

UNCLASSIFIED

AD NUMBER
AD860850
NEW LIMITATION CHANGE
TO Approved for public release, distribution unlimited
FROM Distribution authorized to U.S. Gov't. agencies and their contractors; Critical Technology; 01 AUG 1969. Other requests shall be referred to Space and Missile Systems Organization, Attn: SMON, Los Angeles, CA 90045.
AUTHORITY
SAMSO ltr dtd 19 Jan 1972

THIS PAGE IS UNCLASSIFIED

AIR FORCE REPORT NO.
SAMSO-TR-69-318

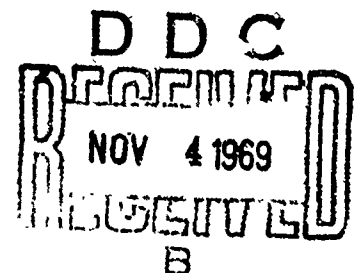
AEROSPACE REPORT NO.
TR-0066(5110-01)-3

AD 860850

Trajectory Analysis Program (TAP)

Prepared by DENNIS C. WALKER
Engineering Science Operations

69 AUG 01



Systems Engineering Operations
THE AEROSPACE CORPORATION

Prepared for SPACE AND MISSILE SYSTEMS ORGANIZATION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
Los Angeles, California

THIS DOCUMENT IS SUBJECT TO SPECIAL EXPORT
CONTROLS AND EACH TRANSMITTAL TO FOREIGN
GOVERNMENTS OR FOREIGN NATIONALS MAY BE MADE
ONLY WITH PRIOR APPROVAL OF SAMSO(SMON).
THE DISTRIBUTION OF THIS REPORT IS LIMITED BE-
CAUSE IT CONTAINS TECHNOLOGY RESTRICTED BY
MUTUAL SECURITY ACTS.

Reproduced by the
CLEARINGHOUSE
for Federal Scientific & Technical
Information Springfield Va 22151

97

Air Force Report No.
SAMSO-TR-69-318

Aerospace Report No.
TR-0066(5110-01)-3

TRAJECTORY ANALYSIS PROGRAM (TAP)

Prepared by
Dennis C. Walker
Flight Mechanics Department
Mathematics and Computation Center
Engineering Science Operations

69 AUG 01

Systems Engineering Operations
THE AEROSPACE CORPORATION

Prepared for
SPACE AND MISSILE SYSTEMS ORGANIZATION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
Los Angeles, California

This document is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of SAMSO(SMON).
The distribution of this report is limited because it contains technology restricted by mutual security acts.

FOREWORD

This report is published by The Aerospace Corporation, El Segundo, California, under Air Force Contract No. F04701-69-C-0066.

This report, which documents research carried out from June 1968 through May 1969, was submitted for review and approval on 2 September 1969 to Colonel Arthur W. Banister, USAF.

Approved

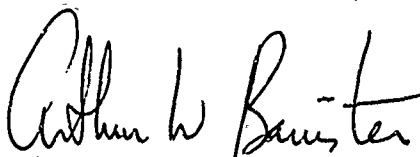


L. J. Lander
Director
Mathematics and Programming Subdivision
Information Processing Division
Engineering Science Operations



W. F. Sampson
Group Director
Satellite Control Directorate
Satellite Systems Division
Systems Engineering Operations

Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.



Arthur W. Banister, Col, USAF
Acting Director for Development
AF Satellite Control Facility

ABSTRACT

The Trajectory Analysis Program (TAP) is designed specifically for use on the CDC 6000 series machine as a computational tool to generate trajectories by different mathematical formulations and to compare the results. In particular, the reference trajectory is the result of numerical integration of the equations of motion in a Cowell formulation with time as the independent variable; whereas, the comparison trajectory selected is achieved by an analytic (Pines, Kyner, or Escobal) or a numerically integrated Encke (classical or modified) formulation. A comprehensive description is presented of the algorithms implemented in TAP with emphasis on characteristics such as logic flow, equations, numerical techniques, comparison differencing, and plotting. A basic usage guide is included with complete instructions for input data preparation and explanations of output tables.

BLANK PAGE

CONTENTS

I.	INTRODUCTION	1
A.	Purpose, Scope, and Limitations	1
B.	Historical Background	1
C.	Summary of Key Features	2
II.	TRAJECTORY GENERATION AND COMPARISON	3
A.	The Basic Problem	3
B.	Trajectory Generation Techniques	3
C.	Trajectory Comparison Techniques	4
III.	REFERENCE SYSTEMS AND EQUATIONS	7
A.	Symbols (Notation)	7
B.	Reference Systems	8
1.	Earth-Centered Inertial System (ECI)	8
2.	Classical Elliptic System	10
3.	Orbit-Plane System	11
C.	Transformation Equations	12
1.	From ECI to the Classical Elliptic System	12
2.	From ECI to the Orbit-Plane System	14
D.	Trajectory Generation Equations	14
1.	Cowell Formulation	14
2.	Encke Formulation	15
3.	Analytic Algorithms	16
a.	Pines Method	16
b.	Kyner Method	17
c.	Escobal Method	21
E.	Miscellaneous Equations	25
IV.	COWELL AND ENCKE IMPLEMENTATION	33

CONTENTS (cont.)

V.	PROGRAM STRUCTURE AND LOGIC FLOW	35
VI.	PROGRAM USAGE	37
A.	Input Data	37
B.	Output Data	41
C.	Plotting Features	46
	REFERENCES	51
	APPENDICES	
A.	NUMERICAL ACCURACY OF THE DEQ NUMERICAL INTEGRATOR	53
B.	KYNER II NON-TWO-BODY NOMINAL TRAJECTORY	63

TABLES

I.	Schematic for Implementing Cowell and Encke Formulation	33
II.	Input Data Description	37
III.	Initial Condition Data	41
IV.	Epoch (TZERO) Data	41
V.	Trajectory Data	42
VI.	Reference Trajectory Data Matrix [E]	43
VII.	Comparison Trajectory Data Matrix [A]	43
VIII.	Difference Trajectory Data Matrix [D]	44
IX.	WKCNC2 Package Output	45
A-I.	Cowell Integration Method	56
A-II.	Test Orbits	58
A-III.	Sample Running Times	62

FIGURES

1.	Earth-Centered Inertial Reference System	9
2.	Classical Elliptic Reference Frame	10
3.	Orbit-Plane Coordinate System	11
4.	Diagram of TAP General Structure and Logic Flow	35
5.	Printer Plot of Variable = (RSS)	47
6.	Printer Plot of Variable = (XI)	48
7.	Printer Plot of Variable = R, C	49

FIGURES (cont.)

8.	Printer Plot of Variable = M, C	50
A-1.	Accuracy of Numerical Integration (Analytic Solution, Keplerian Model)	57
A-2.	Error Variation with Fixed Step Size	59
A-3.	Closure of Numerical Integration for Orbit A ($i = 45^\circ$)	61
A-4.	Closure of Numerical Integration for Orbit C ($i = 95^\circ$)	61

NOMENCLATURE

$[A]$	sum of constant matrices (evaluated at epoch) whose scalar coefficients are slowly varying functions of φ and the rate parameters (η, τ, γ)
$[\dot{A}]$	time derivative of the matrix $[A]$
a	semimajor axis
a_e	mean equatorial radius of earth or semi-diameter of the central mass
a_j	scalar coefficients of rotation matrix $[A]$
$C_D A/W$	drag coefficient
DEQ	numerical integrator used for trajectory generation
DP	double precision
ECI	earth-centered inertial
e	eccentricity
F, G	Keplerian scalar functions
f	true anomaly
$\underline{G}_{LH}$	gravitational acceleration vector due to noncentral force field of earth in local horizontal system, in which coordinate axes are directed Up (along the position vector), East, and North
h	angular momentum
i	inclination
J_2	second zonal harmonic coefficient
$K.E.$	kinetic energy
k	Gaussian or planetary constant
$\underline{L}$	additive secular acceleration vector that accounts for consideration of non-two-body reference orbit

l	true longitude
$\underline{M}, \underline{N}$	rotating vectors always in nominal orbital plane that maintain constant angles with precessing line of apses of nominal ellipse
M	mean anomaly
n	nominal anomalistic mean motion
n	mean motion
PDP	partial double precision
P.E.	potential energy
P_k	Keplerian Period
$P_{\Omega}^{m'}$	derivative of Legendre function with respect to $\sin \phi$
p	semi-parameter of the conic
$[R]$	rotation matrix
RTC	orbit-plane system
r	geocentric distance of vehicle
$\underline{r}$	position vector
$\underline{r}_n(t)$	nominal trajectory position vector at time t
$\dot{\underline{r}}$	velocity vector
$\dot{\underline{r}}_A$	vehicle velocity vector relative to a rotating atmosphere
$\ddot{\underline{r}}$	acceleration vector
$\ddot{\underline{r}}_1$	noncentral gravitational acceleration vector
$\ddot{\underline{r}}_2$	atmospheric drag acceleration vector
rss	root-sum-square
SP	single precision

$S(R, T, C, R', T', C')$	state vector in orbit-plane system coordinates
$S(\underline{r}, \underline{\dot{r}}, \underline{\ddot{r}})$	vehicle state vector of position, velocity, and acceleration
$S(\underline{r}_n, \underline{\dot{r}}_n, \underline{\ddot{r}}_n)$	a non-Keplerian state vector
$S(\underline{r}, \underline{\dot{r}}, t)$	vehicle state vector of position and velocity
TAP	Trajectory Analysis Program
T.E.	total energy (K.E. + P.E.)
$T_{\phi, \lambda}$	earth-fixed local horizontal system transformation matrix
t	time
u	argument of latitude
v	true anomaly
γ	rate parameter of nominal anomalistic mean motion $\bar{n}$
Δ	independent variable difference (time Δt ; true anomaly Δf ; eccentric anomaly ΔE)
δ_{ij}	Kronecker delta function
ϵ_k	convergence criterion for solution of Kepler's equation
$\underline{\zeta}$	vector cross-track direction
η	rate parameter of the major axis of the nominal ellipse
$\underline{\eta}$	vector in-track direction
λ	longitude of vehicle
μ	square of the gravitational constant of the earth
$[\mu]$	sum of celestial object mass and central mass

$\hat{r}$	vector radial direction
τ	rate parameter of the line of nodes of the nominal plane
ϕ	latitude of the vehicle
Ω	longitude of the ascending node
ω	argument of perigee or pericenter

SECTION I

INTRODUCTION

A. PURPOSE, SCOPE, AND LIMITATIONS

This document is intended to serve as a technical reference manual and basic usage guide for the Trajectory Analysis Program (TAP). Trajectory generation and comparison, as defined in Section II, are the result of the evaluation of a particular formulation for the equations of motion, which yields the position and velocity components of a vehicle at some desired time t , and comparison of that solution with one obtained by another method. The mathematical formulations implemented in TAP consist both of numerical integration methods (Cowell and Encke) and analytic algorithms (Pines, Kyner, and Escobal).

A precision numerical integration scheme used in the Cowell and Encke formulations is described in Appendix A. A comprehensive description of all the methods used in TAP is presented with emphasis placed on key characteristics.

Sections III (Reference Systems and Equations), IV (Cowell and Encke Implementation), V (Program Structure and Logic Flow), and VI (Program Usage) are sufficiently detailed to meet the needs of users concerned with trajectory generation and comparison.

Only the class of elliptical geodetic orbits within a specific inertial reference system is considered; force models allow for the noncentral gravitational effects and a basic atmospheric drag effect. Particular emphasis is placed on the use of the Modified Encke formulation for trajectory computation. The algorithm for the Kyner II nominal trajectory is developed in Appendix B.

B. HISTORICAL BACKGROUND

In recent years, trajectory generation has been performed primarily by numerical integration of the equations of motion in the Cowell formulation by the TRACE Orbit Determination Program (Ref. 1) at Aerospace Corporation.

TAP was developed to analyze different computational methods for trajectory generation, and to provide a tool to establish solution accuracy by means of comparison. The program is written entirely in CDC 6000 series FORTRAN and its flexible design permits the addition of new formulations and accuracy tests of interest.

C. SUMMARY OF KEY FEATURES

By a collection of computational algorithms, TAP generates trajectories and associated quantities, which are then compared and illustrated. The key features of TAP can be identified by the following:

1. Methods of Trajectory Generation considered are the Cowell formulation for the reference solution; and the Encke, Pines, Kyner, or Escobal formulation for the comparison solution.
2. Event Detection During Generation consists of:
 - Ascending and descending nodes
 - Apsides
 - Specified print times
 - Crash radius
 - Integration step-size change
3. Comparison of Orbital Parameters, both directly and indirectly (in the form of differences and root-sum-square), are in three basic coordinate systems: Earth-Centered Inertial (ECI), Classical Elliptic, and Orbit-Plane.
4. Printer Plot of prespecified parameters versus time.
5. Integration Closure results.

SECTION II

TRAJECTORY GENERATION AND COMPARISON

A. THE BASIC PROBLEM

Fundamentally, trajectory generation is a numerical process for obtaining the position and velocity components of the vehicle state vector $S(\underline{r}, \dot{\underline{r}}, t)$ at some desired time t . A basic problem of trajectory generation is to choose a process that evaluates a given formulation for the state vector by means of a computational algorithm. In the Trajectory Analysis Program, several mathematical formulations of the equations of motion and associated computational algorithms are considered; both numerical integration methods (Cowell and Encke) and analytic algorithms (Pines, Kyner, and Escobal) are implemented.

Trajectory comparison of different processes or methods of trajectory generation is of particular interest. Typically, the state vector and related quantities resulting from one method are compared at some common time t with those of another. Numerical properties are expressed in exact and difference form. To exhibit a comparison, TAP generates a precise numerical integration method as a reference trajectory, which is then differenced at times of interest with a trajectory generated by another method. In addition, a quick-look plotting feature graphically presents the behavior characteristics of specified quantities.

B. TRAJECTORY GENERATION TECHNIQUES

Solution of the equations of motion in TAP can be expressed in analytical form by an algorithm (Pines, Kyner, or Escobal), or it can be obtained in numerical form by special perturbation methods (Cowell or Encke).

In the case of the Cowell and Encke formulations for numerical integration of the equations of motion or deviation, a predictor/corrector (eighth-order differencing) Runge-Kutta/Gauss-Jackson technique is used (see Appendix A), with time as the independent variable.

Analytic methods available in TAP generate the Keplerian or non-Keplerian state vector at some specified time t by the evaluation of closed-form expressions, as follows.

<u>Method</u>	<u>Closed-Form Representation</u>
Pines	F & G functions (Keplerian only)
Kyner I	Classical elements
Kyner II	F & G functions
Escobal	Classical elements

In addition, Hermitian interpolation techniques (fifth- and third-degree polynomial schemes) are utilized to obtain trajectory quantities at special events (such as nodes, apsides, and crash radius), and specified print times.

C. TRAJECTORY COMPARISON TECHNIQUES

The state vectors and related quantities at time t are obtained from different methods and compared in the form of differences ($\Delta \equiv$ reference minus comparison). Differences are given in three coordinate systems — Earth-Centered Inertial (ECI), Classical Elliptic, and Orbit-Plane — within which the maximum separation between the two state vectors at time t is expressed as the root-sum-square (rss).

The Cowell formulation has been chosen to generate the reference trajectory in TAP, thereby defining the integration step times as comparison times. The comparison trajectory can be specified as the result of

- Encke formulation - Classical (Keplerian nominal) or Modified (non-Keplerian nominal)
- Analytic algorithm - Pines, Kyner I, Kyner II, or Escobal nominals.

That is, both the reference and the comparison trajectories are generated simultaneously for any trajectory generation case. In particular, the following combinations of methods are available.

<u>Formulation</u>	<u>Reference Trajectory</u>	<u>Comparison Trajectory</u>
<u>Cowell</u>	Cowell	Analytic Algorithm PINES KYNER 1 KYNER 2 ESCOBAL
<u>Encke</u>	Cowell	Encke CLASSICAL MODIFIED

Another useful trajectory comparison technique is contrasting integration closure errors obtained from two different methods. Integration closure error is the magnitude of the difference vector between the initial position vector $\underline{r}_0$ and the resulting position vector $\underline{r}(t = \text{epoch})$ after integrating forward to some terminal time and then backward to epoch. Thus

$$\Delta_{\text{CLOSURE}} = |\underline{r}_0 - \underline{r}(t = \text{epoch})| .$$

BLANK PAGE

SECTION III

REFERENCE SYSTEMS AND EQUATIONS

The symbols used throughout this report follow, after which the basic reference systems and equations used in TAP are defined and discussed.

A. SYMBOLS (NOTATION)

- . time differentiation and vector dot product operation
- .. time differentiation
- _ a vector; for example, $\underline{r}$ is the vehicle position vector
- () relates to a functional relationship, such as $S(\underline{r}, \dot{\underline{r}})$ the vehicle state vector, which is a function of $\underline{r}$ and $\dot{\underline{r}}$, or $S(a, e, i, \Omega, \omega, M)$, the state vector expressed in terms of elliptical elements. Also, used to denote row vectors; for example, $\underline{r} = (x, y, z)$
- Δ increment or difference
- $\times$ vector cross product operation
- $||$ absolute value, or vector magnitude operation; for example,
 $r = |\underline{r}|$
- ∇ vector gradient operator
- $\{ \}$ a matrix, or a set of elements
- $I[]$ the integer operator of a scalar
- $[]$ a matrix; for example,

$$\underline{r} = [x, y, z] \text{ (row vector), or } \underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ (column vector)}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ 3 X 3 identity matrix}$$

$$A = [A] \text{ 6 X 6 matrix or array of trajectory quantities.}$$

Superscripts

c	computed or simulated value
i	particular index relative to a set or vector
T	matrix transpose symbol
*	particular value of a predetermined constant

Subscripts

C	comparison solution or value
e	the earth
g	the Greenwich meridian
i, j, k, l	particular index denoting members of a set or components of a vector. In general, i, j, k, l = 1, 2, . . .
n	nominal value
α	a transformation that is a function of α
γ	differentiation of a vector with respect to the parameter γ
Φ	a transformation that is a function of Φ
o	initial value or reference point

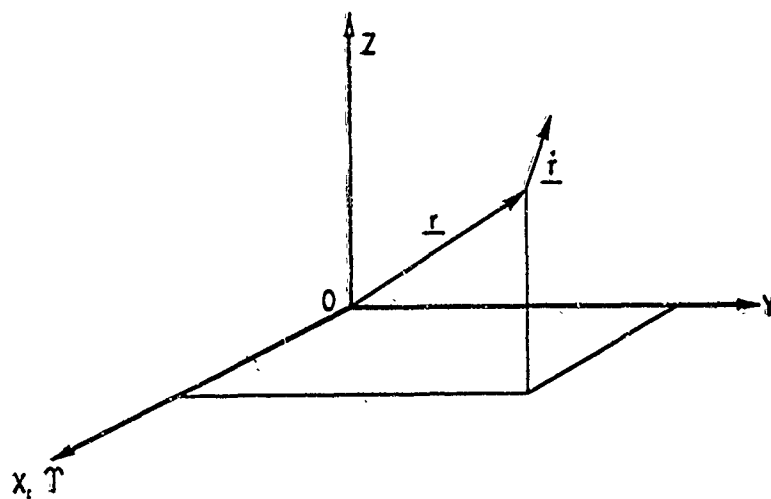
B. REFERENCE SYSTEMS

Trajectory generation and comparison involve coordinate transformations and require selection of the proper reference systems for evaluation and analysis. Several such systems are used in TAP. Basic to all is the earth-centered inertial (ECI) system, which has as its fundamental plane the true equator at epoch; its principal axis is the mean equinox at midnight of the date of epoch. This reference system does not take into account precession and nutation effects.

1. EARTH-CENTERED INERTIAL SYSTEM (ECI)

Inertial position, velocity, and acceleration vectors are oriented in the reference system illustrated in Figure 1.

The position and velocity components of a vehicle's state vector within this reference system may also be expressed in terms of the classical elliptic and orbit-plane systems.



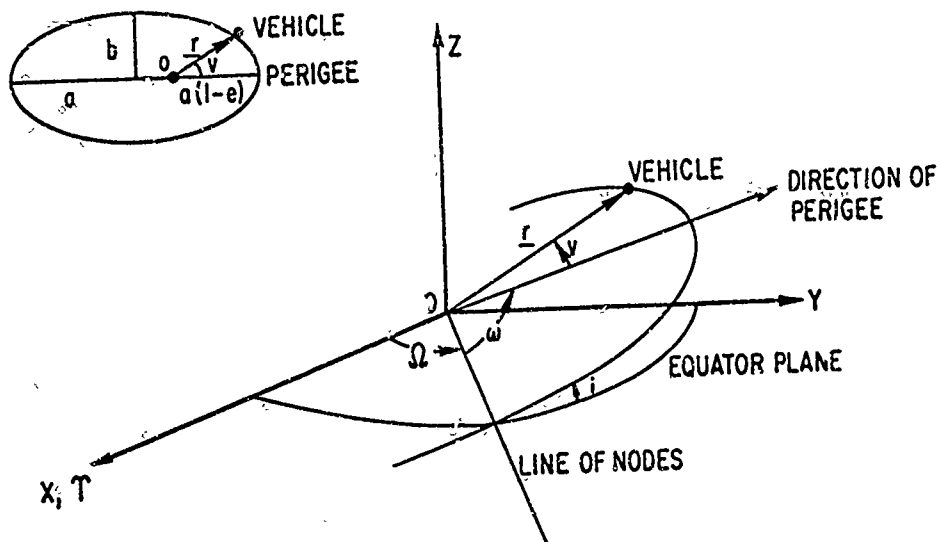
$\underline{r}$	radius vector (x, y, z)
$\dot{\underline{r}}$	velocity vector ($\dot{x}, \dot{y}, \dot{z}$)
X	direction of the vernal equinox T in the equator plane
x	position component in the X direction
$\dot{x}$	velocity component in the X direction (relative to O)
Y	right-hand axis to X and Z
y	position component in the Y direction
$\dot{y}$	velocity component in the Y direction (relative to O)
Z	direction north perpendicular to the equator plane
z	position component in the Z direction
$\dot{z}$	velocity component in the Z direction (relative to O)
O	origin, which coincides with the center of the earth

Figure 1. Earth-Centered Inertial Reference System

2. CLASSICAL ELLIPTIC SYSTEM

The state vector in the classical elliptic system (Fig. 2) is given at time t by

$$S = S(a, e, i, \Omega, \omega, M)$$



- a semimajor axis of the osculating conic
- e eccentricity
- i inclination
- Ω right ascension of the ascending node
- ω angle between the direction of perigee and the line of nodes
- M mean anomaly
- v true anomaly
- b semiminor axis

Figure 2. Classical Elliptic Reference Frame

3. ORBIT-PLANE SYSTEM

The state vector in the orbit-plane system (Fig. 3) is given at time t by

$$S = S(R, T, C, R', T', C')$$

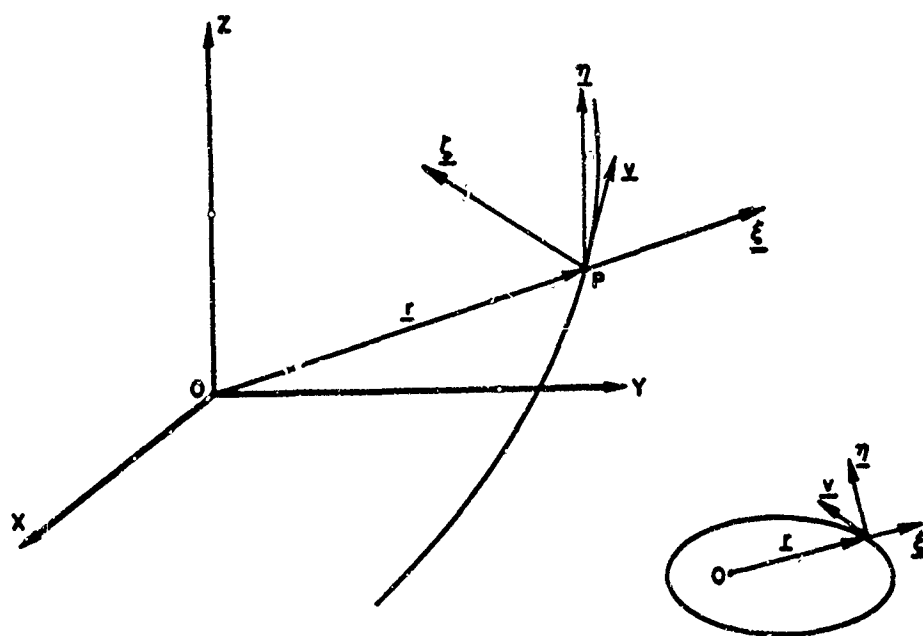


Figure 3. Orbit-Plane Coordinate System

In this instance the point P, position of the vehicle in the ECI system, represents the origin of the orbit-plane system defined by the vectors $\underline{\xi}$, $\underline{\eta}$, $\underline{\zeta}$, which are referred to as the radial, in-track, and cross-track directions, respectively. In the smaller sketch, it should be noted that the $\underline{\xi}$ axis is an extension of the geocentric radius vector. The $\underline{\eta}$ axis is both normal to the $\underline{\xi}$ axis and positive in the same general direction as the inertial velocity vector $\dot{\underline{r}}$. Both lie in the instantaneous orbit plane. The $\underline{\zeta}$ axis is normal to the orbit plane (defined by $\underline{\xi}$ and $\underline{\eta}$) and thus forms a right-hand orthogonal system.

The position and velocity of an alternate point Q relative to point P are then given by

$$Q = Q(R, T, C, R', T', C')$$

where

R = position component in the ξ direction

T = position component in the η direction

C = position component in the ζ direction

R' = velocity component in the ξ direction

T' = velocity component in the η direction

C' = velocity component in the ζ direction

C. TRANSFORMATION EQUATIONS

The transformation of the state vector to the other systems involves the following equations.

1. FROM ECI TO THE CLASSICAL ELLIPTIC SYSTEM

To obtain the classical elements from ECI coordinates, the following equations are solved.

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu}$$

$$r = |\underline{r}|$$

$$v^2 = \dot{\underline{r}} \cdot \dot{\underline{r}}$$

$$C_e = 1 - \frac{r}{a}$$

$$S_e = \dot{r}/a\dot{E} = r \dot{r}/(\mu a)^{1/2}$$

$$\dot{\underline{r}} = (\underline{r} \cdot \dot{\underline{r}})/r$$

$$\dot{E} = (1/r)(\mu/a)^{1/2}$$

$$e^2 = S_e^2 + C_e^2$$

$$\underline{U} = \{ \underline{r}/r \}$$

$$\underline{W} = \{ (\underline{r} \times \dot{\underline{r}}) / | \dot{\underline{r}} \times \dot{\underline{r}} | \}$$

$$\underline{V} = \underline{W} \times \underline{U}$$

$$\sin i = + (U_z^2 + V_z^2)^{1/2}$$

$$\cos i = + [(U_x + V_y)^2 + (U_y - V_x)^2]^{1/2} - 1$$

$$\sin u = U_z / \sin i$$

$$\cos u = V_z / \sin i$$

$$\sin l = (U_y - V_x) / (1 + \cos i)$$

$$\cos l = (U_x - V_y) / (1 + \cos i)$$

$$\Omega = l - u$$

$$C_v = p/r - 1$$

$$S_v = \dot{r}(p/\mu)^{1/2}$$

$$p = a(1 - e^2)$$

$$\cos v = C_v / e$$

$$\sin v = S_v / e$$

$$\omega = u - v$$

$$M = E - e \sin E$$

2. FROM ECI TO THE ORBIT-PLANE SYSTEM

The orbit-plane system is defined, given $\underline{r}$, $\dot{\underline{r}}$, ECI vectors, by the unit vectors

$$\underline{\xi} = \underline{r}/r$$

$$\underline{\zeta} = (\underline{r} \times \dot{\underline{r}})/|\underline{r} \times \dot{\underline{r}}|$$

$$\underline{\eta} = \underline{\zeta} \times \underline{\xi}$$

Any ECI vector can be expressed in the orbit-plane system (RTC) by the following dot products (where $\underline{q}$ and $\dot{\underline{q}}$ are arbitrary ECI position and velocity vectors):

$$R = \underline{q} \cdot \underline{\xi}$$

$$T = \underline{q} \cdot \underline{\eta}$$

$$C = \underline{q} \cdot \underline{\zeta}$$

$$R' = \dot{\underline{q}} \cdot \underline{\xi}$$

$$T' = \dot{\underline{q}} \cdot \underline{\eta}$$

$$C' = \dot{\underline{q}} \cdot \underline{\zeta}$$

D. TRAJECTORY GENERATION EQUATIONS

Several sets of trajectory generation equations are used in TAP to compute the vehicle state vector $S(\underline{r}, \dot{\underline{r}}, \ddot{\underline{r}})$ at some base time t . The mathematical formulations of these sets are now described.

1. COWELL FORMULATION

The direct numerical integration of the following vector equation of motion is referred to as the Cowell method:

$$\ddot{\underline{r}} = -\mu \frac{\underline{r}}{r^3} + \sum \ddot{\underline{r}}_i \quad (i = 1, 2)$$

where

$$\underline{r} = [x, y, z]$$

$$r = |\underline{r}|$$

$$\Sigma \ddot{\underline{r}}_i = \ddot{\underline{r}}_1 + \ddot{\underline{r}}_2$$

$$\ddot{\underline{r}}_1 \equiv \text{noncentral gravitation}$$

$$\ddot{\underline{r}}_2 \equiv \text{atmospheric drag}$$

2. ENCKE FORMULATION

The numerical integration of the following deviation-vector equation, which relates the departure of the actual from some nominal position, is referred to as the Encke method:

$$\begin{aligned} \ddot{\underline{\rho}} &= \ddot{\underline{r}} - \ddot{\underline{r}}_n \\ &= -\mu [\underline{r}/r^3 - \underline{r}_n/r_n^3] - \underline{L} + \sum \ddot{\underline{r}}_i \quad (i = 1, 2) \end{aligned}$$

where

$$\underline{\rho} = \underline{r} - \underline{r}_n$$

$$\underline{r}_n = [x_n, y_n, z_n]$$

$\underline{L} \equiv$ additive secular acceleration vector that accounts for the consideration of a non-two-body reference orbit.

The nominal trajectory $\underline{r}_n$ is the solution of

$$\ddot{\underline{r}}_n = -\mu \underline{r}_n / r_n^3 + \underline{L}$$

If $\underline{L} = \underline{0}$, then the classical Encke solution results.

The μ -term in the above equation can now be manipulated into the O. K. Smith computational form (Ref. 2)

$$-\mu \left[\frac{\underline{r}}{r^3} - \frac{\underline{r}_n}{r_n^3} \right] = \frac{\mu}{r_n^3} \left\{ \frac{\underline{r}}{r^2} \left[1 + \frac{\delta^2}{1+\delta} \right] \left[(\underline{r}_n + \underline{r}) \cdot \underline{\rho} \right] - \underline{\rho} \right\}$$

where $\delta = r_n/r$. Note that only time is expressed as the independent variable in this deviation-vector and that no prespecified rectification criterion or procedure is considered.

3. ANALYTIC ALGORITHMS

Analytic algorithms give rise to closed form solutions for the nominal equation of motion $\ddot{\underline{r}}_n = -\mu \underline{r}_n / r_n^3 + \underline{L}$. The three analytic methods of Pines, Kyner, and Escobal are of interest here.

2. Pines Method

Closed-form expressions for the F & G functions are utilized in the Pines method (Ref. 3) to evaluate the Keplerian state vector $S(\underline{r}_n, \dot{\underline{r}}_n, \ddot{\underline{r}}_n)$ at some specified time t , given the initial state vector $S_0(\underline{r}_0, \dot{\underline{r}}_0)$ at initial time (epoch) t_0 . The method can be summarized by the set of equations

$$\underline{r}_n = f(t, \underline{r}_0, \dot{\underline{r}}_0) \underline{r}_0 + g(t, \underline{r}_0, \dot{\underline{r}}_0) \dot{\underline{r}}_0$$

$$\dot{\underline{r}}_n = \dot{f}(t, \underline{r}_0, \dot{\underline{r}}_0) \underline{r}_0 + \dot{g}(t, \underline{r}_0, \dot{\underline{r}}_0) \dot{\underline{r}}_0$$

$$\ddot{\underline{r}}_n = -\mu \underline{r}_n / r_n^3$$

with scalar functions

$$f = (a/r_0)(\cos\theta - 1) + 1$$

$$g = t - [(\theta - \sin\theta)/n] = [\sin\theta - (\theta - M)]/n$$

$$\dot{f} = -(a^2 n / r r_0) \sin \theta$$

$$\dot{g} = (a/r)(\cos \theta - 1) + 1$$

and Kepler's equation relating the time t to the angular variable given by

$$M = nt = \theta - \sin \theta (1 - r_0/a) - (\cos \theta - 1) \underline{r}_0 \cdot \dot{\underline{r}}_0 / a^2 n$$

where

$$a^{-1} = (2/r_0) - (V_0^2/\mu)$$

$$n = \mu^{1/2} a^{-3/2}$$

$$r_0 = (\underline{r}_0 \cdot \underline{r}_0)^{1/2}$$

$$V_0 = (\dot{\underline{r}}_0 \cdot \dot{\underline{r}}_0)^{1/2}$$

b. Kyner Method

The Kyner method (Ref. 4) evaluates a non-Keplerian state vector $S(\underline{r}_n, \dot{\underline{r}}_n, \ddot{\underline{r}}_n)$, which includes the first-order secular effects of the earth's oblateness. This technique utilizes the three rate parameters (η, τ, γ), in the formulation of nominal $\underline{r}_n$, where

η is the rotation rate of the major axis of the nominal ellipse

τ is the rotation rate of the line of nodes of the nominal plane

γ is defined in terms of the nominal anomalistic mean motion

$\bar{n} = n_0 (1 - \gamma)$, where n_0 is the mean motion at epoch.

(1) Kyner Formulation I (KYNER 1)

The first formulation (Ref. 4) of the nominal $\underline{r}_n$ is given by the vector equation

$$\underline{r}_n = [R(\Omega, i, u)] \begin{bmatrix} r_n \\ 0 \\ 0 \end{bmatrix}$$

that satisfies the differential equation $\ddot{\underline{r}}_n = -\mu \underline{r}_n / r_n^3 + \underline{L}$ where

$$\Omega = \tau(f - f_0) + \Omega_0 \quad (\text{longitude of the ascending node})$$

$$i = i_0 \quad (\text{inclination})$$

$$u = f + \omega \quad (\text{argument of latitude})$$

$$\omega = \eta(f - f_0) + \omega_0 \quad (\text{argument of perigee})$$

$$r_n = a_0 (1 - e_0^2) / (1 + e_0 \cos f) \quad (\text{length of } \underline{r}_n)$$

where

$$a = a_0 \quad (\text{semi-major axis})$$

$$e = e_0 \quad (\text{eccentricity})$$

and the nominal true anomaly f is given as a function of time by the modified Kepler equation

$$n_0 (1 - \gamma) t + M_0 = E - e_0 \sin E.$$

Note that

$$\tan(f/2) = \left[(1 + e_o)/(1 - e_o) \right]^{1/2} \tan(E/2)$$

M_o = initial value of the mean anomaly

$$n_o^2 a_o^3 = \mu$$

$$\text{rotation matrix } [R] = [D(\Omega)][C(i)][B(u)]$$

Thus the nominal trajectory $\underline{r}_n(t)$ depends on nine parameters, which are

The six values $a_o, e_o, i_o, \Omega_o, \omega_o, M_o$

The three arbitrary rate parameters η, τ, γ

If the rate parameters are set equal to zero (that is, if $\underline{L} = \underline{0}$), then the resultant nominal $\underline{r}_n$ is Keplerian.

(2) Kyner Formulation II (KYNER 2)

The second formulation [Ref. 5] of the nominal $\underline{r}_n$ is given by the vector equation

$$\underline{r}_n = F \underline{M} + G \underline{N}$$

where

F, G = Keplerian scalar functions

$\underline{M}, \underline{N}$ = Rotating vectors that are always in nominal orbital plane and maintain constant angles with the precessing line of apses of the nominal ellipse.

In detail, these scalar functions are described as

$$F = 1/r_o(r_n \cos \varphi) - (\dot{r}/h)_o r_n \sin \varphi$$

$$G = (r/h)_o r_n \sin \varphi$$

where

$$\varphi = f - f_o$$

$$h_o = \left| \underline{r}_o \times \dot{\underline{r}}_o \right|$$

and the rotating vectors as

$$\underline{M} = \underline{r}_o + [A] \underline{r}_o$$

$$\underline{N} = \dot{\underline{r}}_o + [A] \dot{\underline{r}}_o$$

The matrix $[A]$ can be represented as a sum of constant matrices $[A_j]$ (evaluated at epoch) whose scalar coefficients are slowly varying functions of φ and the rate parameters η, τ, γ . That is

$$[A] = [A(\varphi, \eta, \tau, \gamma, \Omega_o, i_o, u_o)]$$

$$[A] = \sum_{j=1}^8 a_j(\varphi) [A_j]$$

In summation, the state vector $S_t(\underline{r}_n, \dot{\underline{r}}_n, \ddot{\underline{r}}_n)$ is expressed as

$$\underline{r}_n = F[I+[A]]\underline{r}_0 + G[I+[A]]\dot{\underline{r}}_0$$

$$\dot{\underline{r}}_n = [\dot{F}[I+[A]] + F[\dot{A}]]\underline{r}_0 + [\dot{G}[I+[A]] + G[\dot{A}]]\dot{\underline{r}}_0$$

$$\ddot{\underline{r}}_n = -\mu \underline{r}_n / r_n^3 + \underline{L}$$

where

$$\underline{L} = (1 - (1-\gamma)^2) \mu \underline{r}_n / r_n^3 + (F;[A])\underline{r}_0 + (G;[A])\dot{\underline{r}}_0$$

$$(p;q) = 2 \left(\frac{dp}{dt} \right) \left(\frac{dq}{dt} \right) + p \left(\frac{d^2 q}{dt^2} \right) \text{ (differential operator)}$$

where p and q are the arguments of the operator.

c. Escobal Method

As in the Kyner method, the Escobal method (Ref. 6) evaluates a non-Keplerian state vector $S(\underline{r}_n, \dot{\underline{r}}_n, \ddot{\underline{r}}_n)$, which includes the first-order secular effects of the earth's oblateness. This technique utilizes the secular rates of change in the three elements

$$\dot{\Omega} = -\frac{3}{2} \frac{J_2}{P_0} \frac{\bar{n}}{k} \cos i_0$$

$$\dot{\omega} = +\frac{3}{2} \frac{J_2}{P_0} \left(2 - \frac{5}{2} \sin^2 i_0 \right) \frac{\bar{n}}{k}$$

$$\bar{n} = n_0 (1 - \gamma)$$

where

$$\gamma = -\frac{3}{2} J_2 \left(\frac{a_e}{a_o} \right)^2 \left(\frac{a}{r} \right)_o^3 (1 - 3 \sin^2 i_o \sin^2 u_o) \quad (1)$$

J_2 = second harmonic coefficient

p_o = semi-parameter of the orbit at epoch

i_o = inclination of the orbit epoch

k = Gaussian or planetary constant

$n_o = k[\mu]^{1/2} a_o^{-3/2}$ (mean motion at epoch)

μ = sum of celestial-object mass and central mass

a_e = semi-diameter of the central mass

a_o = semi-major axis of the orbit at epoch

r_o = radius of object at epoch

$u_o = f_o + \omega_o$ (argument of latitude at epoch)

ω_o = argument of pericenter at epoch

f_o = true anomaly at epoch

Note that the formula for $\bar{n}$ is not the one given in Reference 6. To insure that the rate parameter γ for mean motion is consistent with the first-order secular effects for Ω and ω at perigee, the secular effect for the anomalous mean motion should be used; that is, the approximation of the Cunningham integral given in Eq. (1) above.

The state vector $S(\underline{r}_n, \dot{\underline{r}}_n, \ddot{\underline{r}}_n)$ can be expressed as

$$\underline{r}_n = x_\omega \underline{P} + y_\omega \underline{Q}$$

$$\dot{\underline{r}}_n = \left(\frac{d}{d\tau} \right) \left(\frac{d\tau}{dt} \right) \{ \underline{r}_n \} \quad \tau = k(t - t_0)$$

$$= k [\dot{x}_\omega \underline{P} + \dot{y}_\omega \underline{Q} + x_\omega \dot{\underline{P}} + y_\omega \dot{\underline{Q}}]$$

$$\ddot{\underline{r}}_n = \left(\frac{d^2}{d\tau^2} \right) \left(\frac{d\tau}{dt} \right)^2 \{ \underline{r}_n \}$$

$$= k^2 [x_\omega \ddot{\underline{P}} + 2 \dot{x}_\omega \dot{\underline{P}} + \ddot{x}_\omega \underline{P} + y_\omega \ddot{\underline{Q}} + 2 \dot{y}_\omega \dot{\underline{Q}} + \ddot{y}_\omega \underline{Q}]$$

where the unit vectors

$$\underline{P} = \underline{P}(\omega, \Omega, i)$$

$$\underline{Q} = \underline{Q}(\omega, \Omega, i)$$

$$\underline{W} = \underline{W}(\omega, \Omega, i)$$

with

$$\begin{bmatrix} \underline{P} \\ \underline{Q} \\ \underline{W} \end{bmatrix} = \begin{bmatrix} \cos \omega & \sin \omega & 0 \\ -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \times$$

$$\begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ -\sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{I} \\ \underline{J} \\ \underline{K} \end{bmatrix}$$

$$\underline{I} = [1, 0, 0]$$

$$\underline{J} = [0, 1, 0]$$

$$\underline{K} = [0, 0, 1]$$

$$\Omega = \Omega_0 + \dot{\Omega} \tau$$

$$\omega = \omega_0 + \dot{\omega} \tau$$

$$n = \bar{n}$$

and

$$x_\omega = a_0 (\cos E - e_0)$$

$$y_\omega = a_0 [1 - e_0^2]^{1/2} \sin E$$

$$\bar{n} t + M_0 = E - e_0 \sin E \quad (\text{Kepler's equation}).$$

E. MISCELLANEOUS EQUATIONS

Following are equations for specific trajectory quantities.

Angular Momentum, h

$$h = \left| \underline{r} \times \dot{\underline{r}} \right|$$

Root-Sum-Square, rss

$$rss = (R^2 + T^2 + C^2)^{1/2}$$

where

$$R = \Delta \underline{r} \cdot \underline{\xi}$$

$$T = \Delta \underline{r} \cdot \underline{\eta}$$

$$C = \Delta \underline{r} \cdot \underline{\zeta}$$

$$\Delta \underline{r} = \underline{r} - \underline{r}_c$$

Keplerian Period, P_k

$$P_k = \frac{2\pi a^{3/2}}{\mu^{1/2}}$$

Kinetic Energy, K. E.

$$\begin{aligned} K. E. &= V^2/2 \\ &= \dot{\underline{r}} \cdot \dot{\underline{r}}/2 \end{aligned}$$

Potential Energy, P.E. (J_2 Only)

$$P.E. = -(\mu/r) \left[1 + (a_e^2 J_2 / 2r^2) (1 - 3 \sin^2 \delta) \right]$$

where

$$\sin^2 \delta = z^2 / r^2$$

Total Energy, T.E.

$$T.E. = K.E. + P.E.$$

Gravitational Acceleration Due to the Noncentral Force Field
of the Earth, $\ddot{\underline{r}}_1$ (J_2 Effect Only)

$$\ddot{\underline{r}}_1 = \frac{a_e^2}{r^3} J_2 \mu \left[\frac{3}{2} \frac{\underline{r}}{r^2} \left(-1 + \frac{3z^2}{r^2} \right) - \underline{r}^* \right]$$

where

$$a_e = 1 \text{ earth radius}$$

$$\underline{r}^* = -\frac{3}{r^4} \begin{bmatrix} xz^2 \\ yz^2 \\ -r^2 z \left(1 - \frac{z^2}{r^2} \right) \end{bmatrix}$$

Noncentral Earth Gravitational Force, $\ddot{\underline{r}}_1$

The gravitational acceleration $\ddot{\underline{r}}_1$ due to the noncentral force field of the earth is derived from the generalized potential function

$$U = (\mu/r) \sum_{n=2}^{\infty} (a_e/r)^n \sum_{m=0}^n P_n^m(\sin \varphi) (C_{nm} \cos m \lambda + S_{nm} \sin m \lambda)$$

where

r, φ, λ are the geocentric distance, latitude, and East longitude of the vehicle

a_e is the mean equatorial radius of the earth

P_n^m is the Legendre associated function of the first kind of degree n and order m

C_{nm}, S_{nm} are numerical coefficients.

Components of the acceleration vector $\ddot{\underline{r}}_1$ are expressed as

$$\ddot{\underline{r}}_1 = \begin{bmatrix} \ddot{r}_{1x} \\ \ddot{r}_{1y} \\ \ddot{r}_{1z} \end{bmatrix} = T_{\varphi, \lambda} \begin{bmatrix} G_{Up} \\ G_{East} \\ G_{North} \end{bmatrix} = T_{\varphi, \lambda} \underline{G}_{LH}$$

where

$T_{\varphi, \lambda}$ is the earth-fixed local horizontal system transformation matrix

$\underline{G}_{LH}$ is the gravitational acceleration due to the noncentral force field of the earth in the local horizontal system, in which the coordinate axes are directed Up (along the position vector), East, and North.

Note that

$$\underline{G}_{LH} = \begin{bmatrix} G_{Up} \\ G_{East} \\ G_{North} \end{bmatrix}$$

so that

$$G_{Up} = \partial U / \partial r = \sum_{n=2}^{\infty} K_n^1(\varphi, \lambda)$$

$$G_{East} = (1/r \cos \varphi)(\partial U / \partial \lambda) = \sum_{n=2}^{\infty} K_n^2(\varphi, \lambda)$$

and

$$G_{North} = (1/r)(\partial U / \partial \varphi) = \sum_{n=2}^{\infty} K_n^3(\varphi, \lambda)$$

where

$$K_n^1(\varphi, \lambda) = -(\mu/r^2)(n+1)(a_e/r)^n \sum_{m=0}^n P_n^m(\sin \varphi) [C_{nm} \cos m \lambda + S_{nm} \sin m \lambda]$$

$$K_n^2(\varphi, \lambda) = (\mu/r^2 \cos \varphi)(a_e/r)^n \sum_{m=0}^n m P_n^m(\sin \varphi) [C_{nm} \sin m \lambda - S_{nm} \cos m \lambda]$$

$$K_n^3(\varphi, \lambda) = (\mu/r^2)(a_e/r)^n \sum_{m=0}^n P_n^{m'}(\sin \varphi) \cos \varphi [C_{nm} \cos m \lambda + S_{nm} \sin m \lambda]$$

($P_n^{m'}$ is the derivative of the Legendre function with respect to $\sin \varphi$.)

Two different optional normalizations of the Legendre functions are used. The alternate values for C_{nm} and S_{nm} coefficients (Ref. 7) are denoted by

$$\bar{C}_{nm}, \bar{S}_{nm} \text{ (APL normalization)}$$

and

$$\bar{\bar{C}}_{nm}, \bar{\bar{S}}_{nm} \text{ (Kaula normalization)}$$

where

$$(\bar{C}_{nm}, \bar{S}_{nm}) = [(n+m)!/(n-m)!]^{1/2} (C_{nm}, S_{nm})$$

and

$$(\bar{\bar{C}}_{nm}, \bar{\bar{S}}_{nm}) = [(2n-1)(2-\delta_{0m})]^{-1/2} (\bar{C}_{nm}, \bar{S}_{nm})$$

with δ_{ij} being the Kronecker delta function.

Atmospheric Drag Acceleration, $\ddot{\underline{r}}_2$

$$\ddot{\underline{r}}_2 = -\rho(V_A/2)(C_D A/W) \dot{\underline{r}}_A$$

where

ρ = the density at height h above an oblate earth (the only atmosphere model considered is the Lockheed-Jacchia)

$C_D A/W$ = the drag coefficient

$\dot{\underline{r}}_A$ = vehicle velocity vector relative to a rotating atmosphere

$$\dot{\underline{r}}_A = [\dot{x}_A, \dot{y}_A, \dot{z}_A]$$

$$V_A = |\dot{\underline{r}}_A|$$

where

$$\dot{x}_A = \dot{x} + \omega_a y$$

$$\dot{y}_A = \dot{y} - \omega_a x$$

$$\dot{z}_A = \dot{z}$$

ω_a = rotation rate of the atmosphere.

Apsis Event Detection (between times t_i and t_{i-1})

If $\cos \beta_i \cdot \cos \beta_{i-1} > 0$, there is no event. Otherwise, an event has occurred and

$$\partial \cos \beta_i / \partial t = \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i + \underline{r}_i \cdot \ddot{\underline{r}}_i \text{ for all } i.$$

Since $\cos \beta$ is a smooth monotonic function of t and $\cos \beta$ is near zero, then

$$\partial t / \partial \cos \beta_i = (\partial \cos \beta_i / \partial t)^{-1}$$

and allows interpolation for the time t for which $\cos \beta = 0$, given

$$\begin{aligned} &\cos \beta_i, \cos \beta_{i-1}, t_i, t_{i-1}, \\ &\partial t_i / \partial \cos \beta_i, \partial t_{i-1} / \partial \cos \beta_{i-1} \end{aligned}$$

After $t_{\cos \beta=0}$ is obtained, interpolation for the state vector yields

$$S(\underline{r}, \dot{\underline{r}}; t_{\cos \beta=0})$$

given $t_i, t_{i-1}, \underline{r}_i, \underline{r}_{i-1}, \dot{\underline{r}}_i, \dot{\underline{r}}_{i-1}, \ddot{\underline{r}}_i, \ddot{\underline{r}}_{i-1}$.

Node Event Detection

If $z_i \cdot z_{i-1} \geq 0$, there is no event; otherwise, an event has occurred.

Then assume that

$$\partial t_i / \partial z_i = (\partial z_i / \partial t_i)^{-1}$$

and interpolate for the time at which $z = 0$. After $t_{z=0}$ is obtained, interpolation for the state vector yields

$$S(\underline{r}, \underline{\dot{r}}; t_{z=0}).$$

SECTION IV

COWELL AND ENCKE IMPLEMENTATION

Table I indicates the sequence of steps necessary to implement Cowell and Encke formulations.

Table I. Schematic for Implementing Cowell and Encke Formulations

STEP	COWELL	ENCKE
1	Given $S_0(\underline{r}_0, \dot{\underline{r}}_0)$	Given $S_0(\underline{r}_0, \dot{\underline{r}}_0)$ (See Note 2)
2	Initialize $\underline{r} = \underline{r}_0$ $\dot{\underline{r}} = \dot{\underline{r}}_0$ Evaluate $\ddot{\underline{r}}$	Initialize $\underline{r} = \underline{r}_0$ $\dot{\underline{r}} = \dot{\underline{r}}_0$ Evaluate $\underline{r}_n, \dot{\underline{r}}_n, \ddot{\underline{r}}_n$ $\underline{p} = \underline{r} - \underline{r}_n$ $\dot{\underline{p}} = \dot{\underline{r}} - \dot{\underline{r}}_n$ $\ddot{\underline{p}} = \ddot{\underline{r}} - \ddot{\underline{r}}_n$
3	Evaluate $\ddot{\underline{r}}$ Integrate $\underline{r} \leftarrow \int \dot{\underline{r}} + \int \ddot{\underline{r}}$	Evaluate $\underline{r}_n, \dot{\underline{r}}_n, \ddot{\underline{r}}_n$ Integrate $\underline{p} \leftarrow \int \dot{\underline{p}} + \int \ddot{\underline{p}}$ Compute $\underline{r} = \underline{r}_n + \underline{p}$ $\dot{\underline{r}} = \dot{\underline{r}}_n + \dot{\underline{p}}$

Note 1: Subscript n denotes nominal values obtained by some analytic method.

Note 2: If the user chooses to specify rectification criteria ϵ and $\dot{\epsilon}$, then investigate the rectification criteria, $|\underline{p}| \geq \epsilon$, or $|\dot{\underline{p}}| \geq \dot{\epsilon}$. If rectification necessary, set $S_0(\underline{r}_0, \dot{\underline{r}}_0) = S_0(\underline{r}, \dot{\underline{r}})$ and proceed with Step 2.

SECTION V

PROGRAM STRUCTURE AND LOGIC FLOW

The general structure and logic flow are illustrated in Figure 4.

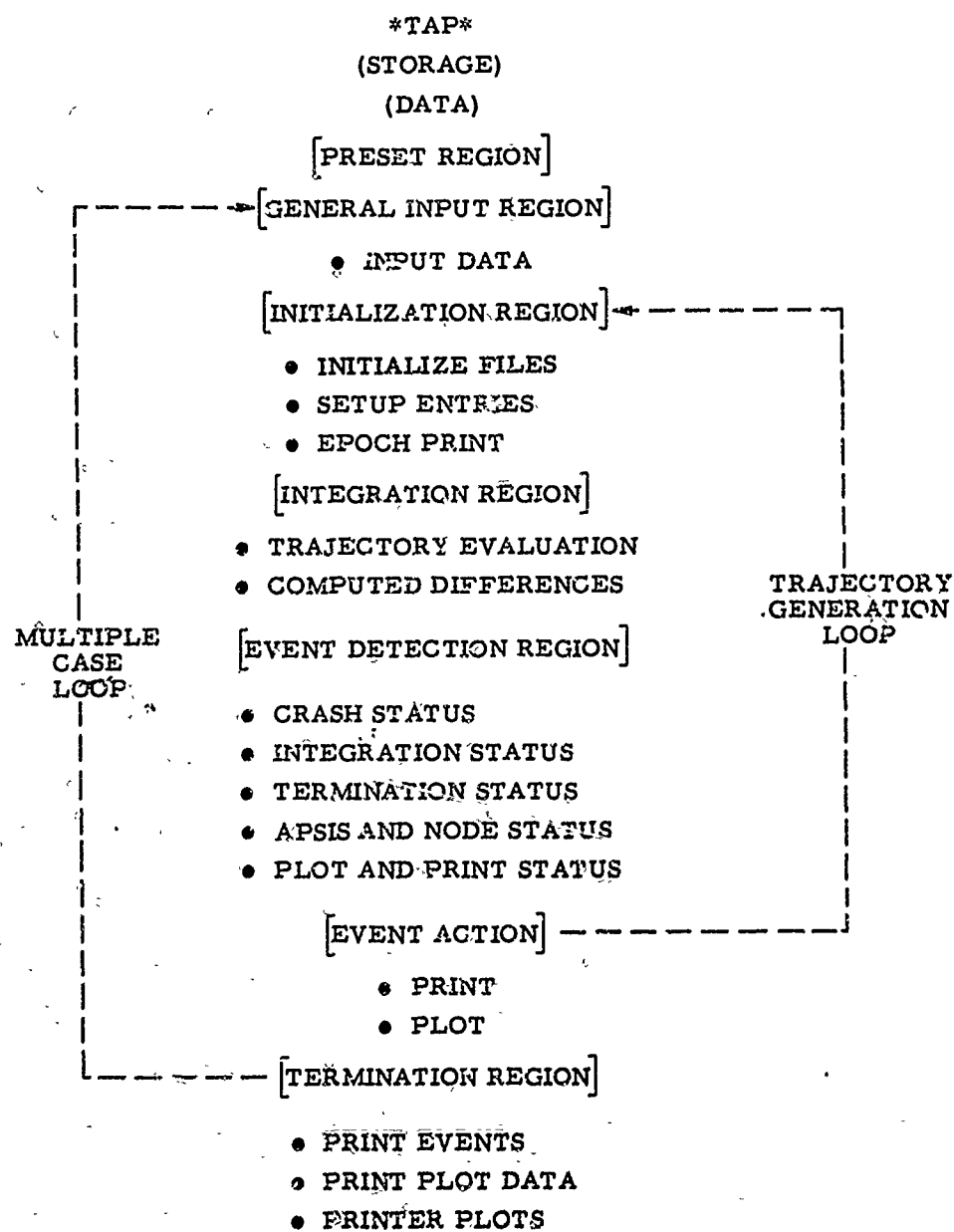


Figure 4. Diagram of TAP General Structure and Logic Flow

SECTION VI

PROGRAM USAGE

The general use of the Trajectory Analysis Program will be shown by describing the specific input data, output tables, and plotting features available to the user.

A. INPUT DATA

All required input is nominally set to values in the basic unit system of feet, degrees, and feet per second. In Table II, each input variable is described and its preset value designated in the value field.

Table II. Input Data Description

Code	Location	Value	Description
I	MSTEP	100	Multi-step print flag
I	MREV	10	Multi-rev print flag
I	IMETH	1	Analytic method flag IMETH = 1 , PINES = 2 , KYNER1 = 3 , KYNER2 = 4 , Escobal
I	IFORM	0	Formulation flag IFORM = 0 , Cowell = 1 , Encke
I	IVAR	1	Independent variable flag IVAR = 1 , Time = 2 , Eccentric anomaly ¹ = 3 , True anomaly ¹
I	IPERT	0	Gravitational perturbation flag IPERT = 0 , No non-central effect = 1 , Non-central (J_2) effect

¹Not available as of 6-1-69.

Table II. Input Data Description (cont.)

Code	Location	Value	Description
I	IPLØT	0	Plot file data generation flag IPLØT = 0 , No plot file = n , Plot file generated every n steps
I	IUNIT	1	Initial units conversion flag IUNIT = 0 , No conversion = 1 , Convert from external units to internal units
I	IPRNT	1	Print flag IPRNT = 0 , Only difference print = 1 , Standard print
	PSTEP	60.	Print step PSTEP ≠ 0 , Trajectory print every PSTEP interval from TZERO
I	IDIFF	0	Integration closure flag IDIFF = 0 , Closure not considered ≠ 0 , Closure error computed
...	...		
	TZERØ	(*)	Reference time (epoch)
	TEND	(*)	Termination time relative to TZERØ = 0.
	X0	(*)	x_0 y_0 z_0 } Initial position vector
	2	(*)	
	3	(*)	
	XDØ	(*)	$\dot{x}_0$ $\dot{y}_0$ $\dot{z}_0$ } Initial velocity vector
	2	(*)	
	3	(*)	

*Required for each trajectory case.

Table II. Input Data Description (cont.)

Code	Location	Value	Description
...	...		
	CRASH	1.	Crash radius
	GM	.55303935E-2	Gravitation constant
	CJ2	1082.3E-6	J_2
	EPS	1.0E-9	Analytic convergence criterion
	DRAD	(57.295779513082)	Degrees per radian constant
	CPI	(3.14159265358979)	π
...	...		
	DF	20925738.	Distance conversion factor
	VF	348762.3	Velocity conversion factor
	AF	5812.705	Acceleration conversion factor
	TF	60.	Time conversion factor
...	...		
	H0	1.	Initial integration step
	HMIN	.015625	Minimum allowed integration step
	HMAX	64.	Maximum allowed integration step
	ER	1.0E-10	Integration truncation control
I	IR	8	Ratio of Runge-Kutta steps to Cowell steps
...	...		
	TDJ0	0.	Julian date of epoch.
	D1	6.83	Coefficients used in the Lockheed-Jacchia atmosphere model for the density expression.
	D2	-15.684	
	FLUX	0.	10.7 cm solar radiation in units of 10^{-20} watt/m ² ; if equal to zero, the FLUX is computed internally.

Table II. Input Data Description (cont.)

Code	Location	Value	Description
	CDAW	(*)	$C_D A/W$: C_D is aerodynamic drag coefficient; A is the average cross-section area of vehicle; W is the weight of vehicle; units are ft^2/lb .

*If option used input required for each trajectory case.

B. OUTPUT DATA

Tables III through IX describe the basic output quantities.

Table III. Initial Condition Data (Internal Units)

x_0	$\left. \begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array} \right\} \underline{r}_0$	$\dot{x}_0$	$\left. \begin{array}{c} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{array} \right\} \underline{\dot{r}}_0$	F_{x_0}	$\left. \begin{array}{c} F_{x_0} \\ F_{y_0} \\ F_{z_0} \end{array} \right\} \underline{F}_0^*$	TZERO
y_0		$\dot{y}_0$		F_{y_0}		TEND
z_0		$\dot{z}_0$		F_{z_0}		GM
DF		VF		AF		TF

* Not available

Table IV. Epoch (TZERO) Data (Internal Units)

T	x	$\left. \begin{array}{c} x \\ y \\ z \end{array} \right\} \underline{r}$	$\dot{x}$	$\left. \begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{z} \end{array} \right\} \underline{\dot{r}}$	$\ddot{x}$	$\left. \begin{array}{c} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{array} \right\} \underline{\ddot{r}}$
NSTEP	y		$\dot{y}$		$\ddot{y}$	
H0	z		$\dot{z}$		$\ddot{z}$	
H (current integration step)	x_C	$\left. \begin{array}{c} x_C \\ y_C \\ z_C \end{array} \right\} \underline{r}_C$	$\dot{x}_C$	$\left. \begin{array}{c} \dot{x}_C \\ \dot{y}_C \\ \dot{z}_C \end{array} \right\} \underline{\dot{r}}_C$	$\ddot{x}_C$	$\left. \begin{array}{c} \ddot{x}_C \\ \ddot{y}_C \\ \ddot{z}_C \end{array} \right\} \underline{\ddot{r}}_C$
N (# of equations)	y_C		$\dot{y}_C$		$\ddot{y}_C$	
	z_C		$\dot{z}_C$		$\ddot{z}_C$	
h_x	$\underline{h} = \underline{r} \wedge \underline{\dot{r}}$	Δx	$\Delta \underline{r} = \underline{r} - \underline{r}_C$	$\Delta \dot{x}$	$\Delta \ddot{x}$	$\Delta \ddot{r}$
h_y		Δy		$\Delta \dot{y}$		
h_z		Δz		$\Delta \dot{z}$		
h						
	$(\underline{h})$					

Table V. Trajectory Data (External Units)

T	NSTEP	H	RSS $\left(\sqrt{\Delta R^2 + \Delta T^2 + \Delta C^2}\right)$
$\left. \begin{matrix} x \\ y \\ z \end{matrix} \right\} \underline{r}$	$\begin{matrix} a \\ e \\ i \end{matrix}$	$\left. \begin{matrix} x_C \\ y_C \\ z_C \end{matrix} \right\} \underline{r}_C$	$\begin{matrix} a_C \\ e_C \\ i_C \end{matrix}$
$\left. \begin{matrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{matrix} \right\} \underline{\dot{r}}$	$\begin{matrix} \Omega \\ \omega \\ M \end{matrix}$	$\left. \begin{matrix} \dot{x}_C \\ \dot{y}_C \\ \dot{z}_C \end{matrix} \right\} \underline{\dot{r}}_C$	$\begin{matrix} \Omega_C \\ \omega_C \\ M_C \end{matrix}$
$\left. \begin{matrix} h_x \\ h_y \\ h_z \end{matrix} \right\} \underline{h}$	$\left. \begin{matrix} R \\ T \\ C \end{matrix} \right\} \begin{matrix} \text{Orbit Plane} \\ \text{Position} \\ \text{Vector} \end{matrix}$	$\left. \begin{matrix} h_x \\ h_y \\ h_z \end{matrix} \right\} \underline{h}_C$	$\begin{matrix} R_C \\ T_C \\ C_C \end{matrix}$
$h \ (\underline{h})$	R'	$h \ (\underline{h}_C)$	R'_C
$R \ (\underline{r})$	T'	$R \ (\underline{r}_C)$	T'_C
$V \ (\underline{\dot{r}})$	C'	$V \ (\underline{\dot{r}}_C)$	C'_C
$\left. \begin{matrix} \Delta x \\ \Delta y \\ \Delta z \end{matrix} \right\} \underline{\Delta R}$	$\begin{matrix} \Delta a \\ \Delta e \\ \Delta i \end{matrix}$	$\begin{matrix} \Delta R \\ \Delta T \\ \Delta C \end{matrix}$	$\left. \begin{matrix} p_x \\ p_y \\ p_z \end{matrix} \right\} \left. \begin{matrix} p = \underline{r} - \underline{r}_C \\ \underline{\dot{p}} = \underline{\dot{r}} - \underline{\dot{r}}_C \end{matrix} \right\} \begin{matrix} \text{(internal} \\ \text{units)} \end{matrix}$
$\left. \begin{matrix} \Delta \dot{x} \\ \Delta \dot{y} \\ \Delta \dot{z} \end{matrix} \right\} \underline{\Delta \dot{r}}$	$\begin{matrix} \Delta \Omega \\ \Delta \omega \\ \Delta M \end{matrix}$	$\begin{matrix} \Delta R' \\ \Delta T' \\ \Delta C' \end{matrix}$	

Table VI. Reference Trajectory
Data Matrix [E] (External Units)

[E]	1	2	3	4	5
1	x	T.E.	a	h_x	R
2	y	P.E.	e	h_y	T
3	z	K.E.	i	h_z	C
4	$\dot{x}$	P _K	Ω	h	R'
5	$\dot{y}$	R	ω	RSSMAX	T'
6	$\dot{z}$	V	M	TMAX	C'

Table VII. Comparison Trajectory Data
Matrix [A] (External Units)

[A]	1	2	3	4	5
1	x_C		a	h_{Cx}	R_C
2	y_C		e	h_{Cy}	T_C
3	z_C		i	h_{Cz}	C_C
4	$\dot{x}_C$		Ω	h_C	R'_C
5	$\dot{y}_C$	R_C	ω	0.	T'_C
6	$\dot{z}_C$	V_C	M	0.	C'_C

Table VIII. Difference Trajectory Data
Matrix [D] (External Units)

[D]	1	2	3	4	5
1	Δx	$\Delta T.E.$	Δa	Δh_x	ΔR
2	Δy	$\Delta P.E.$	Δe	Δh_y	ΔT
3	Δz	$\Delta K.E.$	Δi	Δh_z	ΔC
4	$\Delta \dot{x}$	$\Delta \dot{P}_K$	$\Delta \Omega$	Δh	$\Delta R'$
5	$\Delta \dot{y}$	ΔR	$\Delta \omega$	RSSMAX	$\Delta T'$
6	$\Delta \dot{z}$	ΔV	ΔM	TMAX	$\Delta C'$

Table IX. WKCNC2 Package Output (Internal Units)

INITIALIZATION OUTPUT KEY						
a_o	Semimajor diameter	u_o	Arg. of latitude	γ	n	Mean motion
e_o	Eccentricity	ℓ_o	True longitude	η	$\bar{n}$	Anomalistic mean motion
i_o	Inclination	C_e	$(e \cos E_o)$	τ	h	$ \dot{r} \times \dot{r} $
Ω_o	Longitude of asc. node	S_e	$(e \sin E_o)$	E_o	$\bar{h}$	$h (1-\gamma)$
ω_o	Arg. of perigee	C_f	$(e \cos f_o)$	M_o	p_o	Semi-parameter
f_o	True anomaly	S_f	$(e \sin f_o)$	μ	r_o	Geocentric radius
INDEPENDENT VARIABLE EVALUATION POINT - OUTPUT KEY						
ARG	Independent variable	r	Geocentric radius	x	Position Vector	
ΔM	$M-M_o$	$\dot{r}$	Time rate of change of r	y		
ΔE	$E-E_o$	F	F coefficient	z		
$\dot{\Delta E}$	Time rate of change of ΔE	$\dot{F}$	Time rate of change of F	$\dot{x}$	Velocity Vector	
φ	$t-f_o$	G	G coefficient	$\dot{y}$		
$\dot{\varphi}$	Time rate of change of φ	$\dot{G}$	Time rate of change of G	$\dot{z}$		

NOTE: Subscript o denotes initial (epoch) value.

C. PLOTTING FEATURES

Printer plots are given for prespecified orbital parameters versus time. The parameters considered are:

RSS	Root-sum-square of difference between Reference and Comparison
ENERGY	Difference in total energy ($TE = KE + PE$) between Reference and Comparison
$\xi(XI)$	Radial component of difference between Reference and Comparison
η	In-track component of difference between Reference and Comparison
$r(R)$	Radius of Reference
$a(R)$	Semimajor axis of Reference
$e(R)$	Eccentricity of Reference
$i(R)$	Inclination of Reference
$\Omega(R)$	Right ascension of the ascending node of Reference
$\omega(R)$	Argument of perigee of Reference
$M(R)$	Mean anomaly of Reference
$TE(R)$	Total energy of Reference
$h_z(R)$	Polar component of angular momentum of Reference
$r(C)$	Radius of Comparison
$a(C)$	Semimajor axis of Comparison
$e(C)$	Eccentricity of Comparison
$i(C)$	Inclination of Comparison
$\Omega(C)$	Right ascension of the ascending node of Comparison
$\omega(C)$	Argument of perigee of Comparison
$M(C)$	Mean anomaly of Comparison
$TE(C)$	Total energy of Comparison
$h_z(C)$	Polar component of angular momentum of Comparison
$KE(C)$	Kinetic energy of Comparison

Examples of a subset of these printer plots are given in Figures 5 through 8.

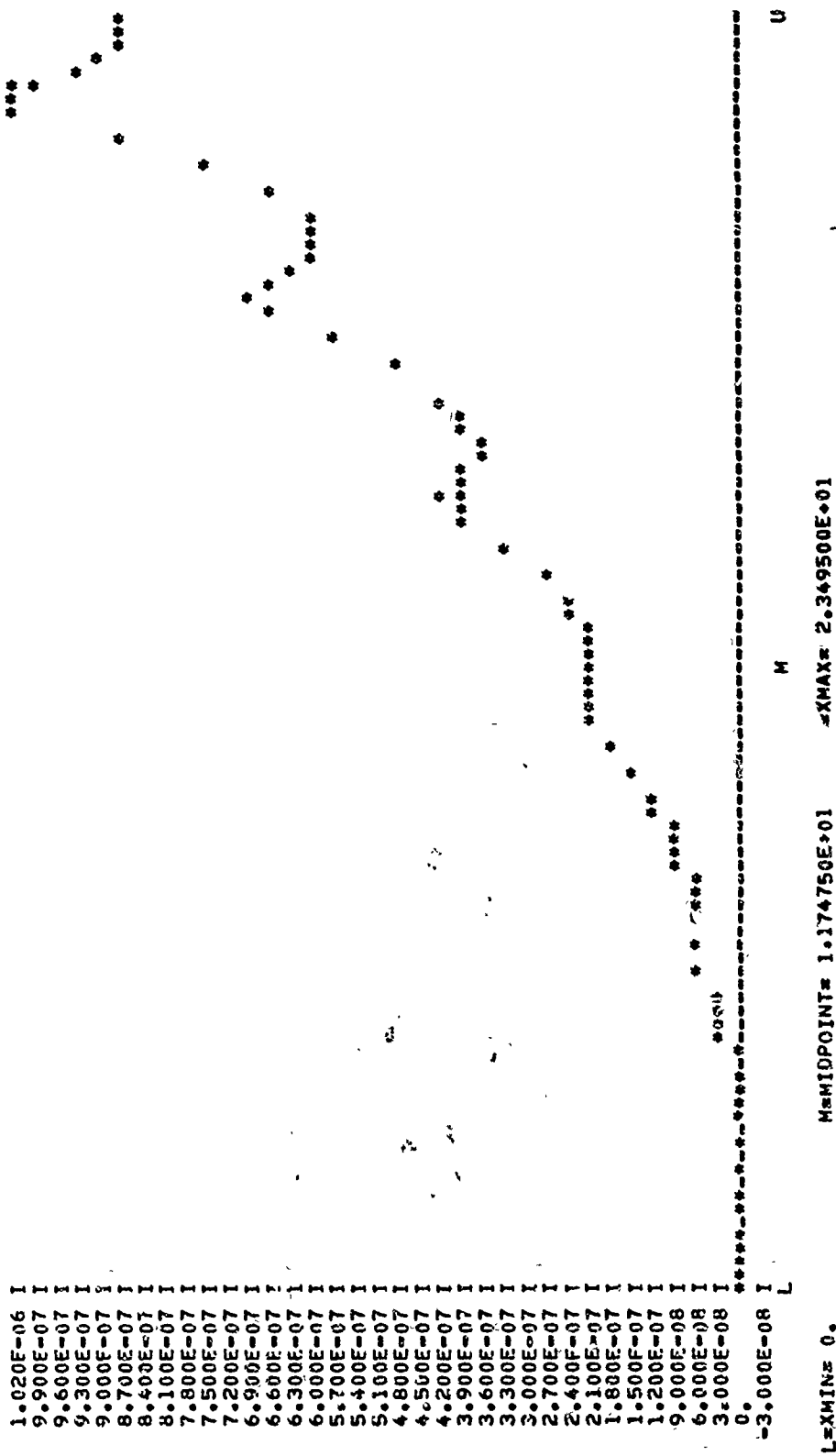


Figure 5. Printer Plot of Variable = (RSS)

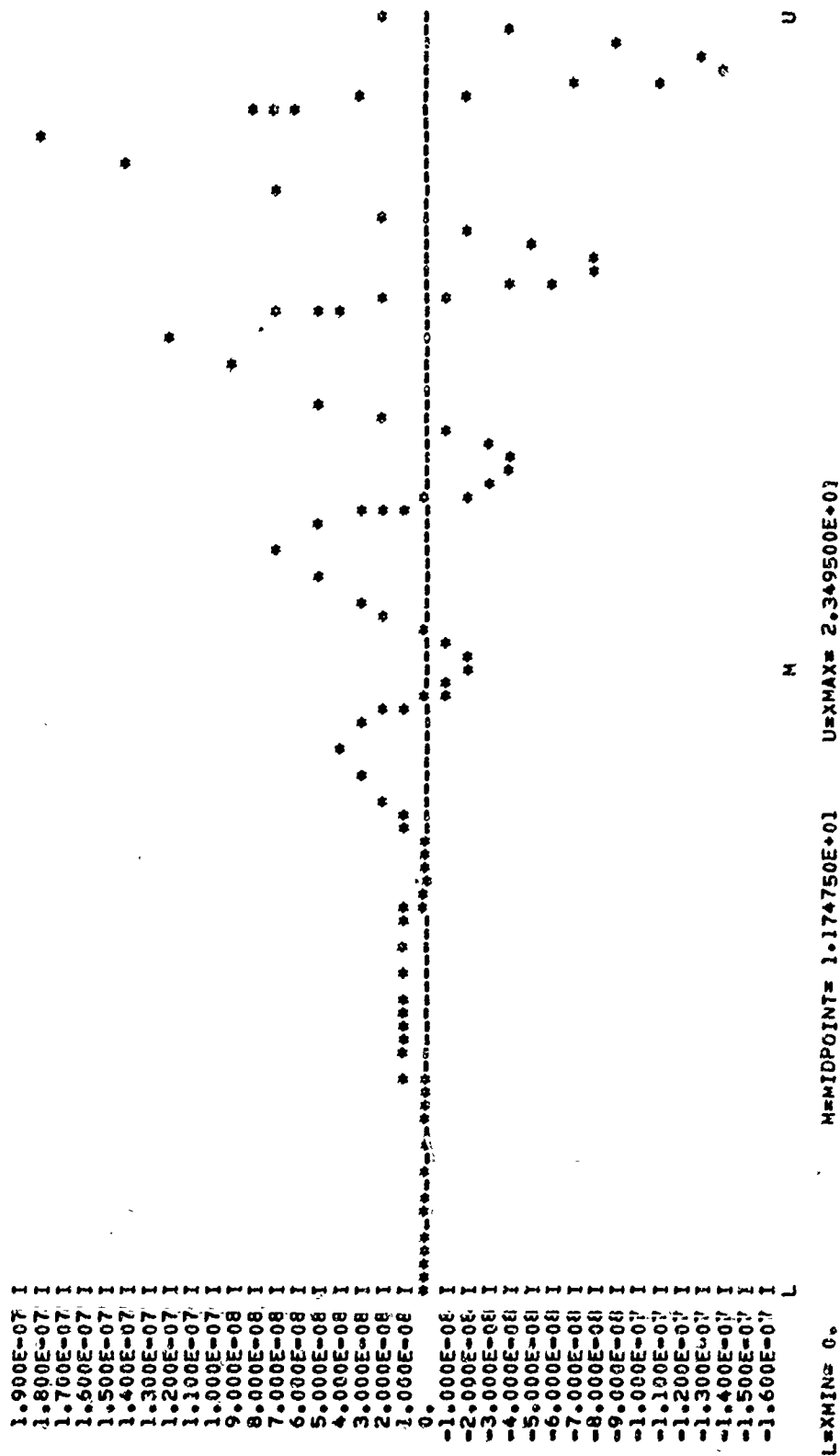


Figure 6. Printer Plot of Variable = (XI)

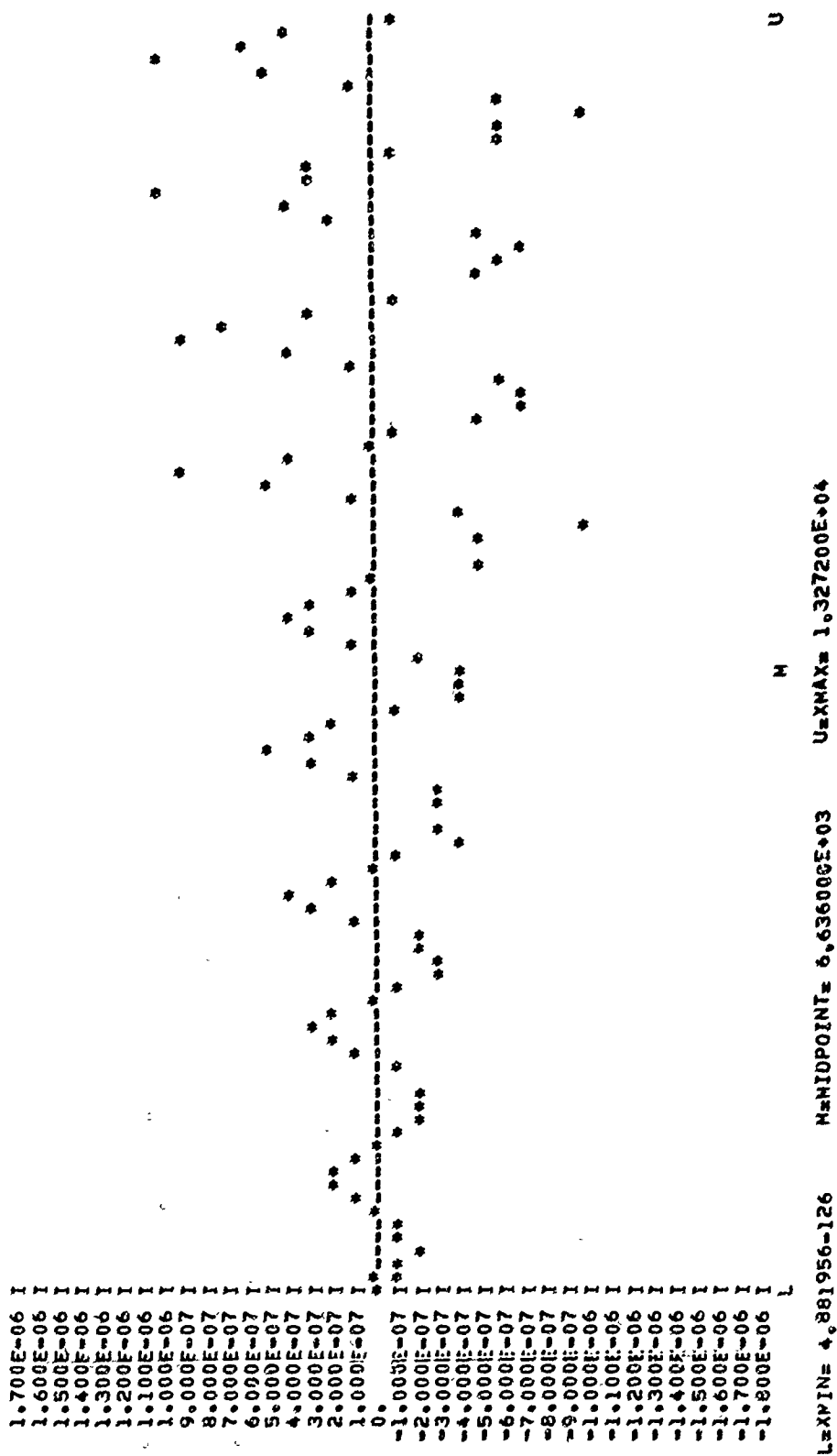


Figure 8. Printer Plot of Variable = M, C

REFERENCES

1. C. C. Tonies, Trace Orbit Determination Program Version D, Report No. TR-669(9990)-3, Aerospace Corp., El Segundo, California, (September 1966).
2. S. I. Conte, "The Computation of Satellite Orbit Trajectories," Advances in Computers, Vol. III, Academic Press Inc., New York, (1962).
3. Samuel Pines, "Variation of Parameters for Elliptic and Near Circular Orbits," Astron. J., 66, (1), No. 1286, (February 1961).
4. W. T. Kyner and M. M. Bennett, Modified Encke Special Perturbation Method, Astron. J., 71, (7), 579-582, (September 1966).
5. W. T. Kyner, Remark on a Modified Encke Perturbation Method, Astron. J., 71, (7) 583-584, (September 1966).
6. P. R. Escobal, Non Two-Body Reference Orbit for Encke's Method, AIAA J. (June 1966).
7. William M. Kaula, Theory of Satellite Geodesy, Blaisdell Publishing Company, Waltham, Mass., (1966).
8. James F. Holt, Floating Point Cowell (Second-Sum) Runge-Kutta Integration of Second-Order Ordinary Differential Equations (Subroutine ASC DEQ4), Report No. TR-0158(3307-02)-9, Aerospace Corp., El Segundo, California, (May 1968).

APPENDIX A
NUMERICAL ACCURACY OF THE DEQ NUMERICAL INTEGRATOR

CONTENTS

1.	INTRODUCTION	55
2.	BASIC MODEL	55
a.	Numerical Accuracy	56
b.	Test Cases	58
(1)	Variable Step Mode	60
(2)	Integration Closure Tests	60
(3)	Constant Energy Considerations	60
(4)	Sample Running Times	60

APPENDIX A

NUMERICAL ACCURACY OF THE DEQ NUMERICAL INTEGRATOR

1. INTRODUCTION

In recent months, much experience has been gained at the Aerospace Mathematics and Computation Center with a particular numerical integrator (Ref.8), referred to as DEQ and used principally for trajectory generation. Developed and written by James F. Holt of Aerospace, it is designed to numerically integrate a set of N simultaneous 2nd-order ordinary differential equations. A fourth-order Runge-Kutta method is used for its starting procedure and a Gauss-Jackson method (with eighth-order differences) is used for normal integration. Additional techniques allow a variable-step mode (halving and doubling) with automatic local truncation-error controls.

In trajectory generation, DEQ is used to integrate the equations of motion expressed in a Cowell and/or Encke formulation. The numerical accuracy of this integrator for typical geodetic orbits is of primary interest here. To investigate the solution accuracy, direct comparisons were made with known analytic solutions and integration closure tests were performed with augmented force models.

2. BASIC MODEL

A Cowell formulation of the equations of motion in vector form was used for testing where the total acceleration vector acting on a vehicle was represented as the sum of a primary gravitational term and total perturbative term. Table A-I shows such a vector in an ECI coordinate system.

Table A-I. Cowell Integration Method

<u>Problem:</u>	Numerically integrate second-order ordinary differential equations of the form
	$\ddot{\underline{r}} = -\frac{\mu \underline{r}}{ \underline{r} ^3} + \sum_{i=1}^N \ddot{\underline{r}}_i^*$
	where $\underline{r}$ = Position vector μ = Newtonian Gravitational Constant $\sum_{i=1}^N \ddot{\underline{r}}_i^*$ = Acceleration vector resulting from perturbing forces
<u>Solution:</u>	DEQ = Floating-point Cowell (second sum) Runge-Kutta integration of second-order equations • 4th-order Runge-Kutta method to start and halve the step-size during the integration • Cowell "second-sum" (Gauss-Jackson) method based on 8th differences to continue the integration.

a. NUMERICAL ACCURACY

All numerical results presented here are as of 12 June 1967.

An example of the numerical accuracy of this integrator on three different computers - IBM 7094, CDC 3600, and CDC 6600 - is given in Figure A-1. Note the following respective word lengths (DP = double precision):

<u>Machine</u>	<u>Word Length</u>	<u>Significant Digits</u>
IBM 7094	36 bits	8 (16 DP)
CDC 3600	48 bits	11 (25 DP)
CDC 6600	60 bits	14

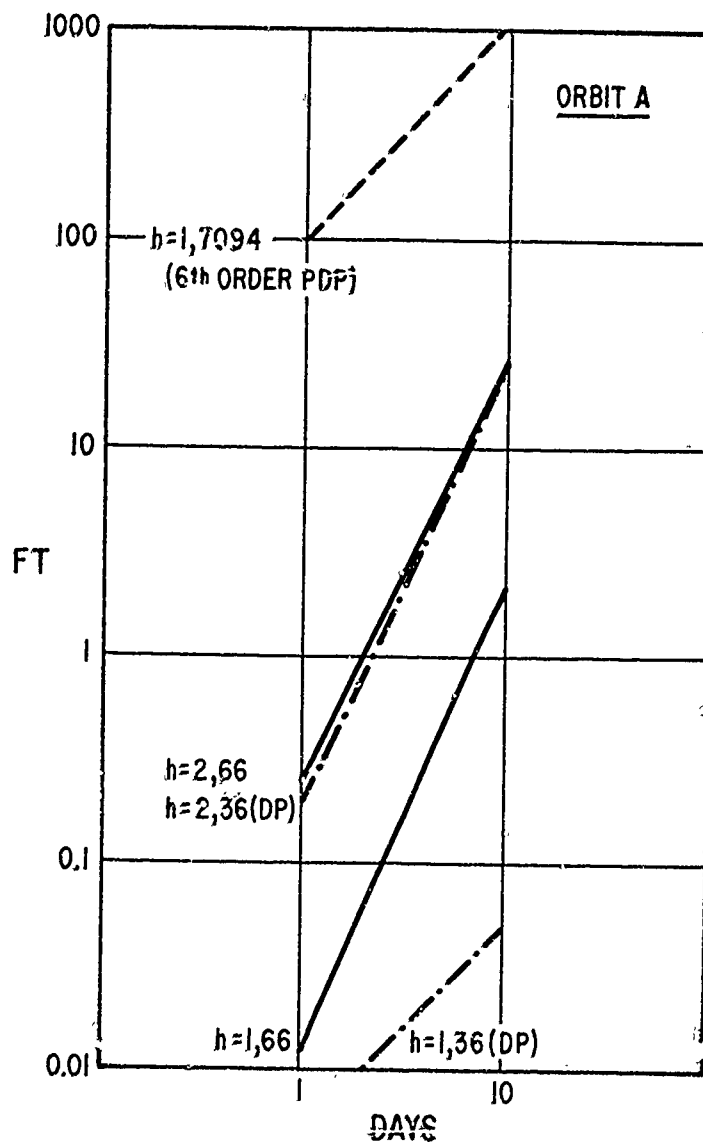


Figure A.1. Accuracy of Numerical Integration
(Analytic Solution, Keplerian Model)

The root-sum-square (rss of the error) is the difference obtained from comparing a known analytic solution at each integration step and is plotted vs time by using a log-log scale. Note that a fixed step of one minute, using 6th-order differences with partial double precision (PDP) on the IBM 7094, gives a 100-ft error after one day; whereas the 8th-order method using single precision (SP) on the CDC 3600, gives an approximate 1/100-ft error, and only 2 ft after 10 days. Comparable results were obtained by using DP on the CDC 3600.

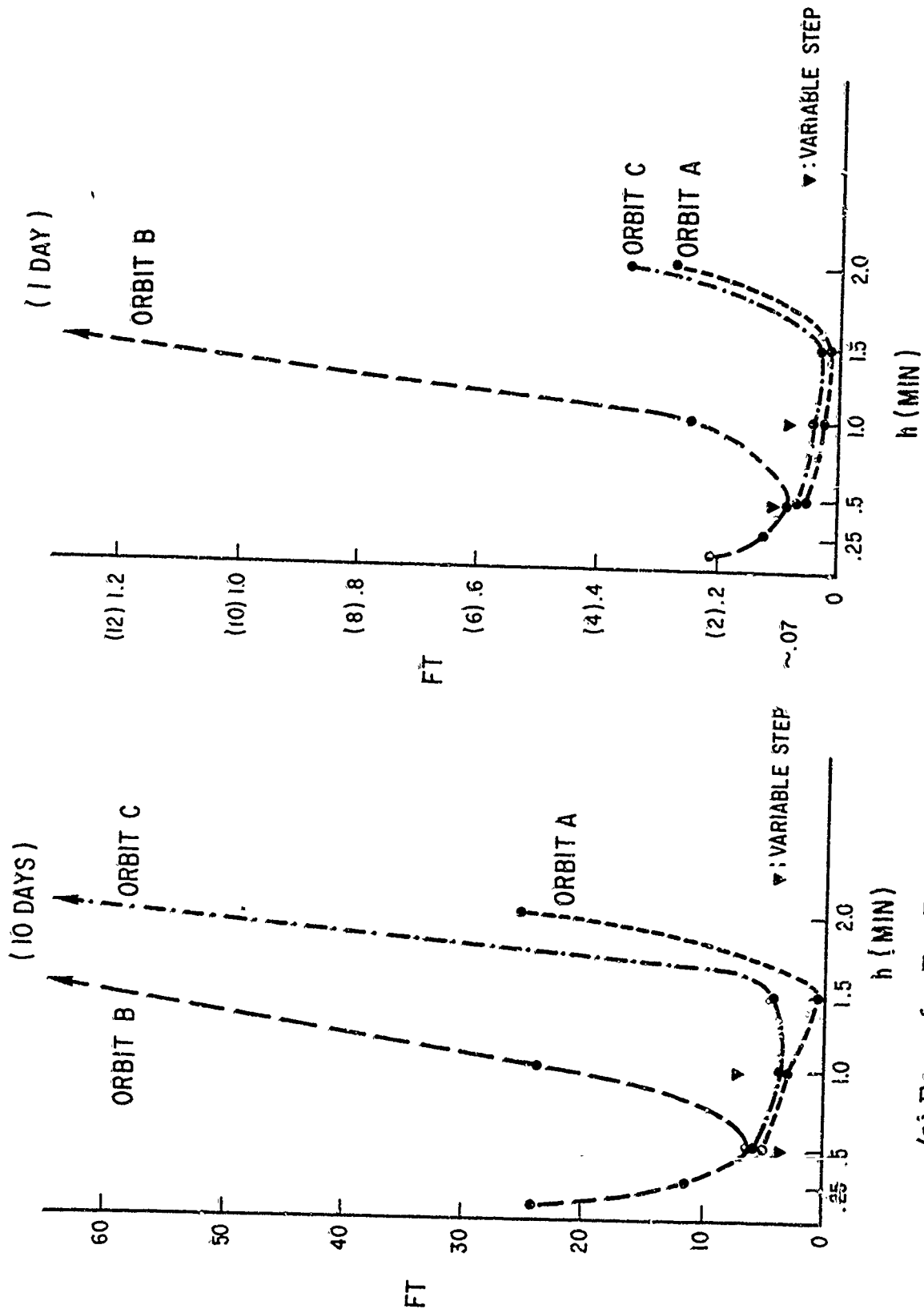
b. TEST CASES

The test cases chosen for this study are listed in Table A-II.

Table A-II. Test Orbits

	Orbit A	Orbit B	Orbit C	
a	21533095.	86767518.	21654273.	(ft)
e	.0000117	.737	.03115	
i	45.	63.4	95.3	(deg)
Ω	0.	177.	351.6	(deg)
ω	180.	270.	295.	(deg)
P_k	88.2	713.4	88.9	(min)
h_p	99.9	317.7	18.3	(n mi)
h_a	100.	21373.	240.3	(n mi)

A parametric relation between the error variation rss and the fixed-step size h for a particular time period can be utilized to investigate optimum step size. In Figure A-2(a) this technique is illustrated by plotting the relative maximum rss error vs the integration-step size for orbits A, B, and C. Note the roundoff error growth for Orbit B to the left of the optimum step (0.5 min) as h tends toward zero. On the other hand, the high eccentricity of this orbit causes sudden changes in acceleration near perigee and a very uniform acceleration at apogee, which results in a greater growth for h than 0.5 minute.



(a) Error for Ten Days
(b) Error for One Day

Figure A-2. Error Variation with Fixed Step Size

Optimum steps for Orbits A and C are 1.5 and 1.0 minutes, respectively. Thus, if the investigation time period is changed, the relative optimum step may also change; for example, Orbit C has a 1.5-minute optimum for a 1-day sample.

(1) Variable Step Mode

In general, the variable-step mode reduces the rss error (for a 10-day period). For example, for Orbit B in Figure A-2 (b), with $h_0 = 0.5$, it decreases the rss; and for $h_0 = 1$, it decreases the error from 25 to 8 ft. The behavior is the same at the end of 1 day.

(2) Integration Closure Tests

With this numerical integration method, closure tests were made on three augmented force models. A generalized potential function expansion for the aspherical earth was assumed, and the symbols Z, T, and R denoted zonal harmonics, zonal-plus-tesseral harmonics, and harmonics-plus-resonance, respectively. Relationships between closure error and fixed-step size permitted the desired "best" step size to be chosen for a particular force model and orbit. Figure A-3 indicates that for a 1-day sample interval, $h = 1.0$ minute is optimum for the Z and T models, and $h = 0.5$ for the R model at the desired level of accuracy. In another example (Figure A-4 for Orbit C), $h = 0.5$ is necessary for both the T and R force models.

(3) Constant Energy Considerations

For simple conservative fields, such as spherical and Z models, both the magnitude of angular momentum and its polar component exhibited linear growth with time, but maintained an acceptable degree of significance as defined by computer word length.

(4) Sample Running Times

Table A-III gives sample running times for trajectory generation by using three augmented force models. All stated times are considered upper bounds.

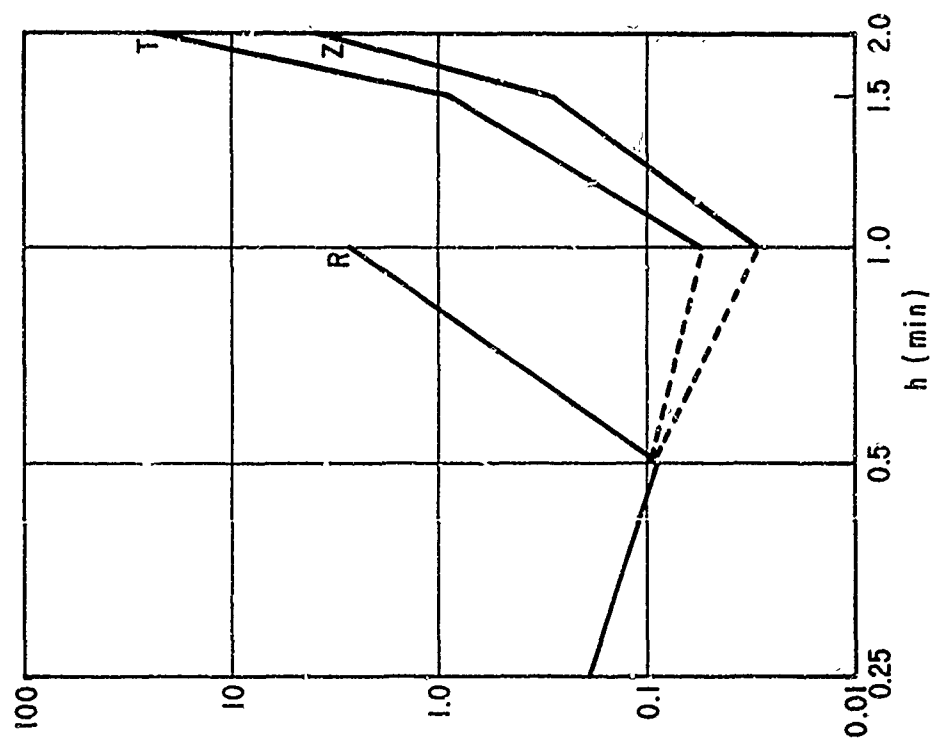


Figure A-3. Closure of Numerical Integration for Orbit A ($i = 45^\circ$)

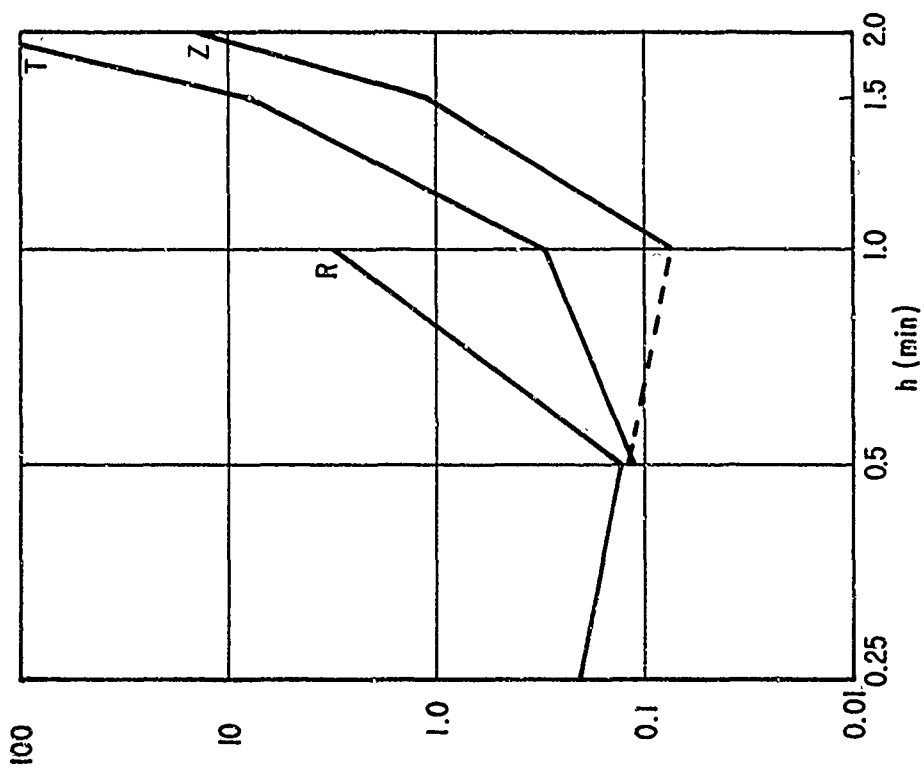


Figure A-4. Closure of Numerical Integration for Orbit C ($i = 95^\circ$)

Table A-III. Sample Running Times

Problem: Numerical Integration of Accelerations Derived from Geopotential Model $U = \sum_{n=0}^N \sum_{m=0}^M U_{nm}$				
<u>Case</u>	<u>h(min)</u>	<u>N</u>	<u>M</u>	<u>CDC 6600 Time (sec)</u>
1 day	2	4	0 (Z)	2
1 day	1	4	0	4
1 day	1	12	6 (T)	12
1 day	0.5	32	32 (R)	27
1 day	0.25	32	32	53

APPENDIX B KYNER II NON-TWO-BODY NOMINAL TRAJECTORY

CONTENTS

1.	FORMULATION	65
2.	COMPUTATIONAL ALGORITHM FOR THE REFERENCE TRAJECTORY	65
	a. Basic Equations of the Formulation	65
	b. Related Formulas	67
	(1) Kepler's Equation in Differenced Form.	67
	(2) True Anomaly and Eccentric Anomaly Relationships	68
	(3) Expressions for r , $\dot{r}$, and Φ	68
	c. Computational Scheme	69
	(1) Initialization Phase	69
	(a) Initial Orbital Parameters	69
	(b) Initial Rate Parameters	72
	(c) F and G Coefficients	73
	(d) Time Invariant Matrices	73
	(2) Evaluation Phase	74
3.	SUBROUTINE WKCNC2	78
	a. Identification	78
	b. Purpose	78
	c. Notation	78
	d. Inputs	79
	e. Outputs	80
	f. Computational Procedure and Equations.	80
	g. Kepler's Equation Solver.	88

BLANK PAGE

APPENDIX B

KYNER II NON-TWO-BODY NOMINAL TRAJECTORY

1. FORMULATION

The formulation of the Kyner II non-two-body nominal trajectory is developed in Reference 5.

2. COMPUTATIONAL ALGORITHM FOR THE REFERENCE TRAJECTORY

To utilize the theoretical concepts presented in Section III.D, a computational algorithm is developed to evaluate the reference trajectory. The algorithm is composed of basic equations of the formulation, related formulas, and the computational scheme.

a. BASIC EQUATIONS OF THE FORMULATION

Given the initial position $\underline{r}_0$ and velocity $\dot{\underline{r}}_0$ vectors at epoch, the reference $\underline{r}$ can be written in terms of the F and G functions as

$$\underline{r} = F \underline{M} + G \underline{N} \quad (B-1)$$

where

$$\underline{M} = \underline{r}_0 + [A] \underline{r}_0 = [I + [A]] \underline{r}_0$$

$$\underline{N} = \dot{\underline{r}}_0 + [A] \dot{\underline{r}}_0 = [I + [A]] \dot{\underline{r}}_0$$

$$F = (r/r_0) \cos \Phi - (\dot{r}/h)_0 r \sin \Phi$$

$$G = (r/h)_0 r \sin \Phi$$

where

$$\Phi = v - v_0$$

$$[A] = \sum_{i=1}^8 a_i(\Phi) [A_i]$$

a_i are slowly varying scalar functions of ϕ

$[A_i]$ are constant matrices evaluated at epoch

Thus Eq. (B-1) becomes

$$\underline{r} = F[I + [A]] \underline{r}_0 + G[I + [A]] \dot{\underline{r}}_0 \quad (B-2)$$

The reference velocity $\dot{\underline{r}}$ is then obtained by differentiating Eq. (B-2) as

$$\dot{\underline{r}} = [\dot{F}[I + [A]] + F[\dot{A}]] \underline{r}_0 + [\dot{G}[I + [A]] + G[\dot{A}]] \dot{\underline{r}}_0 \quad (B-3)$$

where

$$\dot{F} = (r/r_0)(\dot{r} \cos \phi - r \sin \phi \dot{\phi}) - (\dot{r}/n_0)(\dot{r} \sin \phi + r \cos \phi \dot{\phi})$$

$$\dot{G} = (r/h_0)(\dot{r} \sin \phi + r \cos \phi \dot{\phi})$$

$$\dot{\phi} = \frac{\dot{h}}{r^2}$$

$$\dot{r} = a \dot{\Delta E} [C_e \sin \Delta E + S_e \cos \Delta E]$$

$$\Delta E = \bar{n}a/r$$

$$[\dot{A}] = \sum_{i=1}^8 \dot{a}_i(\phi, \dot{\phi}) [A_i]$$

The differential equation satisfied by $\underline{r}$ then becomes

$$\ddot{\underline{r}} = -\mu \frac{\underline{r}}{r^3} + \underline{L} \quad (B-4)$$

where

$$\underline{L} = [1 - (1 - \gamma)^2] \mu \underline{r}/r^3 + (F; [A]) \underline{r}_0 + (G; [A]) \dot{\underline{r}}_0$$

where

$$\gamma = -\frac{3}{2} J_2 \left(\frac{a_e}{a_0} \right)^2 \left(\frac{a}{r} \right)_0^3 (1 - 3 \sin^2 i_0 \sin^2 u_0)$$

$$(p; q) = 2 \frac{dp}{dt} \frac{dq}{dt} + p \frac{d^2 q}{dt^2} \text{ (differential operator)}$$

and p and q are arguments of the operator.

b. RELATED FORMULAS

(1) Kepler's Equation in Differenced Form

The independent variable is expressed in difference form $\Delta = t - t_0$ (time), $\Delta = v - v_0$ (true anomaly), or $\Delta = E - E_0$ (eccentric anomaly). It is therefore convenient to express the fundamental angle-time relationship (Kepler's equation) as

$$M - M_0 = (E - E_0) + 2 S_e \sin^2 (E - E_0)/2 - C_e \sin (E - E_0) \quad (B-5)$$

where

$$M - M_0 = \bar{n} (t - t_0)$$

$$C_e = (e \cos E)_0$$

$$S_e = (e \sin E)_0$$

(2) True Anomaly and Eccentric Anomaly Relationships

Difference forms relating the true anomaly and the eccentric anomaly are necessary to evaluate $\underline{r}$, $\dot{\underline{r}}$, and $\ddot{\underline{r}}$. These are

$$\cos (v - v_o) = 1 - \frac{a^2 (1 - e^2)}{rr_o} [1 - \cos (E - E_o)]$$

$$\sin (v - v_o) = \frac{a^2 (1 - e^2)^{\frac{1}{2}}}{rr_o} [(M - M_o) - (E - E_o) + \sin (E - E_o)]$$

$$\sin (E - E_o) = \frac{r}{(ap)^{\frac{1}{2}}} \sin (v - v_o) - \frac{r}{p} [1 - \cos (v - v_o)] S_e$$

$$\cos (E - E_o) = 1 - \frac{rr_o}{ap} [1 - \cos (v - v_o)]$$

(3) Expressions for r , $\dot{r}$ and Φ

If $\Delta = t - t_o$, or $\Delta = E - E_o$, then the reference position distance is given in difference form by

$$r = [1 - C_e \cos (E - E_o) + S_e \sin (E - E_o)] \quad (B-6)$$

If $\Delta = v - v_o$, then

$$r = p [1 + C_v \cos (v - v_o) - S_v \sin (v - v_o)]^{-1}$$

where

$$C_v = (e \cos v)_o$$

$$S_v = (e \sin v)_o$$

Differentiation of Eq. (B-6) gives

$$\dot{r} = a \Delta \dot{E} [C_e \sin (E - E_o) + S_e \cos (E - E_o)]$$

where

$$\Delta \dot{E} = \bar{n} (a/r)$$

If $\Phi = v - v_0$, then

$$\dot{\Phi} = \bar{h}/r^2$$

where

$$\bar{h} = \bar{n} a^2 (1 - e^2)^{\frac{1}{2}} = (\bar{n}/n_0)$$

c. COMPUTATIONAL SCHEME

If the initial position vector $\underline{r}_0$ and velocity vector $\dot{\underline{r}}_0$ are given at epoch time t_0 , the following procedure will establish a means of determining the reference position $\underline{r}$, velocity $\dot{\underline{r}}$, and acceleration $\ddot{\underline{r}}$ vectors at time t . Two computational phases, initialization and evaluation, are presented in detail.

(1) Initialization Phase

All initial orbital parameters, rate parameters, coefficients, and time invariant matrices are evaluated at epoch by the following equations.

(a) Initial Orbital Parameters

The desired set of orbital parameters is

$$(\underline{r}_0, a, C_e, S_e, e, i, u_0, \Omega_0, C_v, S_v, v_0, \omega_0)$$

where

- r_o is the geocentric radius distance at epoch
- a is semimajor diameter of the conic at epoch
- C_e, S_e are coefficients evaluated at epoch
- e is the eccentricity of the conic at epoch
- i is the inclination of the conic at epoch
- u_o is the argument of latitude at epoch
- Ω_o is the longitude of the ascending node at epoch
- C_v, S_v are coefficients evaluated at epoch
- v_o is the true anomaly at epoch
- ω_o is the argument of perigee at epoch

Equations for the orbital parameters are

$$r_o = (\underline{r}_o \cdot \underline{r}_o)^{\frac{1}{2}}$$

$$\dot{r}_o = (\underline{r}_o \cdot \dot{\underline{r}}_o) / r_o$$

$$1/a = \frac{2}{r_o} - (\dot{\underline{r}}_o \cdot \dot{\underline{r}}_o) / \mu$$

$$C_e = 1 - r_o/a \equiv (e \cos E)_o$$

$$\dot{r}_o = (1/r_o)(\mu/a)^{1/2}$$

$$s_e = \dot{r}_o/a \dot{E}_o \equiv (e \sin E)_o$$

$$e = (C_e^2 + S_e^2)^{1/2}$$

$$\underline{U}_o = \underline{r}_o / r_o$$

$$\underline{W}_o = (\underline{r}_o \times \dot{\underline{r}}_o) / |\underline{r}_o \times \dot{\underline{r}}_o|$$

$$\underline{V}_o = \underline{W}_o \times \underline{U}_o$$

$$\sin i = [U_{oz}^2 + V_{oz}^2]^{1/2}$$

$$\cos i = [(U_{ox} + V_{oy})^2 + (U_{oy} - V_{ox})^2]^{1/2} - 1$$

$$i = \tan^{-1} (\sin i / \cos i)$$

$$\sin u_o = U_{oz} / \sin i$$

$$\cos u_o = V_{oz} / \sin i$$

$$u_o = \tan^{-1} (\sin u_o / \cos u_o)$$

$$\sin l_o = (U_{ox} + V_{oy}) / (1 + \cos i)$$

$$\cos l_o = (U_{oy} - V_{ox}) / (1 + \cos i)$$

$$l_o = \tan^{-1} (\sin l_o / \cos l_o)$$

$$\Omega_o = l_o - u_o \quad \text{(true longitude)}$$

$$p = a(1 - e^2) \quad \text{(semi-parameter)}$$

$$C_v = p/r_o - 1 \equiv (e \cos v)_o$$

$$S_v = \dot{r}_0 (p/\mu)^{\frac{1}{2}} \equiv (e \sin v)_0$$

$$\sin v_0 = S_v / e$$

$$\cos v_0 = C_v / e$$

$$v_0 = \tan^{-1} (\sin v_0 / \cos v_0)$$

If $e = 0$, set $v_0 = 0$. Then

$$\omega_0 = u_0 - v_0$$

(b) Initial Rate Parameters

The initial value of the rate parameters discussed in Reference 4 are evaluated from the equations

$$\gamma = -\frac{3}{2} J_2 (a_e/a)_0^2 (a/r)_0^3 (1 - 3 \sin^2 i_0 \sin^2 u_0)$$

$$\eta = \frac{3}{4} J_2 (a_e/a)_0^2 (1 - e_0^2)^{-2} (4 - 5 \sin^2 i_0)$$

$$\tau = -\frac{3}{2} J_2 (a_e/a)_0^2 (1 - e_0^2)^{-2} \cos i_0$$

$$\bar{n} = n(1 - \gamma)$$

where

$$n = (\mu/a^3)^{\frac{1}{2}}$$

c. F and G Coefficients

The F and G functions have initial coefficients F_1 , F_2 , and G_1 , which are evaluated at epoch as

$$F_1 = (1/r)_0$$

$$F_2 = -(i/h)_0$$

$$G_1 = (r/h)_0$$

d. Time Invariant Matrices

The constant orientation matrices (A_j) ($j = 1, \dots, 8$) discussed in Section III. D. 3. b are given as

$$[A_1] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A_3] = [D(\Omega_0)] [C(i_0)] [A_2] [C(-i_0)] [D(-\Omega_0)]$$

$$[A_4] = [D(\Omega_0)] [C(i_0)] [A_1] [C(-i_0)] [D(-\Omega_0)]$$

where

$$[D(\Omega)] = \begin{bmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[C(i)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix}$$

$$[A_5] = [A_1] [A_3]$$

$$[A_6] = [A_2] [A_4]$$

$$[A_7] = [A_2] [A_3]$$

$$[A_8] = [A_1] [A_4]$$

(2) Evaluation Phase

To obtain the nominal state $S(\underline{r}_n, \underline{\dot{r}}_n, \underline{\ddot{r}}_n, \underline{L})$ for some given value of the independent variable (time t , true anomaly v , or eccentric anomaly E), the following equations are solved.

First determine the value of the true anomaly difference

$$\Phi = v - v_0.$$

If time is the independent variable ($\Delta \equiv$ elapsed time from epoch), then $M - M_0 = n\Delta$, and Kepler's equation

$$M - M_0 = (E - E_0) + 2S_e \sin^2\left(\frac{E - E_0}{2}\right) - C_e \sin(E - E_0)$$

is solved for $E - E_0$. Thus

$$r = a \left(1 - C_e \cos(E - E_0) + S_e \sin(E - E_0) \right) \quad (B-7)$$

$$\cos \Phi = 1 - \frac{a^2 (1 - e^2)}{rr_0} (1 - \cos(E - E_0))$$

$$\sin \Phi = \frac{a^2 (1 - e^2)^{1/2}}{rr_0} ((M - M_0) - (E - E_0) + \sin (E - E_0))$$

$$\Phi = \tan^{-1} \left(\frac{\sin \Phi}{\cos \Phi} \right).$$

Now apply a monotonicity adjustment on Φ (see Section B.3.f of this Appendix). If true anomaly is the independent variable ($\Delta = \Phi \equiv v - v_0$), then compute eccentric anomaly difference $E - E_0$ from

$$\sin (E - E_0) = r/(ap)^{1/2} \sin \Phi - (r/p) (1 - \cos \Phi) S_e$$

$$\cos (E - E_0) = 1 - rr_0 (ap)^{-1} (1 - \cos \Phi)$$

where

$$r = p/(1 + C_v \cos \Phi - S_v \sin \Phi)$$

Thus

$$M - M_0 = (E - E_0) + 2S_e \sin^2 [(E - E_0)/2] - C_e \cos (E - E_0).$$

Now apply a monotonicity adjustment on $M - M_0$ (see Section B.3.f) and compute the corresponding elapsed time from epoch as

$$\Delta = t - t_0 = (M - M_0)/\bar{n}.$$

If eccentric anomaly is the independent variable ($\Delta = E - E_0$), then compute $M - M_0$ and $t - t_0$ as above and proceed with Eq. (B-7).

Evaluate the F & G functions and scalar coefficients a_j of the rotation matrix [A] as follows.

$$F = r/r_0 \cos \Phi - (\dot{r}/h)_0 r \sin \Phi$$

$$G = (r/h)_0 r \sin \Phi$$

$$\dot{F} = (r/r_0) (\dot{r} \cos \Phi - r \sin \Phi \dot{\Phi}) - (\dot{r}/h)_0 (\dot{r} \sin \Phi + r \cos \Phi \dot{\Phi})$$

$$\dot{G} = (r/h)_0 (\dot{r} \sin \Phi + r \cos \Phi \dot{\Phi})$$

where

$$\dot{r} = a (\Delta \dot{E}) (C_e \sin (E - E_0) + S_e \cos (E - E_0))$$

$$(\Delta \dot{E}) = \bar{n} a / r$$

$$\dot{\Phi} = \bar{h} / r^2 \quad \bar{h} = (1 - \gamma) h_0$$

and $\{a_j\}_{j=1}^8$ and $\{\dot{a}_j\}_{j=1}^8$ are as given in Eq. (B-8) in Section B.3.f.

Now

$$[A] = \sum_{j=1}^8 a_j [A_j]$$

and

$$[\dot{A}] = \sum_{j=1}^8 \dot{a}_j [A_j] .$$

Form the $[F]$ and $[G]$ matrices as

$$[F] = F [I + A]$$

$$[G] = G [I + A]$$

$$[\dot{F}] = [\dot{F} [I + A] + F [\dot{A}]]$$

$$[\dot{G}] = [\dot{G} [I + A] + G [\dot{A}]]$$

Evaluate the nominal state $S (\underline{r}_n, \underline{\dot{r}}_n, \underline{\ddot{r}}_n, \underline{L})$ by

$$\underline{r}_n = [F] \underline{r}_o + [G] \underline{\dot{r}}_o$$

$$\underline{\dot{r}}_n = [\dot{F}] \underline{r}_o + [\dot{G}] \underline{\dot{r}}_o$$

$$\underline{\ddot{r}}_n = -\mu \frac{\underline{r}_n}{r_n^3} + \underline{L}$$

$$\underline{L} = \left(\frac{n_o^2 - \bar{n}^2}{n_o^2} \right) \mu \frac{\underline{r}_n}{r_n^3} + (F; [A]) \underline{r}_o + (G; [A]) \underline{\dot{r}}_o .$$

3. SUBROUTINE WKCNC2

a. IDENTIFICATION

WKCNC2: Computational Algorithm for Generating Nominal Trajectory

6600 FORTRAN

Aerospace Corporation

b. PURPOSE

To generate a nominal trajectory $\{\underline{r}, \underline{\dot{r}}, \underline{\ddot{r}}\}$ at time t using a modified F and G formulation.

c. NOTATION

$\underline{r}_0$	(XO)	initial geocentric position vector of the vehicle at epoch
$\underline{\dot{r}}_0$	(XDO)	initial geocentric velocity vector of the vehicle at epoch
μ	(GRAV)	square of the gravitational constant of the earth
J_2	(CJ2)	second zonal harmonic coefficient
π	(PI)	numerical constant ($\pi = 3.1415927\dots$)
ϵ_k	(EPSK)	convergence criterion for solution of Kepler's Equation
Δ	(ARG)	independent variable difference (time Δt , true anomaly Δf , or eccentric anomaly ΔE)
$\{\underline{r}, \underline{\dot{r}}, \underline{\ddot{r}}\}$	(XC)	nominal trajectory point on the conic at time t where $\underline{r}$ is the position vector $\underline{\dot{r}}$ is the velocity vector $\underline{\ddot{r}}$ is the acceleration vector
2π	(TWOPPI)	numerical constant
r_0	(RO)	geocentric radius at epoch

a	(CA)	semimajor axis of the conic at epoch
e	(CE)	eccentricity of the conic at epoch
i	(CI)	inclination at epoch
u_o	(CUO)	argument of latitude at epoch
l_o	(CZO)	true longitude at epoch
Ω	(C ϕ)	longitude of the ascending node at epoch
v_o	(VOARG)	true anomaly at epoch
ω	(ARGPER)	argument of perigee at epoch
γ	(GAMMA)	rate parameter of nominal anomalistic mean motion $\bar{n}$
η	(ETA)	rate parameter of the major axis of the nominal ellipse
τ	(TAU)	rate parameter of the line of nodes of the nominal plane
n_o	(CN)	mean motion at epoch
$\bar{n}$	(CNBAR)	nominal anomalistic mean motion
p	(CP)	semi-parameter of the conic at epoch.

d. INPUTS

ENTRY 1: IENTRY = 1 (initial entry) requires $\underline{r}_o$, $\dot{\underline{r}}_o$, μ , J_2 , π , IARG
where

IARG = 1 time is independent variable
= 2 true anomaly is independent variable
= 3 eccentric anomaly is independent variable

ENTRY 2: IENTRY = 2 (normal entry) requires ϵ_k and Δ

e. OUTPUTS

ENTRY 1: IENTRY = 1 gives no output parameters

ENTRY 2: IENTRY = 2 gives nominal trajectory point at time t ($\underline{r}$, $\dot{\underline{r}}$, $\ddot{\underline{r}}$)

f. COMPUTATIONAL PROCEDURE AND EQUATIONS

Test IENTRY = 1

Yes, go to (100)

No, go to (200)

(100) Set $2\pi = 2. \times \pi$ (SETUP ENTRY)

GMU = μ

Compute initial orbital parameter set

$$\text{ROSQ} = r_o^2 = \underline{r}_o \cdot \underline{r}_o$$

$$\text{RO} = r_o = (r_o^2)^{1/2}$$

$$\text{RORDO} = r_o \dot{r}_o = \underline{r}_o \cdot \dot{\underline{r}}_o$$

$$\text{VOSQMU} = \dot{r}_o^2 / \mu = (\dot{\underline{r}}_o \cdot \dot{\underline{r}}_o) / \mu$$

$$\text{RECIA} = (2/r_o) - \dot{r}_o^2 / \mu$$

$$\text{CA} = a = 1/\text{RECIA}$$

$$\text{CETERM} = C_e = 1 - (r_o/a)$$

$$\text{SETERM} = S_e = \dot{r}_o / a \dot{E}_o$$

$$\text{CE} = e = (C_e^2 + S_e^2)^{1/2}$$

$$\text{UOVEC} = \underline{U}_o = \underline{r}_o / r_o$$

$$\text{WOVEC} = \underline{W}_o = (\underline{r}_o \times \dot{\underline{r}}_o) / |\underline{r}_o \times \dot{\underline{r}}_o|$$

$$\text{VOVEC} = \underline{V}_o = \underline{W}_o \times \underline{U}_o$$

$$\begin{aligned}
\text{SINCI} &= \sin i &= [U_{oz}^2 + V_{oz}^2]^{1/2} \\
\text{C}\phi\text{SCI} &= \cos i &= [(U_{ox} + V_{oy})^2 + (U_{oy} - V_{ox})^2]^{1/2} - 1 \\
\text{CI} &= i &= \tan^{-1} (\sin i / \cos i) \\
\text{SINUO} &= \sin u_o &= U_{oz} / \sin i \\
\text{C}\phi\text{SUO} &= \cos u_o &= V_{oz} / \sin i \\
\text{CUO} &= u_o &= \tan^{-1} (\sin u_o / \cos u_o) \\
\text{SINLO} &= \sin l_o &= (U_{ox} + V_{oy}) / (1 + \cos i) \\
\text{C}\phi\text{SLO} &= \cos l_o &= (U_{oy} - V_{ox}) / (1 + \cos i) \\
\text{CLO} &= l_o &= \tan^{-1} (\sin l_o / \cos l_o) \\
\text{C}\phi &= \Omega &= l_o - u_o \\
\text{SINC}\phi &= \sin \Omega \\
\text{C}\phi\text{SC}\phi &= \cos \Omega \\
\text{CVTERM} &= C_v &= (p/r_o) - 1 \\
\text{SVTERM} &= S_v &= \dot{r}_o (p/\mu)^{1/2} \\
\text{SINVO} &= \sin v_o &= S_v / e \\
\text{C}\phi\text{SVO} &= \cos v_o &= C_v / e \\
\text{VOARG} &= v_o &= \tan^{-1} (\sin v_o / \cos v_o) \\
\text{ARGPER} &= \omega &= u_o - v_o
\end{aligned}$$

Compute initial rate parameter set

$$\begin{aligned}
\text{GAMMA} &= \gamma = -3/2 J_2 (a_e/a_o)^2 (a/r_o)^3 (1 - 3 \sin^2 i_o \sin^2 u_o) \\
\text{ETA} &= \eta = 3/4 J_2 (a_e/a_o)^2 (1 - e_o^2)^{-2} (4 - 5 \sin^2 i_o) \\
\text{TAU} &= \tau = -3/2 J_2 (a_e/a_o)^2 (1 - e_o^2)^{-2} \cos i_o
\end{aligned}$$

$$CN = n = (\mu/a^3)^{1/2}$$

$$CNBAR = \bar{n} = n(1 - \gamma)$$

Initialize F and G coefficients

$$F1 = 1/r_o$$

$$F2 = -(\dot{r}/h)_o$$

$$G1 = (r/h)_o$$

Initialize time invariant matrices

$$\{A_j\}_{j=1}^8 = 0$$

$$[D] = [C] = [D^-] = [C^-] = 0$$

$$[T] = [U] = 0$$

$$A_1(1, 2) = -1$$

$$A_1(2, 1) = 1$$

$$A_2(1, 1) = 1$$

$$A_2(2, 2) = 1$$

$$[U] = I, \text{ the identity matrix}$$

$$[D] = \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D^-] = \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ -\sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix}$$

$$[C^-] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix}$$

$$[A_3] = [D] [C] [A_1] [C^-] [D^-]$$

$$[A_4] = [D] [C] [A_2] [C^-] [D^-]$$

$$[A_5] = [A_1] [A_3]$$

$$[A_6] = [A_2] [A_4]$$

$$[A_7] = [A_2] [A_3]$$

$$[A_8] = [A_1] [A_4]$$

Exit

(200) Test IARG = 1, 2, or 3

If equal to 1, go to (300)

If equal to 2, go to (400)

If equal to 3, go to (500)

(300) Time is the independent variable ($\Delta \equiv$ elapsed time from epoch).

Compute mean anomaly difference $M - M_0$

$$DM = \bar{n} \Delta$$

Call KEQS

Enter with DM, C_e , S_e , ϵ_k

Exit with DE = $E - E_0$, eccentric anomaly difference

Compute true anomaly difference $\phi = v - v_0$

$$\begin{aligned}
(301) \quad \text{SINDE} &= \sin (E - E_0) \\
\text{COSDE} &= \cos (E - E_0) \\
R &= r = a [1 - C_e \cos (E - E_0) + S_e \sin (E - E_0)] \\
\text{C}\phi\text{SPHI} &= \cos \phi = 1 - a^2 (1 - e^2) (1 - \cos (E - E_0)) / rr_0 \\
\text{SINPHI} &= \sin \phi = [a^2 (1 - e^2)^{1/2} / rr_0] \\
&\quad [(M - M_0) - (E - E_0) + \sin (E - E_0)] \\
\text{PHI} &= \phi = \tan^{-1} (\sin \phi / \cos \phi)
\end{aligned}$$

The monotonicity adjustment for $\phi = f - f_0$ is

If $\phi \geq 0$, then $\phi = \phi + \Delta E - \text{MOD} (\Delta E, 2\pi)$, or

If $|\phi| - |\Delta E| \geq \pi$, then $\phi = \phi - 2\pi$

If $\phi < 0$ ($\Delta M < 0$), then $\phi = \phi + \Delta E - \text{MOD} (\Delta E, -2\pi)$, or

If $|\phi| - |\Delta E| \geq \pi$, then $\phi = \phi + 2\pi$

Go to (600).

(400) True anomaly is the independent variable ($\Delta = \phi \equiv v - v_0$).

Compute eccentric difference $E - E_0$

$$\begin{aligned}
R &= r = p [1 + C_v \cos \phi - S_v \sin \phi]^{-1} \\
\text{SINDE} &= \sin (E - E_0) = r(ap)^{1/2} \sin \phi - r/p (1 - \cos \phi) S_e \\
\text{COSDE} &= \cos (E - E_0) = 1 - rr_0 (ap)^{-1} (1 - \cos \phi) \\
\text{DE} &= (E - E_0) = \tan^{-1} ((\sin (E - E_0) / \cos (E - E_0)))
\end{aligned}$$

Compute mean anomaly difference $M - M_0$

$$\begin{aligned}
\text{DM} &= (M - M_0) = (E - E_0) + 2 S_e \sin^2 (E - E_0 / 2) \\
&\quad - C_e \cos (E - E_0)
\end{aligned}$$

Compute corresponding time parameter $(t - t_0)$

$$DT = (t - t_0) = (M - M_0) / \bar{n}$$

Go to (600).

(500) Eccentric anomaly is the independent variable $(\Delta = E - E_0)$.

Compute mean anomaly difference $M - M_0$

$$DM = (M - M_0) = (E - E_0) + 2 S_e \sin^2 (E - E_0/2) - C_e \cos (E - E_0)$$

Compute corresponding time parameter $(t - t_0)$

$$DT = (t - t_0) = (M - M_0) / \bar{n}$$

Go to (301).

(600) Compute reference trajectory point on the conic at time t
 $(\underline{r}, \underline{\dot{r}}, \underline{\ddot{r}})$. Evaluate the F and G function coefficients

$$F = F_1 r \cos \Phi + F_2 r \sin \Phi$$

$$G = G_1 r \sin \Phi$$

Evaluate the scalar coefficients a_j ($j=1, \dots, 8$) of the rotation matrix [A]

$$\left. \begin{aligned} a_1 &= \sin (\tau \Phi) \\ a_2 &= -2 [\sin (\tau \Phi/2)]^2 \\ a_3 &= \sin (\eta \Phi) \\ a_4 &= -2 [\sin (\eta \Phi/2)]^2 \\ a_5 &= a_1 a_3 \\ a_6 &= a_2 a_4 \\ a_7 &= a_2 a_3 \\ a_8 &= a_1 a_4 \end{aligned} \right\}$$

(B-8)

Evaluate the rotation matrix $[A]$

$$[A] = \sum_{j=1}^8 a_j(\Phi) [A_j]$$

Evaluate the $[F]$ and $[G]$ matrices

$$[F] = F[I + A]$$

$$[G] = G[I + A]$$

(620) Evaluate the reference position vector $\underline{r}$

$$\underline{r} = [F] \underline{r}_0 + [G] \underline{\dot{r}}_0$$

Evaluate the F and G function coefficients

$$\dot{\Delta E} = \bar{n} a / r$$

$$\dot{r} = a \dot{\Delta E} [C_e \sin(E - E_0) + S_e \cos(E - E_0)]$$

$$-\sin \Phi \dot{\Phi} = \frac{-a^2 (1 - e^2) [\sin(E - E_0) \dot{\Delta E} r_0 - r_0 \dot{r} (1 - \cos(E - E_0))]}{(r r_0)^2}$$

$$\cos \Phi \dot{\Phi} = \left[\frac{a^2 (1 - e^2)^{1/2}}{(r r_0)^2} \right] \left[\left(\bar{n} - \dot{\Delta E} (1 - \cos(E - E_0)) \right) r \dot{r}_0 - [(M - M_0) - (E - E_0) + \sin(E - E_0)] r_0 \dot{r} \right]$$

$$\dot{\Phi} = [(-\sin \Phi \dot{\Phi})^2 + (\cos \Phi \dot{\Phi})^2]^{1/2}$$

$$\dot{F} = F_1 \dot{r} \cos \Phi + F_1 r (-\sin \Phi \dot{\Phi}) + F_2 \dot{r} \sin \Phi + F_2 r (\cos \Phi \dot{\Phi})$$

$$\dot{G} = G_1 \dot{r} \sin \Phi + G_1 r (\cos \Phi \dot{\Phi})$$

Evaluate the scalar coefficients $\dot{a}_j$ ($j=1, \dots, 8$) of the rotation matrix

$[\dot{A}]$

$$\dot{a}_1 = \tau \dot{\Phi} (a_2 + 1)$$

$$\dot{a}_2 = \tau \dot{\Phi} (-a_1)$$

$$\begin{aligned}
\dot{a}_3 &= \eta \dot{\Phi} (a_4 + 1) \\
\dot{a}_4 &= \eta \dot{\Phi} (-a_3) \\
\dot{a}_5 &= \dot{a}_1 a_3 + a_2 \dot{a}_3 \\
\dot{a}_6 &= \dot{a}_2 a_4 + a_2 \dot{a}_4 \\
\dot{a}_7 &= \dot{a}_2 a_3 + a_2 \dot{a}_3 \\
\dot{a}_8 &= \dot{a}_1 a_4 + a_1 \dot{a}_4
\end{aligned}$$

Evaluate the rotation matrix $[\dot{A}]$

$$[\dot{A}] = \sum_{j=1}^8 \dot{a}_j (\Phi, \dot{\Phi}) [A_j]$$

Evaluate the $[\dot{F}]$ and $[\dot{G}]$ matrices

$$\begin{aligned}
[\dot{F}] &= [\dot{F} [I + A] + F [\dot{A}]] \\
[\dot{G}] &= [\dot{G} [I + A] + G [\dot{A}]]
\end{aligned}$$

(660) Evaluate the reference position vector $\underline{\dot{r}}$

$$\underline{\dot{r}} = [\dot{F}] \underline{r}_0 + [\dot{G}] \underline{\dot{r}}_0$$

Evaluate the differential operator (;)

$$\begin{aligned}
(r \cos \Phi; a_j) & \quad (j = 1, \dots, 8) \\
(r \sin \Phi; a_j) & \quad (j = 1, \dots, 8),
\end{aligned}$$

where $(p; q) = 2 (dp/dt) (dq/dt) + p \, d^2q/dt^2$

Evaluate the $[\ddot{F}]$ and $[\ddot{G}]$ matrices

$$\ddot{F} = \sum_{j=1}^8 [\ddot{F}_1 (r \cos \Phi; a_j) + \ddot{F}_2 (r \sin \Phi; a_j)] [A_j] = (\ddot{F}; [A])$$

$$\ddot{G} = \sum_{j=1}^8 [\ddot{G}_1 (r \sin \Phi; a_j)] [A_j] = (\ddot{G}; [A]).$$

(680) Evaluate the $\underline{L}$ vector

$$\underline{L} = \left(\frac{n_o^2 - \bar{n}^2}{n_o^2} \right) \mu \frac{r}{r^3} + (F; [A]) \underline{r}_o + (G; [A]) \underline{\dot{r}}_o$$

(690) Evaluate the reference acceleration $\underline{\ddot{r}}$

$$\underline{\ddot{r}} = -\mu \underline{r}/r^3 + \underline{L}$$

Go to (900).

(900) Exit.

g. KEPLER'S EQUATION SOLVER

The first approximation for the solution of Kepler's equation is

$$(E - E_o) = (M - M_o) + e \sin (M - M_o) + e^2/2 \sin 2 (M - M_o) \\ + e^3/8 [(3 \sin 3 (M - M_o) - \sin (M - M_o))]$$

Kepler's equation in differenced form is given by

$$(M - M_o) \equiv \bar{n} (t - t_o) = (E - E_o) + 2 S_e \sin^2 [(E - E_o)/2] \\ - C_e \sin (E - E_o)$$

where

$$C_e = 1 - r_o/2$$

$$S_e = r_o \dot{r}_o / (\mu a)^{1/2}$$

$$r_o \dot{r}_o = \underline{r}_o \cdot \underline{\dot{r}}_o$$

An iterative solution of the differenced Keplerian form is computed by the following scheme. Compute

$$\Delta E = (E - E_o)_{i+1} - (E - E_o)_i$$

that is,

$$\Delta E = - f(E - E_0)_i / f'(E - E_0)_i \quad (\text{Newton-Raphson method})$$

Use the first approximation of $(E - E_0)$ given above as an initial guess, where

$$\begin{aligned} f(E - E_0)_i &= (M - M_0) - (E - E_0)_i - 2 S_e \sin^2 [(E - E_0)/2]_i \\ &\quad + C_e \sin (E - E_0)_i \end{aligned}$$

and

$$\begin{aligned} f'(E - E_0) &= -1 - 2 S_e \sin [(E - E_0)/2]_i \cos [(E - E_0)/2]_i \\ &\quad + C_e \cos (E - E_0)_i \\ &= -1 - S_e \sin (E - E_0)_i + C_e \cos (E - E_0)_i \end{aligned}$$

give

$$\Delta E = \frac{(M - M_0) - (E - E_0)_i - 2 S_e \sin^2 [(E - E_0)/2]_i + C_e \sin (E - E_0)_i}{1 + S_e \sin (E - E_0)_i - C_e \cos (E - E_0)_i}$$

Until $\Delta E \leq \epsilon (10^{-8})$, this procedure yields a solution $(E - E_0)$.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)		
1. ORIGINATING ACTIVITY (Corporate author) Aerospace Corporation El Segundo, California		2a. REPORT SECURITY CLASSIFICATION Unclassified
		2b. GROUP
3. REPORT TITLE TRAJECTORY ANALYSIS PROGRAM (TAP)		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)		
5. AUTHOR(S) (Last name, first name, initial) Walker, Dennis C.		
6. REPORT DATE 69 AUG 01	7a. TOTAL NO. OF PAGES 94	7b. NO. OF REFS 8
8a. CONTRACT OR GRANT NO. F04701-69-C-0066	9a. ORIGINATOR'S REPORT NUMBER(S) TR-0066(5110-01)-3	
b. PROJECT NO.		
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d.	SAMSO-TR-69-318	
10. AVAILABILITY/LIMITATION NOTICES This document is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of SAMSO (SMON). The distribution of this report is <u>limited because it contains technology restricted by mutual security acts.</u>		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Space Systems Division Air Force Systems Command Los Angeles, California	
13. ABSTRACT The Trajectory Analysis Program (TAP) is designed specifically for use on the CDC 6000 series machine as a computational tool to generate trajectories by different mathematical formulations and to compare the results. In particular, the reference trajectory is the result of numerical integration of the equations of motion in a Cowell formulation with time as the independent variable; whereas, the comparison trajectory selected is achieved by an analytic (Pines, Kyner, or Escobal) or a numerically integrated Encke (classical or modified) formulation. A comprehensive description is presented of the algorithms implemented in TAP with emphasis on characteristics such as logic flow, equations, numerical techniques, comparison differencing, and plotting. A basic usage guide is included with complete instructions for input data preparation and explanations of output tables. ()		

UNCLASSIFIED

Security Classification

14.

KEY WORDS

Analytic (Pines, Kyner, Escobal) Formulation
Cowell Formulation
Encke (Classic and Modified) Formulation
FORTRAN
Noncentral Gravitation
Printer Plots
Reference Systems and Equations
Trajectory Comparison
Trajectory Generation

Abstract (Continued)

UNCLASSIFIED

Security Classification