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**Intelligent Distributed Systems**

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| 14. ABSTRACT<br><p>We have shown that it is possible to improve the convergence rates for periodic gossiping algorithms by using convex combination rules rather than standard averaging rules. On a ring graph, we have discovered how to sequence the gossips within a period to achieve the best possible convergence rate and we have related this optimal value to the classic edge coloring problem in graph theory. We have developed an algorithm which solves the distributed averaging problem on tree graphs in finite time. We developed an asynchronous, distributed algorithm for solving a linear algebraic equation of the form <math>Ax = b</math> assuming that each processing agent knows a subset of the rows of of the partitioned matrix <math>[A \ b]</math>, current estimates of the solution generated by each of its current neighbors, and nothing more. Necessary and sufficient conditions are derived for all estimates to converge to the same solution. We have shown that the most general class of algorithms for maintaining a rigid formation in two dimension space will go into an unintended circular orbit a constant angular frequency if there is a mismatch in shared data. In three dimensions, such mismatches can cause a formation to exhibit an unintended helical motion. We have developed techniques to eliminate these behaviors.</p> |             |                         |   |   |   |
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The overriding objective of this project has been to develop new tools for analyzing distributed dynamical networks and for controlling them. More specifically, our work has focused on three interrelated objectives. First, we have sought to develop algorithms, exemplified by distributed averaging, for distributing information among the members of a group of mobile autonomous agents. Second we have focused on the crafting of asynchronous distributed algorithms for performing key computations across a network such as finding solutions to linear equations. Third we have studied how to use graph rigidity theory combined with nonlinear control theory to develop techniques for autonomously maintaining the correct relative positions of mobile autonomous agents or sensors in a large agent network. Maintaining correct positions is directly related to maintaining network connectivity, an issue of critical importance in almost any multi-agent network which is to be invulnerable to component failures or attack.

*Distributed averaging* is a special type of consensus seeking for which the goal is for all members of a group of autonomous agents to ultimately compute the average of the initial values of their consensus variables. Because distributed averaging is a special form of consensus seeking, the methods used to obtain a distributed average are more specialized than those needed to reach a consensus. There are three different approaches to distributed averaging: linear iterations, gossiping, and double linear iterations which are also known as push-sum algorithms. Linear iterations for distributed averaging can be modeled as a linear recursion equation in which the possibly time-varying update matrix must be doubly stochastic. The doubly stochastic matrix requirement cannot be satisfied without assuming that each agent knows an upper bound on the number of neighbors of each of its neighbors. Gossiping on the other hand, is a very widely studied approach to distributed averaging in which each agent is allowed to average its consensus variable with at most one other agent at each clock time. Gossiping protocols can lead to deadlock unless specific precautions are taken to insure that they do not and these precautions generally lead to fairly complex algorithms unless one is willing to accept probabilistic or periodic solutions. Push-sum algorithms are based on a quite clever idea proposed more than a decade ago but not fully developed. Such algorithms are somewhat more complicated than linear iterations, and generally require more data to be communicated between agents. They are however attractive because, at least for some implementations, the requirement that each agent know the number of neighbors of each of its neighbors can be avoided.

We have made significant progress in the area of distributed averaging. We have shown that the order in which individual gossips take place in a periodic gossiping process does not effect the rate of convergence if the underlying graph is a tree [26]. We have explained how to solve the distributed averaging problem on tree graphs in finite time [8]. We have demonstrated that it is possible to improve the rate of convergence in periodic gossiping algorithms by using convex combination rules rather than standard averaging rules [2]. For ring graphs, we have explained how to order the gossip in periodic gossiping sequences to achieve the best possible convergence rate and we have related this optimization problem to the classic edge coloring problem from graph theory [18]. We developed several request-based deterministic gossip protocols which are guaranteed not to deadlock [20]. Using a new concept, we derived two simple broadcast-free, asynchronous algorithms of the push-sum type which avoid deadlock and solve the distributed averaging problem [27]. In the area of consensus, we have used a special semi-norm developed in our earlier work, to derive necessary and sufficient conditions for exponential convergence of both discrete-time and continuous time linear consensus processes [10, 24].

We have made a quite important advance in the area of distributed computation by inventing

a distributed algorithm for solving a linear algebraic equation of the form  $Ax = b$  assuming the equation has at least one solution [3, 4, 15]. The equation is simultaneously solved by  $m$  agents assuming each agent knows only a subset of the rows of the partitioned matrix  $[A \ b]$ , the current estimates of the equation's solution generated by its neighbors, and nothing more. Each agent recursively updates its estimate by utilizing the current estimates generated by each of its neighbors. Neighbor relations are characterized by a time-dependent directed graph  $\mathbb{N}(t)$  whose vertices correspond to agents and whose arcs depict neighbor relations. We have shown that for any matrix  $A$  for which the equation has a solution and any sequence of "repeatedly jointly strongly connected graphs"  $\mathbb{N}(t)$ ,  $t = 1, 2, \dots$ , the algorithm causes all agents' estimates to converge exponentially fast to the same solution to  $Ax = b$ . We have also shown that, under mild assumptions, the neighbor graph sequence must actually be repeatedly jointly strongly connected if exponential convergence is to be assured. We've derived worst case convergence rate bound for the case when  $Ax = b$  has a unique solution. We have demonstrated that with minor modification, the algorithm can track the solution to  $Ax = b$ , even if  $A$  and  $b$  are changing with time, provided the rates of change of  $A$  and  $b$  are sufficiently small. We have explained how to modify the algorithm so that it can compute a least squares solution to  $Ax = b$  in a distributed manner, even if  $Ax = b$  does not have a solution. We have devised modifications of the algorithm to solve the least squares problem in an efficient manner [19, 22]. It turns out that this modified algorithm can also be used to solve certain sensor network localization problems. We have shown that exponential convergence to a solution occurs even if the times at which each agent updates its estimates are not synchronized with the update times of its neighbors [5, 23] and we have derived necessary and sufficient conditions for the associated discrete-time dynamical system to be exponentially stable.[11, 24] .

We have significantly broadened our findings in the area of formation control, focusing largely on "undirected rigid formations." By an *undirected rigid formation* of mobile autonomous agents is meant a formation based on "graph rigidity" in which each pair of "neighboring" agents  $i$  and  $j$  is responsible for maintaining the prescribed distance  $d_{ij}$  between them. Recent research by several different groups has led to the development of an elegant potential function based theory of formation control which provides gradient laws for asymptotically stabilizing a large class of rigid, undirected formations in two-dimensional space assuming all agents are described by kinematic point models. This particular methodology is perhaps the most comprehensive currently in existence for maintaining undirected formations based on graph rigidity. In our research we have been able to explain what happens if neighboring agents  $i$  and  $j$  using such gradient controls have slightly different understandings of what the desired distance  $d_{ij}$  between them is suppose to be . The question is relevant because no two positioning controls, whether they use a form of integral control or not, can be expected to move agents to precisely specified positions because of inevitable imprecision in the physical comparators used to compute the positioning errors. What one would hope for is a gradual distortion of the formation from its target shape as discrepancies in desired distances increase. While this is observed for the gradient laws in question, something else quite unexpected happens at the same time. In particular, we have been able to prove that any undirected rigid formation in the plane with mismatching target distances will, almost for certain, go into a circular limit cycle traversed at a constant angular velocity. [12, 17]. This elusive result, which has taken several years to fully develop, makes nontrivial use of rigidity theory and differential geometry. Recently we have been able to show the corresponding algorithm applied to formations in three dimensional space will, under the same conditions, cause such formations to exhibit unwanted helical motion [6, 14]. Our latest efforts along these lines has been to develop algorithms capable of eliminating these unwanted behaviors [7, 13, 28].

Here is a list of our specific accomplishments:

1. A probabilistic explanation was developed of why a previously proposed accelerated gossip algorithm works [1].
2. We were able to prove that the order in which individual gossips take place in a periodic gossiping process do not effect the rate of convergence if the underlying gossip graph is a tree [26].
3. We demonstrated the possible improvement in convergence rate in periodic gossiping algorithms when convex combination rules are used instead of standard averaging rules [2].
4. For ring graphs, we explain how to order the gossip in periodic gossiping sequences to achieve the best possible convergence rate and we relate this optimal value to the classic edge coloring problem [18]
5. We developed several request-based deterministic gossip protocols which are guaranteed not to deadlock [20].
6. Using a new concept, we derived two simple broadcast-free, asynchronous algorithms which avoid deadlock and solve the distributed averaging problem [27].
7. This paper presents an algorithm which solves the distributed averaging problem on tree graphs in finite time [8].
8. Using a suitably defined semi-norm, developed in our earlier work, we derive necessary and sufficient conditions for exponential convergence of both discrete-time and continuous time linear consensus processes [10, 24].
9. We develop a distributed algorithm for solving a linear algebraic equation [3, 4, 15].
10. The preceding algorithm for solving a linear algebraic equation is shown to be capable of function asynchronously [5, 23]
11. Necessary and sufficient conditions are derived for the linear algebraic equation solver just cited, to be an exponentially stable linear system [11, 24].
12. For the case of a fixed neighbor graph, a continuous-time gradient flow algorithm is developed for solving a linear algebraic equation [21].
13. By taking advantage of sparsity, an efficient algorithm is derived for solving a linear algebraic equation [19, 22].
14. Using the concept of a target point, a rigidity based algorithm is developed for maintain the positions of mobile autonomous agents for a large class of acyclic rigid graphs [16].
15. We have been able to show that the most general class of algorithms for maintaining a formation of mobile autonomous agents in two dimensions, will go into a stable limit cycle driven by a sinusoidal frequency if there is either a mismatch in shared data or a bias in at least one agent's measurements [12, 17].
16. The preceding finding are generalized to three dimensional space in [6, 14].

17. Using concepts from adaptive control, algorithms are developed to deal with the robustness issue just described [7, 13, 28]
18. In [9], an algorithm is developed which can cause a group of mobile autonomous agents to move into a rigid formation in finite time.

## References

- [1] J. Liu, B. D. O. Anderson, M. Cao, and A. S. Morse. Analysis of accelerated gossiping algorithms. *Automatica*, pages 873–883, 2013.
- [2] O. Mangoubi, S. Mou, J. Liu, and A. S. Morse. Towards optimal convex combination rules for gossiping. In *Proceedings of the 2013 American Control Conference*, pages 1263–1267, 2013.
- [3] S. Mou and A. S. Morse. A fixed-neighbor, distributed algorithm for solving a linear algebraic equation. In *Proceedings of the 2013 European Control Conference*, pages 2269 – 2273, 2013.
- [4] S. Mou, J. Liu, and A. S. Morse. An distributed algorithm for solving a linear algebraic equation. In *Proceedings of the 51st Annual Allerton Conference on Communication, Control, and Computation*, pages 267–274, 2013.
- [5] J. Liu, S. Mou, and A. S. Morse. An asynchronous distributed algorithm for solving a linear algebraic equation. In *Proceedings of the 2013 IEEE Conference on Decision and Control*, pages 5409–5414, 2013.
- [6] Z. Sun, S. Mou, B. D. O. Anderson, and A. S. Morse. Non-robustness of gradient control for 3-d undirected formations with distance mismatch. In *Proceedings of the 2013 Australian Control Conference*, pages 369–374, 2013.
- [7] S. Mou and A. S. Morse. Fix the non-robustness issue for a class of minimally rigid undirected formations. In *Proceedings of the 2014 American Control Conference*, pages 2348–2352, 2014.
- [8] S. Mou and A. S. Morse. Finite-time distributed averaging. In *Proceedings of the 2014 American Control Conference*, pages 5260–5263, 2014.
- [9] Z. Sun, S. Mou, M. Deghat, B. D. O. Anderson, and A. S. Morse. Finite time distance-based rigid formation stabilization and flocking. In *Proceedings of 2014 IFAC World Congress*, pages 9183–9189, 2014.
- [10] J. Liu, A. S. Morse, A. Nedić, and T. Başar. Internal stability of linear consensus processes. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, pages 922–927, 2014.
- [11] J. Liu, A. S. Morse, A. Nedić, and T. Başar. Stability of a distributed algorithm for solving linear algebraic equations. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, pages 3707–3712, 2014.
- [12] A. Belabbas, S. Mou, A. S. Morse, and B. D. O. Anderson. Undirected rigid formations are problematic. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, pages 637–642, 2014.

- [13] S. Mou, A. S. Morse, and B. D. O. Anderson. Toward the robust control of minimally rigid undirected formations. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, pages 643–647, 2014.
- [14] Z. Sun, S. Mou, B. D. O. Anderson, and A. S. Morse. Formation movements in minimally rigid formation control with mismatched mutual distances. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, pages 6161–6166, 2014.
- [15] S. Mou, J. Liu, and A. S. Morse. An distributed algorithm for solving a linear algebraic equation. *IEEE Transactiona on Automatic Control*, 2013. to appear.
- [16] S. Mou, M. Cao, and A. S. Morse. Target-point formation control. *Automatica*, 2014. to appear.
- [17] A. Belabbas, S. Mou, A. S. Morse, and B. D. O. Anderson. Undirected rigid formations are problematic. *IEEE Transactions on Automatic Control*, 2014. to appear.
- [18] S. Mou, A. S. Morse, and B. D. O. Anderson. Convergence rate of optimal periodic gossiping on ring graphs. In *Proceedings of the 54th IEEE Conference on Decision and Control*. to appear.
- [19] S. Mou, A. S. Morse, Z. Lin, Lili Wang, and D. Fullmer. A distributed algorithm for efficiently solving linear equations. In *Proceedings of the 54th IEEE Conference on Decision and Control*. to appear.
- [20] J. Liu, S. Mou, A. S. Morse, B. D. O. Anderson, and C. Yu. Request-based gossiping without deadlocks. *Automatica*, 2015. submitted.
- [21] B. D. O. Anderson, Shaoshuai Mou, A Stephen Morse, and Uwe Helmke. Decentralized gradient algorithm for solution of a linear equation. *Numerical Algebra, Control and Optimization*, 2015. submitted.
- [22] S. Mou, A. S. Morse, Z. Lin, Lili Wang, and D. Fullmer. A distributed algorithm for efficiently solving linear equations and its applications. *System and Control Letters*, 2015. submitted.
- [23] J. Liu, S. Mou, and A. S. Morse. An asynchronous distributed algorithm for solving a linear algebraic equation. *Automatica*, 2015. to be submitted.
- [24] J. Liu, A. S. Morse, A. Nedic, and T. Basar. Stability of a distributed algorithm for solving linear algebraic equations. *Automatica*, 2015. to be submitted.
- [25] J. Liu, A. S. Morse, A. Nedic, and T. Basar. Internal stability of linear consensus processes. *TAC*, 2015. to be submitted.
- [26] C. Yu, B. D. O. Anderson, S. Mou, J. Liu, F. He, and A. S. Morse. Convergence rate properties for average consensus via periodic gossiping. *IEEE Transactions on Control of Network Systems*, 2015. to be submitted.
- [27] J. Liu and A. S. Morse. Averaging asynchronously using double linear iterations. *Automatica*, 2015. to be submitted.
- [28] S. Mou, A. S. Morse, and B. D. O. Anderson. Toward robust control of undirected formations. in preparation.