$ and $|U^H W|^2$. These spectral-window terms disappear in the maximum-likelihood spectrum as the random noise component vanishes.

In the event that $U = (\alpha V + \beta W)/|\alpha V + \beta W|$,

$$\begin{vmatrix} |U|^2 & U^H V & U^H W \\ V^H U & |V|^2 & V^H W \\ W^H U & W^H V & |W|^2 \end{vmatrix} = 0$$

Note also that

$$\begin{vmatrix} |U|^2 & U^H V \\ V^H U & |V|^2 \end{vmatrix} = \frac{1}{|\alpha V + \beta W|} \begin{vmatrix} U^H(\alpha V + \beta W) & U^H V \\ V^H(\alpha V + \beta W) & |V|^2 \end{vmatrix}$$

$$= \frac{\beta}{|\alpha V + \beta W|} \begin{vmatrix} U^H W & U^H V \\ V^H W & |V|^2 \end{vmatrix}$$

$$= \frac{\beta}{|\alpha V + \beta W|^2} \begin{vmatrix} \left(\alpha^* V^H + \beta^* W^H\right) W & \left(\alpha^* V^H + \beta^* W^H\right) V \\ V^H W & |V|^2 \end{vmatrix}$$

$$= \frac{|\beta|^2}{|\alpha V + \beta W|^2} \begin{vmatrix} |W|^2 & W^H V \\ V^H W & |V|^2 \end{vmatrix}$$

$$= \frac{|\beta|^2}{|\alpha V + \beta W|^2} \begin{vmatrix} |V|^2 & V^H W \\ W^H V & |W|^2 \end{vmatrix}$$

Similarly,

$$\begin{vmatrix} |U|^2 & U^H W \\ W^H U & |W| \end{vmatrix} = \frac{|\alpha|^2}{|\alpha V + \beta W|^2} \begin{vmatrix} |V|^2 & V^H W \\ W^H V & |W|^2 \end{vmatrix}$$

If V is not a scalar multiple of W,

$$\lim_{\rho \to 0} \frac{1}{U^H \Phi^{-1} U} = \frac{|\alpha V + \beta W|^2}{|\alpha|^2 + |\beta|^2}$$

In particular, if $U = V/|V|$,

$$\lim_{\rho \to 0} \frac{1}{U^H \Phi^{-1} U} = |V|^2$$

and, if $U = W/|W|$,

$$\lim_{\rho \to 0} \frac{1}{U^H \Phi^{-1} U} = |W|^2$$

Under the same conditions, the conventional spectrum gives

$$\lim_{\rho \to 0} U^H \Phi U = \lim_{\rho \to 0} \rho|U|^2 + |U^H V|^2 + |U^H W|^2 = |U^H V|^2 + |U^H W|^2$$

If $U = V/|V|$,

$$\lim_{\rho \to 0} U^H \Phi U = |V|^2 + \frac{|V^H W|^2}{|V|^2}$$

If $U = W/|W|$,

$$\lim_{\rho \to 0} U^H \Phi U = |W|^2 + \frac{|W^H V|}{|W|^2}$$

Note that the maximum-likelihood wavenumber spectrum gives spectral values which correctly reflect the magnitude of V and W when the unit-amplitude probe vector U points toward either vector. This property is contrasted with the fact that the conventional spectrum is affected by spectral-window terms in the same situation.

In the event that U is a linear combination of V and W and that W is a scalar multiple of V, the determinant

$$\begin{vmatrix} |V|^2 & V^H W \\ W^H V & |W|^2 \end{vmatrix}$$

vanishes and

$$\lim_{\rho \to 0} \frac{1}{U^H \Phi^{-1} U} = \lim_{\rho \to 0} \frac{\rho^3 + \rho^2 |V|^2 + \rho^2 |W|^2}{\rho^2 |U|^2} = |V|^2 + |W|^2$$

if $|U| = 1$. This particular case is of no additional interest since, if $W = \gamma V$ $WW^H = |\gamma|^2 VV^H$ and the crosspower spectrum matrix could have been written

$$\Phi = \rho I + (1 + |\gamma|^2) VV^H$$

Essentially the same case is covered in subsection B.

D. DISCUSSION

The results of this section are summarized in Table III-1. Of the cases studied, the last is the most interesting. As the proportion of random noise decreases, peaks of magnitude $|V|^2$ and $|W|^2$, respectively, appear in the high-resolution spectrum at $U = V/|V|$ and $U = W/|W|$. At all other points U, which are not a linear combination of V and W, the spectrum goes to zero.

No spectral-window effects are present as the random noise vanishes in the limit, and the spectral values correctly reflect the magnitudes of the plane waves V and W.

The terms $|U^H V|^2$ and $|U^H W|^2$ in the conventional spectrum are spectral-window effects.

Table III-1

SUMMARY OF PROPERTY STUDIES

	Maximum-Likelihood Spectrum	Conventional Spectrum
CASE I $\Phi = \rho I$	$Q = \rho / \lvert U \rvert^2$ $\lim_{\rho \to 0} Q = 0$	$P = \rho \lvert U \rvert^2$ $\lim_{\rho \to 0} P = 0$
CASE II $\Phi = \rho I + VV^H$	$Q = \dfrac{\rho \left\lvert \rho + \lvert V \rvert^2 \right\rvert}{\begin{vmatrix} \lvert U \rvert^2 & U^H V \\ V^H U & \rho + \lvert V \rvert^2 \end{vmatrix}}$ $\lim_{\rho \to 0} Q = \lvert V \rvert^2$ if $U = V / \lvert V \rvert$ $\lim_{\rho \to 0} Q = 0$ if $U \neq \alpha V$	$P = \rho \lvert U \rvert^2 + \lvert U^H V \rvert^2$ $\lim_{\rho \to 0} P = \lvert V \rvert^2$ if $U = V / \lvert V \rvert$ $\lim_{\rho \to 0} P = \lvert U^H V \rvert^2$ if $U \neq \alpha V$
CASE III $\Phi = \rho I + VV^H + WW^H$	$Q = \dfrac{\rho \begin{vmatrix} \rho + \lvert V \rvert^2 & V^H W \\ W^H V & \rho + \lvert W \rvert^2 \end{vmatrix}}{\begin{vmatrix} \lvert U \rvert^2 & U^H V & U^H W \\ V^H U & \rho + \lvert V \rvert^2 & V^H W \\ W^H U & W^H V & \rho + \lvert W \rvert^2 \end{vmatrix}}$ $\lim_{\rho \to 0} Q = \lvert V \rvert^2$ if $U = V / \lvert V \rvert$ $\lim_{\rho \to 0} Q = \lvert W \rvert^2$ if $U = W / \lvert W \rvert$ $\lim_{\rho \to 0} Q = 0$ if $U \neq \alpha V + \beta W$	$P = \rho \lvert U \rvert^2 + \lvert U^H V \rvert^2 + \lvert U^H W \rvert^2$ $\lim_{\rho \to 0} P = \lvert V \rvert^2 + \lvert V^H W \rvert^2 / \lvert V \rvert^2$ if $U = V / \lvert V \rvert$ $\lim_{\rho \to 0} P = \lvert W \rvert^2 + \lvert W^H V \rvert^2 / \lvert W \rvert^2$ if $U = W / \lvert W \rvert$ $\lim_{\rho \to 0} P = \lvert U^H V \rvert^2 + \lvert U^H W \rvert^2$ if $U \neq \alpha V + \beta W$

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SECTION IV

MAXIMUM-ENTROPY SPECTRUM

A. MAXIMUM-ENTROPY SPECTRUM FOR EQUALLY SPACED LINE ARRAY

In the event that data from an equally spaced line array are used to form a crosspower spectrum matrix, a method for computing the maximum-entropy frequency-wavenumber spectrum is known; it was developed by John Burg of Texas Instruments.

In Section II, the maximum-entropy spectrum is shown to be of the form

$$\frac{1}{V^H \Lambda V}$$

where Λ is a Hermitian matrix and V is the column vector

$$\begin{bmatrix} e^{i2\pi\vec{k}\cdot\vec{x}_1} \\ e^{i2\pi\vec{k}\cdot\vec{x}_2} \\ \vdots \\ e^{i2\pi\vec{k}\cdot\vec{x}_N} \end{bmatrix}$$

If the first sensor of an equally spaced line array is chosen as the origin of the sensor-location coordinate system and the positive x-axis runs from the first sensor to the N^{th} sensor, the coordinates of the sensors are 0, Δx, $2\Delta x$, ..., $(N - 1)\Delta x$. Thus, the column vector V can be written as

$$\begin{bmatrix} 1 \\ z \\ \vdots \\ z^{N-1} \end{bmatrix},$$

where

$$z = e^{i2\pi k/U}$$

and

$$U = \frac{1}{\Delta x}$$

Since the matrix Λ is Hermitian, the denominator $V^H \Lambda V$ of the maximum-entropy spectrum can be written as

$$\beta^*_{N-1} z^{-(N-1)} + \beta^*_{N-2} z^{-(N-2)} + \ldots + \beta^*_1 z^{-1} + \beta_0 + \beta_1 z + \ldots + \beta_{N-2} z^{N-2} + \beta_{N-1} z^{N-1}$$

and the term β_0 is real. The equation $V^H \Lambda V = 0$ has $2(N-1)$ roots which may be complex-valued. Note that if z is a root, $(z^*)^{-1}$ is also a root; if

$$\beta^*_{N-1} z^{-(N-1)} + \beta^*_{N-2} z^{-(N-2)} + \ldots + \beta^*_1 z^{-1} + \beta_0 + \beta_1 z^1 + \ldots + \beta_{N-2} z^{N-2} + \beta_{N-1} z^{N-1} = 0$$

then

$$\beta_{N-1} \left(\frac{1}{z^*}\right)^{N-1} + \beta_{N-2} \left(\frac{1}{z^*}\right)^{N-2} + \ldots + \beta_1 \left(\frac{1}{z^*}\right)^1 + \beta_0 + \beta^*_1 \left(\frac{1}{z^*}\right)^{-1} + \ldots + \beta^*_{N-2} \left(\frac{1}{z^*}\right)^{-(N-2)} + \beta^*_{N-1} \left(\frac{1}{z^*}\right)^{-(N-1)} = 0$$

Thus, the number of roots for which $|z| > 1$ is equal to the number of roots for which $|z| < 1$. If no roots lie on the unit circle, there are N - 1 roots outside the unit circle and N - 1 roots inside the unit circle; then, the denominator $V^H \Lambda V$ of the maximum-entropy spectrum can be written as

$$\frac{U}{P}\left[1 + \gamma_2 z^{-1} + \gamma_3 z^{-2} + \ldots + \gamma_N z^{-(N-1)}\right]\left[1 + \gamma_2^* z + \gamma_3^* z^2 + \ldots + \gamma_N^* z^{N-1}\right]$$

All of the roots within the unit circle have been incorporated in the factor at the left, and the constant P is a real number (as is U); thus, the maximum-entropy spectrum is equal to

$$\frac{P}{U}\left(\frac{1}{V^H \Gamma \Gamma^H V}\right)$$

where Γ is the column vector

$$\begin{bmatrix} 1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_N \end{bmatrix}$$

To determine the components $\gamma_2, \gamma_3, \ldots, \gamma_N$ of the column vector Γ, one must satisfy the constraint matrix equation

$$\frac{P}{U}\int_{-\frac{U}{2}}^{\frac{U}{2}} \frac{V V^H}{V^H \Gamma \Gamma^H V}\, dk = \Phi$$

Multiplication of both sides on the right by the column vector Γ gives

$$\frac{P}{U} \int_{-\frac{U}{2}}^{\frac{U}{2}} \frac{V V^H \Gamma}{V^H \Gamma \Gamma^H V} \, dk = \Phi \Gamma$$

or

$$\frac{P}{i2\pi} \oint \frac{V}{z \Gamma^H V} \, dz = \Phi \Gamma \qquad \left(dz = \frac{i2\pi z}{U} \, dk \right)$$

where the contour integration is performed over the unit circle. Looking at this vector equation component-by-component, one obtains

$$\frac{P}{i2\pi} \oint \frac{z^{j-2}}{\Gamma^H V} \, dz = (\Phi \Gamma)_j$$

Since the factor $V^H \Gamma$ contains all of the roots within the unit circle, the other factor $\Gamma^H V$ is never equal to zero within the unit circle. For all j from 2 to N, the power of z is non-negative and the integrand contains no poles within the unit circle; therefore, $(\Phi \Gamma)_j = 0$ for all j from 2 to N but, for j = 1, there is a simple pole at z = 0 for which the residue is

$$\lim_{z \to 0} z \left[\frac{1}{z \left(1 + \gamma_2^* z + \gamma_3^* z^* + \ldots + \gamma_N^* z^{N-1} \right)} \right] = 1$$

so that

$$(\Phi \Gamma)_1 = P$$

by Cauchy's residue theorem. Thus, the maximum-entropy spectrum for an equally spaced line array is

$$S(k) = \frac{1}{U} \frac{P}{V^H \Gamma \Gamma^H V}$$

where P and $\gamma_2, \gamma_3, \ldots, \gamma_N$ are solutions of the matrix equation

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1N} \\ \Phi_{21} & \Phi_{22} & \cdots & \Phi_{2N} \\ \vdots & \vdots & & \vdots \\ \Phi_{N1} & \Phi_{N2} & \cdots & \Phi_{NN} \end{bmatrix} \begin{bmatrix} 1 \\ \gamma_2 \\ \vdots \\ \gamma_N \end{bmatrix} = \begin{bmatrix} P \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

B. ATTEMPTS TO EXTEND MAXIMUM-ENTROPY SPECTRUM TO MULTI-DIMENSIONAL ARRAYS

In an attempt to extend the maximum-entropy spectrum to multidimensional arrays, it was decided to study one of the simplest possible situations involving an array and its associated crosspower spectrum matrix. Figure IV-1 shows the array geometry.

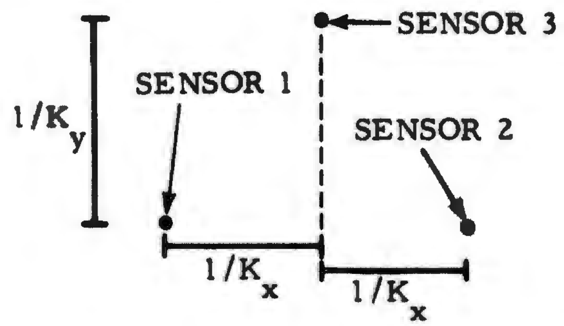

Figure IV-1. Array Geometry for Maximum Entropy Test Case

For suitable choices of $1/K_x$ and $1/K_y$, this array geometry corresponds to an equilateral triangle with sensors at each of its vertices. The associated crosspower spectrum matrix was chosen to be the real symmetric matrix

$$\begin{bmatrix} \phi_0 & \phi_1 & \phi_1 \\ \phi_1 & \phi_0 & \phi_1 \\ \phi_1 & \phi_1 & \phi_0 \end{bmatrix}$$

Regardless of the way in which the sensors are numbered, this matrix remains unchanged. Each autopower spectrum has the value ϕ_0, and each crosspower spectrum has the real value ϕ_1. The maximum-entropy spectrum has the form

$$S(k_x, k_y) = \frac{1}{K_x K_y} \left(\frac{1}{\lambda_0 + \lambda_1 u} \right)$$

where

$$u = e^{-i2\pi\left(-\frac{k_x}{K_x} + \frac{k_y}{K_y}\right)} + e^{-i2\pi\left(-\frac{k_x}{K_x} + \frac{k_y}{K_y}\right)}$$

$$+ e^{-i2\pi\left(\frac{k_x}{K_x} + \frac{k_y}{K_y}\right)} + e^{i2\pi\left(\frac{k_x}{K_x} + \frac{k_y}{K_y}\right)}$$

$$+ e^{-i2\pi\left(\frac{2k_x}{K_x}\right)} + e^{i2\pi\left(\frac{2k_x}{K_x}\right)}$$

$$= 2 \cos \frac{4\pi k_x}{K_x} + 4 \cos \frac{2\pi k_x}{K_x} \cos \frac{2\pi k_y}{K_y}$$

1. Solution Attempt by Direct Integration in Constraint Equations

The following two equations must be satisfied to determine the values λ_0 and λ_1:

$$\phi_0 = \frac{1}{K_x K_y} \int_0^{K_x} \int_0^{K_y} \frac{1}{\lambda_0 + \lambda_1 u} \, dk_y \, dk_x$$

$$\phi_1 = \frac{1}{K_x K_y} \int_0^{K_x} \int_0^{K_y} \frac{\cos 4\pi\left(\frac{k_x}{K_x}\right)}{\lambda_0 + \lambda_1 u} \, dk_y \, dk_x$$

That is, the inverse transform of the spectrum from wavenumber to vector displacement between sensors must agree with the measured crosspower spectrum values at the corresponding vector displacements. Let

$$I_0 = \int_0^{K_y} \frac{dky}{\alpha + \beta \cos 2\pi \left(\frac{k_y}{K_y}\right)}$$

where $\alpha = \lambda_0 + 2\lambda_1 \cos (4\pi k_x/K_x)$ and $\beta = 4\lambda_1 \cos (4\pi k_x/K_x)$. Note that $\alpha > 0$ and $|\alpha|>|\beta|$; otherwise, the spectrum would not be positive. Let

$$z = e^{i2\pi k_y/K_y}$$

Then, $dky = (K_y/i2\pi z)\, dz$ and

$$I_0 = \frac{K_y}{i2\pi} \oint_{|z|=1} \frac{dz}{z\left[\alpha + \frac{\beta}{2}\left(z + \frac{1}{z}\right)\right]}$$

$$= \frac{K_y}{i\pi} \oint_{|z|=1} \frac{dz}{\beta z^2 + 2\alpha z + \beta}$$

$$= \frac{K_y}{i\pi} \oint_{|z|=1} \frac{dz}{\beta (z - z_1)(z - z_2)}$$

where

$$z_1 = \frac{-\alpha + \sqrt{\alpha^2 - \beta^2}}{\beta}$$

and

$$z_2 = \frac{-\alpha - \sqrt{\alpha^2 - \beta^2}}{\beta}$$

Inasmuch as $\alpha > 0$ and $|\alpha| > |\beta|$, then $|z_2| > 1$; furthermore, $z_1 z_2 = 1$ and, therefore, $|z_1| < 1$. Thus, only one pole is inside the unit circle — a simple pole at z_1. The residue at z_1 is

$$\lim_{z \to z_1} (z - z_1) \left[\frac{1}{\beta (z - z_1)(z - z_2)}\right] = \frac{1}{\beta (z_1 - z_2)} = \frac{1}{2\sqrt{\alpha^2 - \beta^2}}$$

so that

$$I_0 = 2\pi i \left(\frac{K_y}{i\pi}\right)\left(\frac{1}{2\sqrt{\alpha^2 - \beta^2}}\right) = \frac{K_y}{\sqrt{(\alpha + \beta)(\alpha - \beta)}}$$

Therefore,

$$\phi_0 = \frac{1}{K_x} \int_{-\frac{K_x}{2}}^{\frac{K_x}{2}} \frac{dk_x}{\sqrt{\left(\lambda_0 + 2\lambda_1 \cos 4\pi \frac{k_x}{K_x} + 4\lambda_1 \cos 2\pi \frac{k_x}{K_x}\right)\left(\lambda_0 + 2\lambda_1 \cos 4\pi \frac{k_x}{K_x} - 4\lambda_1 \cos 2\pi \frac{k_x}{K_x}\right)}}$$

Let $\theta = 2\pi k_x / K_x$. Then, $k_x = K_x \theta / 2\pi$, $dk_x = (K_x / 2\pi)\, d\theta$, and

$$\phi_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{(\lambda_0 + 2\lambda_1 \cos 2\theta + 4\lambda_1 \cos\theta)(\lambda_0 + 2\lambda_1 \cos 2\theta - 4\lambda_1 \cos\theta)}}$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\lambda_0 + 2\lambda_1 \cos 2\theta)^2 - 16\lambda_1^2 \cos^2\theta}}$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\left(\lambda_0 - 2\lambda_1 + 4\lambda_1 \cos^2\theta\right)^2 - 16\lambda_1^2 \cos^2\theta}}$$

inasmuch as $\cos 2\theta = 2\cos^2\theta - 1$. Similarly,

$$\phi_1 = \frac{2}{\pi} \int_0^{\pi/2} \frac{(2 \cos^2 \theta - 1)\, d\theta}{\sqrt{\left(\lambda_0 - 2\lambda_1 + 4\lambda_1 \cos^2 \theta\right)^2 - 16\lambda_1^2 \cos^2 \theta}}$$

Under the transformation $\omega = (\pi/2) - \theta$,

$$\phi_0 = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\omega}{\sqrt{\left(\lambda_0 - 2\lambda_1 + 4\lambda_1 \sin^2 \omega\right)^2 - 16\lambda_1^2 \sin^2 \omega}}$$

and

$$\phi_1 = \frac{2}{\pi} \int_0^{\pi/2} \frac{(2 \sin^2 \omega - 1)\, d\omega}{\sqrt{\left(\lambda_0 - 2\lambda_1 + 4\lambda_1 \sin^2 \omega\right)^2 - 16\lambda_1^2 \sin^2 \omega}}$$

Now,

$$\left(\lambda_0 - 2\lambda_1 + 4\lambda_1 \sin^2 \omega\right)^2 - 16\lambda_1^2 \sin^2 \omega$$

$$= (\lambda_0 - 2\lambda_1)^2 \left[1 + \frac{8\lambda_1(\lambda_0 - 4\lambda_1) \sin^2 \omega}{(\lambda_0 - 2\lambda_1)^2} + \frac{16\lambda_1^2 \sin^4 \omega}{(\lambda_0 - 2\lambda_1)^2}\right]$$

$$= (\lambda_0 - 2\lambda_1)^2 \left\{1 - \frac{4(-\lambda_1)}{(\lambda_0 - 2\lambda_1)^2} \left[\lambda_0 - 4\lambda_1 + 2\sqrt{-\lambda(\lambda_0 - 3\lambda_1)}\right] \sin^2 \omega\right\}$$

$$\cdot \left\{1 - \frac{4(-\lambda_1)}{(\lambda_0 - 2\lambda_1)^2} \left[\lambda_0 - 4\lambda_1 - 2\sqrt{-\lambda_1(\lambda_0 - 3\lambda_1)}\right] \sin^2 \omega\right\}$$

$$= (\lambda_0 - 2\lambda_1)^2 \left[1 - \frac{4(-\lambda_1) \sin^2 \omega}{\lambda_0 - 4\lambda_1 - 2\sqrt{-\lambda_1(\lambda_0 - 3\lambda_1)}}\right]\left[1 - \frac{4(-\lambda_1) \sin^2 \omega}{\lambda_0 - 4\lambda_1 + 2\sqrt{-\lambda_1(\lambda_0 - 3\lambda_1)}}\right]$$

Thus,

$$\phi_0 = \frac{2}{\pi(\lambda_0 - 2\lambda_1)} \int_0^{\pi/2} \frac{d\omega}{\sqrt{\left[1 - \frac{4(-\lambda_1)\sin^2\omega}{\lambda_0 - 4\lambda_1 - 2\sqrt{-\lambda_1(\lambda_0 - 3\lambda_1)}}\right]\left[1 - \frac{4(-\lambda_1)\sin^2\omega}{\lambda_0 - 4\lambda_1 + 2\sqrt{-\lambda_1(\lambda_0 - 3\lambda_1)}}\right]}}$$

and

$$\phi_1 = \frac{2}{\pi(\lambda_0 - 2\lambda_1)} \int_0^{\pi/2} \frac{(2\sin^2\omega - 1)\, d\omega)}{\sqrt{\left[1 - \frac{4(-\lambda_1)\sin^2\omega}{\lambda_0 - 4\lambda_1 - 2\sqrt{-\lambda_1(\lambda_0 - 3\lambda_1)}}\right]\left[1 - \frac{4(-\lambda_1)\sin^2\omega}{\lambda_0 - 4\lambda_1 + 2\sqrt{-\lambda_1(\lambda_0 - 3\lambda_1)}}\right]}}$$

If $\lambda_1 < 0$, the transformation $\sin^2\alpha = (1 - b^2)\sin^2\omega/(1 - b^2\sin^2\omega)$ reduces these integrals to standard elliptic integrals provided that

$$a^2 = \frac{4(-\lambda_1)}{\lambda_0 - 4\lambda_1 - 2\sqrt{-\lambda_1(\lambda_0 - 3\lambda_1)}}$$

and

$$b^2 = \frac{4(-\lambda_1)}{\lambda_0 - 4\lambda_1 + 2\sqrt{-\lambda_1(\lambda_0 - 3\lambda_1)}}$$

Under this transformation, the following identities are valid:

$$\sin^2 \psi = \frac{\sin^2 \alpha}{1 - b^2 + b^2 \sin^2 \alpha}$$

$$\cos^2 \psi = \frac{(1 - b^2)(1 - \sin^2 \alpha)}{1 - b^2 + b^2 \sin^2 \alpha}$$

$$\sin \psi \cos \psi = \frac{\sqrt{1 - b^2}\, \sin \alpha \cos \alpha}{1 - b^2 + b^2 \sin^2 \alpha}$$

$$1 - \alpha^2 \sin^2 \psi = \frac{1 - b^2 - (a^2 - b^2) \sin^2 \alpha}{1 - b^2 + b^2 \sin^2 \alpha}$$

$$1 - b^2 \sin^2 \psi = \frac{1 - b^2}{1 - b^2 + b^2 \sin^2 \alpha}$$

$$d\psi = \frac{\sin \alpha \cos \alpha \,(1 - b^2 \sin^2 \psi)^2 \, d\alpha}{\sin \psi \cos \psi}$$

$$= \frac{(1 - b^2 + b^2 \sin^2 \alpha)\,(1 - b^2)^2 \, d\alpha}{\sqrt{1 - b^2}\,(1 - b^2 + b^2 \sin^2 \alpha)^2}$$

$$= \frac{(1 - b)\sqrt{1 - b}\; d\alpha}{1 - b^2 + b^2 \sin^2 \alpha}$$

Thus,

$$\frac{\pi}{2}(\lambda_0 - 2\lambda_1)\,\phi_0 = \int_0^{\pi/2} \frac{d\omega}{\sqrt{(1 - a^2 \sin^2 \omega)(1 - b^2 \sin^2 \omega)}}$$

$$= \int_0^{\pi/2} \frac{(1-b)\sqrt{1-b}\;\, d\alpha}{\left(1 - b^2 + b^2 \sin^2 \alpha\right)\sqrt{\left[\dfrac{1 - b^2 - (a^2 - b^2)\sin^2 \alpha}{1 - b^2 + b^2 \sin^2 \alpha}\right]\left[\dfrac{1 - b^2}{1 - b^2 + b^2 \sin^2 \alpha}\right]}}$$

$$= \sqrt{1 - b^2} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - \left(\dfrac{a^2 - b^2}{1 - b^2}\right)\sin^2 \alpha}}$$

$$= \sqrt{1 - b^2}\; F\left(\frac{\pi}{2}, \sqrt{\frac{a^2 - b^2}{1 - b^2}}\right)$$

where F is the elliptic integral of the first kind. Similarly,

$$\frac{\pi}{2}\left(\lambda_0 - 2\lambda_1\right)\phi_1 = \int_0^{\pi/2} \frac{\left(2\sin^2\omega - 1\right)d\omega}{\sqrt{\left(1 - a^2\sin^2\omega\right)\left(1 - b^2\sin^2\omega\right)}}$$

$$= \frac{2}{b^2}\int_0^{\pi/2} \frac{\left(b^2\sin^2\omega - \frac{1}{2}b^2\right)d\omega}{\sqrt{\left(1 - a^2\sin^2\omega\right)\left(1 - b^2\sin^2\omega\right)}}$$

$$= \frac{2}{b^2}\int_0^{\pi/2} \frac{\left(1 - \frac{1}{2}b^2\right)d\omega}{\sqrt{\left(1 - a^2\sin^2\omega\right)\left(1 - b^2\sin^2\omega\right)}} - \frac{2}{b^2}\int_0^{\pi/2} \frac{\left(1 - b^2\sin^2\omega\right)d\omega}{\sqrt{\left(1 - a^2\sin^2\omega\right)\left(1 - b^2\sin^2\omega\right)}}$$

$$= \frac{2 - b^2}{b^2}\int_0^{\pi/2} \frac{d\omega}{\sqrt{\left(1 - a^2\sin^2\omega\right)\left(1 - b^2\sin^2\omega\right)}} - \frac{2}{b^2}\int_0^{\pi/2} \frac{d\alpha}{\left(1 + \frac{b^2\sin^2\alpha}{1 - b^2}\right)\sqrt{1 - \left(\frac{a^2 - b^2}{1 - b^2}\right)\sin^2\alpha}}$$

$$= \sqrt{1 - b^2}\left[\frac{2 - b^2}{b^2} F\left(\frac{\pi}{2}, \sqrt{\frac{a^2 - b^2}{1 - b^2}}\right) - \frac{2}{b^2}\Pi\left(\frac{\pi}{2}, \frac{b^2}{1 - b^2}, \sqrt{\frac{a^2 - b^2}{1 - b^2}}\right)\right]$$

where Π is the elliptic integral of the third kind.

Isolating λ_0 and λ_1, in terms of functions of ϕ_0 and ϕ_1 appears to be a very difficult task. This particular attempt to solve the simple test case leads to a pair of constraint equations which are apparently too complicated to be solved by analytic means. It is interesting to speculate to where the general case leads.

2. Solution Attempt by Series Expansion of Constraint Equations

The constraint equations can also be written as

$$\phi_0 = \frac{1}{U} \iint_R \frac{d\vec{k}}{\lambda_0 + \lambda_1 u}$$

$$\phi_1 = \frac{1}{U} \iint_R \frac{e^{i2\pi\vec{k} \cdot (\vec{x}_j - \vec{x}_\ell)} d\vec{k}}{\lambda_0 + \lambda_1 u} \qquad (j \neq \ell)$$

where U is the unit cell's area and R denotes its region. The expression

$$\frac{1}{\lambda_0 + \lambda_1 u} = \frac{1}{\lambda_0} \left[\frac{1}{1 + \left(\frac{\lambda_1}{\lambda_0}\right) u} \right]$$

can be written

$$\frac{1}{\lambda_0} \left[1 - \frac{\lambda_1}{\lambda_0} u + \left(\frac{\lambda_1}{\lambda_0}\right)^2 u^2 - \left(\frac{\lambda_1}{\lambda_0}\right)^3 u^3 + \ldots \right]$$

so that

$$\lambda_0 \phi_0 = \frac{1}{U} \iint_R \left[1 - \frac{\lambda_1}{\lambda_0} u + \left(\frac{\lambda_1}{\lambda_0}\right)^2 u^2 - \left(\frac{\lambda_1}{\lambda_0}\right)^3 u^3 + \ldots \right] d\vec{k}$$

and

$$\lambda_0 \phi_1 = \frac{1}{U} \iint_R \left[1 - \frac{\lambda_1}{\lambda_0} u + \left(\frac{\lambda_1}{\lambda_0}\right)^2 u^2 - \left(\frac{\lambda_1}{\lambda_0}\right)^3 u^3 + \ldots \right] e^{i2\pi\vec{k}(\vec{x}_j - \vec{x}_\ell)} d\vec{k}$$

If the array sensors are located at the vertices of an equilateral triangle, the term u is the wavenumber Fourier transform of a set of Dirac delta functions located as shown in Figure IV-2.

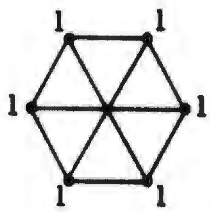

Figure IV-2. Inverse Fourier Transform of u

The values 1 beside each point in Figure IV-2 indicate that each delta function is weighted by 1 before Fourier transformation.

Multiplication of u by itself yields u^2. Its inverse Fourier transform must therefore be the convolution of the inverse Fourier transform of u with itself. The inverse Fourier transform of u^2 is shown in Figure IV-3. Thus, $u^2 = 6 + 2u +$ higher-order terms.

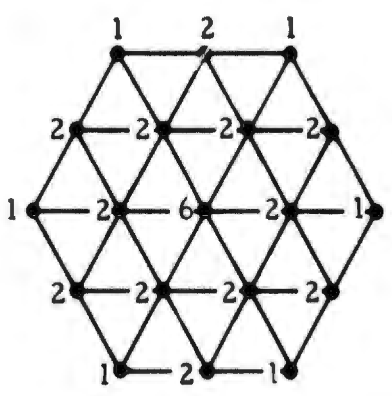

Figure IV-3. Inverse Fourier Transform of u^2

Continuing the process, one obtains

$$u^3 = 12 + 15u + \text{higher-order terms}$$
$$u^4 = 90 + 60u + \text{higher-order terms}$$
$$u^5 = 360 + 340u + \text{higher-order terms}$$
$$u^6 = 2040 + 1660u + \text{higher-order terms}$$

To evaluate $\lambda_0 \phi_0$ and $\lambda_0 \phi_1$, one can use the identity

$$\iint_R e^{-i2\pi \vec{k} \cdot (n\vec{y}_1 + m\vec{y}_2)} \, e^{i2\pi \vec{k} \cdot (n'\vec{y}_1 + m'\vec{y}_2)} \, d\vec{k}$$

$$= \begin{cases} 0 \text{ if } n \neq n' \text{ or } m \neq m' \\ U \text{ if } n = n' \text{ and } m = m' \end{cases}$$

where n, m, n′, and m′ are integers and $\vec{y}_1$ and $\vec{y}_2$ are the basic vector displacements between the array sensors.* In particular,

$$\iint_R 1 \, d\vec{k} = U$$

$$\iint_R u \, e^{i2\pi \vec{k} \cdot (\vec{x}_j - \vec{x}_\ell)} \, d\vec{k} = U$$

$$\iint_R e^{i2\pi \vec{k} \cdot (\vec{x}_j - \vec{x}_\ell)} \, d\vec{k} = 0$$

$$\iint_R u \, d\vec{k} = 0$$

*Texas Instruments Incorporated, 1961: Seismometer Array and Data Processing System, p. 92-94.

The higher-order terms always involve values of n or m which are not equal to -1, 0, or 1, so their integral over the unit cell is also equal to 0. Thus,

$$\lambda_0 \phi_0 = 1 + 6\rho^2 - 12\rho^3 + 90\rho^4 - 360\rho^5 + 2040\rho^6 - \dots$$

and

$$\lambda_0 \phi_1 = -\rho + 2\rho^2 - 15\rho^3 + 60\rho^4 - 340\rho^5 + 1660\rho^6 - \dots,$$

where

$$\rho = \lambda_1 / \lambda_0$$

Note that

$$\begin{aligned} \lambda_0 \phi_0 + 6\lambda_1 \phi_1 &= \lambda_0 \phi_0 + 6\rho \lambda_0 \phi_1 \\ &= 1 + 6\rho^2 - 12\rho^3 + 90\rho^4 - 360\rho^5 + \dots \\ &\quad - 6\rho^2 + 12\rho^3 - 90\rho^4 + 360\rho^5 - \dots \\ &= 1 \end{aligned}$$

and

$$\frac{\phi_1}{\phi_0} = \frac{\lambda_0 \phi_1}{\lambda_0 \phi_0} = \frac{-\rho(1 - 2\rho + 15\rho^2 - 60\rho^3 + 340\rho^4 - 1660\rho^5 + \dots)}{(1 + 6\rho^2 - 12\rho^3 + 90\rho^4 - 360\rho^5 + \dots)}$$

$$= -\rho(1 - 2\rho + 15\rho^2 - 60\rho^3 + 340\rho^4 - 1660\rho^5)(1 - 6\rho^2 + 12\rho^3 - 126\rho^4 + 216\rho^5 - \dots)$$

$$= -\rho(1 - 2\rho + 9\rho^2 - 36\rho^3 + 100\rho^4 - 1372\rho^5 + \dots)$$

$$-\frac{\phi_1}{\phi_0} = \rho - 2\rho^2 + 9\rho^3 - 36\rho^4 + 100\rho^5 - 1372\rho^6 + \dots$$

The series for $\rho = (\lambda_1/\lambda_0)$ is in a form in which ρ can be found in terms of $(-\phi_1/\phi_0)$ by inverting the power series for ρ.* The result is

$$\frac{\lambda_1}{\lambda_0} = \frac{-\phi_1}{\phi_0} + 2\left(\frac{\phi_1}{\phi_0}\right)^2 + \left(\frac{\phi_1}{\phi_0}\right)^3 - 14\left(\frac{\phi_1}{\phi_0}\right)^4 - 43\left(\frac{\phi_1}{\phi_0}\right)^5 + 1564\left(\frac{\phi_1}{\phi_0}\right)^6 + \ldots$$

After the ratio λ_1/λ_0 is found, the relation $\lambda_0\phi_0 + \lambda_1\phi_1 = 1$ can be used to determine the values of λ_0 and λ_1. There is a singular point at $\phi_1/\phi_0 = -\frac{1}{2}$; therefore, the radius of convergence can be no larger than $\frac{1}{2}$. When ϕ_1/ϕ_0 ranges from $-\frac{1}{2}$ to $+1$, the crosspower spectrum matrix is positive-definite. In the range of ϕ_1/ϕ_0 from $+\frac{1}{2}$ to 1, this series, therefore, cannot provide values of λ_0 and λ_1 without resorting to analytic continuation. In view of the large ratio between the fifth and sixth coefficients of the series, it seems prudent to confine the use of this series to cases in which $|\phi_1| << \phi_0$.

These investigations were directed toward the analytic solution of the maximum-entropy spectrum for multidimensional arrays. Although this power series solution for λ_0 and λ_1 is valid for a small range of ϕ_1/ϕ_0 in the specific test case studied, it does not seem to provide any clue about the nature of the general analytic solution. Considerable expertise in the theory of elliptic functions and elliptic integrals would seem to be a prerequisite for further work on this problem.

*James Pierpont, 1914: Functions of a Complex Variable, p 259-262.

SECTION V

DIFFERENCES BETWEEN FREQUENCY-WAVENUMBER SPECTRAL TECHNIQUES

A. RELATIVE RESOLUTION

Every Hermitian form $V^H \Phi V$ can be reduced by a unitary transformation of variables

$$V = TX \qquad (TT^H = I)$$

to the canonical form

$$\sum_{i=1}^{N} \lambda_i x_i x_i^* = \sum_{i=1}^{N} \lambda_i |x_i|^2$$

where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of the matrix Φ.* The scalar quantities x_i are the components of V along the eigenvectors of the matrix Φ.

In the new coordinate system, the axes of which correspond to the eigenvectors of Φ, the matrix Φ can be written as

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{bmatrix}$$

Its inverse in the new coordinate system is

$$\begin{bmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1/\lambda_N \end{bmatrix}$$

*F. R. Gantmacher, The Theory of Matrices, v. I, p. 337.

If U is a unit vector, it can be written as

$$U = x_1 E_1 + x_2 E_2 + \ldots + x_N E_N$$

where vectors $E_1, E_2, \ldots, E_N$ are the normalized eigenvectors of Φ and where $|x_1|^2 + |x_2|^2 + \ldots + |x_N|^2 = 1$.

The beamsteer spectrum is

$$P = U^H \Phi U$$

$$= \lambda_1 |x_1|^2 + \lambda_2 |x_2|^2 + \ldots + \lambda_N |x_N|^2$$

The maximum-likelihood spectrum is

$$Q = \left(U^H \Phi^{-1} U\right)^{-1}$$

$$= \frac{1}{\frac{|x_1|^2}{\lambda_1} + \frac{|x_2|^2}{\lambda_2} + \ldots + \frac{|x_N|^2}{\lambda_N}}$$

The constant-norm spectrum is

$$R = \frac{U^H \Phi^{-1} U}{U^H \Phi^{-2} U}$$

$$= \frac{\frac{|x_1|^2}{\lambda_1} + \frac{|x_2|^2}{\lambda_2} + \ldots + \frac{|x_N|^2}{\lambda_N}}{\frac{|x_1|^2}{\lambda_1^2} + \frac{|x_2|^2}{\lambda_2^2} + \ldots + \frac{|x_N|^2}{\lambda_N^2}}$$

If the unit-amplitude probe vector U lies along the j^{th} eigenvector, then $|x_j|^2 = 1$ and $|x_\ell|^2 = 0$ for $j \neq \ell$. Then,

$$P = \lambda_j$$

$$Q = \lambda_j$$

$$R = \lambda_j$$

Each of these three spectral techniques is equal to the eigenvalue λ_j when the probe vector U lies along the j^{th} eigenvector. Thus, these three spectra have identical values for probe vectors lying along the eigenvectors of Φ.

To determine the relative resolution of the three spectral techniques, one examines the ratios P/Q and Q/R:

$$\frac{P}{Q} = \left(\sum_i \lambda_i |x_i|^2\right)\left(\sum_j \frac{|x_j|}{\lambda_j}\right)$$

$$= \sum_i \sum_j \frac{\lambda_i}{\lambda_j} |x_i|^2 |x_j|^2$$

$$= \sum_i |x_i|^4 + \sum_{i<j}\sum \frac{(\lambda_i^2 + \lambda_j^2)|x_i|^2}{\lambda_i \lambda_j}$$

$$= \sum_i |x_i|^4 + \sum_{i<j}\sum 2|x_i|^2 |x_j|^2 + \sum_{i<j}\sum \frac{\left(\lambda_i^2 - 2\lambda_i \lambda_j + \lambda_j^2\right)|x_i|^2 |x_j|^2}{\lambda_i \lambda_j}$$

$$= \left(\sum_i |x_i|^2\right)^2 + \sum_{i<j}\sum \frac{(\lambda_i - \lambda_j)^2}{\lambda_i \lambda_j} |x_i|^2 |x_j|^2$$

$$= 1 + \sum_{i<j}\sum \frac{(\lambda_i - \lambda_j)^2}{\lambda_i \lambda_j} |x_i|^2 |x_j|^2 \geq 1$$

since the matrix Φ is positive-definite (the values λ_i, λ_j are always non-negative). Similarly,

$$\frac{Q}{R} = \frac{1}{\sum_i \frac{|x_i|^2}{\lambda_i}} \Bigg/ \left[\sum_i \frac{|x_i|^2}{\lambda_i} \Bigg/ \left(\sum_i \frac{|x_i|^2}{\lambda_i^2} \right) \right]$$

$$= \frac{\sum_i \frac{|x_i|^2}{\lambda_i^2}}{\left(\sum_i \frac{|x_i|^2}{\lambda_i} \right)^2}$$

$$= \frac{\sum_i \frac{|x_i|^2}{\lambda_i^2}}{\sum_i \sum_j \frac{|x_i|^2 |x_j|^2}{\lambda_i \lambda_j}}$$

$$= \frac{\sum_i \frac{|x_i|^2}{\lambda_i^2}}{\left(\sum_i \frac{|x_i|^2}{\lambda_i^2} \right) \left(\sum_j |x_j|^2 \right) + \sum_i \sum_j \frac{|x_i|^2 |x_j|^2}{\lambda_i \lambda_j} - \sum_i \sum_j \frac{|x_i|^2 |x_j|^2}{\lambda_i^2}}$$

$$
= \frac{\sum_i \frac{|x_i|^2}{\lambda_i^2}}{\sum_i \frac{|x_i|^2}{\lambda_i^2} - \sum_i \sum_j \left(\frac{\lambda_j^2 - \lambda_i \lambda_j}{\lambda_i^2 \lambda_j^2} \right) |x_i|^2 |x_j|^2}
$$

$$
= \frac{\sum_i \frac{|x_i|^2}{\lambda_i^2}}{\sum_i \frac{|x_i|^2}{\lambda_i^2} - \sum\sum_{i<j} \frac{\left(\lambda_i^2 - 2\lambda_i \lambda_j + \lambda_j^2\right)}{\lambda_i^2 \lambda_j^2} |x_i|^2 |x_j|^2}
$$

$$
= \frac{\sum_i \frac{|x_i|^2}{\lambda_i^2}}{\sum_i \frac{|x_i|^2}{\lambda_i^2} - \sum\sum_{i<j} \frac{\left(\lambda_i - \lambda_j\right)^2}{\lambda_i^2 \lambda_j^2} |x_i|^2 |x_j|^2} \geq 1
$$

Since $P/Q \geq 1$ and $Q/R \geq 1$, $P \geq Q \geq R$; yet, along the principal axes of the matrix Φ, $P = Q = R$. When any unit-amplitude probe vector has nonzero components along principal axes which have unequal corresponding eigenvalues, these formulas for P/R and Q/R show that $P>Q>R$. The corresponding spectra therefore, will look something like Figure V-1. The peaks of the constant-norm spectrum (R) will be sharper than those of the maximum-likelihood spectrum (Q), and the peaks of the maximum-likelihood spectrum will be sharper than those of the beamsteer spectrum (P). Thus, a hierarchy of resolution can be constructed:

(1) Constant-norm spectrum (R)
(2) Maximum-likelihood spectrum (Q)
(3) Beamsteer spectrum (P)

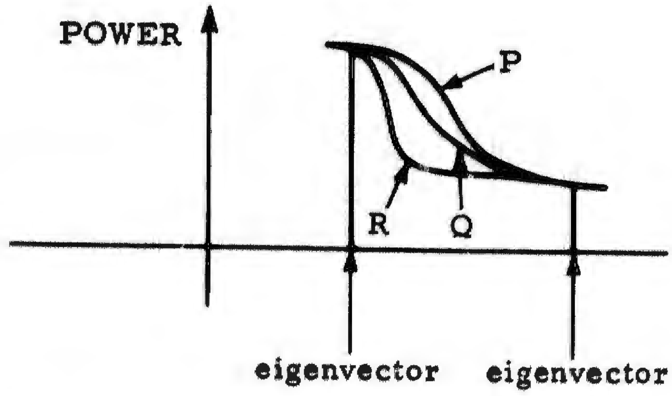

Figure V-1. Relative Resolution of Spectra

These results are valid for spectra in the full N-dimensional space of all possible unit-amplitude probe vectors.

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}$$

such that $|u_1|^2 + |u_2|^2 + \ldots + |u_N|^2 = 1$. Since the set of all probe vectors

$$U = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{i2\pi\vec{k}\cdot\vec{x}_1} \\ e^{i2\pi\vec{k}\cdot\vec{x}_2} \\ \vdots \\ e^{i2\pi\vec{k}\cdot\vec{x}_N} \end{bmatrix}$$

which corresponds to wavenumber vectors $\vec{k}$ is a subset of the full space, the eigenvectors of the crosspower spectrum matrix may possibly lie outside the subset of probe vectors corresponding to wavenumber vectors $\vec{k}$. In this event, the results are not strictly applicable unless the normalized eigenvector corresponding to the maximum eigenvalue is of the form just described; e.g., $\Phi = \rho I + VV^H$, and V reflects a plane wave.

From empirical results, one observes that the relative resolution between the spectra P, Q, and R exhibits a strong tendency to follow the results of this subsection

B. BEHAVIOR FOR SINGULAR CROSSPOWER SPECTRUM MATRICES

If the crosspower spectrum matrix Φ is singular, some of its eigenvalues are zero. Without loss of generality, the M eigenvalues which are zero can be chosen to be the M eigenvalues with the highest subscripts so that

$$\lambda_{N-M+1} = 0, \ \lambda_{N-M+2} = 0, \ \ldots, \text{ and } \lambda_N = 0.$$

Then,

$$P = \lim_{\rho \to 0} U^H \left(\Phi + \rho I\right) U$$

$$= \lim_{\rho \to 0} \left[\left(\lambda_1 + \rho\right) |x_1|^2 + \left(\lambda_2 + \rho\right) |x_2|^2 + \ldots + \left(\lambda_N + \rho\right) |x_N|^2\right]$$

$$= \lambda_1 |x_1|^2 + \lambda_2 |x_2|^2 + \ldots + \lambda_{N-M} |x_{N-M}|^2$$

The beamsteer spectrum will be zero only if the unit-amplitude probe vector U has zero components corresponding to the nonzero eigenvectors of Φ. This means that P = 0 only when U is perpendicular to the subspace spanned by the nonzero eigenvectors of Φ.

Under the same circumstances,

$$Q = \lim_{\rho \to 0} \left[U^H \left(\Phi + \rho I \right)^{-1} U \right]^{-1}$$

$$= \lim_{\rho \to 0} \frac{1}{\frac{|x_1|^2}{\lambda_1 + \rho} + \frac{|x_2|^2}{\lambda_2 + \rho} + \ldots + \frac{|x_{N-M}|^2}{\lambda^2_{N-M} + \rho} + \frac{|x_{N-M+1}|^2 + |x_{N-M+2}|^2 + \ldots + |x_N|^2}{\rho}}$$

$$= \lim_{\rho \to 0} \frac{1}{\frac{|x_1|^2}{\lambda_1} + \frac{|x_2|^2}{\lambda_2} + \ldots + \frac{|x_{N-M}|^2}{\lambda_{N-M}} + \frac{|x_{N-M+1}|^2 + |x_{N-M+2}|^2 + \ldots + |x_N|^2}{\rho}}$$

If any component of U corresponding to the zero eigenvectors of Φ is nonzero, then $Q = 0$. Thus, $Q = 0$ when U lies outside the subspace spanned by the nonzero eigenvectors of Φ.

Again under the same circumstances,

$$R = \lim_{\rho \to 0} \frac{\frac{|x_1^2|}{\lambda_1 + \rho} + \frac{|x_2|^2}{\lambda_2 + \rho} + \ldots + \frac{|x_{N-M}|^2}{\lambda_{N-M} + \rho} + \frac{|x_{N-M+1}|^2 + |x_{N-M+2}|^2 + \ldots + |x_N|^2}{\rho}}{\frac{|x_1|^2}{(\lambda_1 + \rho)^2} + \frac{|x_2|^2}{(\lambda_2 + \rho)^2} + \ldots + \frac{|x_{N-M}|^2}{(\lambda_{N-M} + \rho)^2} + \frac{|x_{N-M+1}|^2 + |x_{N-M+2}|^2 + \ldots + |x_N|^2}{\rho^2}}$$

$$= \lim_{\rho \to 0} \frac{\frac{|x_1|^2}{\lambda_1} + \frac{|x_2|^2}{\lambda_2} + \ldots + \frac{|x_{N-M}|^2}{\lambda_{N-M}} + \frac{|x_{N-M+1}|^2 + |x_{N-M+2}|^2 + \ldots + |x_N|^2}{\rho}}{\frac{|x_1|^2}{\lambda_1^2} + \frac{|x_2|^2}{\lambda_2^2} + \ldots + \frac{|x_{N-M}|^2}{\lambda^2_{N-M}} + \frac{|x_{N-M+1}|^2 + |x_{N-M+2}|^2 + \ldots + |x_N|^2}{\rho^2}}$$

If any of the components $x_{N-M+1}, x_{N-M+2}, \ldots, x_N$ is nonzero, R is zero. Like the maximum-likelihood spectrum, the constant-norm spectrum R is zero whenever the unit-amplitude probe vector U lies outside the subspace spanned by the nonzero eigenvectors of Φ.

These results illustrate the effect of sidelobes in the case of a singular crosspower spectrum matrix. They are restatements and amplifications of the results in Section III for the maximum-likelihood spectrum. The fact that the beamsteer spectrum is not identically zero when the probe vector U lies outside the subspace spanned by the nonzero eigenvectors of Φ is due to sidelobe effects which are traced in detail for the simple cases discussed in Section III. Sidelobe effects appear to be completely missing from the two high-resolution spectra Q and R when the probe vector U lies outside the same subspace.

C. CONSISTENCY WITH CROSSPOWER SPECTRUM MATRIX

The conventional spectrum $P = V^H \Phi V$ is the Fourier transform of a sum of weighted Dirac delta functions located at the vector displacements between the sensors of an array. The weighting at the vector displacement $\vec{y}$ between two sensors is $n_{\vec{y}} \Phi(\vec{y})$, where $n_{\vec{y}}$ is the number of times that vector displacement occurs and $\Phi(\vec{y})$ is the crosspower spectrum between any two sensors at that displacement. Thus, the inverse Fourier transform of the conventional spectrum is equal to the same sum of weighted Dirac delta functions.

If the power in wavenumber space were confined to a unit cell and the spectrum were set equal to $V^H \Phi V$ inside the unit cell but zero outside the unit cell, the spectrum would be equal to $W(\vec{k}) V^H \Phi V$, where $W(\vec{k}) = 1$ inside the unit cell and $W(\vec{k}) = 0$ outside the unit cell. The convolution theorem makes the inverse transform of the altered spectrum equal to the convolution of the inverse transform of $W(\vec{k})$ and the same sum of weighted Dirac delta functions.

The transform of $W(\vec{k})$ is a spatial sampling operator which, upon convolution, makes the inverse transform of the altered power spectrum at any vector displacement $\vec{y}$ between two sensors equal to $n_{\vec{y}} \Phi(\vec{y})$. For an equally spaced line array, for example, the spatial sampling operator is

$$\frac{\sin \pi\left(\frac{y}{\Delta x}\right)}{\pi\left(\frac{y}{\Delta x}\right)}$$

where Δx is the distance between array elements. The spatial sampling operator fills in the points between the array vector displacements by means of a well-known interpolation technique; yet, the resulting inverse Fourier transform is proportional to $n_{\vec{y}} \Phi(\vec{y})$ at the array vector displacements for either $V^H \Phi V$ or the altered spectrum. Thus, the conventional spectrum contains a tapering effect specified by $n_{\vec{y}}$ and is not consistent with the cross-power spectrum matrix.

The maximum-entropy spectrum is consistent with the cross-power spectrum matrix because it satisfies constraint equations which force consistency.

The maximum-likelihood frequency-wavenumber spectrum is not consistent because it is of the same form as the maximum-entropy spectrum but has a coefficient matrix $\Lambda = \Phi^{-1}$ not equal to the maximum-entropy matrix. For example, in the case of an equally-spaced line array, let

$$X_{N+1} = \begin{bmatrix} \Phi^*(1) \\ \vdots \\ \Phi^*(N) \end{bmatrix}, \quad \Phi_{N+1} = \left[\begin{array}{c|c} \Phi(0) & X_{N+1}^H \\ \hline X_{N+1} & \Phi_N \end{array}\right], \text{ and } o_{N+1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then,

$$
\left[\begin{array}{c|c}
\Phi(0) & x^H_{N+1} \\
\hline
x_{N+1} & \Phi_N
\end{array}\right]
\left[\begin{array}{c|c}
\dfrac{1}{\Phi(0) - x^H_{N+1}\,\Phi_N^{-1}\,x_{N+1}} & \dfrac{-x^H_{N+1}\,\Phi_N^{-1}}{\Phi(0) - x^H_{N+1}\,\Phi_N^{-1}\,x_{N+1}} \\
\hline
\dfrac{\Phi_N^{-1}\,x_{N+1}}{\Phi(0) - x^H_{N+1}\,\Phi_N^{-1}\,x_{N+1}} & \dfrac{\Phi_N^{-1} + \Phi_N^{-1}\,x_{N+1}\,x^H_{N+1}\,\Phi_N^{-1}}{\Phi(0) - x^H_{N+1}\,\Phi_N^{-1}\,x_{N+1}}
\end{array}\right]
$$

$$
= \left[\begin{array}{c|c}
\dfrac{\Phi(0) - x^H_{N+1}\,\Phi_N^{-1}\,x_{N+1}}{\Phi(0) - x^H_{N+1}\,\Phi_N^{-1}\,x_{N+1}} & \dfrac{-\Phi(0)\,x^H_{N+1}\,\Phi_N^{-1}}{\Phi(0) - x^H_{N+1}\,\Phi_N^{-1}\,x_{N+1}} + x^H_{N+1}\,\Phi_N^{-1} + \dfrac{\left(x^H_{N+1}\,\Phi_N\,x_{N+1}\right)x^H_{N+1}\,\Phi_N^{-1}}{\Phi(0) - x^H_{N+1}\,\Phi_N^{-1}\,x_{N+1}} \\
\hline
\dfrac{x_{N+1} - \Phi_N\,\Phi_N^{-1}\,x_{N+1}}{\Phi(0) - x^H_{N+1}\,\Phi_N^{-1}\,x_{N+1}} & \dfrac{-x_{N+1}\,x^H_{N+1}\,\Phi_N^{-1}}{\Phi(0) - x_{N+1}\,\Phi_N^{-1}\,x_{N+1}} + I_N + \dfrac{x_{N+1}\,x^H_{N+1}\,\Phi_N^{-1}}{\Phi(0) - x^H_{N+1}\,\Phi_N^{-1}\,x_{N+1}}
\end{array}\right]
$$

$$
= \left[\begin{array}{c|c}
1 & o^H_{N+1} \\
\hline
o_{N+1} & I_N
\end{array}\right] = I_{N+1}
$$

The matrix equation to be solved in the maximum-entropy spectrum is

$$\left[\begin{array}{c|c} \Phi(0) & x_{N+1}^{H} \\ \hline x_{N+1} & \Phi_N \end{array}\right] \left[\begin{array}{c} 1 \\ \hline \Gamma_2 \\ \vdots \\ \Gamma_{N+1} \end{array}\right] = \left[\begin{array}{c} P_{N+1} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]$$

for which the solution is

$$\begin{bmatrix} \Gamma_2 \\ \vdots \\ \Gamma_{N+1} \end{bmatrix} = -\Phi_N^{-1} x_{N+1} \quad \text{and} \quad P_{N+1} = \Phi(0) - x_{N+1}^{H} \Phi_N^{-1} x_{N+1}$$

Now,

$$P_{N+1} \left\{ \Phi_{N+1}^{-1} - \left[\begin{array}{c|c} 0 & 0_{N+1}^{H} \\ \hline 0_{N+1} & \Phi_N^{-1} \end{array}\right] \right\} = \left[\begin{array}{c|c} 1 & -x_{N+1}^{H} \Phi_N^{-1} \\ \hline \Phi_N^{-1} x_{N+1} & \Phi_N^{-1} x_{N+1} x_{N+1}^{H} \Phi_N^{-1} \end{array}\right]$$

$$= \left[\begin{array}{c} 1 \\ \hline -\Phi_N^{-1} x_{N+1} \end{array}\right] \left[\begin{array}{c|c} 1 & -x_{N+1}^{H} \Phi_N^{-1} \end{array}\right] = \begin{bmatrix} 1 \\ \Gamma_2 \\ \vdots \\ \Gamma_{N+1} \end{bmatrix} \begin{bmatrix} 1 & \Gamma_2^{*} \cdots \Gamma_{N+1}^{*} \end{bmatrix}$$

Since

$$S_{N+1}(\vec{k}) = \frac{P_{N+1}}{U\left[V_{N+1}^{H}\,\Gamma\Gamma^{H}\,V_{N+1}\right]} \quad ,$$

$$S_{N+1}(\vec{k}) = \frac{P_{N+1}}{P_{N+1}\,U\left\{V_{N+1}^{H}\,\Phi_{N+1}^{-1}\,V_{N+1} - V_{N+1}^{*}\begin{bmatrix} 0 & o_{N+1}^{H} \\ o & \Phi_{N}^{-1}\end{bmatrix} V_{N+1}\right\}}$$

$$= \frac{1}{U\left(V_{N+1}^{H}\,\Phi_{N+1}^{-1}\,V_{N+1} - e^{-i2\pi k\Delta x}\,V_{N}^{H}\,\Phi_{N}^{-1}\,V_{N}\,e^{i2\pi k\Delta x}\right)}$$

$$= \frac{1}{U}\left(\frac{1}{V_{N+1}^{H}\,\Phi_{N+1}^{-1}\,V_{N+1} - V_{N}^{H}\,\Phi_{N}^{-1}\,V_{N}}\right)$$

Thus, the maximum-entropy spectrum S is not equal to the maximum-likelihood spectrum

$$Q = \frac{1}{\left(V_{N+1}^{H}\,\Phi_{N+1}^{-1}\,V_{N+1}\right)}$$

The constant-norm spectrum

$$R = \frac{V^{H}\,\Phi^{-1}\,V}{V^{H}\,\Phi^{-2}\,V}$$

is not consistent with the measured crosspower spectrum matrix $\Phi = \rho I + WW^{H}$ in the limiting case $\rho \to 0$, for

$$\lim_{\rho \to 0} R \begin{cases} \neq 0 \text{ if } V = \alpha W \\ = 0 \text{ if } V \text{ is not a scalar multiple of } W \end{cases}$$

In the extreme case, an infinitesimal needle of finite height exists at the wavenumber corresponding to the plane wave W. The inverse Fourier transform therefore yields the value 0 at every vector displacement between sensors. It is possible, however, that the complex coherence values $\Phi_{j\ell}/\Phi_{jj}$ obtained by inverse transformation are consistent with those of the original crosspower spectrum matrix. In this case, the reconstructed matrix might be consistent except for a scale factor uniform with respect to each element of the crosspower spectrum matrix Φ.

D. DISCUSSION

A primary goal of this report is to choose a frequency-wavenumber spectral computational technique which best fits the needs of the nuclear explosion detection and location problems. With this in mind, a discussion of the merits of the techniques will be given here.

The beamsteer spectrum $P = V^H \Phi V$ has several disadvantages. First, its resolution is the worst for all techniques studied. Second, its sidelobe effects litter wavenumber space with spurious power created by smearing of the true spectrum. Third, it is not consistent with the original crosspower spectrum matrix. It does have the advantage, however, of being relatively insensitive to errors in the sensor locations and in the crosspower spectral estimates.

The maximum-likelihood spectrum $Q = \left(V^H \Phi^{-1} V\right)^{-1}$ has better resolution and sidelobe characteristics than the beamsteer spectrum. However, it too is not consistent with the original crosspower spectrum matrix. Its stability with respect to measurement errors is worse than that of the beamsteer spectrum, but spectra computed by this technique from real data seldom show serious degradation because of its increased sensitivity to errors. The idea behind it is sensibly motivated — to design an optimum multichannel filter which passes undistorted a unit-amplitude plane wave coming from the directions for which spectral values are desired.

The constant-norm spectrum

$$R = \frac{\left(V^H \Phi^{-1} V\right)}{\left(V^H \Phi^{-2} V\right)}$$

has even better resolution than the maximum-likelihood spectrum and preserves the desirable sidelobe characteristics of that technique. It also is not consistent with the original crosspower spectrum matrix. The chief disadvantage, and perhaps fatal disadvantage, is its extreme sensitivity to measurement errors. When real data is used, spectral peaks in this technique often are noticeably removed from the location of power sources whose position is accurately known. Moreover, the constraint that this technique satisfies does not seem to have any clear physical significance.

Of the three techniques just discussed, the maximum-likelihood spectrum appears to be a suitable compromise which avoids the poor resolution and sidelobe characteristics of the beamsteer spectrum and the extreme sensitivity to measurement errors of the constant-norm spectrum. Like these two techniques, however, it is not consistent with the original crosspower spectrum matrix.

The maximum-entropy spectrum remedies this defect and preserves all of the favorable characteristics of the maximum-likelihood spectrum. The trouble is that the method of computation is known only for an equally-spaced line array.

It appears that the best course of action is to use the maximum-entropy spectrum in those cases where the algorithm is now known or can be found sometime in the future. When the algorithm is not known, it appears that the maximum-likelihood spectrum is the most advantageous alternative available.

APPENDIX A

PROOF OF A DETERMINANT EXPRESSION FOR THE MAXIMUM-LIKELIHOOD FILTER OUTPUT CORRESPONDING TO A RECURSIVELY FORMED CROSSPOWER SPECTRUM MATRIX

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1. Lemma:

$$\begin{vmatrix} \begin{vmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{vmatrix} & \begin{vmatrix} v_{02} & v_{01} \\ v_{12} & v_{11} \end{vmatrix} & \cdots & \begin{vmatrix} v_{0n} & v_{01} \\ v_{1n} & v_{11} \end{vmatrix} \\ \begin{vmatrix} v_{20} & v_{21} \\ v_{10} & v_{11} \end{vmatrix} & \begin{vmatrix} v_{22} & v_{21} \\ v_{12} & v_{11} \end{vmatrix} & \cdots & \begin{vmatrix} v_{2n} & v_{21} \\ v_{1n} & v_{11} \end{vmatrix} \\ \vdots & \vdots & & \vdots \\ \begin{vmatrix} v_{n0} & v_{n1} \\ v_{10} & v_{11} \end{vmatrix} & \begin{vmatrix} v_{n2} & v_{n1} \\ v_{12} & v_{11} \end{vmatrix} & \cdots & \begin{vmatrix} v_{nn} & v_{n1} \\ v_{1n} & v_{11} \end{vmatrix} \end{vmatrix} = v_{11}^{n-1} \begin{vmatrix} v_{00} & v_{01} & v_{02} & \cdots & v_{0n} \\ v_{10} & v_{11} & v_{12} & \cdots & v_{1n} \\ v_{20} & v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{n0} & v_{n1} & v_{n2} & \cdots & v_{nn} \end{vmatrix}$$

Proof:

The proof is by induction. For $n = 1$,

$$\left| \begin{vmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{vmatrix} \right| = \begin{vmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{vmatrix}$$

That is to say, the determinant of the matrix whose one element is

$$\begin{vmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{vmatrix}$$

is equal to the determinant of the matrix

$$\begin{bmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{bmatrix}$$

To complete the proof, it is necessary to show the truth of the lemma for n + 1 under the assumption that the lemma is true for n. This is accomplished by expanding the determinant on the left associated with the statement of the lemma for n + 1:

$$
\begin{vmatrix}
\begin{vmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{vmatrix} & \begin{vmatrix} v_{02} & v_{01} \\ v_{12} & v_{11} \end{vmatrix} & \cdots & \begin{vmatrix} v_{0,n+1} & v_{01} \\ v_{1,n+1} & v_{11} \end{vmatrix} \\
\begin{vmatrix} v_{20} & v_{21} \\ v_{10} & v_{11} \end{vmatrix} & \begin{vmatrix} v_{22} & v_{21} \\ v_{12} & v_{11} \end{vmatrix} & \cdots & \begin{vmatrix} v_{2,n+1} & v_{21} \\ v_{1,n+1} & v_{11} \end{vmatrix} \\
\vdots & \vdots & & \vdots \\
\begin{vmatrix} v_{n+1,0} & v_{n+1,1} \\ v_{10} & v_{11} \end{vmatrix} & \begin{vmatrix} v_{n+1,2} & v_{n+1,1} \\ v_{12} & v_{11} \end{vmatrix} & \cdots & \begin{vmatrix} v_{n+1,n+1} & v_{n+1,1} \\ v_{1,n+1} & v_{11} \end{vmatrix}
\end{vmatrix}
$$

$$
= \left\{ \begin{vmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{vmatrix}
\begin{vmatrix} v_{22} & v_{21} & v_{23} & \cdots & v_{2,n+1} \\ v_{12} & v_{11} & v_{13} & \cdots & v_{1,n+1} \\ v_{32} & v_{31} & v_{33} & \cdots & v_{3,n+1} \\ \vdots & \vdots & & & \vdots \\ v_{n+1,2} & v_{n+1,1} & v_{n+1,3} & \cdots & v_{n+1,n+1} \end{vmatrix}
- \begin{vmatrix} v_{02} & v_{01} \\ v_{12} & v_{11} \end{vmatrix}
\begin{vmatrix} v_{20} & v_{21} & v_{23} & \cdots & v_{2,n+1} \\ v_{10} & v_{11} & v_{13} & \cdots & v_{1,n+1} \\ v_{30} & v_{31} & v_{33} & \cdots & v_{3,n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{n+1,0} & v_{n+1,1} & v_{n+1,3} & \cdots & v_{n+1,n+1} \end{vmatrix} \right.
$$

$$
\left. + \begin{vmatrix} v_{03} & v_{01} \\ v_{13} & v_{11} \end{vmatrix}
\begin{vmatrix} v_{20} & v_{21} & v_{22} & v_{24} & \cdots & v_{2,n+1} \\ v_{10} & v_{11} & v_{12} & v_{14} & \cdots & v_{1,n+1} \\ v_{30} & v_{31} & v_{32} & v_{34} & \cdots & v_{3,n+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ v_{n+1,0} & v_{n+1,1} & v_{n+1,2} & v_{n+1,4} & \cdots & v_{n+1,n+1} \end{vmatrix}
- \ldots + (-1)^{n} \begin{vmatrix} v_{0,n+1} & v_{01} \\ v_{0,n+1} & v_{11} \end{vmatrix}
\begin{vmatrix} v_{20} & v_{21} & v_{22} & \cdots & v_{2n} \\ v_{10} & v_{11} & v_{12} & \cdots & v_{1n} \\ v_{30} & v_{31} & v_{32} & \cdots & v_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{n+1,0} & v_{n+1,1} & v_{n+1,2} & & v_{n+1,n} \end{vmatrix} \right\} v_{11}^{n-1}
$$

$$
= \left\{ \begin{vmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{vmatrix}
\begin{vmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1,n+1} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2,n+1} \\ v_{31} & v_{32} & v_{33} & \cdots & v_{3,n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{n+1,1} & v_{n+1,2} & v_{n+1,3} & \cdots & v_{n+1,n+1} \end{vmatrix}
+ \begin{vmatrix} v_{02} & v_{01} \\ v_{12} & v_{11} \end{vmatrix}
\begin{vmatrix} v_{10} & v_{11} & v_{13} & \cdots & v_{1,n+1} \\ v_{20} & v_{21} & v_{23} & \cdots & v_{2,n+1} \\ v_{30} & v_{31} & v_{33} & \cdots & v_{3,n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{n+1,0} & v_{n+1,1} & v_{n+1,3} & \cdots & v_{n+1,n+1} \end{vmatrix} \right.
$$

$$
\left. - \begin{vmatrix} v_{03} & v_{01} \\ v_{13} & v_{11} \end{vmatrix}
\begin{vmatrix} v_{10} & v_{11} & v_{12} & v_{14} & \cdots & v_{1,n+1} \\ v_{20} & v_{21} & v_{22} & v_{24} & \cdots & v_{2,n+1} \\ v_{30} & v_{31} & v_{32} & v_{34} & \cdots & v_{3,n+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ v_{n+1,0} & v_{n+1,1} & v_{n+1,2} & v_{n+1,4} & \cdots & v_{n+1,n+1} \end{vmatrix}
+ \ldots + (-1)^{n+1} \begin{vmatrix} v_{0,n+1} & v_{01} \\ v_{1,n+1} & v_{11} \end{vmatrix}
\begin{vmatrix} v_{10} & v_{11} & v_{12} & \cdots & v_{1n} \\ v_{20} & v_{21} & v_{22} & \cdots & v_{2n} \\ v_{30} & v_{31} & v_{32} & \cdots & v_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{n+1,0} & v_{n+1,1} & v_{n+1,2} & & v_{n+1,n} \end{vmatrix} \right\} v_{11}^{n-1}
$$

In the first step of this expansion, the determinant of the (n+1) x (n+1) matrix consisting of 2 x 2 determinants was expanded in terms of the minors associated with the top row, and the assumption that the lemma is true for n was used to evaluate the determinant of each minor. In the second step, use was made of the fact that interchanging adjacent rows or columns of a determinant causes a change of sign in the resulting determinant. At this point, it is convenient to introduce some notation. Let $|V_j|$ denote the determinant of the minor obtained by deleting the 0^{th} row and j^{th} column of the matrix

$$\begin{bmatrix} v_{00} & v_{01} & v_{02} & \cdots & v_{0,n+1} \\ v_{10} & v_{11} & v_{12} & \cdots & v_{1,n+1} \\ v_{20} & v_{21} & v_{22} & \cdots & v_{2,n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{n+1,0} & v_{n+1,1} & v_{n+1,2} & \cdots & v_{n+1,n+1} \end{bmatrix}$$

In addition, let $|V_{j\ell}|$ denote the determinant of the minor obtained by deleting the 0^{th} and 1^{st} rows and j^{th} and ℓ^{th} columns of the same matrix. When the (n+1) x (n+1) determinants obtained in the second stage are expanded in terms of the minors associated with the top row of each determinant, the expression obtained in the second stage becomes

$$\left(v_{00}\,|V_0| + v_{02}\,|V_2| - v_{03}\,|V_3| + v_{04}\,|V_4| - \ldots + (-1)^{n+1}\,v_{0,n+1}\,|V_{n+1}|\right) v_{11}^{n}$$

$$+ \Big\{ - v_{01} v_{10} \left[v_{11}\,|V_{01}| - v_{12}\,|V_{02}| + v_{13}\,|V_{03}| - v_{14}\,|V_{04}| + \ldots + (-1)^{n}\,v_{1,n+1}\,|V_{0,n+1}| \right]$$

$$- v_{01} v_{12} \left[v_{10}\,|V_{02}| - v_{11}\,|V_{12}| + v_{13}\,|V_{23}| - v_{14}\,|V_{24}| + \ldots + (-1)^{n}\,v_{1,n+1}\,|V_{2,n+1}| \right]$$

$$+ v_{01} v_{13} \left[v_{10}\,|V_{03}| - v_{11}\,|V_{13}| + v_{12}\,|V_{23}| - v_{14}\,|V_{34}| + \ldots + (-1)^{n}\,v_{1,n+1}\,|V_{3,n+1}| \right]$$

$$- v_{01} v_{14} \left[v_{10}\,|V_{04}| - v_{11}\,|V_{14}| + v_{12}\,|V_{24}| - v_{13}\,|V_{34}| + \ldots + (-1)^{n}\,v_{1,n+1}\,|V_{4,n+1}| \right]$$

$$+ \ldots$$

$$+ (-1)^{n}\,v_{01} v_{1,n+1} \Big[v_{10}\,|V_{0,n+1}| - v_{11}\,|V_{1,n+1}| + v_{12}\,|V_{2,n+1}| - v_{13}\,|V_{3,n+1}|$$

$$+ \ldots + (-1)^{n}\,v_{1,n+1}\,|V_{n,n+1}| \Big] \Big\} \cdot v_{11}^{n-1}$$

Within each bracketed expression, all terms except those multiplied by v_{11} cancel another term within the full expression contained in braces. Thus, the full determinant expansion becomes

$$\left(v_{00}\,|V_0| + v_{02}\,|V_2| - v_{03}\,|V_3| + v_{04}\,|V_4| - \ldots + (-1)^{n+1}\,|V_{n+1}|\right) v_{11}^{n}$$

$$+ \Big\{ - v_{01} \left[v_{10}\,|V_{01}| - v_{12}\,|V_{12}| + v_{13}\,|V_{13}| - v_{14}\,|V_{14}| + \ldots + (-1)^{n+1}\,v_{1,n+1}\,|V_{1,n+1}| \right] \Big\} v_{11}^{n}$$

The bracketed expression is equal to $|V_1|$, so that the result is

$$v_{11}^{n}\left(v_{00}\,|V_0| - v_{01}\,|V_1| + v_{02}\,|V_2| - v_{03}\,|V_3| + v_{04}\,|V_4| - \dots + (-1)^{n+1}\,|V_{n+1}|\right)$$

$$= v_{11}^{n}\begin{vmatrix} v_{00} & v_{01} & v_{02} & \cdots & v_{0,n+1} \\ v_{10} & v_{11} & v_{12} & \cdots & v_{1,n+1} \\ v_{20} & v_{21} & v_{22} & \cdots & v_{2,n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{n+1,0} & v_{n+1,1} & v_{n+1,2} & \cdots & v_{n+1,n+1} \end{vmatrix}$$

This determinant expression is the desired result for $n+1$, so that the lemma is proved for all positive integers n.

2. Theorem:

Let a set of matrices $\left\{\Phi, \Phi_2, \dots, \Phi_n\right\}$ satisfy the following recursive relationship:

$$\Phi_n = \Phi_{n-1} + V_n V_n^H$$

where V_n is a column vector and V_n^H is its conjugate transpose. Then,

$$T^H \Phi_n^{-1} U = \frac{\begin{vmatrix} T^H \Phi_0^{-1} U & T^H \Phi_0^{-1} V_1 & \cdots & T^H \Phi_0^{-1} V_n \\ V_1^H \Phi_0^{-1} U & 1 + V_1^H \Phi_0^{-1} V_1 & \cdots & V_1^H \Phi_0^{-1} V_n \\ \vdots & \vdots & & \vdots \\ V_n^H \Phi_0^{-1} U & V_n^H \Phi_0^{-1} V_2 & \cdots & V_n^H \Phi_0^{-1} V_n \end{vmatrix}}{\begin{vmatrix} 1 + V_1^H \Phi_0^{-1} V_1 & V_1^H \Phi_0^{-1} V_2 & \cdots & V_1^H \Phi_0^{-1} V_n \\ V_2^H \Phi_0^{-1} V_1 & 1 + V_2^H \Phi_0^{-1} V_2 & \cdots & V_2^H \Phi_0^{-1} V_n \\ \vdots & \vdots & & \vdots \\ V_n^H \Phi_0^{-1} V_1 & V_n^H \Phi_0^{-1} V_2 & \cdots & 1 + V_n^H \Phi_0^{-1} V_n \end{vmatrix}}$$

Proof:

The proof is once again by induction. For n = 1, $\Phi_1 = \Phi_0 + V_1 V_1^H$. The inverse matrix for Φ_1 is given by the formula

$$\Phi_1^{-1} = \Phi_0^{-1} - \frac{\Phi_0^{-1} V_1 V_1^H \Phi_0^{-1}}{1 + V_1^H \Phi_0^{-1} V_1}$$

This equation may be verified by multiplying the formula for Φ_1^{-1} by $\Phi_0 + V_1 V_1^H$. Premultiplying this equation by T^H and postmultiplying this equation by U, one obtains

$$T^H \Phi_1^{-1} U = \frac{\left(T^H \Phi_0^{-1} U\right)\left(1 + V_1^H \Phi_0^{-1} V_1\right) - \left(T^H \Phi_0^{-1} V_1\right)\left(V_1^H \Phi_0^{-1} U\right)}{1 + V_1^H \Phi_0^{-1} V_1}$$

$$= \frac{\begin{vmatrix} T^H \Phi_0^{-1} U & T^H \Phi_0^{-1} V_1 \\ V_1^H \Phi_0^{-1} U & 1 + V_1^H \Phi_0^{-1} V_1 \end{vmatrix}}{\left|1 + V_1^H \Phi_0^{-1} V_1\right|}$$

This is the statement of the theorem for n = 1.

Note that the matrix obtained by deleting the row containing the row vector T^H and the column containing the column vector U is the single-element matrix

$$\left[1 + V_1^H \Phi_0^{-1} V_1\right]$$

This is the matrix whose determinant appears in the denominator of the formula for $T^H \Phi^{-1} U$. By adding 1 to the element $T^H \Phi_0^{-1} U$, the numerator is therefore increased by $1 + V_1^H \Phi_0^{-1} V_1$, so that

$$1 + T^H \Phi_1^{-1} U = \frac{\begin{vmatrix} 1 + T^H \Phi_0^{-1} U & T^H \Phi_0^{-1} V_1 \\ V_1^H \Phi_0^{-1} U & 1 + V_1^H \Phi_0^{-1} V_1 \end{vmatrix}}{\left| 1 + V_1^H \Phi_0^{-1} V_1 \right|}$$

Now it is necessary to prove that the truth of the theorem for n implies that the theorem is true for n + 1.

Let $\Phi_{n+1} = \Phi_1 + V_2 V_2^H + \ldots + V_{n+1} V_{n+1}^H$. Under the assumption that the theorem is true for n,

$$T^H \Phi_{n+1}^{-1} U = \frac{\begin{vmatrix} T^H \Phi_1^{-1} U & T^H \Phi_1^{-1} V_2 & \cdots & T^H \Phi_1^{-1} V_{n+1} \\ V_2^H \Phi_1^{-1} U & 1 + V_2^H \Phi_1^{-1} V_2 & \cdots & V_2^H \Phi_1^{-1} V_{n+1} \\ \vdots & \vdots & & \vdots \\ V_{n+1}^H \Phi_1^{-1} U & V_{n+1}^H \Phi_1^{-1} V_2 & \cdots & V_{n+1}^H \Phi_1^{-1} V_{n+1} \end{vmatrix}}{\begin{vmatrix} 1 + V_2^H \Phi_1^{-1} V_2 & V_2^H \Phi_1^{-1} V_3 & \cdots & V_2^H \Phi_1^{-1} V_{n+1} \\ V_3^H \Phi_1^{-1} V_2 & 1 + V_3^H \Phi_1^{-1} V_3 & \cdots & V_3^H \Phi_1^{-1} V_{n+1} \\ \vdots & \vdots & & \vdots \\ V_{n+1}^H \Phi_1^{-1} V_2 & V_{n+1}^H \Phi_1^{-1} V_3 & \cdots & 1 + V_{n+1}^H \Phi_1^{-1} V_{n+1} \end{vmatrix}}$$

$$
= \frac{
\left\| \begin{matrix}
\begin{vmatrix} T^H\Phi_0^{-1}U & T^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}U & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} &
\begin{vmatrix} T^H\Phi_0^{-1}v_2 & T^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_2 & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} & \cdots &
\begin{vmatrix} T^H\Phi_0^{-1}v_{n+1} & T^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_{n+1} & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} \\
\begin{vmatrix} v_2^H\Phi_0^{-1}U & v_2^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}U & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} &
\begin{vmatrix} 1+v_2^H\Phi_0^{-1}v_2 & v_2^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_2 & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} & \cdots &
\begin{vmatrix} v_2^H\Phi_0^{-1}v_{n+1} & v_2^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_{n+1} & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} \\
\vdots & \vdots & & \vdots \\
\begin{vmatrix} v_{n+1}^H\Phi_0^{-1}U & v_{n+1}^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}U & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} &
\begin{vmatrix} v_{n+1}^H\Phi_0^{-1}v_2 & v_{n+1}^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_2 & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} & \cdots &
\begin{vmatrix} 1+v_{n+1}^H\Phi_0^{-1}v_{n+1} & v_{n+1}^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_{n+1} & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix}
\end{matrix} \right\|
\left(1+v_1^H\Phi_0^{-1}v_1\right)^{n-1}
}{
\left\| \begin{matrix}
\begin{vmatrix} 1+v_2^H\Phi_0^{-1}v_2 & v_2^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_2 & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} &
\begin{vmatrix} v_2^H\Phi_0^{-1}v_3 & v_2^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_3 & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} & \cdots &
\begin{vmatrix} v_2^H\Phi_0^{-1}v_{n+1} & v_2^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_{n+1} & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} \\
\begin{vmatrix} v_3^H\Phi_0^{-1}v_2 & v_3^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_2 & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} &
\begin{vmatrix} 1+v_3^H\Phi_0^{-1}v_3 & v_3^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_3 & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} & \cdots &
\begin{vmatrix} v_3^H\Phi_0^{-1}v_{n+1} & v_3^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_{n+1} & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} \\
\vdots & \vdots & & \vdots \\
\begin{vmatrix} v_{n+1}^H\Phi_0^{-1}v_2 & v_{n+1}^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_2 & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} &
\begin{vmatrix} v_{n+1}^H\Phi_0^{-1}v_3 & v_{n+1}^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_3 & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix} & \cdots &
\begin{vmatrix} 1+v_{n+1}^H\Phi_0^{-1}v_{n+1} & v_{n+1}^H\Phi_0^{-1}v_1 \\ v_1^H\Phi_0^{-1}v_{n+1} & 1+v_1^H\Phi_0^{-1}v_1 \end{vmatrix}
\end{matrix} \right\|
}
$$

$$
= \frac{
\begin{vmatrix}
T^H\Phi_0^{-1}U & T^H\Phi_0^{-1}v_1 & T^H\Phi_0^{-1}v_2 & \cdots & T^H\Phi_0^{-1}v_{n+1} \\
v_1^H\Phi_0^{-1}U & 1+v_1^H\Phi_0^{-1}v_1 & v_1^H\Phi_0^{-1}v_2 & \cdots & v_1^H\Phi_0^{-1}v_{n+1} \\
v_2^H\Phi_0^{-1}U & v_2^H\Phi_0^{-1}v_1 & 1+v_2^H\Phi_0^{-1}v_2 & \cdots & v_2^H\Phi_0^{-1}v_{n+1} \\
\vdots & \vdots & \vdots & & \vdots \\
v_{n+1}^H\Phi_0^{-1}U & v_{n+1}^H\Phi_0^{-1}v_1 & v_{n+1}^H\Phi_0^{-1}v_2 & \cdots & 1+v_{n+1}^H\Phi_0^{-1}v_{n+1}
\end{vmatrix}
}{
\begin{vmatrix}
1+v_1^H\Phi_0^{-1}v_1 & v_1^H\Phi_0^{-1}v_2 & v_1^H\Phi_0^{-1}v_3 & \cdots & v_1^H\Phi_0^{-1}v_{n+1} \\
v_2^H\Phi_0^{-1}v_1 & 1+v_2^H\Phi_0^{-1}v_2 & v_2^H\Phi_0^{-1}v_3 & \cdots & v_2^H\Phi_0^{-1}v_{n+1} \\
v_3^H\Phi_0^{-1}v_1 & v_3^H\Phi_0^{-1}v_2 & 1+v_3^H\Phi_0^{-1}v_3 & \cdots & v_3^H\Phi_0^{-1}v_{n+1} \\
\vdots & & \vdots & & \vdots \\
v_{n+1}^H\Phi_0^{-1}v_1 & v_{n+1}^H\Phi_0^{-1}v_2 & v_{n+1}^H\Phi_0^{-1}v_3 & \cdots & 1+v_{n+1}^H\Phi_0^{-1}v_{n+1}
\end{vmatrix}
}
$$

The transition to the final expression for $T^H \Phi_{n+1}^{-1} U$ was made by invoking the lemma proved in subsection 1. The final expression is the one for $T^H \Phi_{n+1}^{-1} U$ which would have been obtained by invoking the theorem for n+1. Thus, the theorem is true for all positive integers n.

3. Corollary:

If $\Phi_n = \rho I + V_1 V_1^H + \ldots + V_n V_n^H$,

$$\frac{1}{U^H \Phi_n^{-1} U} = \frac{\rho \begin{vmatrix} \rho + |V_1|^2 & V_1^H V_2 & \cdots & V_1^H V_n \\ V_2^H V_1 & \rho + |V_2|^2 & \cdots & V_2^H V_n \\ \vdots & \vdots & & \vdots \\ V_n^H V_1 & V_n^H V_2 & \cdots & \rho + |V_n|^2 \end{vmatrix}}{\begin{vmatrix} |U|^2 & U^H V_1 & \cdots & U^H V_n \\ V_1^H U & \rho + |V_1|^2 & \cdots & V_1^H V_n \\ \vdots & \vdots & & \vdots \\ V_n^H U & V_n^H V_1 & \cdots & \rho + |V_n|^2 \end{vmatrix}}$$

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13. ABSTRACT

A theoretical study of several computational techniques for estimating wavenumber spectra is presented. Relationships among the techniques used for computing high-resolution wavenumber spectra are investigated. Stability and resolution of their estimates are compared. Methods for computing 2-dimensional maximum-entropy spectra are found to be impractical, and it is concluded that the maximum-likelihood technique provides the best estimate for most purposes.

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