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SEARCH WHEN FALSE CONTACTS ARE GENERATED BY REAL OBJECTS

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13. ABSTRACT

A theory of search in the presence of false contacts that are generated by real objects was developed by Stone and Stanshine, "Optimal Search Using Uninterrupted Contact Investigation", SIAM J. of Appl. Math. 20 (1971), pp. 241-263. Although they assume that false targets will be marked, when found, so that they will not be investigated again should they be contacted again, no use is made of the known number of found false targets. The search plans that they obtain with their expected-value model are not optimal when all available information is used.

In this paper we formulate the search problem, assuming that the information on the number and location of found false targets is available and will be used to the maximum extent possible. We show that the resulting formulation differs significantly from that of Stone and Stanshine. The optimization problem is more difficult. The solution is outlined for the case in which the number n of false targets is known or has a known finite distribution. An example with n = 1 is solved completely. The problem has not been solved for distributions that are not finite, such as the Poisson distribution.

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SEARCH WHEN FALSE CONTACTS ARE GENERATED BY REAL OBJECTS

1. Introduction and Summary

The first serious attempt to develop a theory of search in the presence of false targets is that of Stone and Stanshine [1]. They assume that false contacts are generated by real objects that can be marked, when located and identified, so that they will not require investigation should they be contacted again. They use an expected-value model for the number of found false targets. They make no reference to (and, hence, no use of) the information afforded by the known number of false targets that have been found by a given search time.

It seems reasonable to assume that the searcher will record the number, as well as the locations, of found false targets in a search for submerged submarines and in most search situations of interest. This information can be used to improve the search plan under most conditions, including those assumed by Stone and Stanshine. Hence, we believe that the Stone and Stanshine formulation of the problem is incorrect.

In this report we formulate the search problem in the presence of false targets, assuming that the information on the number of found false targets is available. We start with the case in which the initial number of false targets is known, and then consider the case in which the number is unknown but the number distribution is known. We exhibit examples for both cases to show that by using the information on the number of found false targets we can find a search plan that yields a smaller expected time to detection than that obtained with the optimal solutions that do not use the information, including the Poisson distribution assumed by Stone and Stanshine.

The changes in the formulation of the problem, required by the assumption that the number of found false targets is known, produce an optimization problem that is considerably more difficult than the optimization problem of Stone and Stanshine. When the initial number of false targets is known or has a finite distribution the solution can be obtained in a finite number of steps by Dynamic Programming. At each step we have a functional to optimize; we use methods from the Calculus of Variations, but other methods could be used. The solution is outlined, and demonstrated for a few examples.

The optimization problem for a non-finite distribution, such as the Poisson, appears to be essentially more difficult than the problem for a finite distribution. We have not solved this problem, although it is likely that it can be solved by an extension of the method that we used for the finite distribution.

2. Target and False Target Number Distributions

We assume that exactly one target is known to be in a region R of Euclidean n-space, with known location density function f(x). During the search, false contacts may be generated by other real objects in R, called false targets, which can't be distinguished from the target except by a close inspection. The number of false targets in R may be known, or unknown with a known number density function. We assume that the false targets are independently distributed with known location density functions. They may be identically distributed with common location density function g(x), or otherwise. Our assumptions on the number distributions are more general than those of Stone and Stanshine, who limit consideration to the Poisson distribution. On the other hand, our assumptions on the location distributions are more specific; Stone and Stanshine introduce the collective density function $\delta(x)$, but say nothing about the individual density functions.

If a contact is made, the search will be interrupted, an investigation will be started, and continued until the contact has been identified. If the contact is the target, the search will be stopped. If the contact is a false target, its location will be recorded and the position marked, perhaps with a buoy, so that another investigation will not be made, should it be contacted again. Then search will be resumed.

When an investigation of a contact has been completed a change occurs in one of the number distributions. If the contact is the target, the number of unlocated targets in R changes from one to zero. We use this information to adjust our search procedure: we stop the search. Similarly, if the contact is a false target, we use this information to adjust our search procedure.

If the number of false targets in R is a known number n, the number of residual (unlocated) false targets in R is n - 1. Hence, we resume the search under the condition that there now are n - 1 false targets. As additional contacts are made, investigated, and found to be false targets, we reduce the number of residual false targets. Finally, if we are unlucky and persevere long enough, all the false targets will have been located and we then continue the search under the assumption that there are no false targets in R.

If the number n of false targets in R is unknown with a known distribution, let

$$p_k = \text{Prob. } \{n = k\}$$

When a contact has been made, investigated, and found to be false, we now know that there is at least one false target in R. If the found false target is eliminated by marking its location, the probability $p_k^{(1)}$ that there are k false targets unlocated in R is

$$p_k^{(1)} = p_{k+1}/(1 - p_0) \text{ , } k = 0, 1, 2, \ldots$$

Continuing in this way, the probability that there are k residual false targets in R, given that i false targets have been located, is

$$p_k^{(i)} = p_{k+i}/(1 - p_0 - p_1 - \ldots - p_{i-1}), \ k = 0, 1, 2, \ldots \quad (1)$$

If we assume that the found false targets are not marked and eliminated from the residual population, the probability that there are k false targets in R, given that i false targets have been located, is

$$p_k^{(i)} = \begin{cases} 0 \text{ , } & k = 0, 1, 2, \ldots, i - 1 \\ p_k/(1 - p_0 - p_1 - \ldots - p_{i-1}), & k = i, i + 1, \ldots \end{cases} \quad (2)$$

Our formulation admits either equation (1) or equation (2). We will use equation (1).

The known fact that i false targets have been found is a condition on the false target distribution in the $(i + 1)^{st}$ stage. The number probability density function is conditional on i, as noted above. Also, the location density function of the residual false targets may be conditional on i and on the known locations of the found false targets. It is for this reason that we need to specify the individual density functions, in addition to the collective density function.

If the false targets are identically, and independently, distributed, the common location distribution of the residual false targets does not depend on i and the locations of the i found false targets. In this case, the collective density function is sufficient. The individual density function can be obtained from the collective density function, and conversely.

If the false targets are not identically distributed, the number and locations of the found false targets have an effect on the location distributions of the residual false targets. The posterior density of false targets depends on the amount of information that is available from the location of the found false targets. Thus, if we know there are exactly n_1 false targets in region I and exactly n_2 false targets in region II of two disjoint regions, the residual numbers of false targets in the two regions, given that i_1 have been located in region I and i_2 in region II, are known precisely. If the probability that a found false target has a particular distribution depends only on the relative density, the probability $r_j(x_1)$ that a found false target is of distribution type j, given that it was found at x_1, is

$$r_j(x_1) = \delta_j(x_1)/\delta(x_1) \tag{3}$$

where

$\delta_j(x)$ = collective density of false targets of type j,

$\delta(x) = \sum_{(j)} \delta_j(x)$ = total collective density of false targets

We then can use $r_j(x_1)$ and equation (1) to obtain the number of residual targets of each type from the probabilities p_{jk} that there are k false targets of type j. For example, if one false target has been located at x_1, the conditional probability $p_{jk}(x_1)$ that there are k residual false targets of type j is

$$p_{jk}(x_1) = r_j(x_1)\left(\frac{p_{j,k+1}}{1-p_{j0}}\right) + (1 - r_j(x_1))\, p_{jk} \tag{4}$$

By repeated use of this equation we can then obtain the probability that there are k residual false targets of type j, given that i false targets have been found at $x_1, x_2, \ldots, x_i$.

3. Effort Distributions and Conditional Detection Functions

It will be convenient to use a notation that permits the search density function to change each time a false target is found. Let

s_1 = search time at which the first contact is made,

$\mu_1(x,s)$ = search density at x by search time s, $0 \le s \le s_1$.

If the contact is a false target and search is resumed, let

s_2 = additional search time in the second stage, measured from the resumption of search until a second contact is made,

$\mu_2(x,s)$ = additional search density at x by the additional search time s in the second stage, $0 \le s \le s_2$.

In general, let

s_i = additional search time to make a contact in the i^{th} stage of the search,

$\mu_i(x,s)$ = additional search density at x by the additional search time s in the i^{th} stage, $0 \le s \le s_i$.

We assume that $\mu_i(x,0) = 0$ and that $\mu_i(x,s)$ is nondecreasing in s for each i and for every x in R. The total effort density at x in the first j stages is

$$M_j(x) = \sum_{i=1}^{j} \mu_i(x,s_i) \tag{5}$$

At additional search time s in the j^{th} stage, after (j - 1) false targets have been found, the total search density at x is

$$m_j(x,s) = M_{j-1}(x) + \mu_j(x,s) \tag{6}$$

We call attention to the fact that $M_j(x) \equiv m_j(x,s_j)$ in our notation. Thus, capital M is used for the value of m when contact occurs. The variable s is used for the additional search time; the total search time S corresponding to the additional search time s in the j^{th} stage is $S = s_1 + s_2 + \ldots + s_{j-1} + s$.

The total effort that has been applied in the additional search time s in the j^{th} stage is

$$w_j(s) = \int_R \mu_j(x,s)\, dx$$

and the total effort $W_j(S)$ from the start of search is

$$W_j(S) = w_j(s) + \sum_{i=1}^{j-1} w_i(s_i) \ , \ S = s + \sum_{i=1}^{j-1} s_i$$

Then we can impose the condition that the effort in the j^{th} stage, or the total effort, is a known function of the search time. For example, the condition that effort is applied at a known rate U_j in the j^{th} stage is

$$w_j(s) = s\, U_j$$

We assume that the conditional probability that a target at x will be contacted when an effort of total density m(x) is applied at x is a function b(m(x)), called the local effectiveness function by Stone and Stanshine, of the total density m(x), however it is applied. We assume that b(0) = 0, lim b(u) = 1 as $u \to \infty$, and that the derivative b'(u) is positive, continuous, and strictly decreasing. Some of these properties can be relaxed or omitted under some conditions. Thus, the limit property is needed if we assume that search will continue until target detection occurs when minimizing the expected time to detection, but could be omitted for some other criteria. Also, the positive and strictly decreasing conditions on $b'(u)$ can be replaced by non-negative and non-increasing conditions.

We obtain the conditional contact function at x for the j^{th} stage, given that the effort of density $M_{j-1}(x)$ was applied in the first j - 1 stages, by using our assumption that b is a function of the total density. Let

$\hat{b}_j(x,s) \equiv b(\mu_j(x,s)|M_{j-1}(x))$ = conditional probability that a target at x will be contacted by the additional effort of density $\mu_j(x,s)$ in the j^{th} stage, given that it hasn't been contacted by the effort of density $M_{j-1}(x)$ that has been applied in the previous stages.

Then, the total probability of contacting a target at x is

$$b(\mu_j(x,s) + M_{j-1}(x)) = b(M_{j-1}(x)) + [1 - b(M_{j-1}(x))]\ \hat{b}_j(x,s), \qquad (7)$$

from which we can obtain $\hat{b}_j(x,s)$. It will be convenient to use the complements of b and $\hat{b}_j$. Let

$$\beta(m(x,s)) = 1 - b(m(x,s)),\ \hat{\beta}_j(x,s) = 1 - \hat{b}_j(x,s)$$

Then from equation (7) we obtain

$$\hat{\beta}_j(x,s) = \frac{\beta(\mu_j(x,s)+M_{j-1}(x))}{\beta(M_{j-1}(x))} \qquad (8)$$

Let $\alpha(u)$ be the conditional contact failure function for false targets, corresponding to $\beta(u)$ for the target. Then the conditional contact failure probability $\hat{\alpha}_j(x,s)$ in the j^{th} stage is obtained from the equation

$$\hat{\alpha}_j(x,s) = \frac{\alpha(\mu_j(x,s)+M_{j-1}(x))}{\alpha(M_{j-1}(x))} \qquad (9)$$

4. Target and False Target Location Distributions

We start with the target location distribution. For the target there are only two stages. In the first stage the number of found targets is zero. Using Bayes' formula, the conditional target density function, given that no target has been found, is

$$\hat{f}_1(x,s) = \frac{f(x)\ \beta\ (m(x,s))}{Q(s)} , \tag{10}$$

where m(x,s) is the search density at x in time s, and

$$Q(s) = \int_R f(x)\beta(m(x,s))dx \tag{11}$$

In the second stage, the conditional target density function, given that one target has been found, is

$$\hat{f}_2(x,s) = 0 \tag{12}$$

Now consider false targets. If the number n of false targets is known, there are (n-j+1) unlocated false targets in the j^{th} stage, j=1,2,...,n+1. Assume that the n targets are identically distributed with common prior density function g(x). The conditional density function for one residual false target in the j^{th} stage is

$$\hat{g}_j(x,s) = \frac{g(x)\alpha(m_j(x,s))}{Q_j(s)} \tag{13}$$

where

$$m_j(x,s) = M_{j-1}(x) + \mu_j(x,s) \tag{6}$$

and

$$Q_j(s) = \int_R g(x)\alpha(m_j(x,s))dx \tag{14}$$

The collective density function of all residual false targets in the j^{th} stage is

$$\hat{\delta}_j(x,s) = (n-j+1)\ \hat{g}_j(x,s) \tag{15}$$

If the false targets are identically distributed and the number is unknown with a known number density probability p_k, we replace the number (n-j+1) of known residual targets by the expected number $\bar{n}_j$, given that j-1 have been found. From equation (1) we have

$$\bar{n}_1 = \sum_{k=1}^{\infty} k\ p_k$$

$$\bar{n}_j = \frac{\sum_{k=1}^{\infty} k p_{k+j-1}}{1-p_0-p_1-\ldots-p_{j-2}} \quad , \; j = 2, 3, \ldots \tag{16}$$

Then the expected value of the collective density function in the j^{th} stage, given that j-1 have been found, is

$$\hat{\delta}_j(x,s) = \bar{n}_j \; \hat{g}_j(x,s) \tag{17}$$

If the false target population contains two or more a priori location distributions, we treat each type separately, using equation (4) and extensions of it to get the number distributions. Then the collective location density function of all residual false targets can be obtained by addition of the separate location density functions.

In writing the collective density of residual false targets in the form (15) when n is known, we are making an obvious extension of the forms (10) and (12) for n = 1. In any particular stage the expected number of residual false targets remains constant as s increases; it changes only when an additional false target is found and we go to the next stage.

We have retained this property in making the extension to other number distributions in equation (17). We assume that failure to contact a false target, while search proceeds in the j^{th} stage, changes the location distribution, but not the number distribution, of residual false targets. The location distribution of a residual false target is obtained from equation (13) and the number distribution from equation (1).

Stone and Stanshine [1] use the expected number of false targets that will be contacted, instead of the observed number. They implicitly assume that the expected density of residual false targets at total search time S, when an effort of total density m(x,S) has been applied, is

$$\hat{\delta}(x,S) = \delta(x)\alpha(m(x,S)) \tag{18}$$

This quantity is the expected density of residual false targets, given no information on the number that have been found. The corresponding value of the expected number of residual false targets is

$$\bar{n}(S) = \int_R \delta(x)\alpha(m(x,S))dx = n\ Q(S), \tag{19}$$

where n is the expected number initially in R and

$$Q(S) = \int_R g(x)\alpha(m(x,S))dx \tag{20}$$

Thus, they use a form that requires the expected residual number $\bar{n}$ to depend on the total search time but not on the observed number of false targets that have been found. It is an expected-value model.

5. Optimization Criteria

It usually is assumed that the general objective of the search is to find the target quickly. The two precise criteria that are obtained from this general objective are:

(1) To maximize the probability P(t) of finding the target by a given elapsed time t;

(2) To minimize the expected time to find the target.

An objection that can be raised to the second criterion is that it assumes implicitly that search will be continued until the target is found, a condition that the searcher will not always adhere to in practice, even when he accepts it as a condition in planning the search.

When no false targets are present we can reconcile the two criteria. If a search plan exists that maximizes P(t) for all t, it minimizes the expected time to find the target. Such a plan exists when no false targets are present. Of course, it is much more difficult to lay out a practical search plan that approximates the distribution of the ideal solution when trying to maximize P(t) for all t than when trying to maximize P(t) for a single value of t.

When false targets are present we have no assurance that a plan exists that maximizes P(t) for all t. Since the expected time to investigate the contact that is the target is independent of the search plan (assuming that the investigation time depends only on the target location), we can use the probability of contacting the target by elapsed time t. Let $G(s|\tau,m)$ be the probability distribution function of the search time s for a given value τ of the time to contact the target and a search density function m. The probability of contacting the target by search time S with the search density function $m(x,S)$ is

$$P_1(m,S) = \int_R f(x)b(m(x,S))dx \tag{21}$$

and the probability of contacting the target by time τ, $\tau \geq S$, is

$$P_2(m,\tau) = \int_{S=0}^{\tau} P_1(m,S)G(dS|\tau,m) \tag{22}$$

The expected time to contact the target is

$$\bar{t}_c(m) = \int_0^{\infty} t\ dP_2(m,t) = \int_0^{\infty} [1-P_2(m,t)]dt \tag{23}$$

In section 5 of their paper [1] Stone and Stanshine exhibit an example for which the search plan that maximizes the probability $P(1+h)$ of finding the target by elapsed time 1+h, for sufficiently small $h > 0$ (with the investigation time $T = 1$), does not coincide with their search plan that minimizes the integral* $\mu(m)$ that they assert is the expected time to contact the target. They do not write an equation for the $P(t)$ function and the reader is left to infer that they mean the function $P_2(m,t-1)$ that we have introduced above. Since they obtain $\mu(m)$ by a procedure that is quite different from that used to get $\bar{t}_c(m)$ from $P_2(m,\tau)$ above, it is not clear whether their example shows that no plan maximizes $P_2(m,t-1)$ for all t, or that $\mu(m)$ and $\bar{t}_c(m)$ are different functionals of m. It seems to us that a proof of the equation

$$\bar{t}_c(m) = \mu(m) \text{ for all } m$$

is needed to complete the argument.

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*The integral $\mu(m)$ of the Stone and Stanshine paper is not the $\mu(x,s)$ used above.

To continue the discussion of criteria, let us assume that there is no plan that maximizes P(t) for all t. Is this a sufficient reason for the selection of the expected-time criterion (2)? In addition to the restriction that it imposes on the searcher to search until the target is found, it also leads to a difficult optimization problem, as does the probability criterion (1). For both criteria it is necessary to look beyond the current stage of the search to determine the optimal procedure for the current stage, as we show below.

6. Expected-Time Criterion

We start with the expected-time criterion (2), but adopt an approach to the problem that differs from that used above in writing equation (23). Our search procedure is as follows:

a. Search until a contact occurs, investigate the contact until it is identified.

b. If the contact is the target, stop the search.

c. If the contact is a false target, record the location and mark the site so that investigation will not be repeated, should this false target be contacted again. Increase the number of found false targets by 1 and use the fact that a false target has been found and eliminated to adjust the number probability density of residual false targets and the location distributions of residual false targets, to the extent possible. Then continue the search.

d. Repeat the above steps until the target is found.

If the initial number n of false targets is known, the number of residual false targets in the j^{th} stage is $n - j + 1$. Let

s_j = additional search time in the j^{th} stage at which contact is made in that stage;

I_j = time to investigate the j^{th} contact, if false;

q_j = conditional probability that the j^{th} contact is false, given that a contact has occurred at x_j in the j^{th} stage.

Then the time t_c to contact the target is

$$t_c = s_1 + q_1(I_1 + s_2 + q_2(I_2 + s_3 + q_3(\ldots + q_n(I_n + s_{n+1}))\ldots) \quad (24)$$

Let

$$\left.\begin{aligned} T_{n+1} &= s_{n+1} \\ T_n &= s_n + q_n(I_n + T_{n+1}) \\ T_{n-1} &= s_{n-1} + q_{n-1}(I_{n-1} + T_n) \\ &\cdot \\ &\cdot \\ &\cdot \\ T_1 &= s_1 + q_1(I_1 + T_2) \end{aligned}\right\} \quad (25)$$

Then $t_c = T_1$ and the elapsed time t to find the target is $t = t_c + I$, where I is the time to investigate the contact that is the target.

The expected time to find the target is $\bar{t} = \bar{t}_c + \bar{I}$. Since $\bar{I}$ is independent of the search plan, we minimize $\bar{t}_c$. The expected time $\bar{t}_c$ to contact the target is obtained by averaging stage by stage in equations (25), starting with the $(n+1)^{st}$ stage and moving (backwards) to the first stage.

In the $(n+1)^{st}$ stage there are no false targets. We find the target density function at the start of this stage from equations (10) and (11), by putting m(x,s) equal to the total density function $M_n(x)$ at the end of the n^{th} stage, where

$$M_n(x) = \sum_{i=1}^{n} \mu_i(x,s_i) \quad (26)$$

Then the density function at the start of search in the $(n+1)^{st}$ stage is

$$f_{n+1}(x) = \frac{f(x)\beta(M_n(x))}{Q_n}, \quad (27)$$

where

$$Q_n = \int_R f(x)\beta(M_n(x))dx \tag{28}$$

We now find the function $\mu^*_{n+1}(x,s)$ that minimizes the expected value of s_{n+1}, which is a solved problem. The optimal function $\mu^*_{n+1}(x,s)$ is the function $\mu_{n+1}(x,s)$ that minimizes the failure probability $Q_{n+1}(s)$ for all s, where

$$Q_{n+1}(s) = \int_R \hat{f}_{n+1}(x) \quad \hat{\beta}_{n+1}(x,s)dx$$

$$= Q_n^{-1} \int_R f(x)\beta(\mu_{n+1}(x,s) + M_n(x)) \, dx \tag{29}$$

We could have written equation (29) directly, without the use of $\hat{f}_{n+1}$ and $\hat{\beta}_{n+1}$.

Let $Q^*_{n+1}(s)$ be the function $Q_{n+1}(s)$ when μ_{n+1} has been replaced by the optimal function μ^*_{n+1}, and put

$$T^*_{n+1} = \int_0^\infty Q^*_{n+1}(s)ds \tag{30}$$

Then T^*_{n+1} is the minimum expected time to contact the target in the (n+1)st stage. From equations (26) and (29) it is seen that T^*_{n+1} is a function of the previous density functions $\mu_j(x,s)$ and the times s_j at which previous contacts were made, j = 1, 2, ... , n.

In the n^{th} stage we minimize the expected time remaining to contact the target, on the assumption that the search procedure in the $(n+1)^{st}$ stage will be optimal. Hence, we minimize the expected value of

$$s_n + q_n(I_n + T^*_{n+1}) \equiv T_n(s_n, \mu_n) \tag{31}$$

That is, we find the function $\mu^*_n(x,s)$ that minimizes $\overline{T}_n(s_n, \mu_n)$ over s_n, given that there is one false target and one real target in R. The expected value is computed over the probability distribution of the false contact time s_n, noting that q_n and T^*_{n+1} (and possibly I_n) depend on s_n and μ_n, in general. This is not a solved problem. We have solved it in special cases used as examples, one of which will be shown later.

Assume that the above problem has been solved and let T_n^* be the minimum value of the expected value of $T_n(s_n, \mu_n)$. Then in the $(n-1)^{st}$ stage we minimize the expected value of

$$s_{n-1} + q_{n-1}(I_{n-1} + T_n^*) - T_{n-1}(s_{n-1}, \mu_{n-1}) \qquad (32)$$

This problem is similar to the optimization problem in the previous stage, except that there now are two false targets. Also, the functional form of T_n^* is more complicated than that of T_{n+1}^*.

Continuing in the way outlined above we find the functions $\mu_{n+1}^*(x,s)$, $\mu_n^*(x,s)$ ... , $\mu_1^*(x,s)$, which constitute the optimal solution, in the sense of minimizing the expected time to find the target when all the available information is used. The function $\mu_j^*(x,s)$ in the j^{th} stage may depend on the functions μ_1^*, μ_2^*, ... , μ_{j-1}^* and the contact times s_1, s_2, ... , s_{j-1} in the previous stages. These functions and times will be known when search is resumed in the j^{th} stage. The optimal function $\mu_j^*(x,s)$ in the j^{th} stage is found under the assumption that search in the later stages will be optimal.

If the initial number n of false targets is not known but has a finite distribution, we can apply a similar procedure to find the solution. If the maximum value of n is K, there are no false targets in the $(K+1)^{st}$ stage. In the K^{th} stage the possible number n_K of residual false targets is 0 or 1. If $n_K = 0$, $q_K = 0$ and there are no later stages. If $n_K = 1$, the problem is the same as the problem when n is known. We find the expected value of $T_K(s_K, \mu_K)$ over the distribution of n_K before proceeding with the optimization; similarly for the other stages. Thus, in principle, we can find the optimal solution in (K+1) steps by applying the general principle of Dynamic Programming.

If the distribution of n is not finite, the problem is more difficult. There now is no convenient starting place. Although we do not know how to solve this problem, there is no reason to doubt that the optimal solution will have the same general properties of the optimal solutions of the problems discussed above, and, in particular, the following properties:

(a) The optimal solution $\mu_j^*(x,s)$ in the j^{th} stage, given that $j - 1$ false targets have been found and eliminated, will depend on j, in general.

(b) In addition, $\mu_j^*(x,s)$ will depend on the optimal functions $\mu_1^*, \mu_2^*, \dots, \mu_{j-1}^*$ and the contact times $s_1, s_2, \dots, s_{j-1}$ in the previous stages, in general.

The statement (a) above is made to emphasize the fact that the optimal solution, in the sense of minimizing the expected time to find the target, is conditional on the number of found false targets when all the available information is used.

The function found by Stone and Stanshine for the expected-value model is not conditional on the number of found false targets. By using the expected number of found false targets, instead of the observed number, they obtain a much easier problem than the problem obtained with the correct formulation. Although their function does not minimize the expected time to find the target, it may be useful as an approximation, provided a bound on the error can be obtained.

7. Solution For a Particular Problem

We start with a simple example to show the general procedure that is used in minimizing $T_n(\mu_n, s_n)$ in (31).

Example 1.

Assume that R consists of two disjoint regions I and II in the plane, each having Lebesque measure 1; the target density function is $f(x) = f_1$ for x in I, $f(x) = 1 - f_1$ for x in II; and there is one false target, known to be in region II. Assume that the time required to identify a contact is 1, that the conditional detection function is $b(z) = 1 - \exp(-z)$, and that effort is applied at a constant rate $U = 1$.

We search with density function $\mu_1(x,s)$ until a contact is made at $s = s_1$. If the contact is the false target, we go to the second stage of the search and search with density function $\mu_2(x,s)$, knowing there are no false targets remaining. At the start of the second stage the conditional target density function has the form

$$f_2(x) = f_2 \text{ for x in I}, \quad f_2(x) = 1 - f_2 \text{ for x in II}$$

where f_2 is independent of x, but depends on μ_1 and s_1. (We can obtain f_2 from equation (27) when needed.)

The search density function $\mu_2^*(x,s)$ that minimizes the expected additional time s_2 to contact the target depends on the magnitude of f_2. If $f_2 \leq 0.5$, μ_2^* has the form:

$$\text{For } 0 < s \leq s_0, \quad \mu_2^*(x,s) = \begin{cases} 0 \text{ for x in I} \\ s \text{ for x in II} \end{cases}$$

$$\text{For } s > s_0, \quad \mu_2^*(x,s) = \begin{cases} (s - s_0)/2 \text{ for x in I} \\ (s + s_0)/2 \text{ for x in II} \end{cases}$$

where

$$s_0 = \ln(f_2^{-1}-1)$$

From equation (29) $Q_2(s)$ for the optimal distribution is

$$Q_2^*(s) = \begin{cases} f_2 + (1-f_2)e^{-s} & , s \leq s_0 \\ 2f_2^{1/2}(1-f_2)^{1/2}e^{-s/2} & , s > s_0 \end{cases}$$

Then

$$T_2^* = \int_0^\infty Q_2^*(s)ds = 1 + f_2(2 + s_0)$$

(If needed, we can get T_2^* when $f_2 > 0.5$ by replacing f_2 by $1 - f_2$ in the above expression.)

Write $\mu_1(x,s)$ in the form

$$\mu_1(x,s) = \begin{cases} y(s) & \text{for x in I} \\ s-y(s) & \text{for x in II} \end{cases}$$

From equation (27), f_2 becomes

$$f_2 = f_1 e^{-y(s_1)} / Q_1(s_1, y(s_1))$$

where s_1 is the observed time at which the first contact was made, and

$$Q_1(s,y) = f_1 e^{-y} + (1 - f_1) e^{-s+y} \qquad (33)$$

Then

$$s_0 = 2y(s_1) - s_1 + \ln (f_1^{-1} - 1)$$

and we now can write T_2^* in terms of s_1 and $y(s_1)$.

To write the equation for T_1 in equation (31) we need the conditional probability q_1 that the first contact is made on the false target, given that the contact is made at search time s_1. The probability that the false target, known to be in region II, will not be contacted by search time s is

$$Q_F(s,y) = e^{-s+y},$$

where it is understood that $y = y(s)$ is a function of s. The probability that a contact will occur by search time s is

$$P_c(s,y) = 1 - Q_1(s,y)\, Q_F(s,y)$$

We note that $P_c(s,y)$ is the probability distribution function of s_1, the time of first contact. The rate of making contact at search time s is

$$P'_c = \frac{d}{ds} P_c(s,y(s)).$$

We can write P'_c in the form

$$P'_c = Q_1 P'_F + Q_F P'_1 \qquad (34)$$

where

$$P_1 = 1 - Q_1, \; P_F = 1 - Q_F$$

and the prime indicates total derivative with respect to s. The two terms in the righthand member of equation (34) are the contact rates on the false target and the target respectively. If the first contact occurs at search time s_1, the conditional probability that the contact is made on the false target is $q_1(s_1,y(s_1))$, where

$$q_1(s,y(s)) = Q_1(s,y(s))\ P'_F(s,y(s))/P'_c(s,y(s))$$

The time $T_1(s_1, \mu_1)$ in equation (31) will be written in the form $T_1(s_1, y(s_1))$. Then

$$T_1(s,y) = s + P'_F(s,y)\ R\ (s,y)/P'_c(s,y),$$

where

$$R(s,y) = f_1 e^{-y}(2y - s + 4 + \ln(f_1^{-1}-1)) + 2(1 - f_1)e^{-s+y} \tag{35}$$

The mean value of $T_1(s_1,y(s_1))$ over s_1 is

$$\overline{T}_1(y) = \int_0^\infty T_1(s,y(s))P'_c(s,y(s))ds = \int_0^\infty F(s,y(s),y'(s))ds \tag{36}$$

where

$$F(s,y,y') = G(s,y) + y'\ H(s,y) \tag{37}$$

and

$$G(s,y) = Q_F(s,y)[Q_1(s,y) + R(s,y)]$$

$$H(s,y) = -Q_F(s,y)\ R(s,y)$$

We now find y(s) to minimize $\overline{T}_1(y)$ in (36). With an integrand (37) that is linear in y' the problem is easy to solve by methods from the Calculus of Variations. The necessary conditions of Legendre and Weierstrass are satisfied. Euler's necessary condition for an extremum,

$$\frac{d}{ds}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y},$$

reduces to the condition

$$\frac{\partial G}{\partial y} = \frac{\partial H}{\partial s}$$

with the integrand F in (37) that is linear in y'. This equation reduces to the equation

$$Q_1 + \frac{\partial Q_1}{\partial y} + \frac{\partial R}{\partial y} + \frac{\partial R}{\partial s} = 0 \tag{38}$$

with the simple form for Q_F above. When the equations for Q_1 and R in (33) and (35) are used, equation (38) becomes

$$2(f_1^{-1}-1)\ e^{2y-s} = 2y - s + 3 + \ln(f_1^{-1}-1)\ ,$$

which becomes

$$2\ e^{u} = u + 3 \qquad (39)$$

with the change of function,

$$u = 2y - s + \ln(f_1^{-1}-1) \qquad (40)$$

There are two solutions of equation (39). Let u_1 be the positive solution, which is approximately $u_1 = 0.583$. The corresponding solution of equation (40) is

$$y_1(s) = \frac{1}{2}\ (s + u_1 - \ln(f_1^{-1}-1)) \qquad (41)$$

We now can show that the solution $y_1(s)$ in equation (41) minimizes the integral $\overline{T}_1(y)$ in equation (36) by showing that the second variation of the integral is non-negative, which is a form of Jacobi's condition. The second variation I" of the integral is

$$I'' = \int_0^\infty (F_{yy}\ \eta^2 + 2\ F_{yy'}\ \eta\eta' + F_{y'y'}\eta'^2)ds,$$

where $\eta = \eta(s)$ is an arbitrary variation from $y_1(s)$ that vanishes at the end points. With the F functional in equation (37) I" becomes

$$I'' = 4 \int_0^\infty [(1-f_1)(3-2y')\ e^{2y-2s}\ \eta^2 - (f_1 e^{-s} + 2(1-f_1)e^{2y-2s})\eta\eta']ds$$

Integrating by parts and using the property that $\eta(s)$ vanishes at 0 and ∞, we have

$$\int_0^\infty (1-y')e^{2y-2s}\ \eta^2\ ds = \int_0^\infty e^{2y-2s}\ \eta\eta'\ ds$$

When we use this equation to eliminate y', the second variation becomes

$$I'' = 4 \int_0^\infty [(1-f_1)e^{2y-2s}\ \eta^2 - f_1 e^{-s}\ \eta\eta']ds$$

Again, integrating $e^{-s}\eta\eta'$ by parts, we obtain

$$I'' = 2 \int_0^\infty [2(1-f_1)e^{2y-2s} - f_1\ e^{-s}]\ \eta^2 ds$$

For $y = y_1(s)$ in equation (41) I'' reduces to

$$I'' = 2f_1(u_1 + 2) \int_0^\infty e^{-s} \eta^2 \, ds,$$

for which $I'' \geq 0$ for all $\eta(s)$, with equality only for trivial $\eta(s)$.

With the negative root u_2 of equation (39), I'' is negative, since $u_2 < -2$. The corresponding function $y_2(s)$ maximizes the integral (35). Of course, it does not maximize the expected time to detection, since the minimum time T_2^* is used in finding $\overline{T}_1(y)$.

The function $y_1(s)$ in equation (41) is the unrestricted minimizing function. We obtain the optimal function $y^*(s)$ by imposing the restrictions $0 \leq y^*(s) \leq s$. The optimal solution depends on the value of f_1. If $f_1 < (1 + e^{u_1})^{-1} \doteq 0.358$,

$$y^* = \begin{cases} 0\ , & 0 < s \leq -u_1 + \ln(f_1^{-1}-1) \\ \frac{1}{2}(s + u_1 - \ln(f_1^{-1}-1)), & s > -u_1 + \ln(f_1^{-1}-1) \end{cases}$$

If $f_1 > (1 + e^{u_1})^{-1}$,

$$y^* = \begin{cases} s\ , & 0 < s \leq u_1 - \ln(f_1^{-1}-1) \\ \frac{1}{2}(s + u_1 - \ln(f_1^{-1}-1)), & s > u_1 - \ln(f_1^{-1}-1) \end{cases}$$

We note that $f_2 = (1 + e^{u_1})^{-1}$, which is less than 0.5 since $u_1 > 0$. This outcome had been anticipated above in obtaining the solution in the second stage. Also, we note that $s_0 = u_1$.

The complete solution is the following:

<u>First Stage</u>

For $0 < s \leq | u_1 - \ln(f_1^{-1}-1)|$,

$$\mu_1^* = \begin{cases} 0\ , & \text{for } x \text{ in I}, \\ s\ , & \text{for } x \text{ in II, if } f_1 < (1 + e^{u_1})^{-1} \end{cases}$$

$$\mu_1^* = \begin{cases} s , & \text{for } x \text{ in I,} \\ 0 , & \text{for } x \text{ in II, if } f_1 > (1 + e^{u_1})^{-1} \end{cases}$$

For $|u_1 - \ln(f_1^{-1}-1)| < s \leq s_1$

$$\mu_1^* = \begin{cases} \frac{1}{2}(s + u_1 - \ln(f_1^{-1}-1)) , & \text{for } x \text{ in I} \\ \frac{1}{2}(s - u_1 + \ln(f_1^{-1}-1)) , & \text{for } x \text{ in II} \end{cases}$$

If $f_1 = (1 + e^{u_1})^{-1}$, the solution is simply $\mu_1^* = \frac{1}{2}s$ for all x.

<u>Second Stage</u>

For $0 < s \leq u_1$, $\mu_2^* = \begin{cases} 0 , & \text{for } x \text{ in I} \\ s , & \text{for } x \text{ in II} \end{cases}$

For $s > u_1$, $\mu_2^* = \begin{cases} \frac{1}{2}(s - u_1), & \text{for } x \text{ in I} \\ \frac{1}{2}(s + u_1), & \text{for } x \text{ in II} \end{cases}$

In the second stage the total search time is $s_1 + s$ and the total effort density function is $\mu_1^*(s_1) + \mu_2^*(s)$.

We start searching in region II when f_1 is less than the critical value, 0.358, and in region I when f_1 is greater than the critical value. The critical value has been changed from 0.5 with no false targets to 0.358 by one false target in region II.

8. Additional Examples

We can use the above method to solve other two-cell problems. We have performed the essential part of the analysis for the cases in which there are 0 and 2 false targets in regions I and II, and 1 false target in each region.

With this background, we have attempted to solve the two-cell problem in which there is a known number n of identically distributed false targets with the common density function: $g(x) = g_1$ for x in I, $g(x) = 1 - g_1$ for x in II. We have written the solution for the $(n+1)^{st}$ stage, the n^{th} stage, and the essential part of the $(n-1)^{st}$ stage. The general form of the solution is the same as that obtained in example 1 above, with some complications. Thus, in the n^{th} stage we get an equation for $y_n(s)$ of the same form as that for $y_1(s)$ in equation (41), but u_1 (now u_n) is a root of the equation

$$f_1(1 - g_1)e^u(2e^u - 3 - u) = g_1(1 - f_1)(1 - 2e^u),$$

which reduces to equation (39) when $g_1 = 0$, as in example 1. It appears to be possible to write out the complete solution for this example. The solution will have the simple form of the solution in example 1; that is, we search in one region exclusively or we divide the effort evenly between the two regions; no other division of the effort is admitted in the optimal solution.

If the number of false targets is not known and the distribution is known and finite, the solution becomes more complicated. We have examined the case in which $p_0 \neq 0$, $p_1 \neq 0$, $p_i = 0$, $i \geq 2$. Again, the solution has the general form described above. It appears to be possible to write out the complete solution for the two-cell problem with a finite distribution.

We also have studied several simple examples for which the number distribution of false targets is the Poisson distribution, as assumed by Stone and Stanshine [1]. Although we do not know how to solve these problems, it is evident that the solution is not the function they obtain from their expected-value model, as shown by the following examples:

Example 2.

Modify the two-cell example 1 by assuming that the number of false targets is not known, that the false target density function $\delta(x)$ is

$$\delta(x) = 0 \text{ for x in I}, \ \delta(x) = 1 \text{ for x in II}$$

and the number distribution has the Poisson probability density function

$$p_k = e^{-1}/k! \ , \ k = 0, 1, 2, \ldots \qquad (42)$$

The expected number of false targets is 1.

The Stone and Stanshine function is as follows:

If $f_1 < 1/3$, start in region II.

$$\text{For } 0 < s \leq \ln\left(\frac{1-2f_1}{f_1}\right), \quad m^*_{SS} = \begin{cases} 0 \ , & \text{for x in I} \\ s \ , & \text{for x in II} \end{cases} \tag{43}$$

For $s > \ln\left(\frac{1-2f_1}{f_1}\right)$,

$$m^*_{SS} = \begin{cases} \ln\ [(f_1 + R(s,f_1))/2(1-f_1)] \ , & \text{for x in I} \\ \ln\ [(-f_1 + R(s,f_1))/2f_1] \ , & \text{for x in II} \end{cases} \tag{44}$$

where $R(s, f_1) = (f_1^{\ 2} + 4f_1(1 - f_1)e^s)^{1/2}$

If $f_1 > 1/3$, start in region I.

$$\text{For } 0 < s \leq \ln\left(\frac{2f_1}{1-f_1}\right), \quad m^*_{SS} = \begin{cases} s \ , & \text{for x in I} \\ 0 \ , & \text{for x in II} \end{cases} \tag{45}$$

For $s > \ln\left(\frac{2f_1}{1-f_1}\right)$, m^*_{SS} as in equation (44).

If $f_1 = 1/3$, divide the effort, as shown in (44), for $s > 0$.

We conjecture that the optimal solution does not contain functions of the form (44), and that it involves only the simple functions of the forms (43), (45), and (46) below, where c is independent of s.

$$m^* = \begin{cases} s/2 + c \ , & \text{for x in I} \\ s/2 - c \ , & \text{for x in II} \end{cases} \tag{46}$$

After a false target has been found, necessarily in region II, the residual density of false targets in region II is

$$\hat{\delta}^{(1)} = \frac{e^{-1}}{1-e^{-1}} = 0.582$$

The sudden decrease in δ from 1.0 before the first contact to 0.582 when the search is resumed requires that a search distribution of form (43) be used for an interval at the start of the second stage, to be followed by a distribution of form (46).

We conjecture that the optimal solution has the following form:

There is a critical value of f_1, say f_1^*. If $f_1 < f_1^*$, we start in region II with a distribution of form (43). If $f_1 > f_1^*$, we start in region I with a distribution of form (45). After a search interval $s^{(1)}$ that depends on f_1 (and is 0 for $f_1 = f_1^*$) we shift to a distribution of form (46). If the first contact is false, we start the second stage in region II with a distribution of form (43). After a search interval $s^{(2)}$ we shift to a distribution of form (46). We conjecture that $s^{(2)}$ will not depend on f_1, but may depend on the time s_1 at which the first contact was made. If a second false target is found, we start the third stage in region II again. After a search interval $s^{(3)}$, which does not depend on f_1 but may depend on s_2, we shift to an even division. After the $(j-1)^{st}$ false target is found, we search in region II for the interval $s^{(j)}$ before shifting to an even division. We conjecture that the sequence $s^{(2)}$, $s^{(3)}$, ... decreases steadily and has the limit zero.

Example 3.

For the two disjoint regions of our earlier examples let f(x) be

$$f(x) = \begin{cases} 0.5, & \text{for } x \text{ in I} \\ 0.5, & \text{for } x \text{ in II} \end{cases}$$

Assume that there are two false target distributions, with density functions as follows:

$$\delta_1(x) = \begin{cases} 1, & \text{for } x \text{ in I} \\ 0, & \text{for } x \text{ in II} \end{cases} \qquad \delta_2(x) = \begin{cases} 0, & \text{for } x \text{ in I} \\ 1, & \text{for } x \text{ in II} \end{cases}$$

Assume that each distribution type has the Poisson distribution (42), and that the assumptions (a) (b) (c) (d) made by Stone and Stanshine [1] on pages 243 and 244 are satisfied. In particular, assumption (c) states that, if B_1 and B_2 are disjoint regions (here regions I and II), the numbers $\Delta(B_1, s)$ and $\Delta(B_2, s)$ of false targets contacted in B_1 and B_2 in search time s are mutually independent random variables.

The Stone and Stanshine function for this example is

$$m^*_{SS} = \begin{cases} s/2 , & \text{for } x \text{ in I} \\ s/2 , & \text{for } x \text{ in II} \end{cases}$$

Obviously, this function is not the optimal solution in the second stage, after one false target has been found and eliminated. Suppose that the first false target is found in region 1. Then the total false target density function changes from the symmetrical case before the contact to the function

$$\hat{\delta}^{(1)}(x) = \begin{cases} 0.582 , & \text{for } x \text{ in I} \\ 1.0 , & \text{for } x \text{ in II} \end{cases}$$

when search is resumed. Since the target density function is the symmetrical function,

$$\hat{f}^{(1)}(x) = \begin{cases} 0.5 , & \text{for } x \text{ in I} \\ 0.5 , & \text{for } x \text{ in II,} \end{cases}$$

we start searching in region I for a non-zero search interval before returning to an equal division.

9. Probability Criterion

In section 6 above we outlined a general procedure for the solution of the problem of minimizing the expected time to find the target when the number n of false targets is known or has a known finite distribution. In section 7 we illustrated the procedure for a simple example. It is evident from the example that the solution for more general problems in which n is known or has a finite distribution will be complicated, even though solvable by the general method of section 6. From our discussion in section 8 of two simple problems in which n has a Poisson distribution it appears that the problem is much more difficult for distributions of n that are not finite.

Is the problem easier when we adopt the criterion (1) discussed in section 5 above, that of maximizing the probability of detection by a given time? We have studied this problem and have concluded that the probability formulation also leads to a difficult optimization problem, one that is likely to be approximately as difficult as, and perhaps more difficult than, the optimization problem obtained from the minimization of the expected time.

The key to the expected-time formulation is the simple sequence (25) that yields the time t_c to contact the target. Then we can minimize the expected value of the remaining time stage-by-stage. Similarly, we can write the probability of contacting the target in the time remaining when search is resumed in the j^{th} stage. It can be written as the sum of the probabilities of the following success modes:

(a) The target is contacted in the remaining time and this contact occurs before a false target has been contacted.

(b) A false target is contacted before the target has been contacted, and the target is then contacted in a later stage, provided there is enough time remaining after the contact occurs.

The probability of contacting the target in the later stages in mode (b) is assumed to be maximized. Mode (b) will not be possible when the time remaining is less than the investigation time. Thus, if the investigation time is a constant I, mode (b) can occur over a time interval that is I less than the time interval over which mode (a) can occur. The integrals for the probabilities that success will occur in modes (a) and (b) will not be over the same interval. This difficulty can be avoided by assuming that the investigation time has a distribution, such as the exponential, that includes $I = 0$.

Again, optimization is made stage-by-stage in the reverse direction. The starting point is not as definite as it was for the expected-time formulation. If the investigation time is constant, we can compute the maximum number of false contacts that could be investigated and thus obtain the number of the latest possible stage. In that stage success in mode (b) is impossible. We then write the probability for mode (a) alone, maximize it, and start the sequence. If the investigation time has an exponential distribution, we start with the condition that all false targets have been found and eliminated; hence, success is restricted to mode (a). Again, we maximize the probability of contacting the target in the remaining time, assuming no false targets; then go to the stage in which there is one false target; etc. This procedure is the same as that used in the expected-time formulation.

The above formulation was made for the problem in example 1 above, but not solved. The optimization problem appears to be at least as difficult as the optimization problem in example 1.

10. Some Alternatives

The probability criterion (1) and the expected-time criterion (2) lead to difficult optimization problems for finite distributions. For distributions that are not finite the problem appears to be difficult to formulate in such a way that we can attack it at all. Are there sufficiently good reasons for restricting attention to these two criteria, or should we consider others?

In addition to the difficulty of the optimization problem, the expected-time criterion contains the implied assumption that search will be continued until the target is found. Aside from the reluctance of the searcher to continue a negative search indefinitely, there also is the possibility in practice that no target is present in the region R. We seldom can be certain that a target is present under conditions in which we do not know its location. Our assumption that there is one and only one target is made for convenience.

For the probability criterion we assume that the total time, either to contact the target or to find (contact and identify) the target, is specified. In practice, the available time is not known precisely, and it may be changed by such factors as weather and conflicting demands on the forces involved. Our assumption that the available time is known precisely is made for convenience.

An alternative procedure is to look for "acceptable" criteria for which the optimization can proceed stage by stage in the forward direction, rather than the backwards direction required with criteria (1) and (2). One such possible criterion is the following:

(3) In each stage maximize the probability of contacting the target before contacting a false target.

With this criterion we automatically move to the next stage when failure occurs. In each stage the conditions depend on the search densities and outcomes for the previous stages, but the optimal search density does not depend on expectations in future stages.

An objection to criterion (3) is that it does not involve time directly, and, in particular, makes no allowance for variations of the investigation time with position x. Assume that the investigation time is a constant I. If the total time t that can be allotted to the operation is given, as in criterion (1), we simply discontinue the search when the remaining time is I or less at the completion of an investigation. If the total time that can be allotted is not specified, we assume that the stop-go decision is made (at a higher level) at the epochs at which the investigations of contacts are completed. Our problem then is to find the density function $\mu_j(x,s)$ for the j^{th} stage that maximizes the probability of contacting the target before contacting a false target.

We intend to study criterion (3), rather than continue with the probability and expected-time criteria. We believe that search plans based on criterion (3) will be adequate, and may have some advantages over those based on criterion (1) or criterion (2).

Another alternative is to look for conditions under which the model of Stone and Stanshine applies, or yields a good approximation. Can it be applied, or modified to apply, to the case in which the number distribution of false targets is not known but the location distribution is known?

Can it be modified to apply to the case in which false contacts are generated by noise, by other random events that affect the detecting instrument, or by moving real objects (such as whales) that can't be identified with certainty. Of course, the assumption that contact investigation is uninterrupted would have to be discarded.

REFERENCES

[1] L. D. Stone and J. A. Stanshine, "Optimal Search Using Uninterrupted Contact Investigation," SIAM J. Appl. Math. 20 (1971), pp. 241-263.