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A THEORETICAL EVALUATION OF VARIOUS SEARCH SALVAGE PROCEDURES FOR USE WITH NARROW-PATH LOCATORS

PART II

LOCATING OBJECTS WHOSE APPARENT PRESENCE AND APPROXIMATE POSITION ARE KNOWN

TECHNICAL REPORT NO. 118

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BY

R. K. REBER

JUNE 1957

ABSTRACT

The present report, representing Part II of a two part study of search salvage procedures for use with narrow-path locators, is concerned with the accurate location of objects whose apparent presence and approximate position have been previously established by wide-path detectors, by intelligence, or by other means. The locator characteristics are specified by an over-all width of searched path W and a probability β of detecting on any single locator pass any object which has a random location within W. It is found that the most effective search procedures differ according to whether the navigational errors are substantially smaller than, or are substantially larger than, or are of the same order of magnitude as the uncertainty in the object position. For each of these three cases practical search procedures are formulated for attaining a searching efficiency close to the theoretical maximum efficiency. Sufficient data are presented for each searching procedure to permit a quick determination of the amount of searching required to obtain sufficient assurance that a contact being investigated does not represent an object if none has been found, and an estimate of the average time required to find an object when one exists. Most of the analysis is concerned with locators that are perfect identifiers, but it is shown how the results can be applied to the location of objects by means of locators with limited classification capabilities.

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PARTIAL LIST OF SYMBOLS

$P(T')$ - Location probability. $P(T')$ is the probability that if there is an object it will be detected if searching is carried out for a time T'.

T - Overall searching time. The percentage of the objects investigated which are detected is $100\,P(T')$ if searching is discontinued when the object is found or after searching for a time T', whichever occurs first.

τ - Average discovery time. If searching is discontinued when an object is found or at time T', whichever occurs first, then τ is the average searching time per contact investigated for those cases where the contact which is investigated is an object.

$\bar{t}$ - Average searching time per contact investigated. When there are no locator false contacts, $\bar{t} = \xi\tau + (1 - \xi)T'$.

ξ - Fraction of the contacts investigated which are objects.

U - Search rate. U is the aggregate area searched per unit time ($U = W\beta V$).

V - Ground speed of locator.

Ω - Aggregate searched area.

W - Over-all width of searched path. It is the minimum width which includes every athwartship position within the locator path for which there is some chance of detecting an object which may be in that athwartship position.

w_a - Aggregate width of searched path, $w_a = \beta W$.

β - Probability of detecting an object at random position within W.

d - Lateral distance. It is the nominal lateral displacement of each search path from the nominal position of the next preceding path.

m - Nominal search-path density.

Δm - Increase in m per coverage.

N - In multiarea searching, N is the total number of coverages made if searching continues until time T'. In single track searching, N is the number of passes made if searching continues until time T'.

$\bar{\sigma}_1$ - Standard deviation of lateral navigational error of locator.

$\bar{\sigma}_2$ - Standard deviation of longitudinal error of locator in placing ends of paths at prescribed positions.

$\bar{\sigma}$ - Equal to $\bar{\sigma}_1$ and $\bar{\sigma}_2$ when $\bar{\sigma}_1 = \bar{\sigma}_2$.

σ_1 - Standard deviation of errors of estimate of object position in the direction of least uncertainty.

σ_2 - Standard deviation of errors of estimate of object position in the direction of greatest uncertainty.

σ - Equal to σ_1 and σ_2 when $\sigma_1 = \sigma_2$.

γ - A parameter which is given by Plates 2 to 5. (The symbol $\bar{\sigma}$ appearing on these plates is to be interpreted as $\bar{\sigma}_1$.)

γ_0 - Limiting value of γ as $d/\bar{\sigma}_1 \longrightarrow 0$.

SUMMARY

1. The present report presents Part II of a two part study of search procedures for use with narrow-path locators. Part I of this study (Reference (1)) is an analysis of general area and channel searching. The procedures considered in Part I do not require any knowledge concerning numbers or positions of objects to be salvaged (objects) which may be in the area to be searched. The present report is concerned with the formulation and evaluation of suitable searching procedures for accurately locating objects whose apparent presence and approximate positions have been previously established by intelligence, by high search-rate detectors, or by other means. The major part of the analysis is based on the assumption that the locator is able to discriminate between objects and clutter, but it is shown how the results which are obtained for highly discriminating locators can be modified to obtain an approximate evaluation of locators with limited discrimination capabilities.

2. For an object or object-like target whose approximate position is known, it has been assumed that it is possible to estimate a most probable position and that, in the most general case, the errors in estimating the object position have a known bivariate normal distribution. In the present report this distribution is usually referred to as an elliptical normal distribution. It is assumed that rectangular x and y-axes are chosen and so oriented that the y-component of the errors of estimate of object position are independent of the x-component of these errors. The standard deviation of the errors of estimate parallel to the x-axis is represented by σ_1 and the standard deviation of the errors of estimate parallel to the y-axis is represented by σ_2, where, in general, the axes are assumed chosen so that $\sigma_2 \geq \sigma_1$. When $\sigma_1 = \sigma_2$, the distribution reduces to a circular normal distribution and the common value of σ_1 and σ_2 is then represented by σ. Methods of determining the x and y-directions and the values of σ_1 and σ_2 are discussed in Appendix E.

3. For a given searching procedure the searching efficiency of a locator depends on the over-all width of searched path W, on the detection probability β, on the ground speed of the locator, and on the navigational error. Here W represents the minimum width which includes all athwartship positions in the path of the locator for which there is some chance of detecting any objects which may be in those positions, and β is the probability of detecting on any single pass an object which has a random location within W. In most cases it is assumed that all search passes are made parallel to the y-axis. Then the navigational error can be characterized by a standard deviation $\bar{\sigma}_1$ of lateral displacements of the actual paths from the nominal or intended paths and by a standard deviation $\bar{\sigma}_2$ of longitudinal displacements of the ends of the actual paths from the nominal or intended longitudinal positions of the ends. If the locator ground speed is V, then the quantity $V\beta W$ is called the search rate and is represented by the symbol U.

4. A primary objective in choosing suitable search procedures is to obtain small values of the time required to either locate an object or to obtain sufficient assurance that the contact which is being investigated does not represent an actual object. For any given locator and searching procedure, and for given σ_1 and σ_2, there will be a probability P(t) that if an object exists it will have been located by time t, measured

from the instant the search begins. In the present report $P(t)$ is called the location probability at time t. In general there may be expected to be some uncertainty that any contact is an object, and that it will be necessary at some point to discontinue the search even though no object has been found. The time at which the search is discontinued, when no object has been found, can logically be taken to be a time T at which the location probability has a specified value $P(T)$. Under conditions where it can be assumed that there is no more than one object in the area being searched and where the locator has sufficient discrimination so that it can be assumed to be known when an object is located, the search can be discontinued as soon as the object is found. Assuming that searching is discontinued when an object is located or at time T, whichever occurs first, the average searching time for those cases where an object exists, whether or not it is found, is called the average discovery time and is represented by τ. The time T is referred to as the overall searching time. If only a fraction ξ of the contacts which are being investigated by narrow-path locators are salvageable objects the average searching time per contact investigated is $\xi\tau + (1 - \xi)T$. The values of T and τ for given values of $P(T)$ serve as useful measures of searching efficiency. For most of the searching procedures considered, the value of τ is found to be only of the order of 1/3 to 1/7 of the value of T for values of $P(T)$ in the range 0.9 to 0.99, the smaller values of the ratio τ/T occurring for the larger values of $P(T)$. Thus the average time required to investigate a real object contact is only of the order of 1/3 to 1/7 of the time required to investigate a false contact.

5. The most suitable procedures for locating an object depend on the relative magnitude of the navigational errors as compared to the uncertainty in the object position. It is convenient to summarize the results separately for the following three cases:

(a) Navigational error small compared to the uncertainty in the object position ($\bar{\sigma}_1 \le \frac{1}{2}\sigma_1$, $\bar{\sigma}_2 \le \frac{1}{2}\sigma_2$).

(b) Navigational error of the same order as the uncertainty in the object position ($\frac{1}{2}\sigma_1 \le \bar{\sigma}_1 \le \frac{3}{2}\sigma_1$, $\bar{\sigma}_2 \le \frac{3}{2}\sigma_2$).

(c) Navigational error substantially larger than the uncertainty in the object position ($\bar{\sigma}_1 \ge \frac{3}{2}\sigma_1$).

Navigational Errors Small Compared to the Uncertainty in the Object Position ($\bar{\sigma}_1 \le \frac{1}{2}\sigma_1$, $\bar{\sigma}_2 \le \frac{1}{2}\sigma_2$)

6. For the case where the navigational errors are substantially smaller than the errors of estimate of the object position the following general types of searching procedures have been considered:

(a) Elliptical Area Searching. This consists of uniform searches of elliptical areas centered at the most probable position of the object with the major axis parallel to the y-axis (i.e. parallel to the direction of greatest uncertainty in object position), and with the dimensions of the ellipses searched on successive coverages varying in a prescribed manner.

(b) Rectangular Area Searching. This consists of uniform searches of rectangular areas centered at the most probable position of the object, with the longer sides of the rectangles parallel to the y-axis and with the dimensions of the rectangles searched on successive coverages varying in a prescribed manner.

(c) Constant Parallel Path Searching. This consists of uniform searches of rectangular areas in parallel paths of constant length, the widths of the rectangular areas searched on successive coverages varying in a prescribed manner. All passes are made parallel to the y-axis and the constant length of the search paths is chosen large enough so there is a sufficiently small chance that the object lies beyond the ends of the search paths.

(d) Circle Diameters Searching. This consists of searches along diameters of a circle centered at the most probable position of the object, with a constant rotation of the path between successive passes. The circle is taken large enough so there is a sufficiently small chance that the object lies outside the circle. This procedure is considered only for circular normal distributions of errors of estimate of object positions.

7. For the limiting case where navigational errors are small compared to the uncertainty in object position but are large compared to the search-path width W, it is possible to determine an optimum searching procedure for locators that are perfectly maneuverable in the horizontal plane and have a fixed ground speed. For an elliptical normal distribution of object position errors with standard deviations σ_1 and σ_2 the procedure which minimizes both T and τ consists of the following:

(a) Elliptical area searching.

(b) Very light and uniform coverages.

(c) Taking a uniform coverage to consist of parallel passes with the first nominal path lying at one edge of the searched area, and with each subsequent nominal path displaced a constant "lateral distance" d relative to the preceding path until the opposite edge is reached, the semi-major axis b and the semi-minor axis a of the elliptical searched areas must increase on successive coverages in such a manner that at the end of the n'th coverage $a/\sigma_1 = b/\sigma_2 = \sqrt{2n\ W\beta/d}$, where n is any integral number. Here d is assumed to have the same value for each coverage.

8. The other types of searching procedures listed in paragraph 6 necessarily give values of T and τ larger than the theoretical minimum values, but for each type procedure the smallest values of T and τ are obtained by making successive coverages very lightly over areas whose dimensions vary for successive coverages in the optimum manner subject to the limitations imposed by the definitions. The procedures for obtaining the conditionally minimum values of T and τ are designated standard procedures.* Table 1 gives values of P(t) as a function of the searching time t for optimum and standard procedures, and Table 2 gives values of T and τ for location probabilities P(T) of 0.90, 0.95 and 0.99. As an indication of the relative efficiencies of the different

*The actual procedures considered in obtaining the data listed under standard procedures in Tables 1 and 2 differ slightly from those which give the theoretically conditional minimum values of T and τ.

standard procedures, the required value of T for $\Gamma(T) = 0.95$ is 8% larger for rectangular area searching than for the optimum searching procedure, and is larger by 15% and 23% respectively for constant parallel path searching and for circle diameters searching than for the optimum procedures. These comparisons are valid only if the search rate U is constant and is the same for all procedures. The comparisons do not hold, for instance, if a substantial time is required for the locator to turn at the end of a pass.

9. The optimum and standard searching procedures are important mainly as theoretical standards of reference. Of more practical importance is multiarea searching, consisting of only a few uniform coverages with the area for each coverage chosen so as to approximate as nearly as possible the search-path distribution obtained with optimum and standard procedures. Table 4 gives values of T and τ for multiarea searching for values of $\Gamma(T)$ of 0.95 and 0.99 and for any total number of coverages in the range 1 to 6. Also given are the required dimensions of the areas to be searched on successive coverages and the required increase in "search-path density" Δm per coverage, where $\Delta m = W\beta/d$. The magnitude of the required increase in search-path density determines the value of the lateral distance d which is to be used. It is found that with a total of 6 coverages the values of T obtained with multiarea searching differ from the values obtained with the corresponding optimum and standard searching procedures by less than 1.6%, and that the values of τ differ by less than 9%. The increase in searching efficiency with increase in the total number of coverages N is quite limited for values of N greater than 3.

10. The values of T and τ are given in terms of $\sigma_1\sigma_2/U$ and were obtained under the assumption that the search rate U is constant. Whether or not U is constant in carrying out any one of the indicated procedures, the location probability will attain the specified values when Ω, the total aggregate area searched, attains the values given by the table. Hence, the indicated values of T will be valid even when the search rate is not constant if U is interpreted as the time average of the search rate. The indicated values of τ are not exact if the search rate is not constant even if U is interpreted as the average search rate, but the indicated values should then represent reasonably good approximations to τ. When a substantial time is required for the locator to turn at the end of each pass this must be taken into account in determining an average search rate. If the search rate is constant except for the turning time required at the end of each pass, and if the required turning time is constant, then for constant parallel path searching and for circle diameters searching the values of both T and τ are adequately given by Table 4 if U is interpreted to be an average search rate equal to the ratio of the aggregate area $\beta W \ell$ searched on one pass to the time required to make one pass, including the time to make one turn, where ℓ is the length of one path. Constant parallel path searching is about as efficient as any of the other procedures if the time spent on turns represents a substantial part of the total searching time and it appears to be the most practical procedure from an operational standpoint.

11. For the carrying out of and the evaluation of multiarea searching it is unnecessary to maintain the restriction that the width of searched path W be small compared to the navigational error. When W is not small compared to $\bar{\sigma}_1$ the required search-path density $W\beta/d$ per coverage is smaller by a factor $1/\gamma$ than it is when $W/\bar{\sigma}_1$ is small, and for a given search rate the values of T and τ are also smaller by a factor of $1/\gamma$.

Here γ has the same significance as it has in Part I and is given by Plates 2 to 5 of the present report. Its value depends on $W/\bar{\sigma}_1$, β and $d/\bar{\sigma}_1$, but it is never less than 1.0. Because of the inclusion of γ in Table 4 as a coefficient of the quantities Λm, T and τ, the data in the table remain valid when $W/\bar{\sigma}_1$ is not small. Because of the dependence of γ on d, the greatest searching efficiency is obtained in some cases with a total of only one or two coverages. These generalizations do not hold for circle diameters searching and the footnotes to Table 4 should be consulted concerning this procedure.

12. In obtaining the data of Table 4 it was assumed that each coverage is made by making the first pass along an edge of the area to be searched on that coverage and that each subsequent path is offset by a fixed lateral distance d from the previous path until the coverage is completed. When not more than two or three coverages are made it is possible to obtain a substantially smaller value of τ by starting the search on the first coverage along a path which is nearer the most probable position of the object. Table 5 gives values of τ for several modified procedures for making the first coverage. These modifications in the procedure for making the first coverage do not effect the value of T.

13. The data presented in Tables 4 and 5 are valid if the navigational error is not greater than roughly one half the uncertainty in object position; that is, provided $\bar{\sigma}_1 \leq \frac{1}{2}\sigma_1$ and $\bar{\sigma}_2 \leq \frac{1}{2}\sigma_2$. For given $\bar{\sigma}_1$ and $\bar{\sigma}_2$ the required search effort, as indicated by the values of T and τ, is approximately proportional to the product $\sigma_1\sigma_2$ for the range of values of σ_1 and σ_2 satisfying the above relation, provided the time spent on turns represents a relatively small part of the total searching time. When a major part of the total searching time is spent on turns, the magnitude of σ_2 is relatively unimportant compared to that of σ_1. Navigational errors of the order of $\frac{1}{2}\sigma_1$ and $\frac{1}{2}\sigma_2$ or less are of no importance except in so far as the lateral navigational errors effect the value of γ.

Navigational Error of the Same Order as the Uncertainty in the Object Position ($\frac{1}{2}\sigma_1 \leq \bar{\sigma}_1 \leq \frac{3}{2}\sigma_1$ and $\bar{\sigma}_2 \leq \frac{3}{2}\sigma_2$)

14. When the navigational error becomes comparable with the uncertainty in the object position the dispersion of the search paths becomes important. If the navigational error is not too large the effect of this dispersion can be largely overcome in multi-area searching by suitably decreasing the dimensions of the nominal areas to be searched on each coverage. The advantage of a large number of coverages largely disappears when there is substantial navigational error and no more than three coverages are considered for this intermediate range of errors. If the number of coverages is restricted to 3 or less when $\frac{1}{2}\sigma_1 \leq \bar{\sigma}_1 \leq \frac{3}{2}\sigma_1$ and $\bar{\sigma}_2 \leq \frac{3}{2}\sigma_2$, and if, in addition, N is not greater than 2 for elliptical area searching and rectangular area searching when either $\bar{\sigma}_1 > \sigma_1$ or $\bar{\sigma}_2 > \sigma_2$, then the efficiency of multiarea searching is either very nearly equal to or is greater than that indicated by Table 4 if the following modifications are made to the search procedure specifications of that table:

(a) Take the over-all width of the area to be searched on the final or Nth coverage to be less than that specified in the table by an amount $\frac{3}{2}\bar{\sigma}_1$.

(b) Take the over-all length of the area to be searched on the N'th coverage to be less than that specified in the table by an amount $\frac{3}{2}\bar{\sigma}_2$.

(c) Reduce the dimensions of the area to be searched on the n'th coverage below the values specified in the table in proportion to the reduction of the dimensions for the N'th area.

(d) Adjust the value of the increase in search-path density Δm per coverage so as to maintain the total amount of searching specified in Table 4.

15. It is not possible to carry out multiarea searching in the prescribed manner when the over-all width of path W is too large. If W is greater than $5(\sigma_1 - \bar{\sigma}_1/2)$ when the required P(T) is 0.95, or if W is greater than $6(\sigma_1 - \bar{\sigma}_1/2)$ when the required P(T) is 0.99, a more practical procedure than multiarea searching, and one giving near maximum efficiency, is to make all passes on the same track with nominal center-line path coinciding with the y-axis. This type of searching procedure, with all passes on the same track and with all nominal paths of the same length and centered at the most probable position of the object, is called "single track" searching. The appropriate length of the search paths depends on the relative magnitude of the time required for the locator to turn at the end of each pass. For negligible or moderate values of the required turning time, Table 7 gives near optimum values of the lengths of search paths and the number of required passes, and also values of T and τ, for P(T) = 0.95 and P(T) = 0.99. For sufficiently large values of the turning time, it is more efficient to choose the path length so large that there is negligible chance of missing an object because of turning too soon. Table 8 gives the required number of passes, and values of T and τ, for this case. Tables 7 and 8 include data for the case where the navigational errors are quite small as well as for the case where these errors are comparable to the uncertainty in the object position.

Navigational Errors Substantially Larger than the Uncertainty in the Object Position ($\bar{\sigma}_1 \geq \frac{3}{2}\sigma_1$)

16. When $\bar{\sigma}_1$ is greater than $\frac{3}{2}\sigma_1$ the most efficient searching procedure is to make all passes so that the nominal center line of each search path falls on the most probable position of the object, and single track searching is quite satisfactory for this case. Tables 9 and 10 give appropriate search data for the case where $\bar{\sigma}_1 \geq \frac{3}{2}\sigma_1$ and $\bar{\sigma}_2 \leq \frac{3}{2}\sigma_2$, and Tables 10 and 11 give the necessary data for the case where $\bar{\sigma}_1 \geq \frac{3}{2}\sigma_1$ and $\bar{\sigma}_2 \geq \frac{3}{2}\sigma_2$. The most suitable choice of length of searched path depends on the time, if any, which is required to make a turn, and can be adequately estimated from the information in the tables. The theoretical effectiveness of circle diameters searching is always equal to or greater than single track searching when the errors of estimate of object position have a circular normal distribution and when navigational errors are independent of the direction of the paths. However, the difference in searching efficiency for these two procedures is not expected to be large, and the data presented in Tables 7 to 11 should be adequate for circle diameters searching under all conditions where both single track searching and circle diameters searching are appropriate procedures.* An example of zero uncertainty in object position (i.e., $\bar{\sigma}_1/\sigma_1 \longrightarrow \infty$, $\bar{\sigma}_2/\sigma_2 \longrightarrow \infty$) is represented by a locator which is guided by sonar on the basis of observed relative positions of the locator and of the object being investigated.

*Circle diameters searching may be substantially more efficient if there is large variation in detection probability with object orientation.

Analysis of More Complex Situations

17. When following the "regular" searching rule which is the basis for most of the analysis of the present report, searching is assumed to be discontinued as soon as an object or apparent object is located or at a specified time T, whichever occurs first. If there is clutter in the area which gives locator signals that are indistinguishable from the object signals, then in following the regular rule the locator search will sometimes be discontinued due to a contact from a locator false target even though the contact being investigated is an object. Hence, if there are locator false targets in the area of operations, the ratio of the expected number of located objects to the number investigated is equal to some quantity $P_e(T)$ which is less than $P(T)$. Then $P_e(T)$ is called the effective location probability.

18. If there is a large locator false-target density it may not be possible to obtain adequate values of $P_e(T)$ with the regular searching rule. Regardless of the false-target density, any desired value of $P_e(T)$ can be obtained if searching always continues for the full time T, since then $P_e(T) = P(T)$. But the average searching time is then T, a relatively large quantity, and the expected number of located false targets may also be large. For a certain intermediate range of false-target densities it is possible to obtain adequate values of $P_e(T)$ by a "modified" searching rule which gives some discrimination by taking advantage of the fact that the relative chances of discovering the object and of discovering a false target during a small time interval of given length is substantially more in favor of the object in the early part of a search than it is for later stages of a search. The modified searching rule consists of a search for some minimum time t_1, regardless of how many targets are located during this time, and an additional search subsequent to time t_1 if, and only if, no target was located during t_1. Any searching subsequent to t_1 is discontinued when a target is located or at time T, whichever occurs first.

19. Table 12 gives approximate values of $P_e(T)$ for the regular searching rule and Table 13 gives values for the modified rule for suitable ranges of the parameters. The values of $P_e(T)$ depend on the quantity $\eta\sigma_1\sigma_2$ where η is the number of locator false targets per unit area of the channel or of the area being cleared. Also given are values of the average searching time $\bar{t}$ and of the number n_f of false targets located per contact investigated by the locator. The quantity n_f is of some importance for the further steps in salvage of objects. Thus, if an attempt is made to salvage each target located, the number n_f represents the number of wasted salvages per contact investigated by the locator. If each object-like target detected by the locator is reinvestigated by an even more discriminating locator or by divers, then n_f determines the added searching effort on reinvestigation due to locator false targets on the initial investigation. The data in Tables 12 and 13 are exact only for the optimum searching procedure under certain limiting conditions, but in the form in which they are presented they should be reasonably adequate for any of the searching procedures considered in this report, provided $\bar{\sigma}_1 \leq \frac{3}{2}\sigma_1$. In order to obtain the necessary generality, the data are given in terms of certain parameters whose values can be obtained from previous tables. The data in Tables 12 and 13 are directly

applicable to the case where any clutter contacts which are investigated by the locator are not locator false targets, but it is shown how the data can be given a more general interpretation if the necessary information on the nature of the clutter is available.

20. Additional problems which have been considered include the possible variation in the detection probability β from object to object, the possibility that β may be zero for some objects, and the use of two locators of different types to investigate a contact in order to obtain sufficient assurance of detecting the target if it is an object. It is shown that the results of the analysis which were based on somewhat idealized assumptions can readily be extended to cover reasonably adequately these more complex situations.

INTRODUCTION

21. Narrow-path locators have been considered especially for pin pointing an object and for identifying it once its apparent presence and approximate position have been determined by intelligence or by high search-rate detectors. However, under some conditions it may be desirable to use narrow-path locators also for general area or channel searching. Corresponding to these two possible general uses for locators, the analysis of searching has been carried out in two parts. Part I of this study, consisting of the analysis of general area searching, is presented in reference (1). Part II, dealing with the problem of accurately locating an object whose approximate position is known, is presented in the present report.

22. For the analysis in Part I it was possible to obtain results of sufficient generality to be applicable to large classes of locators and detectors. The results of that analysis were presented in terms of several operating and equipment parameters whose values probably can be determined or adequately estimated for given operating conditions. The determination and evaluation of suitable procedures for locating objects whose apparent presence and approximate position are known requires a more detailed specification of locator characteristics.

23. The analysis of Part I is concerned with detection only, and is equally applicable to the case where the locator used for area searching is also capable of pin pointing and identifying an object and to the case where it is not. For the analysis of Part II it is assumed that the initial detection has been accomplished previously by wide-path detectors, by intelligence, or by other means, and that subsequent to the initial detection, searching is required by narrow-path locators for accurately fixing the position of and for identifying an object or presumed object as an intermediate step in the salvage of the object. It will be assumed initially that the narrow-path locators have sufficient discrimination to permit a definite conclusion as to when an object has been located. This implies that either the locator is capable of classifying an object which it detects, or that an adequate classifier is immediately available as part of the locator system. More precisely, what is assumed for the initial part of the analysis is that there is no loss of locator searching time due to the necessity of identifying locator false contacts, and that the search for an object can be discontinued as soon as it is contacted. Extension of the results of the initial analysis to cases where these assumptions are not valid is considered in a later section.

FORMULATION OF THE PROBLEM

24. It is assumed in the present study that it is possible to estimate a most probable position for any object to be located and that from past experience or from the nature of the operation by which objects are initially detected the distribution of errors of estimate of the position of objects is known. Consider a system of horizontal rectangular coordinates x,y with origin at the most probable position of the object. Assuming the contact which is being investigated is due to the object, let $f(x, y)\,dx\,dy$ be the probability, before the search begins, that the object is located within an infinitesimal

rectangle of dimensions dx and dy at point (x,y). Conforming to the usual convention f(x,y) may be called the probability density of the object position. To avoid confusion this term will be applied only to the probability distribution of the object position before any searching by locators has taken place and, to emphasize this, it will sometimes be referred to as the initial probability density.

25. Suppose that the locator searches in the vicinity of the estimated object position according to some specified search program and suppose, for the purpose of analysis, that searching is continued until time T regardless of whether or not the object is located before that time. For the specified search procedure, assumed to be carried out until time T, let $\rho(x,y,t)$ be the probability that if the position of the object is at point (x, y) it will have been detected by time t, where t is any searching time less than or equal to T. The quantity $\rho(x,y,t)$ will also be said to be the probability that the point (x,y) is effectively searched in time t. The probability that the object was initially at (x,y) within the area dx dy and is detected by the time searching has been carried out for a time t is $f(x,y)\ \rho(x,y,t)\,dx\,dy$. Then the probability P(t) that the object is discovered during the part of the search which is carried out in time t is

$$P(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ \rho(x,y,t)\,dx\,dy. \tag{1}$$

26. It has been assumed that there is uncertainty concerning the position of the object, but not about its existence. In practice there may always be some chance that the signal by which the object is initially spotted is a false signal. Suppose that from a statistical study of similar previous contacts it is determined that there is a probability ξ, less than 1.0, that there actually is an object. There is then the practical necessity of determining when the area has been sufficiently thoroughly searched to obtain a sufficiently large value for the probability that the signal by means of which the object was initially spotted was a false signal. The probability of finding an object, if there is one, and assuming there is not more than one, is P(T) whether the search is always continued to time T or is continued to time T only when the object is not found before time T, the search being discontinued otherwise when the object is found. For any contact which is being investigated, the probability that it is due to the object and that the object remains undetected by the locator at the end of the search is $\xi\{1 - P(T)\}$. For a given T the value of this expression decreases with decrease in ξ. However, the number of contacts which must be investigated is greater than the salvageable contacts by a factor $1/\xi$. Hence, the fraction of objects which are not detected by the locator is $(1/\xi)\ \xi\{1 - P(T)\}$ or $1 - P(T)$. Thus the probable number of objects located represents a fraction P(T) of the real object contacts of the wide-path locator or other system used in the initial detection. It will be convenient to refer to P(T) as the location probability and to refer to T as the searching time.

27. A primary objective in choosing suitable search procedures is to obtain small values of the time required to either locate an object or to obtain sufficient assurance that the contact which is being investigated does not represent an actual object. If, as assumed, searching is terminated as soon as the object is found, or at the end of the period T which is required to obtain a specified value of the location probability, whichever occurs first, one can consider separately the case where the contact which

is being investigated represents an actual object and the case where it is a false signal. For the latter case the searching time per contact is T. For any case where the contact represents a real object, the probability that an object is found in the interval t_1 to $t_1 + dt$ is $\{(\partial P/\partial t)_{t=t_1}\}dt$. Hence, the mathematical expectation of the searching time in those cases where objects exist is

$$\tau = \int_0^T t \frac{\partial P}{\partial t} dt + \{1 - P(T)\}T. \tag{2}$$

Integrating by parts, this reduces to

$$\tau = T - \int_0^T P(t)\, dt = \int_0^T \{1 - P(t)\}dt. \tag{3}$$

Since τ represents the average or expected search time for cases where an object exists it will be referred to as the average discovery time. However, the term average discovery time does not describe τ exactly, since even for those cases where an object exists, the average searching time includes the term $\{1 - P(t)\}T$ to take account of the fraction of cases where the object is not found even though it exists. For the assumed searching procedure the over-all average searching time $\bar{t}$ is

$$\bar{t} = \mathcal{E}\tau + (1 - \mathcal{E})T. \tag{4}$$

28. If the origin of the x,y-coordinate system is taken at the most probable position of the object, the actual position will normally be at some point (x,y) not at the origin. In the most general case which will be considered it is assumed that the x and y-axes can be chosen so that the errors of estimate of the x-coordinate of the object position are independent of the errors of estimate in the y-coordinate, and that the errors of estimate of each of these coordinates have a normal distribution. Let σ_1 be the standard deviation of the errors of estimate of the x-coordinate of the object and let σ_2 be the standard deviation of the errors of estimate of the y-coordinate. It will be assumed that the x and y-directions are chosen so $\sigma_2 \geq \sigma_1$. Then the probability that the x-coordinate of the object lies between x and x + dx is

$$\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-x^2/2\sigma_1^2} dx$$

and the probability that its y-coordinate lies between y and y + dy is

$$\frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-y^2/2\sigma_2^2} dy.$$

The probability f(x,y) dx dy that the object lies in the area dx dy centered at (x,y) is the product of these quantities, giving

$$f(x,y)\, dx\, dy = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} dx\, dy. \tag{5}$$

Methods of determining the required x and y-directions, and the magnitudes of σ_1 and σ_2, are discussed in Appendix E. A distribution of the type represented by equation (5) will be referred to as an elliptical normal distribution. When $\sigma_1 = \sigma_2$, the function f(x,y) represents a circular normal distribution given by

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$$

where σ_1 and σ_2 have been replaced by σ.

29. Substituting in (1) the value of f(x,y) given by (5), the expression for P(t) for the most general case becomes

$$P(t) = \frac{1}{2\pi\,\sigma_1\,\sigma_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} \rho(x,y,t)\,dt. \tag{6}$$

In the special case where the errors of estimate of object position have a circular normal distribution and where also $\rho(x,y,t)$ is a function of r and t only, where $r = \sqrt{x^2 + y^2}$, the expression for P(t) can be simplified by changing to polar coordinates to obtain

$$P(t) = \frac{1}{\sigma^2} \int_0^{\infty} e^{-r^2/2\sigma^2} \rho(r,t)\ r dr. \tag{7}$$

30. Using the same terminology and symbols as in Part I, let W be the over-all width of searched path of a locator and let β be the probability of detecting an object at random location within W. The over-all width W may be defined as the minimum width which includes all athwartship positions in the locator search path for which there is some chance of detecting an object which may be in that position. The aggregate width of searched path w_a of the locator is defined as the product $W\beta$. The product of the aggregate width of searched path w_a and of the total length of path searched in time t will be called the aggregate searched area Ω or $\Omega(t)$, at time t.

31. Suppose that in searching for an object the attempt is made to move the locator along any prescribed path or paths. The actual search paths will differ from the paths which it is attempted to search because of navigational error. The paths which it is attempted to search will be called the nominal search paths. In the following it is assumed that both the estimated most probable position of the object and the position of the nominal paths are given with respect to the same reference system, and that this reference system is that used by the locator in determining its position at any time in attempting to search prescribed paths. If the probable position of the object was initially determined relative to a different system, then it is necessary to determine the position of this point in terms of the coordinates of the navigation system being used. This transformation from one reference system to another will normally introduce some error and this must be taken into account in determining the values of σ_1 and σ_2. This problem is discussed in Appendix E.

32. To determine $P(t)$ as a function of t by means of equation (6) or equation (7), and also to determine T and τ for given $P(T)$, it is necessary to know the function $\rho(x,y,t)$ or $\rho(r,t)$. In general, ρ depends on the characteristics of the locator, including the values of the parameters W and β, on the accuracy with which the locator can navigate, and on the search procedure. A major part of the analysis in the present report is concerned with the determination first of $\rho(x,y,t)$, and then of T and τ, for specified values of $P(T)$, as functions of the locator parameters and of the navigational accuracy for various searching procedures. Principal consideration is given to searching procedures which appear to be operationally feasible and which give near maximum searching efficiency. As a first step in determining practical searching procedures for obtaining minimum or near minimum values of T and τ, special cases will be considered for which the mathematical analysis is greatly simplified. It will be convenient to consider first the case where navigational errors are small compared to the uncertainty in the position of the object, but are large compared to the width of searched path of the locator.

NAVIGATIONAL ERRORS OF THE LOCATOR SMALL COMPARED TO THE UNCERTAINTY IN THE POSITION OF THE OBJECT BUT LARGE COMPARED TO THE OVER-ALL WIDTH OF SEARCHED PATH OF THE LOCATOR

33. To develop some necessary ideas consider first a uniform search of an area of given constant width in nominal paths parallel to the edges of this area with each nominal path displaced a constant distance d from the position of the preceding or adjacent path. The distance d is called the lateral distance. For each strip of area of width d there is one nominal search path with aggregate width of searched path w_a equal to $W\beta$. Let $m = w_a/d = W\beta/d$. The quantity m is called the average nominal search-path density. Consider the limiting case of very small over-all width W. For a given average nominal search-path density the distance d approaches zero as W approaches zero, and the number of nominal search paths becomes infinite. Let x be the distance of a point from one edge of the nominally searched area. Then for any given m the number of nominal search paths lying in any infinitesimal width Δx between x and $x + \Delta x$ becomes very large as W approaches zero. Suppose now that the lateral distance d varies with x in any continuous manner. Because of the assumed continuity, the ratio of the aggregate width of nominally searched area within Δx to the width Δx approaches a limit as Δx approaches zero. This limit will be called the nominal search-path density at x and can be represented by $m(x)$. More generally consider any distribution of search paths not necessarily parallel to each other but such that the ratio of nominal aggregate searched area in any infinitesimal area ΔA centered at a point (x,y) to the value of ΔA approaches a limit equal to $m(x,y)$ as ΔA approaches zero. Then $m(x,y)$ is the nominal search-path density at (x,y) and is assumed to vary continuously with x and y except, at most, at a finite number of boundaries.

34. The search-path density $\bar{m}(x,y,t)$ at the point (x,y) is defined as the mathematical expectation of the number of times the point is effectively searched in time t. It is

shown in Appendix A that for the assumed small search width W the probability $\rho(x,y,t)$ that a point (x,y) is effectively searched at least once during time t is given by

$$\rho(x,y,t) = 1 - e^{-\bar{m}(x,y,t)}. \quad (8)$$

It is also shown that if the variation in nominal search-path density m(x,y,t) is small over every area whose dimensions are of the order of the standard deviation of navigational error, the search-path density may be taken equal to the nominal search-path density, or

$$\bar{m}(x,y,t) = m(x,y,t).$$

For the searching procedures which will be considered, this condition will be satisfied except, possibly, at a finite number of boundaries, provided the navigational error is small compared to the uncertainty in the object position. Under conditions where the search-path density is equal to the nominal search-path density, m(x,y,t) will be referred to simply as the search-path density.

35. For evaluating searching procedures it is desirable to have some standard for comparison. To obtain such a standard consider the idealized problem of determining the search procedure which minimizes both the overall searching time T and the average discovery time τ, assuming there is a procedure which gives a minimum value for both of these times, for the following assumed conditions:

(a) The locator is perfectly mobile and can follow any prescribed horizontal path within the limits of navigational error which are assumed to be negligible compared to the uncertainty in position of the object.

(b) The water in which the search is conducted is stationary.

(c) The locator has a constant water speed independent of the shape of the prescribed paths.

(d) The over-all width of searched path W is vanishingly small compared to the navigational errors.

(e) The errors of estimate of the position of the object have an elliptical normal distribution with standard deviations σ_1 and σ_2.

36. It may reasonably be expected that the most effective searching procedure is one for which searching is carried out at all times at locations where the object is most likely to be, taking account both of the initial probability distribution of the object position and of the previous searching for the object. On this basis, searching should begin at the most probable position of the object. After carrying out some searching in a small area centered at that point the probability p(x,y) dx dy that the object was initially within a small rectangle dx dy centered at a point (x,y) within this area and remains undetected is

$$p(x,y)\,dx\,dy = f(x,y)\,\{1 - \rho(x,y,t)\}\,dx\,dy.$$

Within a short time the value of p(x,y) at (0,0) would be reduced to the value of the initial probability density f(x,y) at points in adjacent areas. It can be seen that searching at any time would be limited to a central area, the value of p(x,y) being the same for all points of the central area. If the value of p(x,y) were larger at some points of this area than at other points, then, by the assumption that searching be carried out at points where the object is most likely to be, it would be required that searching be carried out at points in that area where p(x,y) was largest until $\rho(x,y,t)$ had been increased sufficiently at those points to reduce the value of p(x,y) there to its value at other points.

37. Consider first the special case where the errors of estimate of object position can be represented by a circular normal distribution. Since no searching procedure will locate an object if none exists, it will be sufficient to consider the case where there is an object to be located. Making use of (5) and (8), with $\bar{m}(x,y,t)$ replaced by m(x,y,t), one obtains for a circular normal distribution of errors of estimate of object position

$$p(x,y) = f(x,y)\ \{1 - \rho(x,y,t)\} = \frac{1}{2\pi\sigma^2}\, e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}\, e^{-m(x,y,t)}.$$

In order for this to be constant over a central area it is necessary that for all points of this area

$$\frac{x^2 + y^2}{2\sigma^2} + m(x,y,t) = k \tag{9}$$

where k is a function of t only. Let U represent the increase in aggregate searched area per unit time. This quantity will be referred to as the search rate. If V is the searching speed, then $U = w_a V$. The total aggregate area searched in time t is $\Omega(t) = Ut$. The aggregate area searched in time t is also given by the surface integral $\int m(x,y,t)dA$, where the integral extends over all parts of the central area in which searching has taken place. It is apparent that, for the assumed circular symmetry of the distribution of the object position, m(x,y,t) must also be circularly symmetric and can be represented by m(r,t). Then for determining k one obtains by substituting for m from (9)

$$Ut = \int_0^R \left(k - \frac{r^2}{2\sigma^2}\right) 2\pi r dr,$$

where R is the radius of the boundary of the central searched area. Here R is the value of r for which m(r,t) is zero. Replacing $x^2 + y^2$ by R^2 in (9) and setting $m = 0$ gives $R = \sqrt{2k}\,\sigma$. Carrying out the integration and solving for k gives

$$k = \sqrt{\frac{Ut}{\pi\sigma^2}}.$$

Then (9) can be written in the form

$$m(r,t) = \sqrt{\frac{Ut}{\pi\sigma^2}} - \frac{r^2}{2\sigma^2}. \tag{10}$$

In terms of R the equation is

$$m(r,t) = \frac{R^2 - r^2}{2\sigma^2}. \tag{11}$$

This gives the required nominal search-path density inside the circle of radius R, where

$$R = \sigma\left(\frac{4Ut}{\pi\sigma^2}\right)^{1/4}. \tag{12}$$

Outside this circle there is no searching previous to time t, so m(r,t) is zero outside this circle.

38. Equation (10) represents a paraboloid of revolution with vertical axis as the axis of symmetry, and with the vertex at the highest point of the paraboloid with coordinates $m = \sqrt{UT/\pi\sigma^2}$, $r = 0$. The quantity m is represented only by the part of the paraboloid which is above the x,y plane. The paraboloid is shifted upward as searching continues. In order to maintain the given distribution, searching must be carried out very lightly and uniformly over the central circular area, this area being increased on each successive coverage sufficiently to give the necessary increase in R, with increase in the distance $\sqrt{Ut/\pi\sigma^2}$ of the vertex above the x,y plane, to maintain the relation (12). Each coverage increases the value of m at any point within the searched area by an infinitesimal amount Δm, where $\Delta m = w_a/d = W\beta/d$. Then taking the differential of both sides of (11), with r assumed to have a fixed value, it is found that if R is the radius of the searched area on one coverage the radius for the next coverage is $R + \Delta R$ with $\Delta R = \sigma^2 \Delta m/R$. That this search procedure does minimize the average discovery time τ and that it gives the required search-path distribution to maximize P(t) for any searching time t is shown more rigorously in Appendix B. The value of m(r,t) given by (10) can be substituted for $\bar{m}(x,y,t)$ in (8) to obtain $\rho(r,t)$, and the latter quantity can then be substituted in (7) to obtain the maximum value of P(t) for any given t, or to find the minimum value of t required to obtain a given value for P(t). Substituting the resulting value of P(t) from (7) into (2) or (3) gives the minimum value of τ for given T or given P(T). The following expressions are obtained for P(t) and τ for the optimum searching procedure and a circular distribution of errors of estimate of object position.

$$P(t) = 1 - \left\{1 + \left(\frac{Ut}{\pi\sigma^2}\right)^{1/2}\right\} e^{-\left(\frac{Ut}{\pi\sigma^2}\right)^{1/2}} \tag{13}$$

$$\frac{\tau}{\left(\frac{\sigma^2}{U}\right)} = 2\pi\left[3 - \left\{3 + 3\left(\frac{UT}{\pi\sigma^2}\right)^{1/2} + \frac{UT}{\pi\sigma^2}\right\} e^{-\left(\frac{UT}{\pi\sigma^2}\right)^{1/2}}\right] \tag{14}$$

39. The results obtained for circular normal distributions of errors of estimate of object position can readily be extended to more general elliptical distributions of errors

of estimate. It is shown in Appendix B that for this more general case the optimum searching procedure is one in which successive elliptical areas are searched very lightly, the semi-axes a and b of the ellipse being searched at any time t being given by the relation

$$\frac{a}{\sigma_1} = \frac{b}{\sigma_2} = \left(\frac{4Ut}{\pi\sigma_1\sigma_2}\right)^{1/4}.$$

The distribution of search-path density at any time t is given for points inside the ellipse of semi-axes a and b by the equation

$$m(x,y,t) = \sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}} - \left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right).$$

This is the equation of an elliptical paraboloid. It is found that P(t) and τ are then given by (13) and (14) if σ^2 is replaced in those equations by $\sigma_1\sigma_2$. The second column of Table 1 gives values of the location probability P(t) obtained with the optimum searching procedure for values of the searching time t given by the first column. Table 2 gives values of the average discovery time τ and of the overall searching time T for location probabilities P(T) of 0.90, 0.95 and 0.99, and also the value of τ corresponding to P(T) = 1.0. The values of t, T, and τ in Tables 1 and 2 are given in terms of the quantity $\sigma_1\sigma_2/U$.

40. The elliptical area searching procedure described above represents the optimum procedure, in the sense of minimizing T and τ, only under the conditions specified in paragraph 35. In particular, it has been assumed that the ground speed of the locator is constant. However, the location probability depends only on the searches which are made, and not on when they are made. That is, the value of the location probability depends only on the value of the aggregate area Ω searched, assuming only that the distribution of the search paths is optimum. On the other hand, the average discovery time τ does depend on when the different search passes are made, and the values given for τ are valid in general only if U is constant. For instance, if the search speed is smaller at the initial part of the search than it is during later stages of the search, the average discovery time will be greater than is indicated by Table 2.

41. For actual locators it is to be expected that not all of the assumptions listed in paragraph 35 are valid. In particular, the assumptions that the locator is perfectly mobile and that its searching speed does not depend on the path it must follow are not likely to be valid for actual locators. Thus in considering how successive uniform coverages are to be obtained for circular areas the most likely procedures are searching in successive circular spirals or in parallel straight paths. But some locators may not be able to search along small circular spirals, and the search speed may depend on the curvature of the path which must be followed. For searching in straight parallel paths, some extra time is likely to be required at the turns, and the determination of when the proper turning point had been reached would likely be a tedious problem. For determining more practical searching procedures it will be useful to first evaluate some additional searching procedures on the basis of the assumptions of paragraph 35. The detailed analysis of these procedures is presented in Appendix B. Following is a brief description of the procedures considered.

TABLE 1

Values of the Location Probability P(t) for Given Values of the Time t for Optimum and Standard Searching Procedures under Conditions where the Navigational Errors are Small Compared to the Uncertainty in the Position of the Object but are Large Compared to the Over-all Width of Searched Path W. (Errors in estimating object position are assumed to have an elliptical normal distribution with standard deviations σ_1 and σ_2 except that for circle diameters searching the distribution is assumed to be circular normal with $\sigma_1 = \sigma_2 = \sigma$.)

$\frac{t}{\frac{\sigma_1\sigma_2}{U}}$ (also $\frac{\Omega}{\sigma_1\sigma_2}$)	P(t)			
	Optimum Elliptical Area Searching	Standard Rectangular Area Searching	Standard* Constant Parallel Path Searching	Standard** Circle Diameters Searching
2	0.190	0.188	0.166	--
4	0.311	0.307	0.209	--
8	0.474	0.466	0.352	0.332
12	0.581	0.571	0.461	--
16	0.659	0.648	0.545	0.525
24	0.763	0.751	0.670	--
32	0.828	0.816	0.754	0.735
48	0.902	0.891	0.856	--
64	0.940	0.931	0.911	0.901
96	0.974	0.969	0.962	0.957
128	0.988	0.984	0.981	0.978
192	0.996	0.995	0.992	0.991
256	0.999	0.998	0.994	--

U = Search rate or aggregate area searched per unit time.
Ω = Aggregate area searched at the time when P(t) has indicated values.

*Note 1: - Ends of paths in constant parallel path searching are assumed to lie on lines whose separation is such as to give a probability of 0.995 that the object lies in the area between them (i.e., length of paths = $5.61\,\sigma_2$).

**Note 2: - Bounding circle of searched area in circle diameters searching is assumed large enough to give a probability of 0.995 that the object lies within this circle (i.e., radius of circle = $3.25\,\sigma$).

TABLE 2

OPTIMUM AND STANDARD SEARCHING PROCEDURES: Values of the Overall Searching Time T and of the Average Discovery Time τ Corresponding to Given Values of the Location Probability P(T) for the Case where Navigational Errors of the Locator are Small Compared to the Uncertainty in Object Positions but are Large Compared to the Over-all Width of Searched Path W. (Errors of estimating object position assumed to have an elliptical normal distribution with standard deviations σ_1 and σ_2, except that, for circle diameters searching, the distribution is assumed to be circular normal with $\sigma_1 = \sigma_2 = \sigma$.)

Search Procedure	P(T) = 0.90			P(T) = 0.95			P(T) = 0.99			P(T) = 1.0	
	$\frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$	$\frac{\tau}{t}$	$\frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$	$\frac{\tau}{t}$	$\frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$	$\frac{\tau}{t}$	$\frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$
Optimum Elliptical Area Searching	47.5	15.02	0.316	70.7	16.68	0.236	138.4	18.30	0.132	∞	18.8
Standard Rectangular Area Searching	50.8	15.84	0.312	76.2	17.66	0.232	152.0	19.47	0.128	∞	20.1
* Standard Constant Parallel Path Searching	53.6	17.74	0.331	81.5	20.70	0.254	166.7	25.56	0.153	--	--
** Standard Circle Diameters Searching	64.0	22.15	0.346	92.0	24.29	0.264	185.0	26.11	0.141	--	--

U = Search rate, or the aggregate area searched per unit time.

*Note 1: - The data for constant parallel path searching are based on values of the path length ℓ of 4.5 σ_2, 5 σ_2, and 6 σ_2 for values of P(T) equal to 0.90, 0.95, and 0.99, respectively.

**Note 2: - Bounding circle of searched area in circle diameters searching is assumed large enough to give a probability of 0.995 that the object lies within this circle (i.e., radius of circle = 3.25 σ).

42. Standard Rectangular Area Searching. This procedure consists of making light uniform searches over rectangular areas centered at the most probable position of theobject, the dimensions of the successive rectangles being determined so as to obtain near optimum variation of search-path density. The simplest method of obtaining uniform coverages is to make all locator passes parallel to one side of the rectangle with a constant lateral distance d between adjacent paths on any one coverage. With this type of searching it is not possible to obtain a constant value of the quantity $p(x,y)$, defined in paragraph 36, within the searched area. But by properly choosing the dimensions of the successive searched rectangular areas it is possible to maintain a constant value of this function along the coordinate axes, provided the latter are taken normal to the sides of the rectangles. Based on this search procedure the search-path density $m(x,0,t)$ along the x-axis in the range $x = 0$ to $x = X$ is

$$m(x,0,t) = \frac{X^2 - x^2}{2\sigma_1^2}$$

and is zero for $x \geq X$, where X is one half the length of one of the smaller sides of the rectangle which is being searched at time t. Similarly along the y-axis, for $y < Y$ the search-path density is

$$m(0,y,t) = \frac{Y^2 - y^2}{2\sigma_2^2}.$$

The rectangle which is being searched at time t has dimensions 2X and 2Y given by

$$\frac{2X}{\sigma_1} = \frac{2Y}{\sigma_2} = 2\left(\frac{Ut}{\sigma_1\sigma_2}\right)^{1/4}.$$

Table 1 includes data on the location probability as a function of the searching time for this searching procedure, and Table 2 includes values of the overall searching time T and of the average discovery time τ for several values of the location probability $F(T)$. From a comparison of these results with those obtained for optimum elliptical area searching it is seen that for location probabilities in the range 0.90 to 0.99 the values of T obtained with rectangular area searching are from 7% to 10% larger than the values obtained with elliptical area searching and the values of τ are roughly 6% larger.

43. Standard Constant Parallel Path Searching. This procedure consists of searching in parallel paths of constant length ℓ, the ends of the paths terminating on two lines which are normal to the direction of the search paths and which are on opposite sides of and equidistant from the most probable position of the object. Searching is carried out so as to minimize T and τ subject to the above limitations. It is found that the optimum procedure consists of light uniform searches of rectangular areas centered at the most probable location of the object and with sides parallel and perpendicular to the x and y-axes, the width of each searched rectangle being slightly larger than the previous one. It is assumed that all paths are parallel to the y-axis, which is the axis of greatest uncertainty in the object position, in accordance with the assumption $\sigma_2 \geq \sigma_1$.

When searching is carried out in the optimum manner the width of the rectangle which is being searched at any time t is 2X where

$$X = \sigma\left(\frac{3}{2}\ \frac{Ut}{\ell\sigma_1}\right)^{1/3}.$$

The resulting search-path density m(x) at any time t is given for values of x less than X and for values of y less than $\ell/2$ by

$$m(x) = \frac{X^2 - x^2}{2\sigma_1^2}.$$

44. Data for constant parallel path searching are included in Tables 1 and 2. The data in Table 1 are for a value of ℓ equal to $5.6\,\sigma_2$. For this value of ℓ there is a probability of 0.995 that if there is an object, it lies between the two parallel lines on which the paths terminate. This choice of ℓ is not necessarily optimum. The optimum value of ℓ depends on the required value of P(t) and different values of ℓ are required to minimize T and τ. The values of ℓ used for the data in Table 2 are indicated in the table. These values of ℓ are shown in Appendix B to give near minimum values of T and τ when there is zero turning time, and to be adequate also when there are relatively long turning times. The values of T and τ obtained with constant parallel path searching are seen to be substantially larger than the values obtained by standard rectangular area searching. This assumes, however, that the search rate U is the same for both searching procedures. For some locators there are likely to be relatively large turning times at the end of each search pass. The assumption of a constant search speed is then not valid, but for constant parallel path searching the numerical data contained in the tables will remain valid if U is interpreted as the average search rate based on a ground speed equal to the ratio of the length of a search path to the time required to make a pass, including the turning time. Thus, it is shown in Appendix B that for long turning times the efficiency of the constant parallel path procedure may be equal to or greater than the rectangular area searching procedure. In addition, the former procedure appears to be simpler to carry out.

45. Standard <u>Circle Diameters Searching Procedure</u>. This procedure is considered only for circular normal distributions of the errors of estimate of object position. It consists of searches along diameters of a circle which is centered at the most probable position of the object. The diameter along which any pass is made is rotated by a specified angle in a specified direction relative to the diameter along which the next previous pass was made. This procedure may have the advantage of being relatively easy to carry out under some conditions, and it may be relatively efficient when there are substantial turning times. It is possible, also, that the detection probability for some locators may depend on the relative object orientation and that it is desirable to approach the object from various directions. Results for this procedure are given in Tables 1 and 2 based on a circle diameter of $6.5\ \sigma$. For this value of the diameter there is a probability of 0.995 that the object is located within the circle, assuming there is an object. The values of T and τ obtained by this procedure are seen to be larger than the values obtained for constant parallel path searching. For this procedure also

a finite turning time can be taken into account in determining T and τ by taking U to be an average search rate, based on an average ground speed equal to the ratio of the length of a diameter to the total time required both to make a pass along one diameter and to make one turn.

46. It is seen that for values of P(T) of 0.9 or greater the average discovery time τ is much smaller than the overall searching time T, the value of the ratio τ/T being of the order of 1/3, 1/4 and 1/7, respectively, for values of P(T) of 0.90, 0.95 and 0.99. The value of τ increases only slowly with P(T) and is only about 20% greater for P(T) = 0.99 than it is for P(T) = 0.90. Thus when there is a high probability that any contact which is being investigated is the object the average searching time, $\xi\tau + (1 - \xi)T$, for a large location probability is not much larger than for obtaining a location of .90, say, but when there is a large probability that any contact which is being investigated is not the object the average searching time, which then depends mainly on T, may be of the order of three times as large for P(T) = 0.99 as for P(T) = 0.90. Although these generalizations are based on the optimum and standard searching procedures it will be seen that they are largely valid also for the more general procedures to be considered in the following sections.

47. The values of the parameters $\frac{T}{\sigma_1\sigma_2/U}$ and $\frac{\tau}{\sigma_1\sigma_2/U}$ given in Tables 1 and 2, and in all other tables of this report, are based on a consistent set of units for T, τ, σ_1, σ_2 and U. For practical purposes it is probably more convenient to express these quantities in units which are not consistent. Let $\left(\frac{t}{\sigma_1\sigma_2/U}\right)_0$ be the value of $\frac{t}{\sigma_1\sigma_2/U}$ for any consistent set of units. Then if σ_1 and σ_2 are given in yards, w_a is the aggregate width of searched path in yards, and V is the locator speed in knots, the time t in hours is

$$t = 4.93(10)^{-4}\left(\frac{t}{\frac{\sigma_1\sigma_2}{U}}\right)_0 \frac{\sigma_1\sigma_2}{w_a V} \text{ hours,}$$

where t may be assumed to represent any of the quantities t, T or τ. For convenience in obtaining quickly rough estimates of the overall searching time and of the average discovery time for values of the parameter $\left(\frac{t}{\sigma_1\sigma_2/U}\right)_0$ listed in the tables, Table 3 gives values of the time in hours for selected values of this parameter and for various values of w_a, V and $\sqrt{\sigma_1\sigma_2}$, where the latter quantities are given in yards, knots, and yards, respectively. Since the values of W listed in Table 3 can not be considered negligible compared to the navigational errors, the data given in Tables 1 and 2 can not be considered exact for these locators. However, the parameter $\left(\frac{t}{\sigma_1\sigma_2/U}\right)_0$ will appear in subsequent tables in which there are less restrictions on the magnitude of W.

48. For any values of P(T) in the range 0.6 to 0.999, reasonably accurate estimates of T and τ can be obtained for the optimum and standard searching procedures by the use

TABLE 3

Values of the Searching Time t* Corresponding to Given Values of $t/(\sigma_1\sigma_2/U)$** for Specified Values of Aggregate Width of Searched Path w_a, of the Locator Speed V, and of the Standard Deviations σ_1 and σ_2 of the Errors of Estimate of Object Positions.

** $\frac{t}{\sigma_1\sigma_2/U}$	Possible Combinations of w_a and V				t (hours)				
	1st comb.		2nd comb.						
	w_a (yd.)	V (knots)	w_a (yd.)	V (knots)	$\sqrt{\sigma_1\sigma_2}$ = 10 yd.	$\sqrt{\sigma_1\sigma_2}$ = 20 yd.	$\sqrt{\sigma_1\sigma_2}$ = 50 yd.	$\sqrt{\sigma_1\sigma_2}$ = 100 yd.	$\sqrt{\sigma_1\sigma_2}$ = 200 yd.
4	10	3	5	6		0.03	0.16	0.66	2.63
	20	3	10	6		0.01	0.08	0.33	1.32
	40	3	20	6		0.007	0.04	0.16	0.66
8	10	3	5	6		0.05	0.33	1.32	5.26
	20	3	10	6		0.03	0.16	0.66	2.63
	40	3	20	6		0.01	0.08	0.33	1.32
16	10	3	5	6		0.11	0.66	2.63	10.53
	20	3	10	6		0.05	0.33	1.32	5.26
	40	3	20	6		0.03	0.16	0.66	2.63
32	10	3	5	6	0.05	0.21	1.32	5.26	21.05
	20	3	10	6	0.03	0.11	0.66	2.63	10.53
	40	3	20	6	0.01	0.05	0.33	1.32	5.26
64	10	3	5	6	0.11	0.42	2.63	10.53	42.10
	20	3	10	6	0.05	0.21	1.32	5.26	21.05
	40	3	20	6	0.03	0.11	0.66	2.63	10.53
96	10	3	5	6	0.16	0.63	3.94	15.78	63.10
	20	3	10	6	0.08	0.32	1.97	7.89	31.55
	40	3	20	6	0.04	0.16	0.99	3.94	15.78
128	10	3	5	6	0.21	0.84	5.26	21.05	84.20
	20	3	10	6	0.11	0.42	2.63	10.53	42.10
	40	3	20	6	0.05	0.21	1.32	5.26	21.05
192	10	3	5	6	0.32	1.26	7.89	31.55	126.21
	20	3	10	6	0.16	0.63	3.94	15.78	63.10
	40	3	20	6	0.08	0.32	1.97	7.89	31.55
256	10	3	5	6	0.42	1.68	10.53	42.10	168.41
	20	3	10	6	0.21	0.84	5.26	21.05	84.20
	40	3	20	6	0.11	0.42	2.63	10.53	42.10

*Note 1: - The data remain valid if t is replaced by either T or τ.

**Note 2: - Consistent units are assumed in fixing the value of the parameter $t/(\sigma_1\sigma_2/U)$. Thus the values of this parameter listed in the first column are equivalent to those listed in Tables 1 and 2. If $(t/\sigma_1\sigma_2/U)_0$ is the value of this parameter in consistent units, and if w_a, σ_1 and σ_2 are in yards, and V is the speed in knots, then the time t in hours is given by

$$t = 4.93(10)^{-4}\left(\frac{t}{\frac{\sigma_1\sigma_2}{U}}\right)_0 \frac{(\sqrt{\sigma_1\sigma_2})^2}{w_a V} \text{ hours.}$$

This formula was used to obtain the data in this table. For circular normal distributions $\sqrt{\sigma_1\sigma_2}$ reduces to σ.

of Table 2 and of Plate 1. Plate 1 gives, for the optimum searching procedure, the values of T for any P(T) in the indicated range in terms either of $T_{0.95}$ or of $T_{0.99}$, where $T_{0.95}$ and $T_{0.99}$ are the values of T for which P(T) is 0.95 and 0.99, respectively. Similarly τ is given for any P(T) in terms of $\tau_{0.95}$ and $\tau_{0.99}$. It is apparent from Tables 1 and 2 that though the magnitudes of T and τ for given P(T) may differ substantially for the different procedures, there is much less difference in the relative rates of increase of T and τ with increase in P(T). Thus for any of the search procedures, values of $T_{0.95}$ and $T_{0.99}$ and the corresponding values of τ can be obtained from Table 2, and T and τ can then be obtained with fair accuracy for any P(T) in the indicated range by use of Plate 1. Also, P(T) can be estimated for specified values of T. For instance, using Plate 1 and Table 2, the value of P(T) obtained with constant parallel path searching is found to be 0.758 for a value of $\frac{T}{\sigma_1\sigma_2/U}$ of 32, whereas the accurate value of P(T) as given in Table 1 is 0.754.

MULTIAREA SEARCHING; NAVIGATIONAL ERROR SMALL COMPARED TO THE UNCERTAINTY IN THE OBJECT POSITION

$(\bar{\sigma}_1 \leq \frac{1}{2}\sigma_1, \ \bar{\sigma}_2 \leq \frac{1}{2}\sigma_2)$

49. The results which have been obtained for very narrow-path locators are based on procedures in which successive areas are searched very lightly so as to give a vanishingly small increase Δm of the nominal search-path density. This procedure would involve the redetermination of the boundaries of the area to be searched after each of very many coverages. Furthermore, a vanishingly small Δm is obviously impossible when the aggregate width of searched path $W\beta$ is not small compared to σ_1 and σ_2. For the formulation of practical searching procedures it is desirable to determine whether it is possible to obtain nearly as good results with only a few coverages using a sufficiently small lateral distance d to obtain a substantial increase in nominal search-path density on each coverage. This will be called "multiarea" searching. For the initial derivations it is assumed that the navigational error is negligible compared to the uncertainty in the position of any object, but is large compared to the over-all width of path W.

50. For a given number of areas, the optimum choice of areas to be searched and of the increments of search-path density to be laid down on each coverage is expected to be one which gives a final search-path density distribution closely approximating the paraboloid discussed in paragraphs 38 and 39. So it may be assumed that the optimum areas are elliptical. While it is possible that, for a given number of areas, best results could be obtained by different choices of Δm for different areas, it seems desirable for reasons of operational convenience to maintain the same lateral distance d, and therefore the same Δm, on each coverage. Several possible procedures are considered in Appendix C for choosing the boundaries of the areas to be searched on successive coverages. It is found that best results are obtained by choosing the elliptical area to be searched on each coverage so that at the completion of that coverage the function p(x,y), defined in paragraph 36, has a value at the center of the area equal to the initial probability density at points just outside the searched area. For multiarea rectangular area searching it has been assumed that for the same number of layers and the same value of Δm the area of the rectangle which is searched on the n'th coverage should be

the same as that of the n'th elliptical area which would be obtained by the above rule. For constant parallel path searching the width of area searched on any coverage is chosen so that at the end of that coverage the one-dimensional function p(x) to which p(x,y) reduces in the region $-\ell/2 < y < \ell/2$ has a value at the center equal to the initial probability density on the x-axis just outside of and on either side of the searched area. For circular diameters searching, multiarea searching consists of sufficiently large rotations of the successive paths so that the specified number of coverages of the circular area is obtained. The detailed analysis of multiarea searching is given in Appendix C. Based on the above criteria it is found that for a given total number of coverages N and a given total aggregate searched area UT', the following are the requirements for the n'th coverage, where n is any number from 1 to N.

(a) Elliptical Area Searching

(1) The required increase Δm in the search-path density on each coverage is given by

$$\Delta m = \sqrt{\left(\frac{UT'}{\sigma_1 \sigma_2}\right) \frac{1}{\pi N(N + 1)}}.$$

(2) The required semi-axes a_n and b_n of the area searched on the n'th coverage are given by

$$\frac{a_n}{\sigma_1} = \frac{b_n}{\sigma_2} = \sqrt{2n\,\Delta m}.$$

(b) Rectangular Area Searching

(1) The required value of Δm is given by

$$\Delta m = \sqrt{\left(\frac{UT}{\sigma_1 \sigma_2}\right) \frac{1}{\pi N(N + 1)}}.$$

(2) The required half lengths X_n and Y_n of the sides of the rectangular area searched on the n'th coverage are given by

$$\frac{X_n}{\sigma_1} = \frac{Y_n}{\sigma_2} = \sqrt{\frac{\pi}{2} n \Delta m}.$$

(c) Constant Parallel Path Searching

(1) The required value of Δm is

$$\Delta m = \left\{ \left(\frac{UT'}{\sigma_1 \sigma_2}\right) \frac{1}{2\sqrt{2}\,\ell \sum_{J=1}^{N} \sqrt{J}} \right\}^{2/3}$$

where ℓ is the length of the search paths.

(2) The required half width X_n of the area searched on the n'th coverage is given by

$$\frac{X_n}{\sigma_1} = \sqrt{2n\,\Delta m}.$$

51. Based on a searching procedure for which the search on any coverage starts along one edge of the area to be searched on that coverage, and proceeds systematically in straight parallel paths with a fixed lateral distance $d = w_a/\Delta m$ until the opposite edge is reached, values of T' and τ have been calculated by methods outlined in Appendix C for each of the multiarea procedures considered. Results are given in Table 4 for values of P(T) of 0.95 and 0.99 and for values of N in the range 1 to 6. Included in the table are the dimensions of each searched area, and the required increment of search-path density Δm in each case. For the case of assumed very small over-all width W the value of the factor γ appearing in the table is to be taken equal to 1.0. It is seen that there is a large decrease in the required value of T', and especially in the value of τ, with increase in the number N of coverages for an increase in N from 1 to 2, with substantially smaller decreases for additional increases in N. The increase in searching efficiency which can be obtained by taking N larger than 3 is quite limited and the values of T' and τ for N = 6 are seen to differ from the values of these parameters for the corresponding optimum and standard searching procedures by not more than a few per cent.

52. In the above analysis it has been assumed that the over-all search-path width W is negligible compared to the navigational error. It was found in Part I of this study (reference (1)) that for a given locator, with a substantial value of the detection probability β, substantially greater searching effectiveness can be obtained when the navigational errors are small compared to the width W than when they are large, provided searching is carried out in a systematic manner with a constant lateral distance d for any given coverage. With multiarea searching the required systematic procedure can readily be carried out for all except circle diameters searching. Indeed, such a systematic procedure has been assumed in obtaining the results for the case where W is assumed small compared to the navigational error. Here, navigational error refers to the deviation of the actual paths searched from the prescribed or nominal paths. The lateral deviations of the actual paths from the nominal paths are assumed to have a normal distribution with standard deviation $\bar{\sigma}_1$. Then the statement that the over-all width of searched path is negligible compared to the navigational error may be taken to be equivalent to the statement that the ratio $W/\bar{\sigma}_1$ is negligible compared to 1.0.

53. When the navigational error is negligible compared to the uncertainty in the position of the object, as has been assumed, and when, in addition, $W/\bar{\sigma}_1$ is small, the value of $\rho(x,y,t)$ is given by equation (8) with $\bar{m}(x,y,t)$ replaced by $m(x,y,t)$. With multiarea searching this gives for $\rho(x,y,t)$, at the completion of the first coverage, a value of $1 - e^{-\Delta m}$ at points within the area A_1 which is searched on that coverage, and zero value for points outside this area. If A_2 is the area searched on the second coverage, then at the end of the second coverage, $\rho(x,y,t)$ has a value $1 - e^{-2\Delta m}$ at points within A_1 and a value $1 - e^{-\Delta m}$ at those points of A_2 which are not within A_1, etc. It is shown in Part I that if exactly similar searches are carried out by locators which have an over-all width of searched path W which is not small compared to $\bar{\sigma}_1$, then each Δm in the above

expressions must be multiplied by a factor γ in order to obtain the true value of $\rho(x,y,t)$ at the indicated points. More exactly, this gives the average value of $\rho(x,y,t)$ within the indicated region over any line segment of length d drawn perpendicular to the direction of the locator paths, where d is the lateral distance; but the distinction is unimportant in cases where multiarea searching would be considered appropriate. Thus for obtaining a given value of the location probability, the required value of Δm, or of $W\beta/d$, for each coverage, is inversely proportional to the factor γ. It then follows that the required values of T and τ are inversely proportional to γ. Thus with the inclusion of γ in Table 4 as a coefficient of the quantities Δm, $\frac{T}{\sigma_1\sigma_2/U}$, and $\frac{\tau}{\sigma_1\sigma_2/U}$, the data in the table remain valid when $W/\bar{\sigma}_1$ is not small. Similarly the relations in paragraph 50 remain valid when $W/\bar{\sigma}_1$ is not small if the quantities Δm and T' appearing in those equations are replaced by $\gamma\Delta m$ and $\gamma T'$, respectively. These results do not apply to circle diameters searching. For the effectiveness of circle diameters searching when $W/\bar{\sigma}_1$ is not small see the footnotes to Table 4.

54. The value of γ depends on the parameters W, β, d, and $\bar{\sigma}_1$. For any given values of these parameters, the value of γ can be obtained from charts which were presented in reference (1) and which, for convenience, have been reproduced as Plates 2, 3, 4 and 5 of the present report. To use these charts the symbol $\bar{\sigma}$ appearing on the charts must be interpreted to be $\bar{\sigma}_1$ when searches are carried out in paths parallel to the y-axis. More generally, the quantity $\bar{\sigma}$ of the charts represents the standard deviation of navigational error normal to the locator path. For sufficiently small values of the lateral distance d, the value of γ can be obtained from Plate 2, the specific conditions for which the data of Plate 2 are valid being indicated on the plate. For values of the lateral distance exceeding the limits specified on Plate 2, but not exceeding $W + 4\bar{\sigma}_1$, the value of γ can be estimated from Plates 3 and 4, which give the ratio γ/γ_0 where γ_0 is the value of γ given on Plate 2 for the same values of W, β and $\bar{\sigma}_1$. For values of d greater than $W + 3\bar{\sigma}_1$, the value of γ is given by Plate 5. The data presented on Plates 2, 3, 4 and 5 are exact only when the detection probability is the same for all transverse positions within W, but it is shown in reference (1) that if β is taken to be the detection probability for mines at random positions within W, the data in the plates represent quite good approximations in most cases, and are always sufficiently conservative. More exact evaluations of γ can be obtained by methods outlined in reference (1), but the conditions under which Plates 2 to 5 substantially underestimate the value of γ are not likely to occur in locating an object whose approximate position is known.

55. Plates 2 to 5 give the value of γ only when navigation on all locator passes is relative to a common fixed reference system, and they are not directly applicable to locator formation searches, with each locator keeping station on the adjacent locator. It is shown in reference (1) that, provided the lateral distance is not too small compared to W, the value of γ can be obtained from Plates 2 to 5 using, for the purpose of reading the charts, a value of $\bar{\sigma}$ equal to $\bar{\bar{\sigma}}/\sqrt{2}$, where $\bar{\bar{\sigma}}$ is the standard deviation of the station-keeping error. Reference (1) can be consulted for details. Because of the relative inflexibility of a formation of locators, this type of operation would likely be limited to constant parallel path searching in which there is either only one or two coverages, or several coverages of a single width. It may be noted that the value of T given in Table 4 for N = 1 is valid also for several coverages of a single area, but the indicated

TABLE 4

MULTI-AREA SEARCHING; ELLIPTICAL (OR CIRCULAR*) NORMAL DISTRIBUTION OF OBJECT POSITION ERRORS: Values of the Overall Searching Time T and of the Average Discovery Time τ Corresponding to Location Probabilities P(T) of 0.95 and 0.99 for the Case where Navigational Errors of Locators are Small Compared to the Uncertainty in the Position of the Object.** (All passes for each coverage are assumed to be made in parallel paths starting at one edge of the Area to be Searched on the given Coverage with Successive Passes Displaced Uniformly by a Constant Distance d Until the Opposite Edge is Reached.)

Search Procedure	Total Number of Cover-ages N	With P(T) = 0.95				With P(T) = 0.99			
		$\gamma\Delta m$	Dimensions† of Searched Area on N'th Coverage	$\gamma \frac{T}{\sigma_1\sigma_2/U}$ $\gamma\left(\frac{\Omega}{\sigma_1\sigma_2}\right)$	$\gamma \frac{\tau}{\sigma_1\sigma_2/U}$	$\gamma\Delta m$	Dimensions† of Searched Area on N'th Coverage	$\gamma \frac{T}{\sigma_1\sigma_2/U}$ $\gamma\left(\frac{\Omega}{\sigma_1\sigma_2}\right)$	$\gamma \frac{\tau}{\sigma_1\sigma_2/U}$
Elliptical Area Searching	1	3.68	$\frac{a_N}{\sigma_1} = \frac{b_N}{\sigma_2} =$ 2.71	84.9	44.6	5.30	$\frac{a_N}{\sigma_1} = \frac{b_N}{\sigma_2} =$ 3.25	176.2	89.0
	2	2.00	2.83	75.3	23.3	2.83	3.37	151.3	34.4
	3	1.39	2.89	73.0	19.7	1.96	3.43	144.9	25.2
	4	1.07	2.93	72.1	18.4	1.50	3.47	142.2	22.1
	5	0.87	2.95	71.6	17.8	1.22	3.50	141.0	20.7
	6	0.73	2.97	71.4	17.5	1.03	3.52	140.7	20.0
Rectangular Area Searching	1	3.81	$\frac{X_N}{\sigma_1} = \frac{Y_N}{\sigma_2} =$ 2.45	91.3	47.9	5.57	$\frac{X_N}{\sigma_1} = \frac{Y_N}{\sigma_2} =$ 2.96	195.0	98.5
	2	2.07	2.55	80.6	24.7	2.96	3.05	164.9	37.1
	3	1.44	2.60	78.0	20.7	2.05	3.11	157.8	26.4
	4	1.11	2.64	76.9	19.3	1.57	3.14	155.1	23.5
	5	0.90	2.66	76.4	18.7	1.28	3.17	153.5	22.0
	6	0.76	2.68	76.2	18.3	1.08	3.18	152.8	21.7
Constant Parallel Path Searching	1	3.49	$\frac{X_N}{\sigma_1} =$ 2.64, $\frac{\ell}{\sigma_2} =$ 5.0	92.2	48.4	5.18	$\frac{X_N}{\sigma_1} =$ 3.22, $\frac{\ell}{\sigma_2} =$ 6.0	200.0	101.0
	2	1.84	2.71 "	85.0	28.0	2.68	3.28 "	180.0	45.9
	3	1.26	2.75 "	82.9	24.0	1.83	3.31 "	174.0	34.7
	4	0.96	2.78 "	82.3	22.6	1.40	3.34 "	172.0	30.8
	5	0.78	2.80 "	82.0	22.1	1.13	3.36 "	171.1	29.0
	6	0.66	2.81 "	81.7	21.5	0.95	3.37 "	169.5	27.9
Circle Diameters Searching	1		$2\frac{R}{\sigma} =$ 6.50	92.0#	48.3#		$2\frac{R}{\sigma} =$ 6.50	185.0#	93.4#
	2		6.50	92.0#	31.9#		6.50	185.0#	51.1#
	3		6.50	92.0#	27.9#		6.50	185.0#	39.1#
	4		6.50	92.0#	26.3#		6.50	185.0#	34.1#

Δm = increase in search-path density per coverage. $\Delta m = w_a/d = W\beta/d$, where W is the over-all width of searched path and β is the probability of detecting an object at random position within W.

σ_1, σ_2 = standard deviation of errors of estimate of the object position parallel to the x- and y-axes, respectively.

U = aggregate area searched per unit time. Data for T are valid if U represents the time average of the rate of searching. Data for τ are valid only if U represents a constant rate of searching, except that for constant parallel path searching and circle diameters the data for τ are approximately valid if the time required per pass is constant and U is taken to be the ratio of the aggregate area searched on one pass to the time required to make a pass, including the turning time.

Ω = the aggregate area which must be searched to obtain the indicated value of P(T). $\Omega = UT$ where U is the time average rate of searching.

γ = a factor which takes account of the increase in efficiency obtained when the standard deviation of navigational error $\bar{\sigma}$ is sufficiently small so that $W/\bar{\sigma}$ is not small compared to 1 (see Note 2).

TABLE 4 (Continued)

*Note 1: - When $\sigma_1 = \sigma_2$ the elliptical distribution of errors of estimate of object position reduces to a circular distribution and $\sigma_1\sigma_2$ can be replaced by σ^2. When $\sigma_1 = \sigma_2$, elliptical area searching reduces to circular area searching and rectangular area searching reduces to square area searching. Data for circle diameters searching are valid only for circular distributions of errors of object position and then only with a special interpretation of the factor γ (see Note 4).

**Note 2: - Navigational error may be considered negligible compared to uncertainty in object position if $\bar{\sigma}_1 \leq \frac{1}{2}\sigma_1$ and $\bar{\sigma}_2 \leq \frac{1}{2}\sigma_2$, where $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are standard deviations of navigational error in x and y-directions respectively. With $\gamma = 1$ the data are valid for the case where the over-all width of searched path W is small compared to the standard deviation of navigational error $\bar{\sigma}_1$. If $W/\bar{\sigma}_1$ is not small, γ must be determined from Plates 2 to 5 (however, see Note 4 for meaning of γ with circular diameters searching).

†Note 3: - Dimensions of all searched areas are as follows:

(a) Elliptic Area Searching

1) a_N and b_N are the semi-axes of the ellipse searched on the N'th or final coverage.
2) The semi-axes a_n and b_n of the area searched on the n'th coverage for $n \leq N$ are

$$a_n = \sqrt{n/N}\ a_N, \quad b_n = \sqrt{n/N}\ b_N.$$

(b) Rectangular Area Searching

1) $2X_N$ and $2Y_N$ are the lengths of the sides of the rectangle searched on the N'th or final coverage.
2) The dimensions of the rectangular area searched on the n'th coverage for $n \leq N$ are

$$2X_n = \sqrt{n/N}\ (2X_N) \quad \text{and} \quad 2Y_n = \sqrt{n/N}\ (2Y_N).$$

(c) Constant Parallel Path Searching

1) X_N is one-half the width of the area searched on the N'th or final coverage.
2) The half width searched on the n'th coverage for $n \leq N$ is $X_n = \sqrt{n/N}\ X_N$.
3) ℓ is the length of search paths for all coverages.

(d) Circle Diameters Searching

1) 2R is the length of the diameters searched on all coverages. Each coverage represents a 180° aggregate rotation of the search path.

‡Note 4: - Data for Circle Diameters Searching are valid only for circular normal distributions of errors of object position and then only for a special interpretation of γ. For this case γ can be taken equal to γ_o, given by Plate 1, when any one of the following conditions is satisfied:

(a) $P(T) = 0.99$, $N = 4$, with either $\beta \leq 0.7$ or $W \leq 2\bar{\sigma}$
(b) $P(T) = 0.99$, $N = 3$, with either $\beta \leq 0.8$ or $W \leq 3\bar{\sigma}$
(c) $P(T) = 0.99$, $N = 2$, with either $\beta \leq 0.9$ or $W \leq 3.8\bar{\sigma}$
(d) $P(T) = 0.99$, $N = 1$, with either $\beta \leq 0.98$ or $W \leq 4.8\bar{\sigma}$
(e) $P(T) = 0.95$, $N = 2$, with either $\beta \leq 0.7$ or $W \leq 2\bar{\sigma}$
(f) $P(T) = 0.95$, $N = 1$, with either $\beta \leq 0.9$ or $W \leq 4\bar{\sigma}$

When these conditions are not satisfied γ is to be considered a factor which may be less than γ_o but which is never less than 1.0. When the over-all width W is larger than the standard deviation σ of the error of estimate of object position, the efficiency of circle diameters searching is somewhat greater than is indicated by the table based on the suggested interpretation of γ.

values of τ are then too large. A possible type of searching operation which is essentially equivalent to formation searching is the following: On the initial pass of each coverage the locator attempts to follow a predetermined path and records continuously its actual path as determined with the aid of a fixed navigation system. The record of its actual path is then used as a basis for determining the desired path for the next path by the same or another locator. A similar record is taken of the actual path searched on the second pass of that coverage, etc. The determination of γ for this case is discussed in paragraph 72 of reference (1).

56. For values of the parameters β, W and $\bar{\sigma}_1$ for which the value of γ_0 given by Plate 2 is greater than 1.5 the theoretically most efficient multiarea searching is likely to be obtained if there are a total of not more than three or four coverages, and in more extreme cases the optimum number of coverages may be only one or two. Since the appropriate value of $\gamma\Delta m$ decreases with increase in total number of coverages, there must be a corresponding increase in the lateral distance. This may result in a decrease in γ which outweighs the theoretical decrease in the values of $\gamma \frac{T}{\sigma_1\sigma_2/U}$ and $\gamma \frac{\tau}{\sigma_1\sigma_2/U}$. The following appears to be a reasonably adequate rule for obtaining near maximum searching efficiency:

(1) Determine the maximum total number of coverages N_1 for which the required lateral distance does not exceed the limits specified on Plate 2. Then proceed as follows:

(2) If N_1 is not less than 3 take N_1 to be the required number of coverages.

(3) If N_1 is equal to 1 or 2, determine the values of $\frac{T}{\sigma_1\sigma_2/U}$ and $\frac{\tau}{\sigma_1\sigma_2/U}$ for the given multiarea searching procedure for a total number of coverages equal to N_1, and also for a total number of coverages equal to $N_1 + 1$, and choose the number of coverages to be used on the basis of these results.

57. To illustrate the proposed procedure, suppose $W/\bar{\sigma}_1 = 4$ and $\beta = 1$. For this combination of parameters γ_0 is 2.14, and on the basis of the limits specified in Plate 1, the value of γ can be taken to be γ_0 provided $d/\bar{\sigma}_1 \leq 3$. The smallest value of $\gamma\Delta m$ or of $\gamma \frac{W\beta}{d}$ which will meet these conditions is $\frac{4}{3}(2.14)$ or 2.85. Assuming constant parallel path searching is to be used, it is apparent that the required condition can be met only if but a single coverage is made. To obtain a P(T) of 0.95 with a single coverage requires $\frac{T}{\sigma_1\sigma_2/U} = \frac{92.2}{\gamma} = 43.8$ and gives a value of $\frac{\tau}{\sigma_1\sigma_2/U}$ of $\frac{48.4}{\gamma} = 22.6$. If there are a total of two coverages, the required value of $\gamma\Delta m$ is only 1.84, and it is apparent that d must be substantially larger than $3\bar{\sigma}_1$. Taking as a first estimate $d/\bar{\sigma}_1 = 4$, Plates 2 and 4 give for γ a value of 1.97. Then $\gamma\Delta m = 1.97$. This is approximately 7% too large. Decreasing the value of Δm by 5%, thus allowing for some decrease in γ, or increasing the value of d by a like amount, gives $W - d/\bar{\sigma}_1 = 0.2$. Then $\gamma = 1.94$ and $\gamma\Delta m = 1.94(0.95) = 1.84$. For this case it is found that

$$\frac{T}{\sigma_1\sigma_2/U} = \frac{85}{1.94} = 43.9 \quad \text{and} \quad \frac{\tau}{\sigma_1\sigma_2/U} = 14.7.$$

It is seen that increasing the number of coverages from 1 to 2 leaves T unchanged, but produces a relatively large decrease in τ. For three coverages the values of both T and τ are substantially larger than for two coverages.

58. The rapid increase in $\gamma\tau$, and to a lesser extent in γT, which occurs in what may be termed the "regular" multiarea searching procedure with decrease in the total number of coverages N, for the range N = 1 to N = 3, tends to limit the improvement in searching efficiency which can be obtained by increasing the navigational accuracy in order to obtain larger values of $W/\bar{\sigma}_1$. For small values of N it is possible to obtain substantially smaller values of τ by modifying the procedure for making the first coverage in carrying out multiarea searching. In the regular multiarea searching procedures it has been assumed that each coverage is made by starting the search at one edge of the area to be searched on that coverage, and that following the initial locator pass at the edge of this area, each succeeding pass is displaced a constant distance d farther from the starting edge than the next preceding locator pass until the opposite edge is reached. Now the choice of areas to be searched on successive coverages has been made so as to obtain as nearly uniform values of p(x,y,t) as possible within the area searched on each coverage subsequent to the first, where p(x,y,t) is the function defined in paragraph 36. Thus, it is expected that for any coverage subsequent to the first, the regular procedure is as efficient as any other practical procedure. But for the first coverage the function p(x,y,t) represents the initial probability density of the object position within the area to be searched on the first coverage, and this has its maximum value at the center and its smallest value within this area at the edges. Thus it may be expected that τ can be substantially reduced by starting the search on the first coverage at the center of the area to be searched on that coverage. On the other hand, the value of UT obviously does not depend on which parts of each area are searched first, provided each area is searched uniformly.

59. For a circular normal distribution of errors of estimate of object position and circular area searching, the theoretical optimum procedure for a given constant value of U is obviously to make the first coverage by searching in a circular spiral, beginning at the center of the initial circular area. Similarly the first coverage in square area searching may logically consist of a square spiral of search paths. For more practical searching procedures, consideration has been given to making the first coverage by searching in straight parallel paths, but with the initial pass made through the interior of the area rather than along an edge. The latter procedures are referred to as modified parallel paths procedures. Two modified parallel path procedures have been considered. Procedure I consists of an initial search path nominally centered at the most probable position of the object, with subsequent widening of the searched area by making alternate passes on opposite sides of the searched area. Procedure II consists of an initial search of the area to be searched on the first coverage in two parts. The first part of the search starts on a track which lies at a specified distance from the most probable position of the object, and subsequent paths are offset a constant lateral distance d in the direction of the most probable position of the object, and then beyond this point until the opposite edge is reached. The second part starts adjacent to the initial pass of the first part, and continues until the first coverage is completed. For any given multiarea procedure, and for any given P(T), there is an optimum position for the first track. However, it is shown in Appendix C for a rectangular area, that taking the first track to be at a distance of $0.4X_1$ from the most probable position

of the object, where $2X_1$ is the width of the area which is searched on the first coverage, gives a value of τ differing from the value obtained with optimum position of the first track by a negligible amount.

60. Table 5 gives values of τ for multiarea searching obtained with the various modified procedures for making the first coverage. For comparison, the table also gives values of τ obtained with the regular procedure. Details of the methods of obtaining the data in the table are given in Appendix C. It is seen that the modified procedures give large reductions in τ in some cases when there is but a single coverage. Data for modified parallel path searching, Method I, has not been included for rectangular area searching for the reason that when substantial turning times are required, constant parallel path searching likely would be used. When the turning time is small compared to the time required to make one pass, Method II is likely to be the more efficient because of the time required to go from one side of the searched area to the other at the end of each pass when using Method I. When the turning times are large, it may be that the turning time is not increased if, during the turn, the locator moves from one side of the searched area to the opposite side. Of course, the turning times must be taken into account in determining the value of U.

61. The results for the modified searching procedures for making the first coverage are exact only when the over-all width W is quite small compared to the uncertainty in the position of the object. For the Spiral Searching Procedures and for the Modified Parallel Path Procedure I, the value of τ is slightly larger than indicated if W is not small compared to σ_1. For in determining τ for these procedures, it has been assumed that the probability density of the object position at all points within the nominal search path is equal to that at the boundary between the nominally searched and unsearched areas, but when there is a substantial width W the average value of the probability density within W is slightly smaller. In contrast the value of τ for Modified Parallel Path Procedure II is slightly smaller than the values given by the table if there is a substantial value of W. Since, in general, the true values of τ for Procedures I and II lie between the values given in Table 5 for these two procedures, it can be seen that the error made in assuming that W is quite small compared to σ_1 is not likely to be important in practical cases.

62. From the above analysis it is clear that it is not possible to realize values of γ substantially greater than 2 except, at most, when the search consists of a single coverage. A single coverage search for an object whose approximate position is known is very similar to the problem of area search considered in Part I. Leaving out of account circle diameters searching, the analysis of Part I can be used to estimate the practical upper limits for γ which it is possible to realize in searching for an object whose approximate position is known. For values of $\gamma\Delta m$ of the order of 3.5 which are required to obtain $P(T) = 0.95$ with a single coverage, it is found that γ is given with sufficient accuracy by γ_0 for values of the parameters W, β and $\bar{\sigma}_1$ for which γ_0 does not exceed 2.5. For values of $\gamma\Delta m$ of the order of 5.3 which are required to obtain $P(T) = 0.99$ with a single coverage, γ is given with sufficient accuracy by γ_0 for values of the parameters for which γ_0 does not exceed 3. For larger values of γ_0 the value of γ can be obtained with the aid of Plates 3, 4 and 5. It is found, however, that only moderate increases in γ above the values of 2.5 and 3 respectively are practically attainable for probabilities of 0.95 and 0.99 regardless of the value of γ_0.

TABLE 5

SUPPLEMENT TO TABLE 4: Values* of the Average Discovery Time τ which are Obtained by Indicated Modifications of Regular Searching Procedure for Making the First Coverage and Their Comparison with the Values Obtained by the Regular Procedure. (The dimensions of the searched areas and the values of $\gamma\Lambda m$ and of $\gamma \frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$ or $\gamma \frac{\Omega}{\sigma_1\sigma_2}$ are as indicated in Table 4, and the footnotes of Table 4 apply to the present table.)

General Search Procedure	Number of Cover-ages N	$\gamma \frac{\tau}{\sigma_1\sigma_2/U}$ for P(T) = 0.95				$\gamma \frac{\tau}{\sigma_1\sigma_2/U}$ for P(T) = 0.99			
		With Regular Procedure as Described in Table 4	Using a Spiral** Searching Procedure on First Coverage	Using Modified Parallel Path Procedure on First Coverage: Procedure I†	Using Modified Parallel Path Procedure on First Coverage: Procedure II‡	With Regular Procedure as Described in Table 4	Using a Spiral** Searching Procedure on First Coverage	Using Modified Parallel Path Procedure on First Coverage: Procedure I†	Using Modified Parallel Path Procedure on First Coverage: Procedure II‡
Circular Area Searching	1	44.58	24.09	---	---	88.99	33.82	---	---
	2	23.32	20.39	---	---	34.36	25.00	---	---
	3	19.66	18.88	---	---	25.17	22.43	---	---
	4	18.38	18.11	---	---	22.12	21.08	---	---
	5	17.78	17.67	---	---	20.75	20.28	---	---
	6	17.46	17.40	---	---	20.01	19.77	---	---
Rectangular Area Searching	1	47.95	25.65	---	34.22	98.47	36.82	---	59.92
	2	24.72	21.46	---	22.72	37.11	26.59	---	30.66
	3	20.71	19.83	---	20.17	26.45	23.30	---	24.51
	4	19.32	19.00	---	19.12	23.52	22.29	---	22.76
	5	18.67	18.53	---	18.58	21.98	21.43	---	21.64
	6	18.31	18.25	---	18.27	21.67	21.39	---	21.50
Constant Parallel Path Searching	1	48.4	---	30.49	33.11	101.0	---	50.84	57.82
	2	28.0	---	24.42	24.97	45.9	---	34.23	35.98
	3	24.0	---	22.80	22.98	34.7	---	30.47	31.12
	4	22.6	---	22.06	22.14	30.8	---	28.85	29.15
	5	22.1	---	21.88	21.92	29.0	---	27.95	28.11
	6	21.5	---	21.39	21.41	27.9	---	27.35	27.44

*Note 1: - The data are exact only when W is quite small compared to σ_1 or σ. For larger values of W the values of τ are slightly larger than indicated for the Spiral Searching Procedures and for the Modified Parallel Path Procedure I, but are slightly smaller than indicated for Modified Parallel Path Procedure II.

**Note 2: - The data on Spiral Searching Procedure are valid only for a circular normal distribution of errors of estimate of object position and for Circular Area and Square Area Searching. For these cases $\sigma_1\sigma_2$ is equal to σ^2. Also the data for Spiral Searching are valid only if U is constant.

†Note 3: - Procedure I consists of making the first coverage by a search in parallel paths with the first pass directly over the most probable position of the object and with the subsequent widening of the searched area by making alternate passes on opposite sides of the searched area. The data are valid if U is taken to be the average search rate, only if the time required to make a pass, including the time to turn, is constant, independent of the distance between successive paths on the first coverage.

‡Note 4: - Procedure II for making the first coverage consists of a search in parallel paths in two parts. The first part of the search starts on a track which lies at a distance $0.4X_1$ from the most probable position of the object at the closest distance of approach, where $2X_1$ is the width of area searched on the first coverage, and subsequent paths are offset a constant lateral distance d in the direction of the most probable position of the object and then beyond this point until the opposite edge is reached. The second part starts adjacent to the initial pass of the first part and continues until the first coverage is completed.

63. A theoretical upper limit for searching efficiency is obtained when both $\beta = 1.0$ and the navigational error is zero. Then neither holidays nor overlap occur between adjacent paths. Table 6 gives results for this case. Since rectangular area searching becomes identical with constant parallel path searching in this case the former has not been listed separately in the table. For reasons mentioned in connection with the data in Table 5 the values of τ given in Table 6 are exact only for values of W small compared to σ_1. Comparison of the values of T in Tables 4 and 6 for equivalent searching procedures shows that reducing the navigational error to zero when $\beta = 1.0$ is approximately equivalent to attaining a value of 4.5 for γ when $P(T) = 0.95$, and a value of roughly 6.2 when $P(T) = 0.99$. Comparison of the values of τ given in Tables 5 and 6, indicate an equivalent γ which is approximately, but not identically, the same as the value obtained on the basis of the comparison of the T's. The reason that there is not exact correspondence between the T's and τ's is that the dimensions of the searched areas for Tables 4 and 5 are slightly different from those for Table 6. It is apparent that for a fixed constant value of U and a circular distribution of errors of object position, the optimum searching procedure is circular spiral searching, and it is interesting to compare optimum results for this case with the optimum results given in Table 2 for the case where navigational error is large compared to W, but is negligible compared to the uncertainty in the position of the object. It is seen that when $\beta = 1.0$ and the required P(T) lies in the range 0.90 to 0.99, reducing the navigational error to zero decreases the value of τ by a factor of slightly less than 1/3 and reduces the value of T by a factor of the order of 0.2 to 0.3, depending on the value of P(T). The comparison is valid, of course, only for circular normal distributions of the errors of estimate of object position.

64. It has been pointed out that zero navigational error is effectively achieved by divers who maintain their radial position by means of a line anchored at the center of the area to be searched, and it has been suggested that locators might be controlled in a similar manner. In considering the use of this procedure, it should be noted that substantial errors may occur in attempting to anchor one end of the line at the most probable position of this object. To take account of this, σ may have to be increased as indicated in Appendix E. It should be observed also that, for given σ, the values of T and τ depend on U; and if the search rate U is substantially less when the locator is held in position by a line of varying length, this must be taken into account in estimating the efficiency of this procedure. For non-circular distributions of errors of estimate of object position a constant parallel path procedure with zero lateral navigational error could similarly be approximated, in principle, by using a sufficiently long line and with the anchor located on the x-axis at a relatively large distance from the estimated object position.

65. Tables 4 and 5 provide data on multiarea searching procedures only for $P(T) = 0.95$ and for $P(T) = 0.99$. However, for any total number of coverages of not less than 2, the appropriate values of T and τ can be obtained for any value of P(T) in the range 0.6 to 0.999 by the use of Plate 1 together with Tables 4 and 5. In using Plate 1 the ordinates should be taken to represent ratios of (γT)'s and $(\gamma\tau)$'s rather than of T's and τ's. For instance, the curve labelled $T/T_{0.95}$ should be taken to represent $\gamma T/(\gamma T)_{0.95}$, where the numerator is the required value of γT for values of P(T) indicated by the ordinates, and $(\gamma T)_{0.95}$ is the value of γT given by Table 4 for $P(T) = 0.95$ for the

TABLE 6

ZERO NAVIGATIONAL ERROR AND β = 1.0: Values of T and τ for Given Values of P(T) when Searching without Overlap or Holidays Using Locator with Small Value of W/σ and with β = 1.0, Assuming a Circular* Normal Distribution in the Errors of Estimate of Object Position.

Search Procedure	P(T) = 0.90			P(T) = 0.95			P(T) = 0.99			P(T) = 1.0	
	$\frac{T}{(\sigma^2/U)}$	$\frac{\tau}{(\sigma^2/U)}$	$\frac{\tau}{T}$	$\frac{T}{(\sigma^2/U)}$	$\frac{\tau}{(\sigma^2/U)}$	$\frac{\tau}{T}$	$\frac{T}{(\sigma^2/U)}$	$\frac{\tau}{(\sigma^2/U)}$	$\frac{\tau}{T}$	$\frac{T}{(\sigma^2/U)}$	$\frac{\tau}{(\sigma^2/U)}$
Circular** Spiral Searching	14.47	5.65	0.390	18.82	5.97	0.317	28.94	6.22	0.215	∞	6.28
Square† Spiral Searching	15.19	5.84	0.384	20.01	6.19	0.309	31.51	6.47	0.205	∞	6.55
Constant Parallel Path Searching#											
Procedure I‡	15.19	6.54	0.431	20.01	7.39	0.369	31.51	9.05	0.287	---	---
Procedure II‡	15.19	6.81	0.449	20.01	7.86	0.393	31.51	10.04	0.317	---	---

*Note 1: - Results for constant parallel path searching are valid also for an elliptic normal distribution of errors of object position provided the values given for $T/\left(\frac{\sigma^2}{U}\right)$ and $\tau/\left(\frac{\sigma^2}{U}\right)$ are interpreted to be values of $T/\left(\frac{\sigma_1\sigma_2}{U}\right)$ and $\tau/\left(\frac{\sigma_1\sigma_2}{U}\right)$, respectively.

**Note 2: - Radius R of circle searched in time T: $R = 2.15\sigma$ for P(T) = 0.90; $R = 2.45\sigma$ for P(T) = 0.95; $R = 3.04\sigma$ for P(T) = 0.99.

†Note 3: - Length 2X of side of square searched in time T: $2X = 3.90\sigma$ for P(T) = 0.90; $2X = 4.47\sigma$ for P(T) = 0.95; $2X = 5.61\sigma$ for P(T) = 0.99.

#Note 4: - Assumed lengths of all paths ℓ for constant parallel path searching are $3.90\sigma_2$, $4.47\sigma_2$, and $5.61\sigma_2$ for P(T) equal to 0.90, 0.95, and 0.99, respectively. Width 2X of area searched in time T is equal to length of path for circular normal distribution, and more generally is equal to $(\sigma_1/\sigma_2)\ell$.

‡Note 5: - Procedures I and II are the same as procedures I and II, respectively, for first coverage defined in notes 3 and 4 of Table 5, and the comment in note 3 concerning turning time applies here also as regards τ.

particular procedure considered. Although the values of $\gamma T'$ and $\gamma\tau$ obtained by this method are not exact, they should be entirely adequate for practical purposes for a total number of coverages N of not less than 2. Even when N = 1 the values of $\gamma T'$ obtained by this method are reasonably adequate, and rough estimates of $\gamma\tau$ can be obtained when N = 1 for either the regular or modified procedures by assuming that $\gamma\tau$ varies linearly with γT. Once the required $\gamma T'$ has been determined by the use of Plate 1, the dimensions of the areas to be searched and the required Δm can be obtained from the formulas of paragraph 50 if Δm and T in those formulas are replaced by $\gamma\Delta m$ and $\gamma T'$.

66. The data presented in Tables 4 and 5 were obtained from formulas which were derived under the assumption that $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ are vanishingly small. When W is very small the only effect of a finite navigational error is to give a search-path density differing from the nominal search-path density. It is to be expected that a small navigational error increases the efficiency of multiarea searching by removing the discontinuities in the search-path density at the boundary of each of the areas searched on successive coverages without appreciably effecting the search-path density at interior parts of the area. This is confirmed by calculations carried out in Appendix D. From the results obtained in Appendix D for various special cases, it appears that for finite navigational error satisfying the relations $\bar{\sigma}_1 \leq \frac{1}{2}\sigma_1$ and $\bar{\sigma}_2 \leq \frac{1}{2}\sigma_2$, the multiarea searching efficiency is slightly greater than is indicated by Table 4 for a total number of coverages less than 4, and that for any number of coverages the data presented in the table are entirely adequate.

67. Most of the calculations in Appendix D on which the above conclusion is based were carried out only for the case $W/\bar{\sigma}_1 \longrightarrow 0$, but from considerations similar to those of paragraph 53 it appears that the result can be extended to finite W if in the expression (8) for $\rho(x,y,t)$, the quantity $\bar{m}(x,y,t)$ can be replaced by $\gamma\bar{m}(x,y,t)$ for points near the boundaries of the nominally searched areas as well as in the interior parts of the areas. Reference (1) shows that, to a good approximation, this is the case if a uniform coverage is taken to be one with constant lateral distance d, and with nominal paths of the first and last passes of the given coverage which overlap the respective nominal edges of the given coverage by an amount equal to one half of the nominal overlap between adjacent paths, assuming there is nominal overlap between adjacent paths. If there is a nominal holiday between the paths searched on two consecutive passes, then the equivalence is maintained if there is a nominal holiday between each of the initial and final paths and its respective edge of magnitude equal to one half the holiday between two adjacent paths. Whether there is nominal overlap or a nominal holiday between adjacent paths, these rules are equivalent to taking the nominal position of the center line of the initial path at a distance d/2 from the starting edge of the area, and ending with a path whose center-line path is a distance d/2 from the edge which is opposite the starting edge. Here the edges of the areas searched on successive coverages are the boundaries specified in Table 4. It may be noted that with this definition of a uniform coverage, the aggregate area searched out on a uniform coverage of a nominal area A is $A\frac{W\beta}{d}$, or $A\Delta m$, whether or not $\frac{W}{\bar{\sigma}_1}$ is negligible. It is considered that Table 5 also is valid, to a sufficient approximation, for values of $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ not exceeding 1/2, though detailed calculations have not been carried out for the modified searching procedures for cases where the navigational error is not negligible.

68. If Δm is the same for all coverages, as is indicated in Table 4, so is d. It is obviously then not possible in general to have the center line of the initial and final paths of each coverage come at a distance d/2 from the edges of the area searched on that coverage. But it is apparent from the many examples considered in Appendices C and D that the searching efficiency is relatively insensitive to moderate deviations of the dimensions of the successive searched areas from those specified in Table 4 provided the magnitude Ω of the total aggregate area searched is equal to that specified in the table. Thus for practical purposes it should be adequate to start each coverage with a nominal path whose center line lies at a distance d/2 from the starting edge, and to make additional passes with the required lateral distance d, taking as the final pass on the given coverage the one whose nominal center-line path falls within the boundaries of the specified area and whose distance from the appropriate edge lies within the range 0 to d. The deviations from the desired distribution may be expected to be smaller, in general, if alternate coverages are started on opposite sides of the most probable object position. An extra pass can be made on the final coverage if this appears to be required to obtain the specified total amount of searching. Accurate calculations for one example using this procedure are carried out in Appendix D. An alternative procedure would be to determine, to the nearest integral value, the number of passes to be made on each coverage, and then determine a new d for each coverage so as to cover each required area uniformly.

69. For a sufficiently large over-all width of path W it is not possible to approximate the procedures specified in Tables 4 and 5 for a large total number of coverages N. Unless W is larger than the dimensions of the area to be searched on the final coverage this presents no new problem if β has a value near 1.0. For if β is near unity and W is large, only a single coverage would be made by the rules proposed in paragraph 56. For smaller values of β the number of coverages may need to be limited for values of W greater than $3\sigma_1$ in order to maintain approximately the dimensions of the searched areas specified in Table 4. When W is greater than $6\sigma_1$, all passes should be made on a single track with the nominal center-line path at the most probable position of the object. Data for this search procedure are presented in Tables 7 and 8. These tables are explained in the next section.

NAVIGATIONAL ERRORS COMPARABLE TO TO THE UNCERTAINTY IN THE OBJECT POSITION

$\left(\frac{1}{2}\sigma_1 \leq \bar{\sigma}_1 \leq \frac{3}{2}\sigma_1 \text{ AND } \bar{\sigma}_2 \leq \frac{3}{2}\sigma_2\right)$

70. The search procedures which are specified in Table 4 are not the most suitable procedures when the navigational errors are not small compared to the uncertainty in object position. It is convenient to consider separately the case where the standard deviation of navigational error $\bar{\sigma}_1$ is not less than $\frac{3}{2}\sigma_1$ and the case where $\bar{\sigma}_1$ lies within the range $\frac{1}{2}\sigma_1$ to $\frac{3}{2}\sigma_1$. The latter case is considered in the present section.

71. When the magnitude of the navigational errors is of the same order as the uncertainty in object position the search-path density $\bar{m}(x,y,t)$ differs substantially from the nominal search-path density $m(x,y,t)$ because of the dispersion of the search paths

resulting from the navigational errors. It is to be expected that $\bar{m}(x,y,t)$ will be less than $m(x,y,t)$ at points near the estimated object position and greater than $m(x,y,t)$ at larger distances. It is apparent that in order to obtain approximately the desired search-path density distribution, the nominal search-path density must be increased in the region near the estimated position of the object at the expense of the nominal density at the larger distances. Based on the analysis of Appendix D it appears that when both $\bar{\sigma}_1 \leq \frac{3}{2}\sigma_1$ and $\bar{\sigma}_2 \leq \frac{3}{2}\sigma_2$, if $\frac{W}{\sigma_1}$ is not too large, then nearly maximum searching efficiency can be obtained by the use of multiarea searching procedures differing from those specified in Table 4 mainly in a suitable reduction of all linear dimensions of the nominally searched areas. Based on the analysis of Appendix D it appears that for a total number of coverages of one or two, and for given $W/\bar{\sigma}_1$, greater searching efficiency is realized when the navigational error is of the same order as the uncertainty in the object position than when it is much smaller than the uncertainty in object position, provided the dimensions of the nominal areas to be searched are properly chosen; but for a total number of coverages of 4 or more, it is not possible to attain efficiencies quite as high as is indicated by Table 4 if $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ are of the order of 1 or more.

72. As the navigational error increases, the searching efficiency becomes less critically dependent on the shape and on the precise dimensions of the nominal areas searched on the various coverages, and it depends less on the total number of coverages N. On the basis of the analysis of Appendix D, it is concluded that when the navigational error becomes comparable to the uncertainty in object position, there is little to be gained in taking N larger than 3, and the apparent theoretical superiority of elliptical area searching and of rectangular area searching over constant parallel path searching is substantially smaller than is indicated by Table 4. Hence, in formulating multiarea searching procedures for the case when both $\frac{1}{2}\sigma_1 \leq \bar{\sigma}_1 \leq \frac{3}{2}\sigma_1$ and $\bar{\sigma}_2 \leq \frac{3}{2}\sigma_2$, it will be adequate to consider only numbers of coverages N for which efficiencies can be realized as large as or, at worst, negligibly less than, are indicated in Table 4.

73. Specifically, it will be assumed that when $\frac{1}{2}\sigma_1 \leq \bar{\sigma}_1 \leq \sigma_1$ and $\bar{\sigma}_2 \leq \sigma_2$; the total number of coverages N is not greater than 3. It is further assumed that when $\sigma_1 < \bar{\sigma}_1 \leq \frac{3}{2}\sigma_1$ and $\bar{\sigma}_2 \leq \frac{3}{2}\sigma_2$, then N does not exceed 3 for constant parallel path searching and for circle diameters searching, and does not exceed 2 for elliptical area searching and rectangular area searching. With these restrictions the values of T and τ given in Table 4 are valid to a sufficient approximation if the multiarea searching procedures are carried out as follows:

(a) Reduce the maximum dimensions in the x and y-directions of the area searched on the final coverage by amounts $\frac{3}{2}\bar{\sigma}_1$ and $\frac{3}{2}\bar{\sigma}_2$, respectively, below the values listed in Table 4.

(1) For elliptical area searching, if $(a_m)_0$ and $(b_m)_0$ are the semi-axes of the N'th area specified in Table 4, take new semi-axes a_N and b_N given by $a_N = (a_N)_0 - \frac{3}{4}\bar{\sigma}_1$, $b_N = (b_N)_0 - \frac{3}{4}\bar{\sigma}_2$.

(2) For rectangular area searching, if $(X_N)_0$ and $(Y_N)_0$ are the half dimensions of the N'th area specified in Table 4, take new dimensions X_N and Y_N, given by $X_N = (X_N)_0 - \frac{3}{4}\bar{\sigma}_1$, $Y_N = (Y_N)_0 - \frac{3}{4}\bar{\sigma}_2$.

(3) For constant parallel path searching, if $2(X_N)_0$ and ℓ_0 are the dimensions of the N'th area specified in Table 4, take new dimensions $2X_N$ and ℓ, given by $2X_N = 2(X_N)_0 - \frac{3}{2}\bar{\sigma}_1$ and $\ell = \ell_0 - \frac{3}{2}\bar{\sigma}_2$.

(4) For circle diameters searching, if R_0 is the radius specified in Table 4, take a new radius R, given by $R = R_0 - \frac{3}{4}\bar{\sigma}$.

(b) Reduce the dimensions of all areas below those specified in the table by the same percentage that the N'th area is reduced. That is, the relation between the dimensions of the n'th area and those of the N'th area are still as indicated in Note 3 of Table 4.

(c) Increase the required value of $\gamma\Delta m$ by the same percentage that the areas are decreased. If $(\gamma\Delta m)_0$ is the value of $\gamma\Delta m$ specified in Table 4, the required new value is as follows:

(1) For elliptical area searching the required value is

$$\frac{(a_N)_0\,(b_N)_0}{a_N\,b_N}\,(\gamma\Delta m)_0.$$

(2) For rectangular area searching, the required value is

$$\frac{(X_N)_0\,(Y_N)_0}{X_N\,Y_N}\,(\gamma\Delta m)_0.$$

(3) For constant parallel path searching, the required value is

$$\frac{(X_N)_0\,\ell_0}{X_N\,\ell}\,(\gamma\Delta m)_0.$$

(4) For circle diameters searching, the required value is

$$\frac{R_0^2}{R^2}\,(\gamma\Delta m)_0.$$

74. It is to be expected that the theoretical advantage of the modified searching procedures over the regular searching procedures will be slightly less than is indicated by Table 5 when the navigational error becomes comparable with the uncertainty in the object position. Nevertheless, the modified searching procedures will give substantial reductions in the value of τ if there are only one or two coverages. An exact

knowledge of the reduction in τ which can be obtained by the modified procedures is not necessary in order to take advantage of these more efficient procedures.

75. The suggested reductions in the dimensions of the nominally searched areas are near optimum for values of $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ of 1.0 or less. For larger values of these ratios, slightly greater theoretical efficiencies could be obtained with greater reductions of the dimensions, assuming zero turning times, but the possible improvement is too small to be important. The decreased size of the areas to be searched places a greater restriction on the values of W/σ_1 for which the multiarea searching procedure can be carried out in the prescribed manner, and it is necessary to consider alternative procedures for use when W/σ_1 is so large that multiarea searching is impractical.

76. A searching procedure which is operationally simple and which gives near maximum searching efficiency when W/σ_1 is so large as to make multiarea searching inappropriate, is "single track" searching. Single track searching consists of making all passes on a single track, parallel to the y-axes, with all nominal search paths of equal length, and with all paths centered at the most probable position of the object. The minimum value of W/σ_1 for which single track searching should be considered in preference to multiarea searching depends on the magnitude of $\bar{\sigma}_1/\sigma_1$. When $\bar{\sigma}_1/\sigma_1 = 3/2$, single track searching and multiarea searching are approximately equally efficient even when W/σ_1 is vanishingly small. At the other extreme, single track searching should not be used when $\bar{\sigma}_1/\sigma_1 \longrightarrow 0$ unless W/σ_1 is of the order of $5\sigma_1$ or more, depending on the value of P(T) required.

77. Tables 7 and 8 give values of T and τ for single track searching, for P(T) = 0.95 and P(T) = 0.99, as a function of the parameters W/σ_1, β, $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$. Details of the calculations are given in Appendix D. The data included in Table 7 are applicable, at most, only when neither $\bar{\sigma}_1/\sigma_1$ nor $\bar{\sigma}_2/\sigma_2$ is greater than 3/2, while the data in Table 8 are valid if $\bar{\sigma}_1/\sigma_1$ does not exceed 3/2, regardless of the value of $\bar{\sigma}_2/\sigma_2$. The smallest listed values of W/σ_1 for given $\bar{\sigma}_1/\sigma_1$ are approximately the smallest values for which single track-searching is about as efficient as multiarea searching and may be used as a guide for the selection of the appropriate procedure. As a general rule, it appears that single track searching will be at least as efficient as multiarea searching if $W > 5(\sigma_1 - \frac{1}{2}\bar{\sigma}_1)$ and P(T) is required to be 0.95, or if $W > 6(\sigma_1 - \frac{1}{2}\bar{\sigma}_1)$ and a value of 0.99 is required for P(T). The data in Table 7 are based on a search path length ℓ equal to $5\sigma_1$ for P(T) = 0.95, and a path length equal to $6\sigma_1$ for P(T) = 0.99. These lengths appear to give near maximum searching efficiency in most cases if turning times are not too large. The data in Table 7 are exact only when $\bar{\sigma}_2/\sigma_2$ is vanishingly small but are adequate for $\bar{\sigma}_2/\sigma_2 \leq 1$, and with proper precautions can be used for $\bar{\sigma}_2/\sigma_2 \leq 3/2$.

78. The data in Table 8 are applicable when the search path length ℓ is sufficiently large so there is negligible chance of missing an object because of turning too soon. As indicated in a footnote to Table 8, the required condition will certainly be satisfied for P(T) = 0.95 if $\ell/\sigma_2 \geq 6\sqrt{1 + (\bar{\sigma}_2/\sigma_2)^2}$, and for P(T) = 0.99 if $\ell/\sigma_2 \geq 6.6\sqrt{1 + (\bar{\sigma}_2/\sigma_2)^2}$. Choosing path lengths to satisfy these conditions may represent a more efficient procedure than that specified in Table 7 if the turning time is quite large or if W and β are so large that only one or two of the longer passes are required to obtain the specified value of P(T). Constant parallel path searching could also be carried out using

search paths so long that there would be negligible chance of missing the object because of turning too soon. Table D2 of Appendix D provides essential data for this case.

79. Included in Tables 7 and 8 are the number of search passes N required to obtain the specified values of P(T). The given values of P(T) are generally not obtained with an integral number of search passes. In practical operations it probably would be desirable to make the smallest integral number of passes required to obtain a value of P(T) of not less than a specified value. However, it is to be expected that the values of the various parameters which apply in particular searching operations will not correspond exactly to any set listed in the table and that the appropriate data must be obtained by interpolation. In order to obtain reasonably adequate estimates of N by such interpolation, it seemed desirable to list non-integral values of N in the table. These non-integral values were obtained by linear interpolation from the values of N', N", $P(T_{N'})$, and $P(T_{N''})$, where N' is the largest integral number of passes for which P(T) is not greater than the specified value, N" is the smallest integral number of passes for which P(T) is not less than the specified value, and $P(T_{N'})$ and $P(T_{N''})$ are the values of P(T) obtained with N' and N" passes, respectively. The values of T given in the tables were obtained from the indicated values of N by the relation

$$N\beta W\ell = UT \text{ or } N\beta\frac{W}{\sigma_1} = \frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}.$$

The values of τ have also been obtained by linear interpolation from the values of τ for $P(T_{N'})$ and $P(T_{N''})$.

80. The analysis of single track searching given in Appendix D is based on the assumption that the probability of detecting an object lying within the over-all path width W is equal to a constant β, independent of the athwartship position within W. If the detection probability varies only moderately with athwartship position within the search path, the data given for single track searching may be expected to be reasonably adequate if β is taken equal to the average detection probability over the width W. But if there are large variations in the detection probability within W, then the use of an average detection probability for β in single track searching may lead to substantial error if W is large compared to σ_1. The principle type of variation of detection probability to be expected for practical locators is one corresponding to a relatively large detection probability over a substantial central part of W, with reduced detection probability over parts of W adjacent to the edges. For this type of variation the data on single track searching should be reasonably adequate if β is interpreted to be the average detection probability. Certainly, the indicated values of T and τ should then be sufficiently pessimistic except, possibly, in the extreme case where the detection probability over substantial parts of the path width near the edges is quite small compared to the detection probability over the central part of the path.

81. The limiting case of zero detection probability near the edges can readily be investigated by considering any of the cases listed in the tables, and supposing the width W is increased by any amount, but with zero detection probability over the added widths. The values of T and τ obtained with a β equal to the average detection probability over

TABLE 7

SINGLE TRACK SEARCHING WITH SEARCH PATH LENGTHS* WHICH ARE NEAR OPTIMUM IN MOST CASES** WHEN TURNING TIME IS ZERO OR SMALL
(For the case where $\bar{\sigma}_1/\sigma_1 \le 3/2$ and $\bar{\sigma}_2/\sigma_2 \le 3/2$)***

$\bar{\sigma}_1/\sigma_1$	W/σ_1	ρ	P(T) = 0.95			P(T) = 0.99		
			N	$T\Big/\left(\frac{\sigma_1\sigma_2}{U}\right)$	$\tau\Big/\left(\frac{\sigma_1\sigma_2}{U}\right)$	N	$T\Big/\left(\frac{\sigma_1\sigma_2}{U}\right)$	$\tau\Big/\left(\frac{\sigma_1\sigma_2}{U}\right)$
0	5	1.0	0.97	24.4	12.2			
		0.9	1.82	41.0	13.5			
		0.75	2.78	52.1	15.4			
		0.5	5.30	66.3	18.4			
		0.25	12.7	79.5	21.4			
		→0	--	91.3	24.1			
	6	1.0	0.96	28.9	14.3	1.00	35.8	17.9
		0.9	1.72	46.4	15.8	2.60	84.1	19.8
		0.75	2.58	58.0	17.9	3.94	106.0	22.5
		0.5	4.87	73.0	21.4	7.81	141.0	27.0
		0.25	11.6	87.3	24.9	18.7	168.0	31.5
		→0	--	100.0	26.8	--	193.0	36.7
	7	1.0	0.96	33.7	16.6	0.99	41.7	20.8
		0.9	1.69	53.3	18.2	2.35	88.8	23.0
		0.75	2.53	66.4	20.7	3.75	118.0	26.0
		0.5	4.80	83.9	24.8	7.25	152.0	31.2
		0.25	11.4	100.0	28.8	17.4	182.0	36.4
		→0	--	115.0	30.8	--	209.0	42.2
0.5	4	1.0	1.91	38.2	11.6			
		0.9	2.57	46.2	14.4			
		0.75	3.59	53.9	15.9			
		0.5	6.45	64.5	18.0			
		0.25	14.8	73.9	20.6			
		→0						
	5	1.0	0.99	24.7	12.5	2.67	80.0	16.3
		0.9	1.85	41.5	14.2	3.35	90.6	17.9
		0.75	2.77	51.8	16.2	4.79	108.0	20.2
		0.5	5.17	64.6	19.6	8.75	131.0	24.0
		0.25	12.2	76.5	22.7	20.3	152.0	27.8
		→0						
	6	1.0	0.97	29.1	14.8	1.00	36.0	18.2
		0.9	1.73	46.8	16.7	2.56	82.9	20.2
		0.75	2.59	58.3	18.9	3.86	104.0	22.9
		0.5	4.86	72.9	22.8	7.48	135.0	27.5
		0.25	11.6	87.0	24.4	17.7	160.0	32.0
		→0						
1.0	2	1.0	7.90	79.0	18.0			
		0.5	17.3	86.4	20.7			
		→0	--	93.8	21.6			
	3	1.0	3.93	58.9	16.0	10.3	185.0	21.2
		0.9	4.62	62.4	16.3	11.6	188.0	22.1
		0.75	5.87	66.0	17.5	14.2	192.0	23.4
		0.5	9.69	72.7	19.3	22.1	199.0	25.9
		0.25	21.0	78.9	21.3	46.1	207.0	28.1
		→0	--	85.1	21.9	--	210.0	28.4
	4	1.0	2.34	46.8	14.0	5.07	122.0	17.9
		0.9	2.86	51.5	15.2	5.88	127.0	19.2
		0.75	3.87	58.0	16.8	7.51	135.0	21.1
		0.5	6.79	67.9	19.6	12.4	148.0	24.1
		0.25	15.4	76.8	22.1	26.9	161.0	27.1
		→0	--	85.2	23.5	--	173.0	29.7
1.0	5	1.0	1.65	41.2	14.6	2.92	87.6	17.9
		0.9	2.00	44.9	15.8	3.69	99.6	19.5
		0.75	2.97	55.8	17.8	4.96	112.0	21.9
		0.5	5.61	70.2	21.1	8.94	134.0	25.7
		0.25	13.0	81.1	24.2	20.6	154.0	29.5
		→0	--	--	--	--	--	--
	6	1.0	1.00	29.9	15.5	1.93	69.6	19.4
		0.9	1.84	49.7	17.0	2.82	91.4	21.2
		0.75	2.73	61.5	19.5	4.06	110.0	24.0
		0.5	5.05	75.7	23.4	7.79	140.0	28.3
		0.25	12.0	89.9	27.2	18.3	165.0	33.5
		→0	--	102.0	29.6	--	190.0	37.1
	7	1.0	0.97	34.1	17.5	1.51	63.3	21.5
		0.9	1.75	55.2	19.5	2.55	96.2	23.7
		0.75	2.60	68.3	22.2	3.83	121.0	26.8
		0.5	4.85	84.8	26.6	7.38	155.0	31.9
		0.25	11.6	102.0	30.9	17.5	184.0	37.5
		→0	--	118.0	33.8	--	214.0	42.2
1.5	→0	All ρ	--	86.0	24.3	--	190.0	32.6
	1.5	1.0	9.87	74.0	20.7	18.3	164.0	25.7
		0.9	11.2	75.3	21.3	20.5	166.0	26.1
		0.75	13.7	77.3	22.1	25.0	169.0	26.9
		0.5	21.4	80.4	22.8	38.6	174.0	28.1
		0.25	44.5	83.5	23.7	79.2	178.0	29.0
		→0	--	87.3	24.1	--	179.0	29.7
	3.0	1.0	4.19	62.8	18.4	7.53	136.0	22.7
		0.9	4.88	65.8	19.4	8.62	140.0	23.7
		0.75	6.24	70.2	20.4	10.8	146.0	25.0
		0.5	10.2	76.8	22.4	17.4	156.0	27.3
		0.25	22.2	83.4	24.3	37.0	166.0	29.6
		→0	--	89.6	24.9	--	177.0	31.4
	6.0	1.0	1.71	51.3	18.0	2.65	95.3	22.1
		0.9	2.03	54.9	19.6	3.34	108.0	23.9
		0.75	2.98	67.1	21.8	4.73	128.0	26.7
		0.5	5.62	84.3	25.7	8.59	155.0	31.2
		0.25	13.0	97.3	29.5	19.8	178.0	35.8
		→0	--	112.0	31.8	--	206.0	39.8
	9.0	1.0	0.97	43.8	22.5	1.44	77.6	27.6
		0.9	1.74	70.6	24.9	2.48	121.0	30.4
		0.75	2.59	87.4	28.5	3.79	154.0	34.5
		0.5	4.86	109.0	34.1	7.31	197.0	41.6
		0.25	11.6	130.0	39.6	17.4	235.0	48.1
		→0	--	142.0	43.0	--	276.0	54.2
	12.0	1.0	0.96	57.8	29.3	0.99	71.5	35.9

TABLE 7 (Continued)

*Note 1: - All search paths are assumed to have a length as follows:

$$\frac{\ell}{\sigma_2} = 5 \text{ for } P(t) = 0.95$$

$$\frac{\ell}{\sigma_2} = 6 \text{ for } P(t) = 0.99.$$

**Note 2: - Where Table 8 indicates that only one or two of the longer passes are required to obtain the specified P(T), the procedure specified in Table 8 is likely to be more efficient and should be used except, at most, when $\bar{\sigma}_2/\sigma_2 \longrightarrow 0$.

***Note 3: - Data are exact for $\bar{\sigma}_2/\sigma_2 \longrightarrow 0$. For larger values of $\bar{\sigma}_2/\sigma_2$ the following applies:

(a) For values of W/σ_1, β, and $\bar{\sigma}_1/\sigma_1$ such that only one or two passes of sufficient length are required, the longer paths should be used (see note 2).

(b) Excluding cases mentioned in (a), the data of the present table are entirely adequate for values of $\bar{\sigma}_1/\sigma_1$ not exceeding 1.0.

(c) Excluding cases mentioned in (a), the searching requirements are likely to be greater than is indicated by the tabulated data if

$$1 < \frac{\bar{\sigma}_2}{\sigma_2} \leq \frac{3}{2},$$

but sufficiently pessimistic searching requirements are obtained for this range if the indicated values of

$$N, \quad \frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)}, \quad \text{and} \quad \frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$$

are increased by 20%.

Note 4: - N is the required number of locator passes to insure the specified value of P(T). That is, if searching continues until the object is found or until N passes have been made, whichever occurs first, then the fraction of objects investigated which are found is the indicated value of P(T).

$$N\beta W\ell = UT.$$

Note 5: - If the search rate U is not constant, then accurate values of T and approximate values of τ are obtained if U is interpreted to be the time average of the search rate. If the ground speed is constant and equal to V_o except on turns, and if t_c is the time required to make each turn, then the time average of the search rate over the interval T is given by

$$U = \beta W \left[\frac{V_o}{1 + \left(1 - \frac{1}{N}\right) \frac{V_o \, t_c}{\ell}} \right].$$

However, T can be obtained most simply in terms of N from the relation

$$T = \frac{N\ell}{V_o} + (N - 1)\, t_c.$$

From this value of T and from the value of τ/T, which can be obtained from the data in the table, an estimate of τ can easily be obtained.

TABLE 8

SINGLE TRACK SEARCHING WITH SEARCH PATHS SO LONG* THERE IS NEGLIGIBLE CHANCE OF MISSING AN OBJECT BECAUSE OF TURNING TOO SOON

(For the case where $\bar{\sigma}_1/\sigma_1 \le 3/2$)**

$\bar{\sigma}_1/\sigma_1$	W/σ_1	β	P(T) = 0.95: N	P(T) = 0.95: T $\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}$	P(T) = 0.95: τ $\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}$	P(T) = 0.99: N	P(T) = 0.99: T $\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}$	P(T) = 0.99: τ $\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}$
0	5	1.0	0.96	4.81	2.43			
		0.9	1.69	7.60	2.71			
		0.75	2.52	9.46	3.09			
		0.5	4.78	12.0	3.71			
		0.25	11.4	14.2	4.33			
		→0	--	16.3	4.95			
	6	1.0	0.95	5.72	2.87	0.99	5.96	2.99
		0.9	1.58	8.55	3.16	2.30	12.4	3.30
		0.75	2.32	10.4	3.59	3.71	16.7	3.76
		0.5	4.48	13.4	4.31	7.13	21.4	4.51
		0.25	10.6	15.9	5.03	17.1	25.7	5.27
		→0	--	18.3	5.75	--	29.5	6.02
	7	1.0	0.95	6.65	3.33	0.99	6.93	3.47
		0.9	1.56	9.83	3.66	2.05	12.9	3.82
		0.75	2.28	12.0	4.18	3.52	18.5	4.34
		0.5	4.41	15.4	5.00	6.78	23.7	5.21
		0.25	10.5	18.3	5.83	16.2	28.3	6.08
		→0	--	21.0	6.66	--	32.6	6.95
0.5	4	1.0	1.60	6.42	2.08			
		0.9	2.05	7.39	2.51			
		0.75	3.01	9.03	2.82			
		0.5	5.69	11.4	3.29			
		0.25	13.1	13.1	3.79			
		→0						
	5	1.0	0.97	4.87	2.49	2.00	9.98	2.66
		0.9	1.73	7.77	2.79	2.89	13.0	2.93
		0.75	2.53	9.50	3.17	4.23	15.9	3.31
		0.5	4.73	11.8	3.74	7.88	19.7	3.94
		0.25	11.1	13.9	4.35	18.4	23.0	4.57
		→0						
	6	1.0	0.96	5.74	2.89	1.00	5.98	3.01
		0.9	1.61	8.67	3.19	2.31	12.4	3.33
		0.75	2.34	10.5	3.62	3.65	16.4	3.78
		0.5	4.49	13.5	4.33	6.92	20.8	4.52
		0.25	10.6	15.9	5.05	16.5	24.8	5.27
		→0						
1.0	2	1.0	6.59	13.2	3.23			
		0.5	14.6	14.6	3.74			
	3	1.0	3.41	10.2	2.88	8.59	25.8	3.21
		0.9	3.94	10.6	3.00	9.71	26.2	3.36
		0.75	5.08	11.4	3.24	11.9	26.9	3.58
		0.5	8.52	12.8	3.60	18.7	28.1	3.96
		0.25	18.6	14.0	3.98	39.1	29.3	4.33
		→0	--	15.1	4.36	--	30.7	4.71
	4	1.0	1.97	7.88	2.70	4.41	17.6	2.93
		0.9	2.59	9.33	2.89	5.08	18.3	3.14
		0.75	3.50	10.5	3.19	6.60	19.8	3.45
		0.5	6.08	12.2	3.71	11.0	22.0	3.95
		0.25	13.9	13.9	4.19	24.2	24.2	4.45
		→0	--	15.4	4.69	--	26.5	4.95
1.0	5	1.0	1.45	7.25	2.81	2.66	13.3	2.97
		0.9	1.91	8.58	3.04	3.29	14.8	3.22
		0.75	2.79	10.5	3.41	4.62	17.3	3.60
		0.5	5.06	12.6	4.03	8.27	20.7	4.22
		0.25	11.8	14.8	4.63	19.0	23.7	4.85
		→0						
	6	1.0	0.98	5.90	3.04	1.84	11.0	3.20
		0.9	1.73	9.35	3.33	2.65	14.3	3.50
		0.75	2.51	11.3	3.78	3.85	17.3	3.95
		0.5	4.68	14.0	4.46	7.26	21.8	4.70
		0.25	11.0	16.4	5.21	17.1	25.6	5.50
		→0	--	18.9	5.93	--	29.1	6.19
	7	1.0	0.96	6.74	3.39	1.28	8.97	3.56
		0.9	1.63	10.3	3.75	2.32	14.6	3.91
		0.75	2.36	12.4	4.25	3.63	19.0	4.44
		0.5	4.49	15.7	5.08	6.87	24.1	5.30
		0.25	10.6	18.6	5.92	16.4	28.7	6.17
		→0	--	21.3	6.75	--	32.9	7.04
1.5	→0	All β		15.4	4.58		28.1	4.88
	1.5	1.0	13.3	13.3	3.93	24.4	24.4	4.20
		0.9	15.0	13.5	4.01	27.5	24.7	4.27
		0.75	18.5	13.9	4.16	33.6	25.2	4.39
		0.5	28.9	14.4	4.30	51.9	26.0	4.57
		0.25	60.1	15.0	4.49	107.0	26.7	4.76
		→0	--	15.6	4.68	--	27.5	4.96
	3.0	1.0	5.69	11.4	3.53	10.1	20.3	3.73
		0.9	6.64	11.9	3.66	11.7	21.0	3.88
		0.75	8.47	12.7	3.89	14.7	22.0	4.10
		0.5	13.9	13.9	4.24	23.7	23.7	4.48
		0.25	30.2	15.1	4.61	50.7	25.4	4.85
		→0	--	16.2	4.97	--	27.0	5.23
	6.0	1.0	2.35	9.39	3.48	3.61	14.4	3.64
		0.9	2.89	10.4	3.77	4.48	16.1	3.94
		0.75	4.21	12.6	4.21	6.60	19.8	4.39
		0.5	7.64	15.3	4.92	11.9	23.9	5.14
		0.25	17.8	17.8	5.63	27.8	27.8	5.88
		→0	--	20.1	6.35	--	31.4	6.62
	9.0	1.0	1.44	8.66	4.38	1.82	10.9	4.57
		0.9	2.43	13.1	4.81	3.36	18.2	5.02
		0.75	3.53	15.9	5.46	5.38	24.2	5.69
		0.5	6.73	20.2	6.52	10.3	30.8	6.80
		0.25	15.9	23.9	7.59	24.4	36.7	7.91
		→0	--	25.6	8.57	--	42.4	9.03
	12.0	1.0	1.43	11.4	5.71	1.49	11.9	5.95

*Note 1: - The required condition will certainly be satisfied if $\ell/\sigma_2 \ge 6\sqrt{1+(\bar{\sigma}_2/\sigma_2)^2}$ for P(T) = 0.95, or if $\ell/\sigma_2 \ge 6.6\sqrt{1+(\bar{\sigma}_2/\sigma_2)^2}$ for P(T) = 0.99.

**Note 2: - Data are valid for all values of $\bar{\sigma}_2/\sigma_2$.

Note 3: - Notes 4 and 5 of Table 7 apply also to this table.

the increased width can then be compared with the accurate values given in the table. From such comparisons, based on data in Tables 7 and 8, and on data in the tables which are presented in the next section, it is apparent that even in this extreme case the use of an average β leads to conservative results except, at most, when $\bar{\sigma}_1/\sigma_1$ is of the order of 1 or less. If one makes the convention that when $\bar{\sigma}_1/\sigma_1 < 1$, the over-all width W is to be taken to be the width for which the detection probability is not less than one third of the value over the central parts of the path, the use of an average detection probability for β should be adequate for all practical cases where the detection probability over the central part of W is not substantially less than for the remaining parts of W. In the unlikely case where the detection probability over the central part of W is substantially smaller than over the outer parts of W, the use of the average detection probability for β may lead to substantial underestimates of N, T', and τ, but conservative results will always be obtained in this case if β is taken equal to the value of the detection probability over the central part of the path.

NAVIGATIONAL ERRORS LARGE COMPARED TO THE UNCERTAINTY IN THE OBJECT POSITION

$$\left(\bar{\sigma}_1 \geq \frac{3}{2}\sigma_1\right)$$

82. When $\bar{\sigma}_1/\sigma_1 > 3/2$, the dispersion of search paths as a result of navigational error, is so large that the most efficient searching procedure is to aim all passes at the most probable position of the object, even when the search path width is quite small. Of the various possible methods of searching so that all nominal search paths are centered at the most probable position of the object, single track searching appears to be easy to carry out and to be about as efficient as any other. Hence, in the present section, numerical data are presented only for single track searching.

83. In considering the location problem when navigational errors are large compared to the uncertainty in the object position, it is convenient to distinguish between the case where neither $\bar{\sigma}_1/\sigma_1$ nor $\bar{\sigma}_2/\sigma_2$ is less than 3/2, and the cases where one of these quantities is less than 3/2. Since it has been assumed that σ_1 and σ_2 have been chosen so that $\sigma_2 \geq \sigma_1$, the case where $\bar{\sigma}_1/\sigma_1 \leq 3/2 < \bar{\sigma}_2/\sigma_2$ is not likely to occur in practice and will not be considered further. It may be noted, however, that Table 8 is valid for this case.

84. In considering single track searching for the case $\bar{\sigma}_2/\sigma_2 \leq 3/2 < \bar{\sigma}_1/\sigma_1$ it is necessary first to consider suitable choices for the length of the search paths. The factors determining this choice are essentially the same as those considered in the last section. Thus, what appears to be required are extensions of Tables 7 and 8 to values of $\bar{\sigma}_1/\sigma_1$ greater than 3/2. Tables 9 and 10 represent such extensions. The lengths of path specified in Table 9 are the same as those specified in Table 7 for the same values of σ_2 and P(T), and are expected to give near maximum searching efficiency in most cases if turning times are not too large. The data in Table 10 are for search path lengths sufficiently large so there is negligible chance of missing an object because

TABLE 9

SINGLE TRACK SEARCHING WITH SEARCH PATH LENGTHS* WHICH ARE NEAR OPTIMUM IN MOST CASES WHEN TURNING TIME IS ZERO OR SMALL**
(For the case where $\bar{\sigma}_1/\sigma_1 \geq 3/2$ and $\bar{\sigma}_2/\sigma_2 \leq 3/2$)***

$\bar{\sigma}_1/\sigma_1$	$W/\bar{\sigma}_1$	β	P(T) = 0.95 N	P(T) = 0.95 $T/\left(\frac{\bar{\sigma}_1\sigma_2}{U}\right)$	P(T) = 0.95 $\tau/\left(\frac{\bar{\sigma}_1\sigma_2}{U}\right)$	P(T) = 0.99 N	P(T) = 0.99 $T/\left(\frac{\bar{\sigma}_1\sigma_2}{U}\right)$	P(T) = 0.99 $\tau/\left(\frac{\bar{\sigma}_1\sigma_2}{U}\right)$
1.5	→0	All β	--	57.3	16.2	--	127.0	21.7
	1	1.0	9.87	49.3	13.8	18.3	110.0	17.1
		0.9	11.2	50.2	14.2	20.5	111.0	17.4
		0.75	13.7	51.5	14.7	25.0	113.0	17.9
		0.5	21.4	53.6	15.2	38.6	116.0	18.7
		0.25	44.5	55.7	15.8	79.2	119.0	19.4
		→0	--	58.2	16.1	--	119.0	19.8
	2	1.0	4.19	41.9	12.3	7.53	90.4	15.2
		0.9	4.88	43.9	12.9	8.62	93.1	15.8
		0.75	6.24	46.8	13.6	10.8	97.1	16.7
		0.5	10.2	51.2	14.9	17.4	104.0	18.2
		0.25	22.2	55.6	16.2	37.0	111.0	19.7
		→0	--	59.7	16.6	--	118.0	20.9
	4	1.0	1.71	34.2	12.0	2.65	63.5	14.7
		0.9	2.03	36.6	13.0	3.34	72.2	16.0
		0.75	2.98	44.7	14.6	4.73	85.1	17.8
		0.5	5.62	56.2	17.2	8.59	103.0	20.8
		0.25	13.0	64.9	19.6	19.8	119.0	23.8
		→0	--	74.4	21.2	--	137.0	26.5
	6	1.0	0.97	29.2	15.0	1.44	51.7	18.4
		0.9	1.74	47.1	16.6	2.48	80.4	20.2
		0.75	2.59	58.3	19.0	3.79	102.0	23.0
		0.5	4.86	72.9	22.8	7.31	131.0	27.7
		0.25	11.6	86.8	26.4	17.4	157.0	32.1
		→0	--	94.8	28.7	--	184.0	36.1
	8	1.0	0.96	38.5	19.5	0.99	47.7	23.9
2.0	→0	All β	--	48.4	14.3	--	92.1	17.3
	1	1.0	8.18	40.9	12.2	13.0	77.9	14.8
		0.9	9.30	41.8	12.5	14.7	79.6	15.1
		0.75	11.5	43.2	12.8	18.2	81.8	15.6
		0.5	18.1	45.2	13.5	28.8	36.5	16.3
		0.25	37.9	47.4	14.1	59.5	89.3	17.1
		→0	--	49.5	14.0	--	92.7	17.6
	2	1.0	3.57	35.7	11.0	5.60	67.2	13.4
		0.9	4.13	37.1	11.5	6.35	68.6	13.9
		0.75	5.40	40.5	12.3	8.42	75.8	14.9
		0.5	8.96	44.8	13.5	13.9	83.5	16.4
		0.25	19.7	49.2	14.8	30.4	91.2	17.9
		→0	--	55.5	15.4	--	97.8	19.2
	4	1.0	1.53	30.7	11.4	2.00	47.9	13.9
		0.9	1.93	34.8	12.6	2.92	63.1	15.2
		0.75	2.87	43.0	14.0	4.32	77.7	17.0
		0.5	5.35	53.5	16.5	7.99	95.9	20.0
		0.25	12.5	62.5	19.0	18.8	113.0	23.0
		→0	--	70.5	20.5	--	130.0	25.7
	6	1.0	0.97	29.1	14.8	1.00	36.0	18.2
		0.9	1.72	46.4	16.5	2.40	77.8	20.0
		0.75	2.56	57.6	18.8	3.76	101.0	22.8
		0.5	4.82	72.4	22.6	7.22	130.0	27.3
		0.25	11.5	86.2	26.2	17.3	155.0	31.8
		→0	--	--	--	--	--	--

$\bar{\sigma}_1/\sigma_1$	$W/\bar{\sigma}_1$	β	P(T) = 0.95 N	P(T) = 0.95 $T/\left(\frac{\bar{\sigma}_1\sigma_2}{U}\right)$	P(T) = 0.95 $\tau/\left(\frac{\bar{\sigma}_1\sigma_2}{U}\right)$	P(T) = 0.99 N	P(T) = 0.99 $T/\left(\frac{\bar{\sigma}_1\sigma_2}{U}\right)$	P(T) = 0.99 $\tau/\left(\frac{\bar{\sigma}_1\sigma_2}{U}\right)$
3	→0	All β	--	43.7	13.2	--	80.0	15.9
	2	1.0	3.13	31.3	10.2	4.79	57.5	12.3
		0.9	3.77	34.0	10.7	5.69	61.5	12.9
		0.75	4.89	36.7	11.4	7.43	66.8	13.8
		0.5	8.35	41.8	12.7	12.5	75.2	15.4
		0.25	18.3	45.8	13.9	27.2	81.5	16.9
		→0	--	50.0	14.5	--	90.0	18.1
	4	1.0	1.36	27.3	11.0	1.93	46.4	13.5
		0.9	1.89	33.9	12.2	2.81	60.8	14.7
		0.75	2.79	41.9	13.4	4.03	72.6	16.5
		0.5	5.18	51.8	16.1	7.79	93.5	19.5
		0.25	12.2	61.0	18.6	18.4	110.0	22.5
		→0	--	--	--	--	--	--
→∞	→0	All β	--	41.0	12.1	--	74.0	14.9
	1	1.0	6.81	34.1	10.2	10.2	61.3	12.6
		0.9	7.78	35.0	10.4	11.7	63.1	12.9
		0.75	9.69	36.3	10.8	14.6	65.5	13.3
		0.5	15.4	38.5	11.4	23.2	69.5	14.1
		0.25	32.5	40.6	12.0	48.9	73.3	14.8
		→0	--	42.7	12.9	--	77.0	15.7
	2	1.0	2.91	29.1	9.34	4.41	52.9	11.5
		0.9	3.55	31.9	9.82	5.23	56.5	12.1
		0.75	4.64	34.8	10.5	6.89	62.0	13.0
		0.5	7.86	39.3	11.7	11.8	70.9	14.5
		0.25	17.5	43.7	12.9	26.3	78.9	16.0
		→0		47.9	14.5	--	86.4	17.6
	3	1.0	1.83	27.4	9.53	2.68	48.3	11.7
		0.9	2.28	30.7	10.2	3.40	55.1	12.6
		0.75	3.18	35.7	11.3	4.79	64.7	13.9
		0.5	5.81	43.6	13.1	8.72	78.5	16.2
		0.25	13.4	50.3	14.9	20.1	90.3	18.4
		→0	--	56.6	17.2	--	102.0	20.8
	4	1.0	1.17	23.4	10.7	1.88	45.1	13.1
		0.9	1.85	33.3	11.6	2.74	59.1	14.3
		0.75	2.74	41.1	13.0	3.95	71.1	16.1
		0.5	5.05	50.5	15.4	7.66	91.9	19.0
		0.25	12.0	60.0	17.8	18.0	108.0	22.0
		→0	--	68.5	20.8	--	124.0	25.2
	5	1.0	0.97	24.4	12.5	1.42	42.5	15.3
		0.9	1.74	39.2	13.7	2.46	66.4	16.8
		0.75	2.59	48.5	15.4	3.78	85.1	19.0
		0.5	4.85	60.6	18.4	7.29	109.0	22.7
		0.25	11.6	72.2	21.4	17.4	130.0	26.4
		→0	--	82.7	25.1	--	149.0	30.5
	6	1.0	0.96	28.9	14.7	1.00	35.8	18.0
		0.9	1.70	45.9	16.1	2.34	75.7	19.8
		0.75	2.54	57.1	18.2	3.73	101.0	22.5
		0.5	4.80	72.0	21.8	7.16	129.0	26.9
		0.25	11.4	85.7	25.4	17.2	154.0	31.4
		→0	--	98.3	29.8	--	177.0	36.2

*Note 1: - All search paths are assumed to have a length ℓ as follows: $\ell/\sigma_2 = 5$ for P(T) = 0.95; $\ell/\sigma_2 = 6$ for P(T) = 0.99.

**Note 2: - Where Table 10 indicates that only one or two of the longer passes are required to obtain the specified P(T), the procedure specified in Table 10 is likely to be more efficient and should be used except, at most, when $\bar{\sigma}_2/\sigma_2 \rightarrow 0$.

***Note 3: - Note 3 Table 7 is applicable to the data of this table.

Note 4: - Notes 4 and 5 of Table 7 apply also to this table.

TABLE 10

SINGLE TRACK SEARCHING WITH SEARCH PATHS SO LONG* THERE IS NEGLIGIBLE CHANCE OF MISSING AN OBJECT BECAUSE OF TURNING TOO SOON

(For the case where $\bar\sigma_1/\sigma_1 \geq 3/2$)**

$\frac{\bar\sigma_1}{\sigma_1}$	$\frac{W}{\bar\sigma_1}$	β	P(T) = 0.95: N	P(T) = 0.95: $T \Big/ \left(\frac{\bar\sigma_1\bar\sigma_2}{U}\right)\frac{\ell}{\bar\sigma_2}$	P(T) = 0.95: $\tau \Big/ \left(\frac{\bar\sigma_1\bar\sigma_2}{U}\right)\frac{\ell}{\bar\sigma_2}$	P(T) = 0.99: N	P(T) = 0.99: $T \Big/ \left(\frac{\bar\sigma_1\bar\sigma_2}{U}\right)\frac{\ell}{\bar\sigma_2}$	P(T) = 0.99: $\tau \Big/ \left(\frac{\bar\sigma_1\bar\sigma_2}{U}\right)\frac{\ell}{\bar\sigma_2}$
1.5	→0	All β	--	10.3	3.05	--	18.7	3.25
	1	1.0	8.84	8.84	2.62	16.3	16.3	2.80
		0.9	9.98	8.98	2.67	18.3	16.5	2.85
		0.75	12.3	9.25	2.77	22.4	16.8	2.93
		0.5	19.3	9.63	2.87	34.6	17.3	3.05
		0.25	40.0	10.0	2.99	71.2	17.8	3.17
		→0	--	10.4	3.12	--	18.3	3.31
	2	1.0	3.79	7.58	2.35	6.76	13.5	2.49
		0.9	4.42	7.96	2.44	7.77	14.0	2.59
		0.75	5.65	8.47	2.59	9.78	14.7	2.74
		0.5	9.28	9.28	2.83	15.8	15.8	2.99
		0.25	20.1	10.1	3.07	33.8	16.9	3.24
		→0	--	10.8	3.31	--	18.0	3.49
	4	1.0	1.56	6.26	2.32	2.40	9.62	2.43
		0.9	1.93	6.93	2.52	2.99	10.8	2.63
		0.75	2.81	8.43	2.80	4.40	13.2	2.93
		0.5	5.09	10.2	3.28	7.95	15.9	3.42
		0.25	11.9	11.9	3.76	18.5	18.5	3.92
		→0	--	13.4	4.23	--	20.9	4.41
	6	1.0	0.96	5.77	2.92	1.21	7.28	3.05
		0.9	1.62	8.76	3.21	2.24	12.1	3.34
		0.75	2.35	10.6	3.64	3.59	16.1	3.79
		0.5	4.49	13.5	4.35	6.83	20.5	4.53
		0.25	10.6	15.9	5.06	16.3	24.4	5.28
		→0	--	17.1	5.71	--	28.3	6.02
	8	1.0	0.95	7.61	3.81	0.99	7.93	3.97
2	→0	All β	--	8.91	2.74	--	14.2	2.86
	1	1.0	7.47	7.47	2.33	12.0	12.0	2.44
		0.9	8.47	7.63	2.37	13.6	12.3	2.49
		0.75	10.5	7.87	2.45	16.8	12.6	2.56
		0.5	16.5	8.24	2.57	26.4	13.2	2.69
		0.25	34.6	8.64	2.69	55.3	13.8	2.81
		→0	--	9.13	2.81	--	14.4	2.94
	2	1.0	3.21	6.42	2.10	5.12	10.2	2.20
		0.9	3.80	6.84	2.20	5.99	10.8	2.30
		0.75	4.89	7.34	2.34	7.94	11.9	2.45
		0.5	8.20	8.20	2.58	12.9	12.9	2.70
		0.25	18.0	8.99	2.82	28.3	14.1	2.95
		→0	--	10.1	3.08	--	15.3	3.20
	4	1.0	1.36	5.42	2.20	1.96	7.82	2.30
		0.9	1.85	6.66	2.40	2.81	10.1	2.50
		0.75	2.68	8.04	2.68	3.97	11.9	2.80
		0.5	4.88	9.76	3.16	7.55	15.1	3.29
		0.25	11.4	11.4	3.63	17.6	17.6	3.79
		→0	--	13.0	3.96	--	20.0	4.29
	6	1.0	0.96	5.74	2.89	1.00	5.98	3.01
		0.9	1.59	8.61	3.18	2.14	11.6	3.31
		0.75	2.31	10.4	3.60	3.54	15.9	3.76
		0.5	4.45	13.3	4.32	6.78	20.3	4.50
		0.25	10.5	15.8	5.03	16.2	24.2	5.24
		→0	--	--	--	--	--	--

$\frac{\bar\sigma_1}{\sigma_1}$	$\frac{W}{\bar\sigma_1}$	β	P(T) = 0.95: N	P(T) = 0.95: $T \Big/ \left(\frac{\bar\sigma_1\bar\sigma_2}{U}\right)\frac{\ell}{\bar\sigma_2}$	P(T) = 0.95: $\tau \Big/ \left(\frac{\bar\sigma_1\bar\sigma_2}{U}\right)\frac{\ell}{\bar\sigma_2}$	P(T) = 0.99: N	P(T) = 0.99: $T \Big/ \left(\frac{\bar\sigma_1\bar\sigma_2}{U}\right)\frac{\ell}{\bar\sigma_2}$	P(T) = 0.99: $\tau \Big/ \left(\frac{\bar\sigma_1\bar\sigma_2}{U}\right)\frac{\ell}{\bar\sigma_2}$
3	→0	All β	--	7.99	2.52	--	12.4	2.63
	2	1.0	2.89	5.78	1.95	4.51	9.01	2.03
		0.9	3.48	6.26	2.04	5.31	9.56	2.12
		0.75	4.53	6.79	2.19	6.91	10.4	2.28
		0.5	7.64	7.64	2.42	11.7	11.7	2.53
		0.25	16.8	8.40	2.66	25.9	12.9	2.77
		→0	--	9.15	2.90	--	14.1	3.02
	4	1.0	1.14	4.58	2.13	1.88	7.53	2.22
		0.9	1.79	6.45	2.32	2.68	9.64	2.42
		0.75	2.60	7.79	2.54	3.86	11.6	2.72
		0.5	4.77	9.54	3.08	7.30	14.6	3.21
		0.25	11.2	11.2	3.56	17.2	17.2	3.71
		→0	--	--	--	--	--	--
∞	→0	All β	--	7.51	2.38	--	11.5	2.48
	1	1.0	6.25	6.25	2.01	9.60	9.60	2.09
		0.9	7.11	6.40	2.05	10.9	9.83	2.14
		0.75	8.87	6.65	2.12	13.6	10.2	2.21
		0.5	14.1	7.05	2.24	21.7	10.8	2.34
		0.25	29.8	7.44	2.36	45.8	11.4	2.46
		→0	--	7.82	2.48	--	12.0	2.59
	2	1.0	2.74	5.47	1.83	4.02	8.04	1.91
		0.9	3.21	5.77	1.93	4.89	8.80	2.01
		0.75	4.23	6.35	2.07	6.51	9.76	2.16
		0.5	7.21	7.21	2.31	11.0	11.0	2.41
		0.25	16.0	8.0	2.55	24.6	12.3	2.66
		→0	--	8.78	2.78	--	13.5	2.90
	3	1.0	1.72	5.17	1.86	2.51	7.52	1.94
		0.9	1.99	5.38	2.01	3.08	8.32	2.09
		0.75	2.91	6.55	2.22	4.52	10.2	2.31
		0.5	5.34	8.00	2.58	8.14	12.2	2.69
		0.25	12.3	9.22	2.93	18.9	14.2	3.06
		→0	--	10.4	3.29	--	15.9	3.43
	4	1.0	1.00	3.98	2.08	1.82	7.27	2.17
		0.9	1.75	6.30	2.27	2.58	9.28	2.37
		0.75	2.53	7.60	2.56	3.79	11.4	2.66
		0.5	4.69	9.38	3.03	7.13	14.3	3.16
		0.25	11.0	11.0	3.51	16.9	16.9	3.65
		→0	--	12.6	3.98	--	19.3	4.15
	5	1.0	0.96	4.81	2.43	1.20	5.99	2.54
		0.9	1.62	7.29	2.67	2.21	9.97	2.78
		0.75	2.35	8.80	3.03	3.58	13.4	3.16
		0.5	4.48	11.2	3.62	6.82	17.1	3.77
		0.25	10.6	13.2	4.22	16.3	20.3	4.39
		→0	--	15.2	4.81	--	23.3	5.01
	6	1.0	0.95	5.72	2.86	0.99	5.96	2.99
		0.9	1.57	8.48	3.15	2.05	11.1	3.28
		0.75	2.28	10.3	3.58	3.50	15.8	3.73
		0.5	4.42	13.3	4.29	6.74	20.2	4.47
		0.25	10.5	15.7	5.00	16.1	24.1	5.21
		→0	--	18.0	5.72	--	27.7	5.96

*Note 1: - The required conditions will certainly be satisfied if $\ell/\bar\sigma_2 \geq 6\sqrt{1+(\sigma_2/\bar\sigma_2)^2}$ for P(T) = 0.95 or if $\ell/\bar\sigma_2 \geq 6.6\sqrt{1+(\sigma_2/\bar\sigma_2)^2}$ for P(T) = 0.99.

**Note 2: - Data are valid for all values of $\bar\sigma_2/\sigma_2$.

Note 3: - Notes 4 and 5 of Table 7 apply also to this table.

of turning too soon. Choosing the path length to satisfy this condition may represent a more efficient procedure than that specified in Table 9 if the turning time is quite large or if W and β are so large that only one or two of the longer passes is required to obtain a specified F(T). In order to include the limiting cases $\sigma_1 \longrightarrow 0$ and $\bar{\sigma}_2 \longrightarrow 0$ in Table 9, the quantities T and τ are expressed in terms of $\bar{\sigma}_1\sigma_2/U$ but can readily be converted into units of $\sigma_1\sigma_2/U$ or of $\bar{\sigma}_1\sigma_2/U$ by multiplying the listed numerical values by $\bar{\sigma}_1/\sigma_1$ or by $\sigma_2/\bar{\sigma}_2$.

85. There remains to be considered the case where neither of the quantities $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ is less than 3/2 and where at least one of them is greater than 3/2. This case seems unlikely to arise in practice except when the position of the object or target being investigated is already known rather accurately, and the term "object location" is then not very appropriate. Presumably the purpose of a searching operation under these conditions is to place the locator within effective range of the target to identify it, or to salvage it, or to do both. An example of effectively no uncertainty in the object position, with corresponding infinite values for $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$, is the investigation of a sonar contact by a locator which is guided by sonar that can show the relative positions of locator and of object or target being investigated. It may be noted that even with pulsed sonar, which is capable of determining the relative positions only at periodic intervals, errors due to too infrequent sighting of the locator and target should be regarded as navigational errors. The essential difference between this case and the case where the position of the object is uncertain, becomes apparent from a comparison of the expected search-path density distributions for the two cases. The greatest search-path density is expected to occur at the position of the object in the former case, whereas when the object position is uncertain, the greatest search-path density is expected to occur at the estimated most probable position of the object and not, in general, at the point where the object is actually located.

86. Since relative navigational errors as large as those considered here are unlikely to occur unless the object position is rather accurately known, it may be expected that when the turning time is appreciable, near maximum searching efficiency can be obtained by using search path lengths so large that there is negligible chance of missing the object because of turning too soon. Table 10 can then be used to determine the search requirements. Even when the required turning time is negligible the search procedure specified in Table 10 may be reasonably adequate. However, for locators with perfect mobility, substantially greater theoretical efficiency can be realized with shorter nominal search paths.

87. Table 11 gives the number of required sweep passes, as well as values of T and τ, as a function of the parameters $W/\bar{\sigma}_1$, β, $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ for various values of the search path length ℓ as specified by the value of the ratio $\ell/\bar{\sigma}_2$. Data are included only for the case where $\bar{\sigma}_1/\sigma_1 = \bar{\sigma}_2/\sigma_2$. When $\bar{\sigma}_1/\sigma_1$ is not equal to $\bar{\sigma}_2/\sigma_2$ the searching efficiency for given values of $W/\bar{\sigma}_1$, β and $\ell/\bar{\sigma}_2$ should, in nearly all cases, be intermediate between the efficiency which would be obtained if both $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ were equal to the smaller of these two quantities and the efficiency which would be obtained if both were equal to the larger of the two quantities. This principle may be sufficient to obtain adequate estimates of the searching requirements for practical purposes. The data presented in Table 11 for the limiting case $\bar{\sigma}_1/\sigma_1 \longrightarrow \infty$, $\bar{\sigma}_2/\sigma_2 \longrightarrow \infty$

TABLE 11

SINGLE TRACK SEARCHING DATA FOR THE CASE OF LARGE NAVIGATIONAL ERROR
(The data of this table supplements that of Table 10.)
(For the case where $\bar{\sigma}_1/\sigma_1 \geq 3/2$ and $\bar{\sigma}_2/\sigma_2 \geq 3/2$)

$\frac{\bar{\sigma}_1}{\sigma_1}$ and $\frac{\bar{\sigma}_2}{\sigma_2}$	$\frac{\ell}{\bar{\sigma}_2}$	β	P(T) = 0.95						P(T) = 0.99					
			$\frac{W}{\bar{\sigma}_1}=2$			$\frac{W}{\bar{\sigma}_1}=4$			$\frac{W}{\bar{\sigma}_1}=2$			$\frac{W}{\bar{\sigma}_1}=4$		
			N	$\frac{T}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	N	$\frac{T}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	N	$\frac{T}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	N	$\frac{T}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$
1.5	→0	All β		37.3*	10.7*		46.0	13.6		72.2	11.5		84.6	14.5
	2	1.0	8.28	33.1	9.63	4.49	35.9	11.0	15.6	62.3	10.3	7.97	63.7	11.6
		0.9	9.38	33.8	9.82	5.13	37.0	11.3	17.5	63.0	10.5	9.12	65.6	12.0
		0.75	11.6	34.7	10.1	6.54	39.2	11.9	21.4	64.2	10.8	11.4	68.5	12.6
		0.5	18.1	36.2	10.6	10.6	42.4	12.8	33.0	66.1	11.3	18.2	72.8	13.6
		0.25	37.7	37.7	11.1	22.7	45.5	13.8	68.0	68.0	11.8	38.6	77.2	14.6
	4	1.0	4.49	35.9	11.0	1.94	31.1	11.4	7.97	63.7	11.6	3.29	52.6	11.9
		0.9	5.13	37.0	11.3	2.52	36.3	12.4	9.12	65.6	12.0	3.95	56.9	12.6
		0.75	6.54	39.2	11.9	3.39	40.7	13.3	11.4	68.5	12.6	5.42	65.0	13.8
		0.5	10.6	42.4	12.8	5.92	47.4	15.1	18.2	72.8	13.6	9.44	75.5	15.8
		0.25	22.7	45.5	13.8	13.5	54.1	17.0	38.6	77.2	14.6	21.3	85.2	17.7
2.0	→0	All β		28.5	8.76		38.1	11.8		46.6	9.20		61.0	12.3
	2	1.0	6.39	25.6	7.96	3.68	29.4	9.54	10.4	41.6	8.34	5.85	46.8	9.89
		0.9	7.27	26.2	8.14	4.26	30.7	9.87	11.8	42.5	8.53	6.81	49.0	10.3
		0.75	9.02	27.1	8.41	5.48	32.9	10.4	14.7	44.0	8.83	8.70	52.2	10.9
		0.5	14.4	28.7	8.90	8.99	36.0	11.3	23.1	46.3	9.32	14.3	57.1	11.9
		0.25	30.2	30.2	9.37	19.6	39.2	12.3	48.5	48.5	9.82	30.9	61.8	12.8
	4	1.0	3.68	29.4	9.54	1.77	28.4	10.8	5.85	46.8	9.89	2.70	43.2	11.0
		0.9	4.26	30.7	9.87	2.10	30.3	11.1	6.81	49.0	10.3	3.36	48.4	11.6
		0.75	5.48	32.9	10.4	2.96	35.6	12.1	8.70	52.2	10.9	4.69	56.3	12.7
		0.5	8.99	36.0	11.3	5.46	43.7	14.1	14.3	57.1	11.9	8.43	67.4	14.6
		0.25	19.6	39.2	12.3	12.5	50.1	15.9	30.9	61.8	12.8	19.4	77.4	16.6
3.0	2	1.0	5.43	21.7	6.99	3.19	25.5	8.60	8.41	33.6	7.26	4.91	39.3	8.90
		0.9	6.20	22.3	7.15	3.79	27.3	8.98	9.63	34.7	7.46	5.82	41.9	9.31
		0.75	7.78	23.3	7.43	4.87	29.2	9.50	12.0	36.1	7.74	7.56	45.3	9.90
		0.5	12.5	25.0	7.90	8.16	32.6	10.4	19.3	38.6	8.24	12.6	50.5	10.9
		0.25	26.5	26.5	8.37	17.9	35.8	11.4	41.0	41.0	8.73	27.7	55.3	11.8
	4	1.0	3.19	25.5	8.60	1.63	26.1	10.5	4.91	39.3	8.90	2.27	36.4	10.3
		0.9	3.79	27.3	8.98	1.94	28.0	13.0	5.82	41.9	9.31	2.95	42.5	10.8
		0.75	4.87	29.2	9.50	2.83	34.0	11.6	7.56	45.3	9.90	4.32	51.8	12.0
		0.5	8.16	32.6	10.4	5.15	41.2	13.4	12.6	50.5	10.9	7.90	63.2	13.9
		0.25	17.9	35.8	11.4	12.0	47.8	15.2	27.7	55.3	11.8	18.4	73.7	15.9
→∞	→0	All β		22.0	6.98		31.5	9.98		33.8	7.27		48.4	10.4
	2	1.0		19.1	6.25		22.7	7.86		29.4	6.52		34.9	8.19
		0.9		19.8	6.44		24.4	8.24		30.5	6.71		37.6	8.59
		0.75		20.9	6.73		26.8	8.81		32.1	7.01		41.2	9.18
		0.5		22.6	7.20		30.4	9.76		34.7	7.51		46.7	10.2
		0.25		24.2	7.68		33.7	10.7		37.2	8.00		51.8	11.2
	4	1.0		22.7	7.86		19.8	9.08		34.9	8.19		30.5	9.47
		0.9		24.4	8.24		25.2	9.84		37.6	8.59		38.7	10.3
		0.75		26.8	8.81		31.3	11.0		41.2	9.18		48.1	11.5
		0.5		30.4	9.76		39.4	12.9		46.7	10.2		60.6	13.4
		0.25		33.7	10.7		46.4	14.8		51.8	11.2		71.3	15.4

*Note 1: - Data given for $\ell/\bar{\sigma}_2 \to 0$, $W/\bar{\sigma}_1 = 2$ are valid also for $W/\bar{\sigma}_1 \to 0$, $\ell/\bar{\sigma}_2 = 2$, and data given for $\ell/\bar{\sigma}_2 \to 0$, $W/\bar{\sigma}_1 = 4$ are valid also for $W/\bar{\sigma}_1 \to 0$, $\ell/\bar{\sigma}_2 = 4$.

Note 2: - Notes 4 and 5 of Table 7 apply also to this table.

are equally valid for circular diameters searching if the navigational error is independent of the orientation of the search paths. In all cases where the errors of estimate of object position have a circular distribution and the navigational errors are independent of path orientation, circular diameters searching is at least as efficient as single track searching for the same length of search path, and, for practical purposes, the data in Tables 7, 8, 9, 10 and 11 may be taken to be sufficiently valid also for circular diameters searching whenever object position and navigational errors have circular distributions.* More generally, with circular distributions of object positions and of navigational errors, wherever single track searching is indicated to be appropriate, then any variations from single track searching which leave all nominal straight search paths of the indicated length centered at the most probable position of the object will be at least as efficient as single track searching, and the indicated data for single track searching may be taken to be reasonably adequate for the modified procedures.

LOCATING OBJECTS BY MEANS OF LOCATORS WITH LIMITED CLASSIFICATION CAPABILITIES

88. In the above analysis it has been assumed not only that there is a probability β of detecting any object which falls within the over-all width of path W of the locator, but also that it will be known with certainty when an object has been detected. In general, the detection element of a locator may respond to clutter as well as objects to be salvaged and in order to discriminate against clutter it is necessary to specify the characteristics of equipment responses which will be taken to indicate salvageable contacts. For a given locator it may not be possible by such criteria to obtain perfect discrimination, and the possibility of clutter contacts due to false targets must then be taken into consideration.

89. It will be convenient to refer to the contacts which are being investigated as pre-search contacts to distinguish them from locator contacts. A pre-search contact represents a contact for an initial sonar search, or for any system used initially to establish the apparent presence of an object which, because of inaccuracy of location or of inadequate discrimination of the initial detector, must be reinvestigated by a narrow-path locator.. In order to simplify the analysis, the problem of locator false targets will be considered for the special case where the targets which give rise to pre-search false contacts do not produce locator false contacts. An example would be the case where the initial pre-search object detections were made by sonar in an area where all real clutter sonar targets have been removed. The assumed condition could also be obtained by removing only clutter which gives rise to both pre-search and locator false contacts.

90. Suppose that in carrying out a search for an object the search is discontinued whenever there is an object-like contact or at a specified time T, whichever occurs first. This will be referred to as the "standard searching rule" to distinguish it from other searching

*The efficiency of circle diameters searching may be considerably greater than that of single track searching in the special case where there is a large variation of detection probability with object orientation.

rules to be considered below. This procedure is the basis of the previous analysis, but the previous results are valid only when object-like locator contacts are obtained only from salvageable objects. It is desirable to consider the effect of possible locator false targets on the results of the search when using the standard searching rule.

91. It is apparent that if there is a substantial probability of contacting a locator false target before the object sought is contacted the chance of locating an object is less than the value P(T) which is obtained on the assumption that there are no false targets, and the average searching time $\bar{t}$ per pre-search contact investigated is smaller than the value $\xi\tau + (1 - \xi)T$ which is obtained when no false targets are present. The location probability P(T) represents the probability that a particular object would be located if searching were always carried out until time T, whether or not false targets were also present. In general, let $P_e(T)$ be the probability that objects which are investigated by locators are found by the locators. The quantity $P_e(T)$ will be called the effective location probability. When searching is carried out in accordance with the standard searching rule, $P_e(T)$ is equal to P(T) if there are no false targets, and is less than P(T) if false targets are present.

92. Let η be the number of locator false targets per unit area. In Appendix F, formulas are derived for $P_e(T)$ and $\bar{t}$ in terms of η, on the assumption that false targets are randomly distributed in the area. Table 12 gives calculated values of $P_e(T)$ and of $\bar{t}$ for various values of $\eta\sigma_1\sigma_2$ and for several values of the overall searching time T, based on the assumption that the detection probability β is the same for false targets as for objects of salvage. The results are based on the optimum searching procedure, assuming a navigational error which is small compared to the uncertainty in the object position, and which is large compared to the over-all width of searched path W. But in view of the likely uncertainty in the value of η, the data are expected to represent sufficiently good approximations for multiarea searching of 2 or more coverages, and should be reasonably adequate for any searching procedures considered provided $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ are not greater than 3/2. The values of the average searching time $\bar{t}$ are seen to depend on the fraction ξ of the investigated pre-search contacts which are salvageable objects. The table also gives the expected number n_f of located false targets per pre-search contact investigated. This number is of some importance for the further steps in salvage of the objects. If an attempt is made to salvage each target which produces an object-like contact, the number n_f represents the number of wasted salvages per pre-search contact investigated. If each object-like target detected by the locator is reinvestigated by an even more discriminating locator or by divers, then n_f determines the wasted searching effort on reinvestigation as a result of the locator false targets. The values of n_f are expected to vary more than $P_e(T)$ and $\bar{t}$ with the particular searching procedures used and with the searching conditions, but the listed values are probably adequate for practical purposes. It may be noted that the number of objects located per pre-search contact investigated is $\xi P_e(T)$.

93. In order to present data on $\bar{t}$ which are sufficiently independent of the particular searching procedure used, the values of $\bar{t}$ are given in terms of $T_{0.95}$, where the latter quantity is the value of T for which P(T) = 0.95 for the particular searching procedure which is being used. For instance, with a constant parallel path procedure and a total of three coverages, Table 4 shows that $T_{0.95} = 82.9 \frac{1}{\gamma}(\sigma_1\sigma_2/U)$. Similarly the values

TABLE 12

EFFECT OF LOCATOR FALSE TARGETS WHEN USING STANDARD SEARCHING RULE

T	$\eta\sigma_1\sigma_2$	$P_e(T)$	$\xi = 1.0$		$\xi = 0.75$		$\xi = 0.5$		$\xi = 0.25$		$\xi \to 0$	
			$\frac{\bar{t}}{T_{0.95}}$	n_f	$\frac{\bar{t}}{T_{0.95}}$	n_f	$\frac{\bar{t}}{T_{0.95}}$	n_f	$\frac{\bar{t}}{T_{0.95}}$	n_f	$\frac{\bar{t}}{T_{0.95}}$	n_f
Equal to												
the Time	0.00	0.900	0.21	0.00	0.33	0.00	0.44	0.00	0.56	0.00	0.67	0.00
Required	0.02	0.804	0.18	0.13	0.27	0.17	0.36	0.22	0.45	0.26	0.54	0.31
to Obtain	0.03	0.762	0.17	0.18	0.25	0.24	0.33	0.30	0.41	0.36	0.49	0.42
P(T) = 0.90												
Equal to	0.00	0.950	0.23	0.00	0.43	0.00	0.62	0.00	0.81	0.00	1.00	0.00
the Time	0.0025	0.930	0.23	0.018	0.41	0.028	0.60	0.038	0.78	0.048	0.97	0.057
Required	0.005	0.922	0.23	0.034	0.40	0.053	0.58	0.072	0.76	0.092	0.93	0.11
to Obtain	0.01	0.890	0.22	0.070	0.38	0.11	0.54	0.14	0.71	0.18	0.87	0.21
P(T) = 0.95	0.015	0.863	0.21	0.10	0.36	0.15	0.51	0.20	0.66	0.25	0.81	0.30
	0.02	0.837	0.20	0.13	0.34	0.19	0.48	0.25	0.62	0.32	0.76	0.38
	0.025	0.812	0.19	0.16	0.32	0.23	0.45	0.30	0.58	0.37	0.71	0.45
	0.03	0.789	0.18	0.19	0.30	0.27	0.43	0.35	0.55	0.43	0.67	0.51
	0.04	0.746	0.17	0.23	0.27	0.33	0.38	0.42	0.48	0.52	0.59	0.61
	0.05	0.707	0.16	0.28	0.25	0.38	0.34	0.49	0.43	0.59	0.52	0.69
Equal to	0.00	0.990	0.26	0.00	0.68	0.00	1.11	0.00	1.53	0.00	1.96	0.00
the Time	0.0025	0.970	0.25	0.019	0.65	0.035	1.05	0.052	1.45	0.068	1.85	0.085
Required	0.005	0.957	0.25	0.035	0.62	0.053	1.00	0.099	1.38	0.13	1.76	0.16
to Obtain	0.01	0.921	0.23	0.072	0.57	0.13	0.91	0.19	1.24	0.24	1.58	0.30
P(T) = 0.99	0.015	0.889	0.22	0.11	0.52	0.22	0.82	0.29	1.12	0.38	1.43	0.47
	0.02	0.860	0.21	0.14	0.48	0.23	0.75	0.32	1.02	0.42	1.29	0.51
	0.025	0.832	0.20	0.16	0.44	0.27	0.69	0.38	0.93	0.48	1.17	0.59
	0.03	0.807	0.19	0.19	0.41	0.31	0.63	0.42	0.84	0.54	1.06	0.66
	0.04	0.759	0.18	0.24	0.35	0.37	0.53	0.50	0.71	0.63	0.88	0.76
	0.05	0.717	0.16	0.28	0.31	0.42	0.45	0.56	0.60	0.69	0.74	0.83

T = Searching time specified in previous tables for the particular searching procedure being used to obtain the indicated value of P(T).

$T_{0.95}$ = Value of T given in previous tables for the particular searching procedure being used to obtain a value of 0.95 for P(T).

of the searching time T specified in the first column are the values of T which give the indicated values of P(T) for the particular searching procedure being used.

94. If η, σ_1, and σ_2 are known, Table 12 will indicate whether adequate coverage can be obtained when using the standard searching rule. It is apparent that for sufficiently large values of $\eta\sigma_1\sigma_2$ it is not possible to obtain adequate coverage by this rule, regardless of the value of T which is used. In considering other searching rules it is desirable to obtain as much discrimination as is practicable by taking advantage of the fact that the relative chances of finding an object and of finding a false target in a short time interval of any given length are substantially more in favor of the object in the early part of a search than in later stages.

95. Modified Searching Rule. The following modified searching rule has been considered:

(a) Always search for some minimum time t_1 regardless of whether or not targets are contacted during time t_1.

(b) Continue the search beyond time t_1 if, and only if, no target has been contacted during time t_1. If the search is continued beyond time t_1, then discontinue the search when there is an object-like contact or at a specified time T, whichever occurs first.

Appendix F gives the analysis for this case and Table 13 gives calculated data for values of t_1/T of 1/4 and 1/2 and for appropriate ranges of the other parameters. Except for the added quantity t_1 all symbols in Table 13 have the same meanings as in Table 12. Any required value of $P_e(T)$ can be obtained with this searching rule by taking sufficiently large values of t_1 and T, but, of course, for given η and ξ, the values of $\bar{t}$ and n_f increase also when $P_e(T)$ is increased.

96. Theoretically it is to be expected that for maximum discrimination between objects and clutter account should be taken of the possible detection of more than one target within specified time intervals and of the exact times at which they are detected. To determine whether substantially increased discrimination is likely to be obtained by such methods a two target searching rule is considered in Appendix F which differs from the above modified searching rule only in the provision that if two targets are detected during the time interval t_1, searching is to be discontinued when the second target is detected. Calculated data for this case are given in Table F of Appendix F. Comparing these results with those for the modified searching rule, it appears that under conditions where approximately the same value of $P_e(T)$ is obtained by the two rules, the values of $\bar{t}$ and n_f are also very nearly the same. It is concluded that very little added discrimination could be obtained by the use of searching rules more complex than the modified searching rule.

97. If searching always continues until time T, then $P_e(T) = P(T)$ and $\bar{t} = T$ The value of n_f is then given, to the same approximation as the values listed in the tables, by equation (8F) of Appendix F. For the values of T which make P(T) equal to 0.95 and 0.99, the approximate values of n_f are $24\eta\sigma_1\sigma_2$ and $35\eta\sigma_1\sigma_2$, respectively. Comparing

TABLE 13

EFFECT OF LOCATOR FALSE TARGETS WHEN USING MODIFIED SEARCHING RULE

T	$\frac{t_1}{T}$	$\eta\sigma_1\sigma_2$	$\Gamma_e(T)$	ξ = 1.0		ξ = 0.75		ξ = 0.5		ξ = 0.25		ξ = 0	
				$\frac{t_1}{T_{0.95}}$	n_f	$\frac{t_1}{T_{0.95}}$	n_f	$\frac{t_1}{T_{0.95}}$	n_f	$\frac{t_1}{T_{0.95}}$	n_f	$\frac{t_1}{T_{0.95}}$	n_f
Equal to the Time Required to Obtain P(T) = 0.95	1/4	0.00	0.95	0.35	0.00	0.51	0.00	0.67	0.00	0.84	0.00	1.00	0.00
		0.01	0.915	0.34	0.11	0.47	0.14	0.61	0.16	0.75	0.19	0.88	0.21
		0.02	0.884	0.32	0.22	0.44	0.26	0.55	0.30	0.67	0.35	0.78	0.39
		0.03	0.858	0.31	0.32	0.41	0.37	0.51	0.43	0.61	0.49	0.70	0.54
		0.04	0.836	0.31	0.42	0.39	0.48	0.47	0.54	0.55	0.61	0.63	0.67
		0.05	0.817	0.30	0.51	0.37	0.58	0.44	0.65	0.51	0.72	0.57	0.78
		0.06	0.800	0.29	0.61	0.35	0.68	0.41	0.75	0.47	0.82	0.53	0.89
		0.08	0.773	0.28	0.79	0.32	0.86	0.37	0.93	0.41	1.00	0.45	1.06
Equal to the Time Required to Obtain P(T) = 0.99	1/4	0.00	0.99	0.56	0.00	0.91	0.00	1.26	0.00	1.61	0.00	1.96	0.00
		0.01	0.962	0.55	0.16	0.82	0.20	1.09	0.23	1.36	0.27	1.63	0.31
		0.02	0.940	0.53	0.31	0.74	0.37	0.95	0.43	1.16	0.49	1.36	0.55
		0.03	0.922	0.53	0.46	0.69	0.53	0.85	0.60	1.01	0.67	1.17	0.74
		0.04	0.908	0.52	0.61	0.64	0.68	0.77	0.76	0.89	0.83	1.02	0.90
		0.05	0.896	0.51	0.76	0.61	0.83	0.71	0.90	0.81	0.98	0.90	1.05
		0.06	0.887	0.51	0.91	0.59	0.98	0.66	1.04	0.74	1.11	0.81	1.18
		0.08	0.873	0.50	1.20	0.55	1.26	0.60	1.31	0.64	1.37	0.69	1.43
Equal to the Time Required to Obtain P(T) = 0.95	1/2	0.00	0.95	0.54	0.00	0.66	0.00	0.77	0.00	0.89	0.00	1.00	0.00
		0.01	0.933	0.54	0.16	0.63	0.18	0.72	0.19	0.82	0.21	0.91	0.22
		0.02	0.918	0.53	0.31	0.61	0.34	0.68	0.36	0.76	0.39	0.84	0.42
		0.03	0.907	0.53	0.46	0.59	0.50	0.65	0.53	0.72	0.56	0.78	0.60
		0.04	0.897	0.52	0.62	0.57	0.65	0.63	0.69	0.68	0.72	0.73	0.76
		0.05	0.889	0.52	0.77	0.56	0.80	0.60	0.84	0.65	0.88	0.69	0.92
		0.06	0.882	0.51	0.92	0.55	0.95	0.59	0.99	0.62	1.03	0.66	1.06
		0.08	0.871	0.51	1.22	0.53	1.25	0.56	1.28	0.58	1.32	0.61	1.35
Equal to the Time Required to Obtain P(T) = 0.99	1/2	0.00	0.99	1.00	0.00	1.24	0.00	1.48	0.00	1.72	0.00	1.96	0.00
		0.01	0.980	1.00	0.24	1.17	0.28	1.35	0.32	1.53	0.36	1.71	0.39
		0.02	0.972	0.99	0.47	1.12	0.50	1.26	0.54	1.39	0.57	1.52	0.60
		0.03	0.966	0.99	0.70	1.09	0.74	1.19	0.78	1.28	0.81	1.38	0.85
		0.04	0.962	0.99	0.94	1.06	0.97	1.13	1.01	1.21	1.05	1.28	1.08
		0.05	0.959	0.98	1.17	1.04	1.20	1.09	1.24	1.15	1.27	1.20	1.31
		0.06	0.956	0.98	1.40	1.02	1.43	1.07	1.46	1.11	1.49	1.15	1.52
		0.08	0.953	0.98	1.86	1.00	1.89	1.03	1.91	1.05	1.93	1.07	1.96

Note: - T and $T_{0.95}$ have same meaning as for Table 12.

the results obtained when searching always continues until time T with the results of Table 13 for the same η and for any combination of t_1 and T giving the same $P_e(T)$, it is found that the differences in the values of $\bar{t}$ and n_f for the two cases are not large enough to be of practical importance except at most, when $\eta\sigma_1\sigma_2$ is less than 0.06. Perhaps the most important advantage of a t_1 different from T is that in the absence of adequate information concerning the magnitude of η, it makes it possible to place on the value of $P_e(T)$ a lower limit of magnitude $P(t_1)$ and yet leave open the possibility of obtaining a substantially larger value of $P_e(T)$ with an average searching time not much larger than t_1.

98. The value of η can be determined for any area by making passes of the locator through the area when it is free of objects and observing the number of object-like contacts obtained in a given total length of searched path. If the detection probability is the same for false targets and for objects, as has been assumed in obtaining the data of Tables 12 and 13, the value of η should be given approximately by

$$\eta = \frac{n}{W\beta L},$$

where L is the total length of path searched and n is the number of object-like contacts obtained in the path of length L. Although the data in Tables 12 and 13 were obtained on the assumption that the detection probability is the same for false targets and for objects, and though the above expression for η is valid only for this case, it is expected that the data of the tables are reasonably adequate even when the detection probability for false targets differs substantially from its value for objects if η is taken to be an apparent false-target density given by this expression. More exact data can be obtained from the formulas of Appendix F for any relative detection probabilities for objects and for false targets. For use with this more exact data, η should be determined as indicated above but with β replaced by the detection probability for false targets.

99. Necessary modifications of the data in Tables 12 and 13 can readily be made to cover adequately the case where some of the targets which produce pre-search false contacts are also locator false targets, if adequate information on these targets is available. If a pre-search false target which is also a locator false target is in the vicinity of an object, and if the initial detection has not distinguished them as separate objects, the effect of the false target on the results of the subsequent locator search is identical with that of a locator false target which is not a pre-search false target. Then for determining how the previous results must be modified to take account of targets which may give rise to both pre-search contacts and locator contacts it is necessary only to consider the effect on subsequent locator investigations of pre-search contacts of false targets which are also locator targets. Since the value of $P_e(T)$ depends only on what occurs during the locator investigations of those pre-search contacts that are due to objects, its value does not depend on whether or not some of the locator false targets are also pre-search false targets. Hence, the values of $P_e(T)$ given in Tables 12 and 13 should be sufficiently valid when some of the locator false targets are also pre-search false targets.

100. To estimate n_f and $\bar{t}$, let ξ_{dlf} be the fraction of pre-search contacts which are pre-search false targets and which are also locator false targets. Let η' be the false-target density when those false targets represented by ξ_{dlf} are excluded, and let $\xi' = \xi + \xi_{dlf}$. Suppose the false targets of the fraction ξ_{dlf} were actually objects. Then the value of n_f would be given by the table if η and ξ in the table were interpreted to be η' and ξ', respectively. Let this value of n_f be $(n_f)_1$. In the actual case, of the fraction ξ_{dlf} of false targets which were considered as objects a number $\xi_{dlf}\ P_e(T')$ per contact investigated would be contacted by the locator, assuming the same detection probability for locator false targets as for objects. Thus the value of n_f for this case is $(n_f)_1 + \xi_{dlf}\ P_e(T')$. It can be seen that the values of $\bar{t}$ given by the tables are valid if the quantities η and ξ are interpreted as η' and ξ', respectively. To illustrate, suppose $\eta\sigma_1\sigma_2 = 0.05$, $\eta'\sigma_1\sigma_2 = 0.04$, $\xi = 0.5$ and $\xi_{dlf} = 0.25$, and suppose that the modified searching rule is used with a value of T such that $P(T') = 0.99$ and with $t_1/T = 1/2$. Then Table 13 shows that $P_e(T') = 0.959$. From the row for which $\eta\sigma_1\sigma_2 = 0.04$ and from the column headed $\xi = 0.75$, it is seen that the value of $\bar{t}/T_{0.95}$ for this case is 1.06. From the same row and column it is seen that $(n_f)_1 = 0.97$. Hence, the correct value of n_f for this case is $0.97 + (0.25)(0.959)$ or 1.23.

101. As previously stated, the data in Tables 12 and 13 represent good approximations only for multiarea searching with a total number of coverages of not less than 2, but they are probably reasonably adequate in all practical cases for which $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ are not greater than 3/2. Large values of $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ are likely to occur only if the position of object or target is quite accurately known, and a locator false target is not likely to be confused with the contact which is being investigated unless the separation between them is no greater than the resolving power of the locator. Then unless the false-target density is quite high, the value of $P_e(T')$ obtained with the standard searching rule may be expected to approach closely the value of P(T), and the value of n_f may be expected to be approximately equal to $\xi_{dlf}\ P(T')$. Also, the value of $\bar{t}$ may be expected to be not much larger than $(\xi + \xi_{dlf})\tau + \{1 - (\xi + \xi_{dlf})\}P(T)$.

APPLICATION OF THE NUMERICAL RESULTS OF THE PRESENT REPORT TO MORE COMPLEX SITUATIONS THAN THOSE CONSIDERED IN THE INITIAL ANALYSIS

102. The major part of the analysis of the present report is based on assumptions concerning locator characteristics and searching conditions which may not always be valid, but the results are believed to be sufficiently general so that only relatively simple modifications are required to make them sufficiently valid under any given practical operating conditions. Although it is not possible to anticipate all situations which may arise in practice, the consideration of a few important special cases will illustrate how the results of the simple analysis can be extended to cover more complex cases. The analysis of the locator false target problem presented in the previous section serves as one illustration.

103. It has been assumed there is an equal detection probability β for all objects. If the detection probability varies only moderately from object to object the results which have been obtained on the basis of a constant detection probability β should be adequate if β is taken equal to the average detection probability. Where there are large variations in β from object to object, an approximate evaluation can be obtained as follows: Divide the expected objects into groups such that the detection probability has only a moderate variation for the objects of any one group. Let ψ_1 be the fraction of all the objects for which the average detection probability is approximately β_1. Let ψ_2 be the fraction for which the average detection probability is β_2, etc. For the given searching procedure let $P_i(T)$ be the location probability for objects of the fraction ψ_i. Then the over-all probability is

$$P(T) = \psi_1 P_1(T) + \psi_2 P_2(T) + \dots = \sum_{i=1}^{n} \psi_i P_i(T)$$

where n is the total number of groups. Similarly if τ_i is the average discovery time for objects of the i'th group, the average searching time is

$$\bar{t} = (1 - \xi)T + \xi \sum_{i=1}^{n} \psi_i \tau_i .$$

Since τ_i would vary only slowly with β_i, it would be sufficient to take the sum equal to the value of τ corresponding to the average value of β provided no β_i is zero. Approximate values of $P_i(T)$ can be obtained with the aid of the appropriate table and Plate 1. It should be noted that the data in the tables and on Plate 1 were obtained on the assumption that the searching procedure parameters would be chosen on the basis of the particular value of P(T) required and it obviously is not possible to satisfy that assumption for each of the fractions ψ_i when there is more than one. However, it has been seen that the results do not depend critically on the precise dimensions of the searched areas, especially if there is substantial navigational error, and if that procedure is chosen which is most suitable for the group for which β_i is nearest the mean value for all the objects, the proposed method of evaluation should be reasonably adequate.

104. An important special case is where some fraction ψ_s cannot be detected by the given locator. Then P(T) is always less than $1 - \psi_s$. If a second locator with different detection characteristics were used to reinvestigate those pre-search contacts where no object was found by the first locator, giving a location probability $\{P(T')\}_2$ for any objects remaining after the investigations by the first locator, the over-all location probability would be $P(T) + [1 - P(T)] \{P(T')\}_2$ and the average searching time for the second locator would be obtained in the same manner as for the first locator. Of course, the average searching time for the second locator means the average for those contacts investigated by the second locator. In estimating ψ_1, ψ_2, etc. for the second locator account would have to be taken of the characteristics of both the first and second locators.

105. No account has been taken of the possible variation of the parameters σ_1, σ_2, $\bar{\sigma}_1$, $\bar{\sigma}_2$, β, W and ξ with position in the area or channel where object location is to be carried

out. Where substantial variations occur, the values of T, τ, $\bar{t}$ and P(T) given in the tables represent expected values for any location for which the necessary parameters are known or can be estimated. If there are relatively large variations in any of the parameters σ_1, $\bar{\sigma}_1$, σ_2, β, W, and ξ with position, the greatest average coverage of an area or channel for a given location effort is likely to be obtained with different values of P(T) for different locations.

106. One example of expected variations in σ_1, σ_2 and ξ is in the location of objects which are initially detected by sonar using the clustering technique. The clustering technique has been proposed to obtain some discrimination against transitory false targets. With this technique it is required that a target be detected at least K times in order to be considered as probably an object, where K is a specified number greater than 1.0. It is found that when sufficient sonar passes have been made to obtain a high probability that any object will be detected at least K times, then most objects will have been detected more than K times, with some objects receiving K + 2 detections, some receiving K + 3 detections, etc. In general, the uncertainty in the position of an object decreases with increase in the number of times it is detected. In general, there will also be chance clusters of K or more contacts due to transitory false targets, but it is found that many more clusters of just K false contacts are to be expected than clusters of more than K false contacts. Thus for a given sonar search, the probability ξ that a cluster is due to an object increases rapidly with the number of contacts in the cluster. Consequently for the clusters of just K contacts the average searching time is likely to be of the order of the overall searching time T, and T is likely to be relatively large because of the relatively large uncertainty in the position of any objects for which there have been only K contacts. For clusters with larger numbers of contacts the average searching time is expected to be of the order of τ if there are no real pre-search false targets in the area. It is apparent that for a given degree of over-all location, to minimize the required searching effort it is necessary that the value of P(T) increase with increase in the number of contacts in a cluster above the minimum number K.

107. The data which have been presented for constant parallel path searching and for single track searching are valid only if the search paths are parallel to the y-axis, where the latter is assumed to be chosen in the direction of maximum uncertainty in the object position.* Also, it is to be expected that elliptical area and rectangular area searching could be carried out most satisfactorily by making all passes parallel to the y-axis, especially if substantial turning times are required. Because of factors such as physical obstructions it may not always be practical to search in paths parallel to the direction of greatest uncertainty in object position. Also, for reasons mentioned in Appendix E, it may be much simpler in some cases to determine the distribution of object position errors in two arbitrary perpendicular directions than to find the distribution for the two perpendicular directions for which the errors are independent. Limiting consideration to constant parallel path searching and single track searching, it will be shown how these procedures can be carried out in any direction. Consider an X,Y

*Of course, the results are valid also if the y-axis is chosen parallel to the direction of minimum uncertainty in object position. If the detection varies substantially with object orientation it may be desirable to rotate the direction of the search paths 90° on successive coverages for elliptical area and rectangular area searching.

coordinate system with origin at the most probable position of the object and with the Y-axis parallel to the direction in which the search paths are to be made. Let the Y-axis make an angle α with the y-axis where the positive direction of Y is assumed chosen so that α does not exceed 90°. Let X and Y be the coordinates of the actual object position. Then the errors in estimating X have a normal distribution with standard deviation σ_X given by $\sigma_X = \sqrt{\sigma_1^2 \cos^2\alpha + \sigma_2^2 \sin^2\alpha}$ and the errors in estimating Y have a normal distribution with standard deviation σ_Y given by $\sigma_Y = \sqrt{\sigma_1^2 \sin^2\alpha + \sigma_2^2 \cos^2\alpha}$. The errors in X and Y are not independent but it is shown in Appendix E that at least for constant parallel path searching and for single track searching, under conditions for which $\bar{\sigma}_1/\sigma_1$ is small, if the search is carried out as though these errors are independent, the results which have been obtained for search paths parallel to the y-axis will be valid for this case to a very good approximation if the parameters σ_1 and σ_2 are replaced in the tables by σ_X and σ_Y.

108. Although constant parallel path searching can be carried out in any direction, it is most efficient when carried out parallel to the y-axis. Provided the navigational error is not substantially larger than the uncertainty in the object position the required searching effort is approximately proportional to the product $\sigma_X\sigma_Y$. From the above relations for σ_X and σ_Y it is found that

$$\sigma_X\sigma_Y = \sigma_1\sigma_2\sqrt{1 + \left\{\left(\frac{\sigma_2^2 - \sigma_1^2}{2\sigma_1\sigma_2}\right) \sin 2\alpha\right\}^2}.$$

This may be much larger than $\sigma_1\sigma_2$ if σ_2 is much larger than σ_1, but it approaches $\sigma_1\sigma_2$ when α approaches zero or 90°.

109. The data presented in the present report are applicable only when it is possible to make an estimate of the most probable position of the object and when the errors of estimate may be assumed to have roughly an elliptical normal distribution. Where nothing is known concerning the position of an object except that it lies within an area whose boundaries can be specified, the appropriate procedure is to carry out an area search by one of the procedures considered in Part I.

111. Acknowledgment. The computations required to obtain the data in Table 11 and part of the data of Table D were carried out by the Applied Mathematics Laboratory of David Taylor Model Basin. The remaining computations were made by the Analysis Section of Code 531 under the direction of Miss Mary McKnight.

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APPENDIX A

DERIVATIONS OF FORMULAS FOR $\rho(x,y,t)$ FOR THE CASE WHERE THE WIDTH OF SEARCHED PATHS IS SMALL COMPARED TO THE UNCERTAINTY IN THE OBJECT POSITION AND COMPARED TO THE NAVIGATIONAL ERROR

A1. In order to determine for any search procedure the location probability $P(T)$, given by equation (6), it is necessary to know the probability $\rho(x,y,t)$ of effectively searching any given point (x,y) during the part of the search which is carried out during time t. By definition, the probability that a point will be effectively searched is the probability that if an object is located at that point it will be detected. Consideration is limited here to the case where the over-all width of searched path W is small compared either to the uncertainty in the position of the object or to the navigational error. It will be convenient to consider separately the case where the navigational error is quite small compared to the uncertainty in the position of the object and the case where it is not.

Navigational Errors Small Compared to the Uncertainty in the Position of the Object.

A2. Consider a locator attempting to carry out searches along any given set of straight lines. Because of navigational error the actual searched paths will differ in general from the paths which the locator attempts to search. It is convenient to refer to the paths which it is attempted to search as the nominal search paths. It is assumed that the lateral deviations of the search paths from the nominal paths follow the normal distribution law with standard deviation $\bar{\sigma}$. It is assumed initially that $\bar{\sigma}$ is independent of the orientation of the nominal search paths. Consider any one pass of the locator and any point located a distance x_0 from the center line of the nominal path. If the ends of the path extend to infinity in both directions the probability that the point lies within the actual searched path is

$$\frac{1}{\sqrt{2\pi}\ \bar{\sigma}} \int_{x_0-\frac{W}{2}}^{x_0+\frac{W}{2}} e^{-x^2/2\bar{\sigma}^2}\, dx.$$

If the width W approaches zero, this expression approaches

$$\frac{W}{\sqrt{2\pi}\ \bar{\sigma}}\, e^{-x_0^2/2\bar{\sigma}^2} .$$

It is now assumed that W is sufficiently small compared to $\bar{\sigma}$ that the probability of the point lying within the given search path is accurately given by the latter expression. If β is the probability that an object will be detected if it falls within the actual search path of over-all width W, the probability that the given point will be effectively searched on the given pass is

$$d\rho = \frac{W\beta}{\sqrt{2\pi}\,\bar{\sigma}} e^{-x_0^2/2\bar{\sigma}^2}$$

Taking the given point as the origin of a rectangular coordinate system, with the y-axis parallel to the given nominal search path, and taking the coordinates of any point on the center line of the nominal search path to be x_0, y_0, divide the line into infinitesimal segments of lengths $\Delta_i y_0$. Noting that

$$\frac{1}{\sqrt{2\pi}\,\bar{\sigma}} \int_{-\infty}^{\infty} e^{-y_0^2/2\bar{\sigma}^2} dy_0 = 1$$

$d\rho$ can be put in the form

$$d\rho = \sum_i \frac{W\beta}{2\pi} \frac{\Delta_i y_0}{\bar{\sigma}^2} \left(e^{-x_0^2/2\bar{\sigma}^2}\right) \left(e^{-y_{0_i}^2/2\bar{\sigma}^2}\right) \tag{1A}$$

where the sum must include all elements of the line which are not far enough from the given point for $e^{-(y_{0_i})^2/2\bar{\sigma}^2}$ to be negligible.

A3. The manner in which the line is divided into segments $\Delta_i y_0$ is unimportant except that all intervals must be vanishingly small. Taking $W\beta\Delta_i y_0 = \delta_i w$ where $\delta_i w$ is the aggregate area of the i'th segment of searched path and taking $r_i = \sqrt{x_0^2 + y_{0_i}^2}$ to be the distance of the given segment from the given point, $d\rho$ can be written

$$d_1\rho = \frac{1}{2\pi\bar{\sigma}^2} \sum_{i=1}^{S_1} \delta_i w \, e^{-r_i^2/2\bar{\sigma}^2} \tag{2A}$$

Similarly the probability that the point is effectively searched during the pass made on a second path is

$$d_2\rho = \frac{1}{2\pi\bar{\sigma}^2} \sum_{i=S_1+1}^{S_2} \delta_i w \, e^{-r_i^2/2\bar{\sigma}^2}$$

Then if there is a total of n search paths the probability $1 - \rho$ that the point will not be effectively searched on any of the n passes is the product of n factors of the type $(1 - d_i\rho)$, or

$$1 - \rho = \prod_J (1 - d_J\rho) = e^{\sum_{J=1}^{n} \ln(1-d_J\rho)}$$

This can be written $$1 - \rho = e^{-\bar{m}} \qquad (3A)$$

where

$$\bar{m} = \frac{1}{2\pi\bar{\sigma}^2}\sum_{i=1}^{S_n}\left\{\delta_i w\, e^{-r_i^2/2\bar{\sigma}^2} + \text{terms of higher order in } \delta_i w\right\}. \qquad (4A)$$

It is seen that for sufficiently small $\delta_i w$ the quantity $\bar{m}$ is $\sum_J d_J \rho$. That is, $\bar{m}$ is the mathematical expectation of the number of times the point is effectively searched during the time t. It will be called the search-path density at the given point.

A4. For any nominal search paths, whether straight or not, if $\bar{\sigma}$ is sufficiently small the paths will be straight paths over lengths of 2r for values of r for which $e^{-r^2/2\bar{\sigma}^2}$ is not negligible. Consider a rectangular coordinate system with origin at an arbitrary point. Suppose the x,y plane is divided into rectangles of dimensions $\Delta_i x'$ and $\Delta_J y'$ with centers at points (x_i', y_J'). Then we may suppose the paths considered above divided into segments so that if any path crosses any of these rectangles the part of the path within that rectangle will constitute one segment. Let the point at which $\bar{m}$ is to be determined be (x,y). Then (4A) becomes

$$\bar{m}(x,y) = \frac{1}{2\pi\bar{\sigma}^2}\sum_{i=1}^{\infty}\sum_{J=1}^{\infty} m(x_i', y_J')\, e^{-\left\{\frac{(x_i'-x)^2 + (y_J'-y)^2}{2\bar{\sigma}^2}\right\}}\Delta_i x' \Delta_J y' + \text{higher order terms} \qquad (5A)$$

where $m(x_i', y_J')\Delta_i x'\Delta_J y'$ represents the sum of all $\delta_i w$ which occur in the rectangle located at (x_i', y_J'). It is now assumed that for the idealized case of vanishingly small and infinitely many paths, searching is carried out sufficiently systematically so that $m(x_i', y_J')$ can be assumed to approach a limit as $\Delta_i x'$ and $\Delta_J y'$ approach zero. Then (5A) reduces to

$$\bar{m}(x,y) = \frac{1}{2\pi\bar{\sigma}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(x', y')\, e^{-\left\{\frac{(x'-x)^2 + (y'-y)^2}{2\bar{\sigma}^2}\right\}} dx'\, dy'. \qquad (6A)$$

The quantity $m(x', y')$ represents the nominal average number of times that points in the vicinity of (x', y') are searched and will be referred to as the nominal search-path density.

A5. We shall be concerned with nominal search-path distributions for which, except possibly at a finite number of boundaries, the variation in nominal search-path density is small over any interval which is small compared to the standard deviation of the errors of estimate of an object's position. Hence, if $\bar{\sigma}$ is sufficiently small compared to σ, or to σ_1 and σ_2, the quantity $m(x', y')$ can be considered to be constant within the range of values of x' and y' for which $e^{-\left\{\frac{(x'-x)^2 + (y'-y)^2}{2\bar{\sigma}^2}\right\}}$ is not negligible. Then $m(x',y')$ can be replaced by $m(x,y)$ and (6A) reduces to $\bar{m}(x,y) = m(x,y)$. During a given searching operation $m(x,y)$ will vary with time of searching t. To indicate the dependence on time, $\bar{m}(x,y)$ and $m(x,y)$ may be written in the form $\bar{m}(x,y,t)$ and $m(x,y,t)$. Similarly ρ can be written in the form $\rho(x,y,t)$. Thus (3A) is equivalent to

$$\rho(x,y,t) = 1 - e^{-\bar{m}(x,y,t)}. \tag{7A}$$

Navigational Errors Not Small Compared to the Uncertainty in the Position of the Object.

A6. If the standard deviation of navigational error $\bar{\sigma}$ is not small compared to σ_1 and σ_2 the search-path density $\bar{m}(x,y,t)$ cannot be taken equal to the nominal search-path density. But for the type of navigational error likely to be encountered, it seems likely that the relationship between these two quantities is given reasonably adequately by equation (6A) if $\bar{\sigma}$ is independent of path orientation, as has been assumed. Thus consider a search procedure in which any desired nominal search-path distribution is obtained by straight path searches in any number of directions with the lengths of path distributed in any manner. Considering any one of these paths of nominal length ℓ, suppose that from a knowledge of the search speed, the desired length of path can be attained, but that because of navigational errors, the beginning and, therefore, also the end of the path may be displaced longitudinally from the desired positions. Assuming the same distribution for the longitudinal displacement errors as for the lateral displacement errors, the probability that a point is included in the effective search path is given by an equation of the type (1A). For if the end points of the nominal path are located at (x_0,y_1) and (x_0,y_2) where $y_1 < y_2$ and the given point is at the origin, we have

$$d\rho = \frac{W\beta}{\sqrt{2\pi}\,\bar{\sigma}} e^{-x_0^2/2\bar{\sigma}^2} \left\{\frac{1}{\sqrt{2\pi}\,\bar{\sigma}} \int_{y_1}^{y_2} e^{-y^2/2\bar{\sigma}^2}\, dy\right\}. \tag{8A}$$

But this can be written in the form of equation (1A), where now the summation includes only elements $\Delta_i y$ which are segments of the line lying between x_0,y_1 and x_0,y_2. We then arrive at equation (6A) without the restriction that $\bar{\sigma}$ is to be so small that the ends of the line can be regarded as being at infinity.

A7. The same result is obtained if instead of attempting to search a fixed length ℓ the beginning and the end of the search on a given pass are assumed to be subject to independent errors. The given point can lie within the effective search path only if simultaneously $y_1' < 0$ and $y_2' > 0$ where y_1' and y_2' are the coordinates of the end points of the actual searched path. Since failure to meet one of these conditions precludes failure to meet the other, these failures are mutually exclusive events and the probability that at least one of the conditions fail is the sum of the probability that the first fail and of the probability that the second fail. Thus the probability that the above conditions are both satisfied is

$$1 - \frac{1}{\sqrt{2\pi}\,\bar{\sigma}} \int_0^{\infty} e^{-(y_1'-y_1)^2/2\bar{\sigma}^2}\, dy_1' - \frac{1}{\sqrt{2\pi}\,\bar{\sigma}} \int_{-\infty}^{0} e^{-(y_2'-y_2)^2/2\bar{\sigma}^2}\, dy_2' .$$

By changing the variables of integration and rearranging, this is found to reduce to

$$\frac{1}{\sqrt{2\pi}\,\bar{\sigma}} \int_{y_1}^{y_2} e^{-y^2/2\bar{\sigma}^2}\, dy.$$

Thus, equation (8A) is valid for this case, also, as is equation (6A).

A8. If the distribution of longitudinal errors differs from that of lateral errors, let $\bar{\sigma}_1$ be the standard deviation of the lateral errors and let $\bar{\sigma}_2$ be the standard deviation of longitudinal errors. Then if all search passes are made parallel to the y direction, it can be seen by repeating the steps in the above analysis that in the place of equation (6A), we obtain

$$\bar{m}(x,y) = \frac{1}{2\pi\,\bar{\sigma}_1\bar{\sigma}_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(x',y')\, e^{-\left\{\frac{(x'-x)^2}{2\bar{\sigma}_1^2} + \frac{(y'-y)^2}{2\bar{\sigma}_2^2}\right\}}\, dx'\, dy' . \qquad (9A)$$

APPENDIX B

OPTIMUM AND "STANDARD" SEARCH PROCEDURES FOR FINDING AN OBJECT WHEN NAVIGATIONAL ERRORS ARE SMALL COMPARED TO THE UNCERTAINTY IN THE OBJECT POSITION BUT ARE LARGE COMPARED TO THE LOCATOR WIDTH OF SEARCHED PATH

B1. The most effective search procedure for finding an object obviously depends on the characteristics of the locator, such as its mobility, its speed limitations, etc. It will be useful for purposes of comparison to determine the optimum search procedure for the simple case where the locator is sufficiently mobile to follow any prescribed paths and has a fixed search-path width and a fixed ground speed. Consideration will be limited here to the case where the width of searched path is small compared to the standard deviation $\bar{\sigma}$ of navigational errors of the locator and where $\bar{\sigma}$ is small compared to the standard deviation of the errors of estimate of object positions. It will be assumed initially that the errors of estimate of the object position have a circular normal distribution, with standard deviation σ.

<u>Optimum Procedure.</u>

B2. Consider first the search-path distribution which maximizes the probability that the object will have been detected after a specified total search time, assuming searching is carried out for the given time regardless of whether or not an object is detected at any time during this period. Equation (6), which applies to this case, together with equation (8), gives the probability P(t) that a given object has been detected after a search time t for any specified search-path distribution. In accordance with the results of Appendix A the search-path density $\bar{m}(x,y,t)$ is assumed to be replaced by the nominal search-path density $m(x,y,t)$. The problem is to maximize F(t) subject to the condition that the aggregate area searched have a specified value. If w_a is the aggregate width of searched path, V is the search speed and t is the searching time, the latter condition is equivalent to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(x,y)\,dx\,dy = w_a Vt = Ut,$$

where the search rate U is equal to $w_a V$.

B3. For the assumed circular normal distribution of errors of estimate of the object position it seems apparent that the optimum search-path distribution will have circular symmetry and initially we make this assumption. Then F(t) is given by equation (7). Thus the problem is to maximize

$$P(t) = \frac{1}{\sigma^2}\int_0^\infty e^{-r^2/2\sigma^2}\left\{1 - e^{-m(r)}\right\} rdr \tag{1B}$$

subject to the condition

$$\int_0^\infty m(r)\, 2\pi\, rdr = Ut. \tag{2B}$$

There is the additional requirement that from the nature of m(r) it cannot be negative. This leads to difficulties in attempting a direct solution of the problem. To get around this difficulty consider first the problem of finding the optimum search-path distribution within a circle of radius R', centered at the most probable position of the object, and assume searching is carried out only in this area. Then the infinite limits in (1B) and (2B) are replaced by R'. We attempt first to restrict R' to sufficiently small values so that negative values of m(r) do not arise in minimizing (1B).

B4. This is a problem in the calculus of variations. It is shown in reference (1) that a necessary condition for a maximum of the integral in (1B), subject to the condition (2B), is that the following differential equation be satisfied:

$$\frac{\partial(E - \lambda H)}{\partial m} = 0,$$

where E is the integrand of the integral in (1B), H is the integrand of the integral in (2B), and λ is a parameter whose value is fixed by the requirement that the solution m(r) must satisfy (2B) with upper limits replaced by R'. Carrying out the differentiation, and substituting in (2B) with upper limit replaced by R' to determine λ, we find

$$m(r) = \frac{Ut}{\pi(R')^2} + \frac{\frac{1}{2}(R')^2 - r^2}{2\sigma^2}. \tag{3B}$$

This satisfies the requirement that m(r) not be negative for any r in the given circle only if

$$\left(\frac{R'}{\sigma}\right)^4 \leq \frac{4}{\pi}\frac{Ut}{\sigma^2}. \tag{4B}$$

Considering all circles with radius R' satisfying (4B), the largest value of P(t) is obtained if R' is taken to be the largest radius satisfying (4B), for if R' is taken to have this value the search-path density corresponding to the optimum distribution (3B) is not zero at any point within this circle. Hence, subject to the limitations of (4B), P(t) is a maximum when R' = R, where

$$\left(\frac{R}{\sigma}\right)^4 = \frac{4}{\pi}\frac{Ut}{\sigma^2}. \tag{5B}$$

Making this substitution in (3B), we have

$$m(r) = \frac{R^2 - r^2}{2\sigma^2} = \sqrt{\frac{Ut}{\pi\sigma^2}} - \frac{r^2}{2\sigma^2} \tag{6B}$$

where R is a function of t and is given by (5B). This solution is given in reference (2).

B5. That the distribution (6B) does make P(t) a maximum can be easily shown by considering the effect of removing a small amount of searching from a small area within the circle of radius R and adding an equal amount to some other area inside or outside this circle. The probability that the object lies within a small area dA at any point r and remains undetected after a given amount of searching is

$$\frac{1}{2\pi\sigma^2}\left(e^{-r^2/2\sigma^2}\right)\left(e^{-m(r)}\right) dA.$$

Inside the circle of radius R the coefficient of dA is the same for all points and is equal to

$$\frac{1}{2\pi\sigma^2} e^{-\sqrt{Ut/\pi\sigma^2}}.$$

Thus, if, over one area dA within the circle of radius R, the searching is reduced by an amount Δm and is increased in another area dA in this circle by the same amount, the net change in probability that the object is detected is

$$-\frac{dA}{2\pi\sigma^2} e^{-\sqrt{Ut/\pi\sigma^2}} [(e^{-\Delta m} - 1) + (e^{\Delta m} - 1)] = -\frac{dA}{\pi\sigma^2} e^{-\sqrt{Ut/\pi\sigma^2}} [\cosh \Delta m - 1].$$

This is negative for all values of Δm. If the small amount of searching is transferred to an area dA at $r = r'$ outside of the circle of radius R, the net change in F(t) is

$$-\frac{dA}{2\pi\sigma^2}\left[e^{-(r')^2/2\sigma^2}(e^{-\Delta m} - 1) + e^{-R^2/2\sigma^2}(e^{\Delta m} - 1)\right]$$

where $r' > R$. This can be put in the form

$$-\frac{dA}{\pi\sigma^2}\left[e^{-(r')^2/2\sigma^2}(\cosh \Delta m - 1) + \frac{1}{2}\left(e^{-R^2/2\sigma^2} - e^{-(r')^2/2\sigma^2}\right)(e^{\Delta m} - 1)\right]$$

which is seen to be negative for all values of Δm.

B6. Substituting the value of m(r) from (6B) into (1B) with the upper limits of the integral in the latter expression replaced by R, we obtain

$$P(t) = \frac{1}{\sigma^2}\int_0^R e^{-r^2/2\sigma^2}\left(1 - e^{-\left(\frac{R^2-r^2}{2\sigma^2}\right)}\right) rdr$$

where R is a function of t, given by (5B). Carrying out the integration we obtain for the location probability corresponding to optimum search-path distribution

$$P(t) = 1 - \left(1 + \frac{R^2}{2\sigma^2}\right) e^{-R^2/2\sigma^2} \qquad (7B)$$

or

$$P(t) = 1 - \left\{1 + \left(\frac{Ut}{\pi\sigma^2}\right)^{½}\right\} e^{-\left(\frac{Ut}{\pi\sigma^2}\right)^{½}} \qquad (8B)$$

B7. If, as has been assumed in the derivation, searching is carried out for a time T, regardless of whether the object is located sooner, and is then discontinued, it is unimportant how searching is carried out provided only the final search-path distribution is that given by (6B) with t replaced by T. If, however, searching can be discontinued as soon as the object is located, the manner in which the required search-path density is laid down becomes important. We now determine the optimum search procedure to minimize the average discovery time τ, where τ is defined as the mathematical expectation of the searching time for an object, when one exists, if searching is discontinued when the object is located or at the end of a specified period T, whichever occurs first. Normally T may be assumed to be the time required to attain a given value for the location probability P(T).

B8. Equation (3) shows that to minimize τ, searching must be carried out so as to maintain at all times t the search-path density distribution which maximizes P(t). The required distribution at any time t is given by (6B). This is the equation of a paraboloid of revolution with axis of symmetry at r = 0, and with vertex located at the top of this surface at the point r = 0, m = $\sqrt{Ut/\pi\sigma^2}$. Only the part of the surface which is above the plane m = 0 represents a real distribution. The paraboloid representing the required distribution at t + dt is obtained from that representing the distribution at time t by shifting all points of the surface upward a distance

$$\Delta m = \sqrt{\frac{U(t + dt)}{\pi\sigma^2}} - \sqrt{\frac{Ut}{\pi\sigma^2}}.$$

Thus the required search procedure is one which gives successive uniform very light coverages of the central area, with the radius of the area searched increasing on successive coverages so as to maintain at all times the relation (5B). It may be noted that with the indicated search procedure the function p(x,y) given by $p(x,y) = f(x,y)\ \{1 - \rho(x,y)\}$, where f(x,y) is the function defined in paragraph 28, is a constant equal to

$$\frac{1}{2\pi\sigma^2}\, e^{-\sqrt{Ut/\pi\sigma^2}}$$

at any given time for points of the central area of radius R, and is less than this for points outside this area. The quantity $p(x,y)\,dx\,dy$ is the probability that the object was initially located in an area of magnitude $dx\,dy$ centered at (x,y), and that it remains there undetected at time t. Thus the optimum procedure is one for which searching is always carried out at points where the chance of finding an object is at least as great as it is at any other points.

B9. The derivation of the optimum search procedures has been based on probability theory, taking account only of information which is available before the search begins, and the question arises whether in an actual search the procedure cannot be improved by taking into account, at any given time, the observed results of the search up to that time. If the analysis has been properly carried out it is apparent that such improvement is not possible since, whether or not searching is stopped when the object is located, to minimize the time required to find the object, the searching procedure should be that which is best for the case where the object has not been located up to that time, the method of searching subsequent to the finding of the object being of no importance. A more formal proof of this, based on Bayes' theorem, is given in reference (2).

B10. Substituting the value of $P(t)$ given by (8B) into equation (2) gives for the average discovery time, with optimum searching procedure,

$$\tau = \int_0^T \left\{1 + \left(\frac{Ut}{\pi\sigma^2}\right)^{1/2}\right\} e^{-\left(\frac{Ut}{\pi\sigma^2}\right)^{1/2}} dt.$$

This can be integrated by making the substitution $z = (Ut/\pi\sigma^2)^{1/2}$ and integrating by parts. We find

$$\frac{\tau}{\left(\frac{\sigma^2}{U}\right)} = 2\pi\left[3 - \left\{3 + 3\left(\frac{UT}{\pi\sigma^2}\right)^{1/2} + \frac{UT}{\pi\sigma^2}\right\} e^{-\left(\frac{UT}{\pi\sigma^2}\right)^{1/2}}\right]. \qquad (9B)$$

If searching always continues until the object is found, we must take T to be infinite. Equation (9B) then reduces to

$$\frac{\tau}{\left(\frac{\sigma^2}{U}\right)} = 6\pi.$$

B11. For an elliptic normal distribution of errors of estimate of the position of objects it may be expected, on the basis of the results for the circular normal distribution, that the optimum search procedure is one which makes the quantity $f(x,y)\ \{1 - \rho(x,y)\}$ a constant over a central area with no searching outside this area, where now $f(x,y)$ is given by equation (5). On the basis of this criterion we must have

$$e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)}\ e^{-m(x,y)}$$

constant over the central area. This condition is equivalent to

$$m(x,y) = k - \left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2} \right)$$

where k depends only on the time t. At the boundary of the searched area we must have m = 0. So the boundary is an ellipse with semi-axes $\sqrt{2k}\,\sigma_1$ and $\sqrt{2k}\,\sigma_2$. To find k in terms of the time t we have the requirement that, after searching a time t, the surface integral of m(x,y) taken over the area of this ellipse must equal Ut. We have

$$4 \int_0^{\sqrt{2k}\,\sigma_2} \int_0^{\sigma_1\sqrt{2k - y^2/\sigma_2^2}} \left(k - \frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2} \right) dx\,dy = Ut.$$

This gives

$$k = \sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}}.$$

That this distribution does maximize P(t), for any given t, follows from an analysis essentially identical with that of paragraph B5.

B12. Substituting the above value of k in the equation for m(x,y), we find for the optimum distribution of search-path density

$$m(x,y) = \sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}} - \left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2} \right) \qquad (10B)$$

where all points (x,y) are excluded for which the right hand side is negative. Substituting the value of m(x,y) for $\bar{m}$ in equation (8), and carrying out the resulting integral in (6) only over the part of the x,y plane included in the applicable ellipse, we obtain for the location probability at time t

$$P(t) = \frac{1}{2\pi\sigma_1\sigma_2} \int_E e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} \left\{ 1 - e^{-\left(\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}} - \frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2}\right)} \right\} dx\,dv$$

where $\int_E$ indicates a surface integral over the area of the ellipse. This can be written as the sum of two integrals

$$P(t) = \frac{1}{2\pi\sigma_1\sigma_2} \int_E e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} dx\,dy - \frac{1}{2\pi\sigma_1\sigma_2}\, e^{-\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}}} \int_E dx\,dy\,.$$

The second integral represents the area of the ellipse and is equal to

$$\pi\left(\sqrt{2k}\ \sigma_1\right)\left(\sqrt{2k}\ \sigma_2\right) \quad \text{or} \quad 2\pi\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}}\ \sigma_1\sigma_2.$$

We evaluate the first integral by a method suggested in reference (3). The integrand has the same value at all points for which

$$\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2} = L,$$

where L is any constant. These points lie on an ellipse whose area is $2\pi\sigma_1\sigma_2 L$. The area between the two ellipses represented by L and by $L + \Delta L$ is $2\pi\sigma_1\sigma_2\Delta L$. Hence, the value of the integral is given by

$$2\pi\sigma_1\sigma_2\int_0^{\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}}} e^{-L}\, dL = 2\pi\sigma_1\sigma_2\left(1 - e^{-\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}}}\right).$$

Hence, the value of the location probability for the optimum searching procedure is

$$P(t) = 1 - \left(1 + \sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}}\right)e^{-\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}}}. \qquad (11B)$$

B13. Equation (11B) differs from (8B) only in the occurrence of the product $\sigma_1\sigma_2$ in the place of σ^2. From an analysis similar to that by which (9B) was obtained from (8B) it follows that for the present case the minimum average discovery time τ is given by (9B) if σ^2 in that expression is replaced by $\sigma_1\sigma_2$.

B14. The conditions which were assumed to hold in deriving the above optimum search procedures for circular and elliptical normal distributions, probably do not hold in most practical cases, and even when they do hold, it should be possible to approximate the theoretical optimum procedures sufficiently by procedures which are more easily carried out. In general, it will be assumed in considering various search procedures, that the relative search rate U is constant, though it will be shown in some cases how variations in U can be taken into account.

"Standard" Square and Rectangular Area Searching.

B15. We attempt to formulate square or rectangular area search procedures which give search-path distributions approximating fairly closely those obtained with circular or elliptical area searches. For convenience in carrying out the search, it seems desirable to obtain the required distributions by uniform searches of the square or rectangular areas. It will be sufficient to consider rectangular area searching, square area searches being considered as a special case of rectangular searches.

B16. With rectangular area searches it is not possible to maintain the same value of $p(x,y)$ at all points of the searched area, but it is possible to maintain the same value at all points on the x and y-axes. For $p(x,y)$ to be the same for all points along the x-axis it is necessary that the search-path density distribution be of the form $m(x,0) = A_1 - x^2/2\sigma_1^2$ where A_1 is a quantity independent of x. Similarly, along the y-axis we must have $m(0,y) = A_2 - y^2/2\sigma_2^2$. Let the vertices of the successive rectangles lie along the curves $y = \pm \phi(x)$. Then for a given x the search-path density is the same for all values of y between $-\phi(x)$ and $\phi(x)$ as it is on the x-axis. Similarly, for a given y the density is the same for all values of x between $-\phi^{-1}(y)$ and $\phi^{-1}(y)$ as it is on the y-axis, where ϕ^{-1} is the inverse of ϕ. Thus at any point on the curve $y = \phi(x)$ we must have

$$A_1 - \frac{x^2}{2\sigma_1^2} = A_2 - \frac{\{\phi(x)\}^2}{2\sigma_2^2}.$$

For this to hold for the initial infinitesimal rectangle, we must have $A_1 = A_2$. To hold at all other points on the curve, we must then have

$$\phi(x) = x\,\frac{\sigma_2}{\sigma_1}.$$

Hence, the curves containing the vertices are straight lines

$$y = \pm \frac{\sigma_2}{\sigma_1}\,x,$$

and the ratio of the two dimensions of any of these rectangles is σ_2/σ_1.

B17. If X and Y are the dimensions of the rectangle being searched at time t we have

$$A_1 - \frac{X^2}{2\sigma_1^2} = A_2 - \frac{Y^2}{2\sigma_2^2} = 0$$

or

$$A_1 = \frac{X^2}{2\sigma_1^2} = \frac{Y^2}{2\sigma_2^2}.$$

Thus for

$$-\frac{\sigma_2}{\sigma_1}\,x \le y \le \frac{\sigma_2}{\sigma_1}\,x$$

we have

$$m(x,y) = \frac{X^2 - x^2}{2\sigma_1^2}.$$

For

$$-\frac{\sigma_1}{\sigma_2}\,y \le x \le \frac{\sigma_1}{\sigma_2}\,y$$

we have

$$m(x,y) = \frac{Y^2 - y^2}{2\,\sigma_2^2}.$$

B18. To determine P(t) it is convenient to divide the area of the x,y plane into two parts, one part consisting of the area which lies between the lines of vertices and includes the x-axis, and a second part which is similarly situated with respect to the y-axis. We have, taking account of symmetry,

$$P(t) = \frac{4}{2\pi\sigma_1\sigma_2}\int_0^X \int_0^{\frac{\sigma_2}{\sigma_1}x} e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)} \left\{1 - e^{-\left(\frac{X^2-x^2}{2\,\sigma_1^2}\right)}\right\} dy\,dx$$

$$+ \frac{4}{2\pi\sigma_1\sigma_2}\int_0^Y \int_0^{\frac{\sigma_1}{\sigma_2}y} e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)} \left\{1 - e^{-\left(\frac{Y^2-y^2}{2\,\sigma_2^2}\right)}\right\} dx\,dy$$

where X and Y are functions of t. If each of the integrals is divided into two, corresponding to the two terms in braces, the two resulting integrals which have

$$e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)}$$

as the integrand, can be combined to obtain an integral over a quadrant. Then

$$P(t) = \frac{4}{2\pi\sigma_1\sigma_2}\int_0^X \int_0^Y e^{-\left(\frac{x^2}{2\sigma_2}+\frac{y^2}{2\sigma_2^2}\right)} dy\,dx$$

$$- \frac{4}{2\pi\sigma_1\sigma_2}\int_0^X \int_0^{\frac{\sigma_2}{\sigma_1}x} e^{-\frac{X^2}{2\sigma_1^2}}\, e^{-\frac{y^2}{2\sigma_2^2}}\, dy\,dx$$

$$- \frac{4}{2\pi\sigma_1\sigma_2}\int_0^Y \int_0^{\frac{\sigma_1}{\sigma_2}y} e^{-\frac{Y^2}{2\sigma_2^2}}\, e^{-\frac{x^2}{2\sigma_1^2}}\, dx\,dy.$$

Changing the variables of integration to x/σ_1 and y/σ_2 and noting that $X/\sigma_1 = Y/\sigma_2$, the last two integrals are seen to be equal. We then obtain

$$P(t) = \left\{ \frac{2}{\sqrt{2\pi}} \int_0^Z e^{-z^2/2}\, dz \right\}^2 - \frac{8}{2\pi} e^{-Z^2/2} \int_0^Z \left\{ \int_0^z e^{-u^2/2}\, du \right\} dz$$

or

$$P(t) = \{F(Z)\}^2 - \frac{4}{\sqrt{2\pi}} Z e^{-Z^2/2} F(Z) + \frac{4}{\pi} e^{-Z^2/2} - \frac{4}{\pi} e^{-Z^2} \qquad (12B)$$

where $Z = X/\sigma_1 = Y/\sigma_2$ and $F(Z)$ is the probability integral defined by

$$F(Z) = \frac{1}{\sqrt{2\pi}} \int_{-Z}^{Z} e^{-z^2/2}\, dz. \qquad (13B)$$

It may be noted that

$$\frac{dF(z)}{dz} = \frac{2}{\sqrt{2\pi}} e^{-z^2/2}$$

$$\int F(z)\, dz = zF(z) + \frac{2}{\sqrt{2\pi}} e^{-z^2/2}.$$

Also

$$\frac{1}{\sqrt{2\pi}} \int_{Z_1}^{Z_2} e^{-z^2/2}\, dz = \frac{1}{2}\left\{F(Z_2) - F(Z_1)\right\} = \frac{1}{2}\left\{F(Z_2) + F(-Z_1)\right\}.$$

Extensive tables for $F(Z)$ are given in reference (4).

B19. To find Z as a function of t we have the requirement

$$\int m(x,y)\, dx\, dy = Ut$$

where the integral is evaluated over all parts of the x,y plane for which m is not zero. It is convenient to divide the area into two parts, as before, giving

$$Ut = 4 \int_0^X \left(\frac{X^2 - x^2}{2\sigma_1^2} \right) \left(x \frac{\sigma_2}{\sigma_1} \right) dx + 4 \int_0^Y \left(\frac{Y^2 - y^2}{2\sigma_2^2} \right) \left(y \frac{\sigma_1}{\sigma_2} \right) dy$$

where the second parenthesis in each integrand represents the length of the perpendicular from the point on the line of vertices to the indicated axis. This gives

$$\frac{Ut}{\sigma_1 \sigma_2} = \left(\frac{X}{\sigma_1} \right)^4 = \left(\frac{Y}{\sigma_2} \right)^4 = Z^4. \qquad (14B)$$

This can be substituted for Z in (12B) to find $P(t)$ as an explicit function of t.

B20. To evaluate the average discovery time τ, it is convenient to use equation (2) in the form

$$\tau = \int_0^{Z_1} t \frac{dP}{dZ} dZ - \{1 - P(T)\}T \qquad (15B)$$

where Z_1, is the value of Z corresponding to t = T. Substituting for dP/dZ from (12B) and for t from (14B) the resulting integral can be integrated by parts to obtain

$$\frac{\tau}{(\sigma_1\sigma_2/U)} = \{1 - P(T)\} \frac{T}{(\sigma_1\sigma_2/U)} + 15 \{F(Z_1)\}^2$$

$$- \frac{4}{\sqrt{2\pi}} e^{-Z_1^2/2} F(Z_1) \{Z_1^5 + 5Z_1^3 + 15Z_1\}$$

$$- \frac{4}{\pi} e^{-Z_1^2} \left\{Z_1^4 + \frac{9}{2} Z_1^2 + 12\right\} \qquad (16B)$$

$$+ \frac{4}{\pi} e^{-Z_1^2/2} \{Z_1^4 + 4Z_1^2 + 8\}$$

$$+ \frac{16}{\pi}.$$

Assuming the search is continued until the object is found, we obtain, by letting Z_1 become infinite,

$$\frac{\tau}{(\sigma_1\sigma_2/U)} = 15 + \frac{16}{\pi}.$$

"Standard" Constant Parallel Path Searching.

B21. Where turning time is an important factor in the time required to find an object the optimum lengths of the rectangles are substantially larger in the early part of the search than the values given by the "standard" rectangular area searching. We do not attempt to find this optimum distribution, but consider a search which is likely to be reasonably satisfactory when the turning time is large. Consider a procedure in which successive searches are made in parallel paths, and all search paths have the same length. Let the end points of the paths lie on two straight lines which are normal to the direction of the search paths, and are equidistant from and on opposite sides of the most probable position of the object. Let the separation of the lines, or the length ℓ of the search paths, be sufficient so that the chance that the object does not lie between the lines is a small quantity μ. If there is an elliptical distribution of the errors of estimate of the position of objects we choose the direction of the search path parallel to the direction of greatest uncertainty. Taking the y-axis in this direction we have

$$F\left(\frac{\ell}{2\sigma_2}\right) = 1 - \mu.$$

B22. If the length ℓ is taken so large that μ is negligible, it is easily found by the methods used in connection with circular area searching that the optimum search-path distribution relative to x, is the distribution which makes $p(x) = e^{-x^2/2\sigma^2} e^{-m(x)}$ a constant over the width being searched at a given time, with $m(x)$ equal to zero outside this region. This requires that

$$m(x) = k - \frac{x^2}{2\sigma_1^2} \tag{17B}$$

where k depends only on t. Taking this distribution to be adequate even when μ is not negligible we obtain for the location probability from (6) and (8)

$$P(t) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \int_{-X}^{X} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} \left\{1 - e^{-\left(k - \frac{x^2}{2\sigma_1^2}\right)}\right\} dx\, dy$$

where the boundaries of the searched area are at -X and at X. Since $m = 0$ at $x = X$, we have $X = \sqrt{2k}\,\sigma_1$. We obtain

$$P(T) = F\left(\frac{\ell}{2\sigma_2}\right)\left[F\left(\frac{X}{\sigma_1}\right) - \frac{2}{\sqrt{2\pi}}\frac{X}{\sigma_1}e^{-k}\right]. \tag{18B}$$

To find k and X in terms of t we have

$$\ell \int_{-\sqrt{2k}\,\sigma_1}^{\sqrt{2k}\,\sigma_1} m(x)\, dx \doteq Ut.$$

Substituting for $m(x)$ from (17B) we find

$$k = \frac{1}{2}\left\{\frac{3}{2}\left(\frac{\sigma_2}{\ell}\right)\left(\frac{Ut}{\sigma_1\sigma_2}\right)\right\}^{2/3}$$

and

$$\frac{X}{\sigma_1} = \left\{\frac{3}{2}\left(\frac{\sigma_2}{\ell}\right)\left(\frac{Ut}{\sigma_1\sigma_2}\right)\right\}^{1/3} \tag{19B}$$

These can be substituted in (18B) to find $P(t)$ as an explicit function of t.

B23. To find the average discovery time τ it is convenient to start with equation (15B), where now $Z = X/\sigma_1$. In terms of Z equation (18B) gives

$$P(t) = F\left(\frac{\ell}{2\sigma_2}\right)\left[F(Z) - \frac{2}{\sqrt{2\pi}} Z\, e^{-Z^2/2}\right]. \tag{20B}$$

Substituting for dP/dZ and t in (15B) in terms of Z and carrying out the integration by parts gives

$$\frac{\tau}{(\sigma_1\sigma_2/U)} = \frac{T}{(\sigma_1\sigma_2/U)} \{1 - P(T)\} + \frac{4}{3\sqrt{2\pi}} \frac{\ell}{\sigma_2} F\left(\frac{\ell}{2\sigma_2}\right) \left[8 - e^{-Z_1^2/2} \{Z_1^4 + 4Z_1^2 + 8\}\right] \quad (21B)$$

where

$$Z_1 = \left\{\frac{3}{2}\left(\frac{\sigma_2}{\ell}\right)\left(\frac{UT}{\sigma_1\sigma_2}\right)\right\}^{1/3}.$$

B24. For the assumed search-path distribution along the x-axis, and for a given required value of $P(T)$, there is a value of ℓ which gives the minimum value of T, and another value of ℓ which gives the minimum value of τ. However, in considering a suitable choice of ℓ, it is desirable to take account of possible finite turning times between successive passes. In all the derivations for standard searching procedures, it is assumed that the ground speed is constant, and this assumption obviously is not justified if there is an appreciable turning time. Provided the assumption of a constant ground speed falls down only because of a finite turning time, the results can readily be modified to take account of this if the turning time has the same value for all passes. The formulas which have been obtained for constant parallel path searching are valid in this case if U is interpreted to be the average search rate, equal to the ratio of the aggregate area searched on one pass to the time required to make one pass and one turn.

B25. Let U_0 be the actual search rate while making a pass, and let the turning time be $\bar{k}$ times the time required for the locator to move a distance σ_2 when its speed is $U_0/W\beta$. Then the turning time is $\bar{k}(\sigma_2/U_0)W\beta$, and the average search rate, taking into account the turning time is

$$U = \frac{\ell\, W\beta}{\frac{\ell}{U_0} W\beta + \bar{k}\frac{\sigma_2}{U_0} W\beta} = \frac{U_0}{1 + \bar{k}\frac{\sigma_2}{\ell}}.$$

Table B gives calculated values of T and τ in terms of $\sigma_1\sigma_2/U_0$ for values of $P(T)$ of 0.95 and 0.99, for various values of ℓ, and for several values of the turning time, as indicated by the values of $\bar{k}$. It is seen that different values of ℓ are required to minimize T and τ and that the required values depend on $\bar{k}$. However a choice of $\ell/\sigma_2 = 5$ for $P(T) = 0.95$, and of $\ell/\sigma_2 = 6$ for $P(T) = 0.99$, gives values of both T and τ very close to the minimum values for the range of values of $\bar{k}$ considered.

B26. That constant parallel path searching may be more efficient than the standard rectangular area searching procedure, is easily demonstrated. For either procedure, the value of T depends only on the number of required passes when the turning time becomes very large. In both cases the search-path distribution along the x-axis is given by an equation of the form $m = (X^2 - x^2)/2\sigma_1^2$, and the number of search passes is the integral of this expression from $-X$ to X, divided by $W\beta$. The quantity X is given in terms of T by (14B) for rectangular area searching and by (19B) for constant parallel

TABLE B

$\frac{\ell}{2\sigma_2}$	$F\left(\frac{\ell}{2\sigma_2}\right)$	$P(T)$	$\frac{U_0T}{\sigma_1\sigma_2}$				$\frac{U_0\tau}{\sigma_1\sigma_2}$			
			$\bar{k}=0$	$\bar{k}=2$	$\bar{k}=5$	$\bar{k}=10$	$\bar{k}=0$	$\bar{k}=2$	$\bar{k}=5$	$\bar{k}=10$
2.25	.9756	0.95	84.4	121.8	178	272	19.9	28.8	42.0	64.1
2.5	.9876		81.5	114.1	163	244	20.7	29.0	41.4	62.1
2.75	.9940		84.3	115.0	161	238	22.1	30.1	42.2	62.3
3.0	.9973		89.4	119.2	164	238	23.7	31.7	43.5	63.3
3.25	.9989		95.6	125.0	169	243	25.5	33.4	45.2	64.9
3.5	.9995		102.4	131.6	175	249	27.4	35.3	47.0	66.6
3.75	.9998		109.4	138.5	182	255	29.4	37.2	48.9	68.5
2.75	.9940	0.99	178	243	340	501	24.1	32.9	46.1	68.1
3.0	.9973		167	222	306	445	25.6	34.1	46.9	68.2
3.25	.9989		172	224	303	435	27.4	35.8	48.4	69.5
3.5	.9995		181	233	310	440	29.3	37.7	50.3	71.3
3.75	.9998		192	243	320	448	31.4	39.7	52.3	73.2

path searching. Substituting for T from Table 2, and carrying out the integration, it is found that for $P(T) = 0.95$, the number of paths required for rectangular area searching is larger by a factor of 1.02 than the number required for constant parallel path searching with the lengths of path specified in Table 2, and for $P(T) = 0.99$ the corresponding factor is 1.04. Obviously the turning time might need to be rather large for the constant parallel path procedure to be theoretically the more efficient. To obtain a more accurate comparison for moderate turning times we need to know the average length of path for rectangular area searching. This can be obtained by noting that the average length of path, multiplied by the number of paths and by $W\beta$, is UT. This gives for the average length of path a value $4.38\sigma_2$ for $P(T) = 0.95$, and a value of $5.27\sigma_2$ for $P(T) = 0.99$. These may be compared with the values $5\sigma_2$ and $6\sigma_2$, respectively, for constant parallel path searching. The average time to make one pass, including the turning time, is $(\bar{\ell}/U_0 + \bar{k}\,\sigma_2/U_0)\,W\beta$, where $\bar{\ell}$ is the average length of path. Thus, for $P(T) = 0.99$, the two methods give the same value of T when

$$\frac{6\sigma_2 + \bar{k}\,\sigma_2}{U_0} = 1.04\left(\frac{5.27\,\sigma_2 + \bar{k}\,\sigma_2}{U_0}\right).$$

This is satisfied when $\bar{k} = 18.2$, or when the turning time is approximately 3 times as large as the time required to make the straight run part of a pass. However, even for a turning time equal to the average time required to make the straight run part of a pass, the value of T for constant parallel path searching is only 2-1/2% larger than for rectangular area searching to obtain $P(T) = 0.99$, and is 4-1/2% larger to obtain $P(T) = 0.95$. There would be slightly larger differences in the values of τ.

"Standard" Circle Diameters Searching.

B27. Another type of search procedure which may be useful when there are relatively long turning times and when also the errors of estimate of object position have a circular normal distribution, consists of searches in straight paths along diameters of a circle centered at the most probable position of the object, there being a small rotation of the direction of the path for each successive path. It is assumed that the lengths of the parts of the paths consisting of circumferential arcs are negligible compared to the straight parts of the paths and that the errors of estimating the position of the object have a circular normal distribution. For this case the nominal search-path density within the given circle is given by a function of the form $m(r) = k/r$ where k will depend on the search rate and on the search time. We have

$$\int_0^R m(r)\ 2\pi r\, dr = Ut$$

where R is the radius of the given circle. Substituting the above expression for $m(r)$, we find $k = Ut/2\pi R$ and $m(r) = Ut/2\pi rR$. Substituting in (7) we obtain for $P(t)$

$$P(t) = \frac{1}{\sigma^2}\int_0^R e^{-\frac{r^2}{2\sigma^2}}\left\{1 - e^{-\left(\frac{Ut}{2\pi Rr}\right)}\right\} r\, dr.$$

Putting $Z_1 = R/\sigma$, and carrying out the integration for the first term of the integrand, we obtain

$$P(t) = 1 - e^{-\frac{1}{2}Z_1^2} - \int_0^{Z_1} e^{-\left\{\frac{z^2}{2} + \frac{1}{zZ_1}\left(\frac{1}{2\pi}\frac{Ut}{\sigma^2}\right)\right\}} z\, dz. \qquad (22B)$$

The integral must be evaluated by numerical methods. If we take the circle large enough so that the probability of the object being outside the circle is some small quantity μ, we have

$$\mu = \frac{1}{\sigma_2}\int_R^\infty e^{-r^2/2\sigma^2}\, r\, dr = e^{-R^2/2\sigma^2}$$

or

$$Z_1 = \frac{R}{\sigma} = \sqrt{-2\log_e \mu}.$$

Substituting from (22B) into (3), we obtain for the average discovery time τ

$$\tau = T e^{-\frac{Z_1^2}{2}} + \int_0^T\left[\int_0^{Z_1} e^{-\left\{\frac{z^2}{2} + \frac{1}{zZ_1}\left(\frac{1}{2\pi}\frac{Ut}{\sigma^2}\right)\right\}} z\, dz\right] dt.$$

Reversing the order of integration and simplifying, this leads to

$$\frac{\tau}{\left(\frac{\sigma^2}{U}\right)} = \frac{T}{\left(\frac{\sigma^2}{U}\right)} e^{-Z_1^2/2} - 2\pi Z_1^2 e^{-Z_1^2/2} + \pi\sqrt{2\pi}\, Z_1\, F(Z_1)$$

$$- 2\pi Z_1 \int_0^{Z_1} z^2 e^{-\left\{\frac{z^2}{2} + \frac{1}{zZ_1}\left(\frac{1}{2\pi}\frac{UT}{\sigma^2}\right)\right\}} dz .$$

APPENDIX C

DERIVATION OF FORMULAS FOR THE EVALUATION OF MULTIAREA SEARCHING WHEN THE NAVIGATIONAL ERROR IS SMALL COMPARED TO THE UNCERTAINTY IN THE OBJECT POSITION

C1. The formulas obtained in Appendix B for the overall searching time T and the average discovery time τ are exact only when each successive coverage adds only a very small amount to the aggregate searched area. In practice it would not be possible to search in this manner in view of the finite width of searched path. Before considering the effect of finite width of searched path it will be useful to consider first successive coverages with a locator of vanishingly small width of searched path, but with enough passes on each coverage to increase the nominal search-path density per coverage by a substantial amount Δm.

Width of Searched Path Small Compared to the Navigational Errors.

C2. Leaving for later consideration the circle diameters searching procedure, let w_a be the aggregate width of searched path of the locator, and let the lateral distance between all adjacent paths on a given coverage be a constant d. Then for that coverage we have $\Delta m = w_a/d$, a constant. Consider the general problem of determining P(t), T, and τ when areas A_1, A_2 .... A_N are searched successively, each area A_i being searched uniformly with a lateral distance on the i'th coverage to give an increase of the search-path density for all points in A_i of $\Delta_i m$. In order to take account of possible turning times, it will be assumed that the average search rate for the i'th area is U_i and that this may be different for different areas. It will be assumed that the boundaries of any area A_i do not extend beyond any area A_J if $i < J$.

C3. Suppose that at time T_n the search of the n'th area A_n has just been completed. There have then been n coverages of the area A_1, n - 1 coverages of that part of the area A_2 not included in A_1, etc. For an elliptical distribution of errors of estimate of the position of the object we have for the location probability $P(T_n)$ at time T_n

$$P(T_n) = \frac{1}{2\pi\sigma_1\sigma_2}\int_{A_1} e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)}\left(1 - e^{-\sum_{i=1}^{n}\Delta_i m}\right) dx\,dy$$

$$+ \frac{1}{2\pi\sigma_1\sigma_2}\int_{A_2-A_1} e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)}\left(1 - e^{-\sum_{i=2}^{n}\Delta_i m}\right) dx\,dy$$

$$+ -$$

$$+ \frac{1}{2\pi\sigma_1\sigma_2}\int_{A_n-A_{n-1}} e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)}\left(1 - e^{-\Delta_n m}\right) dx\,dy$$

where the integral $\int_{A_i - A_{i-1}}$ is to be taken over that part of the area A_i which is not included in A_{i-1}. Taking $\Delta_i A = A_i - A_{i-1}$ for $i > 1$ and $\Delta_1 A = A_1$, the expression for $P(T_n)$ can be written

$$P(T_n) = \sum_{J=1}^{n} \left\{ 1 - e^{-\sum_{i=J}^{n} \Delta_i m} \right\} \frac{1}{2\pi\sigma_1\sigma_2} \int_{\Delta_J A} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} dx\, dy. \qquad (1C)$$

Setting $\Delta_i T' = T'_i - T_{i-1}$ we have, in general,

$$\Delta_i m = \frac{U_i \Delta_i T}{A_i}. \qquad (2C)$$

Hence,

$$T'_n = \sum_{i=1}^{n} \Delta_i T' = \sum_{i=1}^{n} \frac{\Delta_i m A_i}{U_i}. \qquad (3C)$$

If a total of N coverages are made, the location probability $P(T')$, with $T' = T_N$, is obtained from (1C) by replacing n by N.

C4. It is now assumed that each coverage is made in straight parallel paths starting at one edge of the area which is to be searched on that coverage, and that searching proceeds uniformly toward the opposite edge, each path being displaced a fixed distance d farther from the initial path than the path of the next preceding pass. To determine the average discovery time τ for an overall searching time $T' = T_N$, we can write equation (2) in the form

$$\tau - T_N \{1 - P(T_N)\} = \int_0^{T_N} t \frac{dP}{dt} dt = \sum_{n=1}^{N} \int_{T_{n-1}}^{T_n} t \frac{dP}{dt} dt.$$

Consider the n'th term

$$\int_{T_{n-1}}^{T_n} t \frac{dP}{dt} dt.$$

The quantity dP/dt will vary during the interval T_{n-1} to T_n, but, for the symmetrical areas which are considered in this study and for the assumed searching procedure, during the n'th coverage dP/dt is symmetrical with respect to time about the time

$$t = T_{n-1} + \frac{\Delta_n T'}{2}.$$

Changing the variable of integration by making the substitution

$$t = T_{n-1} + \frac{\Delta_n T}{2} + t'$$

we obtain for the n'th term

$$\int_{-\frac{\Delta_n T}{2}}^{\frac{\Delta_n T}{2}} \frac{dP}{dt}\left(T_{n-1} + \frac{\Delta_n T}{2} + t'\right) dt'$$

where dP/dt is symmetrical with respect to t' about t' = 0. Then dP/dt t' is an odd function for the interval $-\Delta_n T/2$ to $\Delta_n T/2$ and the integral reduces to

$$\left(T_{n-1} + \frac{\Delta_n T}{2}\right)\{P(T_n) - P(T_{n-1})\}.$$

The first factor can be written

$$\frac{T_n + T_{n-1}}{2}.$$

Then

$$\tau - T_N\{1 - P(T_N)\} = \sum_{n=1}^{N}\left\{\frac{T_n + T_{n-1}}{2}\right\}\{P(T_n) - P(T_{n-1})\}$$

$$= \frac{1}{2}\sum_{n=1}^{N} P_n T_n - \frac{1}{2}\sum_{n=2}^{N} P_{n-1} T_{n-1} - \frac{1}{2}\sum_{n=2}^{N} P_{n-1} T_n + \frac{1}{2}\sum_{n=1}^{N} P_n T_{n-1}$$

where P_n has been written for $P(T_n)$, etc., and where use has been made of the fact that $P_0 = 0$ and $T_0 = 0$. By suitable changes of the summation indices the second and third sums become $-\frac{1}{2}\sum_{n=1}^{N-1} P_n T_n$ and $-\frac{1}{2}\sum_{n=1}^{N-1} P_n T_{n+1}$, respectively. Hence,

$$\tau - T_N\{1 - P(T_N)\} = \frac{1}{2} P(T_N)\, T_N - \frac{1}{2}\sum_{n=1}^{N-1} P(T_n)\{T_{n+1} - T_{n-1}\} + \frac{1}{2} P(T_N)\, T_{N-1}$$

or

$$\tau = T_N - \frac{1}{2} P(T_N)\{T_N - T_{N-1}\} - \frac{1}{2}\sum_{n=1}^{N-1} P(T_n)\{T_{n+1} - T_{n-1}\}. \qquad (4C)$$

C5. The result is valid for any choice of areas A_n, satisfying the condition that A_{n-1} shall be included in A_n, and for any degrees of coverage $\Delta_n m$, and for any rate of searching U_n for each A_n. It applies to the case where a number of coverages are made on each area, some of the values of A_n then being equal for different values of n. For the case where all $\Delta_n m$ are equal to a given value Δm and all U_n are equal to a given value U, the results can be put in a simpler form. Taking

$$I_J = \frac{1}{2\pi\sigma_1\sigma_2} \int_{\Delta_J A} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} dx\, dy \tag{5C}$$

equation (1C) becomes for this case

$$P(T_n) = \sum_{J=1}^{n} \left\{1 - e^{-(n-J+1)\Delta m}\right\} I_J. \tag{6C}$$

Then

$$\tau = T_N - \frac{\Delta m}{2U} A_N P(T_N) - \frac{\Delta m}{2U} \sum_{n=1}^{N-1} (A_{n+1} + A_n) P(T_n) \tag{7C}$$

and

$$T = T_N = \frac{\Delta m}{U} \sum_{n=1}^{N} A_n. \tag{8C}$$

C6. For the types of multiarea searching considered in the present study, A_n and I_n have the following values:

(a) For circular areas we have if R_n is the radius of the n'th area A_n that

$$A_n = \pi R_n^2 \quad \text{and} \quad I_n = e^{-\frac{R_{n-1}^2}{2\sigma^2}} - e^{-\frac{R_n^2}{2\sigma^2}}.$$

For elliptical areas, σ^2 is replaced in the above equation by $\sigma_1\sigma_2$ and R_n^2 is replaced by $a_n b_n$, where a_n and b_n are the semi-axes of the n'th elliptical area.

(b) For rectangular areas with a width $2X_n$ and a length $2Y_n$ for the n'th area A_n and with $X_n/Y_n = \sigma_1/\sigma_2$, we have

$$A_n = 4X_n Y_n = 4X_n^2 \frac{\sigma_2}{\sigma_1} = 4Y_n^2 \frac{\sigma_1}{\sigma_2}$$

and

$$I_n = F\left(\frac{X_n}{\sigma_1}\right) F\left(\frac{Y_n}{\sigma_2}\right) - F\left(\frac{X_{n-1}}{\sigma_1}\right) F\left(\frac{Y_{n-1}}{\sigma_2}\right)$$

$$= \left\{F\left(\frac{X_n}{\sigma_1}\right)\right\}^2 - \left\{F\left(\frac{X_{n-1}}{\sigma_1}\right)\right\}^2$$

where F is the probability integral defined by (13B) of Appendix B.

(c) For rectangular areas of fixed length ℓ and width $2X_n$ we have

$$A_n = 2X_n\ell$$

and

$$I_n = F\left(\frac{\ell}{2\sigma_2}\right)\left\{F\left(\frac{X_n}{\sigma_1}\right) - F\left(\frac{X_{n-1}}{\sigma_1}\right)\right\}.$$

C7. These results will be used to evaluate multiarea searching procedures which are quite similar to the procedures considered in Appendix B but with successive uniform coverages which add a substantial and fixed amount Δm to the search-path density. Consider first the optimum searching procedure determined in Appendix B. We consider several possible procedures for approximating the search-path distributions obtained with the optimum procedure when the distribution of errors of object position is circular normal.

C8. Method I. One possible approach is to assume that at the end of each coverage the boundary of the searched area is to be the same as it would be, for the same aggregate searched area, or for the same searching time, if successive coverages were made as implied in the derivation in Appendix B. For the multiarea procedure, assuming a constant increase Δm in search-path density per coverage, the aggregate area searched at the end of n coverages is

$$UT'_n = \pi\Delta m\left[R_n^2 + \sum_{i=1}^{n-1} R_i^2\right] \tag{9C}$$

where T'_n is the time required for n coverages and R_i is the radius of the boundary of the area searched on the i'th coverage. Then from (5B) and (9C) we have

$$\left(\frac{R_n}{\sigma}\right)^4 = 4\Delta m\left[\frac{R_n^2}{\sigma^2} + \sum_{i=1}^{n-1}\frac{R_i^2}{\sigma^2}\right]$$

Taking $v_i = R_i/2\sigma\sqrt{\Delta m}$, the above equation reduces to

$$v_n^4 - v_n^2 - \sum_{i=1}^{n-1} v_i^2 = 0.$$

We have $\nu_1 = 1$ and

$$\nu_n^2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\sum_{i=1}^{n-1} \nu_i^2}$$

The required values of R_n can then be obtained from the relation

$$R_n = 2\sigma\sqrt{\Delta m}\ \nu_n .$$

For a total of N coverages we have

$$UT_N = 4\pi\sigma^2(\Delta m)^2 \sum_{n=1}^{N} \nu_n^2 .$$

C9. Method II. Another possible approach, one which is used in reference (2), is to assume that at the end of each coverage the search-path density in the vicinity of the initially most probable position of the object is to be the same as it would be for the same aggregate searched area if searching were carried out as assumed in Appendix B. This requires that in searching the n'th area of radius R_n the aggregate area $\pi R_n^2\ \Delta m$ searched on that coverage be equal to that searched by the procedure considered in Appendix B while increasing the search-path density at the center of the paraboloid from $(n - 1)\ \Delta m$ to $n\Delta m$. To find the latter quantity we set $r = 0$, $m(r) = n\Delta m$ in equation (6B) of Appendix B. The total aggregate searched area at a time when the search-path density is $n\Delta m$ at $r = 0$ is seen to be $UT_n = \pi\sigma^2(n\Delta m)^2$. Since this must be equal to $\pi\Delta m \sum_{i=1}^{n} R_i^2$ we have

$$\pi R_n^2\ \Delta m = \pi\sigma^2\left[(\Delta m)^2\ \{n^2 - (n - 1)^2\}\right]$$

or

$$R_n^2 = (2n - 1)\ \sigma^2\ \Delta m .$$

It is seen that the radius of the first searched area obtained by this procedure is substantially smaller than the radius of the first area searched by Procedure I. If N is the total number of coverages we have, for determining Δm,

$$UT = UT_N = \pi\sigma^2 N^2\ (\Delta m)^2 .$$

C10. Method III. From a comparison of graphs representing the variation in search-path density with r for Methods I and II with the optimum distribution found in Appendix B, it appears that the searched areas obtained with Method I are larger than optimum and that the areas for Procedure II are too small. For a third approach to the problem of approximating the desired distribution, consider the function $p(x,y)$ given by $p(x) = f(x,y)\ \{1 - \rho(x,y,t)\}$. The quantity $p(x,y)\,dx\,dy$ is the probability that the object was initially within a given small area of dimensions dx and dy in the vicinity of the point (x, y) and that it remains there undetected after time t. Suppose now that the n'th circular area is chosen so that the value of p(x, y) just outside the boundary of this area is equal to the value of p(x, y) at the center x = 0, y = 0. At points just outside the searched area, p(x, y) represents the initial probability density of the object f(x, y). If this procedure is followed for n = 1, 2, etc., then at the end of the n'th coverage there is equal probability of the undetected existence of the object in any area of dimensions dx dy at the center and at all of the boundaries of the first n circular areas. It is easily proven that if the complete search consists of a single coverage this method of choosing the boundary of the searched area minimizes T and τ for a given location probability P(T). Equating p(x, y) or p(r) at the center at the end of the n'th coverage, and the initial probability density just outside the boundary of the n'th area, gives

$$e^{-n\Delta m} = e^{-\frac{R_n^2}{2\sigma^2}},$$

or

$$R_n^2 = 2n\,\sigma^2\,\Delta m. \qquad (10C)$$

The aggregate area searched in making a total of N coverages is

$$UT = UT_N = \Delta m \sum_{n=1}^{N} \pi R_n^2 .$$

Substituting for R_n from (10C) and summing gives

$$UT_N = \pi\,\sigma^2 N(N + 1)\,(\Delta m)^2 . \qquad (11C)$$

It may be noted that equation (10C) and the corresponding relation obtained by Method II become identical in the limit as $\Delta m \rightarrow 0$, and each of these methods gives the optimum distribution for this limiting case.

C11. Calculations of P(T) and of τ have been carried out for each of the three methods for given aggregate searched areas and given numbers of coverages. The results are given in Table C1. Also given are values of the radius R_N of the final coverage. It is seen that Method III gives best results in all cases. In addition to giving the best theoretical results Method III is also clearly superior to Method II in giving larger searched areas on successive coverages so that a smaller total number of search passes are required by Method III than by Method II. Applying the rule which forms the basis for Method III to other multilayer search procedures we find for an elliptical distribution of errors of estimate of object position and for elliptical searched areas that the edge of the n'th searched area is an ellipse with semiaxes a_n, b_n, given by

$$\frac{a_n\,b_n}{\sigma_1\,\sigma_2} = \frac{a_n^2}{\sigma_1^2} = \frac{b_n^2}{\sigma_2^2} = 2n\,\Delta m \qquad (12C)$$

and

$$UT_N = \sum_{n=1}^{N} \pi a_n b_n \Delta m = \pi N(N + 1)\ \sigma_1\sigma_2\ (\Delta m)^2 . \qquad (13C)$$

TABLE C1

CIRCULAR AREA MULTIAREA SEARCHING

$\frac{T}{\left(\frac{\sigma^2}{U}\right)}$	N	Method I			Method II			Method III		
		$\frac{R_N}{\sigma}$	$P(T_N)$	$\frac{\tau}{\left(\frac{\sigma^2}{U}\right)}$	$\frac{R_N}{\sigma}$	$P(T_N)$	$\frac{\tau}{\left(\frac{\sigma^2}{U}\right)}$	$\frac{R_N}{\sigma}$	$P(T_N)$	$\frac{\tau}{\left(\frac{\sigma^2}{U}\right)}$
70.7	1	3.08	0.899	38.9	2.18	0.899	38.9	2.59	0.931	37.8
	2	3.08	0.931	24.6	2.67	0.940	23.5	2.78	0.944	22.6
	3	3.08	0.940	20.8	2.81	0.946	19.9	2.87	0.947	19.4
138.4	1	3.64	0.963	71.8	2.58	0.963	71.8	3.06	0.982	70.5
	2	3.64	0.980	37.2	3.16	0.985	33.3	3.29	0.987	32.6
	3	3.64	0.985	28.0	3.33	0.988	25.3	3.39	0.989	24.7

C12. For rectangular area searching we cannot satisfy the suggested criterion for the assumed uniform coverages but can expect to approach it closely by taking the area of the n'th rectangle to be equal to the area of the n'th equivalent ellipse for the same UT_n, with $X_n/Y_n = a_n/b_n = \sigma_1/\sigma_2$. Then the relation between T_N and Δm is given by (13C) and the half lengths X_n and Y_n of the sides of the n'th rectangle are given by

$$\frac{X_n^2}{\sigma_1^2} = \frac{Y_n^2}{\sigma_2^2} = \frac{X_n Y_n}{\sigma_1\sigma_2} = \frac{\pi}{4}\ \frac{a_n b_n}{\sigma_1\sigma_2} = \frac{\pi}{2}\ n\,\Delta m. \qquad (14C)$$

C13. For constant parallel path searching we need to consider the distribution p(x) in one dimension only. To obtain a value of the initial probability density at the edge of the n'th searched area equal to p(x) at the center after n coverages requires

$$X_n^2 = 2\sigma_1^2\ n\,\Delta m . \qquad (15C)$$

The relation between UT_N and Δm is given by

$$UT_N = \sum_{n=1}^{N} 2X_n\ \ell\,\Delta m = 2\sqrt{2}\ \sigma_1\sigma_2\ \frac{\ell}{\sigma_2}\left\{\sum_{n=1}^{N} \sqrt{n}\right\}(\Delta m)^{3/2} . \qquad (16C)$$

C14. Multiarea Circle Diameters Searching. For this searching procedure, assuming all coverages are made with the same length of search path, the final search-path density depends only on the total aggregate area searched and not on the aggregate area searched on each coverage. Hence, P(T) does not depend on whether coverages are very light as assumed in Appendix B or whether a substantial amount of searching is carried out on each coverage. However, the average discovery time τ corresponding to a given P(T), does depend on the amount of searching carried out on each coverage. Suppose that there are a total of N coverages and that the total aggregate area searched on each coverage is UT/N. Then (2) can be written

$$\tau - T \{1 - P(T)\} = \sum_{n=1}^{N} \int_{(n-1)\frac{T}{N}}^{n\frac{T}{N}} t \frac{dP}{dt} dt .$$

Since dP/dt is constant during any one coverage and for the n'th coverage is

$$\left[P\left(n\frac{T}{N}\right) - P\left\{(n - 1)\frac{T}{N}\right\}\right]\Big/\frac{T}{N}$$

the summation becomes

$$\frac{1}{2}\sum_{n=1}^{N}\left(\frac{T}{N}\right)(2n - 1)\left[P\left(n\frac{T}{N}\right) - P\left((n - 1)\frac{T}{N}\right)\right].$$

Then

$$\tau - T \{1 - P(T)\} = \frac{1}{2}\frac{T}{N}\sum_{n=1}^{N-1} P\left(n\frac{T}{N}\right)\{(2n - 1) - (2n + 1)\} + \frac{1}{2}T\left(\frac{2N - 1}{N}\right)P(T)$$

$$= -\frac{T}{N}\sum_{n=1}^{N-1} P\left(n\frac{T}{N}\right) + \frac{1}{2}\left(\frac{2N - 1}{N}\right) T\, P(T)$$

or

$$\tau = T - \frac{T}{2N} P(T) - \frac{T}{N}\sum_{n=1}^{N-1} P\left(\frac{nT}{N}\right) \qquad (17C)$$

where P(nT/N) is given by (22B) of Appendix B if t in that expression is replaced by nT/N.

Modified Searching of the First Layer.

C15. All of the formulas which have been obtained for τ in multiarea searching except for circle diameters searching are based on the assumption that all search passes on any coverage are parallel to each other and that for each coverage searching begins at one edge of the area to be searched on that coverage and that additional passes are made at increasing distance from this initial edge until the opposite edge is reached.

Searching in this manner appears to be about as effective theoretically as any other for all layers except the first. For the first coverage, in circle or square area searching, a more efficient procedure, based on the assumptions of paragraph 35, is a search in spiral paths beginning at the most probable position of the objectand continuing until the area of the first coverage has been uniformly searched. Considerable theoretical improvement can also be obtained by retaining searching in parallel paths but modifying the order in which different portions of the initial area are searched. We determine what reductions can be obtained in the values of τ by these modifications of the searching procedure on the first coverage. The values of T do not depend on whether the initial coverage is made by the standard parallel path procedure or by one of the modified procedures.

C16. Let t_1 be the time required to make the first coverage. The value of t_1 does not depend on whether searching is in parallel paths or in spirals. From (3C) we have $t_1 = A_1 \Delta m/U$. Then equation (2) can be written

$$\tau = \tau_1 + \tau_2 \tag{18C}$$

where

$$\tau_1 = \int_0^{t_1} t \frac{dP}{dt} dt \text{ and } \tau_2 = \int_{t_1}^{T} t \frac{dP}{dt} dt + T\{1 - P(T)\} . \tag{19C}$$

For the case where the search is carried out in parallel paths in the standard way it follows from paragraph C4 that

$$\tau_1 = \frac{t_1}{2} P(t_1) = \frac{A_1 \Delta m}{2U} P(t_1). \tag{20C}$$

We wish to compare this with the value of τ_1 obtained with spiral searching and with modified parallel path searching.

C17. Circular Spiral Searching. It is assumed that successive paths in the circular spiral are separated by the required distance d to give the desired value of Δm, where $\Delta m = w_a/d$. For the assumed small search width compared to the uncertainty in the object position, the area searched in time t is essentially circular. After searching for a time t, the outer path of the spiral has a radius R given by

$$\pi R^2 \Delta m = Ut .$$

Then the location probability P(t) is given by

$$P(t) = \frac{1}{2\pi\sigma^2} \int_0^R e^{-\frac{r^2}{2\sigma^2}} (1 - e^{-\Delta m})\, 2\pi r\, dr$$

or

$$P(t) = (1 - e^{-\Delta m}) \left(1 - e^{-\frac{R^2}{2\sigma^2}}\right) = (1 - e^{-\Delta m}) \left(1 - e^{-\frac{Ut}{2\pi \Delta m \sigma^2}}\right). \tag{21C}$$

Substituting in (19C) and integrating by parts we obtain for circular spiral searching

$$\tau_1 = (1 - e^{-\Delta m}) \left[-t_1 e^{-\frac{Ut_1}{2\pi\Delta m\sigma^2}} + 2\pi\Delta m \left(\frac{\sigma^2}{U}\right) \left\{ 1 - e^{-\frac{Ut_1}{2\pi\Delta m\sigma^2}} \right\} \right]$$

or

$$\frac{\tau_1}{\left(\frac{\sigma^2}{U}\right)} = \pi\Delta m(1 - e^{-\Delta m}) \left[2 - \left\{ 2 + \left(\frac{R_1}{\sigma}\right)^2 \right\} e^{-R_1^2/2\sigma^2} \right], \tag{22C}$$

where R_1 is the radius of the area A_1 searched on the first coverage. For comparison (20C) gives, for the regular circular area searching on the first coverage,

$$\frac{\tau_1}{\left(\frac{\sigma^2}{U}\right)} = (1 - e^{-\Delta m}) \frac{\pi}{2} \left(\frac{R_1}{\sigma}\right)^2 \Delta m \left(1 - e^{-R_1^2/2\sigma^2}\right)$$

where $P(t_1)$ in (20C) has been obtained from (21C) by replacing R by R_1. The difference between these two values of τ_1 represents the improvement in τ which can be attained by changing the searching procedure from all parallel path searching to spiral searching on the first coverage and parallel searching on the remaining coverages. Of course, the results for circular spiral searching, and those obtained in the next section for square area searching, are valid only for circular normal distribution of the errors of estimate of object positions.

C18. <u>Square Spiral Searching</u>. In this case we have

$$4 \left(\frac{X}{\sigma}\right)^2 \Delta m = \frac{U}{\sigma^2} t \tag{23C}$$

and

$$P(t) = \frac{1}{2\pi\sigma^2} \int_{-X}^{X} \int_{-X}^{X} e^{-\frac{x^2 + y^2}{2\sigma^2}} (1 - e^{-\Delta m})\, dx\, dy = (1 - e^{-\Delta m}) \left\{ F\left(\frac{X}{\sigma}\right) \right\}^2 \tag{24C}$$

where 2X is the length of one side of the square being searched at time t. For this case it is convenient to change the variable in (19C) to obtain for τ_1

$$\tau_1 = \int_0^{Z_1} t \frac{dP}{dz} dz \tag{25C}$$

where $z = X/\sigma$ and $Z_1 = X_1/\sigma$, and where t is replaced by its equivalent in terms of z from (23C). The resulting integral can be integrated by parts to obtain

$$\frac{\tau_1}{\left(\frac{\sigma^2}{U}\right)} = 4\Delta m(1 - e^{-\Delta m}) \left[- \frac{4}{\sqrt{2\pi}} Z_1 F(Z_1) e^{-Z_1^2/2} + \{F(Z_1)\}^2 + \frac{2}{\pi}\left(1 - e^{-Z_1^2}\right)\right]. \quad (26C)$$

For comparison, (20C) gives for parallel path searching on the first coverage

$$\frac{\tau_1}{\left(\frac{\sigma^2}{U}\right)} = 2Z_1^2 \Delta m (1 - e^{-\Delta m}) \{F(Z_1)\}^2.$$

C19. Modified Parallel Path Procedure I. If there is a substantial turning time between successive passes, and if the turning time is independent of the separation between any two successive paths as long as both tracks are within the width $2X_1$ of the area to be searched on the first coverage, then the theoretically most efficient procedure is obviously to make the first pass at the center of this width and then to search on alternate sides of the searched area on alternate passes until all of the initial area A_1 has been searched. For this case we have

$$2X \ell \Delta m = Ut \quad (27C)$$

where 2X is the width of the area searched in time t. Also, assuming an elliptical distribution of errors in estimating the object position

$$P(t) = (1 - e^{-\Delta m}) \frac{1}{2\pi\sigma_1\sigma_2} \int_{-X}^{X} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} dy\, dx = (1 - e^{-\Delta m}) F\left(\frac{X}{\sigma_1}\right) F\left(\frac{\ell}{2\sigma_2}\right). \quad (28C)$$

Taking $z = X/\sigma_1$ and $Z_1 = X_1/\sigma$, and substituting in (25C), we obtain

$$\frac{\tau_1}{\left(\frac{\sigma_1\sigma_2}{U}\right)} = \frac{4}{\sqrt{2\pi}} \frac{\ell}{\sigma_2} F\left(\frac{\ell}{2\sigma_2}\right) \Delta m (1 - e^{-\Delta m}) (1 - e^{-Z_1^2/2}). \quad (29C)$$

For comparison, (20C) gives for the standard searching procedure beginning at one edge

$$\frac{\tau_1}{\left(\frac{\sigma_1\sigma_2}{U}\right)} = Z_1 \frac{\ell}{\sigma_2} \Delta m (1 - e^{-\Delta m}) F(Z_1) F\left(\frac{\ell}{2\sigma_2}\right).$$

This modified parallel path procedure will be called Modified Parallel Path Procedure I.

C20. Modified Parallel Path Procedure II. If the turning time for passes made on adjacent tracks is too small to be important, or if the turning time increases substantially with increase in distance between successive tracks, a likely more efficient procedure than that considered in the previous paragraph is to make the first pass at some distance from the center of the area to be searched and then to search uniformly toward the center and beyond to the opposite side; and then to search the remaining area, beginning at the edge of the remaining unsearched area which is adjacent to the path searched on the initial pass. Let X' be the absolute value of the distance of the initial pass from the center of the area. Let the first pass be made at $X = -X'$. Then during the first part of the search

$$\{z + z'\} \frac{\ell}{\sigma_2} \Delta m = \frac{U}{\sigma_1 \sigma_2} t \tag{30C}$$

where $z' = X'/\sigma_1$. For the second part of the search, we have, with change of variable corresponding to a 180° rotation of the coordinate axis

$$(z + Z_1) \frac{\ell}{\sigma_2} \Delta m = \frac{U}{\sigma_1 \sigma_2} t. \tag{31C}$$

Then for any t during the first part of the search, the location probability is

$$P(t) = (1 - e^{-\Delta m}) F\left(\frac{\ell}{2\sigma_2}\right) \frac{1}{\sqrt{2\pi}} \int_{-z'}^{z} e^{-v^2/2} \, dv .$$

During the second part of the search, with $z \geq z'$, it is

$$P(t) = (1 - e^{-\Delta m}) F\left(\frac{\ell}{2\sigma_2}\right) \frac{1}{\sqrt{2\pi}} \int_{-Z_1}^{z} e^{-v^2/2} \, dv .$$

Changing the variable in (19C) and integrating separately for the two parts of the search we find

$$\frac{\tau_1}{\left(\frac{\sigma_1 \sigma_2}{U}\right)} = \frac{1}{\sqrt{2\pi}} \left(\frac{\ell}{\sigma_2}\right) F\left(\frac{\ell}{2\sigma_2}\right) \Delta m (1 - e^{-\Delta m}) \left[\int_{-z'}^{Z_1} (z + z') e^{-z^2/2} \, dz + \int_{z'}^{Z_1} (z + Z_1) e^{-z^2/2} \, dz\right]$$

or

$$\frac{\tau_1}{\left(\frac{\sigma_1 \sigma_2}{U}\right)} = \frac{\ell}{\sigma_2} F \frac{\ell}{2\sigma_2} \Delta m (1 - e^{-\Delta m}) \left[\frac{2}{\sqrt{2\pi}} \left(e^{-(z')^2/2} - e^{-Z_1^2/2}\right) + \frac{1}{2} (z' + Z_1) F(Z_1) + \frac{1}{2} (z' - Z_1) F(z')\right]. \tag{32C}$$

By differentiation with respect to z' it is found that τ_1 is a minimum when z' is chosen to satisfy the relation

$$(z' + Z_1) e^{-(z')^2/2} = \frac{\sqrt{2\pi}}{2} \{F(z') + F(Z_1)\} . \tag{33C}$$

Table C2 gives required values of X' as a function of X_1. Considered as a function of X'/X_1 the minima for τ are quite broad and, in the interest of simplicity, it seems desirable to choose a single value of the ratio X'/X_1 in formulating practical searching procedures. Table 5 contains values of τ obtained with the second modified procedure for making the first coverage based on the values of X'/X_1 given by Table C2. For constant parallel path searching the values of τ were recalculated using $X'/X = 0.4$. The values of τ obtained by this method were found to differ from the minimum τ by negligible amounts in all cases except for $P(T) = 0.99$ when there is but a single coverage, and for the latter case the difference was less than 1/2%.

TABLE C2

$\frac{X_1}{\sigma_1}$	Optimum $\frac{X'}{\sigma_1}$	Optimum $\frac{X'}{X_1}$
1.0	.4818	.4818
1.4	.6516	.4654
1.8	.8016	.4454
2.2	.9305	.4230
2.6	1.0393	.3997
3.0	1.1307	.3769
3.4	1.2081	.3553

C21. Obviously, parallel path rectangular area searching can be modified in the same manner as that just considered for constant parallel path searching. The results obtained for the latter case are applicable to the former cases if $\ell/2\sigma_2$ is replaced by Y_1/σ_2.

Width of Searched Path Not Small Compared to Navigational Errors.

C22. The results obtained for multiarea searching on the assumption that the overall width of searched path W is vanishingly small compared to the navigational error, can readily be extended to the case where W is not small compared to the standard deviation $\bar{\sigma}$ of navigational error, provided both W and $\bar{\sigma}$ are small compared to the standard deviation of the errors of estimate of the position of the object. For this purpose use can be made of the results of Part I of this report. It is found in Part I that for any number of similar systematic coverages of an area during which the average search-path density has been increased by an amount m, the probability ρ that a random point in the area has been effectively searched is $1 - e^{-\gamma m}$ where γ is a constant greater than 1 whose value depends on $W\beta$ and d. If there are n coverages and each coverage increases the average search-path density by $\Delta m = W\beta/d$, then $m = n\Delta m$. Unless $W/\bar{\sigma}$ is less than 2 the result is strictly valid for n coverages only if the grids of searched paths laid down on successive coverages are randomly displaced relative to each other, as they normally would be in multiarea searching in which different size areas are searched on different coverages.

C23. To take account of the finite width of searched path it may be noted that equation (6C) was based on the fact that for very small widths of searched path the probability that any point is effectively searched while increasing the search-path density at that point by an amount $(n - J + 1)\Delta m$ is $1 - e^{-(n-J+1)\Delta m}$. This expression must now be replaced by $1 - e^{-\gamma(n-J+1)\Delta m}$. Consider the value of $P(T')$ obtained with a given γ, for a given Δm and a given choice of areas A_n. To obtain the same values of $P(T')$ with locators of vanishingly small search path width requires a value of Δm larger by a factor γ. It follows that the required value of UT_n or UT' to obtain a given value of the location probability $P(T_n')$ or $P(T')$ is also larger by a factor of γ when the path width is vanishingly small. Equation (7C) shows that if both Δm, and UT_n' and UT_N' are larger by a factor of γ, so is $U\tau$. Hence, all the results which have been obtained from equation (6C), (7C) and (8C) are seen to be valid in this more general case if Δm is replaced by $\gamma\Delta m$, UT' or UT_N' is replaced by $\gamma UT'$ or $\gamma UT_N'$, and $U\tau$ is replaced by $\gamma U\tau$. The value of γ is given by Plates 2, 3 and 4 if $d - W/\bar{\sigma} \leq 4$. If $d - W/\bar{\sigma}_1 \geq 3$, γ is given by Plate 5.

C24. Although theoretically γ can be quite large if $\beta = 1$ and $W/\bar{\sigma}$ is large, in order to realize a large value of γ the value of Δm must be equal to or greater than 1. The value of $\gamma\Delta m$ is then large and multiarea searching then reduces to a single coverage. Thus the most efficient searching in this case is likely to be spiral searching or a modified parallel path searching. Consider the limiting case $\gamma \to \infty$ which is obtained when $\bar{\sigma} \to 0$ and $\beta = 1$, with $\Delta m \geq 1$. Consider circular spiral searching. In general, for finite search path widths we have in the place of (21C)

$$P(T') = (1 - e^{-\gamma\Delta m})\left(1 - e^{-\frac{UT\gamma}{2\pi\Delta m\gamma\sigma^2}}\right).$$

The limit of this as γ becomes infinite, assuming $\Delta m = 1$, is

$$P(T') = 1 - e^{-\frac{UT}{2\pi\sigma^2}}. \qquad (34C)$$

This gives T for any required $P(T')$. Similarly replacing $U\tau_1$ by $\gamma U\tau_1$ and Δm by $\gamma\Delta m$ in (22C), and cancelling the common factor γ, we obtain in the limit when $\gamma \to \infty$ and $\Delta m = 1$

$$\frac{\tau_1}{\left(\frac{\sigma^2}{U}\right)} = \pi\left[-\left(\frac{R_1}{\sigma}\right)^2 e^{-\frac{1}{2}\left(\frac{R_1}{\sigma}\right)^2} + 2\left\{1 - e^{-\frac{1}{2}\left(\frac{R_1}{\sigma}\right)^2}\right\}\right]$$

where $\pi R_1^2 = UT$. To this we must add τ_2, which in this case is $T\{1 - P(T')\}$. Hence,

$$\frac{\tau}{\left(\frac{\sigma^2}{U}\right)} = 2\pi\left\{1 - e^{-\frac{UT}{2\pi\sigma^2}}\right\} = 2\pi\, P(T'). \qquad (35C)$$

This result could be obtained quite simply without introducing γ by noting that when $\bar{\sigma} = 0$, $\beta = 1$, and $W = d$, all objects are detected in the area nominally searched. This leads at once to equation (21C) with the factor $(1 - e^{-\Delta m})$ missing. Then τ can be obtained from (3). In the same manner that (34C) and (35C) were derived, formulas can be readily obtained for the limiting case $\gamma \to \infty$ for square spiral searching and for modified parallel path searching by simple modifications of the formulas previously obtained.

C25. The results which have been obtained for widths of searched path W not small compared to $\bar{\sigma}$ are strictly valid only if both $\bar{\sigma}$ and W are small compared to σ or compared to σ_1 and σ_2.

APPENDIX D

EVALUATION OF GIVEN SEARCHING PROCEDURES WHEN NAVIGATIONAL ERRORS ARE NOT SMALL COMPARED TO UNCERTAINTY IN THE OBJECT POSITION

D1. When navigational errors are not small compared to the uncertainty in the position of objects the differences between the nominal position of the search paths and the actual positions may be quite important. Hence, searching procedures which are satisfactory when navigational errors are small are not necessarily so when navigational errors are large. It will be convenient to consider this problem first under the restriction that the over-all width of path W is assumed small compared to the standard deviation of navigational error.

ANALYSIS OF PROBLEM UNDER CONDITIONS WHERE THE OVER-ALL WIDTH OF SEARCHED PATH IS SMALL COMPARED TO STANDARD DEVIATION OF NAVIGATIONAL ERRORS

D2. Under conditions where the navigational errors are not small compared to the uncertainty in the position of the object and are large compared to the over-all width of searched path W, the location probability P(t) is still given by (6) and (8) but the search-path density $\bar{m}(x,y,t)$ cannot be assumed to be equal to the nominal search-path density. The direct solution of the problem of finding the most suitable searching procedures in this general case might consist of the following two steps:

(a) Determination of the nominal search-path density distribution $m(x,y,t)$ which gives as nearly as possible the desired actual search-path density $\bar{m}(x,y,t)$.

(b) Determination of suitable searching procedures to obtain the required nominal search-path distributions.

Because of mathematical difficulties, such a direct attack on the problem will not be attempted. Instead, searching procedures which appear likely to be reasonably satisfactory will be evaluated, and the results will be compared with those obtained for the case where the navigational errors are small.

D3. It cannot be expected that the optimum search-path density can be obtained, even theoretically, by any searching procedure when there are appreciable navigational errors, and the difference between the theoretical optimum distribution and the best attainable distribution is likely to increase with increase in the navigational error. It is apparent that with large navigational errors relatively large search-path densities must be expected in areas where there is little chance of finding the object with a corresponding undesirable decrease in search-path density in the area where the object is most likely to be located.

D4. To obtain some idea of how large the navigational errors may be without seriously effecting the searching efficiency which can be attained by suitable searching procedures, we consider the theoretical searching procedure which consists of searching at a uniform search rate while attempting to stay as near as possible to the most probable position of the object. To obtain the search-path density $\bar{m}(x,y,t)$ for this case consider first the more general case where searching consists of repeated light coverages of a rectangular area of dimensions 2X and 2Y and let the standard deviation of navigational errors parallel to the x and y-directions be $\bar{\sigma}_1$ and $\bar{\sigma}_2$ respectively. The nominal search-path density m at points within this rectangle varies only with time and is given by $4XYm = Ut$. Then equation (9A) of Appendix A gives for the search-path density $\bar{m}(x,y,t)$

$$\bar{m}(x,y,t) = \frac{m(t)}{2\pi\bar{\sigma}_1\bar{\sigma}_2} \int_{-Y}^{Y} \int_{-X}^{X} e^{-\left(\frac{(x'-x)^2}{2\bar{\sigma}_1^2} + \frac{(y'-y)^2}{2\bar{\sigma}_2^2}\right)} dx'\, dy'. \tag{1D}$$

We now suppose X and Y so small that the integrand is effectively constant for points within the rectangle. The double integral then becomes in the limit as X and Y approach zero,

$$4XY\, e^{-\left(\frac{x^2}{2\bar{\sigma}_1^2} + \frac{y^2}{2\bar{\sigma}_2^2}\right)}$$

and $\bar{m}(x,y,t)$ reduces to

$$\bar{m}(x,y,t) = 2\,\frac{m(t)}{\pi}\,\frac{X}{\bar{\sigma}_1}\,\frac{Y}{\bar{\sigma}_2}\, e^{-\left(\frac{x^2}{2\bar{\sigma}_1^2} + \frac{y^2}{2\bar{\sigma}_2^2}\right)}$$

or

$$\bar{m}(x,y,t) = \frac{Ut}{2\pi\sigma_1\sigma_2}\, e^{-\left(\frac{x^2}{2\bar{\sigma}_1^2} + \frac{y^2}{2\bar{\sigma}_2^2}\right)}. \tag{2D}$$

D5. We now suppose that both the errors of estimate of object position and the navigational errors have circular normal distributions with standard deviations σ and $\bar{\sigma}$, respectively. Then substituting $\bar{m}(x,y,t)$ from (2D) into (8) and using (7), we obtain for the location probability

$$P(t) = 1 - \frac{1}{\sigma^2} \int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}}\, e^{-\left(\frac{Ut}{2\pi\bar{\sigma}^2}\, e^{-r^2/2\bar{\sigma}^2}\right)}\, r\, dr. \tag{3D}$$

In general, P(t) must be evaluated by numerical integration, but for the special case $\bar{\sigma} = \sigma$ the expression for P(t) reduces to

$$P(t) = 1 - \frac{2\pi\sigma^2}{Ut}\left(1 - e^{-\frac{Ut}{2\pi\sigma^2}}\right),$$

valid for $\sigma = \bar{\sigma}$. Substituting P(t) from (3D) into (3) and reversing the order of integration we obtain for the average discovery time

$$\tau = \frac{T}{\sigma^2} \int_0^{\infty} e^{-r^2/2\sigma^2} \left\{ \frac{1 - e^{-\left(\frac{UT}{2\pi\bar{\sigma}^2} e^{-r^2/2\bar{\sigma}^2}\right)}}{\frac{UT}{2\pi\bar{\sigma}^2} e^{-r^2/2\bar{\sigma}^2}} \right\} r\,dr. \tag{4D}$$

This can be simply evaluated for the case where searching always continues until the object is found. Letting $T \longrightarrow \infty$, we obtain

$$\tau = 2\pi \left(\frac{\bar{\sigma}^2}{U}\right) \frac{\bar{\sigma}^2}{\bar{\sigma}^2 - \sigma^2},$$

provided $\bar{\sigma} > \sigma$. The integral does not converge if $\bar{\sigma} \leq \sigma$, indicating an infinite average discovery time for this case. The above expression for τ is a minimum for given σ and U when $\bar{\sigma} = \sqrt{2}\,\sigma$. That is, based on the average discovery time corresponding to $T \longrightarrow \infty$ or to $P(T) = 1$, this type of searching is most efficient when $\bar{\sigma}$ is about 41% greater than σ. The minimum value of τ is $\tau_{min} = 8\pi\,\sigma^2/U$. This may be compared with the value $6\pi\,\sigma^2/U$ obtained for the optimum searching procedure with negligible navigational error.

D6. If instead of attempting to keep the locator as close as possible to the most probable location of the object, searching consists of repeated light uniform searches of the rectangular area considered in paragraph D4, then (1D) gives for the search-path density

$$\bar{m}(x,y,t) = \frac{m(t)}{4} \left\{F\left(\frac{X-x}{\bar{\sigma}_1}\right) + F\left(\frac{X+x}{\bar{\sigma}_1}\right)\right\} \left\{F\left(\frac{Y-y}{\bar{\sigma}_2}\right) + F\left(\frac{Y+y}{\bar{\sigma}_2}\right)\right\}$$

$$= tE(x,y)$$

where

$$E(x,y) = \frac{U}{16XY} \left\{F\left(\frac{X-x}{\bar{\sigma}_1}\right) + F\left(\frac{X+x}{\bar{\sigma}_1}\right)\right\} \left\{F\left(\frac{Y-y}{\bar{\sigma}_2}\right) + F\left(\frac{Y+y}{\bar{\sigma}_2}\right)\right\} \tag{5D}$$

and F is the probability function defined by (13B) of Appendix B. Then from (6) we obtain, on taking account of symmetry about the x and y-axes

$$P(t) = 1 - \frac{2}{\pi\sigma_1\sigma_2} \int_0^{\infty} \int_0^{\infty} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} e^{-tE(x,y)}\,dx\,dy. \tag{6D}$$

From (3) we obtain, by reversing the order of integration

$$\tau = \frac{2}{\pi\sigma_1\sigma_2} \int_0^{\infty} \int_0^{\infty} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} \left\{\frac{1 - e^{-TE(x,y)}}{E(x,y)}\right\} dx\,dy. \tag{7D}$$

If searching is in parallel paths, with each path aimed at the position of the object, E(x,y) reduces to

$$E(x,y) = \frac{U}{\frac{\bar{\sigma}_1\bar{\sigma}_2}{4\sqrt{2\pi}}}\frac{\bar{\sigma}_2}{Y}\left\{F\left(\frac{Y - y}{\bar{\sigma}_2}\right) + F\left(\frac{Y + y}{\bar{\sigma}^2}\right)\right\} e^{-\frac{x^2}{2\bar{\sigma}_1^2}} . \tag{8D}$$

D7. If the rectangle is so long that there is negligible chance that the object would be missed because of turning too soon on any pass, then $F(\{Y - y\}/\bar{\sigma}_2)$ and $F(\{Y + y\}/\bar{\sigma}_2)$ each become unity and F(x,y) is given by

$$E(x,y) = \frac{U}{4X\ell}\left\{F\left(\frac{X - x}{\bar{\sigma}_1}\right) + F\left(\frac{X + x}{\bar{\sigma}_1}\right)\right\} \tag{9D}$$

where ℓ is the length of each path and is equal to 2Y. If, in addition, each of the parallel paths is aimed at the most probable position of the object we have

$$E(x,y) = \frac{U}{\frac{\bar{\sigma}_1\bar{\sigma}_2}{\sqrt{2\pi}}}\left(\frac{\bar{\sigma}_2}{\ell}\right) e^{-\frac{x^2}{2\bar{\sigma}_1^2}} . \tag{10D}$$

In this case the integrations with respect to y in (6D) and (7D) can be easily carried out and numerical integration is required only with respect to x. From (7D) and (10D) we find for the case $T \longrightarrow \infty$ and with all parallel paths aimed at the most probable position of the object

$$\tau = \sqrt{2\pi}\,\frac{\ell}{\sigma_2}\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\bar{\sigma}_1^2/\sigma_1}{\sqrt{\bar{\sigma}_1^2 - \sigma_1^2}} .$$

The value of τ is a minimum for given σ_1 when $\bar{\sigma}_1 = \sqrt{2}\,\sigma_1$. The minimum value of τ is

$$2\sqrt{2\pi}\,\frac{\ell}{\sigma_2}\left(\frac{\sigma_1\sigma_2}{U}\right),$$

which may be compared to the value

$$\frac{32}{3\sqrt{2\pi}}\,\frac{\ell}{\sigma_2}\,\frac{\sigma_1\sigma_2}{U}$$

which is obtained with the optimum path distribution in the x-direction and negligible navigational error. The latter value is approximately 85% of the former.

D8. The case where there is no uncertainty in the position of the object is of some practical and theoretical interest. The values of P(t) and of τ for this case can be obtained by considering the limits of the above integrals as σ_1 and σ_2 approach zero, but they can be obtained more simply from the definition of $\rho(x,y,t)$. By definition,

$\rho(x,y,t)$ is the probability that the object has been detected in time t if it is located at (x, y). Hence, for an object known to be at x = 0, y = 0 we have

$$P(t) = \rho(0,0,t) = 1 - e^{-\bar{m}(0,0,t)}.$$

Hence, for the cases considered in paragraphs D6 and D7 we have when $\sigma_1 = \sigma_2 = 0$

$$P(t) = 1 - e^{-tE(0,0)}. \tag{11D}$$

Then substitution in (3) gives

$$\tau = \frac{1}{E(0,0)} \{1 - e^{-TE(0,0)}\}. \tag{12D}$$

D9. If the repeated coverages of given rectangular areas are not made very lightly but sufficiently intensively to give a substantial increase Δm in nominal search-path density on each coverage the value of P(t) remains unchanged, but τ will be increased. Suppose there is only a single coverage with total searching time T. Comparing with the derivation in paragraph (C4) of Appendix C, it is not true in the present case as it was for the case considered there that dP/dt is symmetrical with respect to t about the time t = T/2. However, it can readily be seen that for the present case there is a larger probability that the object will be discovered during the first interval T/2 than during the second. For though the expected distribution of actual search paths laid down during the first interval T/2 and that laid down during the second interval T/2 are symmetrical with respect to the probable position of the object, due to navigational error they overlap, and hunting is less fruitful on the average during the second half interval than during the first. Furthermore, for searching carried out during equal time intervals dt at T/2 - t' and T/2 + t', where t' is any time interval less than T/2, the hunting is better during the first interval than during the second. This is apparent if it is noticed that the expected search-path density at the position of the nominal path then being searched is at least as large at T/2 + t' as at T/2 - t', and that, whereas at T/2 - t' deviations from the nominal path in the direction of decreased search-path density is also in the direction of greater initial probability density for the object, at T/2 + t' the converse is true. Let $(dP/dt)_1$ be the value of dP/dt at t = T/2 - t' and let $(dP/dt)_2$ be the value of dP/dt at t = T/2 + t'. Then

$$\int_0^T t\left(\frac{dP}{dt}\right)dt = \int_{-T/2}^{0} (T/2 + t'') \left(\frac{dP}{dt}\right)_1 dt'' + \int_0^{T/2} (T/2 + t') \left(\frac{dP}{dt}\right)_2 dt'$$

$$= T/2 \; \{P(T) - P(0)\} + \int_0^{T/2} t' \left\{ -\left(\frac{dP}{dt}\right)_1 + \left(\frac{dP}{dt}\right)_2 \right\} dt'$$

or

$$\int_0^T t\left(\frac{dP}{dt}\right)dt < (T/2) \; P(T).$$

Thus, whereas under the conditions considered in Appendix C the value of τ for a single coverage was found to be $T - (T/2)\,P(T)$, we obtain from (2) for the present case of not negligible navigational error

$$\tau < T - (T/2)\,P(T). \tag{13D}$$

D10. If the searching procedure consists of making N successive coverages, the result (13D) is obviously valid for the first coverage if T is replaced by T_1. It follows by the same arguments that for any succeeding coverage, say the n'th coverage, there is a greater chance of locating the object during the first half of the time interval T_n than during the second half. While it does not appear possible to prove in general that for the n'th coverage the corresponding inequality holds for each pair of time elements dt at $T_n/2 - t'$ and $T_n/2 + t'$, it will certainly hold if the navigational errors are large compared to the uncertainty in the position of the object. Thus we may expect that for the n'th coverage, in the place of the equality

$$\int_{T_{n-1}}^{T_n} t\,\frac{dP}{dt}\,dt = \left(\frac{T_n + T_{n-1}}{2}\right)\left\{P(T_n) - P(T_{n-1})\right\}$$

obtained in paragraph (C4) of Appendix C for negligible navigational error, we have more generally that the value of the integral is smaller than the expression on the right. Then in the place of equation (4C) we obtain for the case where there is substantial navigational error, the inequality

$$\tau < T_N - 1/2\,P(T_N)\left\{T_N - T_{N-1}\right\} - 1/2\sum_{n=1}^{N-1} P(T_n)\left\{T_{n+1} - T_{n-1}\right\} \tag{14D}$$

where $P(T_n)$ is given by (6D) or (11D) with t replaced by T_n and with F given by one of the equations (5D), (8D), (9D) or (10D). In the only applications which will be made of this result for rectangular area searching each coverage will be assumed to add a fixed amount Δm to the nominal search-path density and the search rate U will be assumed to be constant. We then have for repeated searches of a single area

$$T_n U = 4\,n\,X\,Y\,\Delta m = \frac{n}{N}\,T_N\,U,$$

where use has been made of the fact that

$$4N\Delta m XY = UT_N, \quad \text{or} \quad \Delta m = \frac{UT_N}{4NXY}.$$

If N is large the calculations of all the required $P(T_n)$ becomes laborious. The problem can be reduced to one double integration by substituting in (14D) the value of $P(T_n)$ given by (6D) and interchanging the summation and the integration. This gives for τ when there are repeated searches of the same area

$$\tau < \frac{T_N}{2N} + \frac{2\,T_N/N}{\pi\,\sigma_1\,\sigma_2}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)}\left[\frac{1}{2}\,e^{-NQ(x,y)} + \sum_{n=1}^{N-1} e^{-nQ(x,y)}\;dx\,dy\right]$$

where

$$Q(x,y) = \frac{\Delta m}{4}\left\{F\left(\frac{X-x}{\bar{\sigma}_1}\right) + F\left(\frac{X+x}{\bar{\sigma}_1}\right)\right\}\left\{F\left(\frac{Y-y}{\bar{\sigma}_2}\right) + F\left(\frac{Y+y}{\bar{\sigma}_2}\right)\right\}.$$

Summing the geometric series, the quantity in square brackets becomes

$$1/2\;e^{-NQ} + \frac{e^{-Q} - e^{-NQ}}{1 - e^{-Q}}.$$

If the rectangle is so long that there is negligible chance that the object lies beyond the ends of the rectangle we obtain for Q

$$Q = \frac{\Delta m}{2}\left\{F\left(\frac{X-x}{\sigma_1}\right) + F\left(\frac{X+x}{\sigma_1}\right)\right\}.$$

D11. If successive coverages are made over different rectangles, (9A) gives for the search-path density at the end of the n'th coverage

$$\bar{m}(x,y,T_n) = \sum_{i=1}^{n}\frac{\Delta m}{2\pi\bar{\sigma}_1\bar{\sigma}_2}\int_{-Y_i}^{Y_i}\int_{-X_i}^{X_i} e^{-\left(\frac{(x'-x)^2}{2\bar{\sigma}_1^2}+\frac{(y'-y)^2}{2\bar{\sigma}_2^2}\right)}dx'\,dy'$$

$$= \sum_i Q_i(x,y)$$

where

$$Q_i(x,y) = \frac{\Delta m}{4}\left\{F\left(\frac{X_i - x}{\bar{\sigma}_1}\right) + F\left(\frac{X_i + x}{\bar{\sigma}_1}\right)\right\}\left\{F\left(\frac{Y_i - y}{\bar{\sigma}_1}\right) + F\left(\frac{Y_i + y}{\bar{\sigma}_2}\right)\right\}$$

and where the dimensions of the i'th rectangle are $2X_i$ and $2Y_i$. We then have for $P(T_n)$

$$P(T_n) = 1 - \frac{2}{\pi\sigma_1\sigma_2}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)}\,e^{-\sum_{i=1}^{n}Q_i(x,y)}\,dx\,dy.$$

Then τ is given by (14D) with T_n given by the relation

$$UT_n = 4\Delta m\sum_{i=1}^{n} X_i\,Y_i.$$

D12. In general, the evaluation of P(T) and of τ when $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ are not small involves double integration, and the numerical computations become quite laborious. For the purpose of determining suitable search procedures we can consider the simpler one-dimensional problem to which constant parallel path searching reduces when the length of path is taken sufficiently large so there is negligible chance of missing an object because of turning too soon. The results obtained in paragraphs D5 and D7 illustrate the close relation between the one-dimensional and the two-dimensional problems.

D13. From general theoretical considerations and from the results of paragraphs D5 and D7 it may be expected that as $\bar{\sigma}_1$ and $\bar{\sigma}_2$ increase relative to σ_1 and σ_2, the nominal search-path density should be increased in the vicinity of the most probable position of the object and decreased at the larger distances. It is also to be expected that the results of the search are less critically dependent on the nominal search-path distribution. Table D1 gives computed data for the one-dimensional case where searching consists of making repeated light coverages of a single specified width. The table gives values of T and τ corresponding to P(T) = 0.95 and P(T) = 0.99 for various values of the nominally searched width 2X and for various values of the ratio $\bar{\sigma}_1/\sigma_1$. For comparison, the first row of numerical data gives T and τ for the case where $\bar{\sigma}_1/\sigma_1 \rightarrow 0$ and where searching is carried out so as to maintain always the optimum lateral search-path distribution.

D14. It is seen that for values of $\bar{\sigma}_1/\sigma_1$ of 3/2 or greater the minimum values of τ are obtained with $X \rightarrow 0$. Also, for this range of values of $\bar{\sigma}_1/\sigma_1$, the values of T are a minimum for $X \rightarrow 0$ with the one exception that for $\bar{\sigma}_1/\sigma_1 = 3/2$ and P(T) = 0.99 a slightly smaller value of T is obtained with a larger value of X. From these results it appears that for this one-dimensional problem, all paths should be aimed at the most probable position of the object if $\bar{\sigma}_1/\sigma_1$ is greater than 3/2. This type of searching, where all passes are made on a single track with nominal search paths all of the same length and centered at the most probable position of the object, will be called "single track" searching. The above conclusion is based on results of repeated uniform searches over a single width and it is conceivable that multiarea searching would give better results. However, the value of T obtained with single track searching for $\bar{\sigma}_1/\sigma = 3/2$ is seen to be only 7% greater than the value obtained with optimum nominal path distribution when $\bar{\sigma}_1 = 0$. Calculations of T and τ for P(T) = 0.95 with $\bar{\sigma}_1/\sigma_1 = 3/2$, using a searching procedure consisting of some passes on a single track and additional searching consisting of uniform coverages of a finite width, indicate that T and τ cannot be decreased by this method below the values obtained by the single track procedure. On the basis of Table D1 it may be expected that when there is negligible turning time and $\bar{\sigma}_1/\sigma_1 \gg 3/2$, single track searching with $W/\bar{\sigma} \rightarrow 0$ is most effective when the path length ℓ is quite small and this is confirmed by calculations which have been made for various path lengths. When there is appreciable turning time the most effective single track searching is obtained with substantial path lengths. Data for single track searching when $\bar{\sigma}_1 \geq 3/2\, \sigma_1$ are presented in Tables 9, 10 and 11, which will be explained in a later section.

D15. For the intermediate case, defined by the condition $0 < \bar{\sigma}_1 < 3/2\, \sigma_1$, Table D1 shows that single-track searching is not the most efficient, at least not when W is small compared to $\bar{\sigma}_1$. However, for the one-dimensional problem it is seen that for

TABLE D1

SEARCHING A SINGLE RECTANGULAR AREA CENTERED AT THE MOST PROBABLE POSITION OF THE OBJECT BY REPEATED VERY LIGHT COVERAGES WITH OVER-ALL PATH WIDTH W SUCH THAT $W/\bar{\sigma} \to 0$ AND WITH PARALLEL SEARCH PATHS OF CONSTANT LENGTH ℓ SO LARGE THAT THERE IS NEGLIGIBLE CHANCE OF MISSING AN OBJECT BECAUSE OF TURNING TOO SOON

$\frac{\bar{\sigma}_1}{\sigma_1}$	$\frac{X^*}{\sigma_1}$	$\frac{X^*}{\bar{\sigma}_1}$	P(T) = 0.95				P(T) = 0.99			
			$\frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{T}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)\frac{\ell}{\bar{\sigma}_2}}$	$\frac{\tau}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)\frac{\ell}{\bar{\sigma}_2}}$	$\frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{T}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)\frac{\ell}{\bar{\sigma}_2}}$	$\frac{\tau}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)\frac{\ell}{\bar{\sigma}_2}}$
$\to 0$	Optimum Distribution**		[14.5]	[3.91]			[25.48]	[4.18]		
$\to 0$	2.0	--	21.4	4.77						
	2.5	--	16.3	4.95						
	2.58	--	16.5	5.07			46.5	5.57		
	3.0	--	18.3	5.75			29.5	6.02		
	3.32	--					31.2	6.60		
	3.5	--	21.0	6.66			32.6	6.95		
0.5	1.5	3.0	18.5	4.32	37.0	8.64	163.0	6.81	326.0	13.6
	2.0	4.0	14.7	4.34	29.4	8.68	34.7	4.74	69.4	9.48
	2.5	5.0					26.0	5.20	52.0	10.4
	2.58	5.16	16.5	5.08	33.0	10.2				
	3.0	6.0					28.4	6.01	56.8	12.0
	3.32	6.64					30.9	6.60	61.7	13.2
1.0	0.0	0.0	19.4	4.45	19.4	4.45	73.6	5.56	73.6	5.56
	1.0	1.0	16.1	4.27	16.1	4.27				
	1.5	1.5	15.1	4.36	15.1	4.36	30.7	4.71	30.7	4.71
	2.0	2.0	15.4	4.69	15.4	4.69	26.5	4.95	26.5	4.95
	2.58	2.58	17.2	5.34	17.2	5.34	27.0	5.57	27.0	5.57
	3.0	3.0	18.9	5.93	18.9	5.93	29.1	6.19	29.1	6.19
	3.32	3.32					31.5	6.72	31.5	6.72
	3.5	3.5	21.3	6.75	21.3	6.75	32.9	7.04	32.9	7.04
1.5	0.0	0.0	15.4	4.58	10.3	3.05	28.1	4.87	18.7	3.25
	0.5	0.33	15.5	4.61	10.3	3.07				
	0.75	0.5	15.6	4.68	10.4	3.12	27.5	4.96	18.3	3.31
	1.0	0.67	15.7	4.74	10.5	3.16	27.2	5.01	18.2	3.34
	1.5	1.0	16.2	4.97	10.8	3.31	27.0	5.23	18.0	3.49
	2.0	1.33	17.3	5.32	11.5	3.55	27.5	5.56	18.4	3.71
	3.0	2.0	20.1	6.35	13.4	4.23	31.4	6.62	20.9	4.41
	4.5	3.0	25.6	8.57	17.1	5.71	42.4	9.03	28.3	6.02
2.0	0.0	0.0	17.8	5.47	8.91	2.74	28.5	5.72	14.2	2.86
	0.5	0.25	17.9	5.50	8.96	2.75	28.4	5.75	14.2	2.88
	1.0	0.5	18.3	5.62	9.13	2.81	28.9	5.88	14.4	2.94
	1.5	0.75	18.8	5.83	9.41	2.92	29.5	6.09	14.7	3.05
	2.0	1.0	20.1	6.15	10.1	3.08	30.6	6.39	15.3	3.20
	4.0	2.0	26.1	7.91	13.0	3.96	40.1	8.57	20.0	4.29
3.0	0.0	0.0	24.0	7.57	7.99	2.52	37.2	7.89	12.4	2.63
	3.0	1.0	27.5	8.69	9.15	2.90	42.4	9.06	14.1	3.02
$\to\infty$	--	0.0			7.51	2.38			11.5	2.48
$(\sigma_1 \to 0)$	--	0.5			7.82	2.48			12.0	2.59
	--	1.0			8.78	2.78			13.5	2.90
		1.5			10.4	3.29			16.0	3.43
		2.0			12.6	3.98			19.3	4.15
		2.5			15.2	4.81			23.3	5.01
		3.0			18.0	5.72			27.7	5.96

*Note 1: - 2X is the width of the nominally searched area.

**Note 2: - For comparison, the first row with $\bar{\sigma}/\sigma \to 0$ gives values of T and τ for the case where the search-path density distribution in the x-direction is optimum.

$\frac{1}{2}\sigma_1 \leq \bar{\sigma}_1 \leq \frac{3}{2}\sigma_1$, if the width 2X is properly chosen, a search consisting of repeated uniform coverages of the single width 2X gives values of T for given P(T) which are within six per cent of the values obtained when the one-dimensional search-path distribution is optimum. In considering possible more efficient searching procedures for this intermediate case, consideration has been limited mainly to multiarea searching in which the dimensions of the successive searched areas have the same ratio as they have in Table 4, but, possibly, with all dimensions decreased to counteract the dispersion of search paths due to the navigational error. Table D2 gives computed data for the one-dimensional problem for various values of the total number of coverages N and for various values of the width $2X_N$ of the area searched on the final coverage. Included are data for the limiting case $\bar{\sigma}_1 \longrightarrow 0$, with values of X_N which were found in Appendix C to be near optimum for this case. It is seen that for these same values of X_N the values of T and of τ obtained with $\bar{\sigma}_1/\sigma_1 = 1/2$ are either less than the values which are obtained when $\bar{\sigma}_1 \longrightarrow 0$ or differ from the latter by amounts which are negligible for practical purposes. For values of $\bar{\sigma}_1/\sigma_1$ larger than 1/2 substantially better results are obtained with values of X_N somewhat smaller than the values which are optimum for $\bar{\sigma}_1 \longrightarrow 0$, but the dependence of the results on X_N is not at all critical for values of $\bar{\sigma}_1/\sigma_1$ in the range 1.0 to 3/2. For purposes of formulating general rules it may be noted that taking $(X_N)_0$ to be the value of the optimum X_N for given N when $\bar{\sigma}_1 \longrightarrow 0$, then for values of $\bar{\sigma}_1/\sigma_1$ not exceeding 3/2 a value of X_N equal to $(X_N)_0 - 3/4\,\bar{\sigma}_1$ gives values of T and τ which are less than or, at most, only slightly greater than the values obtained for the same N when $\bar{\sigma}_1 \longrightarrow 0$, provided either $\bar{\sigma}_1/\sigma_1 \leq 1$ and $N \leq 3$, or $\bar{\sigma}_1/\sigma_1 \leq 3/2$ and $N \leq 2$. For values of $\bar{\sigma}_1/\sigma_1$ in the range 1 to 3/2 the results are considerably less dependent on N than for smaller values of this ratio, the improvement with decreasing N being limited mainly to decreases in the average discovery time τ.

D16. Considering general multiarea searching, if the over-all dimensions of all the areas searched are reduced in a manner similar to the suggested reduction in the widths of the searched area for the one-dimensional problem, it may be expected that the values of T and τ given by Table 4 are conservative under conditions comparable to those for which conservative values of T and τ are obtained for the one-dimensional problem. One demonstration of the close relation between these two problems has already been cited. Indeed, for rectangular area searching and especially for constant parallel path searching, the effectiveness may be expected to be somewhat better than is indicated by this comparison because the finite navigational error tends to remove the sharp corners from the actual path distribution, thus giving a search-path distribution more closely resembling an elliptical distribution.

D17. Based on the above, the following multiarea procedures are suggested when either $\bar{\sigma}_1/\sigma_1$ or $\bar{\sigma}_2/\sigma_2$ is greater than 1/2 but neither of these quantities is greater than 3/2.

(a) Take the maximum dimensions in the x and y-directions of the area searched on the final coverage to be less than those specified in Table 4 by amounts $3/2\,\bar{\sigma}_1$ and $3/2\,\bar{\sigma}_2$ respectively and decrease the dimensions of the areas searched on the other coverages by the same percentage.

(b) Increase Δm sufficiently so as to obtain the same total amount of searching as required by Table 4.

TABLE D2

MULTIAREA* CONSTANT PARALLEL PATH SEARCHING WITH OVER-ALL WIDTH W** SUCH THAT $W/\bar{\sigma}_1 \longrightarrow 0$ BUT WITH $\bar{\sigma}_1$ NOT NECESSARILY NEGLIGIBLE COMPARED TO σ_1. LENGTH OF SEARCHED PATH ℓ IS ASSUMED TO BE SO LONG THAT THERE IS NEGLIGIBLE CHANCE OF MISSING THE OBJECT BECAUSE OF TURNING TOO SOON.*** THE TOTAL NUMBER OF COVERAGES IS N.

		$\bar{\sigma}_1 \longrightarrow 0$			$\bar{\sigma}_1 = \frac{1}{2}\sigma_1$				$\bar{\sigma}_1 = \sigma_1$				$\bar{\sigma}_1 = \frac{3}{2}\sigma_1$			
$P(T_N)$	N	$\frac{X_N}{\sigma_1}$	$\frac{T_N}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{X_N}{\sigma_1}$	$(X_N)_0 - X_N$	$\frac{T_N}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{X_N}{\sigma_1}$	$(X_N)_0 - X_N$	$\frac{T_N}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{X_N}{\sigma_1}$	$(X_N)_0 - X_N$	$\frac{T_N}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$	$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}$
0.95	1												0.0	$1.69\bar{\sigma}_1$	15.43	<8.10
													0.5	$1.36\bar{\sigma}_1$	15.49	<8.13
													0.75	$1.19\bar{\sigma}_1$	15.58	<8.18
									1.0	$1.54\bar{\sigma}_1$	16.08	<8.44	1.0	$1.03\bar{\sigma}_1$	15.74	<8.26
					1.5	$2.08\bar{\sigma}_1$	18.52	<9.72	1.5	$1.04\bar{\sigma}_1$	15.10	<7.93	1.5	$0.69\bar{\sigma}_1$	16.24	<8.53
		2.0	21.43	11.25	2.0	$1.08\bar{\sigma}_1$	14.72	<7.73	2.0	$0.54\bar{\sigma}_1$	15.40	<8.09	2.0	$0.36\bar{\sigma}_1$	17.31	<9.09
		2.58	16.54	8.68	2.58	$-0.08\bar{\sigma}_1$	16.48	<8.65	2.58	$-0.04\bar{\sigma}_1$	17.21	<9.04				
		3.0	18.29	9.60					3.0	$-0.46\bar{\sigma}_1$	18.86	<9.90	3.0	$-0.31\bar{\sigma}_1$	20.1	<10.55
		3.5	21.03	11.04					3.5	$-0.96\bar{\sigma}_1$	21.3	<11.18				
	2												1.485	$0.75\bar{\sigma}_1$	15.98	<5.63
									1.61	$1.0\bar{\sigma}_1$	15.17	<5.13				
													1.86	$0.5\bar{\sigma}_1$	16.5	<5.80
									2.11	$0.5\bar{\sigma}_1$	15.24	<5.23				
		2.61	15.21	5.16	2.61	0.0	15.12	<5.12	2.61	0.0	16.12	<5.54				
	3								1.65	$1.0\bar{\sigma}_1$	15.23	<4.60				
									1.90	$0.75\bar{\sigma}_1$	15.11	<4.62				
									2.15	$0.5\bar{\sigma}_1$	15.20	<4.69				
		2.65	14.89	4.48	2.65	0.0	14.94	<4.50	2.65	0.0	15.93	<4.91				
	4				2.18	$1.0\bar{\sigma}_1$	15.05	<4.23								
		2.68	14.76	4.24	2.68	0.0	14.85	<4.28								
0.99	1												0	$2.0\bar{\sigma}_1$	28.08	<14.18
													0.75	$1.5\bar{\sigma}_1$	27.48	<13.88
													1.0	$1.33\bar{\sigma}_1$	27.23	<13.75
					1.5	$3.0\bar{\sigma}_1$	163.0	<82.32	1.5	$1.5\bar{\sigma}_1$	30.66	<15.48	1.5	$1.0\bar{\sigma}_1$	27.03	<13.65
					2.0	$2.0\bar{\sigma}_1$	34.71	<17.52	2.0	$1.0\bar{\sigma}_1$	26.53	<13.39	2.0	$0.67\bar{\sigma}_1$	27.53	<13.90
					2.5	$1.0\bar{\sigma}_1$	26.0	<13.13								
		2.58	46.53	23.50					2.58	$0.42\bar{\sigma}_1$	27.0	<13.63				
		3.0	29.50	14.90	3.0	0.0	28.41	<14.34	3.0	0.0	29.13	<14.71	3.0	0.0	31.4	<15.86
		3.1	29.89	15.09												
		3.32	31.20	15.76	3.32	$-0.64\bar{\sigma}_1$	30.87	<15.58	3.32	$-0.32\bar{\sigma}_1$	31.46	<15.89				
		3.5	32.57	16.45					3.5	$-0.50\bar{\sigma}_1$	32.85	<16.59				
	2								1.67	$1.5\bar{\sigma}_1$	31.1	<7.65				
													2.045	$0.75\bar{\sigma}_1$	27.25	<7.31
									2.17	$1.0\bar{\sigma}_1$	26.85	<6.92				
													2.42	$0.5\bar{\sigma}_1$	28.0	<7.52
									2.67	$0.5\bar{\sigma}_1$	26.4	<6.96				
									3.12	$0.05\bar{\sigma}_1$	27.64	<7.32				
		3.17	27.06	7.10	3.17	0.0	26.71	<6.99								
	3								2.21	$1.0\bar{\sigma}_1$	27.05	<5.55				
									2.46	$0.75\bar{\sigma}_1$	26.4	<5.54				
									2.71	$0.5\bar{\sigma}_1$	26.29	<5.60				
		3.21	26.29	5.50	3.21	0.0	26.12	<5.46	3.21	0.0	27.4	<5.87				
	4	3.23	25.96	4.93	3.23	0.0	26.0	<4.93								

*Note 1: - Nominal width of area searched on n'th coverage is assumed to be $\sqrt{n/N}\,(2X_N)$ where $2X_N$ is the nominal width of the area searched on the last coverage, and $\Delta m = \left(\frac{T_N}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}\right) \Big/ \left(\frac{X_N}{\sigma_1}\frac{2}{\sqrt{N}}\sum_{n=1}^{N}\sqrt{n}\right)$ for each coverage.

**Note 2: - The data are valid to a sufficient approximation for any value of $W/\bar{\sigma}_1$ if the values given for Δm, $\frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$ and $\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$ are taken instead to represent the quantities $\gamma\Delta m$, $\gamma\left(\frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}\right)$, and $\gamma\left(\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)\frac{\ell}{\sigma_2}}\right)$.

***Note 3: - This condition is certainly satisfied to a sufficient approximation for $P(T) = 0.95$ if $\ell/\sigma_2 \geq 6\sqrt{1 + (\bar{\sigma}_2/\sigma_2)^2}$, and for $P(T) = 0.99$ if $\ell/\sigma_2 \geq 6.6\sqrt{1 + (\bar{\sigma}_2/\sigma_2)^2}$.

Note 4: - $(X_N)_0$ is the approximate optimum value of X_N for given N when $\bar{\sigma}_1/\sigma_1 \longrightarrow 0$.

D18. Accurate calculations were made for constant parallel path searching based on the above procedure with N = 3. Table D3 compares the results obtained for $\bar{\sigma}_1/\sigma_1 = \bar{\sigma}_2/\sigma_2 = 1$ and for $\bar{\sigma}_1/\sigma_1 = \bar{\sigma}_2/\sigma_2 = 3/2$ with the data which is given in Table 4 for $\bar{\sigma}_1/\sigma_1 \longrightarrow 0$, $\bar{\sigma}_2/\sigma_2 \longrightarrow 0$, assuming the same total amount of searching as indicated by the values of $T/(\sigma_1\sigma_2/U)$.

TABLE D3

	$\frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)} = 82.9$			$\frac{T}{\left(\frac{\sigma_1\sigma_2}{U}\right)} = 174.0$	
	$\frac{\bar{\sigma}_1}{\sigma_1} \rightarrow 0, \frac{\bar{\sigma}_2}{\sigma_2} \rightarrow 0$	$\frac{\bar{\sigma}_1}{\sigma_1} = \frac{\bar{\sigma}_2}{\sigma_2} = 1$	$\frac{\bar{\sigma}_1}{\sigma_1} = \frac{\bar{\sigma}_2}{\sigma_2} = \frac{3}{2}$	$\frac{\bar{\sigma}_1}{\sigma_1} \rightarrow 0, \frac{\bar{\sigma}_2}{\sigma_2} \rightarrow 0$	$\frac{\bar{\sigma}_1}{\sigma_1} = \frac{\bar{\sigma}_2}{\sigma_2} = \frac{3}{2}$
P(T) for N = 3		0.957	0.945		0.992
P(T) from Table 4 for N = 3	0.95			0.99	
$\frac{\tau}{\left(\frac{\sigma_1\sigma_2}{U}\right)}$				34.7	<34.6

D19. It is seen that with the use of the proposed operating rules the efficiency of two-dimensional, constant parallel path searching is somewhat higher for the indicated range of navigational error than would be indicated by the one-dimensional analog. It is also apparent that with N = 3 and values of $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ not exceeding 3/2, the searching efficiency is greater than is indicated by Table 4 or, at worst, is negligibly smaller. It will be shown in the next section that a similar statement may be made concerning circle diameters searching. Without detailed calculations for elliptical area and rectangular area searching, it seems apparent from the previous analysis that the data in Table 4 may also be taken to give adequately the efficiency for those procedures if N is restricted to values not exceeding 3 when $1/2 \leq \bar{\sigma}_1/\sigma_2 \leq 1$ and $\bar{\sigma}_2/\sigma_1 \leq 1$, and is restricted to values not exceeding 2 when $1 < \bar{\sigma}_1/\sigma_1 \leq 3/2$ and $\bar{\sigma}_2/\sigma_2 \leq 3/2$.

WIDTH OF PATH NOT SMALL COMPARED TO NAVIGATIONAL ERRORS

D20. In considering multiarea searching, it was seen in Appendix C that when the navigational error is small compared to the uncertainty in position of an object the probability ρ that a point has been effectively searched when the nominal search-path density is m may be taken to be $1 - e^{-\gamma m}$ where γ depends on $W/\bar{\sigma}_1$ and on β, and is 1.0 when $W/\bar{\sigma}_1 \longrightarrow 0$. From the results of reference (1) it follows more generally for multiarea searching that provided the navigational error is not so large as to give a substantial probability of detecting any object located at one edge of an area being searched when nominally searching the opposite edge, the probability that a point has been searched after a given amount of searching may be taken to depend on γ and m only in the combination γm. It follows that under conditions for which multiarea searching is appropriate, results which have been obtained on the assumption that $W/\bar{\sigma}_1 \longrightarrow 0$ can be extended with sufficient accuracy to larger values of $W/\bar{\sigma}_1$ by replacing m by γm and by replacing T and τ by γT and $\gamma\tau$, respectively. Thus the generalizations of the previous section concerning multiarea searching when $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$ are not negligible but do not exceed 3/2, may be expected to be sufficiently valid for any values of $W/\bar{\sigma}_1$ for which multiarea searching is appropriate.

D21. For an evaluation of relatively wide-path locators under more general conditions it is assumed that passes are made on parallel paths in the y-direction and that each nominal path is symmetrical about the x-axis, the origin being at the most probable position of the object. Suppose that the uncertainty in the position of any path in the longitudinal direction is an uncertainty as to the point at which the search on that pass begins and that there is no uncertainty in the length of the pass. Let the over-all width of any searched path be W and let the length of the i'th path be $2Y_i$. Let x_i and 0 be the coordinates of the center of the i'th nominal path and let x'' and y' be the coordinates of the center of the actual path. Then a point x,y lies within the boundaries of the actual i'th path if, and only if, x'' lies in the range $x - W/2 < x'' < x + W/2$ and also y' lies in the range $y - Y_i < y' < y + Y_i$. The first relation can be written

$$x - W/2 - x_i < x'' - x_i < x + W/2 - x_i .$$

If $\bar{\sigma}_1$ is the standard deviation of the lateral navigational errors and $\bar{\sigma}_2$ is the standard deviation of longitudinal errors the probability that both of these conditions will be satisfied is

$$\frac{1}{2\pi\bar{\sigma}_1\bar{\sigma}_2} \int_{y-Y_i}^{y+Y_i} \int_{x-x_i-W/2}^{x-x_i+W/2} e^{-(x'^2/2\bar{\sigma}^2 + y'^2/2\bar{\sigma}_2^2)}\, dx'\, dy'$$

where $x'' - x_i$ has been replaced by x'. From the results of Appendix A it follows that the same result is obtained if instead of the length of searched paths being fixed, the y position of the end points of the searched paths are determined independently and each is subject to errors with standard deviation $\bar{\sigma}_2$. The probability that the point (x,y) is effectively searched on the i'th pass is the above expression multiplied by β. Then the probability $g_i(x,y)$ that the point (x,y) is not effectively searched on the i'th pass is

$$g_i(x,y) = 1 - \frac{\beta}{4}\left[\left\{F\left(\frac{\frac{W}{2} + x - x_i}{\bar{\sigma}_1}\right) + F\left(\frac{\frac{W}{2} - x + x_i}{\bar{\sigma}_1}\right)\right\}\left\{F\left(\frac{Y_i + y}{\bar{\sigma}_2}\right) + F\left(\frac{Y_i - y}{\bar{\sigma}_2}\right)\right\}\right].$$

Then the probability that the point (x,y) is not effectively searched in n passes is the product of the n factors $g_i(x,y)$, or is

$$\prod_{i=1}^{n} g_i(x,y).$$

Hence, at the end of n passes,

$$\rho(x,y) = 1 - \prod_{i=1}^{n} g_i(x,y),$$

and the location probability $P(T_n)$ at the end of n passes is

$$P(T_n) = 1 - \frac{2}{\pi\sigma_1\sigma_2} \int_0^{\infty} \int_0^{\infty} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} \prod_{i=1}^{n} g_i(x,y)\, dx\, dy. \qquad (15D)$$

Because of symmetry about the x-axis there is an equal chance that an object will be discovered on the first half and on the second half of any pass; so τ is given by (4C) rather than by (14D). The value of T_n can be obtained from the relation

$$UT_n = 2\beta W \sum_{i=1}^{n} Y_i .$$

D22. The above expression for $\rho(x,y)$ is strictly valid only if the $g_i(x,y)$ are all independent. In practice there is likely to be some dependence due to the fact that the starting point for one pass will depend to some extent on the position of the locator at the end of the previous pass. If there are a substantial number of passes the effect of this is likely to be negligible. Under conditions where only two or three passes are required, the variations in the positions of the starting points of the passes from the nominal positions are not likely to be an important factor in searching efficiency.

D23. When the time required to make a pass is the same for all passes, as it is likely to be in constant parallel path searching, the time τ is given by

$$\tau = \frac{T_N}{N} - \frac{T_N}{2N} P(T_N) + \frac{T_N}{N} \frac{2}{\pi\sigma_1\sigma_2} \int_0^{\infty} \int_0^{\infty} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} \sum_{n=1}^{N-1} \prod_{i=1}^{n} g_i(x,y)\, dx\, dy \qquad (16D)$$

where N is the total number of passes. In the special case where all passes are made with center line at the most probable position of the object and all nominal paths are of the same length, the product $\prod_{i=1}^{n} g_i$ in the integrands of (15D) and (16D) becomes g^n where $g = g_1 = g_2 \cdots = g_n$. Then equation (16D) reduces to

$$\tau = \frac{T_N}{N} - \frac{T_N}{2N} P(T_N) + \frac{T_N}{N} \frac{2}{\pi\sigma_1\sigma_2} \int_0^{\infty} \int_0^{\infty} e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}\right)} \left(\frac{g - g^N}{1 - g}\right) dx\, dy. \qquad (17D)$$

If, in addition, there is no uncertainty in the position of the object, equations (15D) and (16D) reduce to

$$P(T_n') = 1 - \{g(0,0)\}^n \qquad (18D)$$

$$\tau = \frac{T_N'}{N}\left[1/2 + 1/2\,\{g(0,0)\}^N + \frac{g(0,0) - \{g(0,0)\}^N}{1 - g(0,0)}\right]. \qquad (19D)$$

It is apparent that if the navigational errors are independent of path orientation, equations (18D) and (19D) also apply to a circle diameters searching procedure if, in the expression for g, the half path length Y is replaced by the radius R of the nominally searched area.

D24. As a check of the conclusions in paragraph D20 concerning the effect of a non-negligible value of $W/\bar{\sigma}_1$, equations (15D) and (16D) have been used to obtain accurate data for several examples with substantial values of $W/\bar{\sigma}_1$ and with $\bar{\sigma}_1/\sigma_1 = 1$. The length of path was assumed to be so large that there would be negligible chance of missing the object because of turning too soon. One example considered was a locator with $W/\bar{\sigma}_1 = 3$ and $\beta = 1$ for obtaining $P(T') = 0.99$ with a single coverage. Using a value of $2\sigma_1$ for X, Table D2 gives 26.53 for the required value of $\gamma T/\frac{\sigma_1\sigma_2}{U}\frac{\ell}{\sigma_2}$ when $W/\sigma_1 \rightarrow 0$. For the present case in which there is only one coverage, it is apparent that γ is given by Plate 2. The value of γ is found to be 1.74, giving a required value of 15.25 for $T/\frac{\sigma_1\sigma_2}{U}\frac{\ell}{\sigma_2}$ when $W/\sigma_1 = 3$ and $\beta = 1$. Since 5 locator passes give a value of 15 for this parameter, calculations were made on the basis of 5 passes with a lateral distance of 2X/5. This places the center lines of the initial and final paths a distance d/2 from the edges of the width 2X as is required. The computed value of $P(T')$ was .9893 and the value of τ/T' was found to be 0.386, which may be compared to the value of 0.50 which can be obtained from Table D2 as an upper limit for this ratio.

D25. A second example has been considered consisting of three coverages with a locator for which $W/\bar{\sigma}_1 = 1.85$, $\beta = 1$, and $X_N/\sigma_1 = 2.15$. Table D2 gives for N = 3 and $X_N = 2.15$ a required value $\gamma T/\frac{\sigma_1\sigma_2}{U}\frac{\ell}{\sigma_2}$ equal to 15.2 to obtain P(T) = 0.95. This requires a value of $\gamma\Delta m$ of 1.486, and the resulting value of γ is found to be 1.375. Then d must be $1.703\,\sigma_1$, and a total of 6 locator passes is required to give the indicated value of T'. The position of the various paths was selected in accordance with a procedure suggested in paragraph 68. This consists of starting successive coverages on alternate sides of the most probable position of the object, of making the first pass on each coverage with center line at a distance d/2 from the edge of the area of width $2X_i$, and of taking the final pass on each coverage to be that pass whose center line falls within the area of width $2X_i$ and at a distance 0 to d from the edge of this area which is opposite the starting edge for that coverage. Taking the negative x-direction to be in the direction of the starting edge of the first coverage, this gives for the first coverage a single pass with center line at $x/\sigma_1 = -0.384$. Similarly the second coverage is found to consist of two passes with center lines at $x/\sigma_1 = 0.894$ and at $x/\sigma_1 = -0.819$. The third coverage consists of passes with center lines at $x/\sigma_1 = -1.294$, at $x/\sigma_1 = 0.419$ and at 2.132. The computed value of P(T) for this case was 0.948 compared to the value 0.95 given by Table D2. Thus, in spite of the lack of symmetry in the distribution of the search paths on the different coverages, the agreement appears to be quite adequate. The value of τ/T was found to be 0.316 compared to an upper limit of 0.309 for this ratio given by Table D2.

D26. It is apparent that multiarea searching cannot be carried out as specified in previous paragraphs if W/σ_1 is quite large. Considering an increasing W/σ_1 it is apparent that for some sufficiently large value of W/σ_1, multiarea constant parallel path searching reduces essentially to single-track searching, and it is apparent from the previous analysis that for all larger values of W, single-track searching gives near maximum attainable efficiency. The values of W/σ_1 for which single-track searching should be considered in preference to multiarea searching depend on the magnitude of $\bar{\sigma}_1/\sigma_1$. From a comparison of Tables D1 and D2 it can be seen that for $\bar{\sigma}_1/\sigma_1 = 3/2$, single-track searching is about as efficient as multiarea searching even for $W/\sigma_1 \longrightarrow 0$, the values of T being slightly larger for single-track searching and the values of τ being substantially smaller. On the other hand when $\bar{\sigma}_1/\sigma_1 \longrightarrow 0$, multiarea searching will give better results than single-track searching unless W is about as large as the width of the N'th area indicated in Table 4 for constant parallel path searching with $N = 1$. To provide data for single-track searching, T and τ have been computed by means of the above formulas for values of P(T) of 0.95 and 0.99 for appropriate values of W/σ_1, β, $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$. Tables 8 and 10 present single-track data for the one-dimensional problem where the paths are taken sufficiently long so there is negligible chance of missing an object because of turning too soon. Table 8 presents the data for the case where $\bar{\sigma}_1/\sigma_1$ does not exceed 3/2 and Table 10 presents the data for the case where $\bar{\sigma}_1/\sigma_1$ is not less than 3/2. In Table 8 the smallest values of W listed for given values of $\bar{\sigma}_1/\sigma_1$ are values for which single-track searching and multiarea searching are approximately equally efficient.

D27. In addition to giving T and τ, Tables 8 and 10 also give the number of passes required to obtain given values of P(T). In general the specified values of P(T) are not obtained with an integral number of passes. No doubt, in practice one would consider the minimum number of passes which would insure a value of P(T) of not less than the specified value. But, in general, the appropriate values of W and β would not be exactly those listed in the table, and in order to estimate the number of passes N required, extrapolation of the data in the table would be necessary. For this reason the tables list fractional values of N which were obtained by determining the smallest value of N for which P(T) is not less than a specified value and the largest value of N for which P(T) is not greater than the specified value. From these two values of N and the corresponding computed values of P(T), the appropriate N to give the specified value of P(T) was determined by linear interpolation. The indicated values of T were then found in terms of the computed values of N.

D28. In considering the length of the paths which are necessary so there will be negligible chance of missing an object because of turning too soon, the most unfavorable case is that in which but a single pass is required. In this case the probability that the object is missed because of turning too soon is less than 0.0027 if $\ell/\sigma_2 \leq 6\sqrt{1 + (\bar{\sigma}_2/\sigma_2)^2}$, and is less than 0.001 if $\ell/\sigma_2 \leq 6.6\sqrt{1 + (\bar{\sigma}_2/\sigma_2)^2}$. Thus, if ℓ/σ_2 is required to satisfy the first relation for obtaining P(T) = 0.95 and is required to satisfy the second relation for obtaining P(T) = 0.99, then the value of P(T) will be not less than 0.9475 where it is indicated in the table that the value of P(T) is 0.95 and its value will be not less than 0.989 where the table indicates a value of 0.99. These required conditions are specified in footnotes in the tables. If the required turning time is quite large, or if W and β are sufficiently large so only one or two passes is required, then the length of search paths specified in Tables 8 and 10 are expected to be near optimum, but under other conditions greater searching efficiency can be obtained with shorter search paths. The problem is similar to that for constant parallel path searching. But whereas for the latter problem it was possible to choose all dimensions of the nominal areas, only the length of path is open to choice in single-track searching with a given locator. For cases when the navigational error in the y-direction is small compared to the uncertainty in the position of the object it is reasonably obvious from previous analysis that a length of path equal to that specified for constant parallel path searching is near optimum if the turning time is not too large. Consequently calculations for the case $\bar{\sigma}_2/\sigma_2 \longrightarrow 0$ have been based on values for ℓ_2/σ_2 of 5 and 6 respectively for values of P(T) of 0.95 and 0.99. Tables 7 and 9 present data for this limiting case based on these lengths of path.

D29. For cases of substantial navigational error, use can be made of the data presented in Tables D4 and 11 to estimate suitable lengths of path. It is apparent that for values of $\bar{\sigma}_2$ of the same order as σ_2, the optimum lengths of path are less than the values $5\sigma_2$ and $6\sigma_2$ if W/σ_2 is small, but are of the order of $5\sigma_2$ and $6\sigma_2$ or larger if W/σ_2 is sufficiently large. However, the results do not depend very critically on the choice of ℓ, and values of 5 and 6 for ℓ/σ_2, depending on P(T), appear to be adequate in all cases for $\bar{\sigma}_2/\sigma_2 \leq 3/2$ except when the turning time is large compared to the time required to make the straight run part of a pass or when only one or two sufficiently long passes are required to obtain the required value of P(T). With this choice of the values of ℓ, the data in Tables D4 and 11 indicate that the required number of passes to obtain given values of P(T) may be smaller or larger than the values listed in Tables 7 and 9, but that for values of $\bar{\sigma}_2/\sigma_2$ not greater than 1.0, the searching requirements to obtain given values of P(T) do not exceed those specified in the tables by more than a few per cent, with the exceptions mentioned above. For the range $1 < \bar{\sigma}_2/\sigma_2 \leq 3/2$, the searching requirements are somewhat greater than is indicated by Tables 7 and 9, being greater by about 5 to 20% for $\bar{\sigma}_2/\sigma_2 = 3/2$. For this case, reasonable estimates of the searching requirements may be obtained from Table 11, or the values of N, T and τ given in Tables 7 and 9 may be increased by 20% to obtain reasonably conservative estimates of the searching requirements.

D30. For values of $\bar{\sigma}_2/\sigma_2$ substantially greater than 3/2 the methods of estimating searching requirements considered in the previous paragraph are not adequate. Since it is highly unlikely that $\bar{\sigma}_1/\sigma_1 < 3/2 < \bar{\sigma}_2/\sigma_2$ it is sufficient to consider the case when both $\bar{\sigma}_2/\sigma_2 \geq 3/2$ and $\bar{\sigma}_1/\sigma_1 \geq 3/2$. Of course, Table 10 is valid for all values of $\bar{\sigma}_2/\sigma_2$. Table 11 gives, for single-track searching, computed values of T and τ and of the number N of required passes for various values of $W/\bar{\sigma}_1$, β, ℓ/σ_2, $\bar{\sigma}_1/\sigma_1$, and $\bar{\sigma}_2/\sigma_2$. The fractional values of N were determined by the method described in paragraph D27. Data are included only for the case $\bar{\sigma}_1/\sigma_1 = \bar{\sigma}_2/\sigma_2$. To make estimates for other cases it is probably sufficient to note that the values of either T or τ for any given values of $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$, should normally lie between the value obtained when $\bar{\sigma}_1/\sigma_1$ and $\bar{\sigma}_2/\sigma_2$

TABLE D4

COMPUTED DATA ON SINGLE TRACK SEARCHING WHICH IS NOT INCLUDED IN OTHER TABLES

					P(T) ≅ 0.95									P(T) ≅ 0.99			
$\frac{\bar{\sigma}_1}{\sigma_1}$	$\frac{\bar{\sigma}_2}{\sigma_2}$	$\frac{l}{\sigma_2}$	$\frac{W}{\bar{\sigma}_1}$	β	P(T)	N	$\frac{T}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	$\frac{\bar{\sigma}_1}{\sigma_1}$	$\frac{\bar{\sigma}_2}{\sigma_2}$	$\frac{l}{\sigma_2}$	$\frac{W}{\bar{\sigma}_1}$	β	P(T)	N	$\frac{T}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$
1.0	1.0	4.0	2.0	1.0	.9494	9	72.0	17.2	1.0	1.0	5.0	2.0	1.0	.9894	22	220.0	
					.9566	10	80.0	17.6						.9901	23	230.0	
				0.5	.9478	19	76.0	19.1					0.5	.9900	47	235.0	24.3
					.9519	20	80.0	19.3						.9903	48	240.0	24.3
1.0	1.0	4.0	4.0	1.0	.9414	3	48.0	15.0	1.0	1.0	5.0	4.0	1.0	.9874	5	100.0	17.2
					.9653	4	64.0	15.7						.9913	6	120.0	17.4
				0.5	.9449	8	64.0	19.1					0.5	.9895	13	130.0	22.3
					.9574	9	72.0	19.5						.9913	14	140.0	22.4
1.0	1.0	6.0	4.0	1.0	.9418	2	48.0	17.2	1.0	1.0	6.60	4.0	1.0	.9869	4	105.6	20.0
					.9739	3	72.0	18.2						.9918	5	132.0	20.3
				0.5	.9408	6	72.0	23.1					0.5	.9890	11	145.2	26.7
					.9595	7	84.0	23.7						.9913	12	158.4	26.9
1.5	1.0	6.0	2.0	1.0	.9061	3	36.0	14.0	1.5	1.0	6.60	4.0	1.0	.9803	2	52.8	16.5
					.9506	4	48.0	14.8						.9949	3	79.2	16.8
				0.5	.9388	9	54.0	17.5					0.5	.9885	8	105.6	23.2
					.9531	10	60.0	17.8						.9931	9	118.8	23.3
1.5	1.0	6.0	4.0	1.0	.8733	1	24.0	13.5	1.5	1.5	4.5	2.0	1.0	.9898	10	90.0	15.8
					.9756	2	48.0	15.3						.9922	11	99.0	15.9
				0.5	.9387	5	60.0	20.4					0.5	.9896	22	99.0	18.3
					.9638	6	72.0	21.0						.9909	23	103.5	18.1
2.0	1.0	4.0	2.0	1.0	.9310	4	32.0	10.9				4.0	1.0	.9846	4	72.0	16.7
					.9586	5	40.0	11.3						.9924	5	90.0	16.9
				0.5	.9373	10	40.0	12.9					0.5	.9869	11	99.0	21.2
					.9502	11	44.0	13.2						.9905	12	108.0	21.3
2	1.0	4.0	4.0	1.0	.9318	2	32.0	12.1	1.5	1.5	6.60	2.0	1.0	.9876	7	92.4	18.1
					.9709	3	48.0	12.8						.9916	8	105.6	18.2
				0.5	.9341	6	48.0	16.0					0.5	.9893	17	112.2	21.5
					.9548	7	56.0	16.5						.9911	18	118.8	21.5
→∞	0.5	5.0	2.0	1.0	.9565	3	30.0					4.0	1.0	.9679	2	52.8	17.8
														.9914	3	79.2	18.3
→∞	0.5	5.0	4.0	1.0	.9303	1	20.0						0.5	.9847	8	105.6	24.7
					.9867	2	40.0							.9905	9	118.8	24.9
				0.5	.9506	5	50.0										
					.9712	6	60.0		2.0	1.0	5.0	2.0	1.0	.9873	6	60.0	12.6
														.9923	7	70.0	12.7
→∞	0.5	5.0	6.0	1.0	.9720	1	30.0						0.5	.9889	15	75.0	15.1
														.9912	16	80.0	15.2

TABLE D4 (Continued)

					P(T) ≈ 0.95			
$\frac{\bar{\sigma}_1}{\sigma_1}$	$\frac{\bar{\sigma}_2}{\sigma_2}$	$\frac{\ell}{\sigma_2}$	$\frac{W}{\bar{\sigma}_1}$	β	P(T)	N	$\frac{T}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$
→∞	1.0	5.0	2.0	1.0	.9426	3	30.0	
				0.5	.9462	8	40.0	
→∞	1.0	5.0	4.0	1.0	.8809	1	20.0	
					.9758	2	40.0	
				0.5	.9393	5	50.0	
					.9638	6	60.0	
→∞	1.5	5.0	1.0	1.0	.9277	7	35.0	
					.9495	8	40.0	
→∞	1.5	5.0	2.0	1.0	.9146	3	30.0	
					.9602	4	40.0	
				0.5	.9269	8	40.0	
					.9466	9	45.0	
→∞	1.5	5.0	4.0	1.0	.9508	2	40.0	
				0.5	.9476	6	60.0	
				0.25	.9399	13	65.0	
					.9512	14	70.0	
→∞	1.5	5.0	6.0	1.0	.8322	1	30.0	
					.9634	2	60.0	
				0.5	.9270	5	75.0	
					.9556	6	90.0	
→∞	1.5	3.5	1.0	1.0	.9382	10	35.0	
					.9520	11	38.5	
→∞	1.5	3.5	2.0	1.0	.9413	5	35.0	
				0.5	.9459	12	42.0	
→∞	1.5	3.5	4.0	1.0	.9393	3	42.0	
					.9711	4	56.0	
				0.5	.9437	8	56.0	

					P(T) ≈ 0.99			
$\frac{\bar{\sigma}_1}{\sigma_1}$	$\frac{\bar{\sigma}_2}{\sigma_2}$	$\frac{\ell}{\sigma_2}$	$\frac{W}{\bar{\sigma}_1}$	β	P(T)	N	$\frac{T}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$	$\frac{\tau}{\left(\frac{\bar{\sigma}_1\bar{\sigma}_2}{U}\right)}$
2.0	1.0	5.0	4.0	1.0	.9894	3	60.0	13.6
					.9952	4	80.0	13.8
				0.5	.9893	9	90.0	18.7
					.9928	10	100.0	18.8
→∞	0.5	6.0	2.0	1.0	.9954	5	60.0	
			4.0	1.0	.9952	2	48.0	
				0.5	.9931	8	96.0	
			6.0	1.0	.9900	1	36.0	
→∞	1.0	6.0	2.0	1.0	.9939	5	60.0	
				0.5	.9906	12	72.0	
			4.0	1.0	.9901	2	48.0	
				0.5	.9916	8	96.0	
→∞	1.5	6.0	1.0	1.0	.9892	11	66.0	
			2.0	1.0	.9897	5	60.0	
				0.5	.9866	12	72.0	
			4.0	1.0	.9764	2	48.0	
					.9946	3	72.0	
				0.5	.9874	8	96.0	
				0.25	.9888	19	114.0	
			6.0	1.0	.9015	1	36.0	
					.9851	2	72.0	
				0.5	.9902	8	144.0	
→∞	1.5	4.5	1.0	1.0	.9901	14	63.0	
			2.0	1.0	.9851	6	54.0	
				0.5	.9900	16	72.0	
				1.0	.9761	3	64.0	
					.9904	4	72.0	
				0.5	.9909	11	99.0	

are assumed to be equal to the larger of these two quantities and the value obtained when both are assumed to be equal to the smaller of the two quantities.

CIRCLE DIAMETERS SEARCHING PROCEDURE

D31. For a circular normal distribution of errors of estimate of object position the location probability can be written

$$P(t) = \frac{1}{\sigma^2} \int_0^{\infty} \int_0^{2\pi} e^{-r^2/2\sigma^2} \rho(r,\theta,t)\, r d\theta\, dr \qquad (20D)$$

where $\rho(r,\theta,t)$ is the probability that any point (r,θ) has been effectively searched in time t. If $\bar{\rho}(r,t)$ is the average value of $\rho(r,\theta,t)$ with respect to θ for given r and t, (20D) can be written

$$P(t) = \frac{1}{\sigma^2} \int_0^{\infty} e^{-r^2/2\sigma^2} \bar{\rho}(r,t)\, r\, dr. \qquad (21D)$$

D32. Consider a single pass of a locator, searching a path of over-all width W and length 2R. Let the nominal searched path be centered at the most probable position of the object. Choose a rectangular coordinate system with origin at the most probable position of the object and with the x-axis coinciding with the nominal center line of the searched path. Let (x,y) be the center of the actual searched path and consider any other point (x_1,y_1). The probability that (x_1,y_1) will be included within the searched path is the combined probability that x lies between $x_1 - W/2$ and $x_1 + W/2$ and that y lies between $y_1 - R$ and $y_1 + R$. This is

$$\frac{1}{2\pi\bar{\sigma}_1\bar{\sigma}_2} \int_{y_1-R}^{y_1+R} \int_{x_1-W/2}^{x_1+W/2} e^{-(x^2/2\bar{\sigma}_1^2 + y^2/2\bar{\sigma}_2)}\, dx\, dy$$

where $\bar{\sigma}_1$ is the standard deviation of the lateral displacement errors and $\bar{\sigma}_2$ is the standard deviation of the longitudinal displacement errors. As shown in Appendix A, the same result is obtained if instead of assuming a constant length 2R, it is assumed that the position of each of the two end points are subject to the uncertainty characterized by standard deviation $\bar{\sigma}_2$. The probability g_1 that the point (x_1,y_1) is not effectively searched during the given pass is

$$g_1 = 1 - \frac{\beta}{2\pi\bar{\sigma}_1\bar{\sigma}_2} \int_{y_1-R}^{y_1+R} \int_{x_1-W/2}^{x_1+W/2} e^{-(x^2/2\bar{\sigma}_1^2 + y^2/2\bar{\sigma}_2^2)}\, dx\, dy. \qquad (22D)$$

D33. Let the origin of the above x,y coordinate system be taken as the pole of a polar coordinate system r,θ with polar axis coinciding with the x-axis. Let the polar coordinates of the point (x_1,y_1) be r,θ_1. Then we have $x_1 = r\cos\theta_1$ and $y_1 = r\sin\theta_1$. Then g_1 becomes

$$g_1 = g(r,\theta_1) = 1 - \left[\frac{\beta}{4}\left\{F\left(\frac{R + r\sin\theta_1}{\bar{\sigma}_2}\right) + F\left(\frac{R - r\sin\theta_1}{\bar{\sigma}_2}\right)\right\}\right.$$
$$\left.\left\{F\left(\frac{\frac{W}{2} + r\cos\theta_1}{\bar{\sigma}_1}\right) + F\left(\frac{\frac{W}{2} - r\cos\theta_1}{\bar{\sigma}_1}\right)\right\}\right]. \quad (23D)$$

Suppose that this is the first pass of the locator and that successive passes are made in slightly different directions so that the direction of the i'th path makes an angle $-i\Delta\theta$ with the direction of the first path. Then the radius vector to the point x_1, y_1 makes an angle $\theta_1 + i\Delta\theta$ with the center line of the nominal i'th searched path. It follows that the probability that the point at x_1, y_1 is not effectively searched during the i'th pass is g_i where

$$g_i = g(r,\theta_1 + i\Delta\theta). \quad (24D)$$

Then the probability that the point will have been effectively searched at the end of n passes is

$$\rho(r,\theta_1,t) = 1 - \prod_{i=1}^{n} g_i \quad (25D)$$

where $\prod_i^n g_n$ indicates the product of the n quantities $g_1, g_2 \ldots g_n$ and t is the time required to make the n passes. Then

$$\bar{\rho}(r,t) = 1 - 1/2\pi \int_0^{2\pi} \prod_{i=1}^{n} g_i d\theta_1.$$

Because each path is symmetrical about the pole we also have

$$\bar{\rho}(r,t)\ 1 - 1/\pi \int_0^{\pi} \prod_{i=1}^{n} g_i d\theta_1. \quad (26D)$$

If there is a single coverage and $\Delta\theta$ is chosen so that $\pi/\Delta\theta$ is an integer the integrand is periodic with period $\Delta\theta$. Then the integration need be carried out over one period only. We have

$$\bar{\rho}(r,T_1) = 1 - 1/\Delta\theta \int_0^{\Delta\theta} \prod_i^n g_n\, d\theta_1 \quad (27D)$$

where T_1 is the time for one coverage. If the starting path of succeeding coverages is chosen at random orientation we have for $\bar{\rho}$ at the end of the n'th coverage

$$1 - \bar{\rho}(r,T_n) = \{1 - \bar{\rho}(r,T_1)\}^n. \quad (28D)$$

The values of $\bar{\rho}(r,T_n)$ for given r can be substituted in (21D) to obtain $P(T_n)$ and the inequality (14D) can then be used to obtain a reasonable estimate of the time τ.

D34. To obtain τ exactly we can use (4C) if T_N and T_n are replaced by T_n and T_i, where n is the total number of passes and i is the number of passes made by time T_i. Since for any i we have $T_i - T_{i-1} = T_n/n$, then (4C) gives

$$\tau = T_n \left\{1 - 1/2n\ P(T_n) - 1/n \sum_{i=1}^{n-1} P(T_i)\right\}.$$

$P(T_i)$ is given by (20D) if $\rho(r,\theta,t)$ is taken to be the expression (25D) with g_i given by (24D). Because each nominal path is symmetrical about the origin the upper limit for θ in (20D) can be replaced by π if the integral is multiplied by 2.

D35. We now consider the limiting case of very small widths of searched path. It will also be assumed that $\bar{\sigma}_1$ and $\bar{\sigma}_2$ have a common value $\bar{\sigma}$. Then in the limit as $W/\bar{\sigma} \longrightarrow 0$, $\rho(r,\theta,t)$ reduces to

$$\rho(r,\theta,t) = 1 - e^{\sum_{i=1}^{n} \ln g_i}$$

$$= 1 - e^{-\left[\frac{\beta W}{2\pi\bar{\sigma}^2} \sum_{i=1}^{n} \int_{-R}^{R} e^{-\left(\frac{\{r\cos(\theta_1 + i\Delta\theta)\}^2 + \{y + r\sin(\theta_1 + i\Delta\theta)\}^2}{2\bar{\sigma}^2}\right)} dy\right]}$$

$$= 1 - e^{-\left[\frac{\beta W}{2\pi\bar{\sigma}^2} \int_{-R}^{R} \sum_{i=1}^{n} e^{-\left(\frac{r^2 + y^2 + 2yr\sin(\theta_1 + i\Delta\theta)}{2\bar{\sigma}^2}\right)} dy\right]}$$

The integrand can be written

$$e^{-\frac{r^2+y^2}{2\bar{\sigma}^2}} \sum_{i=1}^{n} e^{-\frac{yr}{\bar{\sigma}^2}\sin(\theta_1 + i\Delta\theta)}.$$

Now as W becomes quite small the number of search paths becomes quite large if there is an appreciable aggregate searched area. Hence, $\Delta\theta$ may be assumed to approach zero, and at the end of any integral number of coverages the summation can be replaced by the integral

$$\frac{n_1}{n\pi} \int_{\theta_1}^{\theta_1 + n\pi} e^{-\frac{yr}{\bar{\sigma}^2}\sin\theta}\, d\theta$$

where now n_1 is the number of passes and n is the number of coverages at time T_n, there being a rotation of the path of 180° for each coverage. If the integration with respect to y is separated into two parts corresponding to the intervals -R to 0 and

0 to R and if a change of variable is made in the first integral by substituting -y for y, we find for $\rho(r,\theta_1,t)$

$$\rho(r,\theta_1,T_n) = 1 - e^{-\left[\frac{n_1\beta W}{\pi\bar{\sigma}^2}\int_{-R}^{R} e^{-\left(\frac{r^2+y^2}{2\bar{\sigma}^2}\right)}\left\{\frac{1}{n\pi}\int_{\theta_1}^{\theta_1+n\pi}\cosh\left(\frac{ry}{\bar{\sigma}^2}\sin\theta\right)d\theta\right\}dy\right]}.$$

The integrand of the integral with respect to θ is periodic with period π. Hence, the quantity in braces represents the average value of the integrand over an integral number of periods. Since this is equal to the integral over any interval of length equal to a period divided by the period, it may be replaced by

$$\frac{1}{\pi}\int_0^{\pi}\cosh\left(\frac{ry}{\bar{\sigma}^2}\sin\theta\right)d\theta$$

showing that ρ does not depend on θ_1. Hence $\bar{\rho}(r,T_n) = \rho(r,\theta,T_n)$.

D36. To evaluate the integral with respect to θ we can make use of Bessels' integral relation

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi}\cos(z\sin\theta)\,d\theta$$

where J_0 is the Bessel function of zero order. We have

$$\frac{1}{\pi}\int_0^{\pi}\cosh\left(\frac{ry}{\bar{\sigma}^2}\sin\theta\right)d\theta = \frac{1}{\pi}\int_0^{\pi}\cos\left(\frac{iry}{\bar{\sigma}^2}\sin\theta\right)d\theta$$

$$= J_0\left(\frac{iry}{\bar{\sigma}^2}\right) = I_0\left(\frac{ry}{\bar{\sigma}^2}\right)$$

where $i = \sqrt{-1}$. Tabulated values of the function I_0 are given in reference (6). Noting that $2n_1\beta WR = UT_n$ we now have

$$\bar{\rho}(r,T_n) = 1 - e^{-\left[\frac{UT_n}{2\pi\bar{\sigma}^2} e^{-\left(\frac{r^2}{2\bar{\sigma}^2}\right)}\left\{\frac{1}{R}\int_0^{R} e^{-\left(\frac{y^2}{2\bar{\sigma}^2}\right)} I_0\left(\frac{ry}{\bar{\sigma}^2}\right)dy\right\}\right]}.$$

This can be put in the simpler form

$$\bar{\rho}(r,T_n) = 1 - e^{-\left[\frac{UT_n}{2\pi\bar{\sigma}^2} e^{-\left(\frac{r^2}{2\bar{\sigma}^2}\right)}\left\{\left(\frac{\bar{\sigma}}{R}\right)\int_0^{R/\bar{\sigma}} e^{-\frac{z^2}{2}} I_0\left(z\frac{r}{\bar{\sigma}}\right)dz\right\}\right]}. \qquad (29D)$$

The integral can be evaluated for given values of r. The resulting values of $\bar{\rho}$ can then be substituted in (21D) to obtain $P(T_n')$ and (14D) can then be used to obtain estimates of τ.

D37. If successive coverages are made very lightly, T_n in (29D) can be replaced by t. Then from (21D) and (3) we obtain

$$\frac{\tau}{T} = \frac{1}{\sigma^2} \int e^{-r^2/\sigma^2} \left\{ \frac{1 - e^{-TM(r)}}{TM(r)} \right\} rdr \qquad (30D)$$

where

$$TM(r) = \frac{UT}{2\pi\bar{\sigma}^2} e^{-r^2/2\bar{\sigma}^2} \left\{ \frac{\bar{\sigma}}{R} \int_0^{R/\bar{\sigma}} e^{-z^2/2} I_0\left(z \frac{r}{\bar{\sigma}}\right) dz \right\}.$$

D38. It has been assumed that the nominal path is rotated by the same angle between each pair of successive passes and that the overall searching time T is the time required to make an integral number of coverages. Considering the more general problem of making all passes along straight nominal paths of fixed length and centered at the most probable position of the object, it may reasonably be expected that the assumed procedure is the most efficient for circular distributions of object position and navigational errors. Considering all possible angular distributions the distribution which maximizes P(t) is one which maximizes $\bar{\rho}(r,t)$ in the integrand of (21D), provided the same distribution maximizes $\bar{\rho}(r,t)$ for all values of r. For the general case, $\bar{\rho}(r,t)$ is given by (26D) and can be written

$$\bar{\rho}(r,t) = 1 - \frac{1}{\pi} \int_0^{\pi} e^{\sum_{i=1}^{n} \ln g_i} \, d\theta . \qquad (31D)$$

The quantities g_i are periodic in θ with period π and the average value of any g_i over a period is independent of the path orientation. Hence

$$\int_0^{\pi} \sum_{i=1}^{n} \ln g_i \, d\theta = n \int_0^{\pi} \ln g_i \, d\theta = c_1$$

where c_1 is a constant. It was shown in Reference (1) that if u(x) is an arbitrary function of x subject only to the condition that $\int_a^b u(x)\,dx$ is a given constant, then the value of the integral $\int_a^b e^{-u(x)}\,dx$ is a minimum when u is a constant. The exponent in the integrand of (31D) cannot be made constant in general by a suitable distribution of the path orientations, but the variations in the value of the exponent may be expected to be smallest when there is a constant path rotation between successive passes and to be largest when all paths have the same orientation as they do in single-track searching. By analysis similar to that presented in Appendix B of reference (1) it can be shown that the exponent in (31D) does become essentially constant with respect to θ if the rotation between successive passes is constant and sufficiently small.

D39. Under conditions where both single track searching and circle diameters searching may be considered appropriate, the difference in the efficiencies of the two procedures obviously becomes negligible when the uncertainty in the object position is negligible compared to the navigational error, and also when the over-all width W is of the same order as the length of the search passes. For a given value of W less than ℓ, the theoretical superiority of the circle diameters searching increases with increasing uncertainty in object position compared to the navigational error. Taking account of the fact that the minimum value of W for which single track searching is appropriate increases fairly rapidly with decrease in $\bar{\sigma}/\sigma$ for values of $\bar{\sigma}/\sigma$ less than 3/2, the greatest difference in efficiencies of the two procedures under conditions where both may be considered appropriate procedures is when $W/\bar{\sigma} \rightarrow 0$ and $\bar{\sigma}/\sigma = 3/2$, where $\bar{\sigma}$ is the standard deviation of lateral navigational errors and is assumed to be independent of orientation. Computations for this case have been carried out for circle diameters searching with the aid of equations (21D) and (29D) with T_n replaced by t, and using a value for 2R of $6.5\sigma - .75\bar{\sigma}$ or 4.25σ, as suggested in an earlier section. These give a value of 38.9 for $\frac{T'}{\frac{\bar{\sigma}^2}{U}}$ to obtain a P(T') of 0.95. Similar computations for single track searching with $W/\bar{\sigma} \rightarrow 0$, and with $\bar{\sigma}_1/\sigma_1 = \bar{\sigma}_2/\sigma_2 = 3/2$, and a value of ℓ equal to the above value of 2R, give a value of 39.8 for $\frac{T'}{\frac{\sigma_1\bar{\sigma}_2}{U}}$ to obtain $P(T') = 0.95$. Thus it appears that the data obtained for single track searching is quite adequate also for circle diameters searching under conditions where both of these procedures are appropriate. A value of 38.9 for $\frac{T}{\left(\frac{\bar{\sigma}^2}{U}\right)}$ is equivalent to a value of 87.5 for $\frac{T'}{\left(\frac{\sigma^2}{U}\right)}$ when $\bar{\sigma}/\sigma = 3/2$. This compares with a value of 92 given by Table 4 for $\frac{T}{\left(\frac{\sigma^2}{U}\right)}$ to obtain $P(T') = 0.95$ when the navigational error is negligible, and illustrates the fact that a more effective search-path distribution is obtained with circle diameters when there is substantial navigational error than when there is negligible navigational error, provided the nominal radius is suitably reduced, depending on the relative magnitude of the navigational error.

APPENDIX E

THEORETICAL DETERMINATION OF THE DISTRIBUTION FUNCTION FOR THE ERRORS OF ESTIMATE OF THE OBJECT POSITION

E1. To use the results of the analysis of the present report it is necessary to have an estimate of the most probable position of the object and of the distribution function of the errors of estimate of object position. It is assumed that this distribution is of the form indicated by equation (5) if the x and y-directions are properly chosen. It may be that where there has been sufficient experience with the systems used for the initial location of objects in given areas, the best estimate of the distribution of object position errors is one based on observations on previous objects. But initially it may be necessary to base the distribution of errors on estimates of the accuracy with which measurements of various quantities are made in fixing the approximate position of an object.

E2. We wish to find a general method of finding the distribution of position errors of any object in terms of the errors in measurements of the parameters which determine the position of the object. Consider any horizontal rectangular coordinate system in which the coordinates of any point are represented by X and Y. In general, the coordinates X, Y of any object are assumed to be determined indirectly by the measurement of certain other quantities. Let M_i be the magnitude of the i'th quantity which is measured and let the measured value be $M_i + \Delta M_i$, where ΔM_i is the error of measurement. Let X and Y be given in terms of n quantities M_i by the functions

$$X = X(M_1, M_2 \ldots M_n)$$

$$Y = Y(M_1, M_2 \ldots M_n).$$

Then expanding in a Taylor's series and retaining only terms of the first order in ΔM_i, we obtain approximately

$$\Delta X = \sum_{i=1}^{n} a_i \, \Delta M_i, \quad \Delta Y = \sum_{i=1}^{n} b_i \, \Delta M_i \tag{1E}$$

where ΔX and ΔY are the errors in X and Y respectively, and

$$a_i = \left(\frac{\partial X}{\partial M_i}\right)_{M_i + \Delta M_i}, \quad b_i = \left(\frac{\partial Y}{\partial M_i}\right)_{M_i + \Delta M_i}. \tag{2E}$$

To this approximation the errors in X and Y are linear functions of the independent errors ΔM_i. Then, according to a well known theorem, if the errors for each M_i have a normal distribution with standard deviation σ_i, the quantities ΔX and ΔY are also normally distributed with standard deviations σ_X and σ_Y of magnitudes

$$\sqrt{\sum_{i=1}^{n} a_i^2 \sigma_i^2} \quad \text{and} \quad \sqrt{\sum_{i=0}^{n} b_i^2 \sigma_i^2},$$

respectively.

E3. For arbitrary choices of the perpendicular directions of X and Y, the errors ΔX and ΔY are not independent. To obtain such independence, consider a rotation of the axes by an angle θ and let the coordinates of the given object relative to the new axes be X' and Y'. Then

$$\begin{aligned} \Delta X' &= \Delta X \cos\theta + \Delta Y \sin\theta \\ \Delta Y' &= -\Delta X \sin\theta + \Delta Y \cos\theta \end{aligned} \tag{3E}$$

where $\Delta X'$ and $\Delta Y'$ are the errors in the values found for the coordinates X' and Y'. If ΔX and ΔY in (3E) are replaced by their values from (1E), the expressions for $\Delta X'$ and $\Delta Y'$ can be put in the form of a sum of independent terms, and the standard deviations of the errors in $\Delta X'$ and $\Delta Y'$ are found to be

$$\begin{aligned} \sigma_{X'} &= \sqrt{\sum_{i=1}^{n} (a_i \cos\theta + b_i \sin\theta)^2 \sigma_i^2} \\ \sigma_{Y'} &= \sqrt{\sum_{i=1}^{n} (-a_i \sin\theta + b_i \cos\theta)^2 \sigma_i^2}. \end{aligned} \tag{4E}$$

E4. Suppose that θ has been chosen so that $\Delta X'$ and $\Delta Y'$ are independent, and consider a second rotation of the axis through an angle ϕ. Then relative to the new axes the errors are

$$\begin{aligned} \Delta X'' &= \Delta X' \cos\phi + \Delta Y' \sin\phi \\ \Delta Y'' &= -\Delta X' \sin\phi + \Delta Y' \cos\phi. \end{aligned} \tag{5E}$$

Making use of (1E) and (3E), $\Delta X''$ and $\Delta Y''$ can be found in terms of the ΔM_i, and the variance of $\Delta X''$ is then found to be

$$\sigma_{X''}^2 = \sum_{i=1}^{n} \sigma_i^2 \{(a_i \cos\theta + b_i \sin\theta)\cos\phi + (-a_i \sin\theta + b_i \cos\theta)\sin\phi\}^2.$$

Expanding the square of the sum of the two terms in braces and making use of (4E), we find

$$\sigma_{X''}^2 = \sigma_{X'}^2 \cos^2\phi + \sigma_{Y'}^2 \sin^2\phi + 2\sum_{i=1}^{n} \sigma_i^2 (a_i \cos\theta + b_i \sin\theta)(-a_i \sin\theta + b_i \cos\theta)\sin\phi\cos\phi. \tag{6E}$$

But if $\Delta X'$ and $\Delta Y'$ are independent, we also have from equations (5E), and from the theorem previously mentioned, that

$$\sigma_{X''}^2 = \sigma_{X'}^2 \cos^2 \phi + \sigma_{Y'}^2 \sin^2 \phi . \tag{6Ea}$$

Both of these expressions for $\sigma_{X''}^2$ can be valid only if the summation in (6E) is identically zero, independent of ϕ. Thus a necessary condition for the independence of $\Delta X'$ and $\Delta Y'$, is that

$$2 \sum_{i=1}^{n} \sigma_i^2 \{(a_i \cos \theta + b_i \sin \theta) (-a_i \sin \theta + b_i \cos \theta)\} = 0 . \tag{7E}$$

By simplifying the expression within braces (7E) can be written

$$\sum \sigma_i^2 \{(b_i^2 - a_i^2) \sin 2\theta + 2a_i b_i \cos 2\theta\} = 0 .$$

This requires that

$$\tan 2\theta = \frac{2 \sum_{i=1}^{n} a_i b_i \sigma_i^2}{\sum_{i=1}^{n} (a_i^2 - b_i^2) \sigma_i^2} . \tag{8E}$$

E5. It has been shown only that (7E) or (8E) is a necessary condition for the independence of the normally distributed variables $\Delta X'$ and $\Delta Y'$, and not that it is sufficient. But since two independent normally distributed variables $\Delta X'$ and $\Delta Y'$, with standard deviations $\sigma_{X'}$ and $\sigma_{Y'}$, give exactly the same distributions for $\Delta X''$ and $\Delta Y''$ for an infinite number of different values of ϕ as are obtained with the actual variables $\Delta X'$ and $\Delta Y'$, it appears that (7E) is also a sufficient condition.

E6. To illustrate the application of these results consider the simple example of locating an object such as the mast of a ship or an object splash by measuring, at two fixed stations which are a known distance c apart, the angles ϕ_1 and ϕ_2 made by the object with the baseline. Taking the X and Y-directions as illustrated in Figure E, reference (6) gives X and Y as functions of ϕ_1, ϕ_2, and c. Assuming c has been determined exactly, the errors in ΔX and ΔY can be found as a function of the errors $\Delta\phi_1$ and $\Delta\phi_2$ by means of (1E) and (2E). Suppose the measured values of ϕ_1 and ϕ_2 are 90° and 60° respectively. Then from reference (5) we obtain

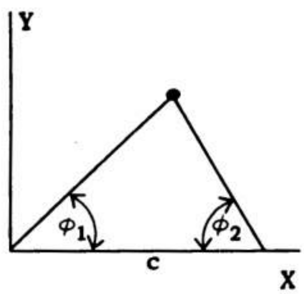


Figure E

$$\Delta X = -\sqrt{3}\, c \Delta\phi_1$$

$$\Delta Y = c(3 \Delta\phi_1 + 4 \Delta\phi_2).$$

In terms of the notation of paragraph E2 we have

$$a_1 = -\sqrt{3}c, \quad a_2 = 0$$

$$b_1 = 3c, \quad b_2 = 4c.$$

If measurements of ϕ_1 and ϕ_2 are equally accurate, and if σ_ϕ is the standard deviation of the errors of measurement of each of these quantities, the condition (8E) gives for the required angle of rotation of the axes, θ,

$$\tan 2\theta = \frac{3\sqrt{3}}{11}$$

or

$$\theta = 12°, 39'.$$

Then equations (4E) give

$$\sigma_{X'} = 1.35\, c\, \sigma_\phi \quad \text{and} \quad \sigma_{Y'} = 5.12\, c\, \sigma_\phi .$$

In the notation of paragraph 28, σ_2 is always the larger of the values $\sigma_{X'}$ and $\sigma_{Y'}$ and the corresponding axis is taken as the y-axis. Hence, in the present case, $\sigma_1 = 1.35\, c\, \sigma_\phi$ and $\sigma_2 = 5.12\, c\, \sigma_\phi$.

E7. The standard deviations σ_1 and σ_2 represent the minimum and maximum values of $\sigma_{X'}$ and $\sigma_{Y'}$ given by (4E), and the condition (8E) for independent errors could have been obtained by determining the value of θ at which $\sigma_{X'}$ and $\sigma_{Y'}$ have their maximum or minimum values. In the above example the values of σ_X and σ_Y for the initial choice of coordinate axes are found to be $\sigma_X = 1.73\, c\, \sigma_\phi$ and $\sigma_Y = 5\, c\, \sigma_\phi$. These may be compared with the values found for σ_1 and σ_2 in the preceding paragraph.

E8. The above method may be used to determine the distribution of errors of estimate of object position if the coordinates X and Y of the object are known as functions of parameters which are measured or which can be estimated. In practice it may be convenient to determine the distribution of errors of position of an object splash or the position of a ship at the time an object is detected, and to combine these errors with the errors of location of the object relative to the position of the splash or relative to the position of the ship, to obtain the over-all errors of estimate of object position. We consider the general problem of determining the distribution of errors of estimate of the vector sum of two position vectors when the errors of estimate of each position vector have a known elliptical normal distribution.

E9. Let the first position vector be represented by coordinates X_0 and Y_0 and let the errors of estimate of these coordinates be ΔX_0 and ΔY_0, where the directions of X_0 and Y_0 are chosen so that the errors ΔX_0 and ΔY_0 are independent, with the Y_0-axis in the direction of expected greatest error. Similarly let X_1 and Y_1 be coordinates representing the components of the second position vector so chosen that the errors ΔX_1 and ΔY_1 are independent, and with the Y_1-axis in the direction of expected greatest error. Let the Y_1-axis make an angle α with respect to the direction of the Y_0-axis. Then the resultant errors, resolved in the X_0, Y_0-directions, are

$$\Delta X'_0 = \Delta X_0 + \Delta X_1 \cos\alpha - \Delta Y_1 \sin\alpha$$
$$\Delta Y'_0 = \Delta Y_0 + \Delta X_1 \sin\alpha + \Delta Y_1 \cos\alpha . \tag{9E}$$

Consider a rotation of the axes by an angle θ to a position such that the errors ΔX and ΔY parallel to these new axes are independent. Then

$$\Delta X = \Delta X'_0 \cos\theta + \Delta Y'_0 \sin\theta$$
$$\Delta Y = -\Delta X'_0 \sin\theta + \Delta Y'_0 \cos\theta . \tag{10E}$$

Substituting for $\Delta X'_0$ and $\Delta Y'_0$ from (9E) and simplifying, these relations become

$$\Delta X = \Delta X_0 \cos\theta + \Delta Y_0 \sin\theta + \Delta X_1 \cos(\alpha - \theta) - \Delta Y_1 \sin(\alpha - \theta)$$
$$\Delta Y = -\Delta X_0 \sin\theta + \Delta Y_0 \cos\theta + \Delta X_1 \sin(\alpha - \theta) + \Delta Y_1 \cos(\alpha - \theta). \tag{11E}$$

Since the quantities ΔX_0, ΔY_0, ΔX_1, and ΔY_1 are independent, normally distributed errors, the variances of ΔX and ΔY are

$$\sigma_X^2 = \sigma_{X_0}^2 \cos^2\theta + \sigma_{Y_0}^2 \sin^2\theta + \sigma_{X_1}^2 \cos^2(\alpha - \theta) + \sigma_{Y_1}^2 \sin^2(\alpha - \theta)$$
$$\sigma_Y^2 = \sigma_{X_0}^2 \sin^2\theta + \sigma_{Y_0}^2 \cos^2\theta + \sigma_{X_1}^2 \sin^2(\alpha - \theta) + \sigma_{Y_1}^2 \cos^2(\alpha - \theta) . \tag{12E}$$

If ΔX and ΔY are independent the variance of the sum $\Delta X + \Delta Y$ is given by

$$\sigma_{(X+Y)}^2 = \sigma_X^2 + \sigma_Y^2 = \sigma_{X_0}^2 + \sigma_{Y_0}^2 + \sigma_{X_1}^2 + \sigma_{Y_1}^2 . \tag{13E}$$

Adding the two equations (11E) the sum $\Delta X + \Delta Y$ is found to be

$$\Delta X + \Delta Y = \Delta X_0(\cos\theta - \sin\theta) + \Delta Y_0(\sin\theta + \cos\theta)$$
$$+ \Delta X_1 \{\cos(\alpha - \theta) + \sin(\alpha - \theta)\}$$
$$+ \Delta Y_1 \{-\sin(\alpha - \theta) + \cos(\alpha - \theta)\}.$$

Since the four terms on the right hand side are independent we have for $\sigma_{(X+Y)}^2$, whether or not ΔX and ΔY are independent,

$$\sigma_{(X+Y)}^2 = \{1 - \sin 2\theta\} \sigma_{X_0}^2 + \{1 + \sin 2\theta\} \sigma_{Y_0}^2 + \{1 + \sin 2(\alpha - \theta)\} \sigma_{X_1}^2$$
$$+ \{1 - \sin 2(\alpha - \theta)\} \sigma_{Y_1}^2 . \tag{14E}$$

If ΔX and ΔY are independent, the two expressions (13E) and (14E) for $\sigma_{(X+Y)}^2$ must be equal, and this requires that

$$\sin 2\theta \left(\sigma_{Y_0}^2 - \sigma_{X_0}^2\right) = \sin 2(\alpha - \theta)\left(\sigma_{Y_1}^2 - \sigma_{X_1}^2\right) .$$

Thus the required angle of rotation θ to obtain independent errors ΔX and ΔY is given by

$$\cot 2\theta = \cot 2\alpha + \csc 2\alpha \left(\frac{\sigma_{Y_0}^2 - \sigma_{X_0}^2}{\sigma_{Y_1}^2 - \sigma_{X_1}^2}\right) .$$

This has only one solution for values of θ less than 90°, and if α is chosen less than 90°, the required value of θ lies between 0 and α. Then σ_X and σ_Y can be obtained from (12E), and these are the required values of σ_1 and σ_2, respectively.

E10. If either of the initial two distributions is circular normal, the above relations can be seen to reduce to the following simple rule: If one of the distributions is elliptical normal with characteristic standard deviations σ'_1 and σ'_2 and if the second distribution is circular normal with standard deviation σ_c, then the distribution of the combined errors is elliptical normal with characteristic standard deviations σ_1 and σ_2 given by

$$\sigma_1 = \sqrt{(\sigma'_1)^2 + \sigma_c^2}, \quad \sigma_2 = \sqrt{(\sigma'_2)^2 + \sigma_c^2}$$

where the axes directions for independence of errors are the same as for the initial elliptical distribution.

E11. One possible application of the above results is in transforming the charted position of an object from one reference system to another. Thus suppose that an initial estimate of the most probable position of an object is obtained relative to the position of sighting posts, and that it is required to specify coordinates of this point in terms of coordinates of the navigational system used to guide the locator in the reinvestigation of the object. In general the absolute accuracy of these systems may be substantially less than the relative accuracy as determined by the repeatability of any one system in determining the position of a fixed object. If the position of the object is initially determined relative to the system used for navigation of the locator it is the error of repeatability which is of importance, but where a position is transformed from one system to another the absolute errors in position fixing must be taken into account. By suitable comparison experiments in simultaneously fixing the position of objects in an area, systematic errors probably could be largely eliminated, but some errors would be expected to remain which probably could be considered to be random errors for random points in the area. If a reasonable estimate of the distribution of these errors can be made, these errors can be combined with the initial errors of estimate of object position, by the above explained method, to obtain the resultant distribution of errors which must be used in determining the appropriate locator search procedure.

E12. One example is mentioned in paragraphs 63 and 64. In order to carry out a circular spiral searching procedure with the position of the locator determined by a line of varying length fixed to an anchor, it is necessary to place the anchor as nearly as possible at the most probable position of the object. Suppose that the errors in determining the position of the ship planting the anchor have a circular normal distribution with standard deviation σ_s, and suppose that the deviations of anchor positions from the positions at which the ship attempts to plant them, on the basis of its estimated position, have a circular normal distribution with standard deviation σ_a. Assuming the ship navigates relative to the same system used initially in estimating the position of the object, and that the errors of estimate of object position are normally distributed with standard deviation σ', then the deviations of the object positions from the anchor positions are normally distributed with standard deviation σ given by

$$\sigma = \sqrt{\sigma_s^2 + \sigma_a^2 + \sigma'^2}.$$

It is the latter standard deviation which must be used in Table 6 for this case to determine the appropriate T and τ for the circular spiral searching procedure.

E13. Returning to equation (8E) it is apparent that the determination of the directions for which these errors are independent may involve a substantial volume of calculations, and it is of some interest to determine whether the results obtained in terms of the characteristic standard deviations σ_1 and σ_2 would hold in terms of σ_X and σ_Y for arbitrary choices of X, Y-axes if the search were carried out as though the errors in the X, Y-directions were independent. Consideration will be limited to the constant parallel path and single-track searching procedures, and to the case where the navigational errors are small compared to the uncertainty in the object position.

E14. Consider constant parallel path or single-track searching with path lengths so large that there is negligible chance that an object is missed because of turning too soon. The location probability then depends only on the transverse distribution of errors of estimate of object position and on the transverse distribution of search paths. Thus for a given σ_X, the location probability does not depend on whether or not the errors in the X and Y-directions are independent. Let P_ω be the location probability for this case for a given aggregate width of searched area ω and for a given nominal search path distribution. Consider the area bounded by two lines which are normal to the direction of the search paths and are equidistant from the most probable position of the object. Let $(P_\omega)_1$ be the probability that an object has been detected in this area by the time an aggregate width ω has been searched and let $(P_\omega)_2$ be the probability that in the same time the object has been detected outside this area. Then

$$(P_\omega)_1 = P_\omega - (P_\omega)_2.$$

E15. If the navigational error is negligible compared to the uncertainty in the position of the object $(P_\omega)_1$ represents the location probability which would be obtained for a constant parallel path or single track procedure, with path length ℓ equal to the distance between the lines, for the given aggregate width ω and for the given nominal search-path distribution in the X-direction. Limiting consideration to the case of small navigational error, we have in all cases $(P_\omega)_2 < \mu$ where μ is the probability that the object's position was outside the area bounded by the two lines. If the errors in the

X and Y-directions are independent, the quantity $(F_\omega)_2$ is equal to μP_ω. Hence, the value of the location probability obtained with a constant parallel path or single track procedure for given σ_X and σ_Y either is greater than the value which is obtained on the assumption that errors in the X and Y-directions are independent, or differs from the latter value by less than $(1 - P_\omega)\mu$ and, therefore, by less than $\{1 - P(t)\}\mu$, where $P(t)$ is the value of the location probability which is obtained on the basis of the assumption of independence. For the constant parallel path procedures specified in Table 4 the value of $\{1 - P(t)\}\mu$ is 0.00062 for $P(t) = 0.95$ and is 0.00003 for $P(t) = 0.99$. Differences of this magnitude are entirely negligible.

E16. On the basis of the above result it may be concluded that if σ_X and σ_Y are known, then carrying out a constant parallel path or single-track searching procedure as though the errors in the X and Y-directions were independent gives, with negligible error, a location probability at all times during the search as large as would be obtained if the errors in the X and Y-directions were actually independent. This has been shown only for the case where the navigational errors are small compared to the uncertainty in the position of the object, but it obviously is true also in all cases for which the path length is taken sufficiently large so there is negligible chance of missing an object because of turning too soon. Qualitatively, it appears that for shorter path lengths the assumption of independence would lead to underestimates of the searching efficiency but no simple rigorous proof of this is known.

E17. Although it has been shown that, at least under some conditions, one may proceed as if σ_X and σ_Y are σ_1 and σ_2 even if they are not, the resulting search procedure may be quite inefficient. If the navigational errors are less than the uncertainty in object position and if turning times are not too large the required searching effort is approximately proportional to $\sigma_X\sigma_Y$. From equation (6Ea) and from a similar equation for $\sigma_{Y''}^2$, it is easily shown that

$$\sigma_X \sigma_Y = \sqrt{\sigma_1^2 \ \sigma_2^2 + \left\{\left(\frac{\sigma_2^2 - \sigma_1^2}{2}\right) \sin 2\varphi\right\}^2}$$

where φ is the angle between the Y and the y-axes. This may be much larger than $\sigma_1\sigma_2$ if σ_2 is much larger than σ_1 unless the angle between the Y-axis and the axis of σ_2 is small or near 90°.

APPENDIX F

LOCATION OF OBJECTS BY MEANS OF LOCATORS WHICH ARE NOT PERFECT IDENTIFIERS

F1. The analysis presented in the present report is concerned mainly with determining the overall searching time T and the average discovery time τ for given values of the location probability P(T) under conditions where the locator can be assumed to be capable of identifying any object which it can detect as an object or clutter. Under the assumed conditions, P(T) represents the expected fraction of the salvageable objects contacts being investigated which are detected. Under these conditions the value of the quantities P(T), T and τ do not depend on what fraction of the contacts being investigated is represented by salvageable objects, but the value of the average searching time does. If ξ is the fraction of the contacts being investigated which are objects, the average searching time $\bar{t}$ is given by equation (4).

F2. For present purposes a perfect identifier may be considered to be a locator which has a finite detection probability for every object over some width of path W, but has zero detection probability for all clutter. Since present locators are not perfect identifiers it is necessary to consider what can be accomplished by means of a locator which detects some false targets which it cannot differentiate from object targets.

F3. It will be convenient to refer to the contact which is being investigated as a pre-search contact. It represents a contact for an initial sonar search, or for any system used to establish initially the presence of an object or apparent object, which, because of the inaccuracy of location or inadequate discrimination of the detector, must be reinvestigated by a narrow-path locator. The term locator contact will be used to designate a contact by the locator while carrying out a search for an object whose probable existence and approximate location have been established previously by a pre-search contact or contacts. The analysis in the present Appendix will be limited to the case where the targets which give rise to pre-search false contacts do not produce locator false contacts. An example might be the case where the initial pre-search object detections were made by sonar in an area where all clutter sonar targets in the channel had been removed.

F4. Suppose that searching by the locator is carried out according to the "standard searching rules;" namely, that searching is continued until a contact is obtained or until time T, whichever occurs first. There will then be not more than one target located for each pre-search contact investigated and if the locator is subject to object-like contacts from clutter the target which is located may not be an object, even if one is present. For practical purposes we wish to know the probability that objects detected through pre-search contacts are located under these conditions, what is the average searching time, and how many false targets are located per pre-search contact investigated.

F5. For the purpose of analysis, suppose first that searching is carried out for a time T regardless of whether or not there are locator contacts during this time. Let $P'_r(t)$ be the probability that exactly r locator false targets are detected in time t, where t is any time less than T. Then the probability that no false targets are detected in time t is $P'_0(t)$. If the pre-search contact being investigated is an object, the probability that it will have been detected in time t is $P(t)$, whether or not false targets are also detected by the locator, and $dP/dt\, dt$ is the probability that the object is first contacted by the locator during the time interval t to t + dt. The quantity $dP/dt\, dt$ can be written as the product of two probabilities $p_1(t)\ \{p_2(t)dt\}$, where $p_1(t)$ is the probability that the object is not discovered while searching for an initial time t, and $p_2(t)dt$ is the conditional probability that if the object is not discovered by time t it will be discovered between t and t + dt. The product $P'_0(t)\ p_1(t)$ is the probability that there is neither a false contact nor an object contact in time t in cases where there is an object.

F6. Suppose now that searching continues only until there is a contact if this occurs before time T, and that otherwise the search is discontinued at time T. Then $P'_0(t)\,p_1(t)$ is also the probability that searching is not discontinued before time t if there is a object, and the product $\{P'_0(t)\,p_1(t)\}\ p_2(t)dt$ or $P'_0(t)\ dP(t)/dt\ dt$ represents the unconditional probability that if the pre-search contact is an object it will be discovered in the interval t to t + dt. The probability $P_e(T)$ that the object is discovered during the locator search is obtained by summing for all intervals dt between 0 and T. Thus

$$P_e(T) = \int_0^T P'_0(t)\,\frac{dP(t)}{dt}\,dt. \tag{1F}$$

The quantity $P_e(T)$ will be called the effective location probability. This is smaller than $P(T)$ for a given T if there is some chance of detecting false targets, but if the locator false-target density is not too large, it may be possible to obtain the required location probability by taking T somewhat larger than would be required if there were no false targets.

F7. The average searching time $\bar{t}$ is given by

$$\bar{t} = \mathcal{E} \int_0^T P'_0(t)\ \{1 - P(t)\}\ dt + (1 - \mathcal{E}) \int_0^T P'_0(t)dt.$$

The quantity $P'_0(t)\ \{1 - P(t)\}$ is the probability that there has been no contact in time t, due either to an object or a false target, assuming an object is present. This expression takes the place of $\{1 - P(t)\}$ in equation (3). Similarly the quantity $P'_0(t)$ in the second integral represents the probability of no contact in time t when there is no object. The expression for $\bar{t}$ can be put in the simpler form

$$\bar{t} = \int_0^T P'_0(t)dt - \mathcal{E} \int_0^T P'_0(t)\ P(t)dt. \tag{2F}$$

F8. In considering the further steps to salvage an object, once its apparent location has been determined by a narrow-path locator, it is of some importance to know the relative number of locator false targets which must be dealt with. Thus if an attempt is made to salvage the object or possible object, it is desirable to know the relative number of wasted salvage attempts due to inability to distinguish between objects and false targets. The expected number n_{f+m} of targets discovered per pre-search contact investigated, including both locator false targets and objects, is

$$\xi[1 - P_0'(T) \{1 - P(T)\}] + (1 - \xi) \{1 - P_0'(T)\}$$

or

$$n_{f+m} = 1 - P_0'(T) + \xi P_0'(T) P(T)$$

where, of course, n_{f+m} is a fraction less than 1.0. The expected number n_m of objects located per pre-search contact investigated is $\xi P_e(T)$. Then the expected number n_f of false targets per contact investigated is $n_{f+m} - n_m$ or

$$n_f = 1 - P_0'(T) + \xi \{P_0'(T) P(T) - P_e(T)\} . \tag{3F}$$

F9. For an approximate evaluation of the false target problem it should be sufficient to consider the case where $\bar{\sigma}/\sigma_1$, $\bar{\sigma}_2/\sigma_2$ and $W/\bar{\sigma}$ are quite small and where optimum elliptical area searching is carried out. Suppose that the locator false targets are distributed at random in the channel or area being searched, and that there are an average of η locator false targets per unit area. Let the aggregate width of searched path for false targets be $\bar{k}W\beta$ where $W\beta$ is the aggregate width of searched path for objects. Considering the element of elliptical area $2\pi\sigma_1\sigma_2\, dL$, where L is defined in paragraph B12 of Appendix B, the fraction of this element of area which is effectively searched for false targets during time t is $\rho'(L,t)$, where $\rho'(L,t)$ is the probability that any point of this area is effectively searched for false targets. From equation (8) and from paragraph B12 of Appendix B it follows that

$$\rho'(L,t) = 1 - e^{-\bar{k}\left\{\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}} - L\right\}} .$$

Then the total effectively searched area for false targets is

$$A(t) = \int_0^{\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}}} 2\pi\sigma_1\sigma_2 \left\{1 - e^{-\bar{k}\left(\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}} - L\right)}\right\} dL$$

$$= 2\pi\sigma_1\sigma_2 \left[\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}} - \frac{1}{\bar{k}}\left\{1 - e^{-\bar{k}\sqrt{\frac{Ut}{\pi\sigma_1\sigma_2}}}\right\}\right]. \tag{4F}$$

By the assumption of a random distribution of locator false targets over an area which is large compared to $A(t)$, the probability $P_r'(t)$ that exactly r false targets are detected in an area $A(t)$ is

$$P_r'(t) = \frac{e^{-\eta A(t)} \{\eta A(t)\}^r}{r!} . \tag{5F}$$

For the case considered here, $P(t)$ is given by equation (11B) of Appendix B.

F10. With the aid of equations (F1) to (F5), values of $P_e(T)$, of $\bar{t}$, and of n_f have been determined for the case $\bar{k} = 1$ and for various values of T, η and ξ. The results are presented in Table 12. In all cases the average searching time $\bar{t}$ is given in terms of $T_{0.95}$, where the latter quantity represents the time which makes P(T) = 0.95. As shown by Table 2 the value of $T_{0.95}$ is 70.7 $\sigma_1\sigma_2/U$ for the optimum searching procedure considered in deriving equation (11B). Although the data are exact only for the optimum searching procedure with $\bar{\sigma}/\sigma_1$, $\bar{\sigma}_2/\sigma_2$, and $W/\bar{\sigma}$ small, they should be reasonably adequate in all cases for which multiarea searching is appropriate. However, to apply the results to any multiarea procedure, T must be chosen to give the indicated value of P(T) under the existing conditions for the particular procedure used, and $T_{0.95}$ is the value of T which gives P(T) = 0.95 for that procedure.

F11. It can be seen that if all pre-search contacts are objects, corresponding to $\xi = 1.0$, then if it is possible to choose a T to give the required value of $P_e(T)$, the average searching time depends almost entirely on $P_e(T)$, regardless of the value of η. Thus with $\eta\sigma_1\sigma_2 = .006$, and with T chosen to give P(T) = 0.99, then the value of $P_e(T) = 0.95$, and $\bar{t}$ is the same, within the accuracy of the calculations, as it is when $\eta = 0$ with T chosen to give $P_e(T) = P(T) = 0.95$. On the other hand, when ξ is not 1.0 the average searching time $\bar{t}$ corresponding to a given value of $P_e(T)$ increases with increase in η. The number n_f of false targets detected per pre-search contact investigated is seen to increase fairly rapidly with decrease in ξ.

F12. It is apparent that the value of η determines an upper limit for the value of $P_e(T)$ which can be realized using the standard searching rules. Thus with $\eta = .03/\sigma_1\sigma_2$ a value of 0.76 is obtained with a value of T which gives P(T) = 0.90, and $P_e(T)$ is increased only to 0.81 if T is increased to the value required to give P(T) = 0.99. To obtain values of $P_e(T)$ larger than those listed in the lower block of rows in Table 12 it is necessary to use other than the standard searching rules. The standard searching rules give substantial discrimination in favor of the object by virtue of the fact that the search begins at the most probable position of the object, so the relative chance of discovering the object, when one exists, and of discovering a false target during a time interval of magnitude ΔT, is considerably more in favor of the object during intervals at the beginning of a search than it is after searching has continued for a substantial fraction of the time T. In considering alternative searching rules in order to obtain larger values of $P_e(T)$, it is desirable to make use of this principle to obtain as much discrimination as possible subject to the requirement that $P_e(T)$ is to have given values. One possible modification of the standard searching rules is to require that searching always continue for an initial interval t_1 which is less than T, regardless of whether one or more targets have been detected during time t_1, and to continue the search during time t_1 to T only if, and only as long as, no target has been detected. These will be referred to as modified searching rules. Some additional discrimination could be obtained by taking account of possible multiple targets when more than one is detected and of the exact times at which they are detected, but the improvement in discrimination which can be attained by making use of all available information on contacts during the search over that which can be attained with fairly simple rules of procedure is not expected to be very important.

F13. Two Target Searching Rules. Keeping in mind the necessity of relatively simple rules of procedure it appears the following rules should give near maximum discrimination for practical operating procedures:

(a) Discontinue searching after searching a specified time T if there has not been a contact before time T.

(b) Discontinue searching when a first contact is made provided this occurs after a specified searching time t_1 where t_1 has some value less than T.

(c) Discontinue searching at time t_1 if one and only one target has been detected during time t_1.

(d) If two targets are detected during time t_1, searching is to be discontinued when the second target is detected.

F14. If each target could be removed as soon as it is detected, both the modified searching rules and the two target-searching rules would be equally applicable to the case where the error of position fix by the locator is as great as the initial uncertainty in the position of the pre-search contact and to the case where the accuracy of location by the locator is much greater. But if targets are not removed when a contact is made, the rules are meaningful only when the location accuracy of the locator is sufficient so that contacts from different targets can be distinguished. It will be convenient to consider first the two target-searching rules and then the modified searching rules.

F15. Based on the two target-searching rules given in paragraph F13, and using the notation of paragraph F5, the combined probability that searching is not discontinued before time t and that the object has not been detected by that time is $\{P_0'(t) + P_1'(t)\}\ p_1(t)$ for values of t less than t_1. Then the probability that the object is discovered between t and t + dt is this expression multiplied by $p_2(t)\,dt$. The corresponding probability for values of t greater than t_1 is obtained from the latter expression by omitting $P_1'(t)$. Hence, the effective location probability is

$$P_e(T) = \int_0^{t_1} \{P_0'(t) + P_1'(t)\} \frac{dP(t)}{dt} dt + \int_{t_1}^{T} P_0'(t) \frac{dP(t)}{dt} dt . \qquad (6F)$$

The location probability does not depend on what fraction of the pre-search contacts are objects though the required average searching time does.

F16. To find the average required searching time let $p'(t)$ be the probability that searching has been terminated by time t for values of t between 0 and t_1, and let $p''(t)$ be the corresponding probability for values of t between t_1 and T. Then the probability that searching is terminated just at time t_1 is $p''(t_1) - p'(t_1)$ and the probability that it is terminated at the instant T is $1 - p''(T)$. Then the average required searching time is

$$\int_0^{t_1} t \frac{d}{dt} p'(t)dt + t_1 \{p''(t_1) - p'(t_1)\} + \int_{t_1}^{T} t \frac{d}{dt} p''(t)dt + T\{1 - p''(T)\} .$$

Integrating by parts and rearranging, this reduces to

$$T - \int_0^{t_1} p'(t)dt - \int_{t_1}^{T} p''(t)dt \quad \text{or} \quad \int_0^{t_1} \{1 - p'(t)\}\, dt + \int_{t_1}^{T} \{1 - p''(t)\}\, dt.$$

If the pre-search contact represents a salvageable object, then

$$1 - p'(t) = \{P_0'(t) + P_1'(t)\}\{1 - P(t)\} + P_0'(t)\, P(t)$$

and

$$1 - p''(t) = P_0'(t)\{1 - P(t)\}.$$

If no object is present, $P(t)$ must be set equal to zero in the above expressions. Hence, the average required searching time $\bar{t}$ is

$$\bar{t} = \mathcal{E}\left[\int_0^{t_1} \{P_0'(t) + P_1'(t) - P_1'(t)\, P(t)\}\, dt + \int_{t_1}^{T} P_0'(t)\{1 - P(t)\}\, dt\right]$$

$$+ (1 - \mathcal{E})\left[\int_0^{t_1} \{P_0'(t) + P_1'(t)\}\, dt + \int_{t_1}^{T} P_0'(t)dt\right].$$

This may be written

$$\bar{t} = \int_0^{t_1} P_1'(t)dt + \int_0^{T} P_0'(t)dt - \mathcal{E}\left[\int_0^{t_1} P_1'(t)\, P(t)dt + \int_{t_1}^{T} P_0'(t)\, P(t)dt\right]. \qquad (7F)$$

When $t_1 = 0$, this expression reduces to (2F) as it should.

F17. Let $\bar{P}_{\geq 1}(t)$ be the probability that at least one target is discovered in time t, assuming that searching continues until time t, and let $\bar{P}_{\geq 2}(t)$ be the probability that at least two targets are discovered in time t, assuming searching continues to time t. Then for the actual two target search procedure, the probability that exactly one target is discovered during the search is $\bar{P}_{\geq 1}(T) - \bar{P}_{\geq 2}(t_1)$, and the probability that exactly two targets are discovered during the search is $\bar{P}_{\geq 2}(t_1)$. Then the expected number of targets discovered during the search for any pre-search target is

$$n_{f+m} = \left\{\bar{P}_{\geq 1}(T) - \bar{P}_{\geq 2}(t_1)\right\} + 2\bar{P}_{\geq 2}(t_1)$$

where the factor 2 in the last term is due to the fact that when two targets are contacted each is counted separately in obtaining the expected number of targets. We have

$$\bar{P}_{\geq 1}(T) = \mathcal{E}\,\{1 - P_0'(T)\,(1 - P(T))\} + (1 - \mathcal{E})\,\{1 - P_0'(T)\}$$

$$\bar{P}_{\geq 2}(t_1) = \mathcal{E}\left[\left\{1 - \left(P_0'(t_1) + P_1'(t_1)\right)\right\}\{1 - P(t_1)\} + \{1 - P_0'(t_1)\}\, P(t_1)\right]$$

$$+ (1 - \mathcal{E})\left[\left\{1 - \left(P_0'(t_1) + P_1'(t_1)\right)\right\}\right].$$

This gives

$$n_{f+m} = 2 - P_0'(t_1) - P_1'(t_1) - P_0'(T) + \xi [P_1'(t_1)\ P(t_1) + P_0'(T)\ P(T)]$$

and

$$n_f = n_{f+m} - \xi P_e(T) ,$$

where $P_e(T)$ is given by (6F).

F18. Modified Searching Rules. From the analysis of paragraph F15 it follows that when using the modified searching rules, $P_e(T)$ is given by (6F) if the first term is replaced by $P(t_1)$. Thus for this case

$$P_e(T) = P(t_1) + \int_{t_1}^{T} P_0'(t) \frac{dP(t)}{dt} dt .$$

Similarly from the analysis of paragraph F16 it follows that with the modified searching rules, the average or expected searching time, for cases where an object is present, is

$$t_1 [1 - P_0'(t_1) \{1 - P(t_1)\}] + \int_{t_1}^{T} t \frac{d}{dt} p''(t) + T\{1 - p''(T)\} .$$

Integrating by parts and substituting for $p''(t)$, this reduces to

$$t_1 + \int_{t_1}^{T} P_0'(t) \{1 - P(t)\} dt .$$

For cases where there is no object, the average searching time is obtained from the latter expression by setting $P(t)$ equal to zero. Thus the over-all average searching time is

$$\bar{t} = t_1 + \int_{t_1}^{T} P_0'(t)dt - \xi \int_{t_1}^{T} P_0'(t)\ P(t)dt .$$

F19. The average number of targets detected during time t_1 is obviously $\eta A(t_1) + \xi P(t_1)$ and, proceeding as in paragraph F17, the expected number discovered between t_1 and T is seen to be $\bar{P}_{\geq 1}(T) - \bar{P}_{\geq 1}(t_1)$. This gives

$$n_{f+m} = \eta A(t_1) + P_0'(t_1) - P_0'(T) + \xi [P_0'(T)\ P(T) - P_0'(t_1)\ P(t_1) + P(t_1)]$$

and

$$n_f = n_{f+m} - \xi P_e(T) .$$

F20. Taking $P_r'(t)$ to be given by equations (4F) and (5F), values of $P_e(T)$, of $\bar{t}$, and of n_f have been determined on the basis of the modified searching rules for various values of T, η, and ξ, for values of t_1 equal to (1/4)T and (1/2)T. The results are presented in Table 13. It is seen that these searching rules make it possible to obtain relatively high values of the effective location probability but only at the expense of relatively high values of the average searching time and of the number of false targets. By taking T and t_1 sufficiently large, values of $P_e(T)$ can be obtained as near 1.0 as desired, but it is apparent that for values of η greater than about $.06/\sigma_1\sigma_2$, or to obtain values of $P_e(T)$ greater than those given in the table, there is little to be gained indiscrimination by taking t_1 different from T.

F21. Table F gives similar data based on the two target-searching rules. Although these searching rules appear to give slightly more discrimination than the modified searching rules in cases where the same values of $P_e(T)$ are obtained by the two procedures, the difference is too small to be of practical importance, and if η is large it is not possible to obtain adequate values of $P_e(T)$ by the two target-searching rules. It seems apparent that still more complex searching rules would not lead to any significant gain in discrimination.

F22. Where larger values of $P_e(T)$ are required for given η than those listed in Table 13 or when η is greater than $.08/\sigma_1\sigma_2$ there appears to be no useful alternative to searching the full time T. Then $P_e(T) = P(T)$. The average searching time then obviously is T, and $n_f = \eta A(T)$. Based on equation (4F) with $\bar{k} = 1$, this gives

$$n_f = 2\pi\eta\ \sigma_1\sigma_2 \left\{ \sqrt{\frac{UT}{\pi\sigma_1\sigma_2}} - \left(1 - e^{-\sqrt{\frac{UT}{\pi\sigma_1\sigma_2}}}\right) \right\} \tag{8F}$$

This gives approximately $n_f = 24\,\eta\,\sigma_1\sigma_2$ for P(T) = 0.95, and $n_f = 35\,\eta\,\sigma_1\sigma_2$ for P(T) = 0.99.

TABLE F

EFFECT OF LOCATOR FALSE TARGETS WHEN USING TWO TARGET SEARCHING RULE

T	$\frac{t_1}{T}$	$\eta\sigma_1\sigma_2$	$P_e(T)$	$\xi = 1.0$		$\xi = 0.75$		$\xi = 0.5$		$\xi = 0.25$		$\xi \to 0$	
				$\frac{\bar{t}}{T_{0.95}}$	n_f	$\frac{\bar{t}}{T_{0.95}}$	n_f	$\frac{\bar{t}}{T_{0.95}}$	n_f	$\frac{\bar{t}}{T_{0.95}}$	n_f	$\frac{\bar{t}}{T_{0.95}}$	n_f
Equal to	1/4	0.00	0.950	0.35	0.00	0.51	0.00	0.67	0.00	0.84	0.00	1.00	0.00
the Time		0.02	0.882	0.31	0.21	0.43	0.25	0.55	0.30	0.66	0.35	0.78	0.39
Required		0.03	0.853	0.29	0.30	0.40	0.36	0.50	0.42	0.60	0.48	0.70	0.54
to Obtain		0.04	0.827	0.28	0.38	0.37	0.45	0.45	0.52	0.54	0.59	0.63	0.66
P(T) = 0.95		0.05	0.803	0.26	0.46	0.34	0.54	0.42	0.62	0.49	0.69	0.57	0.77
		0.06	0.781	0.25	0.53	0.32	0.62	0.38	0.70	0.45	0.78	0.52	0.86
		0.08	0.741	0.23	0.66	0.28	0.75	0.33	0.84	0.38	0.93	0.43	1.02
Equal to	1/4	0.00	0.990	0.56	0.00	0.91	0.00	1.21	0.00	1.61	0.00	1.96	0.00
the Time		0.01	0.955	0.52	0.16	0.79	0.19	1.07	0.23	1.34	0.27	1.62	0.31
Required		0.02	0.934	0.48	0.28	0.70	0.35	0.92	0.41	1.14	0.48	1.36	0.54
to Obtain		0.03	0.909	0.45	0.40	0.62	0.48	0.80	0.57	0.98	0.65	1.15	0.73
P(T) = 0.99		0.04	0.886	0.42	0.51	0.56	0.60	0.70	0.69	0.85	0.78	0.99	0.88
		0.05	0.864	0.39	0.60	0.51	0.70	0.63	0.80	0.75	0.90	0.86	1.00
		0.06	0.843	0.36	0.69	0.46	0.79	0.56	0.90	0.66	1.00	0.76	1.10
		0.08	0.803	0.32	0.83	0.40	0.94	0.47	1.05	0.54	1.16	0.61	1.27
Equal to	1/2	0.00	0.950	0.54	0.00	0.66	0.00	0.77	0.00	0.89	0.00	1.00	0.00
the Time		0.02	0.912	0.47	0.28	0.56	0.32	0.65	0.35	0.74	0.38	0.83	0.41
Required		0.03	0.893	0.44	0.40	0.52	0.45	0.60	0.49	0.68	0.54	0.76	0.58
to Obtain		0.04	0.874	0.41	0.51	0.49	0.56	0.56	0.62	0.63	0.68	0.70	0.73
P(T) = 0.95		0.05	0.855	0.39	0.60	0.45	0.67	0.52	0.74	0.58	0.80	0.65	0.87
		0.06	0.837	0.36	0.69	0.42	0.76	0.48	0.84	0.54	0.91	0.60	0.99
		0.08	0.801	0.32	0.84	0.37	0.92	0.42	1.01	0.47	1.10	0.52	1.19
Equal to	1/2	0.00	0.990	1.00	0.00	1.24	0.00	1.48	0.00	1.72	0.00	1.96	0.00
the Time		0.01	0.971	0.89	0.15	1.09	0.18	1.29	0.20	1.49	0.23	1.70	0.25
Required		0.02	0.960	0.79	0.39	0.97	0.44	1.14	0.49	1.31	0.54	1.48	0.59
to Obtain		0.03	0.942	0.71	0.54	0.86	0.61	1.01	0.67	1.16	0.74	1.31	0.81
F(T) = 0.99		0.04	0.922	0.64	0.66	0.77	0.75	0.90	0.83	1.03	0.91	1.17	1.00
		0.05	0.902	0.58	0.77	0.69	0.86	0.81	0.96	0.93	1.06	1.04	1.15
		0.06	0.881	0.52	0.85	0.63	0.96	0.73	1.07	0.83	1.18	0.94	1.29
		0.08	0.838	0.43	0.99	0.52	1.12	0.60	1.24	0.69	1.37	0.77	1.50

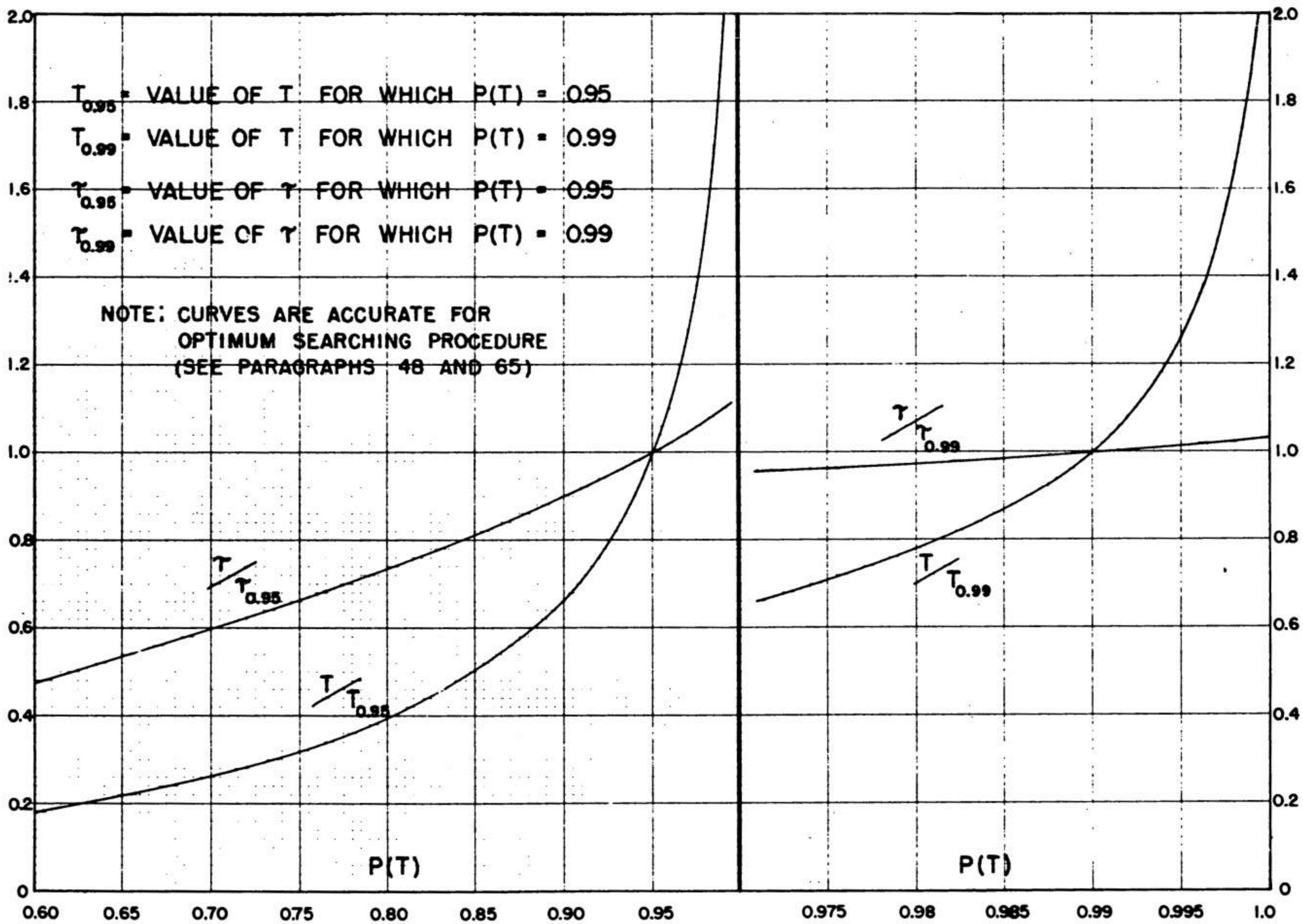

$T_{0.95}$ = VALUE OF T FOR WHICH P(T) = 0.95
$T_{0.99}$ = VALUE OF T FOR WHICH P(T) = 0.99
$\tau_{0.95}$ = VALUE OF τ FOR WHICH P(T) = 0.95
$\tau_{0.99}$ = VALUE OF τ FOR WHICH P(T) = 0.99
NOTE: CURVES ARE ACCURATE FOR OPTIMUM SEARCHING PROCEDURE (SEE PARAGRAPHS 48 AND 65)
$\tau/\tau_{0.95}$
$T/T_{0.95}$
$\tau/\tau_{0.99}$
$T/T_{0.99}$
P(T)
P(T)
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95
0.975
0.98
0.985
0.99
0.995
1.0
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0

Plate 1

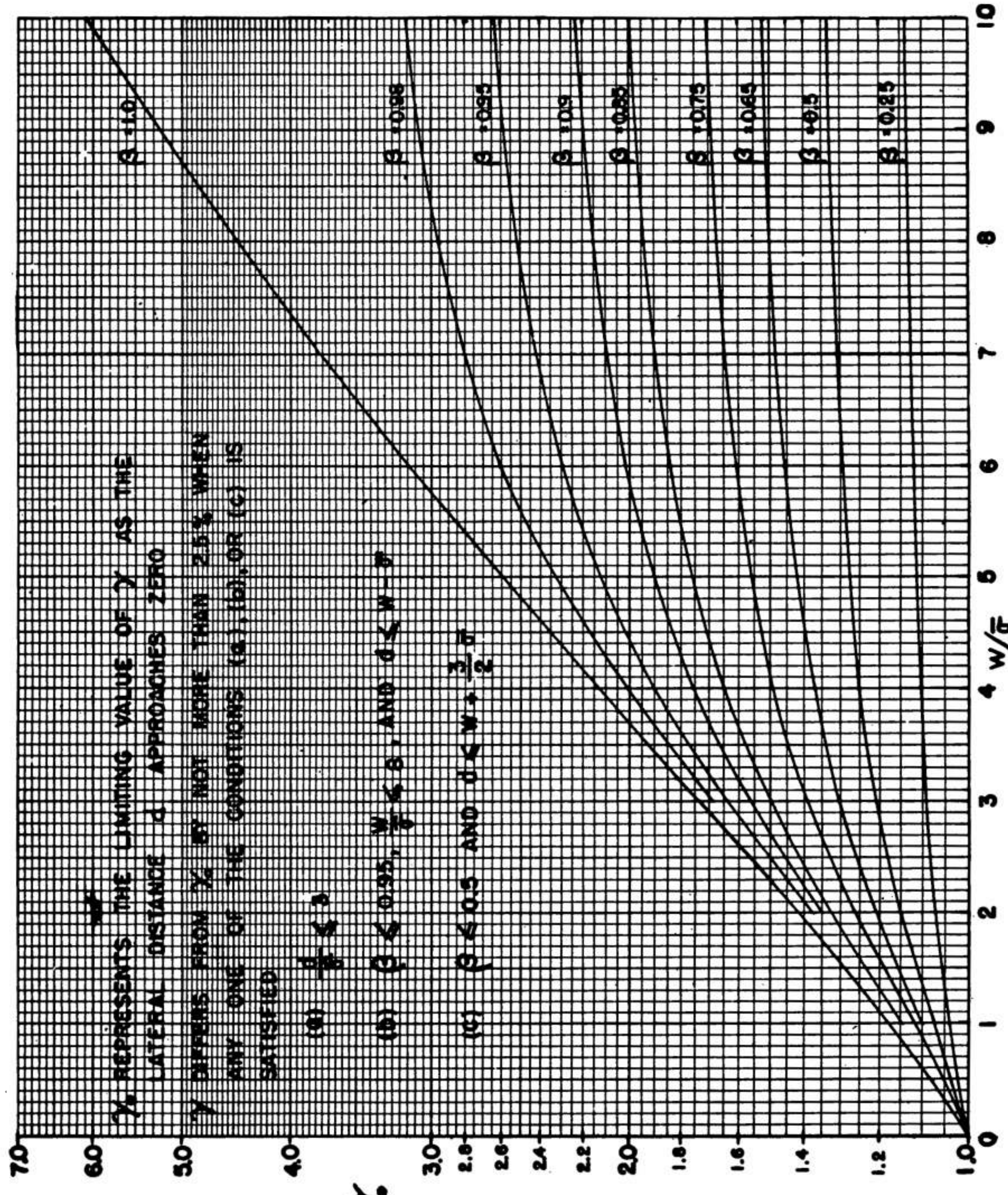
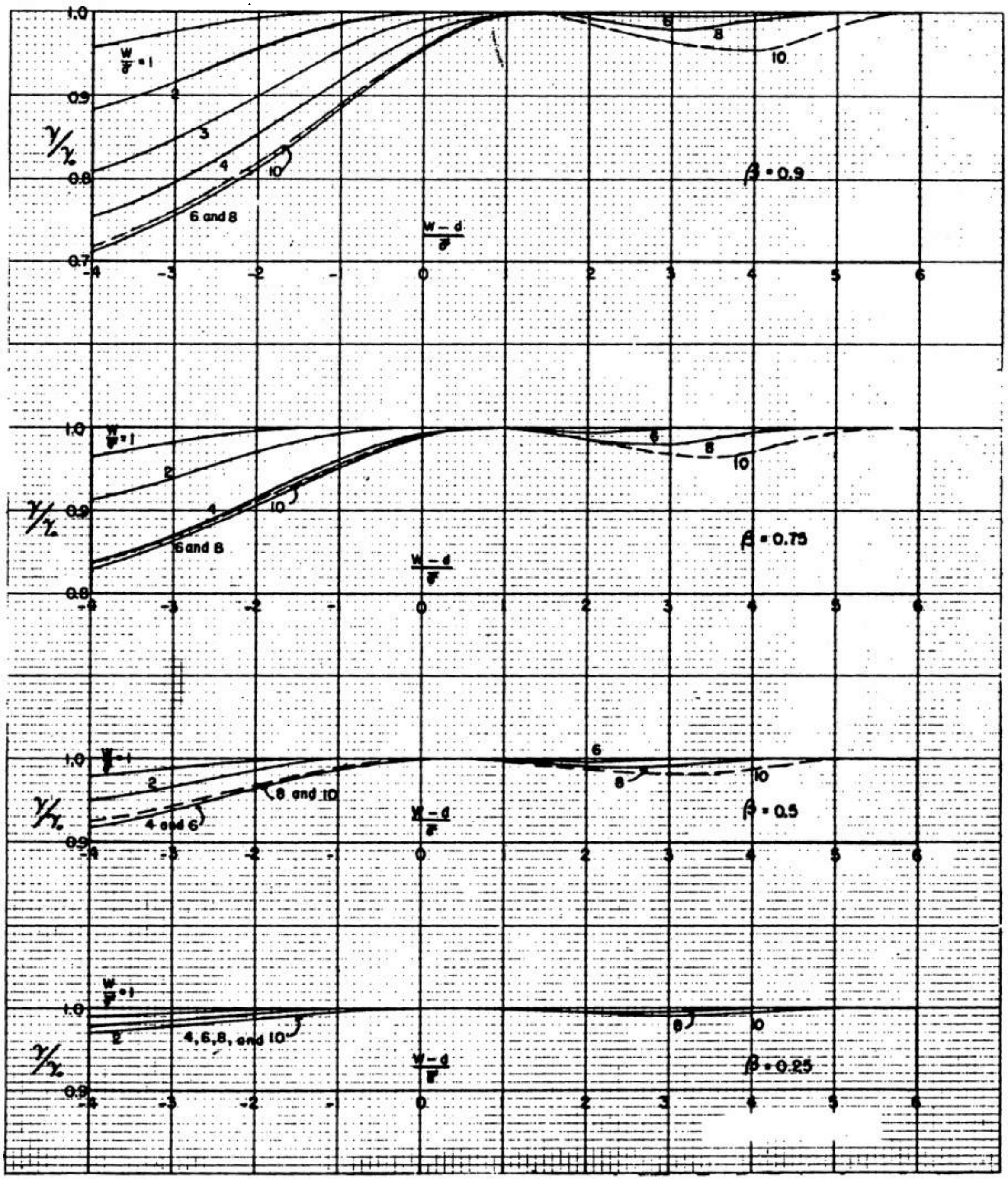

γ_o REPRESENTS THE LIMITING VALUE OF γ AS THE LATERAL DISTANCE d APPROACHES ZERO
γ DIFFERS FROM γ_o BY NOT MORE THAN 2.5% WHEN ANY ONE OF THE CONDITIONS (a), (b), OR (c) IS SATISFIED
(a) $\frac{d}{\sigma} \leq 3$
(b) $\beta \leq 0.95$, $\frac{W}{\sigma} \leq 8$, AND $d \leq W - \sigma$
(c) $\beta \leq 0.5$ AND $d \leq W + \frac{3}{2}\sigma$
γ_o
W/σ
$\beta = 1.0$
$\beta = 0.98$
$\beta = 0.95$
$\beta = 0.9$
$\beta = 0.85$
$\beta = 0.75$
$\beta = 0.65$
$\beta = 0.5$
$\beta = 0.25$
7.0
6.0
5.0
4.0
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0
1
2
3
4
5
6
7
8
9
10

Plate 2



W/σ = 1
2
3
4
6 and 8
10
β = 0.9
W − d / σ
γ/γ₀
β = 0.75
β = 0.5
8 and 10
4 and 6
β = 0.25
4, 6, 8, and 10

Plate 3

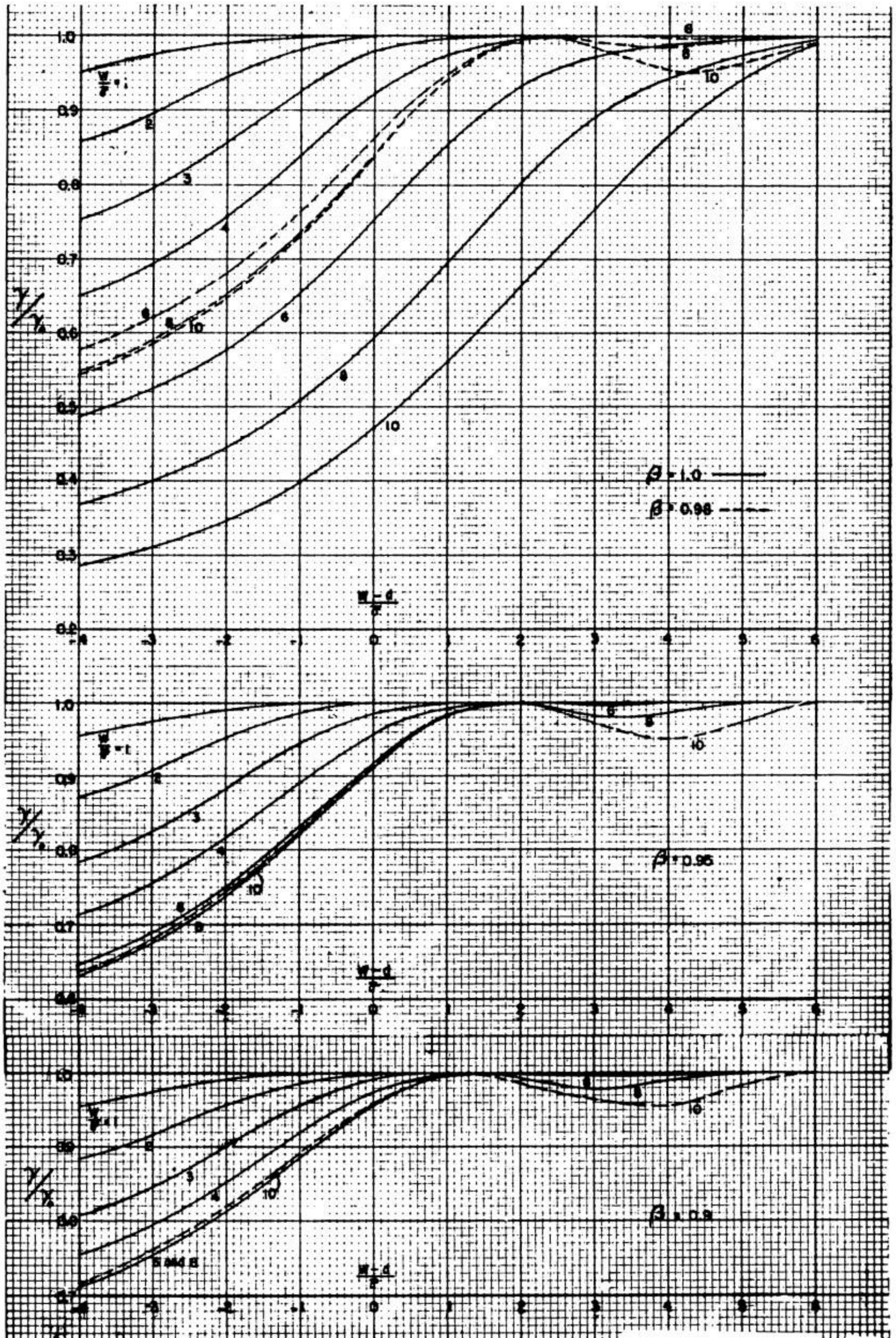
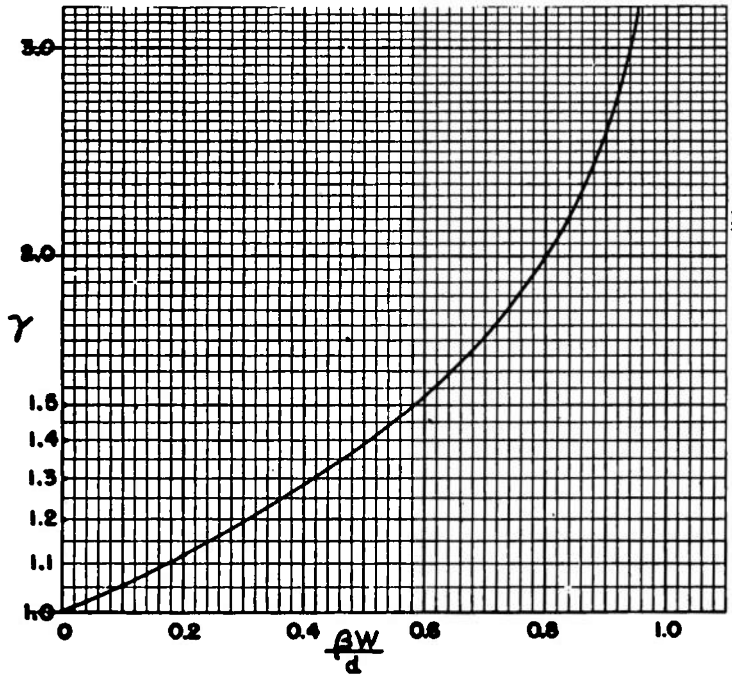


Plate 4



APPROXIMATE RANGE OF PARAMETERS
W AND d FOR WHICH CHART IS VALID

Range of Parameters	Per Cent Error
$d \geq W + 4\bar{\sigma}$	Negligible
$d \geq W + 3\bar{\sigma}$	Less than 1%
$\frac{W}{\bar{\sigma}} > 5$ and $d \geq W + 2\bar{\sigma}$	Less than 2%

Plate 5