



Distributed RF systems for close-in sensing and imaging

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UNIVERSITY OF ILLINOIS

04/01/2016
Final Report

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REPORT DOCUMENTATION PAGE				Form Approved OMB No. 0704-0188	
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1. REPORT DATE (DD-MM-YYYY) 30-03-2016		2. REPORT TYPE Final		3. DATES COVERED (From - To) 01-04-2012/31-12-2016	
4. TITLE AND SUBTITLE Distributed RF systems for close-in sensing and imaging				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER FA9550-12-1-0174	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S) Erricolo, Danilo				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) University of Illinois at Chicago Dept. of ECE 851 S. Morgan St Chicago, IL 60607				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) USAF, AFRL DUNS 143574726 AF Office of Scientific Research 875 North Randolph Street Arlington, Va 22203				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION / AVAILABILITY STATEMENT DISTRIBUTION A					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT The goal of this research is to improve close-in sensing and imaging applications through a new formulation of RF tomography, which is considered because of its flexibility and advantages when dealing with distributed sensors. This research is motivated by the removal of some limitations in the current formulation of RF tomography. One limitation is the first order Born approximation that is physically equivalent to neglecting multiple scattering phenomena. This limitation is overcome with the quadratic forward model. A second limitation is associated with the contrast function, which represents the unknown quantity to be reconstructed. In fact, while the initial formulations of RF Tomography have been based upon a scalar contrast function, it is possible to extract more information by using a dyadic contrast function, which takes advantage of the vector nature of both the incident and scattered electromagnetic field. A third limitation is the current existing inversion algorithms that do not allow for the introduction of prior knowledge to compute the solution of imaging problems, which is overcome with the introduction of physical bounds into iterative solution methods.					
15. SUBJECT TERMS Radio Frequency Tomography, Distributed Sensing,					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES 175	19a. NAME OF RESPONSIBLE PERSON Danilo Erricolo
a. REPORT	b. ABSTRACT	c. THIS PAGE			19b. TELEPHONE NUMBER (include area code) (312) 996 5771

Distributed RF systems for close-in sensing and imaging

Program Manager: Dr. Arje Nachman

Research Report for the period April 1, 2012 - December 31, 2015

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March 30, 2016

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Chapter 1

Summary

The goal of this research is to improve close-in sensing and imaging applications through a new formulation of RF tomography, which is considered because of its flexibility and advantages when dealing with distributed sensors. This research is motivated by the removal of some limitations in the current formulation of RF tomography. One limitation is the first order Born approximation that is physically equivalent to neglecting multiple scattering phenomena. Another limitation is associated with the contrast function, which represents the unknown quantity to be reconstructed. In fact, while the initial formulations of RF Tomography have been based upon a scalar contrast function, it is possible to extract more information by using a dyadic contrast function, which takes advantage of the vector nature of both the incident and scattered electromagnetic field. The other limitation is the current existing inversion algorithms that do not allow for the introduction of prior knowledge to compute the solution of imaging problems.

Therefore, a research plan consisting of the following 5 tasks has been executed to improve RF tomography.

1.1 Task 1: New formulation of RF tomography

Studies on a forward model that takes advantage of a dyadic contrast function to account for depolarization in the case of wire-shaped targets were presented in [1], [2], [3]. However, the inversion of the forward model requires handling a larger number of unknowns. The resulting contrast dyadic is decomposed into its eigenvectors, which indicate the orientation of the target. In addition, information about the target material can be extracted because the rank of the contrast dyadic matrix must be one for metallic targets while it has to be full for dielectric targets [4], [5]. Additional details are provided in Section 2.1.

An experimental validation of the quadratic forward model has been published in [6], extending the preliminary results obtained in [7, 8, 9, 10] to improve the Born approximation, which was introduced to simplify the scattering problem. In fact, by using an exact Neumann series representation of the scattered field, the Born approximation corresponds to the first term of the series. This approximation is valid when the scattered field is weak and is equivalent to neglecting scattering mechanisms with other objects. Additional details are provided in Section 2.2.

The effect of the choice of the Green's function used in the forward model has been investigated for the application to the study of the presence of cracks within reinforced concrete structures and preliminary results have been presented in [11, 12, 13, 14].

1.2 Task 2: Fast solvers for the forward model

The forward model of RF tomography uses a dyadic Green's function that depends on the geometry of the scene. Evaluating the dyadic Green's function is equivalent to finding a radiated field due to an infinitesimal dipole. In particular, the dyadic Green's function for a complex environment, where an analytical evaluation is not practical, can be found numerically by computing the radiated field due to a infinitesimal dipole in the desired environment. Research efforts were devoted to developing fast methods to compute electromagnetic fields and validation approaches.

Fast methods were focused on high-frequency ray-based techniques because they are fast, they are particularly advantageous for electrically large problems and they allow also to provide physical insights. Another advantage is that ray-methods afford one approach to construct Green's functions, which are fundamental for the forward model. Accordingly, first we completed the derivation of the double-diffraction mechanism for the Incremental Theory of Diffraction (ITD) [15], for which preliminary results were presented in [16]. Additional details are provided in Section 3.1.

Validation approaches were developed in a twofold way. First, new exact analytical solutions that serve as benchmarks for comparison were developed. Specifically, the scattering from simple shapes consisting of elliptical [17, 18, 19, 20, 21] or spheroidal [22, 23, 24, 25, 26, 27, 28] objects filled with metamaterials was investigated. Additional details for one sample geometry are provided in Section 3.2. A byproduct of the investigations on the scattering from elliptical shapes was Algorithm 934, the first software publicly available to compute Mathieu functions when a parameter is a complex variable [29, 30]. Additional details are provided in Section 3.3.

A byproduct of the investigations on the scattering from spheroidal shapes filled with metamaterials was a symmetry property of the spheroidal functions [31]. Additional details are provided in Section

1.3 Task 3: Physics-based inversion schemes

Alternative approaches to the classical Truncated Singular Value Decomposition (TSVD) method to invert the linear matrix equation that links the (measured) scattered field to the unknown contrast function have been investigated due to two main factors. One factor is obtaining sharper images because the transition between the background material and the object is too smooth with TSVD, which is not physical. The other factor is to introduce physical bounds to enforce that the returned values of the contrast function correspond to what is possible to find in nature. In fact, TSVD does not allow to enforce such condition. In particular, we focused on iterative inversion methods such as the conjugate gradient method (CG) and the Algebraic Reconstruction Technique which are faster to invert ill-conditioned matrix and simple to manipulate rational physical constraints. The comparison of TSVD, CG and ART has been discussed in [32, 33, 34]. TSVD, CG and ART provide equivalent results in general except that CG and ART do not require large memory usage and execution time. Furthermore, ART has resulted high resolution when physical constraints are introduced. In addition, dielectric and metallic targets can be distinguished by value of contrast function [4, 35]. Additional details are provided in Chapter 4 of this report.

1.4 Task 4: System implementation

Various aspects of system implementation issues have been investigated. Specifically, irregular terrains have been discussed in [36], the use of magnetic sources was presented in [37], a dyadic target model was introduced in [38], sidelobe suppression was discussed in [39] and a comprehensive forward model that also takes into account irregular terrain was published in [40]. A book chapter that describes a possible application of RF Tomography for homeland security and surveillance was also published in [41].

Related to the problem of distributed sensing, we considered the exploitation of prior information about the environment with the purpose of predicting multipath and using this information to obtain better models of the received signal to improve, for example, the probability of detection of a target. Our results have been published in [42], preliminary results were presented in conferences [43, 44, 45, 46], and additional details are given in Chapter 5 of this report. In addition, we also contributed to another research related to the exploitation of multipath in [47, 45].

Moreover, the coexistence of the spectrum of radar and communication systems has been addressed in [48], [49], [50], [51], [52].

1.5 Task 5: Experimental validation

The budget included funds for the purchase of a vector network analyzer, which was purchased at the beginning of the funding period. As part of this task, various measurements in the anechoic room of the Andrew

Electromagnetics Laboratory at the University of Illinois at Chicago were carried out. The experiments focused on the verification of various aspects of RF Tomography techniques.

First, we investigated which approaches provide better results to reconstruct dielectric and metallic targets in free space [32, 5, 53, 33, 4, 54]. We investigated various reconstruction algorithms, such as truncated singular values decomposition (TSVD), conjugate gradient (CG), Algebraic Reconstruction Technique (ART) and the least squares by Lanczos bidiagonalization (LSQR). In general, iterative methods such as CG and ART, rather than classical inversion techniques such as TSVD, are better for both accuracy and speed. In addition, CG and ART are easy to introduce rational physical bounds which helps to reconstruct better contrast. We found that in CG is the most reliable, while ART provides the sharper images.

Second, we proceeded the experiments to non-trivial scenario, which involves buried targets and targets at non line of sight [55, 56, 57, 58]. In general, the essential limitation is to find Green's function as explained in task 2. However, we could find Green's function of not large geometry with Method of Moment. We could identify the location of buried targets in sand or gravel in a box as preliminary experimental demonstration for ground penetrating radar. Details will be provided in next report.

Third, laboratory experiments were carried out to verify the recently developed triple diffraction mechanism for the Uniform Theory of Diffraction [59, 60, 61], additional details are provided in Section 6.3.

1.6 Broader impacts

1.6.1 Training and Education of graduate and undergraduate students

Two Ph.D. students were involved in the theoretical and experimental research activities. One Ph.D. student graduated and now is employed by Schlumberger, while the other Ph.D. student completed his Ph.D. preliminary defense in Dec. 2015. In addition, all the experimental activities performed at UIC involved one visiting scholar and 19 undergraduate students, who were supported by funds internal to UIC.

1.6.2 Distribution of research results

The research activities supported by this program have results in 11 journal publications [6, 15, 17, 29, 35, 36, 37, 40, 42, 47, 59], two in preparation [31, 1], 47 conference presentations [2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 32, 33, 34, 38, 39, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 60, 61] and one book chapter [41]. In addition, as part of the broader impact of the research supported by this grant, special sessions on distributed radar systems and underground sensing have been organized/co-organized at major international scientific conferences. Specifically,

- Dr. Erricolo, as General Chair of the IEEE International Symposium on Antennas and Propagation/USNC-URSI National Radio Science Meeting in Chicago, IL, July 8-14, 2012, invited Lorenzo Lo Monte and Gary Scalzi, both from AFRL, to organize a special session on "Future Trends in Radar."
- Dr. Erricolo organized an invited session on "Electromagnetic sensing for underground applications" at the URSI Commission B International Symposium on Electromagnetic Theory, Hiroshima, Japan, May 20-24, 2013.
- Dr. Erricolo organized the special session on "Distributed Radar Sensing" at the International Conference on Electromagnetics in Advanced Applications (ICEAA), Torino, Italy, Sept. 9-13, 2013
- Dr. Erricolo co-organized a special session on "Synergistic approaches for novel radar systems: electromagnetism, signal processing, and communications" at the International Conference on Electromagnetism in Advanced Applications and IEEE-APS Topical Conference on Antennas and Propagation in Wireless Communications, Palm Beach, Aruba, Aug. 3-9, 2014
- Dr. Erricolo co-organized a special session on "Electromagnetic Modeling and Applications for Underground Imaging" at the XXXI URSI General Assembly and Scientific Symposium, Aug 16-23, 2014, Beijing, China.

1.7 Honors

Dr. Erricolo, was elevated to Fellow of IEEE, with effective date of Jan. 1, 2016.

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Chapter 2

Task 1: New formulation of RF tomography

Preliminary studies on an anisotropic forward model using a dyadic contrast function has been developed in [3]. This new forward model accounts for depolarization in the case of wire-shaped targets. However, the inversion of the forward model increases the number of unknowns. The resulting contrast dyadic is decomposed into its eigenvectors, which indicate the orientation of the target. In addition, information about the target material can be extracted because the rank of the contrast dyadic matrix must be one for metallic targets while it has to be full for dielectric targets. Additional details are provided in Section 2.1.

A quadratic forward model has been researched in [10, 9] to improve on the Born approximation, which is introduced to simplify the scattering problem. In fact, by using an exact Neumann series representation of the scattered field, the Born approximation corresponds to the first term of the series. This approximation is valid when the scattered field is weak and is equivalent to neglecting scattering mechanisms with other objects. Additional terms of the series account for multiple scattering mechanisms. The quadratic model uses the first two terms of the Neumann series so that the electric field in the region of investigation partially accounts for multiple scattered fields. Additional details are provided in Section 2.2.

2.1 A dyadic contrast function for RF Tomography

A dyadic contrast function in the RF Tomography forward model is introduced by applying it to the scattering from thin elongated objects and describing how there exists a fundamental difference between the scattering originated by metallic versus non-metallic targets. The dyadic contrast function thus introduced allows extracting information previously unavailable without changing the way data are collected. Therefore, in principle, the new dyadic contrast function could be applied to data already available.

RF Tomography generates images of the contrast function, i.e. the difference between the dielectric permittivity constant of the background medium and the dielectric permittivity of the target. The space is partitioned in voxels (or pixels) and each voxel is assigned a (complex) scalar numerical value. An image is obtained by mapping these values to a color scale. The information returned is therefore scalar and the shape of the target is then inferred by simple visual inspection of the resulting image. It would be interesting to have the possibility of reconstructing the shape of an object in terms of actual 3D vectors, pointing in space along a direction corresponding to the orientation of the edges of the target. This would allow to obtain a “wire frame” model of the target, and such wire frame model would be more informative than a simple color plot. Another added benefit would be that this supplemental information could augment the colored image particularly in the presence of clutter, or when limits in the system resolution do not allow to get a good idea of the shape of the target.

By analyzing the scattering from thin and elongated objects, one notices that there exists a fundamental difference between scattering originated by metallic versus non-metallic targets. An analytic explanation of this phenomenon can be given using a modal analysis. This finding motivates a change in the RF Tomography forward model so as to include a dyadic (second-order tensor) contrast function. This new contrast function

would be able to represent preferred scattering directions and therefore orientation of objects.

Interestingly, a parallel can be drawn between this approach and Diffusion Tensor Imaging (DTI), an imaging technique based on Magnetic Resonance, used to obtain 3D maps of the structure of the brain of a subject [62, 63, 64].

2.1.1 De-polarization of the scattered field

Consider a canonical scattering problem, depicted in Fig. 2.1.

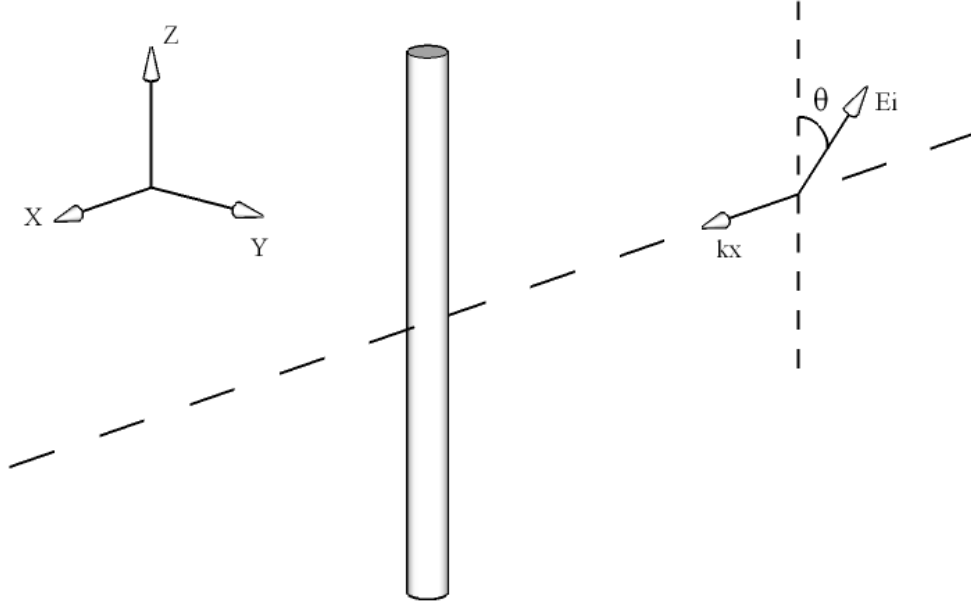


Figure 2.1: Coordinate system and scattering problem representation.

A plane wave is impinging upon a cylinder of negligible radius ($r \ll \lambda$) and infinite length. The cylinder is aligned with the z axis. The incident plane wave propagates along the x axis and is linearly polarized in the zy plane, with an angle θ with respect to the z axis. The problem can be decomposed as the linear superposition of two orthogonal components: one corresponds to the part of the incident field parallel to the cylinder, the other corresponds to its perpendicular counterpart. In practice the problem is decomposed into a TM^z and a TE^z mode, which can be studied separately. In addition, the analysis is further divided into the cases of Perfect Electric Conductor (PEC) and dielectric (non-conductive) cylinder. Although this derivation is a classical EM procedure, it serves to provide a rigorous analytical explanation for the more extensive simulation results described later on.

Modal analysis: thin PEC cylinder

When the cylinder is made of Perfect Electric Conductor, the analysis is based on a classical Mode Matching technique which is reported here almost in its entirety, to better understand its consequences [65, 66, 67].

TM mode The TM mode occurs when the incident field is oriented along the z axis, traveling in the positive x direction with wave number $k = 2\pi/\lambda$. The incident field therefore is:

$$\mathbf{E}^i = E_0 e^{-jkx} \hat{\mathbf{z}}. \quad (2.1)$$

This can be expanded in cylindrical coordinates in order to facilitate the imposition of the boundary conditions:

$$\mathbf{E}^i = \hat{\mathbf{z}}E_0 \sum_{n=-\infty}^{+\infty} j^{-n} J_n(k\rho) e^{jn\phi}, \quad (2.2)$$

where J_n represents a Bessel function of the first kind of order n . This wave impinges upon the cylinder and generates a scattered wave. The scattered wave is assumed of the same polarization of the incident field and it travels outwards. Therefore¹

$$\mathbf{E}^s = \hat{\mathbf{z}}E_0 \sum_{n=-\infty}^{+\infty} c_n H_n^{(2)}(k\rho). \quad (2.3)$$

The application of the boundary conditions on the surface of the cylinder allows to determine the value of the coefficients c_n . Since the object is made of PEC, the total tangential electric field on its surface (of radius a) must be identically zero:

$$\begin{aligned} \mathbf{E}^t &= \mathbf{E}^i + \mathbf{E}^s = 0|_{\rho=a} \\ \Rightarrow \hat{\mathbf{z}}E_0 \sum_{n=-\infty}^{+\infty} \left[j^{-n} J_n(ka) e^{jn\phi} + c_n H_n^{(2)}(ka) \right] &= 0, \quad \forall \phi, z. \end{aligned} \quad (2.4)$$

It immediately follows that:

$$c_n = -j^{-n} \frac{J_n(ka) e^{jn\phi}}{H_n^{(2)}(ka)}. \quad (2.5)$$

In conclusion:

$$\mathbf{E}^s = \hat{\mathbf{z}}E_0 \sum_{n=-\infty}^{+\infty} -j^{-n} \frac{J_n(ka) e^{jn\phi}}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho). \quad (2.6)$$

The current impressed on the surface of the cylinder by this wave can be computed by finding the total magnetic field on the surface of the cylinder first:

$$\mathbf{H}^t = -\frac{1}{j\omega\mu_0} \nabla \times \mathbf{E}^s. \quad (2.7)$$

Since the total electric field only has components in the z direction, the total magnetic field will have components in the ρ and ϕ directions only. However, only the tangential ϕ component will contribute to the creation of the induced current density, since:

$$\mathbf{J}_{TM} = \hat{\mathbf{n}} \times \mathbf{H}^t \quad (2.8)$$

$$= \hat{\boldsymbol{\rho}} \times (\hat{\boldsymbol{\rho}} H_\rho^t + \hat{\boldsymbol{\phi}} H_\phi^t) \quad (2.9)$$

$$= \hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} H_\phi^t, \quad \rho = a \quad (2.10)$$

$$= \hat{\mathbf{z}} H_\phi^t, \quad \rho = a. \quad (2.11)$$

The resulting impressed current is parallel to both incident and scattered electric fields. After some algebra:

$$\mathbf{J}_{TM} = \hat{\mathbf{z}} \frac{2E_0}{a\pi\omega\mu_0} \sum_{n=-\infty}^{+\infty} j^{-n} \frac{e^{jn\phi}}{H_n^{(2)}(ka)}. \quad (2.12)$$

¹The notation is important. Here the H with the superscript (2) indicates the Bessel function of the second kind, $H_i^{(2)}(x) = J_i(x) + jY_i(x)$ and should not be confused with the magnetic field $\mathbf{H}$.

For a given frequency and radius of the cylinder, the magnitude of the current depends solely on the specific position around the cylinder, i.e. ϕ . In addition, if the cylinder is very small the dominant term of the summation is n equal to zero. The small radius approximation leads to:

$$\mathbf{J}_{TM} \approx \hat{\mathbf{z}} \frac{2E_0}{a\pi\omega\mu_0 H_0^{(2)}(ka)}. \quad (2.13)$$

As a result (2.13) has a constant value for a given radius and frequency.

TE mode The TE mode corresponds to the portion of the incident field that is oriented along the y direction, traveling in the positive x direction with wave number $k = 2\pi/\lambda$. The derivation is very similar to the TM case, but with converse electric and magnetic fields. Briefly, the incident magnetic field is:

$$\mathbf{H}^i = H_0 e^{-jkx} \hat{\mathbf{z}} \quad (2.14)$$

$$= \hat{\mathbf{z}} H_0 \sum_{n=-\infty}^{+\infty} j^{-n} J_n(k\rho) e^{jn\phi}, \quad (2.15)$$

which causes a scattered magnetic field:

$$\mathbf{H}^s = \hat{\mathbf{z}} H_0 \sum_{n=-\infty}^{+\infty} c_n H_n^{(2)}(k\rho). \quad (2.16)$$

Application of the boundary conditions to the tangential electric field (thus only E_ϕ^s component, with $\mathbf{E}^s = \frac{1}{j\omega\epsilon} \nabla \times \mathbf{H}^s$) leads to:

$$c_n = -j^{-n} \frac{J'_n(ka)}{H_n^{(2)}(ka)} e^{jn\phi}, \quad (2.17)$$

with

$$J'_n(ka) = \frac{\partial J_n(k\rho)}{\partial k\rho}, \quad \rho = a \quad (2.18)$$

$$H_n^{(2)}(ka) = \frac{\partial H_n^{(2)}(k\rho)}{\partial k\rho}, \quad \rho = a. \quad (2.19)$$

Therefore the total magnetic field on the surface of the cylinder will be:

$$\mathbf{H}^t|_{\rho=a} = \mathbf{H}^i + \mathbf{H}^s \quad (2.20)$$

$$= -\hat{\mathbf{z}} j \frac{2H_0}{\pi ka} \sum_{n=-\infty}^{+\infty} j^{-n} \frac{1}{H_n^{(2)}(ka)} e^{jn\phi}. \quad (2.21)$$

Finally, the impressed current on the surface of the cylinder is:

$$\mathbf{J}_{TE} = \hat{\mathbf{n}} \times \mathbf{H}^t \quad (2.22)$$

$$= \hat{\boldsymbol{\rho}} \times \hat{\mathbf{z}} H_z^t \quad (2.23)$$

$$= -\hat{\phi} H_z^t, \quad \rho = a \quad (2.24)$$

$$= \hat{\phi} \frac{2H_0}{\pi ka} \sum_{n=-\infty}^{+\infty} j^{-n} \frac{1}{H_n^{(2)}(ka)} e^{jn\phi}. \quad (2.25)$$

Unlike the TM case, (2.25) has non-negligible terms when $n = -1, 0, +1$. As a result, even in the case of small radius, the impressed current will depend on the position around the cylinder, i.e. ϕ . In any case, (2.25) can be rewritten as

$$\mathbf{J}_{TE} \approx \hat{\phi} \frac{2H_0}{\pi ka} \sum_{n=-1}^{+1} j^{-n} \frac{1}{H_n^{(2)}(ka)} e^{jn\phi}, \quad (2.26)$$

which does not involve an infinite summation and is easier to compute.

Comparison A direct comparison of (2.13) and (2.26) allows understanding that there is a significant difference between the two. From Fig. 2.2 it is clear that the magnitude of the impressed current in the

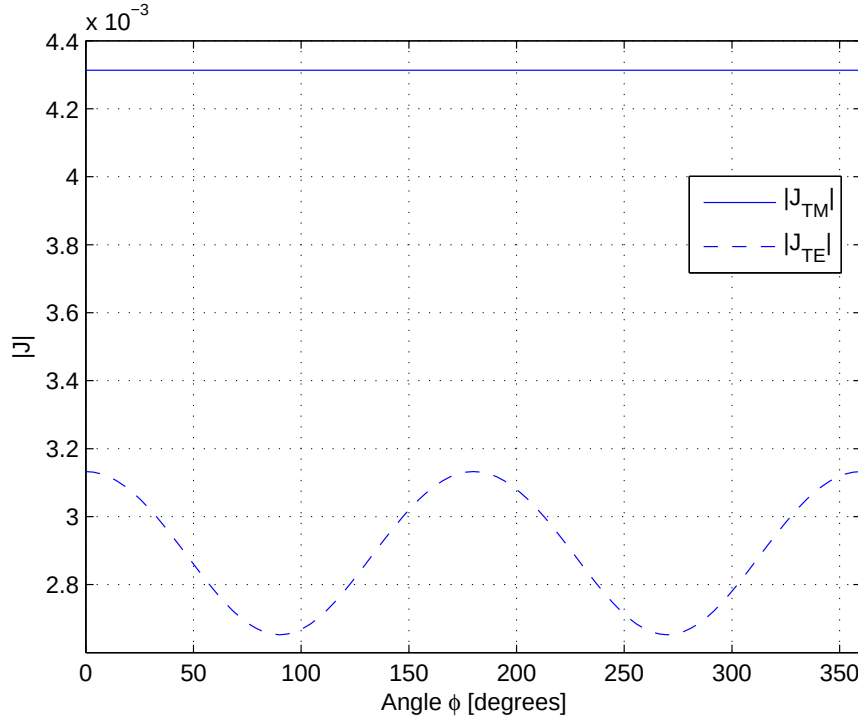


Figure 2.2: Comparison of the magnitude of the impressed currents for the TM and TE cases.

TM case is larger than the magnitude of the impressed current in the TE case. Therefore the TM case is dominant. In turn, this means that when the plane wave impinges on the thin cylinder at an angle with respect to the z axis, the impressed current will be mostly aligned with the thin cylinder itself. As a consequence, the scattered field will also be mostly aligned with the thin cylinder and create a *depolarization* effect.

Modal analysis: thin dielectric cylinder

The same procedure used for a PEC cylinder can be applied to a dielectric cylinder. Also this case is a classical EM problem, which has been studied in the past [68]. The derivation follows the same principles of the previous case, but the result is much simpler.

For the TM mode the total electric field inside the cylinder is oriented in the z direction and is equal to²

$$\mathbf{E}^t = \hat{\mathbf{z}} E_0 \sum_{n=-\infty}^{+\infty} a_n j^{-n} J_n(k_d \rho) e^{jn\phi}, \quad \rho \leq a, \quad (2.27)$$

where the coefficients a_n are equal to:

$$a_n = j^{-n} \frac{J_n(k_0 a) H_n^{(2)'}(k_0 a) - J_n'(k_0 a) H_n^{(2)}(k_0 a)}{J_n(k_d a) H_n^{(2)'}(k_0 a) - \sqrt{\epsilon_r / \mu_r} J_n'(k_d a) H_n^{(2)}(k_0 a)}. \quad (2.28)$$

For the TE mode the total magnetic field is equal to

$$\mathbf{H}^t = \hat{\mathbf{z}} H_0 \sum_{n=-\infty}^{+\infty} b_n j^{-n} J_n(k_d \rho) e^{jn\phi}, \quad \rho \leq a, \quad (2.29)$$

²Everywhere the subscript 0 indicates free space and the subscript d indicates the dielectric inside the cylinder

where the coefficients b_n are equal to:

$$b_n = j^{-n} \frac{J_n(k_0 a) H_n'^{(2)}(k_0 a) - J_n'(k_0 a) H_n^{(2)}(k_0 a)}{J_n(k_d a) H_n'^{(2)}(k_0 a) - \sqrt{\mu_r/\varepsilon_r} J_n'(k_d a) H_n^{(2)}(k_0 a)}. \quad (2.30)$$

Comparing (2.27) with (2.29) and (2.28) with (2.30) it can be seen that they are remarkably similar. For a weak scatterer ε_r is small, so the coefficients a_n and b_n will be almost identical. Also, only the coefficients with $n = -1, 0, 1$ will have a significant magnitude for cylinders of small radius. A plot of the coefficients a_n and b_n is shown in Fig. 2.3

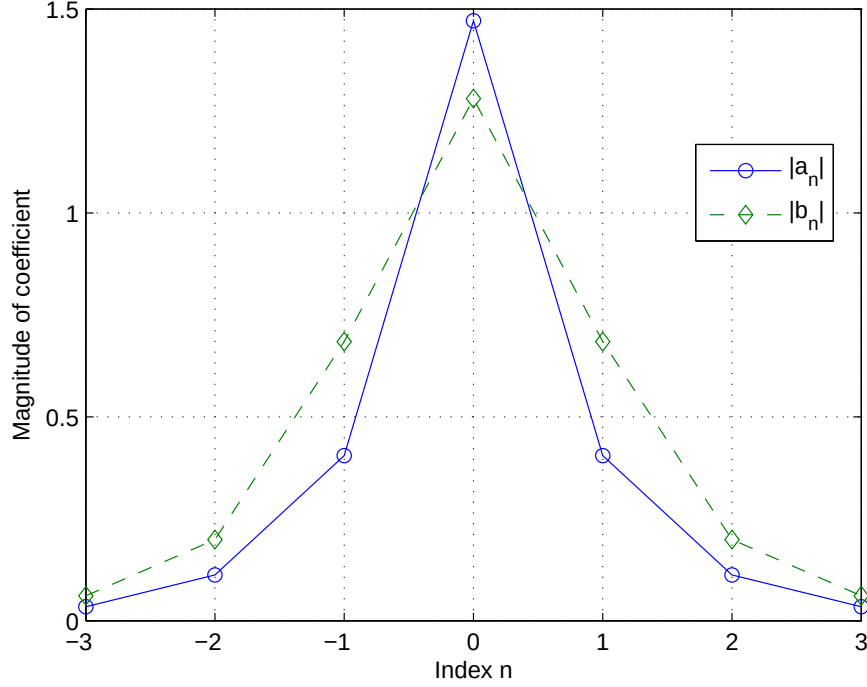


Figure 2.3: Magnitude of coefficients a_n and b_n of 2.28 and 2.30 as a function of the mode index n .

As a result, the impressed currents for the TM and TE cases will be very similar in magnitude, unlike what happened for a PEC cylinder. In the case of oblique incidence, the overall impressed current will be a combination of the TE and TM case, but since their magnitude is similar, the impressed current will have almost the same direction of the incident field. The weaker the scatterer, the more correct is this conclusion.

Therefore, it exists a fundamental difference between scattering from PEC and dielectric objects, which affects greatly the scattered field, both in magnitude and, this is the main point, in its orientation in space. These simple analytical results can be generalized to encompass a much wider variety of cases, using numerical simulations.

Method of Moments simulation

A Method of Moments simulation allows to analyze cases that are not as simple as the ones for which an analytic description could be derived.

First, consider a $\hat{z}$ -polarized plane wave impinging upon a thin cylinder. The cylinder is located in the zy -plane and forms a 45 degrees angle with the $\hat{z}$ axis. The radius of the cylinder is equal to $\lambda/50$ and its length is equal to λ . The distribution of the currents impressed on the surface of the cylinder, for both dielectric and PEC case, is computed with a Method of Moments simulation.

Fig. 2.4 shows the distribution of the currents on a PEC cylinder, while Fig. 2.5 shows the distribution of the impressed currents on a dielectric cylinder with $\varepsilon_r = 5$. For the PEC case the currents are always mostly

aligned with the thin cylinder, while for the dielectric case the currents are always mostly aligned with the incident field.

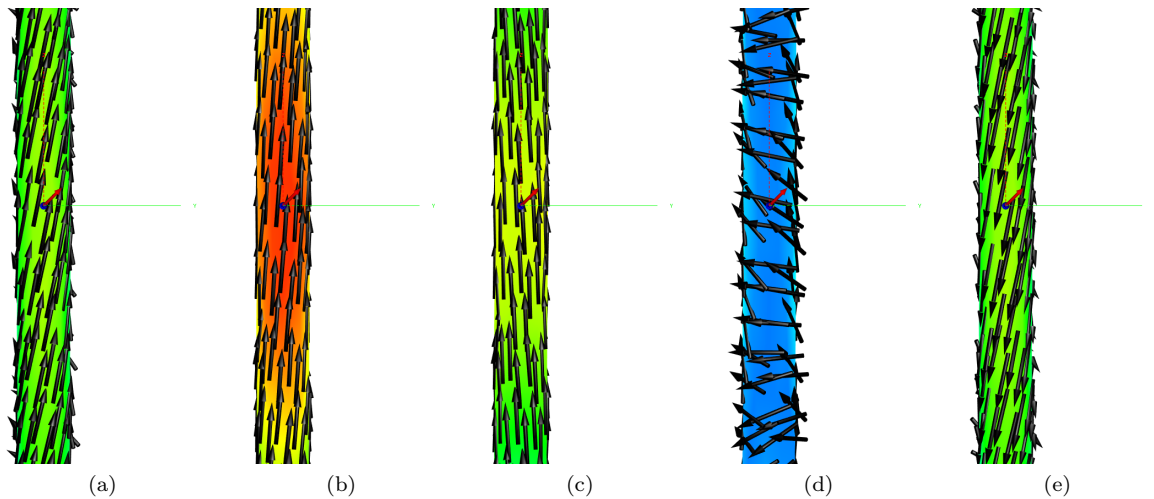


Figure 2.4: Impressed currents (black arrows) on a PEC cylinder of radius $\lambda/50$ and length $\lambda/2$. The incident field is impinging at a 45° angle with respect to the z axis. The figure shows the instantaneous distribution at phase ωt equal to (a) 0° , (b) 45° , (c) 90° , (d) 135° and (e) 180° .

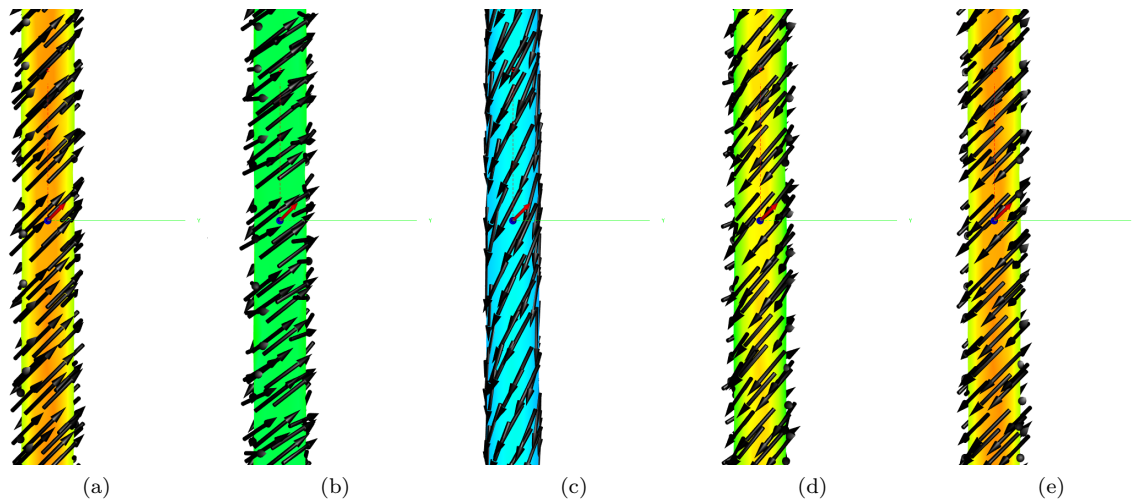


Figure 2.5: Impressed currents (black arrows) on a dielectric cylinder ($\epsilon_r = 5$) of radius $\lambda/50$ and length $\lambda/2$. The incident field is impinging at a 45° angle with respect to the z axis. The figure shows the instantaneous distribution at phase ωt equal to (a) 0° , (b) 45° , (c) 90° , (d) 135° and (e) 180° .

A similar effect can be observed when plotting the far field radiation pattern of the scattered-only part of the electric field. This is shown in Fig. 2.6. For a metallic cylinder the scattering pattern is tilted toward the main axis of the cylinder; instead, for the dielectric object the pattern is tilted toward the direction of the incident field.

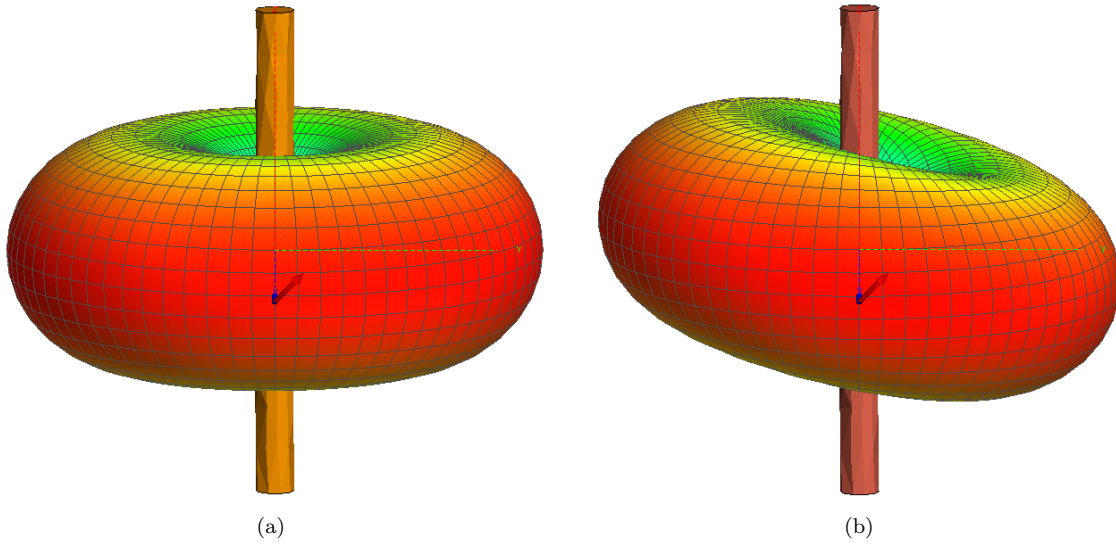


Figure 2.6: Far field radiation pattern of the scattered only part of the electric field, for a metallic (a) and dielectric (b) cylinders placed at a 45° angle with respect to the polarization of an incident plane wave.

The same effect can be observed in the near field distribution of the scattered only part of the field, as shown in Fig. 2.7.

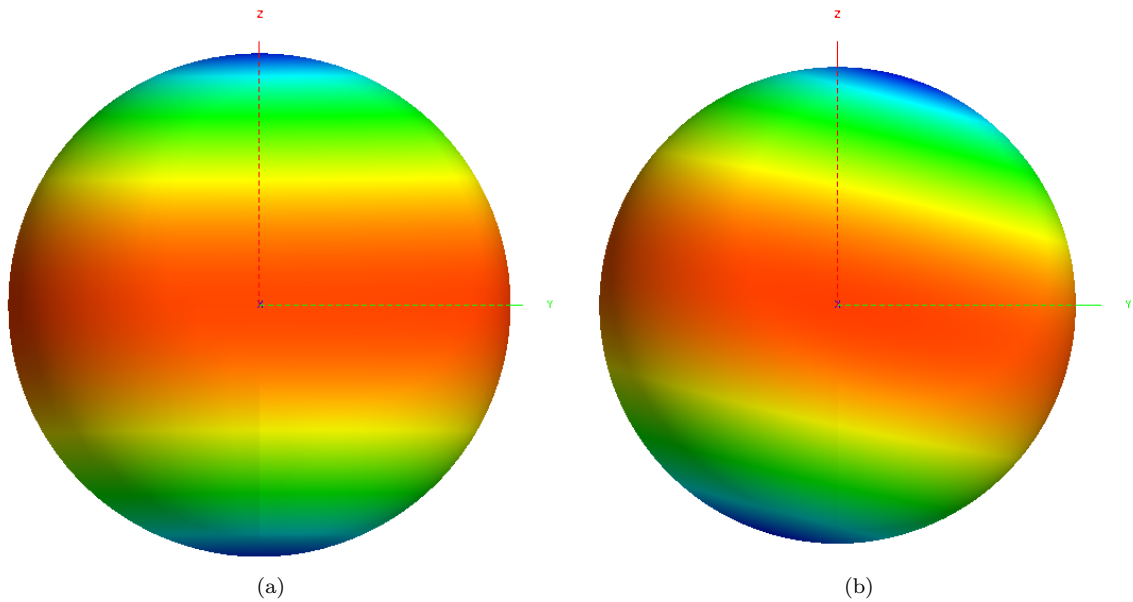


Figure 2.7: Near field radiation pattern of scattered only part of the electric field, for a metallic (a) and dielectric (b) cylinders placed at a 45° angle with respect to the polarization of an incident plane wave, at a distance equal to one wavelength from the center of the cylinder.

Finally, it can be observed that the de-polarization effect occurs also for a generic shape composed of thin cylinders. Fig. 2.9a shows the instantaneous currents impressed on the structure made of PEC at time

instants $\omega t = 0^\circ$, while 2.9b shows the currents at $\omega t = 90^\circ$. A similar result is obtained at all other time instants. The currents are aligned with the cylinders, regardless of their orientation with respect to the incident field.

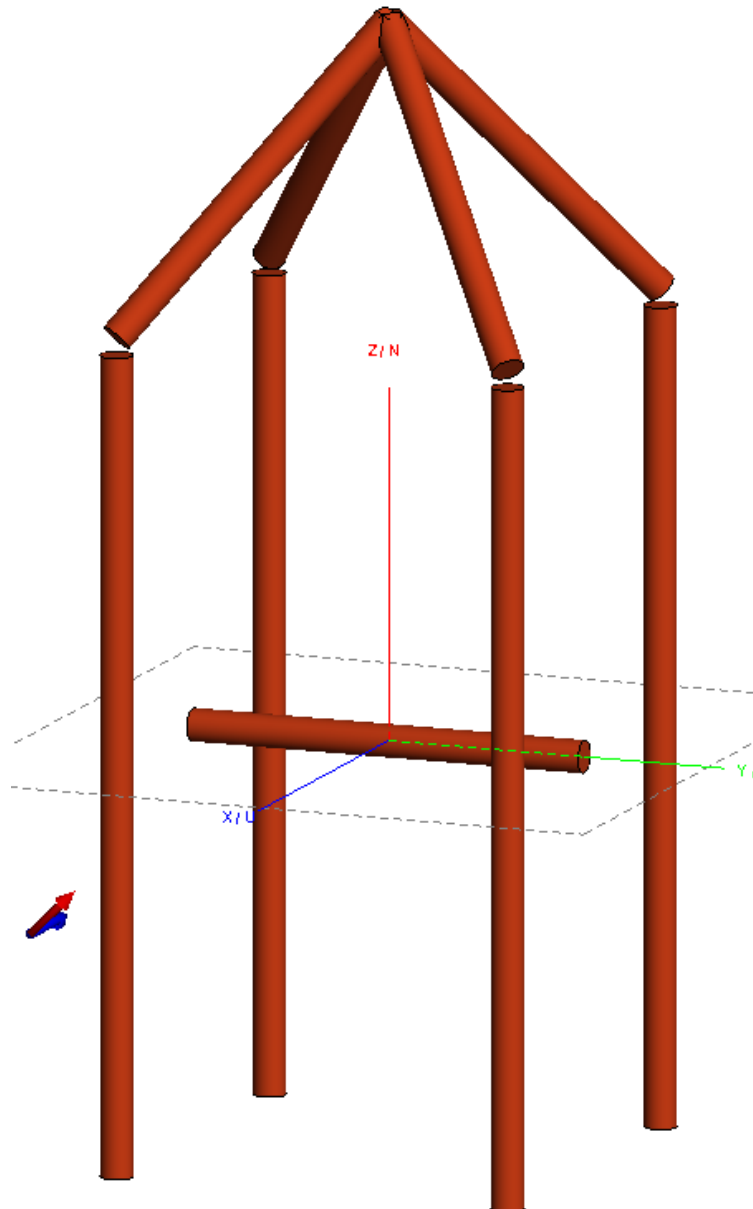


Figure 2.8: Geometrical shape composed of thin cylinders. The incident field is indicated by the red arrow and is impinging upon the structure at an angle of 45° with respect to the vertical z axis.

Furthermore, simulation results seem to suggest that the thin elements composing a complex structure do not interfere very much with each other, even when in close proximity. The current distributions on each thin cylinder is basically independent from the distribution on its neighbor. In the structure, the elements on top are in contact, thus currents are free to flow: this seems to have an effect on the current distribution only in a region extremely close to the point of contact itself.

Instead, Fig. 2.10a shows the instantaneous currents impressed on the structure made of a dielectric material ($\epsilon_r = 5$) at time instants $\omega t = 45^\circ$, while Fig. 2.10b shows the currents at $\omega t = 135^\circ$. In this case the currents are aligned with the incident field, regardless of the orientation of the pieces of the structure with respect to the incident field.

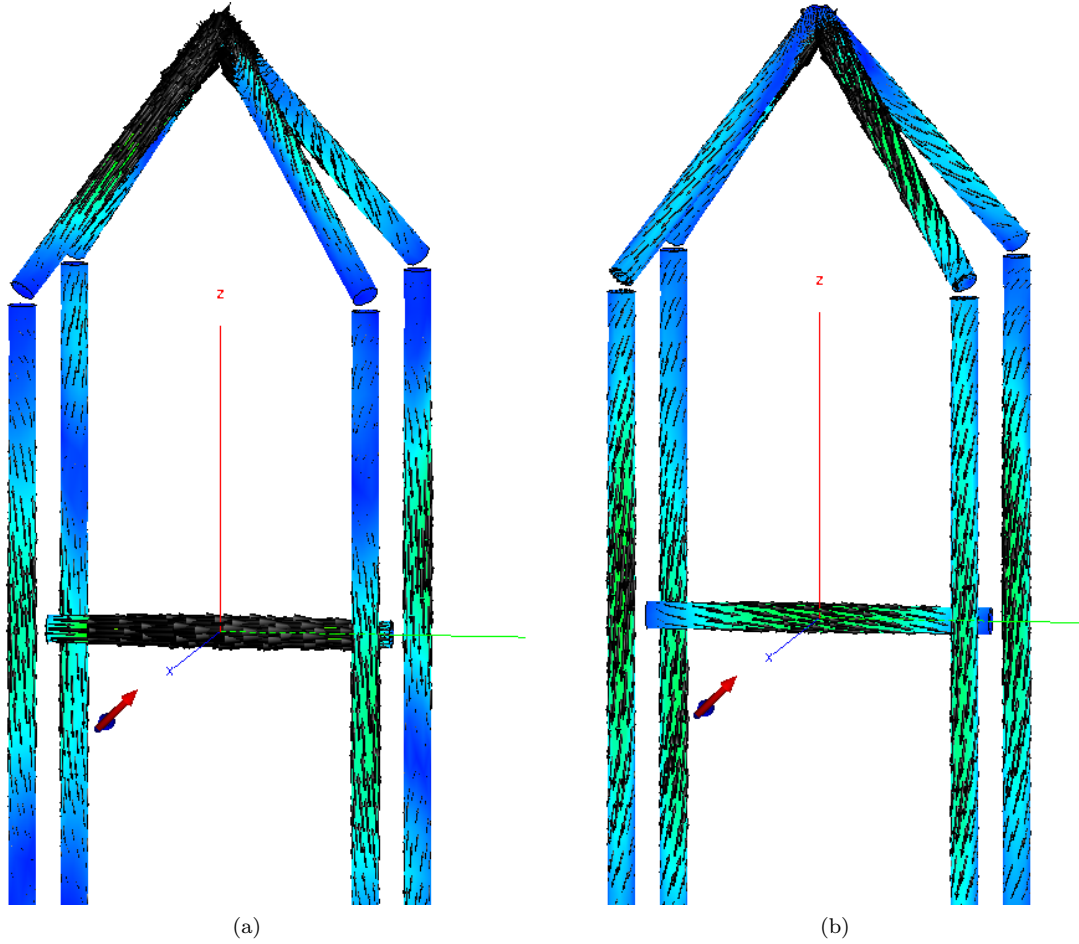


Figure 2.9: Instantaneous impressed currents on a geometrical structure made of PEC, at $\omega t = 0^\circ$ (a) and $\omega t = 90^\circ$ (b).

In conclusion, it is observed that there is a fundamental difference in the scattering generated by complex structures, depending on the material of which they are made of. This difference could be properly exploited in order to gain additional information on the geometrical structure of metallic objects. The information retrieved would then be integrated with the classic images returned by RF Tomography, and result in an increased situational awareness.

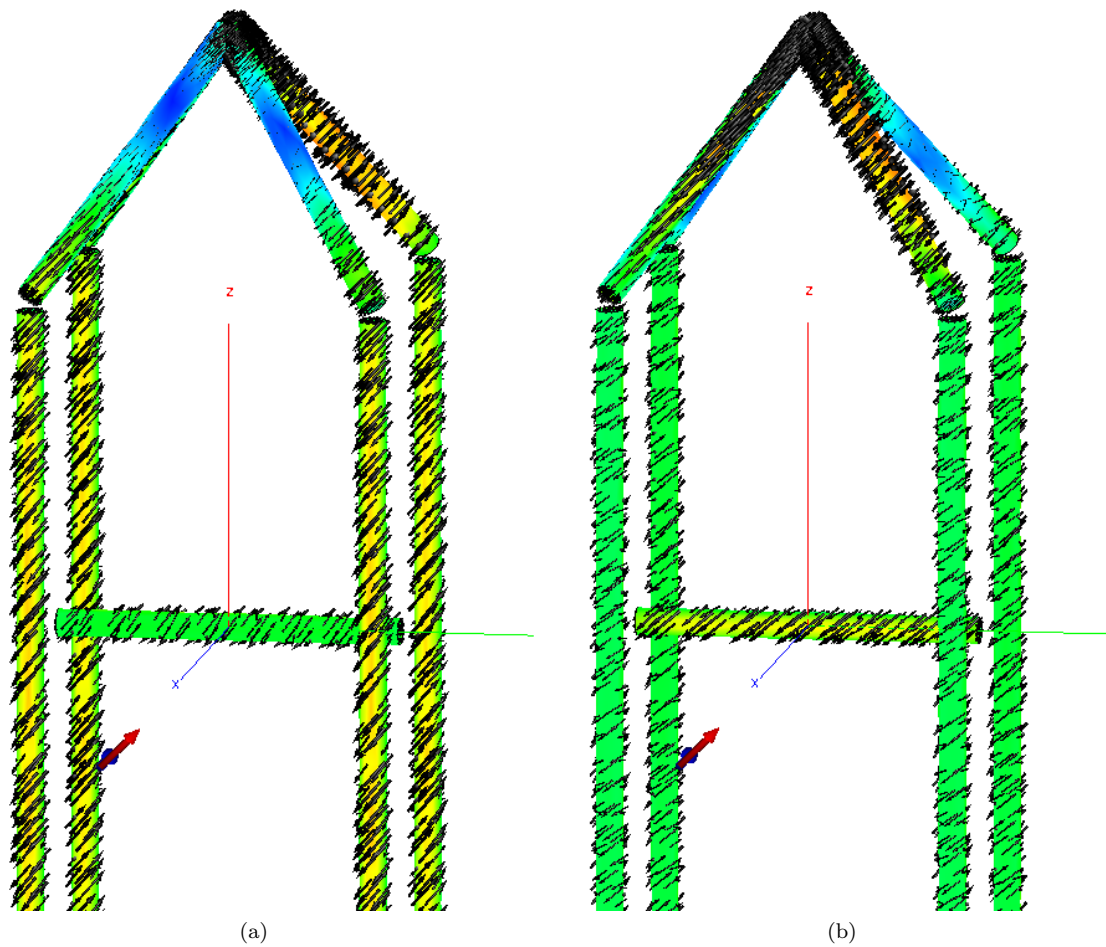


Figure 2.10: Instantaneous impressed currents on a geometrical structure made of dielectric material ($\epsilon_r = 5$), at $\omega t = 45^\circ$ (a) and $\omega t = 135^\circ$ (b).

2.1.2 Forward model

Derivation of a vectorial forward model with a dyadic contrast function

Consider the basic vectorial RF Tomography formulation, in a 3D homogeneous space:

$$E^s(\mathbf{r}^t, \mathbf{r}^r) = k_0^2 \iiint_D \mathbf{a}^r \cdot \mathbf{G}(\mathbf{r}^r, \mathbf{r}') v(\mathbf{r}') \mathbf{G}(\mathbf{r}', \mathbf{r}^t) \cdot \mathbf{a}^t d\mathbf{r}', \quad (2.31)$$

where:

- $\mathbf{r}^t$ and $\mathbf{r}_n^r$ indicate the location of the transmitting and receiving antenna;
- $\mathbf{a}^t$ and $\mathbf{a}^r$ indicate the spatial orientation of the transmitting and receiving antennas, which are assumed to be small dipoles³;
- $\mathbf{G}$ is the dyadic Green's function for the homogeneous space;
- $v(\mathbf{r}')$ is the scalar contrast function being reconstructed;
- $\mathbf{r}'$ is the variable of integration which spans the domain of investigation D .

The objective is to replace the scalar contrast function $v(\mathbf{r}')$ with a vector quantity so as to be able to represent the de-polarization observed in thin, elongated metallic objects. In order to be able to perform matrix-vector multiplications correctly, the contrast function must be dyadic as well. Eq. (2.31) therefore becomes:

$$E^s(\mathbf{r}^t, \mathbf{r}^r) = k_0^2 \iiint_D \mathbf{a}^r \cdot \mathbf{G}(\mathbf{r}^r, \mathbf{r}') \cdot \mathbf{V}(\mathbf{r}') \cdot \mathbf{G}(\mathbf{r}', \mathbf{r}^t) \cdot \mathbf{a}^t d\mathbf{r}', \quad (2.32)$$

where

$$\mathbf{V}(\mathbf{r}') = \begin{bmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{bmatrix}. \quad (2.33)$$

The elements of (2.32) can be further analyzed individually to understand their physical meaning. This is best done proceeding from right to left. The term $\mathbf{G}(\mathbf{r}', \mathbf{r}^t) \cdot \mathbf{a}^t$ has the meaning of electric field incident in the domain D :

$$\mathbf{E}^i = \mathbf{G} \cdot \mathbf{a}^t = \begin{bmatrix} E_x^i \\ E_y^i \\ E_z^i \end{bmatrix} = \begin{bmatrix} G_{xx}a_x^t + G_{xy}a_y^t + G_{xz}a_z^t \\ G_{yx}a_x^t + G_{yy}a_y^t + G_{yz}a_z^t \\ G_{zx}a_x^t + G_{zy}a_y^t + G_{zz}a_z^t \end{bmatrix}. \quad (2.34)$$

The incident field multiplies the dyadic contrast function, and an equivalent impressed surface current term is obtained.

$$\mathbf{J} = \mathbf{V} \cdot \mathbf{E}^i = \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} V_{xx}E_x^i + V_{xy}E_y^i + V_{xz}E_z^i \\ V_{yx}E_x^i + V_{yy}E_y^i + V_{yz}E_z^i \\ V_{zx}E_x^i + V_{zy}E_y^i + V_{zz}E_z^i \end{bmatrix}. \quad (2.35)$$

It is evident that while a scalar contrast function will produce an induced current $\mathbf{J}$, which is always parallel to $\mathbf{E}^i$, a dyadic contrast function can correctly represent de-polarization effects, thanks to its off-diagonal terms. This equivalent current produces a scattered field through the Green's function, which is projected onto the receiving antenna orientation.

³In order to simplify the notation, the transpose operator has been omitted. It is implied that $\mathbf{a}^t$ is a column-vector and that $\mathbf{a}^r$ is a row vector instead.

To simplify the notation the auxiliary row vector $\mathbf{p}$ can be introduced, which groups the terms related to the receiver side:

$$\mathbf{p} = \mathbf{a}^r \cdot \mathbf{G} = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix} = \begin{bmatrix} a_x^r G_{xx} + a_y^r G_{yx} + a_z^r G_{zx} \\ a_x^r G_{xy} + a_y^r G_{yy} + a_z^r G_{zy} \\ a_x^r G_{xz} + a_y^r G_{yz} + a_z^r G_{zz} \end{bmatrix}^T. \quad (2.36)$$

The scattered field due to the contribution of the infinitesimal scatterer at location $\mathbf{r}'$ can be called $E^s(\mathbf{r}')$, it is a scalar quantity which depends on both illumination and observation, and can be written as:

$$\begin{aligned} E^s(\mathbf{r}') &= \mathbf{p} \cdot \mathbf{J} = p_x J_x + p_y J_y + p_z J_z \\ &= p_x (V_{xx} E_x^i + V_{xy} E_y^i + V_{xz} E_z^i) \\ &\quad + p_y (V_{yx} E_x^i + V_{yy} E_y^i + V_{yz} E_z^i) \\ &\quad + p_z (V_{zx} E_x^i + V_{zy} E_y^i + V_{zz} E_z^i). \end{aligned} \quad (2.37)$$

All the known terms in (2.37), which correspond to a given configuration of transmitting and receiving antennas, can now be grouped into a linear operator $\mathbf{l}$, namely:

$$\begin{aligned} \mathbf{l}(\mathbf{r}^t, \mathbf{a}^t, \mathbf{r}^r, \mathbf{a}^r) &= \begin{bmatrix} l_{xx} & l_{xy} & l_{xz} & l_{yx} & l_{yy} & l_{yz} & l_{zx} & l_{zy} & l_{zz} \end{bmatrix} \\ &= \begin{bmatrix} p_x E_x^i & p_x E_y^i & p_x E_z^i & p_y E_x^i & p_y E_y^i & p_y E_z^i & p_z E_x^i & p_z E_y^i & p_z E_z^i \end{bmatrix}. \end{aligned} \quad (2.38)$$

Notice that in (2.38) the order in which the terms are arranged into a row is irrelevant, as long as it matches the order used to represent the contrast function, as described in the next section.

Since normally more than one antenna and/or polarization is used, multiple row vectors like the one described in (2.38) can be stacked to form the full matrix $\mathbf{L}$. The number of transmitting and receiving antennas is arbitrary, indicated by M and N respectively. Theoretically, also the number of polarizations is arbitrary. However, in practice it often makes sense to consider only three orthogonal polarizations for each antenna, for a total number of measurements equal to $M \times N \times 9$.

$$\mathbf{L} = \begin{bmatrix} \mathbf{l}(\mathbf{r}_1^t, \mathbf{a}_1^t, \mathbf{r}_1^r, \mathbf{a}_1^r) \\ \vdots \\ \mathbf{l}(\mathbf{r}_M^t, \mathbf{a}_3^t, \mathbf{r}_N^r, \mathbf{a}_3^r) \end{bmatrix}. \quad (2.39)$$

Building the dyadic contrast function

Now, it arises the need to fill out the elements of the matrix $\mathbf{V}$ (2.33). This matrix can be written as the sum of two terms. The need to use two terms originates from the physics of the problem. If there are no de-polarization effects, $\mathbf{V}$ should show elements different from zero only on its main diagonal. Instead, when de-polarization occurs also off-diagonal elements become different from zero. Ideally, if de-polarization were perfect $\mathbf{V}$ could be represented as a *linear* dyad, i.e. the product of two vectors forming a 3-by-3 matrix of rank 1. However, in reality de-polarization is not perfect and a complete dyad (i.e. full-rank matrix) is expected from the inversion.

As a result, it is expedient to represent $\mathbf{V}$ as the sum of two terms. The first term is a diagonal matrix representing the effects of the target when de-polarization effects are not strong. The second term is obtained as a dyadic product. In conclusion:

$$\mathbf{V}(\mathbf{r}') = \begin{bmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{bmatrix} = \epsilon_\delta \mathbf{I} + \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}. \quad (2.40)$$

The terms of the dyad $\underline{\mathbf{V}}$ need to be rearranged to match the same order used in (2.38). To avoid confusion, the rearranged version of $\mathbf{V}$ can be called $\mathbf{t}$.

$$\mathbf{t}(\mathbf{r}') = \begin{bmatrix} v_{xx} & v_{xy} & v_{xz} & v_{yx} & v_{yy} & v_{yz} & v_{zx} & v_{zy} & v_{zz} \end{bmatrix}^T. \quad (2.41)$$

The domain of investigation D is then discretized into pixels and one vector $\mathbf{t}$ is created for each pixel. All these vectors are then stacked into a larger column vector $\mathbf{T}$. If the number of pixels is equal to P the vector $\mathbf{T}$ will have size $9P \times 1$ and will be given by:

$$\mathbf{T} = \begin{bmatrix} \mathbf{t}(\mathbf{r}'_1) \\ \vdots \\ \mathbf{t}(\mathbf{r}'_P) \end{bmatrix}. \quad (2.42)$$

It is then finally possible to write the forward model equation as a matrix-vector multiplication.

$$\mathbf{E}^s = \mathbf{L} \cdot \mathbf{T}. \quad (2.43)$$

2.1.3 Inversion

The forward model obtained in (2.43) must be inverted to obtain the dyadic contrast function. The simplest way to do so is to apply any regularized inversion routine, either direct (e.g. Truncated Singular Value Decomposition) or iterative (Conjugate Gradient, Algebraic Reconstruction Technique) to (2.43).

After retrieving the vector $\mathbf{T}$, its elements need to be rearranged so as to create 9 reconstructed images versions, each one corresponding to a different orientation, i.e. xx , xy and so forth. The result can be visualized as a 3-by-3 collection of images.

The following examples show this method at work. In all examples images are obtained using the Conjugate Gradient method described in Chapter 4.2, without enforcing any physical bound.

Example 1

In the first example a thin, elongated cylinder is placed at a 45-degree angle in the xz plane, as shown in Fig. 2.11. The object crosses the xy plane at $x = 5$ cm and $y = 5$ cm.

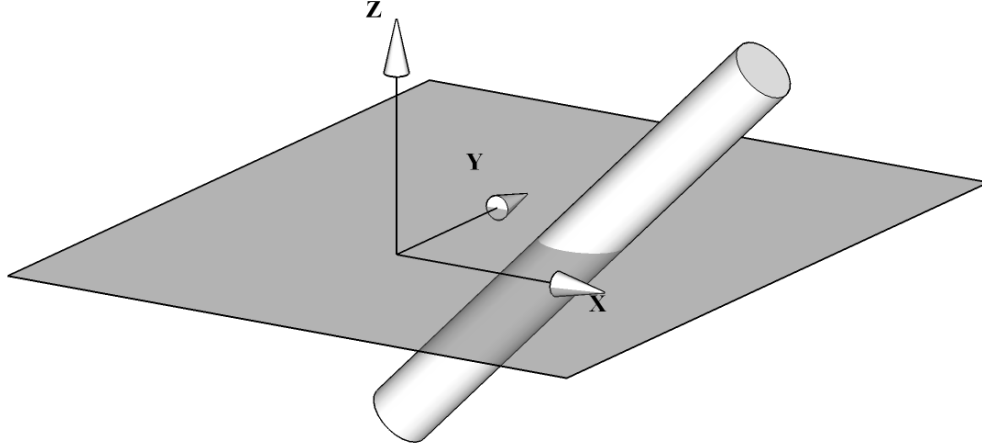


Figure 2.11: Geometry for Example 1. The imaging target is a thin PEC cylinder which intersects the xy plane at a 45° angle.

Antennas are placed in the xy plane at $z = 0$. In this example 21 transmitters and 40 receivers are employed, all of which transmit and receive in all 3 polarizations ($\hat{x}$, $\hat{y}$ and $\hat{z}$). The antennas are placed along a radius of 44.5 cm for the transmitters and 23.1 cm for the receivers. The frequency of operation is 3.16 GHz ($\lambda \approx 9.5$ cm). Overall, $21 \times 40 \times 9 = 7560$ measurements are collected.

The area under investigation is a square centered in the origin, of side equal to 20 cm, partitioned into pixels of size $\lambda/10$. This corresponds to a total of $21 \times 21 \times 9 = 3969$ unknowns. The problem is therefore over-determined.

Two cases are simulated. In the first one the object is made of Perfect Electric Conductor, in the second it is made of a dielectric material with $\epsilon_r = 3$.

PEC object The result of the inversion for the PEC object is shown in Fig. 2.12. The image shows that only xx , xz , zx and zz components of the image produce pixels significantly different from zero (the actual value of the pixel is of no importance because of the Born approximation). This means that the object is placed in the xz plane.

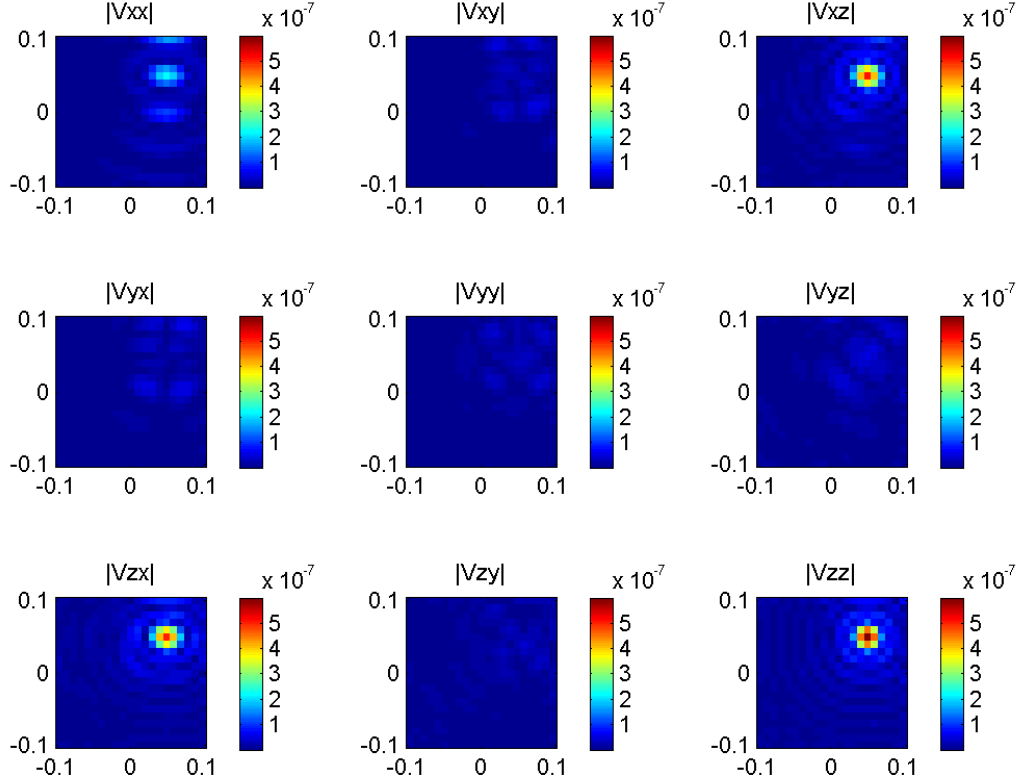


Figure 2.12: Result of the inversion for a PEC object. The figure shows the 9 versions of the image, one per spatial component.

The image also shows that the zz response is stronger than the others. This is because of the pattern of the antennas. Since the small dipole are placed in the xy plane, only when transmitter and receiver are oriented in the $\hat{z}$ direction there are no nulls pointing towards the object. The other polarizations will be affected by nulls and therefore will produce a weaker response.

Dielectric object The imaging result for the dielectric object is shown in Fig. 2.13. The image shows a clear response only in the zz component, with weaker responses in the xx and yy components. This is because, as expected, the dielectric object does not cause a strong depolarization of the incident field. Only the diagonal terms should be different than zero. However, because of the way the samples are taken, the zz case is predominant and tends to mask the result of the other cases.

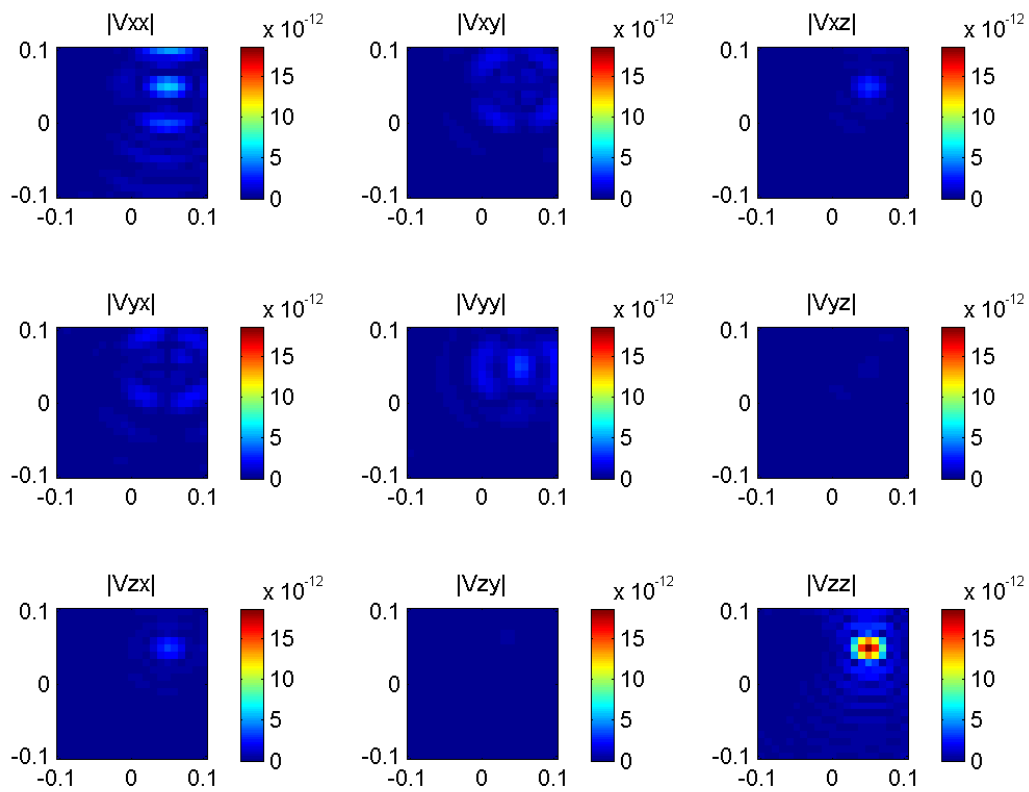


Figure 2.13: Result of the inversion for a dielectric object. The figure shows the 9 versions of the image, one per spatial component.

Example 2

In the second example a thin, elongated cylinder is placed parallel to the $\hat{x}$ axis, lying in the xy plane, as shown in Fig. 2.14. The object stretches in the left semi-plane, at $y = 5\text{cm}$.

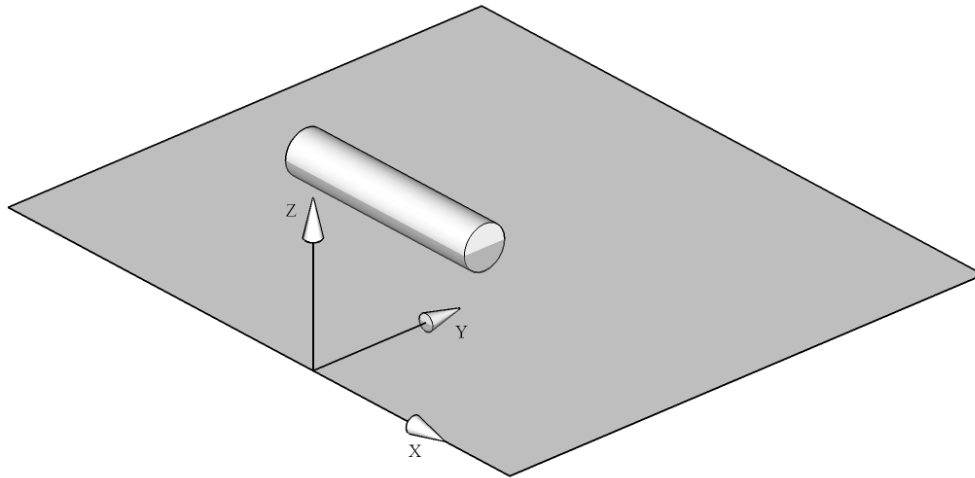


Figure 2.14: Geometry for Example 2.

This case is more difficult to reconstruct than the previous, due to the orientation of the object, which lies in the plane that gave a weak response in the previous example, due to the pattern of the antennas.

Two cases are simulated. In the first one the object is made of Perfect Electric Conductor, in the second it is made of a dielectric material with $\epsilon_r = 3$. The simulation conditions are the same as before and so is the domain under investigation.

PEC object The result of the inversion for the PEC object is shown in Fig. 2.15. The image shows that, as expected, only the top-left terms produce an observable response. The xx response is the strongest, although affected by visible artifacts. The artifacts are due to the small size of the object, which is placed in the same plane of measurement, thus offering an incomplete view from the antennas. In addition, the pattern effect is expected to be stronger in this case, because the zz response is almost null.

Dielectric object The imaging result for the dielectric object is shown in Fig. 2.16. The dielectric object shows a clear response only in the zz component, with weaker responses in the xx and yy components. The analysis in this case is the same as before. As expected, the dielectric object does not cause a strong depolarization of the incident field. Only the diagonal terms should be different than zero. However, because of the way the samples are taken, the zz case is predominant and tends to mask the result of the other cases.

These results show how a change in the way the contrast function is defined can bring additional information regarding the spatial orientation of the target. However, there is still an interpretation step that must be performed by a user in order to fully understand the results. In the next section it is shown how the images obtained in these examples can be further processed to gain a more direct visualization of the orientation of the objects.

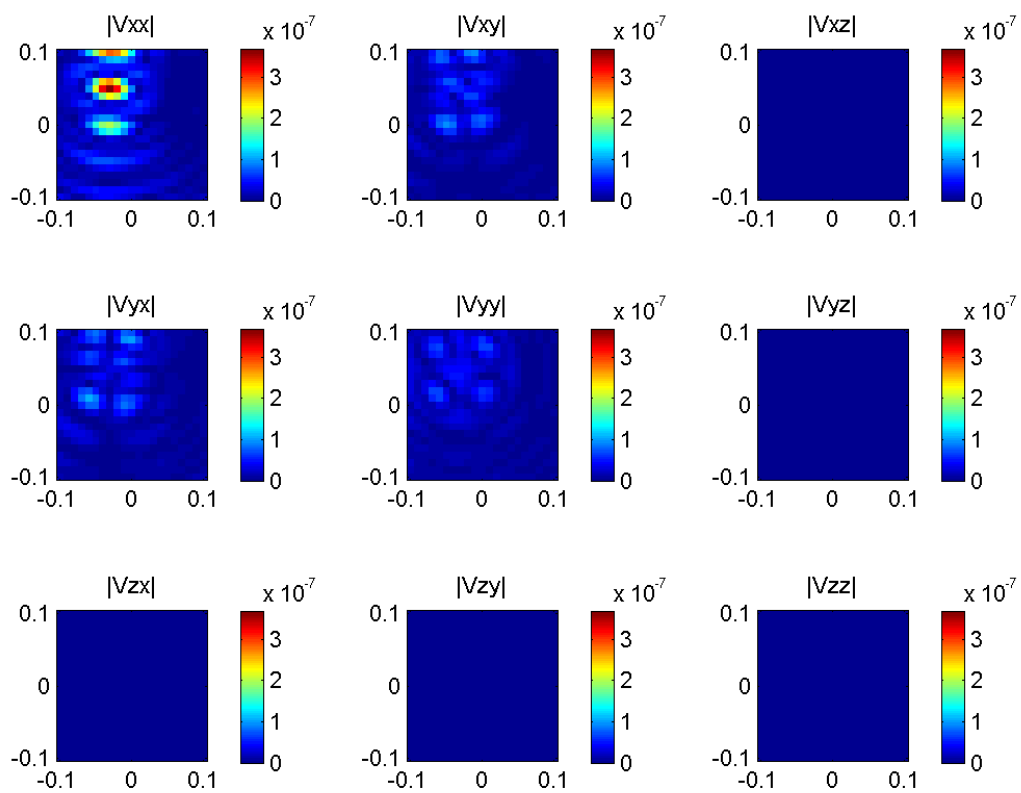


Figure 2.15: Result of the inversion for a PEC object.

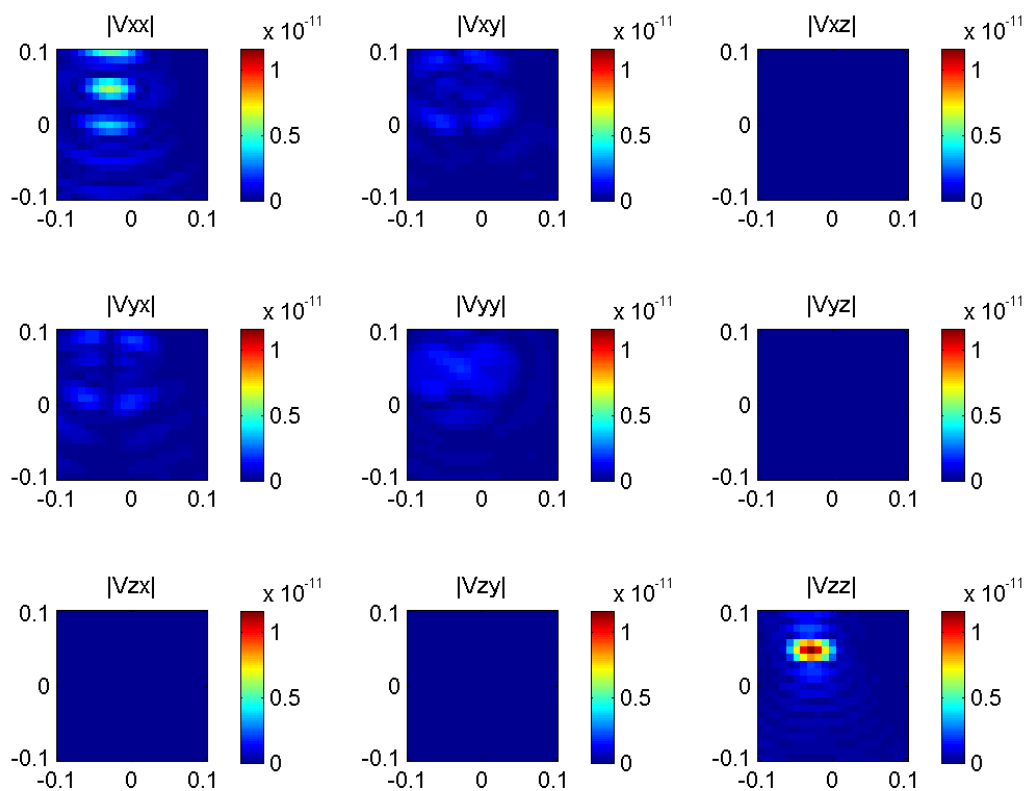


Figure 2.16: Result of the inversion for a dielectric object.

2.1.4 Eigenvalues-eigenvectors analysis

The goal of this analysis is to provide a different representation of the results, to complement the set of 9 images generated in the previous examples. This can be done by making a change in the data representation.

Instead of building a set of 3-by-3 images, each one made of P pixels and representing a given orientation in space, P matrices are constructed, each one made up of 3-by-3 elements. The first matrix collects the xx , xy , xz , yx , yy , yz , zx , zy and zz elements of the first pixel in the domain under investigation. The second matrix collects again the xx , xy , etc. elements of the second pixel in the domain under investigation, and so forth until P matrices are obtained.

The eigenvalues and eigenvectors associated with each of the P matrices are found by solving the problem:

$$\mathbf{V} \cdot \mathbf{u} = \lambda \mathbf{u}, \quad (2.44)$$

where $\mathbf{u}$ is an auxiliary vector. The vectors $\mathbf{u}$ satisfying (2.44) are called eigenvectors and their corresponding values of λ are called eigenvalues. For a 3-by-3 matrix there will be at most 3 eigenvalues and 3 eigenvectors.

If de-polarization is strong, because of the way the matrix $\mathbf{V}$ was built, there will be one large eigenvalue (λ_1) and two smaller eigenvalues (λ_2 and λ_3). Therefore, the eigenvector $\mathbf{u}_1$ associated with λ_1 will give $\mathbf{v}$, i.e. the orientation of the pixel in space (see (2.40)); λ_1 will give a measure of the magnitude of the contrast. The other eigenvalues do not provide useful information so they are discarded.

The eigenvalue decomposition can be used to build a *quiver* plot of the contrast function. For each pixel the eigen-decomposition is performed. Then, only the eigenvalue with largest magnitude is kept, along with its corresponding eigenvector. The auxiliary vector $\mathbf{d}$ is built:

$$\mathbf{d} = \lambda_1 \mathbf{u}_1. \quad (2.45)$$

The collection of vectors $\mathbf{d}$ (one per pixel), can be plot as arrows pointing in a 3D space. To make the visualization easier, the quiver plot can be superimposed to the image representing the sum of all 9 components, as used in the previous section. This allows to generate images which are extremely easy to interpret, as can be shown by looking at the result for the same examples showed before.

Single target: example 1

PEC object The same result obtained in Fig. 2.12 can be processed with the eigen-decomposition just described. The result of this analysis is shown in Fig. 2.17.

The quiver plot provides a very insightful description of the object. The location of the object is indicated by the magnitude of the arrows, which indicate by their direction how the target is oriented in the 3D space.

Dielectric object For the dielectric object the result of the eigen-decomposition is shown in Fig. 2.18.

In this case the quiver plot indicates that the location of the object is the same, but that the orientation is mostly towards the $\hat{z}$ direction. This is again an effect of the different direction of the currents impressed on the target.

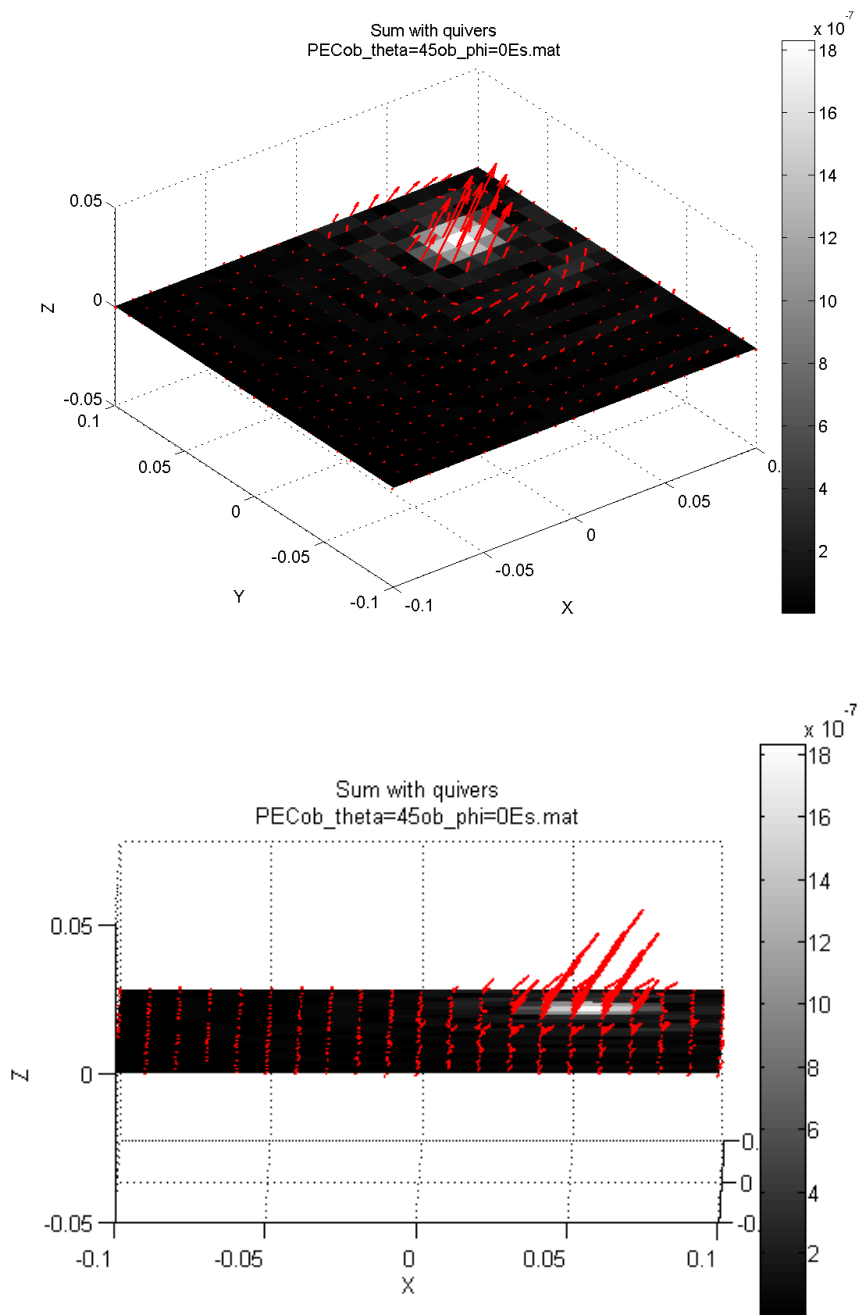


Figure 2.17: Quiver plot obtained with eigen-decomposition for the PEC target. The gray image is the absolute value of the sum of all components of the image. Two views are provided.

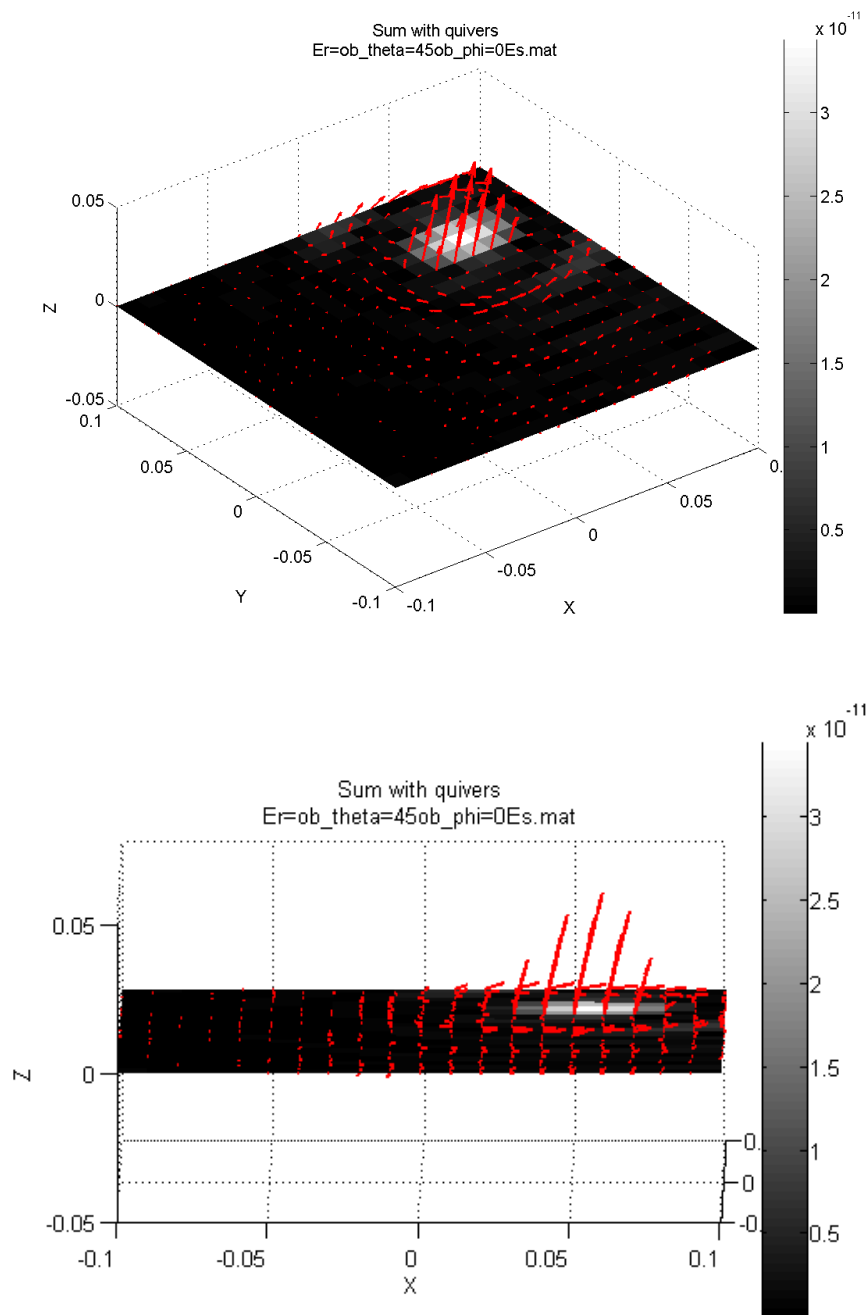


Figure 2.18: Quiver plot obtained with eigen-decomposition for the dielectric target. The gray image is the absolute value of the sum of all components of the image. Two views are provided.

Single target: example 2

PEC object The results for the second example, in the case of PEC, is shown in Fig. 2.19.

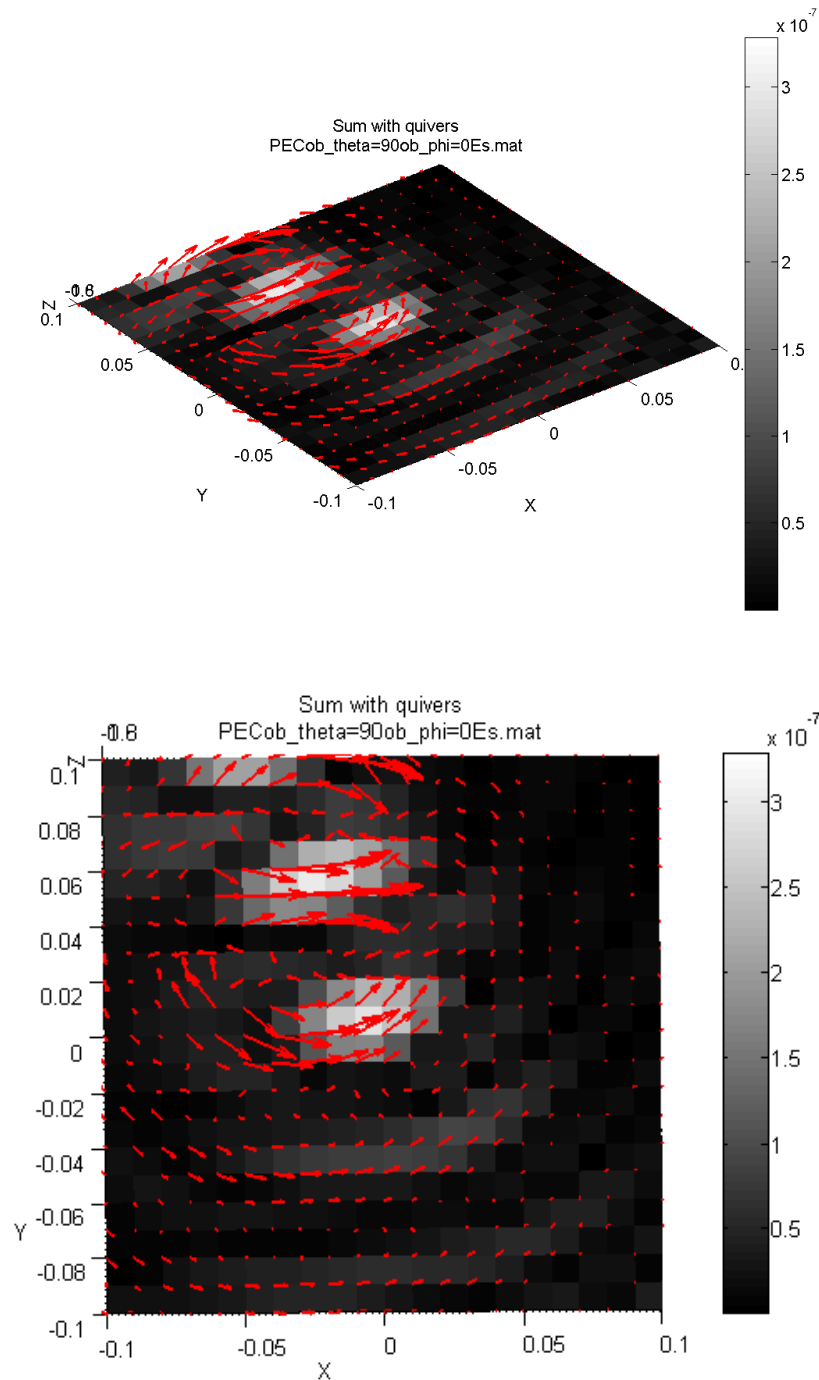


Figure 2.19: Quiver plot obtained with eigen-decomposition for the PEC target. The gray image is the absolute value of the sum of all components of the image.

For the PEC target the quiver plot indicates that the object is oriented along the $\hat{x}$ direction. Artifacts are present, and their orientation is nonetheless pointing in the same direction. Because of the artifacts it is difficult to estimate the actual position of the target. This problem cannot be solved unless the spatial

sampling is changed.

Dielectric object The dielectric target provides a more difficult interpretation, as shown in Fig. 2.20. Due to the absence of the depolarization effects, the pixels significantly different from zero provide different clues to the orientation of the object. The strongest pixels are oriented along $\hat{x}$, but the ringing artifacts point to all directions in the xy plane.

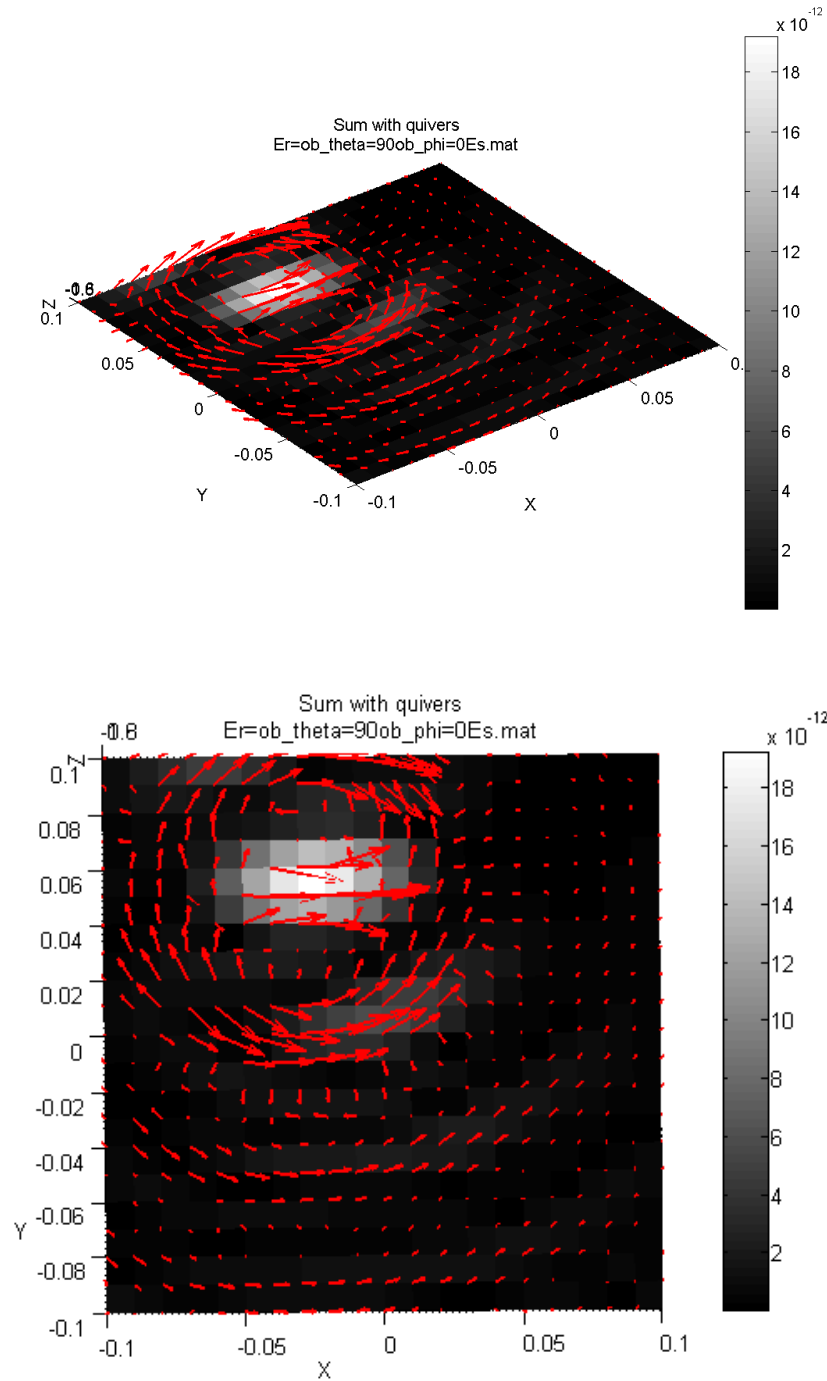


Figure 2.20: Quiver plot obtained with eigen-decomposition for the dielectric target. The gray image is the absolute value of the sum of all components of the image. Two views are provided.

Multiple targets

This last example shows reconstruction results when multiple objects are present in the domain of investigation. The geometry under consideration is depicted in Fig. 2.21.

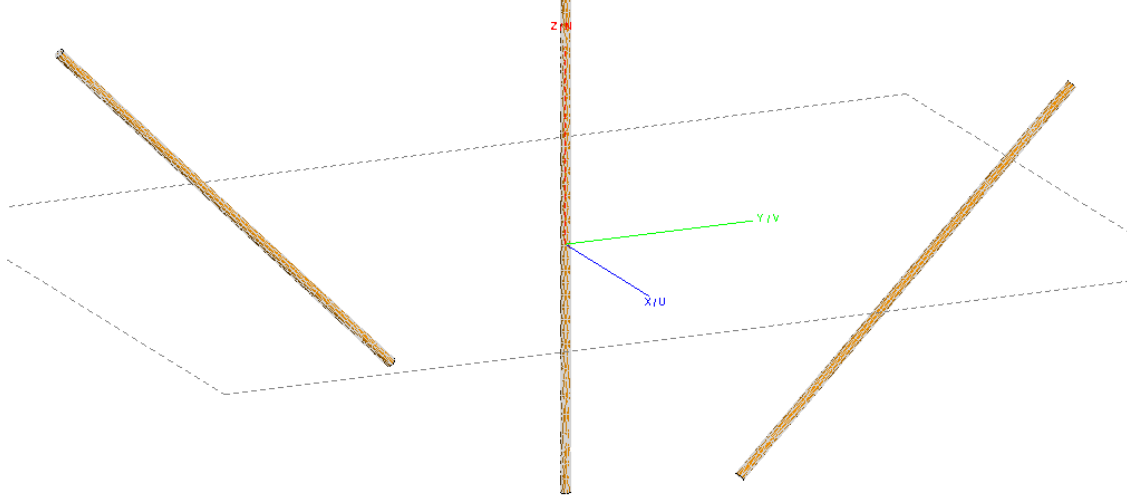


Figure 2.21: Geometry for testing of the dyadic contrast function. Three metallic thin cylinders are present in the scene, with three different spatial orientations.

Three objects are present in the scene. All cylinders are made of PEC and have radius equal to 0.1 cm and length equal to 10 cm. The first cylinder intersects the $z = 0$ plane at coordinates $x = y = 5$ cm and is oriented at an angle toward the positive y axis, i.e. in a standard spherical coordinates system it is oriented toward $\theta = 45^\circ$ and $\phi = 90^\circ$. The second cylinder intersects the $z = 0$ plane at coordinates $x = y = 0$ cm and is oriented toward the positive z axis, i.e. in a standard spherical coordinates system it is oriented toward $\theta = 0^\circ$ and $\phi = 0^\circ$. The third cylinder intersects the $z = 0$ plane at coordinates $x = y = -5$ cm and is oriented at an angle toward the negative y axis, i.e. in a standard spherical coordinates system it is oriented toward $\theta = 45^\circ$ and $\phi = -90^\circ$.

The result of the inversion using the dyadic contrast function is shown in the following figures. Fig. 2.22 shows the absolute value of the 9 components of the dyadic contrast. The strongest response is given by the zz component, for the same reasons explained for the other examples. In the zz image, the object in the center clearly stands out, while it does not in any other image.

The other significant responses are given by yz , zy and yy , with the latter being weaker again because of the advantage that the antennas orientation give to the z components.

From the reconstruction of the absolute value image it is evident that the objects in the corner positions are oriented in the yz or zy direction, but it is impossible to say which one of the two. The phase plot shown in Fig. 2.23 only partially helps clearing the doubt.

Fig. 2.23 is of difficult interpretation because the phase is shown for all pixels, including the ones whose magnitude is small, so non-interesting. To make the plot easier to read it is possible to plot the phase scaled by the magnitude, i.e.

$$\Omega = |\mathbf{V}| \cdot \underline{\mathbf{V}} \quad (2.46)$$

The plot of Ω is shown in Fig. 2.24.

Looking at the yz and zy components it can be observed that the phase flips sign for the two objects. In the yz image it is positive for the bottom-left object and negative for the top-right object. For the zy image the situation is the same. Without some sort of prior knowledge, it is hard to understand how the two objects are oriented into space.

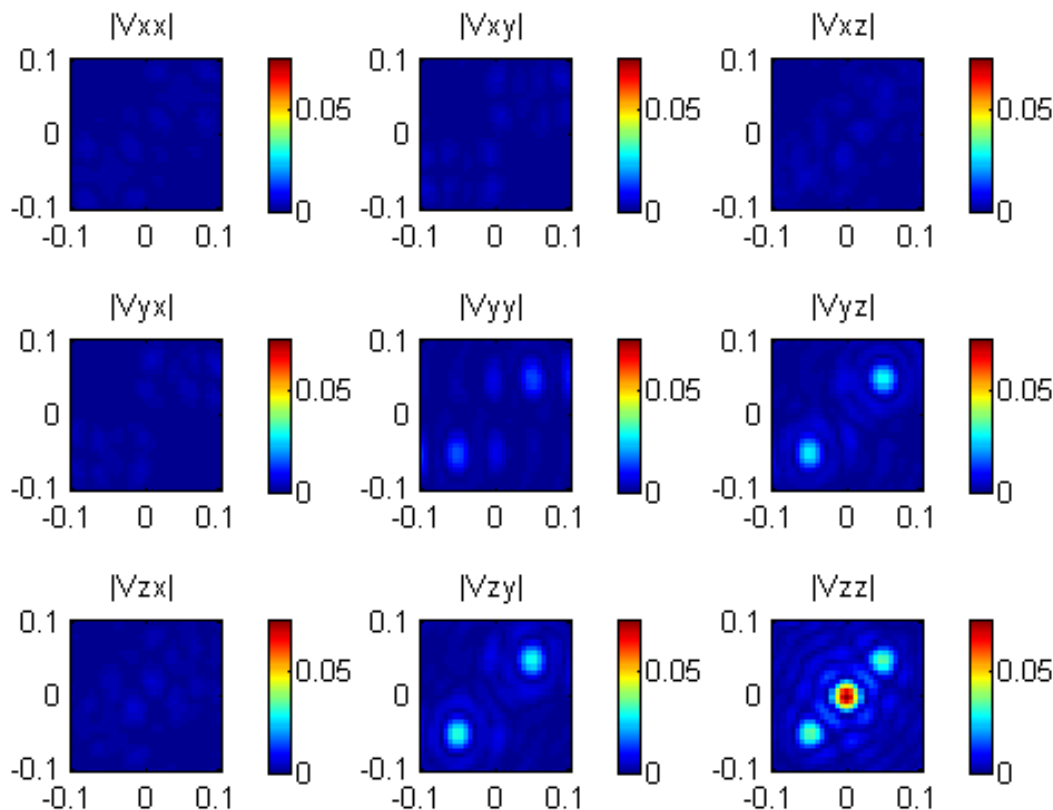


Figure 2.22: Absolute value of components of dyadic contrast function.

The quiver plot obtained with the eigen-decomposition clears the dilemma. As shown in Fig. 2.25 the eigenvectors multiplied by their corresponding eigenvalues, correctly point in the direction toward which the object is oriented.

In conclusion, the formulation described in this chapter allows to correctly reconstruct thin metallic objects oriented in different directions in space. A dyadic contrast function is developed and introduced into the classical RF Tomography forward model. Usual inversion routines can be employed to reconstruct the dyadic contrast, thanks to a data organization which makes the problem larger but does not change its structure (linear system of equations). The result of the inversion in the form of 2D color plots is augmented with a 3D visualization based on an eigen-decomposition, which allows to give a straightforward representation of the orientation of the objects in space, providing overall more actionable results.

2.2 Quadratic Forward Model

A quadratic forward model is developed to remove some of the limitations associated with the Born approximation, which was initially introduced to linearize and solve the volume integral equation for the scattered field.

Although this approximation is used extensively in the remote sensing literature and, under certain conditions, it is indeed a valid approximation, there are cases where the Born approximation does not work well. When more than one target is present in the volume under investigation, multiple scattering phenomenon can indeed generate artifacts. Also, when the target is not a weak scatterer, quantitative reconstruction is impossible. This is also why in the study of iterative algorithms as inversion methods

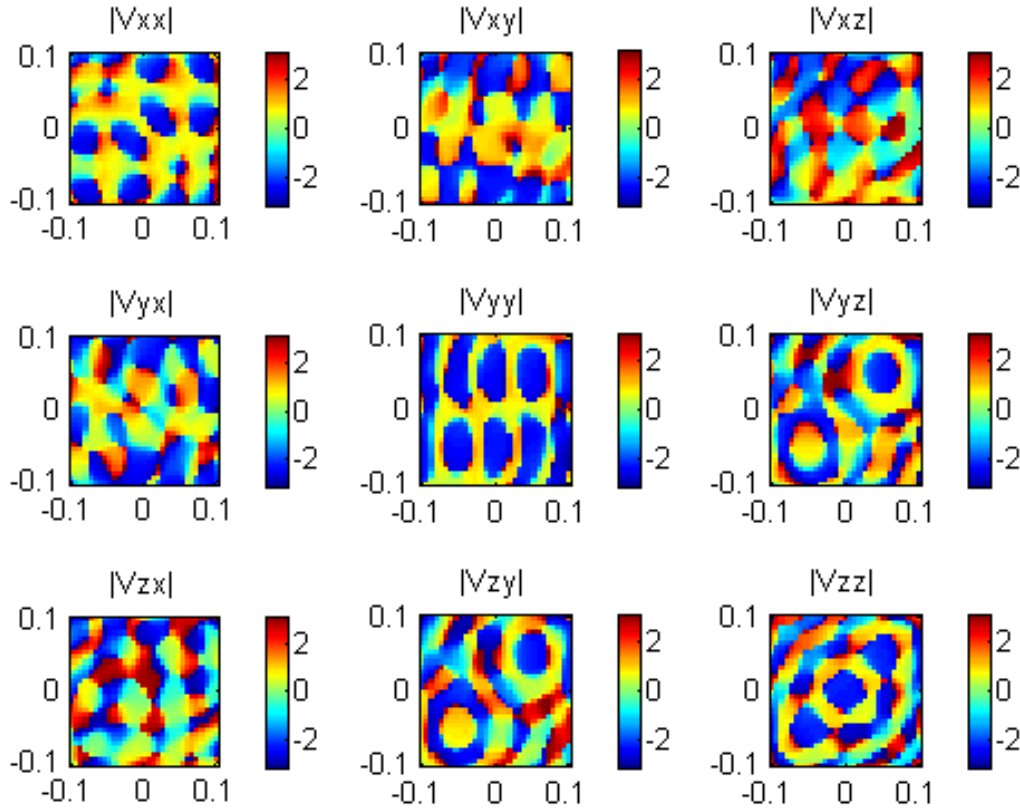


Figure 2.23: Phase of components of dyadic contrast function.

alternative to TSVD presented in section 4.2, the physical bounds have been introduced only on the sign of the real and imaginary parts of the contrast vector, and images are always normalized: the Born approximation does not make it possible to achieve good quantitative reconstruction.

In addition, it can be shown that the linear operator $\mathbf{L}$, developed under the Born approximation, acts as a filter on the spatial harmonics of the reconstructed contrast function. As a result, sharp edges cannot be normally imaged correctly. In fact, the Born approximation is one of the biggest obstacles in increasing the resolution of an RFT system.

In the literature, a number of methods have been developed to overcome these limitations. These algorithms are employed particularly in applications where quantitative reconstruction is necessary in order to consider imaging successful, such as medical imaging. The two most popular methods are the Born Iterative and Distorted Born methods [69, 70].

The Born Iterative method consists in a numerical algorithm which involves the solution of multiple inverse and forward problems. First, the Born approximation is used to obtain an estimation of the contrast function, solving an *inverse* problem; this estimation is then used to calculate the scattered electric field that such a contrast function would cause⁴, by solving a *forward* problem, which, incidentally, is usually much more computationally demanding than the inverse one. The field estimated in this method is compared to the one actually measured: if the difference between the two is too large (according to some user-imposed rule) it is used to estimate a new contrast function, and so on.

Although numerically heavy, the Born Iterative method has gained wide popularity, due to its straight-

⁴Since these are ill-posed problems, the solution to the inverse problem is never the one that would explain exactly the starting data from which it was obtained, but it is always an approximation.

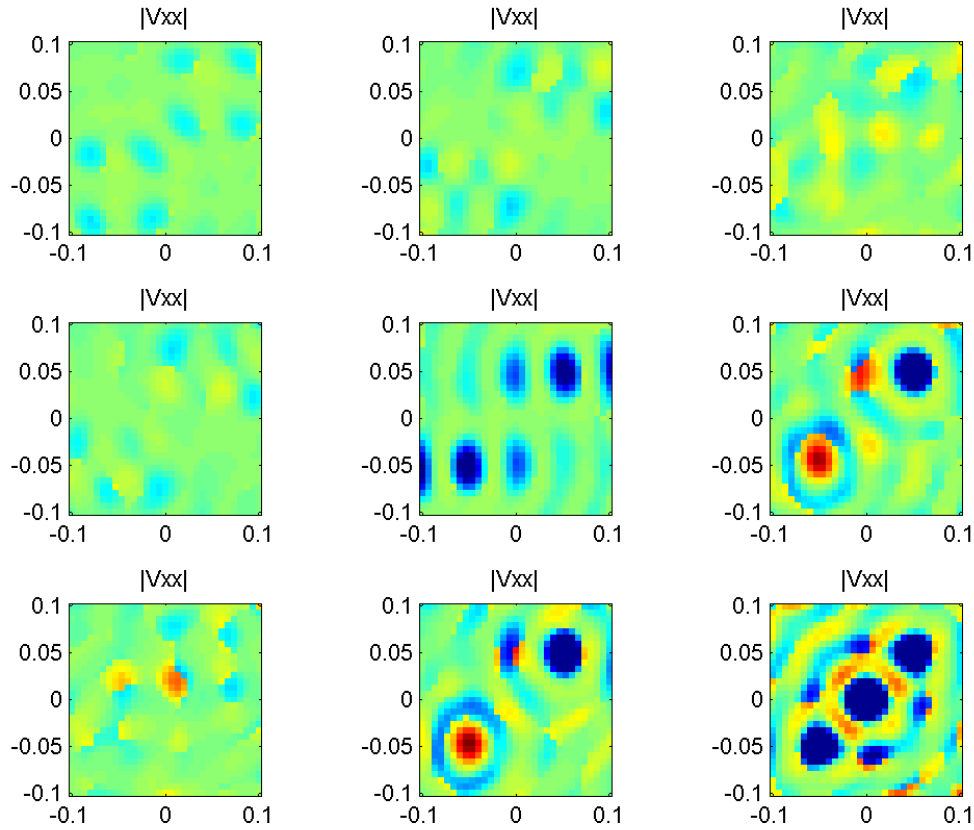


Figure 2.24: Phase of components of dyadic contrast function scaled by the relative magnitude at each pixel. The plot highlights the phase information only where the magnitude is sufficiently big to represent an actual object instead of noise.

forward structure and to the good quantitative results that can be obtained using it [69].

The Distorted Born (or Distorted Born Iterative) method is a further advance on the Born Iterative method [70]. In addition to the iterative procedure, at every step the Green's function is updated. This allows for faster convergence, which is beneficial because it reduces, as a side effect, the number of times the forward problem has to be solved, hence, it reduces the computational load.

Like every iterative algorithm, Born and Distorted Born Iterative methods could fail to converge to the right solution, depending on their starting point. Therefore, a different approach can be considered.

This chapter deals with the development of a forward model which overcomes some limitations imposed by the Born approximation. A few papers [71, 72, 73] provide motivation for this work. All of them are based on the development of an analytical solution, based on a quadratic forward model, which promises to overcome the limitations given by the Born approximation, while at the same time avoiding the computational burden given by the Born and Distorted Born Iterative methods.

In order to better understand the origin of this idea, in this chapter the RF Tomography forward model is derived again, to highlight where and how the Born approximation comes into play, and how it can be replaced with a quadratic formulation.

In addition, in order to limit the size of the problem and make the inversion more treatable from a computational point of view, the inversion is recast into an optimization problem involving contrast functions which can be represented with a limited number of spatial harmonics. A detailed description of this method is provided.

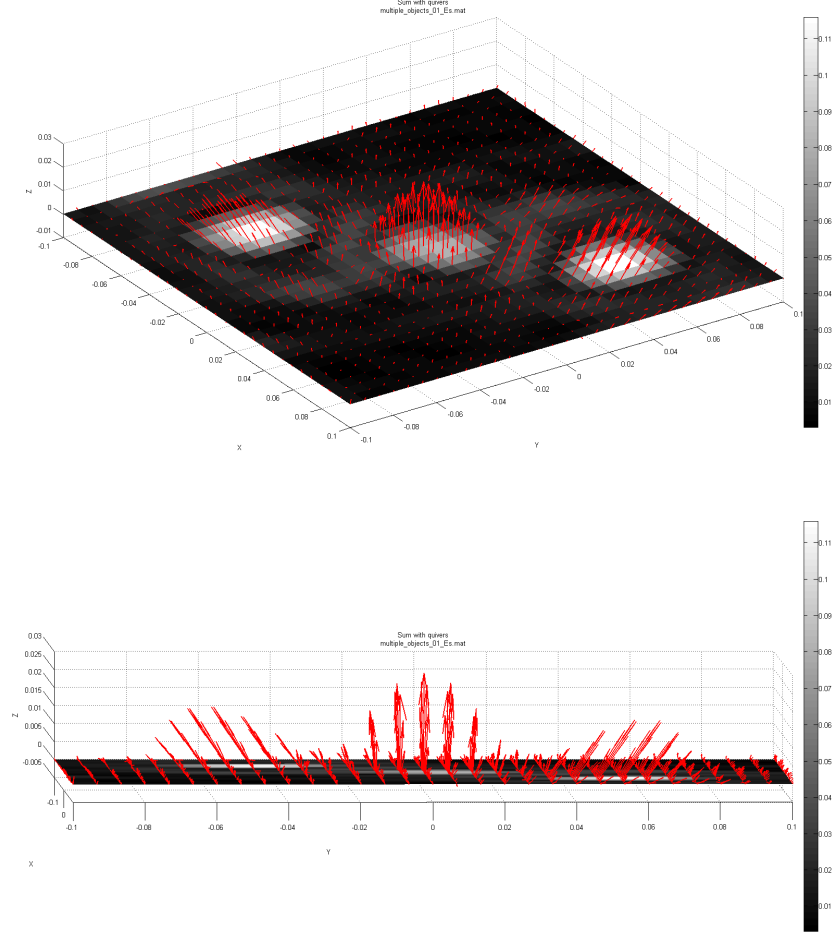


Figure 2.25: Quiver plots, highlighting the different orientation of the objects being reconstructed.

In the end, simulation results are present to show advantages and limitations of the quadratic model with respect to the Born approximation.

2.2.1 Forward model derivation

To understand how to develop a quadratic model, it is convenient to start from the basic Volume Integral Equation around which RF Tomography is based [74]. This equation expresses the scattered field as an integral equation involving the total electric field, the contrast function, and the Green's function appropriate to the environment where sensing is being performed. In order to simplify this derivation as well as the following ones, the model is derived in 2D space, where the objects are assumed to extend to infinity in the z direction. Also, the TX antennas are assumed to be infinite current lines. As a result:

$$E^s(\mathbf{r}^r) = k_0^2 \iiint_D \chi(\mathbf{r}') E^t(\mathbf{r}') g_e(\mathbf{r}', \mathbf{r}^r) d\mathbf{r}', \quad (2.47)$$

where the contrast function is indicated with the variable χ , in order to highlight the difference between the scalar contrast function in a 3D space (standard RF Tomography formulation) and the dyadic contrast developed in section 2.1. Also, the scalar Green's function is indicated with g_e , where the symbol e indicates *external*. The need for this will be explained in the following.

Equation (2.47) is *exact* and the reason for the introduction of the Born approximation is that the total field inside (2.47) includes also the scattered field, making the problem non-linear.

In (2.47) the term $E^t(\mathbf{r}')$ can be expanded using another Volume Integral Equation inside the same domain of investigation, as

$$E^t(\mathbf{r}') = E^i(\mathbf{r}') + k_0^2 \iiint_D \chi(\mathbf{r}') E^t(\mathbf{r}') g_i(\mathbf{r}', \mathbf{r}') d\mathbf{r}', \quad (2.48)$$

where this time the symbol g_i indicates *internal* Green's function and the auxiliary variable $\mathbf{r}'$ is introduced.

The difference between external and internal Green's functions can be now explained with the help of Fig. 2.26.

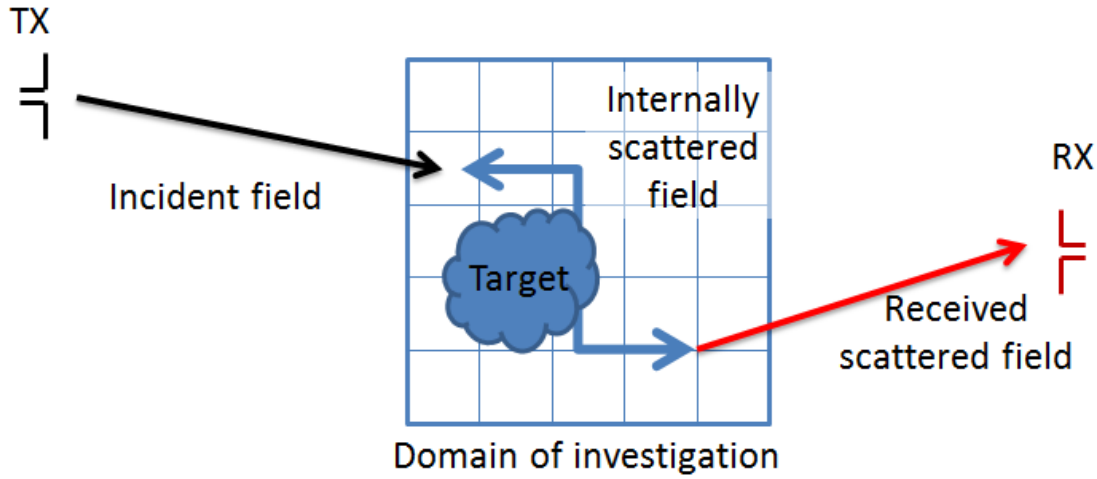


Figure 2.26: Pictorial representation of the scattering phenomenon.

The transmitter generates the incident field, which travels to the domain D and impinges upon the target(s). The total field is the sum of the incident one plus all the contributions due to interactions inside the domain of investigation. All these interactions (which are non-linear) originate the impressed currents, which in turn will give rise to the scattered field. Therefore, the scattering phenomenon can be divided in two stages. First, the incident field generates the total field inside the domain D ; this is represented with the internal Green's function g_i . Then, the domain can be seen as a “black box”: assuming that the total field inside D was somehow generated, the scattered field at the location of the receiving antenna is its consequence, and it is explained with the external Green's function g_e .

If the domain under investigation is partitioned into many pixels, g_i explains the pixel-to-pixel interactions, while g_e explains the pixel-to-RX interactions.

The Born approximation consists in assuming that the second term of (2.48) is so small compared to the incident field that it can be neglected. Therefore, in (2.47) the total field is replaced with the incident field and a linear equation is obtained.

A formulation equivalent and alternative to the one of (2.47) and (2.48) is the one using an operator notation to replace the integrals.

In (2.47) the term $\chi(\mathbf{r}')E^t(\mathbf{r}')$ represents an equivalent impressed current, therefore the volume integral equation is an operator that receives as input an equivalent current, and returns an electric field, by applying the appropriate Green's function. This can be written as:

$$E^s(\mathbf{r}') = \mathbf{A}_e [\chi(\mathbf{r}')E^t(\mathbf{r}')] . \quad (2.49)$$

Similarly, (2.48) can be rewritten using an operator notation as:

$$E^t(\mathbf{r}') = E^i(\mathbf{r}') + \mathbf{A}_i [\chi(\mathbf{r}')E^t(\mathbf{r}')] . \quad (2.50)$$

The domain of investigation is spanned by the vectors $\mathbf{r}'$ and $\mathbf{r}'/$. Since the domain does not change, $\mathbf{r}'$ and $\mathbf{r}'/$ are pointing in substance to the same pixels. When organizing the data on a computer, it is therefore convenient to organize the pixels pointed by $\mathbf{r}'$ and $\mathbf{r}'/$ in the same order, so that in the end $E^t(\mathbf{r}') = E^t(\mathbf{r}'/)$. In short, assuming that the pixels are organized in the same way, the position vectors $\mathbf{r}'$ and $\mathbf{r}'/$ can be dropped and it is possible to write simply

$$E^s = \mathbf{A}_e(\chi E^t) \quad (2.51)$$

and

$$E^t = E^i + \mathbf{A}_i(\chi E^t). \quad (2.52)$$

Now (2.51) and (2.52) can be combined into a single non-linear equation. Performing the usual division of the domain D into a finite number of pixels, and introducing a matrix-vector notation as in [75], it is possible to write:

$$\begin{aligned} E^t &= E^i + \mathbf{A}_i(\chi E^t) \\ E^t - \mathbf{A}_i(\chi E^t) &= E^i \\ (I - \mathbf{A}_i\chi)E^t &= E^i \\ E^t &= (I - \mathbf{A}_i\chi)^{-1}E^i, \end{aligned} \quad (2.53)$$

Then, substituting (2.53) into (2.51):

$$E^s(\mathbf{r}^r) = \mathbf{A}_e [\chi(I - \mathbf{A}_i\chi)^{-1}E^i], \quad (2.54)$$

which is non-linear in χ .

The important term in (2.54) is the one involving the inverse operation. It is possible to rewrite this term as the result of the summation of a geometrical series of argument $\mathbf{A}_i\chi$. Introducing the auxiliary variable p , the sum of a geometrical series is written as:

$$(1 - p)^{-1} = \frac{1}{1 - p} = 1 + p + p^2 + \dots, \quad \|p\| < 1. \quad (2.55)$$

Similarly:

$$(\mathbf{1} - \mathbf{A}_i\chi)^{-1} = \mathbf{1} + \mathbf{A}_i\chi + (\mathbf{A}_i\chi)^2 + \dots, \quad \|\mathbf{A}_i\chi\| < 1. \quad (2.56)$$

Therefore an alternative way to explain the Born approximation is obtained. In (2.56) the Born approximation is obtained by keeping only the first term in the summation.

The next logical step is to keep an additional term, thus obtaining a better approximation of the scattered field. Hence:

$$(\mathbf{1} - \mathbf{A}_i\chi)^{-1} \approx \mathbf{1} + \mathbf{A}_i\chi, \quad \|\mathbf{A}_i\chi\| < 1 \quad (2.57)$$

which leads to:

$$E^s(\mathbf{r}^r) \approx \mathbf{A}_e\chi E^i + \mathbf{A}_e(\chi\mathbf{A}_i\chi E^i), \quad (2.58)$$

Eq. (2.58) establishes a quadratic relationship between the contrast χ and the scattered field $E^s(\mathbf{r}^r)$.

Numerical calculation of the operators

The calculation of the external operator on a computer is substantially equivalent to the calculation of the operator $\mathbf{L}$. The only difference is that $\mathbf{L}$ already includes the incident field, therefore it depends on both transmitters and receivers; on the other hand, $\mathbf{A}_e$ only includes the Green's function between pixels and receivers, therefore it does not depend on the transmitters.

Singular points are normally not a concern, as long as no receivers are located inside the domain of investigation. This is usually easily avoided by choosing an appropriate domain D . As a result the operator $\mathbf{A}_e$ is simply the collection of the values of the external Green's function in a 2D space:

$$g_e = H_0^{(2)}(\mathbf{r}', \mathbf{r}^r), \quad (2.59)$$

where $H_0^{(2)}$ is the Hankel function of the second kind of order 0.

The derivation of the internal operator $\mathbf{A}_i$, instead, is slightly more complicated. The internal operator describes the pixel-to-pixel iterations, therefore it will surely include singular points, because it is computed for each pixel with respect to every other pixel, including the starting pixel itself.

The numerical calculation of the integral of the Green's function in a 2D geometry in the presence of singularities has been derived in [68]. An explicit solution for square pixels is not available, but it can be reasonably approximated with the solution for round pixels of diameter dx equal to the length of the side of the square pixels.

As a result:

$$\mathbf{A}_i(\mathbf{r}'_1, \mathbf{r}'_2) = \begin{cases} -\frac{j\pi k_0 dx}{4} H_0^{(2)}(k_0(\mathbf{r}'_1 - \mathbf{r}'_2)) J_1(k_0 dx/2), & \mathbf{r}'_1 \neq \mathbf{r}'_2 \\ -\frac{j\pi k_0 dx}{4} H_1^{(2)}(k_0 dx/2) - 2j, & \mathbf{r}'_1 = \mathbf{r}'_2 \end{cases} \quad (2.60)$$

2.2.2 Harmonic contrast function

The calculation of the contrast function using the quadratic model can be computationally intensive. The numerical load can be made lighter by changing the way the contrast function is defined.

Normally, in RF Tomography, the domain under investigation is partitioned into pixels, which are then organized into a large vector and thus reconstructed. This normally leads to a large under-determined problem, because it is easy to have much fewer measurements than unknowns. Using the linear model (Born approximation) this is not a serious issue, because very efficient algorithms such as Conjugate Gradient can be employed. These can not only return a solution with a reasonable use of time and memory, but can also tackle ill-conditioning.

Using the quadratic forward model the inversion is not so simple. Apart from the larger computational requirements, due to the term involving $\mathbf{A}_i$, the problem is more complicated by the possible presence of local minima. In a linear model this is not a concern, because Ψ^{lin} is convex. Instead, Ψ^{quad} is quartic in χ , thus may have local minima in addition to an absolute minimum.

In order to make the problem easier to solve, one step is to make it over-determined instead of under-determined. One way of doing this is to express the contrast as a finite summation of spatial harmonics, then reconstruct the harmonics rather than the contrast itself.

The contrast χ can be written as a Discrete Fourier Transform:

$$\chi(x, y) = \sum_{m, n \in \mathbf{Z}} c_{m, n} e^{j2\pi m x / L_x} e^{j2\pi n y / L_y}, \quad (2.61)$$

where:

$$c_{m, n} = \frac{1}{L_x L_y} \iint \chi(x, y) e^{-j2\pi m x / L_x} e^{-j2\pi n y / L_y} dx dy. \quad (2.62)$$

The problem now is to reconstruct $c_{m, n}$ and then obtain χ in a second step. This can be done in two ways: the first way requires re-writing the whole forward model under the assumption of an harmonic contrast function, changing the definition of the operators and so forth. Alternatively, the second, easier way simply requires to change the optimization problem. In practice one will solve:

$$\begin{aligned} \underset{c_{m, n}}{\text{minimize}} \quad & \Psi^{\text{quad}} = \|E^s(\mathbf{r}^r) - \mathbf{A}_e [\chi \mathbf{A}_i \chi E^i]\|^2 \\ \text{with} \quad & \chi(x, y) = \sum_{m, n \in \mathbf{Z}} c_{m, n} e^{j2\pi m x / L_x} e^{j2\pi n y / L_y}. \end{aligned} \quad (2.63)$$

which in a modern computational tool such as MATLAB is done very simply by adding the definition of χ in terms of $c_{m,n}$ to the definition of the functional.

Eq. (2.63) is solved using a Levenberg-Marquardt minimization algorithm [76, 77, 78]. The algorithm has been chosen because it can work with complex numbers and can deal with both under- and over-determined problems. In addition, the algorithm can compute the gradients necessary to minimize the functional itself, by approximation with the finite-difference method, and does not require user-supplied derivatives in explicit form, which would require cumbersome calculations. The algorithm chosen is implemented in MATLAB as `lsqnonlin`.

The Levenberg-Marquardt minimization algorithm will be given as input an initial guess of the coefficients $c_{m,n}$ (usually the null vector) and will return as output the calculated $c_{m,n}$ which will be translated from Fourier-domain to space-domain using (2.61) for plotting.

Using the harmonic contrast function thus defined, the number of unknown is greatly reduced. In fact, as few as a dozen harmonics can represent sufficiently well real-world objects such as cylinders.

Reconstruction of harmonics by linear and quadratic models

The harmonic representation of the contrast also aids in understanding why the quadratic model can outperform the linear one. The reason lies in the harmonics representation of the linear model itself [79].

The linear model consists in building the operator $\mathbf{A}_e$, which essentially represents a (linear) map from the target space to the scattered electric field space. This mapping operation can be analyzed by looking at the singular value decomposition of $\mathbf{L}$.

In the Truncated Singular Value Decomposition method (described in Section 4.1) generalized eigenvalues with small magnitude have been associated with ill-conditioning, which in turn translates into poor reconstruction quality. To overcome this issue, only k generalized eigenvalues which are larger than a certain user-selected threshold are used, along with the corresponding k columns of the matrices containing left- and right-singular vectors. Therefore, only the k singular vectors which are maintained take part in the image formation. Remembering the definition

$$\mathbf{L} = \mathbf{U}\mathbf{S}\mathbf{V}^H \quad (2.64)$$

it is observed that the matrix $\mathbf{V}$ has as many rows and columns as the number of pixels into which the domain of investigation has been partitioned. This is the matrix of interest, because it spans the target space.

Each column of $\mathbf{V}$ contains as many elements as the number of pixels. Each of these columns can therefore be represented in Fourier space by performing the Discrete Time Fourier Transform, as done when representing the target into Fourier domain.

Since only k columns participate in the process of image formation, an overall Fourier representation of the space spanned by $\mathbf{V}_k$ can be written as

$$\mathbf{S}_\mathbf{V}(m, n) = \sum_k |e^{-jk_x X_0} e^{-jk_y Y_0} \text{FFT}(\mathbf{V}_k)(m, n)|, \quad (2.65)$$

where k_x and k_y should not be confused with k and represent the horizontal and vertical spatial frequencies, found as

$$k_x = \frac{2\pi}{dxM} m \quad (2.66)$$

$$k_y = \frac{2\pi}{dyN} n. \quad (2.67)$$

The Fourier representation obtained with (2.65) can be plotted against k_x and k_y . The plot depends on all the variables which affect the linear operator $\mathbf{L}$, i.e. the choice of domain of investigation (through X_0 , Y_0 , dx and dy), the location of transmitters and receivers, and the frequency of operation (which all affects $\mathbf{L}$ and therefore $\mathbf{V}$ itself).

Increasing the frequency of operation enlarges the domain encompassed by $\mathbf{S}_\mathbf{V}$ and the position and number of antennas affect how uniformly and densely the k -space is filled. When imaging a certain target

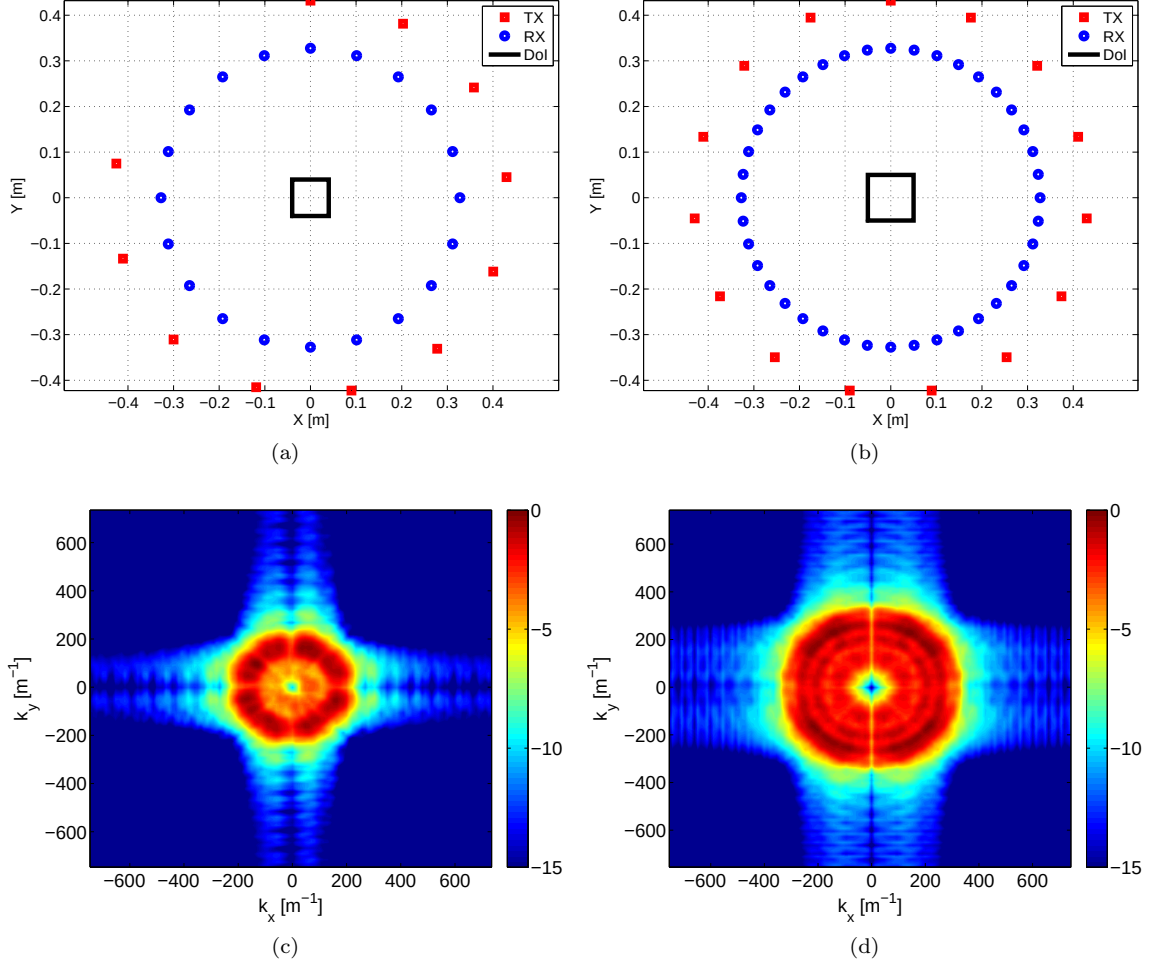


Figure 2.27: k -space representation of the linear operator $\mathbf{L}$ for two different antennas configurations. (a) 11 TX ($0 - 280^\circ$), 20 RX ($0 - 360^\circ$) at 3 GHz lead to the k -space shown in (c); (b) 15 TX ($0 - 360^\circ$), 20 RX ($0 - 360^\circ$) at 6 GHz lead to the k -space shown in (d).

using the linear model, only those harmonics which fall inside the space spanned by $\mathbf{S}_\mathbf{V}$ can be correctly reconstructed.

Fig. 2.27 shows two examples of this plot which highlight how $\mathbf{S}_\mathbf{V}$ changes when the number and position of antennas is changed. In Fig. 2.27a, 11 transmitters spanning 280° and 20 receivers spanning 360° , operating at 3 GHz are used. In Fig. 2.27b, instead, 15 transmitters spanning 360° and 20 receivers spanning 360° , operating at 6 GHz are used. The image shows that while in the first case spatial frequencies up to $|k_x| = |k_y| = 200$ can be reconstructed, in the second case spatial with $|k_x| = |k_y| > 300$ can be correctly imaged. In addition, the k -space in Fig. 2.27b is a lot fuller than the one in Fig. 2.27a. These two configurations will be later used in numerical examples which will highlight further this difference.

These plots also explain why the TSVD also has a filtering effect on the reconstructed image: harmonics that are outside the space spanned by $\mathbf{S}_\mathbf{V}$ are effectively filtered out, leaving only lower harmonics.

The quadratic model expands the number of harmonics that can be correctly reconstructed. This can be seen from the approximated scattered field equation in the quadratic case. Recalling

$$E^s \approx \mathbf{A}_e \chi E^i + \mathbf{A}_e (\chi \mathbf{A}_i \chi E^i), \quad (2.68)$$

it is clear that χ contributes to the scattered field not only through $\mathbf{A}_e$ like in the linear model (first term of the equation), but also through the internal operator $\mathbf{A}_i$ (second term of the equation). In practice $\mathbf{A}_i$ operates a transformation to the spatial components of χ before the multiplication by $\mathbf{A}_e$ occurs, i.e. before

the spatial components of χ are truncated by $\mathbf{S}_V$.

In the same way as $\mathbf{S}_V$ has been obtained, it is possible to plot a similar k -space, spanned this time by $\mathbf{A}_i$, for the same two scenarios described in Fig. 2.27.

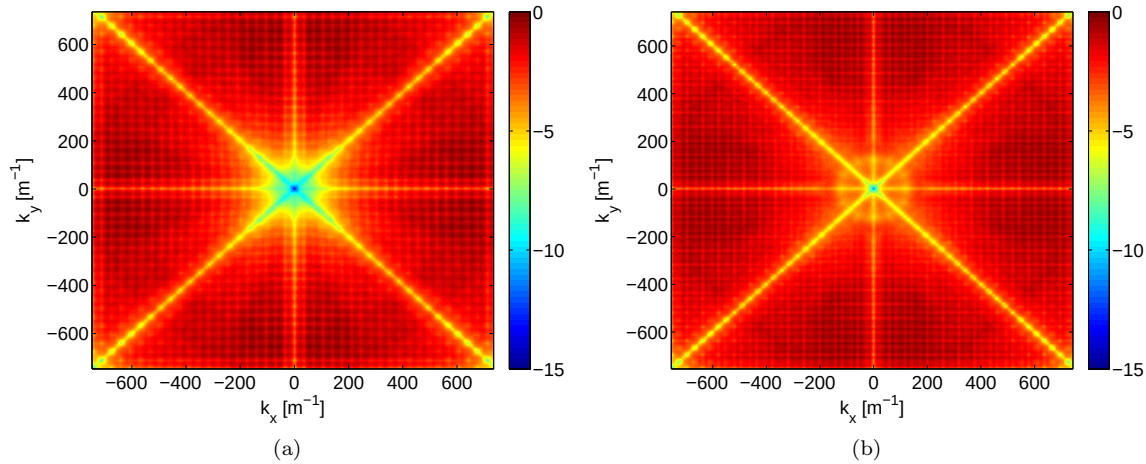


Figure 2.28: k -space representation of the internal operator $\mathbf{A}_i$ for two different frequencies: (a) 3 GHz and (b) 6 GHz.

Notice that the internal operator $\mathbf{A}_i$ is only affected by the frequency of operation and not by the position of the antennas. Fig. 2.28 makes it obvious that the k -space spanned by $\mathbf{A}_i$ is a lot larger than the one spanned by the linear operator, therefore it affects many more spatial frequencies of χ .

In any case it will be the external operator $\mathbf{A}_e$ that determines how many spatial frequencies end up actually affecting the scattered electric field. As a result, the fact that $\mathbf{A}_i$ operates on a larger set of spatial frequencies could have no effect whatsoever, because those would be filtered by $\mathbf{A}_e$, leading to no advantage. This, however, is not what happens thanks to the specifics of the multiplication of $\mathbf{A}_i$ and χ .

It can be shown [72, 73] that **the effect of the internal operator is to “beat” high-harmonics of the contrast function into lower harmonics**, much like a modulation effect. This is crucial, because it implies that spatial components that previously were inaccessible now can be reconstructed because they are beat into lower harmonics and therefore are not affected by the filtering effect of the linear operator.

This is the fundamental advantage of the quadratic forward model over the linear model, which has been tested by means of numerical examples.

Method of Moment algorithm to compute the scattered field

Expressing the contrast as a summation of a finite number of harmonics presents a new challenge: the retrieval of the scattered field needed for the inversion. The field can come from two sources: a laboratory measurement or a computer simulation. If the second way is adopted, it is necessary to find a simulator that allows to define smooth, continuous variations of ϵ_r , pixel by pixel. This is not the case in most commercial software. Some educational Finite Difference Time Domain implementation allow complete control over the domain of investigation, letting the user set the properties of every pixel individually. These routines, however, lack in performance, requiring exceedingly long times to obtain the solution.

Interestingly, a clever solution to this problem lies into the problem itself. The calculation of the scattered electric field from the knowledge of the contrast function is the definition of the forward model. In particular, this very chapter described the exact solution to the forward problem while deriving the formulation of the quadratic model.

Equations (2.51) and (2.52) represent the forward problem in its entirety. The problem is nonlinear when χ must be recovered from the knowledge of E^s . However, the problem is much simpler for the computation of E^s if χ is known.

The solution is found using the exact same minimization algorithm employed for the retrieval of the

contrast function. First, a functional is defined using (2.52):

$$\Psi^{Et} = \|E^t - E^i - \mathbf{A}_i \chi E^t\|^2. \quad (2.69)$$

Then, (2.69) is minimized with E^t as an unknown. The functional defined by Ψ^{Et} is convex in E^t , therefore the minimum solution will be found without worrying about local minima.

After E^t has been found, the scattered field is given by (2.51).

This method is basically a Method of Moments algorithm. It is therefore accurate when the meshing of the scattering object – in this case, the discretization of the domain under investigation – is sufficiently small. In virtually all cases, a discretization of $\lambda/15$ or finer proves to be sufficient to obtain the scattered field correctly.

The algorithm just described has been employed to calculate the scattered electric field in all the cases simulated in this chapter.

2.2.3 Inversion

Due to the quadratic nature of the forward model, the inversion has to be thought from scratch. All the methods described in the previous chapters assume to work with a linear system of equations that is expressed as a matrix-vector multiplication. This is not the case anymore.

The most straightforward way to tackle the inversion is to recast it as an optimization problem. Doing that, a variety of tools for solving both linear and non-linear optimization problems becomes available.

First, the problem is expressed as a functional. In this case the functional is immediately written starting from (2.58) as:

$$\Psi^{\text{quad}} = \|E^s(\mathbf{r}^r) - \mathbf{A}_e [\chi \mathbf{A}_i \chi E^i]\|^2. \quad (2.70)$$

Equation (2.70) is solved using the same Levenberg-Marquardt minimization algorithm employed earlier to calculate the scattered field.

In order to compare the results obtained with the quadratic forward model to the ones obtained with the linear forward model, it is important that the inversion algorithm be the same for both cases. Therefore another functional is defined for the linear problem, and it is then inverted using the same algorithm and the same parameters adopted for the quadratic model. Namely:

$$\Psi^{\text{lin}} = \|E^s(\mathbf{r}^r) - \mathbf{A}_e \chi E^i\|^2. \quad (2.71)$$

When expressed in matrix form $\mathbf{A}_i$ is much larger than $\mathbf{A}_e$. While $\mathbf{A}_e$, just like $\mathbf{L}$, has as many rows as the number of measurements and as many columns as the number of unknowns, $\mathbf{A}_i$ is square and has as many rows and columns as the number of unknowns (i.e. pixels); it is therefore expected that the minimization of Ψ^{quad} be heavier than the minimization of Ψ^{lin} .

2.2.4 Numerical results

The procedure detailed in this chapter has been applied to numerical data. In the following, a few examples are used to demonstrate the differences between linear and quadratic forward model in solving the same inversion problem.

Imaging of a dielectric cylinder

In the first test imaging of a dielectric cylinder is attempted. The simulation setup is visualized in Fig. 2.29.

The target is illuminated by 11 transmitters, placed along a circumference of radius $r_t = 43.2$ cm; the transmitters span 280° . The resulting scattered field is sampled by a set of receivers, placed along a circle of radius $r_r = 32.8$ cm; the receivers span all 360° . The target is sampled with a sinusoidal signal of frequency equal to 3 GHz.

The Domain of Investigation (DoI) is a square of side equal to 8 cm, divided into square pixels of side equal to 4.2 mm, or approximately $\lambda/24$. The total number of pixels is therefore 400.

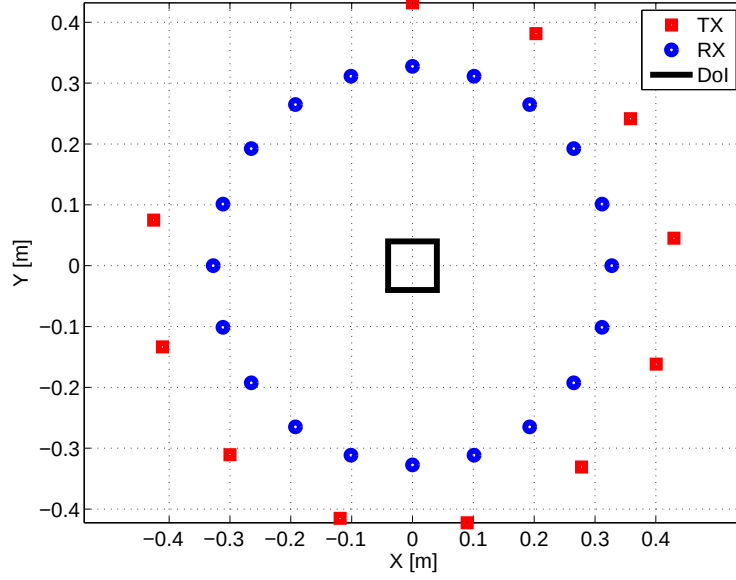
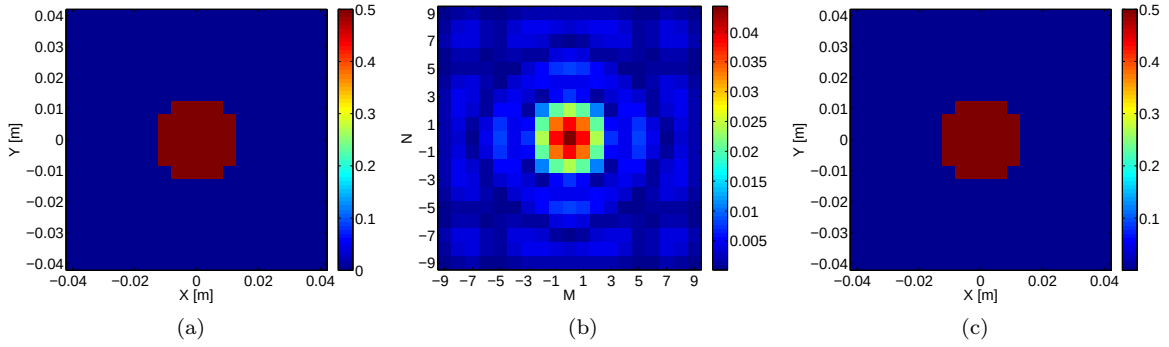


Figure 2.29: Simulation setup.

Figure 2.30: Target representation: (a) pixel-based, (b) Fourier coefficients, $|c_{m,n}|$ and (c) corresponding pixel representation with only the finite number of coefficients shown in (b).

The target is a cylinder of radius equal to 1.27 cm (1/2 in), and its contrast value is initially set to 0.5 (Fig. 2.30a). The cylinder is located at the origin of the coordinate system. The target is represented using a finite set of harmonics, with $M = N = 9$ (Fig. 2.30b). The finite number of harmonics chosen represents the target with substantially no distortion (Fig. 2.30c). Notice that by choosing a representation with a finite number of coefficients the total number of unknowns of the problem decreases significantly: from 400 pixels to $(2M + 1) \times (2N + 1)$ in the harmonic representation, where M and N usually are smaller than 7.

From the knowledge of the domain of investigation both external and internal operators can be computed. In addition, the incident field is calculated analytically since the source is the well-known infinite current line in a 2D geometry. Lastly, after creating the target and the domain of investigation, the scattered electric field is computed using the algorithm described in Section 2.2.2.

The scattered field thus calculated is used for the inversion. The main variable when performing the inversion is the number of unknowns (i.e. harmonics) that want to be reconstructed. Due to ill-conditioning and to the approximations given by both the linear and the quadratic models, it is unrealistic to reconstruct all the 10 components which fully represent the target. Therefore, the desired number of harmonics to reconstruct is given as input to the inversion. The larger the number of Fourier coefficients required, the larger is the size of the problem, and the more both models are put to the test.

As a first test, only $M = N = 2$ harmonics are reconstructed. The result is shown in Fig. 2.31. Since

only three harmonics are reconstructed, both the linear and the quadratic model provide good reconstruction results. Due to the very small differences in the reconstruction of the Fourier coefficients, the pixel representation obtained from them is almost identical in the two cases.

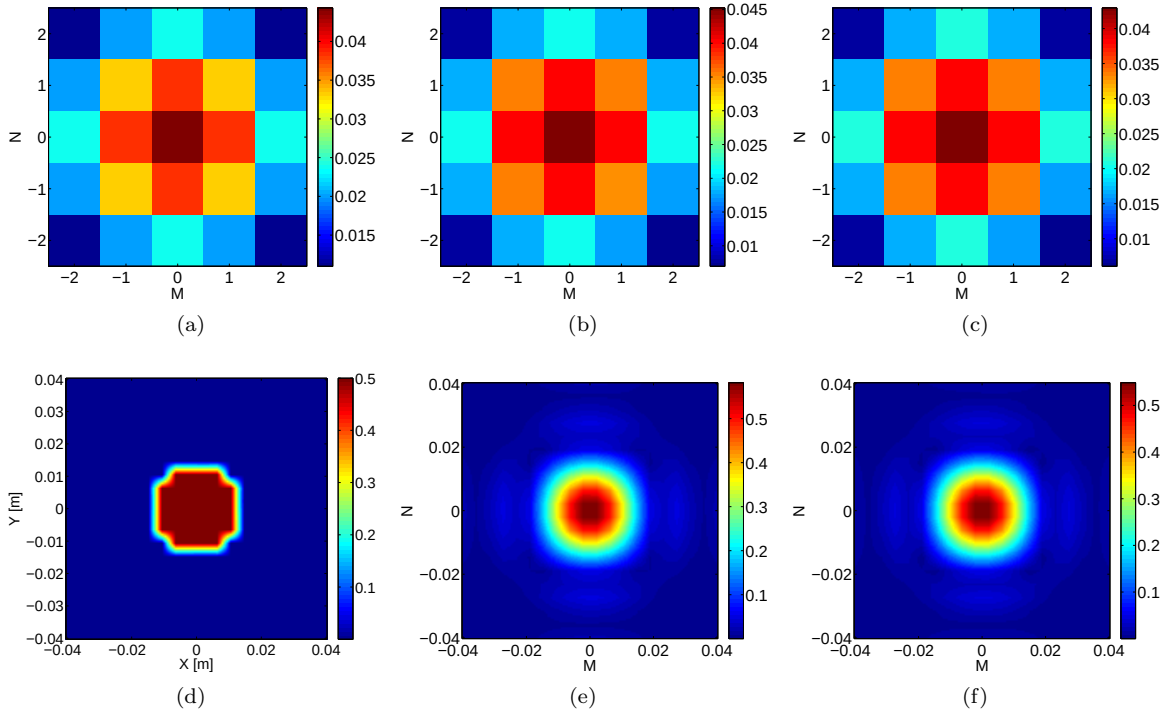


Figure 2.31: Reconstruction result for $M = N = 2$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

In order to give a quantitative measurement of the differences between the reconstruction obtained using the linear and the quadratic models, the Mean Squared Error (MSE) is computed as:

$$\text{MSE} = \sqrt{\frac{\sum |\chi^{\text{rec}} - \chi^{\text{true}}|^2}{\sum |\chi^{\text{rec}}|^2}}, \quad (2.72)$$

where χ^{rec} and χ^{true} represent the pixel representation of the reconstructed and true contrasts, respectively.

In this first test, the linear model produces an $\text{MSE} = 0.51838$ and the quadratic model an $\text{MSE} = 0.50058$, which represents an improvement of 3.4%.

Additional insight on the quality of the reconstruction can be gained by taking section cut plots of the target, as shown in Fig. 2.32.

The section cuts shows that the absolute value of the contrast is more closely reconstructed by the quadratic model; the linear model tends to over-estimate the peak value. Linear and quadratic models show almost identical side-lobe artifacts.

Increasing the number of Fourier coefficients to reconstruct is possible to stress the differences between linear and quadratic models. In order to give a detailed description of the inversion, the results corresponding to values of M and N ranging from 3 to 7 are shown in Fig. 2.33 through Fig. 2.40

When $M = N = 3$ the performance gap between linear and quadratic models widens, although both models performs overall better than the previous case. The linear models produces an $\text{MSE} = 0.48637$, while the quadratic model returns an $\text{MSE} = 0.43695$ (10.2% improvement). Fig. 2.33 summarizes the results. Looking at the plot of the absolute values of the coefficients $c_{m,n}$ for both models there is not a significant difference from the true values. The same applies to the pixel representation.

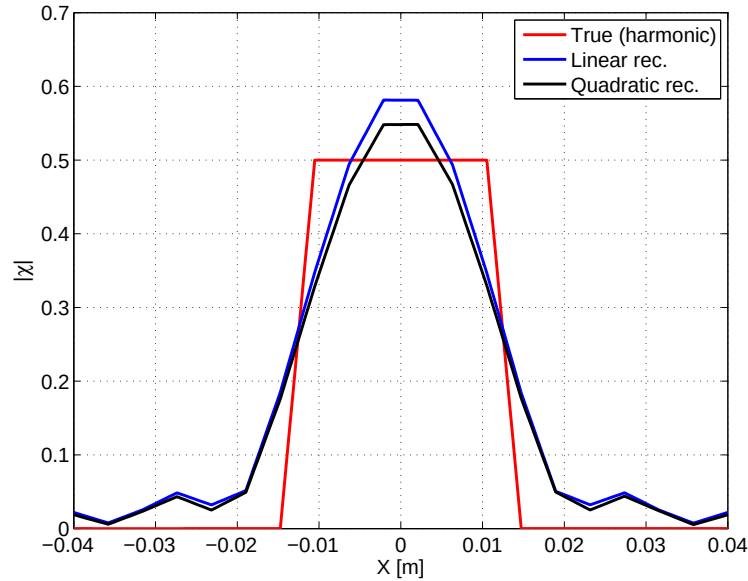


Figure 2.32: Section cuts of the true and reconstructed targets at $y = 0$ for $M = N = 2$.

Fig. 2.34 shows the section cuts. In this case both quadratic and linear models visibly over-estimate the peak value of the contrast function, with the linear model producing the least accurate reconstruction.

When $M = N = 4$ the linear model performs better than the quadratic model. Fig. 2.35 shows that the quadratic model produces high-frequency artifacts that interfere with the true components. In this case $\text{MSE} = 0.49485$ for the linear model. The quadratic model performs worse, with $\text{MSE} = 0.6793$, 37% worse) because the reconstruction is affected by high-frequency components. Looking at the Fourier coefficients plot (Fig. 2.35) it is clear that strong high-frequency artifacts alter the reconstruction, but that the true information contained in the lower spatial frequencies is still preserved.

The section cuts plot (Fig. 2.36) highlights the side-lobe artifacts which affect the quadratic model.

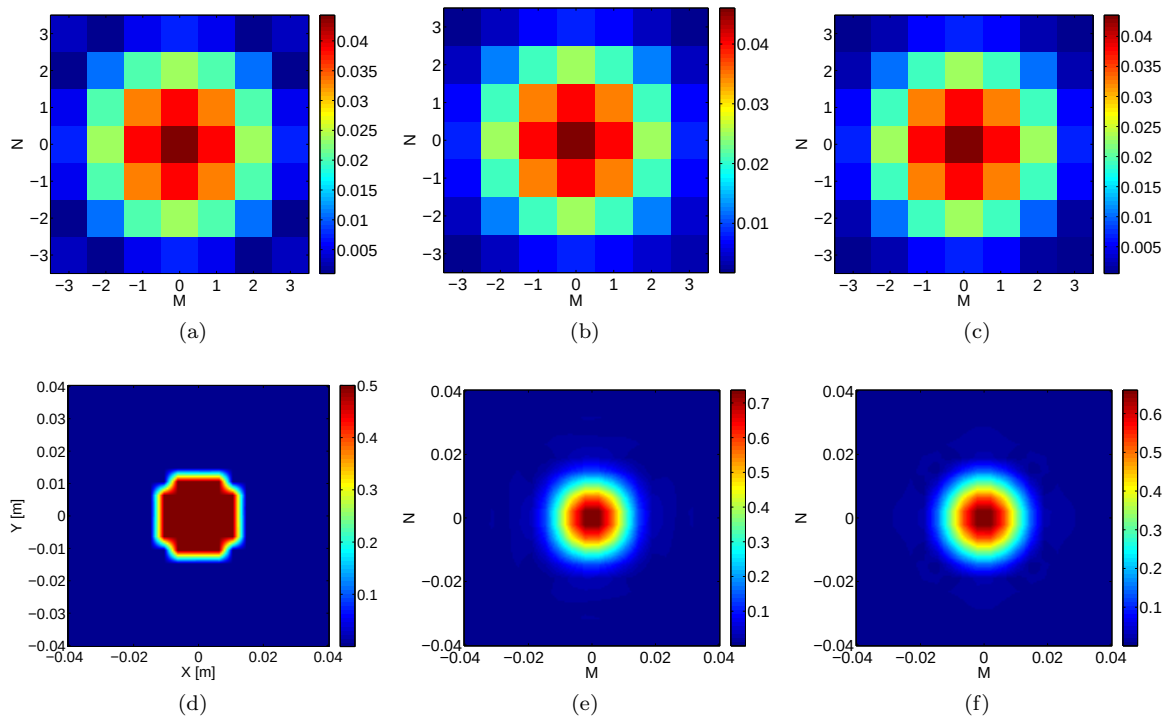


Figure 2.33: Reconstruction result for $M = N = 3$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

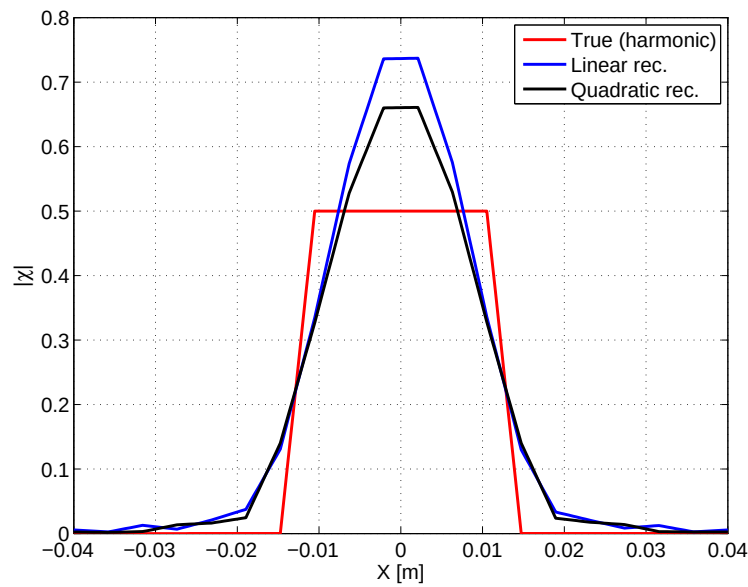


Figure 2.34: Section cuts of the true and reconstructed targets at $y = 0$ for $M = N = 3$.

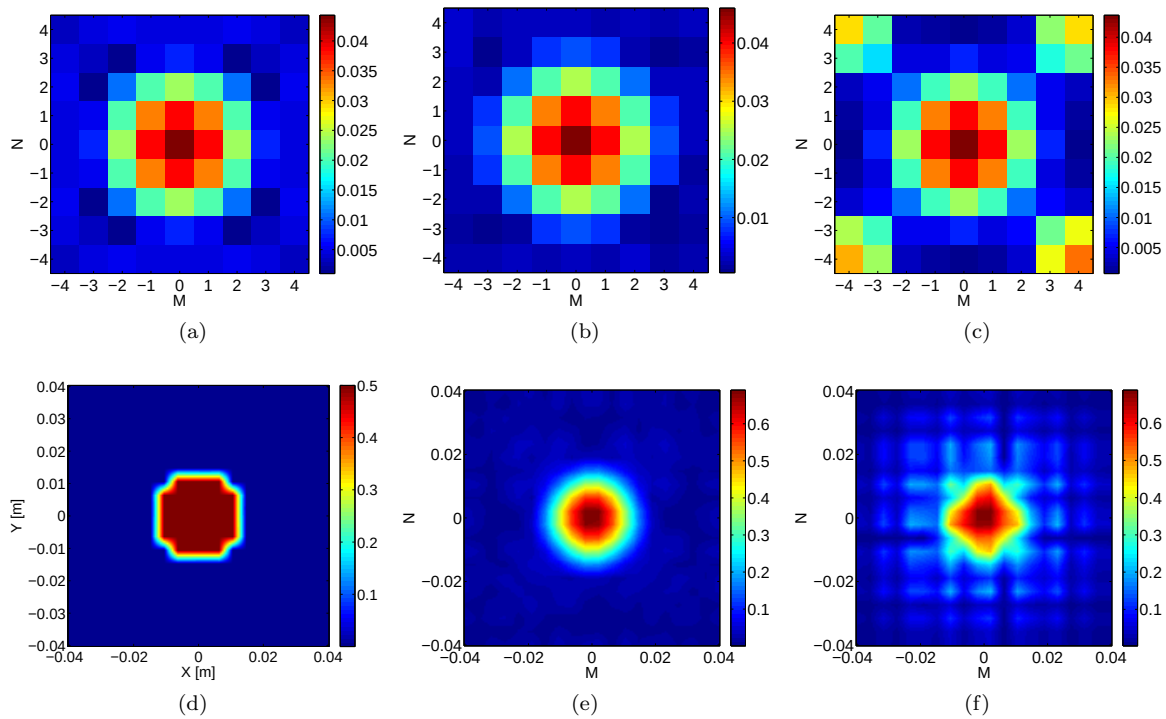


Figure 2.35: Reconstruction result for $M = N = 4$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

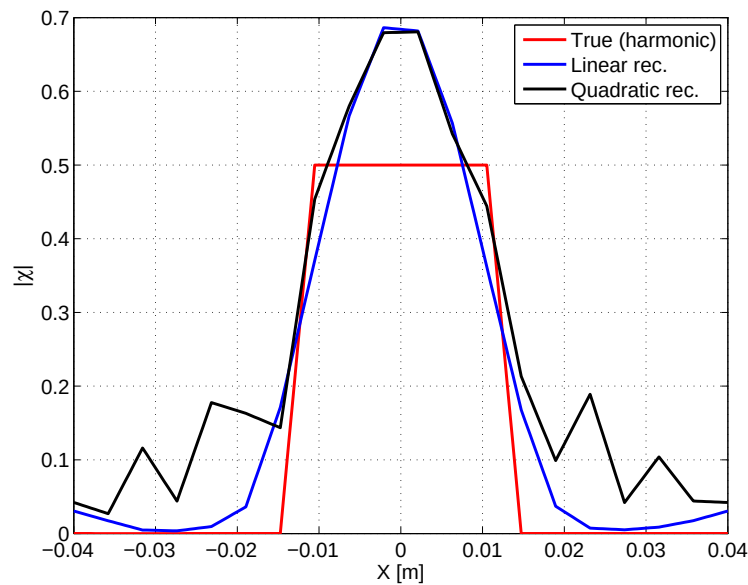


Figure 2.36: Section cuts of the true and reconstructed targets at $y = 0$ for $M = N = 4$.

The reconstruction with $M = N = 5$ is shown in Fig. 2.37.

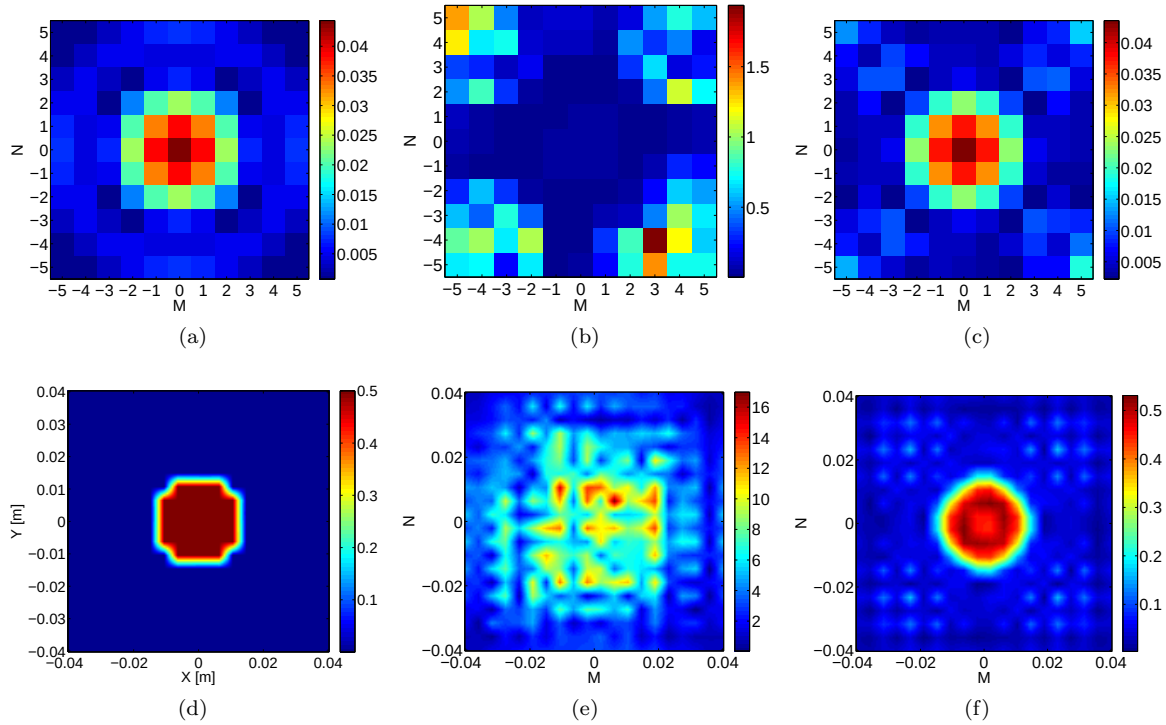


Figure 2.37: Reconstruction result for $M = N = 5$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

The linear reconstruction is completely disrupted by noise, with the high spatial frequency artifacts completely ruining the reconstruction. The value of $\text{MSE} = 1.0006$ is not very informative, since it is obtained as an overall sum, therefore cannot correctly represent the type of noise observed in the image. The quadratic case, instead, performs better than the previous case. The high frequency noise is spread over a larger number of pixels, making the reconstruction of the meaningful components more accurate. The MSE for the quadratic case is equal to 0.47866, i.e. improved with respect to the case with $M = N = 4$, but still not as low as when $M = N = 3$. Also in this case, though, the MSE indicator can be misleading, because it hides the fact that the edge of the cylinder is reconstructed with much greater accuracy in this case than in the case with $M = N = 3$.

The section cuts plot (Fig. 2.38a) is not very informative, since the linear model produces artifacts so large that a change of scale would be needed in order to compare linear and quadratic reconstructions. When the scale is adjusted, Fig. 2.38b, it is possible to appreciate the good quality of reconstruction offered by the quadratic model. The comparison of Fig. 2.38b and Fig. 2.34 makes it clear that in this case the cylinder edge is reconstructed with superior accuracy.

Further increasing the number of reconstructed harmonics confirms this trend. The linear model performs extremely poorly for both $M = N = 6$ and $M = N = 7$, while the quadratic model, although more noisy, still produces very good images. The accuracy of reconstruction is evident in the coefficient plots of both Fig. 2.39 and Fig. 2.40, which translate in noisy but substantially accurate pixel representations in both cases. For the quadratic models the MSE is equal to 0.55254 when $M = N = 6$ and 0.55642 when $M = N = 7$.

Importantly, for the quadratic model in all cases the absolute value of the reconstructed contrast is never higher than 0.7 (compared to 0.5 for the representation of the true contrast using 9 harmonics, which has been used to compute the scattered electric field), providing an estimation of the relative dielectric permittivity of the cylinder with sufficiently good approximation.

In the next section the effect of changing the value of the contrast is shown.

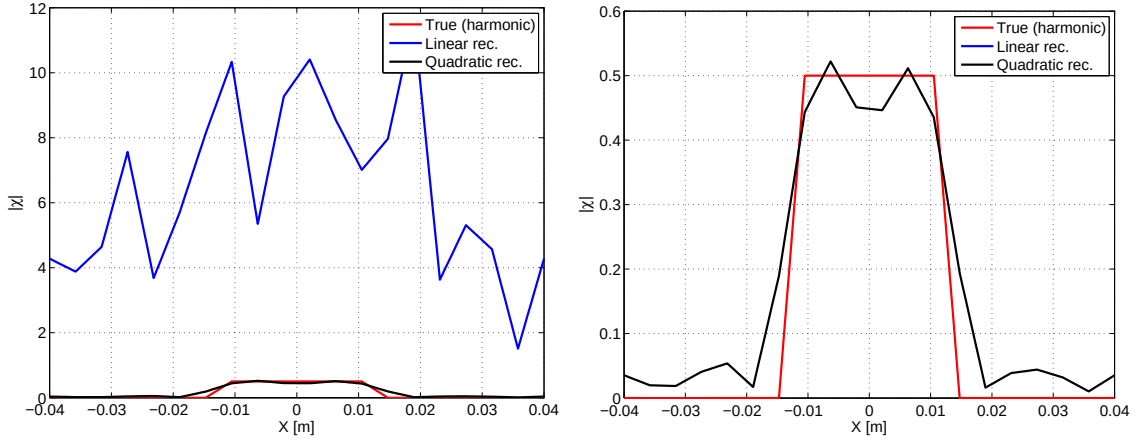


Figure 2.38: Section cuts of the true and reconstructed targets at $y = 0$ for $M = N = 5$.

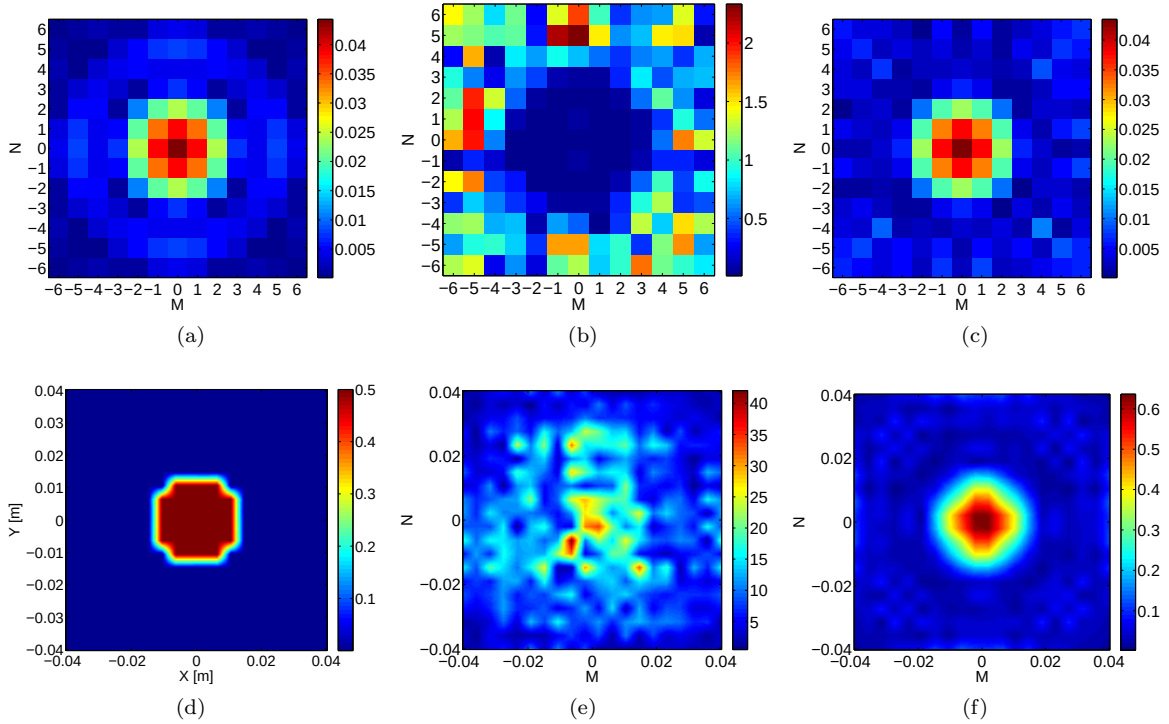


Figure 2.39: Reconstruction result for $M = N = 6$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

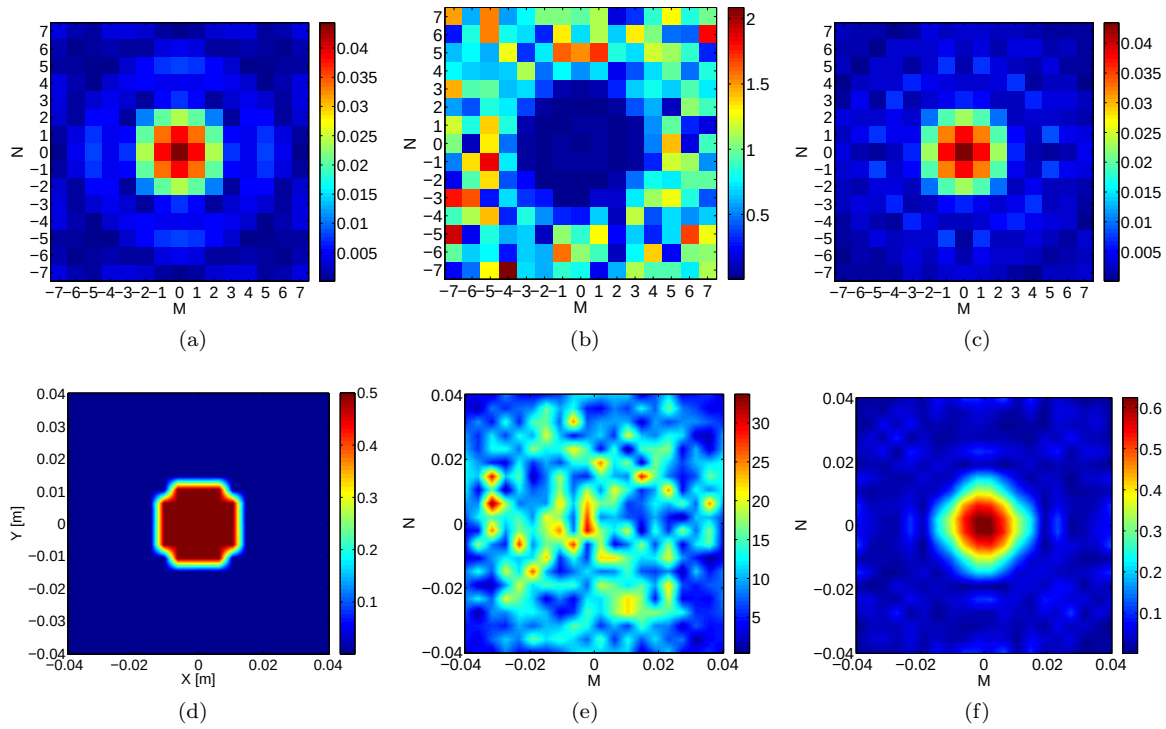


Figure 2.40: Reconstruction result for $M = N = 7$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

Effect of increasing the contrast value

When the value of the contrast is increased, it is more difficult for the forward model to correctly represent the physics of the problem. In fact, it has been described how both the linear and the quadratic model derive from the assumption that $\|\mathbf{A}_i\chi\| < 1$. While in practice even when this condition is not met, it is often possible to obtain images, there is no guarantee of performance.

Increasing the contrast is important because in real-world conditions a value of dielectric permittivity equal to 1.5, as simulated in the previous section, is not common. Most materials have ϵ_r significantly larger.

In a laboratory environment, under controlled conditions, it is important to operate with well-known materials as targets. This excludes for example wood, whose dielectric permittivity is a function of a large number of variables: tree of origin, water content, age, humidity of the laboratory, local anisotropy due to the grain all play important roles in determining the actual value of permittivity, and are not easily estimated. Most plastics, instead, due to their regular molecular structures, show values of ϵ_r which is well contained between certain known ranges. In addition, most plastics do not show large variations with the frequency of operation, or with local conditions such as humidity. Many plastics found for retail have values of ϵ_r smaller than 3 [80, 81]. The values of ϵ_r for some easily acquirable plastics are shown in 2.1.

Molecule	Common Name	ϵ_r at 1 GHz
Fluorinated ethylene-propylene	Teflon FEP	2.05
Polytetrafluoroethylene	Teflon PFA	2.1
Polycaprolactam	Nylon-6	3.0 @ 1 MHz
Polyethyleneterephthalate	PET, Mylar	2.8
Low-Density Polyethylene	LDPE	2.2
High-Density Polyethylene	HDPE	2.3
Polypropylene	PP	2.2
Polyvinylchloride	PVC	2.8

Table 2.1: Dielectric properties of some plastics.

The actual values for material acquired through retail distributors can show differences, but these are usually not large, due to the very regular nature of these molecules and their well-standardized manufacturing techniques.

After these considerations, it is likely that in most cases a value of ϵ_r no larger than 2.5, corresponding to a contrast of 1.5 in free-space, can be expected. Therefore, the imaging case used in the previous section has been replicated, with the only difference of changing the value of contrast, this time equal to 1.5.

For this case, only the results for some representative values of M and N are chosen, in order to shorten the description.

The imaging result with $M = N = 3$ is represented in Fig. 2.41 and shows that increasing the contrast from 0.5 to 1.5 did not severely affects the reconstruction quality. The linear model performs worse than the quadratic model; both images are not affected by noise, but they lack resolution due to the low number of harmonics reconstructed. The linear model returns $\text{MSE} = 0.87711$, while the quadratic produces an $\text{MSE} = 0.61539$, which is almost 30% lower (hence better).

The section cut plot shown in Fig. 2.42 confirms that the quadratic reconstruction, although closer to the peak value of the contrast shows larger side-lobes artifacts.

Increasing the number of harmonics to $M = N = 5$ makes the linear model fail completely as shown in Fig. 2.43. The quadratic model instead returns a fair image, significantly better than the one produced with the linear model. For the quadratic model $\text{MSE} = 0.6354$ or about 5% worse than the previous case.

The section cut plot is provided in Fig. 2.44. In the figure the linear reconstruction is not visible because it is off-scale by a large amount. The quadratic reconstruction is noisy but correctly follows the outline of the true contrast.

Fig. 2.45 shows the result of the inversion when $M = N = 7$. The linear model still completely fails. The quadratic reconstruction instead does not significantly change. The MSE for the quadratic case is worse, equal to 0.7539. Looking at the reconstruction of the Fourier coefficients (Fig. 2.45c) is evident that there is some high-frequency noise, but that the main components in the Fourier space are reconstructed correctly.

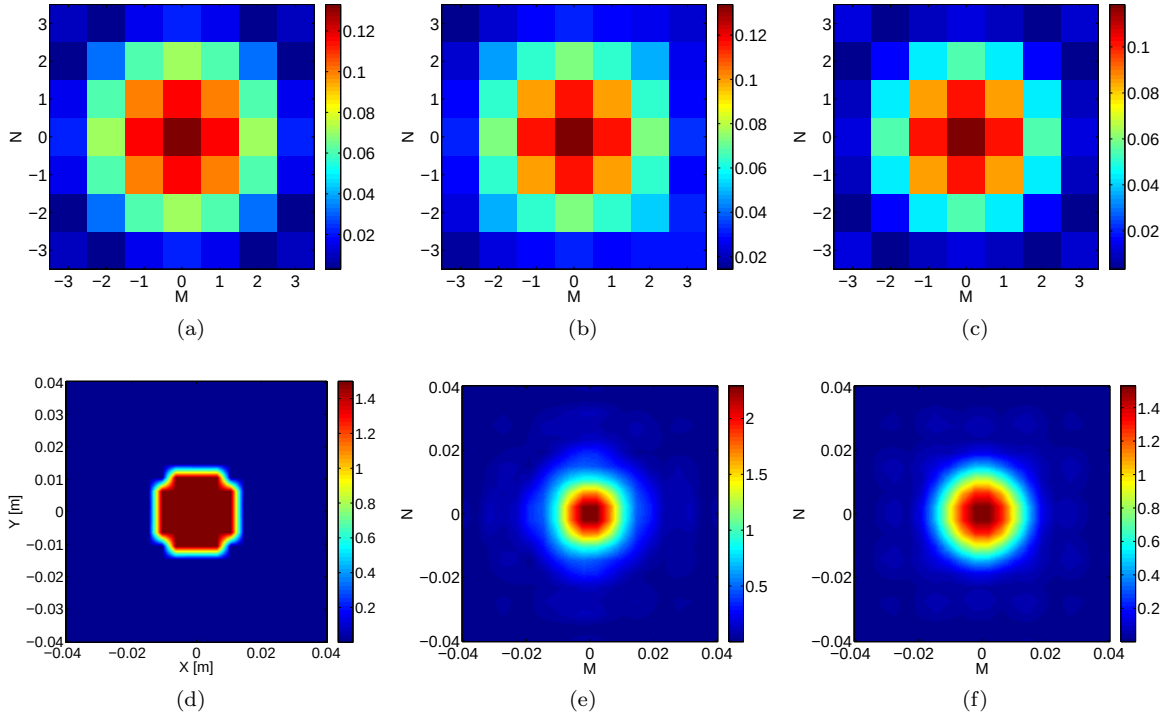


Figure 2.41: Reconstruction result for $M = N = 3$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

The section cut shown in Fig. 2.46 has been adjusted by rescaling the vertical axis. The linear reconstruction is not visible because it is much larger than the plot scale. The quadratic reconstruction is very noisy but fairly close to the actual numerical value of the contrast.

In conclusion, increasing the value of the contrast has detrimental effects on the reconstruction. However, even under these difficult conditions, without changing at all the experiment conditions the quadratic model is able to return a solution that is close to the true contrast in terms of absolute of χ , and with fair reconstruction of the Fourier coefficients, which are affected by noise, but not to the point of rendering the reconstruction useless. The linear model instead fails as soon as more than 3 harmonics are requested.

This reconstruction example can be complicated by adding more objects in the domain of investigation, changing significantly the Fourier coefficients that need to be reconstructed.

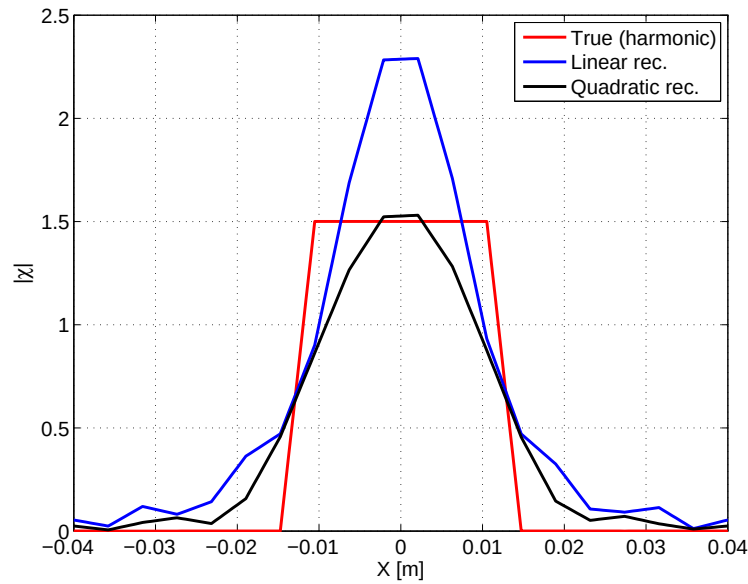


Figure 2.42: Section cuts of the true and reconstructed targets at $y = 0$ for $M = N = 3$.

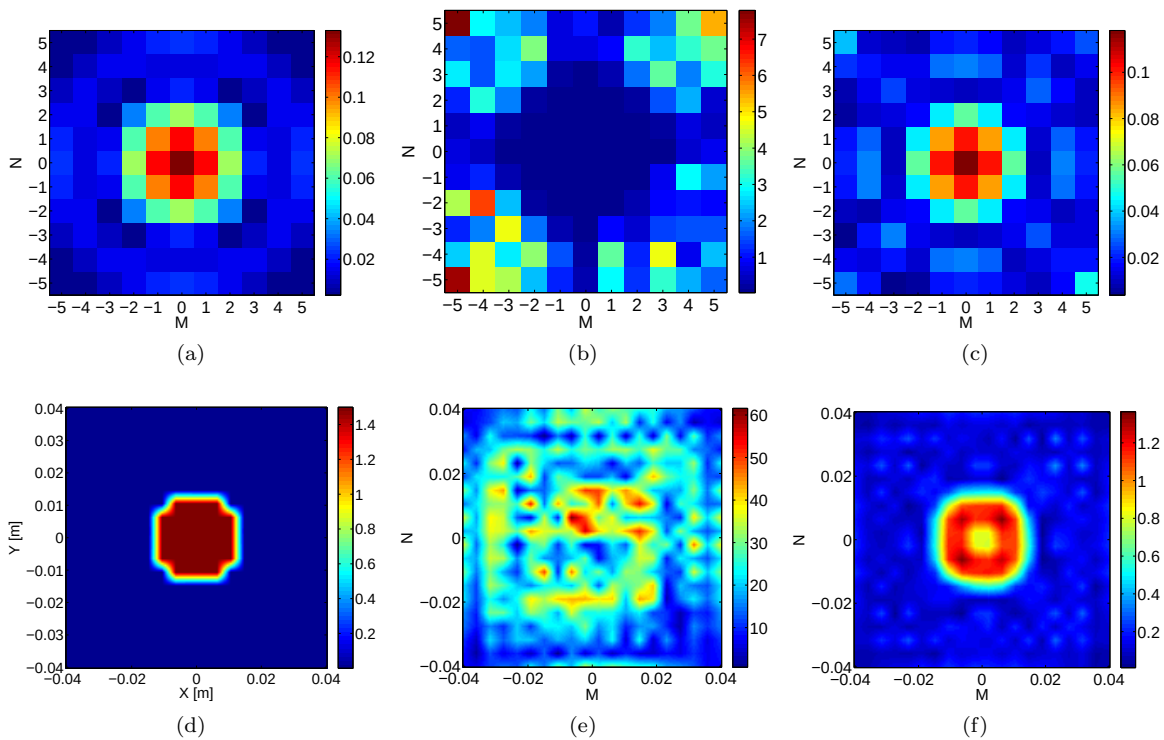


Figure 2.43: Reconstruction result for $M = N = 5$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

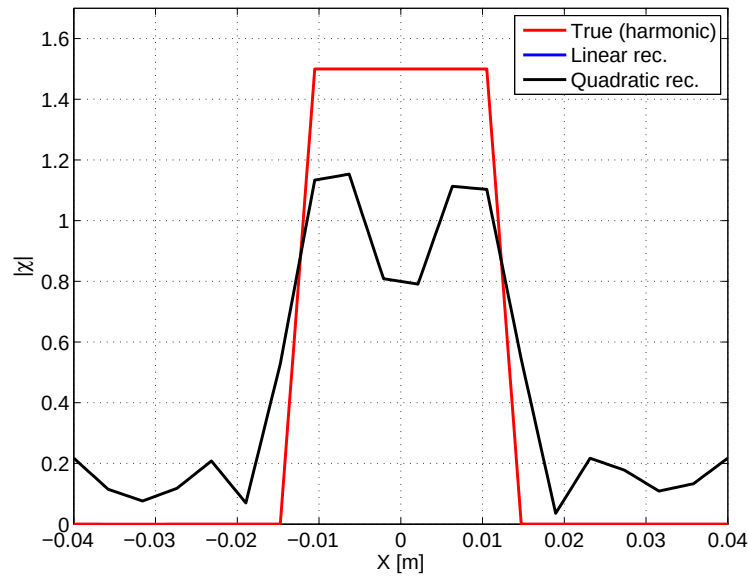


Figure 2.44: Section cuts of the true and reconstructed targets at $y = 0$ for $M = N = 5$.

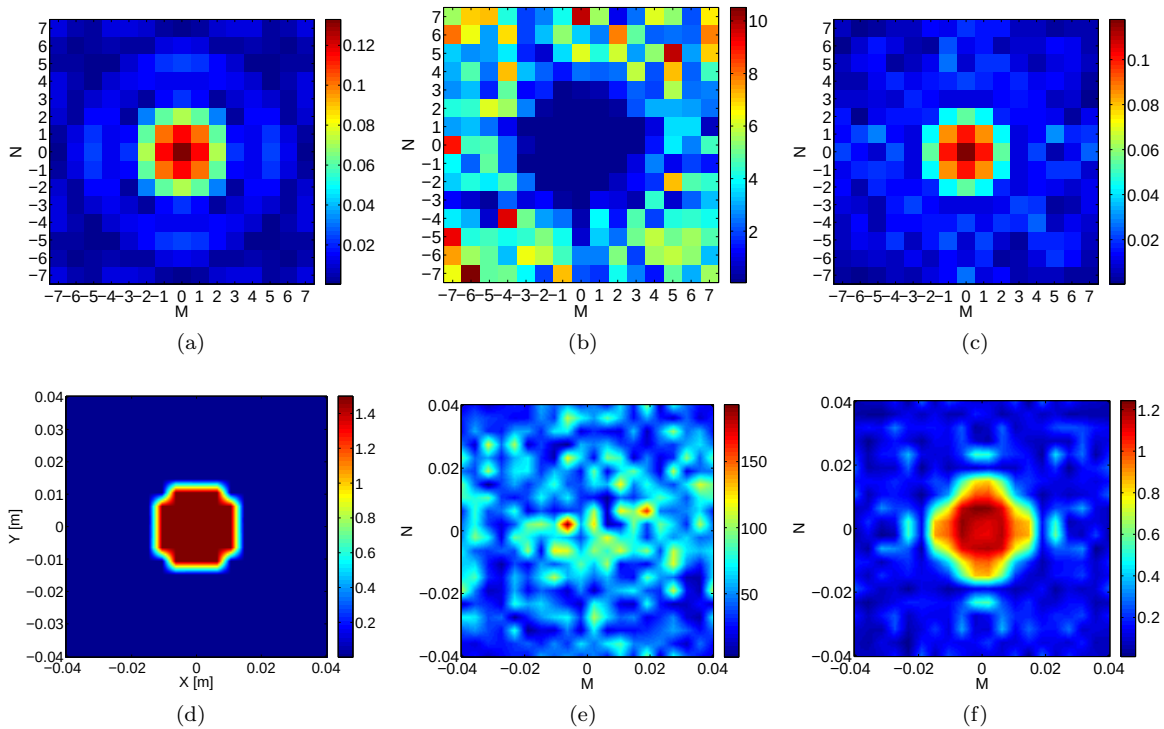


Figure 2.45: Reconstruction result for $M = N = 7$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

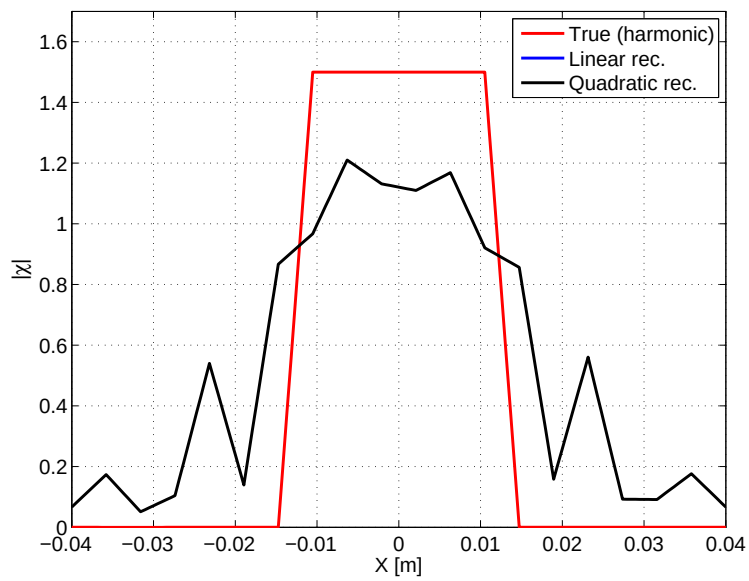


Figure 2.46: Section cuts of the true and reconstructed targets at $y = 0$ for $M = N = 7$.

Imaging of two dielectric cylinders

In this section imaging of two dielectric cylinders is studied. Two cylinders of radius 0.95 cm (diameter 3/4 in) are placed at a distance of 3.4 cm from each other along the x axis. The distance is measured with respect to the centers of the two cylinders and corresponds to a separation of 1.5 cm measured from edge to edge of the cylinders. The targets are represented with $M = N = 24$ and their dielectric constant is set to 2.5 (i.e. $\chi = 1.5$) as in the latest example. The targets are shown in Fig. 2.47.

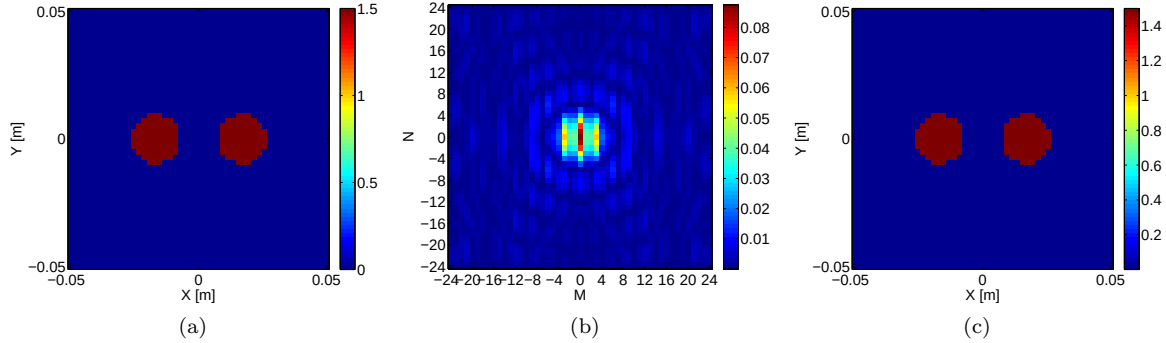


Figure 2.47: Target representation: (a) pixel-based, (b) Fourier coefficients, $|c_{m,n}|$ and (c) corresponding pixel representation with only the finite number of coefficients shown in (b).

This setup is more difficult to image than the previous one. The smaller targets mean that more harmonics are significantly different from zero and the presence of two targets close to each other causes mutual interaction phenomena. For this reason, the antenna setup has been changed.

The antenna setup is shown in Fig. 2.48. 15 TX and 40 RX are employed, operating at 4.5 GHz ($\lambda \approx 6.7$ cm). The domain of investigation is a square of side 10 cm. The pixel size has been decreased to 2 mm ($\approx \lambda/33$). The reason for a smaller pixel size has been dictated by simple visual reasons: the targets employed in this case are small, so that large pixels would have made cylinders appear as squares. From an electromagnetic point of view, however, there is no need to choose such a small pixel size; the electric field can be computed with reasonable accuracy using pixels of side equal to 6 mm as before.

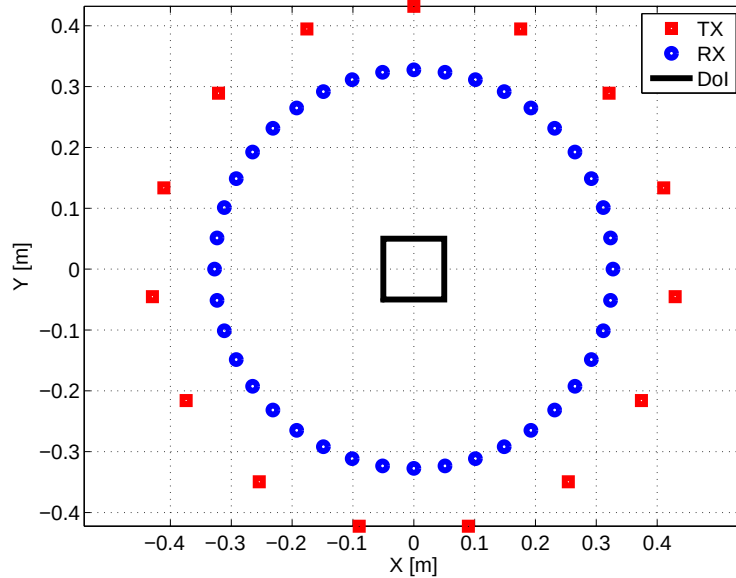


Figure 2.48: Simulation setup.

The result of the reconstruction setting $M = N = 3$ is shown in Fig. 2.49. The result shows that the

quadratic model is superior to the linear one. The image is less noisy and shows no artifact between the two cylinders. For the linear model the MSE is equal to 1.1735, while for the quadratic model MSE= 0.87538, a 24% improvement.

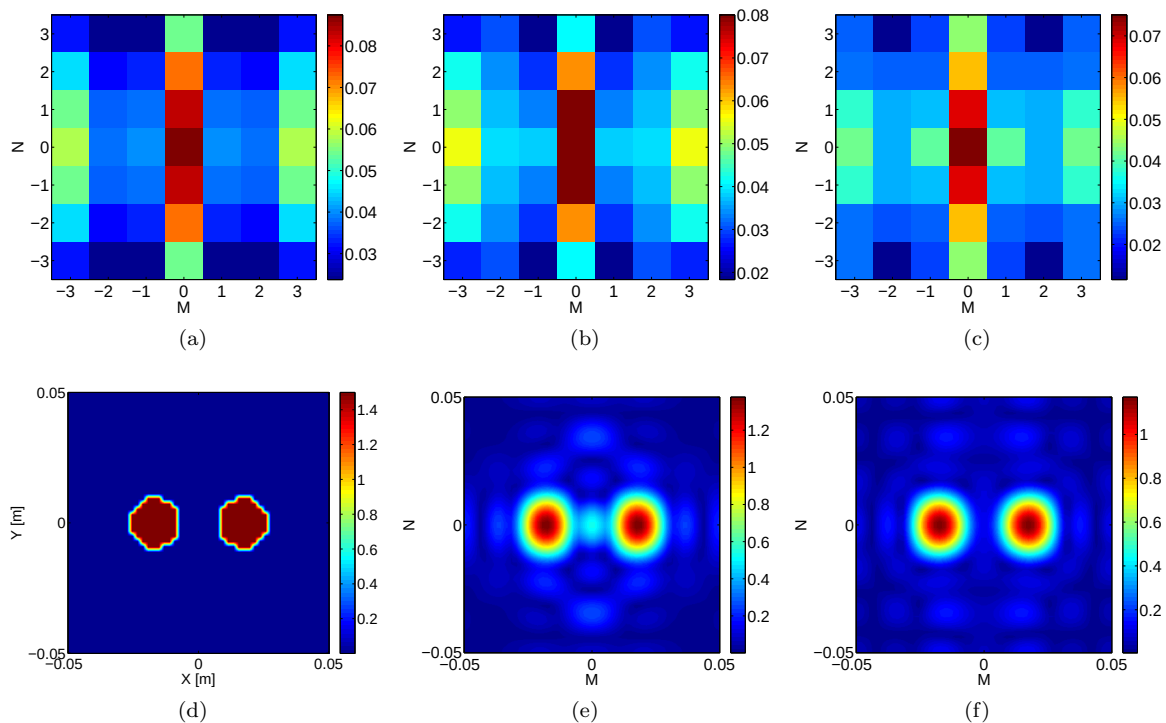


Figure 2.49: Reconstruction result for $M = N = 3$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

The section cut shown in Fig. 2.50 highlights this difference. Although the linear model approximates more closely the peak value of the contrast, the quadratic model produces reduced side-lobes artifacts as well as no artifact between the two targets.

The fact that imaging is more difficult is highlighted by the fact that both methods fail when reconstruction of just one more harmonic ($M = N = 4$) is attempted. Fig. 2.51 shows that both methods introduce very large noise in high-frequency components that completely disrupt the final reconstruction.

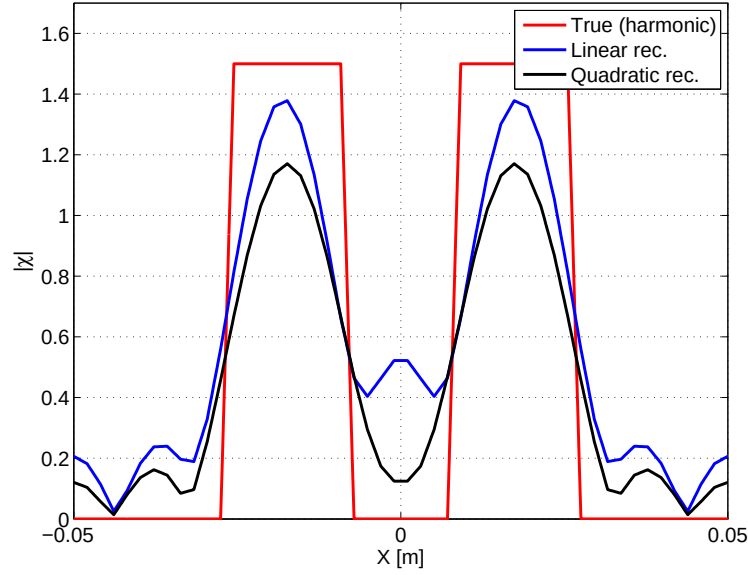


Figure 2.50: Section cuts of the true and reconstructed targets at $y = 0$ for $M = N = 3$ at 4.5 GHz.

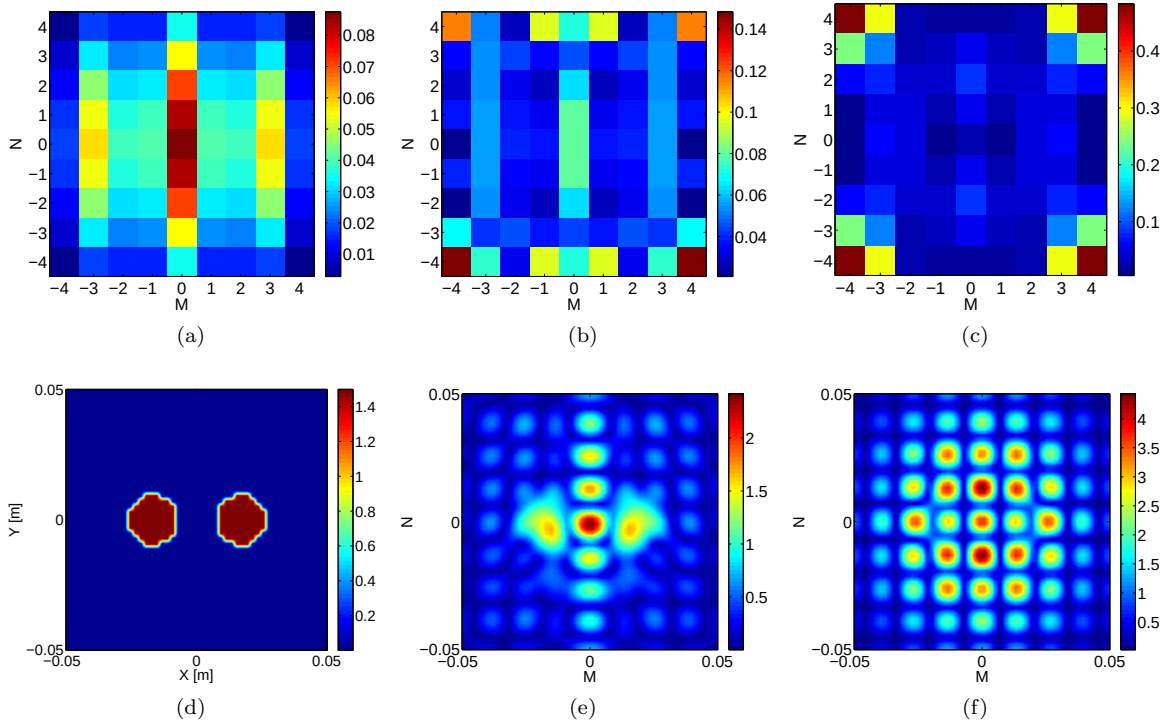


Figure 2.51: Reconstruction result for $M = N = 4$. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

In order to overcome this difficulty the frequency of operation is increased. This means that for both linear and quadratic models the set of spatial frequencies that fall into the reconstructible domain is enlarged. However, the quadratic model is still expected to outperform the linear model, because of its superior ability to reconstruct higher spatial harmonics.

The simulation is performed again, with the only change of increasing the frequency from 4.5 GHz to 6 GHz. The result of the inversion is shown in Fig. 2.52.

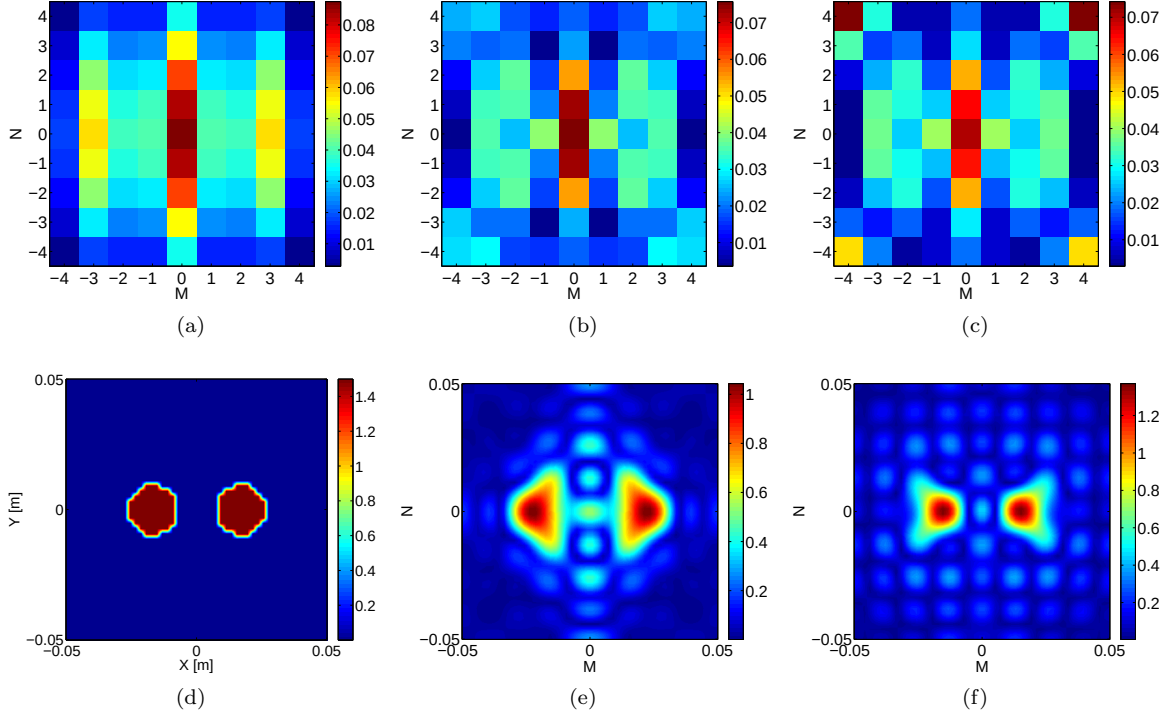


Figure 2.52: Reconstruction result for $M = N = 4$ at 6 GHz. (a) True Fourier coefficients $|c_{m,n}|$, (d) True contrast function, (b) and (e) Linear model reconstruction, (c) and (f) Quadratic model reconstruction. The resolution of (d)–(f) has been increased through linear interpolation.

In this case, although both models are noisy, the quadratic model clearly outperforms the linear one. For the linear model periodic artifacts are created between the two objects and the shape of the targets is not correctly reconstructed. In this case $\text{MSE} = 1.3636$ and the peak value of the contrast is estimated to be around 1, instead of the expected 1.5.

For the quadratic model, instead, $\text{MSE} = 0.91725$, i.e. almost 33% better. Periodic noise still affects the reconstruction, but the shape and size of the targets can be more correctly inferred. In addition, the peak value of the contrast is estimated to be around 1.4, a significantly better estimation than the linear case. These considerations are all made very clear in the section cuts depicted in Fig. 2.53.

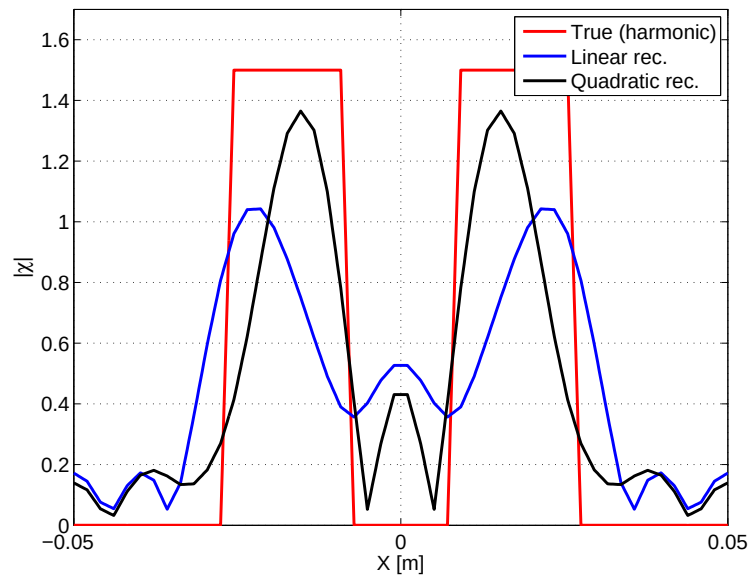


Figure 2.53: Section cuts of the true and reconstructed targets at $y = 0$ for $M = N = 4$ at 6 GHz.

Imaging of an L-shaped object

The second test-case refers to an L-shaped object made of plexiglass ($\varepsilon_t \approx 2.6\varepsilon_0$). The object is 6.9 cm (1.15λ) and 5.6 cm (0.93λ) long along x and y respectively, and the thickness of each arm is about 0.5 cm (0.08λ). The most accurate solution is attained when $M = N = 4$ (see Figs. 2.54 and 2.55). In spite of the small thickness of the L arms, which is lower than system resolution, the reconstructions of Figs. 2.54 and 2.55 confirm the superior focusing of the quadratic model with respect to the linear one. The model error is not negligible also in this case ($\|\mathbf{A}_i\chi\| = 9.9$), thus the retrieved contrast does not allow to estimate the target permittivity.

The resolution improvement provided by the quadratic model is quantified in terms of the normalized sum of squared intensity:

$$C = \frac{\sum_{q=1}^Q |\chi(x_q, y_q)|^4}{\left(\sum_{q=1}^Q |\chi(x_q, y_q)|\right)^4} \quad (2.73)$$

where (x_q, y_q) are the coordinates of pixel q in the image, and Q is the total number of pixels. Note that a greater value of C correspond to a higher image sharpness. The C^{quad}/C^{Born} ratio summarized in Tab. 2.2 confirm that the tomographic images obtained with the quadratic model always have higher sharpness than those achieved inverting the linear model, save for the the L-shaped target when $M = N = 5$ because the quadratic solution (not shown) is approaching instability.

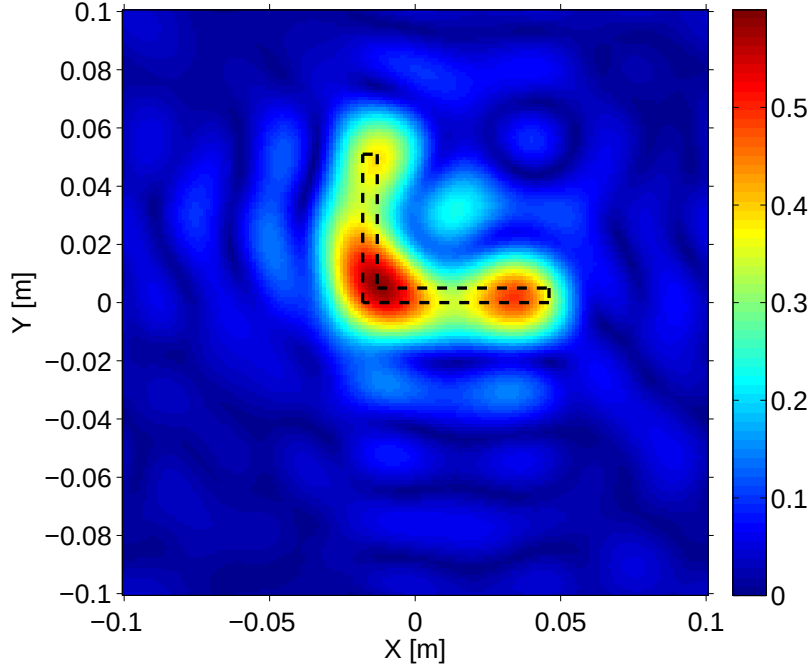


Figure 2.54: Tomographic reconstructions of the L-shaped plexiglass object at 5 GHz. Linear model using $M = N = 4$.

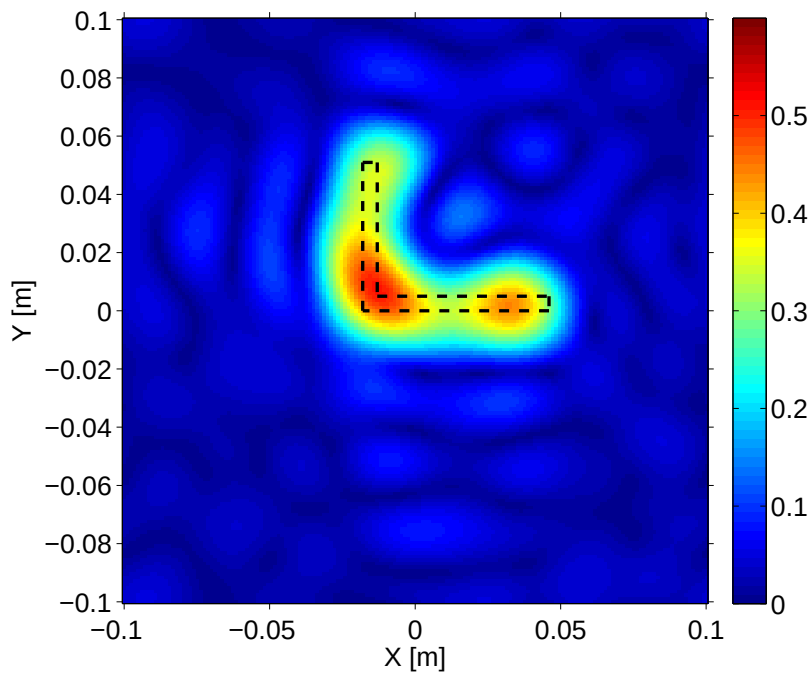


Figure 2.55: Tomographic reconstructions of the L-shaped plexiglass object at 5 GHz. Quadratic model using $M = N = 4$.

Table 2.2: Normalized sum of squared intensity ratio

Test case	C^{quad}/C^{Born}
Two HDPE cylinders (M=N=4)	1.57
Two HDPE cylinders (M=N=5)	4.55
L-shaped plexiglass object (M=N=4)	1.22
L-shaped plexiglass object (M=N=5)	0.45

Chapter 3

Task 2: Fast solvers

The forward model of RF tomography uses a dyadic Green's function that depends on the geometry of the scene. Evaluating the dyadic Green's function is equivalent to finding a radiated field due to an infinitesimal dipole. In particular, the dyadic Green's function for a complex environment, where an analytical evaluation is not practical, can be found numerically by computing the radiated field due to a infinitesimal dipole in the desired environment. Research efforts were devoted to developing fast methods to compute electromagnetic fields and validation approaches.

Fast methods were focused on high-frequency ray-based techniques because they are fast, they are particularly advantageous for electrically large problems and they allow also to provide physical insights. Another advantage is that ray-methods allow one approach to construct Green's functions, which are fundamental for the forward model. Accordingly, first we completed the derivation of the double-diffraction mechanism for the Incremental Theory of Diffraction (ITD) [15], for which preliminary results were presented in [16]. Additional details are provided in Section 3.1.

Validation approaches were developed in a twofold way. First, new exact analytical solutions that serve as benchmarks for comparison were developed. Specifically, the scattering from simple shapes consisting of elliptical [21] or spheroidal [28, 27] objects filled with metamaterials was investigated. Second, Algorithm 934, the first software publicly available to compute Mathieu functions when a parameter is a complex variable [29] was developed.

3.1 ITD Double diffraction coefficient

The new high-frequency Incremental Theory of Diffraction (ITD) formulation for the double diffraction by metallic wedges when illuminated by Complex Source Points was published in [15] and summarized in the following. The main motivation for this research was the extension of the class of problems that can be studied using asymptotic (i.e. ray-based and incremental) methods by providing double diffraction descriptions for complex source points (CSP), which are considered because they are efficient to analyze electrically large structures. The new formulation provides an accurate asymptotic description of the interaction between two edges in an arbitrary configuration, including slope diffraction contributions. Advantages of the ITD formulation for CSP illumination include avoiding the typical ray-caustic impairments of the GTD/UTD ray techniques and not requiring ray tracing in complex space.

Efficient techniques to represent the illuminating field are often employed in the accurate prediction of the far field radiated or scattered by large structures, such as large reflector antennas. As an example, a Complex-Source-Beam (CSB) expansion is successfully used to represent the radiation from a given realistic source, such as a feed or a feed array, as well as a more general arbitrary directional wave field. An exact representation of scalar fields in terms of Complex Source Points (CSP) launched from a point was first proposed in [82]. An alternative formulation for vector electromagnetic fields has been proposed in [83, 84, 85], where the field is expanded in terms of a magnetic and an electric CSP-dipole distribution on a sphere enclosing the actual sources. The expansion coefficients may be obtained by matching the CSB expansion to the known far-field pattern of the feed, or to its vector spherical wave representation obtained from measured data.

In the application of these techniques, the actual source is replaced with a discrete set of CSBs, whose beams are outward radially directed from the enclosing sphere [84]. Each complex beam reaches a large object where it may undergo both a reflection and a diffraction by the surface and the edges, respectively. The overall radiation can therefore be estimated from the superposition of the contributions from only those CSBs that are strongly intercepted by the structure.

However, for certain configurations where more than one metallic edge is present, and for grazing aspects of incidence and/or observation, the effects of multiple interactions between edges can not be overlooked, whether the wedges are either located on the same plane, or they are skewed with respect to each other, or they share a common PEC face [86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96]. In fact, at high frequency the dominant field associated with a single-diffracted ray is discontinuous due to shadowing by a second wedge. Then, the introduction of a double-diffracted field becomes necessary to compensate for discontinuities occurring mostly at grazing incidence and observation aspects for the diffracted field.

The novelty of this work is the introduction of an incremental double-diffraction formulation for CSP illumination, with the aim to overcome the problem of the discontinuities in the first-order incremental diffracted field contributions under CSP illumination of complex structures. We would like to emphasize that the use of CSPs is motivated by the need to efficiently describe directional wave fields in many practical electromagnetic antenna and scattering problems, where a distribution of CSP on a closed regular surface [97, 98, 82], or collapsed to a single real point, provides an efficient representation of an arbitrary radiating wave field. ITD double diffraction coefficients are derived by solving a proper canonical problem for the ITD applied to Complex Source Points[99].

As pointed out in other existing UTD formulations available in the literature for real source illumination [100, 87, 101, 93, 89, 90], the cascaded application of ordinary UTD coefficients fails when the edge of the second wedge lies within the transition region of the field diffracted by the first wedge [100, 87]. This behavior is due to the rapid spatial variation and to the non ray-optical behavior of the field diffracted by the first edge when it illuminates the second wedge. The same limitation is expected to affect the Incremental Theory of Diffraction representation [102, 99, 103, 104], thus preventing a simple cascaded application of the ITD coefficients for single diffraction. Therefore, it is necessary to develop an ITD double-diffraction coefficient that uniformly accounts for the different transitions that may occur.

Hence, a formulation based on the application of the reaction principle for the scattering by metallic object is proposed. Equivalent filamentary currents are first introduced; in free space such currents radiate a field that is equivalent to the incremental single-diffracted field by one edge. Then the incremental double-diffracted field is represented as the reaction between the incremental field diffracted by one of the wedges and the filamentary current sources associated with the diffraction by the other wedge. The total double-diffracted field representation requires a two-fold numerical integration in the space domain on each edge of the complex structures. It is found that this formulation provides an accurate asymptotic description of the interaction between two edges, which is valid both for skewed separate wedges and for edges joined by a common PEC face. It also explicitly satisfies reciprocity and includes a double incremental slope diffraction augmentation, which provides the correct dominant high-frequency incremental contribution at grazing aspect of incidence and observation.

Incremental Double Diffraction Formulation

The geometry of the problem consists of a pair of wedges with curved edges l_1 and l_2 , arranged in an arbitrary configuration as in Fig. 3.1, and illuminated by a CSP incident beam. The edges may or may not be coplanar, and they may or may not share a common face. The double diffracted field between the two wedges is represented in incremental form on the basis of the ITD formulation for single CSP diffracted fields [99]. In principle, the incremental diffracted field arising at each local point Q_{l_1} , illuminated by a CSP beam, impinges on each local point Q_{l_2} on the second edge l_2 , where it causes a second incremental diffracted field. At high frequency, it is supposed that both the wedges and the local radius of curvature of the two edges l_1 and l_2 are large in terms of wavelength. Then, the incremental fields at each local point Q_{l_1} and Q_{l_2} may be obtained as those arising from two locally tangent canonical infinite wedges, as sketched out with dashed lines in Fig. 3.1.

It is now useful to refer to four local spherical edge-fixed coordinate systems: $(\hat{r}'_1, \hat{\beta}'_1, \hat{\phi}'_1)$ and $(\hat{r}_{12}, \hat{\beta}_1, \hat{\phi}_1)$ associated with the incident and diffracted rays at Q_{l_1} on l_1 , respectively; $(\hat{r}_{12}, \hat{\beta}'_2, \hat{\phi}'_2)$ and $(\hat{r}_2, \hat{\beta}_2, \hat{\phi}_2)$ asso-

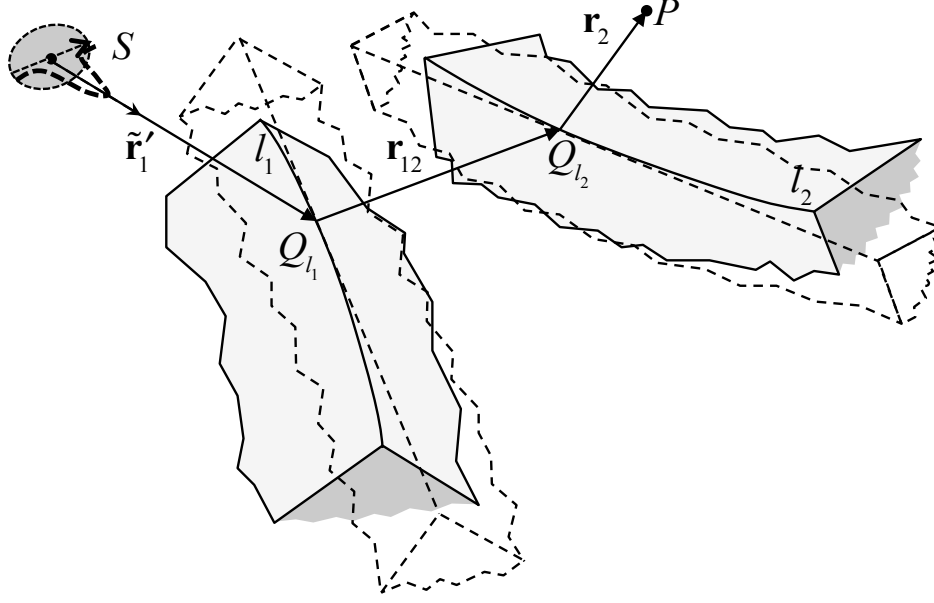


Figure 3.1: The geometrical configuration for the Incremental Theory of Double Diffraction.

ciated with the incident and diffracted rays at Q_{l_2} on l_2 . With respect to these local reference systems we denote with $S \equiv (\tilde{r}'_1, \tilde{\beta}'_1, \tilde{\phi}'_1)$ the complex coordinates of the illuminating CSP with respect to Q_{l_1} and with $P \equiv (r_2, \beta_2, \phi_2)$ the coordinates of the observation point with respect to Q_{l_2} . The double diffraction mechanism between edge l_1 and l_2 can be associated with the incremental ray-path contribution $S - Q_{l_1} - Q_{l_2} - P$, where (Q_{l_1}, Q_{l_2}) are all the possible pairs of incremental diffraction points along l_1 and l_2 , respectively. The total double-diffracted field $\mathbf{E}_{12}^{dd}(P)$ at P is then reconstructed by superimposing in integral form all the incremental double-diffracted field contributions $d\mathbf{E}_{12}^{dd}(P, Q_{l_2}, Q_{l_1})$ at P , and namely

$$\mathbf{E}_{12}^{dd}(P) = \int_{l_1} \int_{l_2} d\mathbf{E}_{12}^{dd}(P, Q_{l_2}, Q_{l_1}). \quad (3.1)$$

It is obvious that a complete double diffraction description of the phenomena includes also the corresponding incremental ray-path contribution $S - Q_{l_2} - Q_{l_1} - P$ between edges l_2 and l_1 . However, this contribution is not considered here, since it is easily obtained by swapping Q_{l_1} with Q_{l_2} .

Incremental formulation

According to a well-established locality principle for high-frequency phenomena, the double-diffracted incremental field contribution $d\mathbf{E}_{12}^{dd}(P, Q_{l_2}, Q_{l_1})$ is deduced from the canonical problem of double diffraction between two infinite uniform cylindrical wedges locally tangent at Q_{l_1} and Q_{l_2} , with exterior angles $n_1\pi$ and $n_2\pi$, respectively, as shown in Fig. 3.2. To obtain a useful expression of the double-diffracted field contribution $d\mathbf{E}_{12}^{dd}(P, Q_{l_2}, Q_{l_1})$, we refer to a particular formulation of the reaction principle for the scattering by metallic objects. Accordingly, we start by representing the incremental field at an observation point P diffracted by a canonical metallic wedge, with straight edge l illuminated by a source at S , as the field radiated in free space by a set of filamentary equivalent electric (e) and magnetic (m) current sources $I_{e,m}(P, Q_{l_i}, S)$ located at each incremental point Q_{l_i} , $i = 1, 2$, and flowing along the line l of the edge. Next, we consider the canonical metallic wedge tangent at Q_{l_2} as the scatterer, which is illuminated by two set of sources. One is the equivalent filamentary current sources distributed on the edge l_1 (labeled as a) that radiate the field diffracted by the first wedge; the other is a local $\hat{z}$ -directed unit dipole located at the observation point P (labeled as b). If we limit our analysis to the field generated by the equivalent filamentary electric currents $\mathbf{I}_e$ flowing along the canonical edges and by the illuminating electric dipoles, the reaction principle states

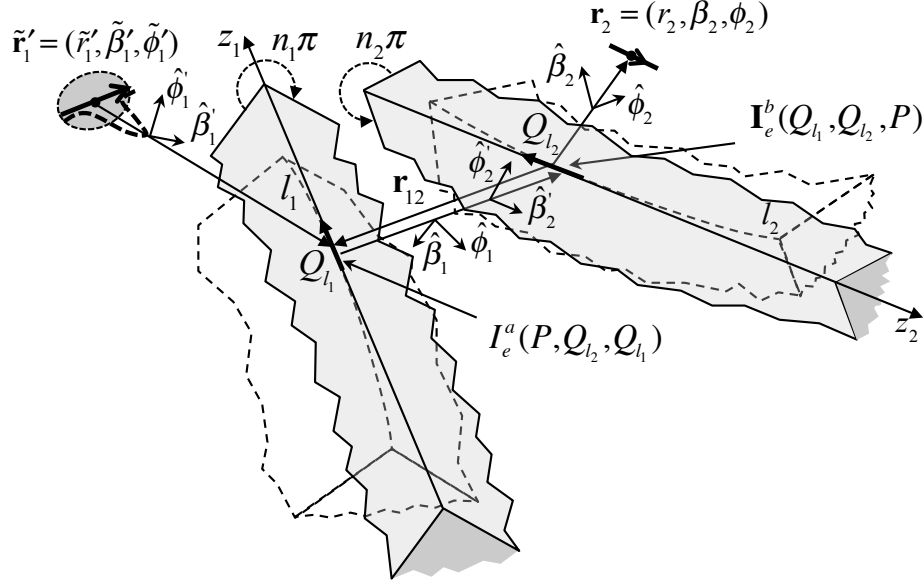


Figure 3.2: The canonical configuration for the Incremental Theory of Double Diffraction.

that

$$\mathbf{E}_{12}^{dd}(P) = \int_{-\infty}^{\infty} [\mathbf{E}_i^b(Q_{l_2}) \cdot \mathbf{I}_e^a(P, Q_{l_2}, Q_{l_1})] dz_2 = \int_{-\infty}^{\infty} [\mathbf{E}_i^a(Q_{l_2}) \cdot \mathbf{I}_e^b(Q_{l_1}, Q_{l_2}, P)] dz_2, \quad (3.2)$$

where $\mathbf{E}_i^a(Q_{l_2})$ is the diffracted field incident at Q_{l_2} due to the equivalent currents at Q_{l_1} , $\mathbf{I}_e^a(P, Q_{l_2}, Q_{l_1})$ is the equivalent free-space electric currents on l_2 that radiate the diffracted field at P when the second wedge is illuminated by $\mathbf{E}_i^a(Q_{l_2})$, $\mathbf{E}_i^b(Q_{l_2})$ is the incident field at Q_{l_2} due to a local $\hat{z}$ -directed unit electric dipole placed at P , and $\mathbf{I}_e^b(Q_{l_1}, Q_{l_2}, P)$ is the equivalent free-space currents on l_2 that radiate the diffracted field at Q_{l_1} when the second wedge is illuminated by $\mathbf{E}_i^b(Q_{l_2})$.

The analysis is limited to the local $\hat{z}$ -component of the electric field (soft component), although a similar procedure can be repeated for the local $\hat{z}$ -component of the magnetic field (hard component). The single-diffracted field by the first edge, incident at Q_{l_2} , is represented as

$$\mathbf{E}_i^a(Q_{l_2}) = \int_{-\infty}^{\infty} \underline{\underline{D}}(\tilde{\nu}_1; \phi_1, \tilde{\phi}'_1) \cdot \mathbf{E}^i(Q_{l_1}) \frac{e^{-jkr_{12}}}{2\pi r_{12}} dz_1, \quad (3.3)$$

where $\mathbf{E}^i(Q_{l_1})$ is the incident field at Q_{l_1} by the CSP at S and

$$\underline{\underline{D}}(\tilde{\nu}_1; \phi_1, \tilde{\phi}'_1) = D_s(\tilde{\nu}_1; \phi_1, \tilde{\phi}'_1) \hat{\beta}_1 \hat{\beta}'_1 + D_h(\tilde{\nu}_1; \phi_1, \tilde{\phi}'_1) \hat{\phi}_1 \hat{\phi}'_1 \quad (3.4)$$

is the incremental diffraction dyadic. Explicit expressions of D_s and D_h together with the corresponding arguments are summarized in Appendix I. To apply the reaction principle in (3.2), it is supposed that this field is generated by equivalent filamentary free-space sources distributed along the edge l_1 that radiate in the presence of the second wedge. The component of the above incremental field aligned along $\hat{\beta}'_2$ (in the Q_{l_2} reference system) is

$$E_{i,\beta'_2}^a(Q_{l_2}) = \mathbf{M}_\beta(\gamma_{12}) \cdot \mathbf{E}_i^a(Q_{l_2}), \quad (3.5)$$

where $\mathbf{M}_\beta(\gamma_{12})$ is the first row of the transformation matrix $\underline{\underline{M}}(\gamma_{12})$ between the two local spherical systems at Q_{l_1} and Q_{l_2} , and γ_{12} is the rotation angle defined between the ray-fixed unit vectors $(\hat{\beta}_1, \hat{\phi}_1)$ and $(\hat{\beta}'_2, \hat{\phi}'_2)$, as illustrated in [90]. As a consequence, the $\hat{z}_{l_2}$ -component of the incident field at Q_{l_2} is then

$$E_{i,z_{l_2}}^a(Q_{l_2}) = E_{i,\beta'_2}^a(Q_{l_2})(\hat{\beta}'_2 \cdot \hat{z}_{l_2}) = \sin \beta'_2 \mathbf{M}_\beta(\gamma_{12}) \cdot \int_{-\infty}^{\infty} \underline{\underline{D}}(\tilde{\nu}_1; \phi_1, \tilde{\phi}'_1) \cdot \mathbf{E}^i(Q_{l_1}) \frac{e^{-jkr_{12}}}{2\pi r_{12}} dz_1. \quad (3.6)$$

The above field component is parallel to the local equivalent filamentary current source $I_{e,z_{l_2}}^b(Q_{l_1}, Q_{l_2}, P)$ at Q_{l_2} , that radiates at Q_{l_1} the incremental field

$$E_{z_{l_2}}^b(Q_{l_1}) = E_{\beta'_2}^b(Q_{l_1})(\hat{\beta}'_2 \cdot \hat{z}_{l_2}) = jk\zeta \sin \beta'_2 \frac{e^{-jkr_{12}}}{4\pi r_{12}} I_{e,z_{l_2}}^b(Q_{l_1}, Q_{l_2}, P) dz_2 \sin \beta'_2. \quad (3.7)$$

Since the latter represents the incremental field at Q_{l_1} single-diffracted by the edge l_2 when illuminated by the unit electric dipole in P , we obtain

$$I_{e,z_{l_2}}^b(Q_{l_1}, Q_{l_2}, P) = \frac{2D_s(-\nu_2; \phi'_2, \phi_2) [\mathbf{E}_i^b(Q_{l_2}) \cdot \hat{\beta}_2]}{jk\zeta \sin \beta'_2}. \quad (3.8)$$

By using (3.6) and (3.8), the last integral in (3.2) can be written as

$$\begin{aligned} & \int_{-\infty}^{\infty} E_{i,z_{l_2}}^a(Q_{l_2}) I_{e,z_{l_2}}^b(Q_{l_1}, Q_{l_2}, P) dz_2 = \\ & = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{M}_{\beta}(\gamma_{12}) \cdot \underline{\underline{D}}(\tilde{\nu}_1; \phi_1, \tilde{\phi}'_1) \cdot \mathbf{E}^i(Q_{l_1}) \frac{e^{-jkr_{12}}}{2\pi r_{12}} \sin \beta_2 \frac{e^{-jkr_2}}{2\pi r_2} D_s(-\nu_2; \phi'_2, \phi_2) dz_1 dz_2 s. \end{aligned}$$

A similar procedure exists for the hard polarization. When the notation is compacted in matrix form, the final formulation is

$$\mathbf{E}_{12}^{dd}(P) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{\underline{D}}_{12}^{dd}(\tilde{\nu}_1, \nu_2; \tilde{\phi}'_1, \phi_1, \phi'_2, \phi_2) \cdot \mathbf{E}^i(Q_{l_1}) \frac{e^{-jkr_{12}}}{2\pi r_{12}} \frac{e^{-jkr_2}}{2\pi r_2} dz_2 dz_1. \quad (3.9)$$

Comparing the above equation with (3.1), we obtain that the incremental double-diffracted field contribution at P is

$$d\mathbf{E}_{12}^{dd}(P, Q_{l_2}, Q_{l_1}) = \underline{\underline{D}}_{12}^{dd}(\tilde{\nu}_1, \nu_2; \tilde{\phi}'_1, \phi_1, \phi'_2, \phi_2) \cdot \mathbf{E}^i(Q_{l_1}) \frac{e^{-jkr_{12}}}{2\pi r_{12}} \frac{e^{-jkr_2}}{2\pi r_2} dz_2 dz_1, \quad (3.10)$$

where the double diffraction dyadic $\underline{\underline{D}}_{12}^{dd}(\tilde{\nu}_1, \nu_2; \tilde{\phi}'_1, \phi_1, \phi'_2, \phi_2)$ is found as

$$\underline{\underline{D}}_{12}^{dd}(\tilde{\nu}_1, \nu_2; \tilde{\phi}'_1, \phi_1, \phi'_2, \phi_2) = \underline{\underline{D}}(-\nu_2; \phi'_2, \phi_2) \cdot \underline{\underline{M}}(\gamma_{12}) \cdot \underline{\underline{D}}(\tilde{\nu}_1; \phi_1, \tilde{\phi}'_1), \quad (3.11)$$

and, similarly to (3.4),

$$\underline{\underline{D}}(-\nu_2; \phi'_2, \tilde{\phi}_2) = D_s(-\nu_2; \phi'_2, \tilde{\phi}_2) \hat{\beta}_2 \hat{\beta}'_2 + D_h(-\nu_2; \phi'_2, \tilde{\phi}_2) \hat{\phi}_2 \hat{\phi}'_2. \quad (3.12)$$

The total high-frequency diffracted field by two arbitrary wedges with edges l_1 and l_2 is then obtained by distributing and integrating the incremental contribution in (3.10) as shown in (3.1).

Even and odd spectral contributions

Similar to the case of real sources illuminating the wedges [89, 90] it is best to retain in the spectral formulation the product of both the even and the odd parts of each cotangent term associated with the spectral Green's Function of a single wedge [105]. This procedure gives a local double diffraction description that provides, when integrated, the proper asymptotic order in all the different transitions that may occur. As a result, the non-vanishing contributions to the solution for the canonical problem of a pair of wedges are provided by the even parts of the final spectra, i.e. the even parts of $\underline{\underline{D}}_{12}^{dd}$. The latter are obtained from the product of both the two even and odd parts of each cotangent term in the ITD formulation [99]. On the other hand, the mixed terms that result from products between an even and an odd part of the spectrum provide a vanishing contribution when integrated into the symmetrical spectral domain. The final form of the dyadic double diffraction coefficient is

$$\underline{\underline{D}}_{12}^{dd}(\nu_2, \tilde{\nu}_1; \phi_2, \phi'_2, \phi_1, \tilde{\phi}_1) = \underline{\underline{D}}_{12}^{e,e}(\nu_2, \tilde{\nu}_1; \phi_2, \phi'_2, \phi_1, \tilde{\phi}_1) + \underline{\underline{D}}_{12}^{o,o}(\nu_2, \tilde{\nu}_1; \phi_2, \phi'_2, \phi_1, \tilde{\phi}_1), \quad (3.13)$$

where

$$\underline{\underline{D}}_{12}^{e,e}(\nu_2, \tilde{\nu}_1; \phi_2, \phi'_2, \phi_1, \tilde{\phi}_1) = \underline{\underline{D}}^e(-\nu_2; \phi'_2, \phi_2) \cdot \underline{\underline{M}}(\gamma_{12}) \cdot \underline{\underline{D}}^e(\tilde{\nu}_1; \phi_1, \tilde{\phi}'_1), \quad (3.14)$$

is the even part of the ITD double diffraction coefficient and

$$\underline{\underline{D}}_{12}^{o,o}(\nu_2, \tilde{\nu}_1; \phi_2, \phi'_2, \phi_1, \tilde{\phi}_1) = \underline{\underline{D}}^o(-\nu_2; \phi'_2, \phi_2) \cdot \underline{\underline{M}}(\gamma_{12}) \cdot \underline{\underline{D}}^o(\tilde{\nu}_1; \phi_1, \tilde{\phi}'_1) \quad (3.15)$$

is the odd part. It is worth pointing out that expression (3.10) together with (3.13)-(3.15) is not a merely analytic continuation in complex space of that for real illuminating sources [90]. Here, the presence of the two terms $\tilde{\nu}_1$ and $-\nu_2$ assures the symmetrical behavior of the incremental double diffraction contribution with respect to the source and observation coordinates, and when (3.10) is integrated along the actual structure, provides the correct dominant high-frequency contribution at grazing aspect of incidence and observation.

Application to the case of a Perfectly conducting square plate

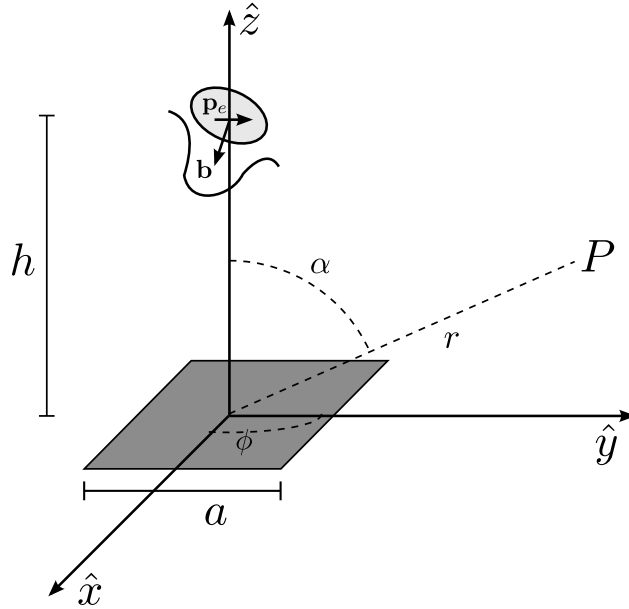


Figure 3.3: Geometry for the scattering by a square plate illuminated by a CSP.

As an example of the application of the new ITD double diffraction mechanism, we consider the scattering from a perfectly conducting square plate illuminated by a single CSP, as shown in Fig. 3.3. A $\hat{y}$ -directed electric CSP dipole is placed at $(0, 0, h = 8\lambda)$ above a square plate with sides $a = 5\lambda$, and located in the xy plane. The vector associated with the beam is $\mathbf{b} = \lambda(\sin(8\pi/9), 0, \cos(8\pi/9))$, and the beam points towards one of the edges the plate. Numerical results obtained from the Method of Moments (MoM), Physical Optics (PO) and the incremental fringe method (PO+IFF) explained in [103] are presented in Fig. 3.4 and Fig. 3.5, for observations at a distance $r = 11\lambda$ in the yz plane and xz plane, respectively. Specifically, in Fig. 3.4 the $\hat{\theta}$ -component of the scattered electric field through the IFF formulation is plotted for $\alpha \in [0, 2\pi]$ in the E-plane (yz plane). This curve is found in good agreement with MoM calculations, except for the region close to grazing observation. As expected, at these observation aspects, the first-order ITD incremental diffraction coefficients exhibit a discontinuity that is removed with the introduction of the new ITD double diffraction contributions, as shown in Fig. 3.6. Additionally, the accuracy of PO calculations is significantly improved when the incremental fringe contributions are added. The same considerations also apply for the $\hat{\phi}$ -component of the scattered electric field shown in Fig. 3.5 for $\alpha \in [0, 2\pi]$ in the H-plane (xz plane). The computations including the ITD double diffraction coefficient contributions are shown in Fig. 3.7. However, in this case, the

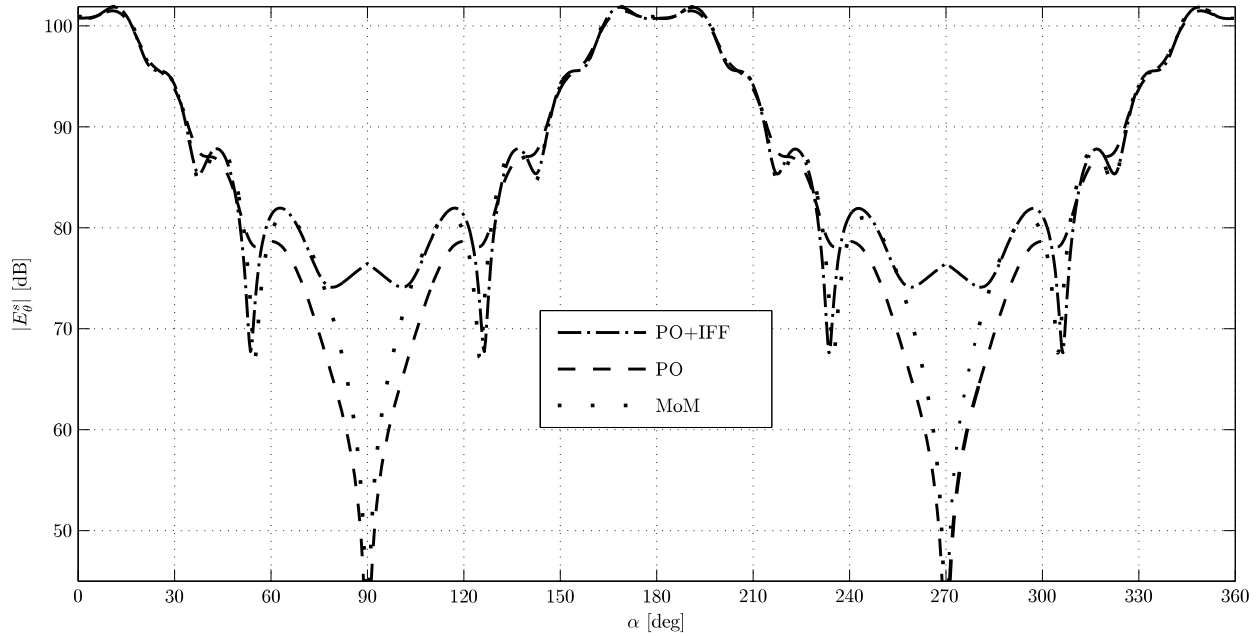


Figure 3.4: $|E_\theta^s|$ component of the scattered electric field for the square plate illuminated by a single CSP electric dipole. Observations are made in the E-plane.

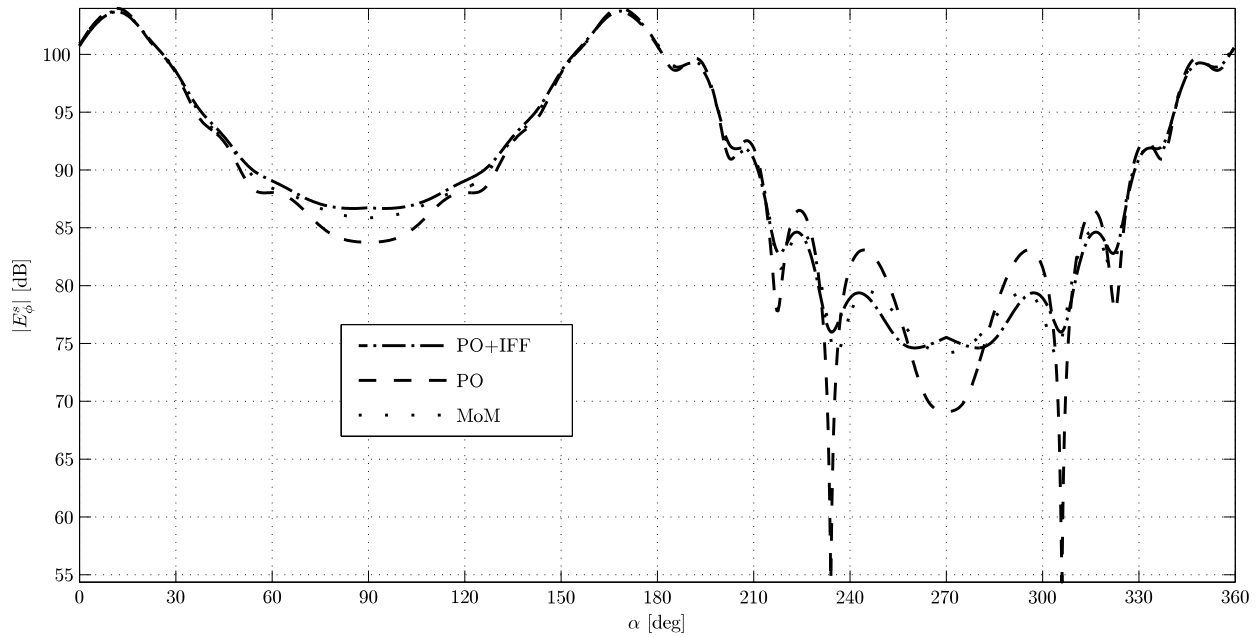


Figure 3.5: $|E_\phi^s|$ component of the scattered electric field for the square plate illuminated by a single CSP electric dipole. Observations are made in the H-plane.

ITD double diffraction contribution is not as important, because, at grazing observation aspects, the fringe calculation do not exhibit first-order discontinuities in the ITD field. Furthermore, correct contributions from vertices in this illumination condition become important, but they are not properly taken into consideration in this formulation since they are reconstructed only by the truncation of the IFF integration along the edges of the plate. For both polarizations, once the double diffracted field contribution is added to the PO+IFF

formulation, a much closer result to the MoM computations is achieved, especially in the region near to the xy -plane.

The effects of the double-diffracted field are more evident in the E-plane cut: double diffraction contributes to recover the solution for the electric field close to the xy -plane. In the H-plane cut, however, double diffraction plays a negligible role: we believe that the discrepancies found with the MoM are due to the fact that the corner diffraction contributions play a significant role.

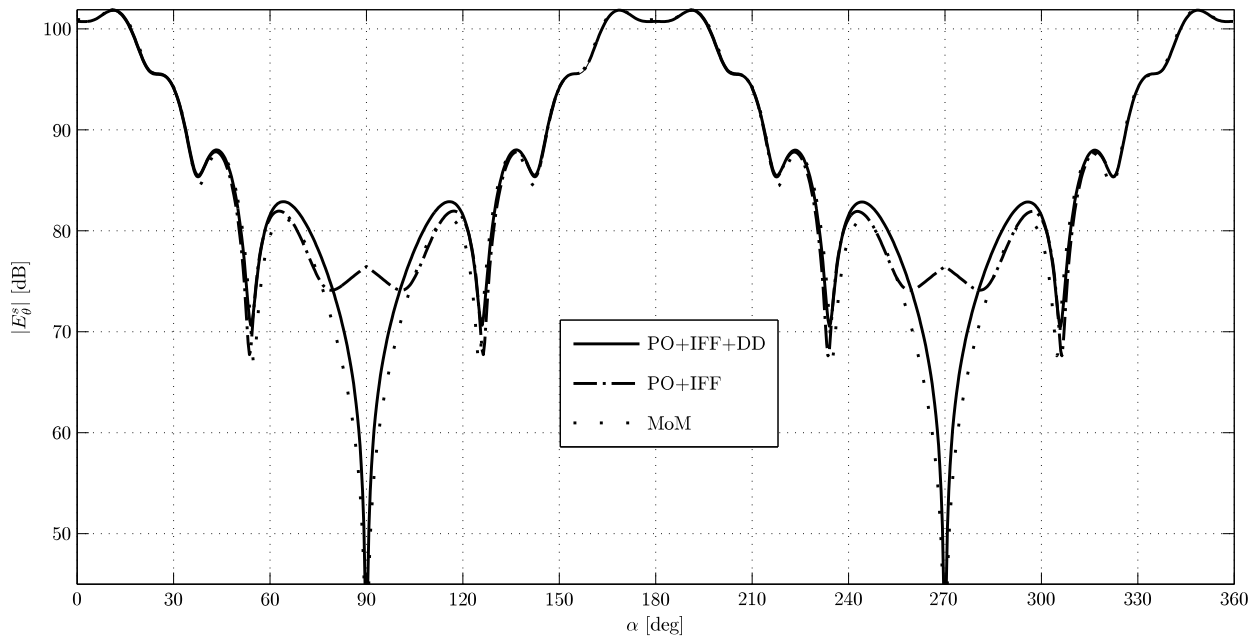


Figure 3.6: $|E_{\theta}^s|$ component of the scattered electric field for the square plate illuminated by a single CSP electric dipole. Observations are made in the E-plane.

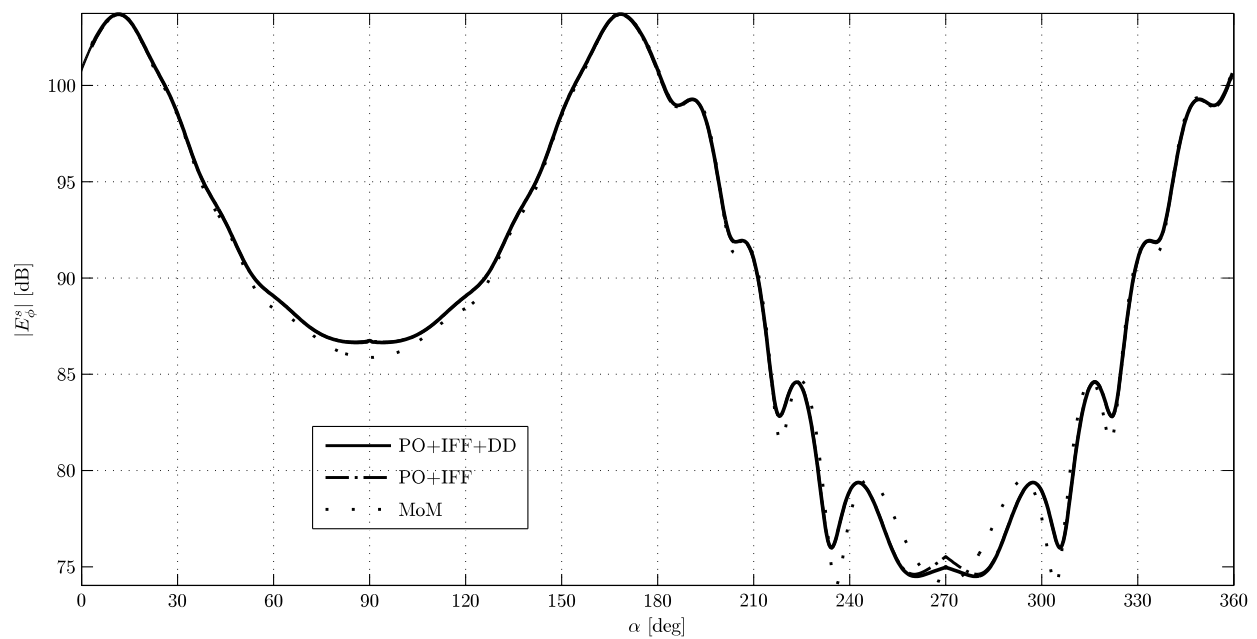


Figure 3.7: $|E_\phi^s|$ component of the scattered electric field for the square plate illuminated by a single CSP electric dipole. Observations are made in the H-plane.

3.2 Metamaterial elliptical cavity

Introduction

This investigation focuses on the use of metamaterials to modify the distribution of the electromagnetic field due to a source, however its main difference compared to previous approaches [106, 107, 108, 109, 110] is that the source is not embedded in the metamaterial. In fact, our geometry consists of a dipole source immersed in an ordinary double positive (DPS) material with positive dielectric permittivity and positive magnetic permeability, illuminating a semi-oblate spheroidal cavity which is filled with a DNG metamaterial layer with negative dielectric permittivity and negative magnetic permeability, and with a layer made of the same DPS material outside the cavity. The cavity is built underneath an infinite metallic plane and connected to the outside through an aperture, as shown in the cross-section of Fig. 3.8.

An exact analytical solution for a dipole source located on the axis of symmetry and axially oriented is obtained in terms of infinite series containing oblate spheroidal functions. This geometry is relevant because it contains features such as sharp edges, an aperture, a cavity, and different materials. All these features make this new exact solution important as a canonical reference to validate the accuracy of numerical approaches, such as CAD software, to solve electromagnetic scattering problems.

The analytical results show that with a specific combination of the DPS and DNG layers inside the semi-oblate spheroidal cavity, the radiation of the dipole source is significantly more directive compared to the one obtained with related geometries such as those discussed by Berardi *et al.* in [111], Valentino and Erricolo [112], and Askarpour and Uslenghi in [113]. Preliminary results were presented in [27].

The time dependence $\exp(-i\omega t)$ is assumed and suppressed throughout.

Geometry of the problem

The geometry of the problem has symmetry of rotation around the z axis and is shown in the cross section of Fig. 3.8. All surfaces are coordinate surfaces in the oblate spheroidal system, (η, ξ, φ) related to the rectangular coordinates (x, y, z) by $x = (d/2)\sqrt{\xi^2 + 1}\sqrt{1 - \eta^2} \cos \varphi$, $y = (d/2)\sqrt{\xi^2 + 1}\sqrt{1 - \eta^2} \sin \varphi$, $z = (d/2)\xi\eta$, where $\xi \geq 0$, $-1 \leq \eta \leq 1$, $0 \leq \varphi \leq 2\pi$ and d is the distance between the focal points $F_1 \equiv (x = -d/2, y = 0, z = 0) \equiv (\eta = 0, \xi = 0, \varphi = \pi)$ and $F_2 \equiv (x = d/2, y = 0, z = 0) \equiv (\eta = 0, \xi = 0, \varphi = 0)$. Referring to Fig. 3.8, there is a metallic plane at $z = 0$ with a circular aperture of radius $d/2$ that corresponds to the coordinate surface $\eta = 0$. The half-space $z > 0$ is filled by DPS material with dielectric permittivity ε and magnetic permeability μ . Below the aperture, there is a cavity limited by the metallic semi-oblate spheroidal surface at $\xi = \xi_1$. The interior of the cavity contains two regions. Region 1 is between the aperture $\xi = 0$, the metallic plane $\eta = 0$ and the oblate spheroidal surface $\xi = \xi_2$. Region 2 is between the oblate spheroidal surface at $\xi = \xi_2$, the plane $\eta = 0$, and the metallic wall of the cavity along the oblate spheroidal surface $\xi = \xi_1$.

Electric dipole source

For an electric dipole source located along the z axis above the ground plane at $(\xi_0 > 0, \eta_0 = 1)$ and axially oriented, the rotational symmetry of the problem requires that the field components be

$$\begin{aligned} \mathbf{H} &= H_\varphi(\eta, \xi) \hat{\varphi}, \quad H_\eta = H_\xi = 0 \\ \mathbf{E} &= E_\eta(\eta, \xi) \hat{\eta} + E_\xi(\eta, \xi) \hat{\xi}, \quad E_\varphi = 0 \end{aligned} \quad (3.16)$$

where

$$E_\eta = \frac{iZ}{c} \sqrt{\frac{\xi^2 + 1}{\xi^2 + \eta^2}} \left(\frac{\partial}{\partial \xi} + \frac{\xi}{\xi^2 + 1} \right) H_\varphi \quad (2a)$$

$$E_\xi = -\frac{iZ}{c} \sqrt{\frac{1 - \eta^2}{\xi^2 + \eta^2}} \left(\frac{\partial}{\partial \eta} - \frac{\eta}{1 - \eta^2} \right) H_\varphi \quad (2b)$$

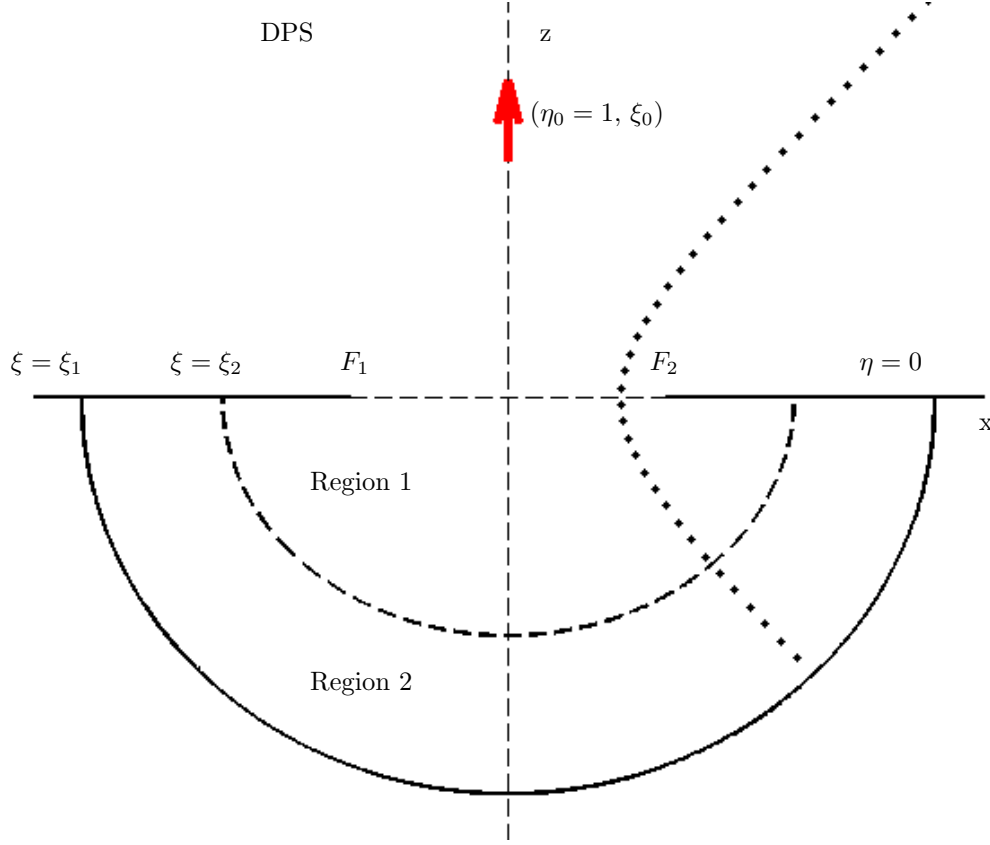


Figure 3.8: Cross-section of the geometry of the problem. A dipole source located at $(\xi_0, \eta_0 = 1)$ illuminates the infinite metallic plane $\eta = 0$ and the aperture $\xi = 0$. The cavity has a diameter d corresponding to the interfocal distance between F_1 and F_2 . The dotted line represents a sample coordinate surface with $|\eta| = 0.7$, a hyperboloid of revolution.

Z is the wave impedance of the medium, and $c = kd/2$ is the product of the wave number k and the inter-focal radius $d/2$. The constant c is replaced by $-c$ in the DNG medium.

If the electric Hertz potential of the dipole is $\mathbf{\Pi}_e = \hat{\mathbf{z}} \exp(ikr)/(kr)$, where r is the distance between the dipole $(\eta_0 = 1, \xi_0)$ and the field point (η, ξ) , then the corresponding magnetic field is [114]:

$$H_\varphi^i = \frac{2k^2}{Z_1 \sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-i)^n}{\tilde{\rho}_{1,n} \tilde{N}_{1,n}} \times R_{1,n}^{(1)}(-ic, i\xi_{<}) R_{1,n}^{(3)}(-ic, i\xi_{>}) S_{1,n}(-ic, \eta). \quad (3.18)$$

where $R_{1,n}^{(1)}$ and $R_{1,n}^{(3)}$ are radial oblate spheroidal functions of the first and third kind, respectively, $S_{1,n}$ are angular oblate spheroidal functions, and $\tilde{\rho}_{1,n}$ and $\tilde{N}_{1,n}$ are normalization factors, according to the notation of Flammer [115]. The symbol $< (>)$ refers to the smaller (greater) between ξ and ξ_0 .

Outside the cavity, the total magnetic field may be written as the superposition of the incident field (3.18), the field reflected by the infinite metallic plane evaluated assuming that there is no aperture, and a

diffracted field representing the perturbation due to the aperture and satisfying the radiation condition:

$$\begin{aligned}
 H_\varphi^t &= H_\varphi^i + H_\varphi^r + H_\varphi^d = \frac{-4ik^2}{Z_1\sqrt{\xi_0^2+1}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\tilde{\rho}_{1,2\ell+1}\tilde{N}_{1,2\ell+1}} \\
 &\times S_{1,2\ell+1}(-ic, \eta) \left[a_\ell^{(e)} R_{1,2\ell+1}^{(3)}(-ic, i\xi) + \right. \\
 &\left. R_{1,2\ell+1}^{(1)}(-ic, i\xi_{<}) R_{1,2\ell+1}^{(3)}(-ic, i\xi_{>}) \right]. \tag{3.19}
 \end{aligned}$$

Inside the cavity, the total magnetic field in the DPS region is:

$$\begin{aligned}
 H_{\varphi, \text{DPS}} &= \frac{-4ik^2}{Z_1\sqrt{\xi_0^2+1}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\tilde{\rho}_{1,2\ell+1}\tilde{N}_{1,2\ell+1}} S_{1,2\ell+1}(-ic, -\eta) \\
 &\times \left[b_\ell^{(e)} R_{1,2\ell+1}^{(1)}(-ic, i\xi) + c_\ell^{(e)} R_{1,2\ell+1}^{(3)}(-ic, i\xi) \right], \tag{3.20}
 \end{aligned}$$

whereas inside the DNG region is:

$$\begin{aligned}
 H_{\varphi, \text{DNG}} &= \frac{-4ik^2}{Z_2\sqrt{\xi_0^2+1}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\tilde{\rho}_{1,2\ell+1}\tilde{N}_{1,2\ell+1}} S_{1,2\ell+1}(ic, -\eta) \\
 &\times \left[B_\ell^{(e)} R_{1,2\ell+1}^{(1)}(ic, i\xi) + C_\ell^{(e)} R_{1,2\ell+1}^{(3)}(ic, i\xi) \right] \tag{3.21}
 \end{aligned}$$

In the DPS region, $\varepsilon_1 > 0$, $\mu_1 > 0$ are the electric permittivity and magnetic permeability, respectively, $k_1 = k = \omega\sqrt{\varepsilon_1\mu_1} > 0$ the wavevector and $Z_1 = \sqrt{\mu_1/\varepsilon_1}$ the material impedance. In the DNG region, $\varepsilon_2 < 0$, $\mu_2 < 0$ are the electric permittivity and magnetic permeability, respectively, $k_2 = \omega\sqrt{\varepsilon_2\mu_2} < 0$ the wavevector and $Z_2 = \sqrt{\mu_2/\varepsilon_2}$ the material impedance. The DPS and DNG materials satisfy the anti-isorefractive condition, i.e. $k_1 = -k_2$, while the impedances Z_1 and Z_2 are both positive, but different in general, hence we introduce the ratio $\zeta = Z_1/Z_2$.

The unknown modal coefficients $a_\ell^{(e)}$, $b_\ell^{(e)}$, $c_\ell^{(e)}$, $B_\ell^{(e)}$ and $C_\ell^{(e)}$ are found by imposing the boundary conditions, i.e. the continuity of the total tangential component of the magnetic and electric field across the aperture $\xi = 0$, across the surface at ξ_2 and the vanishing of the electric field at the surface ξ_1 . Note that in the application of the boundary conditions, the property that $S_{1,2\ell+1}(-ic, \eta) = S_{1,2\ell+1}(\pm ic, -\eta)$ and $S_{1,2\ell}(-ic, \eta) = -S_{1,2\ell}(\pm ic, -\eta)$ should be used.

If region 1 is DNG and region 2 is DPS, the modal coefficients are:

$$a_\ell^{(e,m)} = \frac{t^{(e,m)}}{|M|} \begin{vmatrix} M_{2,2}^{(e,m)} & M_{2,3}^{(e,m)} \\ M_{3,2}^{(e,m)} & M_{3,3}^{(e,m)} \end{vmatrix} \tag{3.22}$$

$$b_\ell^{(e,m)} = \frac{-t^{(e,m)}}{|M|} \begin{vmatrix} M_{2,1}^{(e,m)} & M_{2,3}^{(e,m)} \\ M_{3,1}^{(e,m)} & M_{3,3}^{(e,m)} \end{vmatrix} \tag{3.23}$$

$$B_\ell^{(e,m)} = \frac{t^{(e,m)}}{|M|} \begin{vmatrix} M_{2,1}^{(e,m)} & M_{2,2}^{(e,m)} \\ M_{3,1}^{(e,m)} & M_{3,2}^{(e,m)} \end{vmatrix}, \tag{3.24}$$

where $M_{m,n}^{(e,m)}$ is an element of the matrix

$$M^{(e,m)} = \begin{pmatrix} M_{1,1}^{(e,m)} & M_{1,2}^{(e,m)} & M_{1,3}^{(e,m)} \\ M_{2,1}^{(e,m)} & M_{2,2}^{(e,m)} & M_{2,3}^{(e,m)} \\ M_{3,1}^{(e,m)} & M_{3,2}^{(e,m)} & M_{3,3}^{(e,m)} \end{pmatrix} \quad (3.25)$$

$$M_{1,1}^{(e)} = R_{1,2\ell+1}^{(3)}(-ic, 0) - \zeta R_{1,2\ell+1}^{(3)}(ic, 0) \frac{R_{1,2\ell+1}^{(3)'}(-ic, 0)}{R_{1,2\ell+1}^{(3)'}(ic, 0)} \quad (3.26)$$

$$M_{1,2}^{(e)} = 0, \quad M_{1,3}^{(e)} = -\zeta R_{1,2\ell+1}^{(1)}(ic, 0) \quad (12a, b)$$

$$M_{2,1}^{(e)} = -\zeta R_{1,2\ell+1}^{(3)}(ic, i\xi_2) \frac{R_{1,2\ell+1}^{(3)'}(-ic, 0)}{R_{1,2\ell+1}^{(3)'}(ic, 0)} \quad (3.13)$$

$$M_{2,2}^{(e)} = R_{1,2\ell+1}^{(1)}(-ic, i\xi_2) - R_{1,2\ell+1}^{(3)}(-ic, i\xi_2) \frac{D_{2\ell+1}^{(1)}(-ic, i\xi_1)}{D_{2\ell+1}^{(3)}(-ic, i\xi_1)} \quad (3.14)$$

$$M_{2,3}^{(e)} = -\zeta R_{1,2\ell+1}^{(1)}(ic, i\xi_2) \quad (3.15)$$

$$M_{3,1}^{(e)} = \frac{R_{1,2\ell+1}^{(3)'}(-ic, 0)}{R_{1,2\ell+1}^{(3)'}(ic, 0)} D_{2\ell+1}^{(3)}(ic, i\xi_2) \quad (3.16)$$

$$M_{3,2}^{(e)} = D_{2\ell+1}^{(1)}(-ic, i\xi_2) - D_{2\ell+1}^{(3)}(-ic, i\xi_2) \frac{D_{2\ell+1}^{(1)}(-ic, i\xi_1)}{D_{2\ell+1}^{(3)}(-ic, i\xi_1)} \quad (3.17)$$

$$M_{3,3}^{(e)} = D_{2\ell+1}^{(1)}(ic, i\xi_2) \quad (3.18)$$

$$t^{(e)} = -R_{1,2\ell+1}^{(1)}(-ic, 0) R_{1,2\ell+1}^{(3)}(-ic, i\xi_0) \quad (3.19)$$

$$D_n^{(h)}(\pm ic, i\xi) = R_{1,n}^{(h)'}(\pm ic, i\xi) + \frac{\xi}{\xi^2 + 1} R_{1,n}^{(h)}(\pm ic, i\xi), \quad (h = 1, 3) \quad (3.20)$$

where the prime indicates partial derivative with respect to the argument ξ

$$C_l^{(e)} = \frac{R_{1,2\ell+1}^{(3)'}(-ic, 0)}{R_{1,2\ell+1}^{(3)'}(ic, 0)} a_l^{(e)}, \quad c_l^{(e)} = -\frac{D_{2\ell+1}^{(1)}(-ic, i\xi_1)}{D_{2\ell+1}^{(3)}(-ic, i\xi_1)} b_l^{(e)} \quad (21a, b)$$

The far field behavior in the half space $z > 0$ is obtained using the asymptotic expressions $c\xi \sim kr$, $\eta \sim \cos \theta$ as $c\xi \rightarrow \infty$, yielding

$$H_\varphi^t|_{c\xi \rightarrow \infty} \sim \frac{e^{ikr}}{kr} \frac{4ik^2}{Z_1 \sqrt{\xi_0^2 + 1}} \times \sum_{l=0}^{\infty} \frac{R_{1,2\ell+1}^{(1)} + a_l^{(e)}}{\tilde{\rho}_{1,2\ell+1} \tilde{N}_{1,2+1}} S_{1,2\ell+1}(-ic, \cos \theta). \quad (3.22)$$

If region 1 is DPS and region 2 is DNG, the field expressions (3.19 - 3.21) and (3.22) in each region are still valid, however the modal coefficients change because the materials for regions 1 and 2 have been interchanged. The new modal coefficients are still formally expressed through eqs. (3.22-3.24) with $\sim$ superscript, where all

symbols, except $t^{(e,m)}$, are now given by:

$$\tilde{M}_{1,1}^{(e)} = 2R_{1,2\ell+1}^{(3)}(-ic, 0) \quad (3.23)$$

$$\tilde{M}_{1,2}^{(e)} = -R_{1,2\ell+1}^{(1)}(-ic, 0) \quad (3.24)$$

$$\tilde{M}_{1,3}^{(e)} = 0, \quad \tilde{M}_{2,1}^{(e)} = R_{1,2\ell+1}^{(3)}(-ic, i\xi_2) \quad (25a, b)$$

$$\tilde{M}_{2,2}^{(e)} = -R_{1,2\ell+1}^{(1)}(-ic, i\xi_2) \quad (3.26)$$

$$\tilde{M}_{2,3}^{(e)} = \zeta \left[R_{1,2\ell+1}^{(1)}(ic, i\xi_2) - \frac{D_{2\ell+1}^{(1)}(ic, i\xi_1)}{D_{2\ell+1}^{(3)}(ic, i\xi_1)} R_{1,2\ell+1}^{(3)}(ic, i\xi_2) \right] \quad (3.27)$$

$$\tilde{M}_{3,1}^{(e)} = D_{2\ell+1}^{(3)}(-ic, i\xi_2), \quad \tilde{M}_{3,2}^{(e)} = -D_{2\ell+1}^{(1)}(-ic, i\xi_2) \quad (28a, b)$$

$$\tilde{M}_{3,3}^{(e)} = -D_{2\ell+1}^{(1)}(ic, i\xi_2) + \frac{D_{2\ell+1}^{(1)}(ic, i\xi_1)}{D_{2\ell+1}^{(3)}(ic, i\xi_1)} D_{2\ell+1}^{(3)}(ic, i\xi_2) \quad (3.29)$$

$$\tilde{C}_l^{(e)} = -\frac{D_{2\ell+1}^{(1)}(ic, i\xi_1)}{D_{2\ell+1}^{(3)}(ic, i\xi_1)} \tilde{B}_l^{(e)}, \quad \tilde{c}_l^{(e)} = -\tilde{a}_l^{(e)} \quad (30a, b)$$

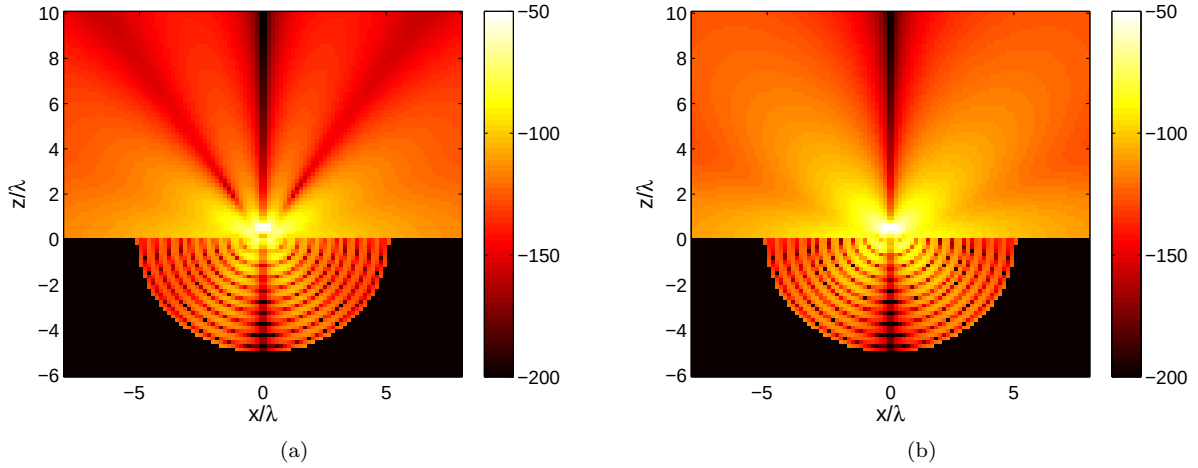


Figure 3.9: Total magnetic field $|H_\varphi|$ due to an electric dipole located at $(z_0/\lambda = 1/2, \xi_0 = 1/2)$ for an oblate spheroidal cavity with $c = 2\pi$, $d/\lambda = 2$, $\xi_1 = 5$, $\xi_2 = 3$, $\zeta = 0.5$. (a) Region 1 DNG and Region 2 DPS; and, (b) Region 1 DPS and Region 2 DNG.

Magnetic dipole source

For a magnetic dipole source located above ground plane on the z axis at $(\eta_0 = 1, \xi_0)$ and axially oriented, the rotational symmetry of the problems requires that the field components be

$$\begin{aligned} \mathbf{E} &= E_\varphi(\eta, \xi) \hat{\varphi}, \quad E_\eta = E_\xi = 0 \\ \mathbf{H} &= H_\eta(\eta, \xi) \hat{\eta} + H_\xi(\eta, \xi) \hat{\xi}, \quad H_\varphi = 0 \end{aligned} \quad (3.31)$$

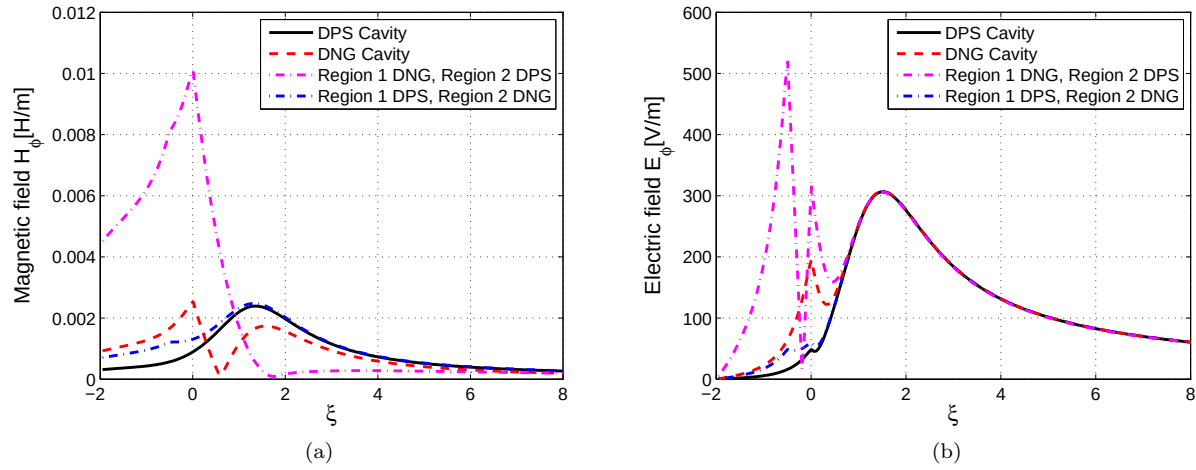


Figure 3.10: (a) Total magnetic field $|H_\varphi|$ due to an electric dipole and (b) total electric field $|E_\varphi|$ due to magnetic dipole located at $(\xi_0 = 1.5, \eta_0 = 1)$ and evaluated along the coordinate line $|\eta| = 0.7$ for a cavity with $c = 1$, $d = \lambda/\pi$, $\zeta = 0.5$, $\xi_1 = 2$ and $\xi_2 = 0.5$. Negative values of ξ correspond to locations within the cavity. Also, the fields are evaluated with $\xi_1 = \xi_2 = 2$ corresponding to an all DNG and an all DPS filled cavity.

where

$$H_\eta = -\frac{i}{cZ} \sqrt{\frac{\xi^2 + 1}{\xi^2 + \eta^2}} \left(\frac{\partial}{\partial \xi} + \frac{\xi}{\xi^2 + 1} \right) E_\varphi, \quad (32a)$$

$$H_\xi = \frac{i}{cZ} \sqrt{\frac{1 - \eta^2}{\xi^2 + \eta^2}} \left(\frac{\partial}{\partial \eta} - \frac{\eta}{1 - \eta^2} \right) E_\varphi. \quad (32b)$$

If the magnetic Hertz potential of the dipole is $\mathbf{\Pi}_m = \hat{\mathbf{z}} \exp(ikr)/(kr)$ then the corresponding electric field is [114]:

$$E_\varphi^i = -\frac{2k^2 Z_1}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-i)^n}{\tilde{\rho}_{1,n} \tilde{N}_{1,n}} \times R_{1,n}^{(1)}(-ic, i\xi_{<}) R_{1,n}^{(3)}(-ic, i\xi_{>}) S_{1,n}(-ic, \eta). \quad (3.33)$$

Outside the cavity, the total electric field may be written as the superposition of the incident field, given by (3.33), the field reflected by the infinite metallic plane evaluated assuming that there is no aperture, and a diffracted field representing the perturbation due to the aperture.

$$E_\varphi^t = E_\varphi^i + E_\varphi^r + E_\varphi^d = \frac{-4ik^2 Z_1}{\sqrt{\xi_0^2 + 1}} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\tilde{\rho}_{1,2\ell} \tilde{N}_{1,2\ell}} \times S_{1,2\ell}(-ic, \eta) \left[a_\ell^{(m)} R_{1,2\ell}^{(3)}(-ic, i\xi) + R_{1,2\ell}^{(1)}(-ic, i\xi_{<}) R_{1,2\ell}^{(3)}(-ic, i\xi_{>}) \right]. \quad (3.34)$$

Inside the cavity, the total electric field in the DPS region is:

$$E_{\varphi, \text{DPS}} = \frac{-4ik^2 Z_1}{\sqrt{\xi_0^2 + 1}} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\tilde{\rho}_{1,2\ell} \tilde{N}_{1,2\ell}} S_{1,2\ell}(-ic, -\eta) \times \left[b_\ell^{(m)} R_{1,2\ell}^{(1)}(-ic, i\xi) + c_\ell^{(m)} R_{1,2\ell}^{(3)}(-ic, i\xi) \right]. \quad (3.35)$$

whereas the total electric field inside the DNG region is:

$$E_{\varphi, \text{DNG}} = \frac{-4ik^2 Z_2}{\sqrt{\xi_0^2 + 1}} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\tilde{\rho}_{1,2\ell} \tilde{N}_{1,2\ell}} S_{1,2\ell}(ic, -\eta) \times \left[B_l^{(m)} R_{1,2\ell}^{(1)}(ic, i\xi) + C_l^{(m)} R_{1,2\ell}^{(3)}(ic, i\xi) \right] \quad (3.36)$$

If region 1 is DNG and region 2 is DPS, the modal coefficients are given by eqs. (3.22-3.24) and matrix (3.25) with new matrix elements:

$$t^{(m)} = -R_{1,2\ell}^{(1)'}(-ic, 0) R_{1,2\ell}^{(3)}(-ic, i\xi_0) \quad (3.37)$$

$$M_{1,1}^{(m)} = R_{1,2\ell}^{(3)'}(-ic, 0) - \zeta R_{1,2\ell}^{(3)'}(ic, 0) \frac{R_{1,2\ell}^{(3)}(-ic, 0)}{R_{1,2\ell}^{(3)}(ic, 0)} \quad (3.38)$$

$$M_{1,2}^{(m)} = 0, \quad M_{1,3}^{(m)} = R_{1,2\ell}^{(1)'}(ic, 0) \quad (39a, b)$$

$$M_{2,1}^{(m)} = \zeta \frac{R_{1,2\ell}^{(3)}(-ic, 0)}{R_{1,2\ell}^{(3)}(ic, 0)} R_{1,2\ell}^{(3)}(ic, i\xi_2) \quad (3.40)$$

$$M_{2,2}^{(m)} = \zeta \left[R_{1,2\ell}^{(1)}(-ic, i\xi_2) - \frac{R_{1,2\ell}^{(1)}(-ic, i\xi_1)}{R_{1,2\ell}^{(3)}(-ic, i\xi_1)} R_{1,2\ell}^{(3)}(-ic, i\xi_2) \right] \quad (3.41)$$

$$M_{2,3}^{(m)} = -R_{1,2\ell}^{(1)}(ic, i\xi_2) \quad (3.42)$$

$$M_{3,1}^{(m)} = -\zeta \frac{R_{1,2\ell}^{(3)}(-ic, 0)}{R_{1,2\ell}^{(3)}(ic, 0)} D_{2\ell}^{(3)}(ic, i\xi_2) \quad (3.43)$$

$$M_{3,2}^{(m)} = D_{2\ell}^{(1)}(-ic, i\xi_2) - \frac{R_{1,2\ell}^{(1)}(-ic, i\xi_1)}{R_{1,2\ell}^{(3)}(-ic, i\xi_1)} D_{2\ell}^{(3)}(-ic, i\xi_2) \quad (3.44)$$

$$M_{3,3}^{(m)} = D_{2\ell}^{(1)}(ic, i\xi_2) \quad (3.45)$$

$$C_l^{(m)} = -\zeta \frac{R_{1,2\ell}^{(3)}(-ic, 0)}{R_{1,2\ell}^{(3)}(ic, 0)} a_l^{(m)} \quad (3.46)$$

$$c_l^{(m)} = -\frac{R_{1,2\ell}^{(1)}(-ic, i\xi_1)}{R_{1,2\ell}^{(3)}(-ic, i\xi_1)} b_l^{(m)} \quad (3.47)$$

The far field behavior in the half space $z > 0$ is obtained using the asymptotic expressions $c\xi \sim kr$, $\eta \sim \cos \theta$ as $c\xi \rightarrow \infty$, yielding

$$E_{\varphi}^t \big|_{c\xi \rightarrow \infty} \sim \frac{e^{ikr}}{kr} \frac{4ik^2 Z_1}{\sqrt{\xi_0^2 + 1}} \times \sum_{\ell=1}^{\infty} \frac{R_{1,2\ell}^{(1)} + a_l^{(m)}}{\tilde{\rho}_{1,2\ell} \tilde{N}_{1,2\ell}} S_{1,2\ell+1}(-ic, \cos \theta). \quad (3.48)$$

If region 1 is DPS and region 2 is DNG, the field expressions (3.34 - 3.36) and (3.48) in each region are still valid, however the modal coefficients change because of different boundary conditions.

The new modal coefficients are still formally given by eqs. (3.22-3.24), where all symbols, except $t^{(e,m)}$, have changed and are given below:

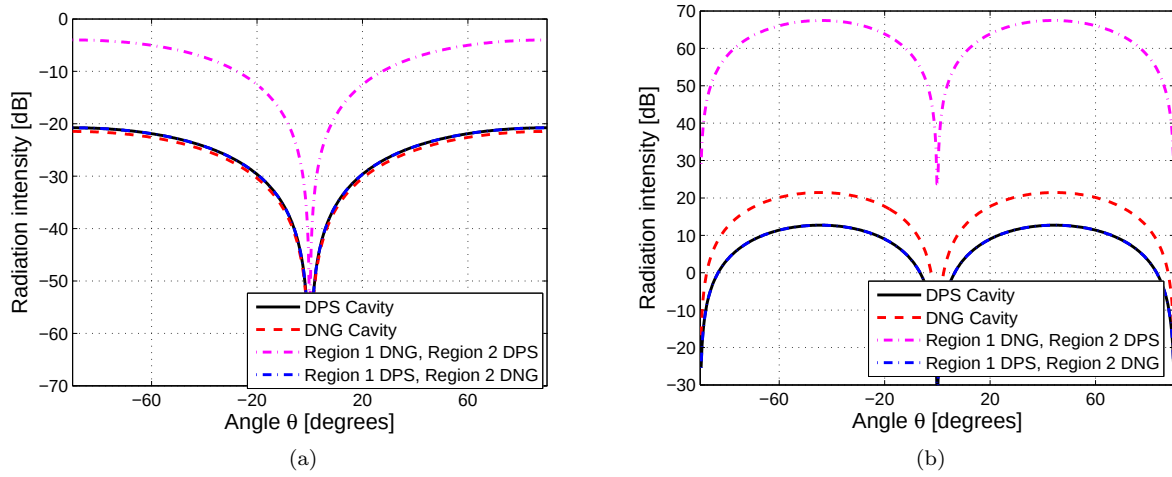


Figure 3.11: (a) Radiation intensity $U^{(e)}$ due to an electric dipole and (b) radiation intensity $U^{(m)}$ due to a magnetic dipole located at $(z_0/\lambda = 0.01, x_0 = y_0 = 0)$ for $c = 1$, $d/\lambda = 0.3183$, $\zeta = 0.5$, $\xi_1 = 1.205$ and $\xi_2 = 0.50$.

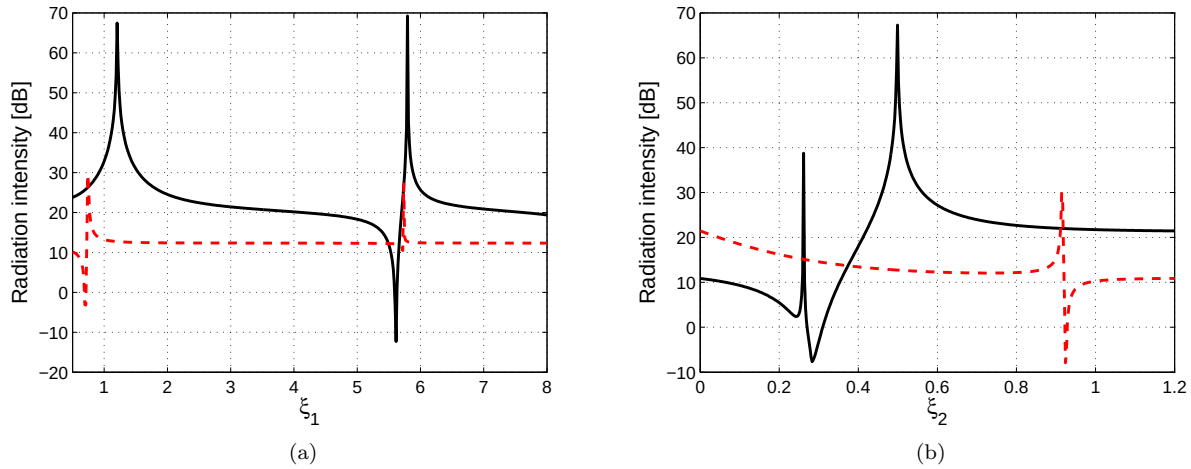


Figure 3.12: Maximum radiation intensity $U^{(m)}(\theta_{max})$ due to a magnetic dipole located at $(z_0/\lambda = 0.01, x_0 = y_0 = 0)$ for $c = 1$, $d/\lambda = 0.3183$, $\zeta = 0.5$, (a) $\xi_2 = 0.5$ and variable ξ_1 (b) $\xi_1 = 1.205$ and variable ξ_2 . The solid black line refers to Region 1 DNG and Region 2 DPS, while the dashed red line refers to Region 1 DPS and Region 2 DNG.

$$\tilde{M}_{1,1}^{(m)} = 2R_{2\ell}^{(3)'}(-ic, 0), \quad \tilde{M}_{1,2}^{(m)} = -R_{2\ell}^{(1)'}(-ic, 0) \quad (49a, b)$$

$$\tilde{M}_{1,3}^{(m)} = 0, \quad \tilde{M}_{2,1}^{(m)} = -\zeta R_{1,2\ell}^{(3)}(-ic, i\xi_2) \quad (50a, b)$$

$$\tilde{M}_{2,2}^{(m)} = \zeta R_{1,2\ell}^{(1)}(-ic, i\xi_2) \quad (3.51)$$

$$\tilde{M}_{2,3}^{(m)} = -R_{1,2\ell}^{(1)}(ic, i\xi_2) + \frac{R_{1,2\ell}^{(1)}(ic, i\xi_1)}{R_{1,2\ell}^{(3)}(ic, i\xi_1)} R_{1,2\ell}^{(3)}(ic, i\xi_2) \quad (3.52)$$

$$\tilde{M}_{3,1}^{(m)} = D_{2\ell}^{(3)}(-ic, i\xi_2), \quad \tilde{M}_{3,2}^{(m)} = -D_{2\ell}^{(1)}(-ic, i\xi_2) \quad (53a, b)$$

$$\tilde{M}_{3,3}^{(m)} = -D_{2\ell}^{(1)}(ic, i\xi_2) + \frac{R_{1,2\ell}^{(1)}(ic, i\xi_1)}{R_{1,2\ell}^{(3)}(ic, i\xi_1)} D_{2\ell}^{(3)}(ic, i\xi_2) \quad (3.54)$$

$$\tilde{c}_l^{(m)} = -\tilde{a}_l^{(m)}, \quad \tilde{C}_l^{(m)} = -\frac{R_{1,2\ell}^{(1)}(ic, i\xi_1)}{R_{1,2\ell}^{(3)}(ic, i\xi_1)} \tilde{B}_l^{(m)} \quad (55a, b)$$

Numerical results

Numerical results are provided only for the components of the magnetic and electric fields H_φ , E_φ along the azimuth direction using eqs. (3.19-3.21) and (3.34-3.36), since all other polarization can be evaluated from these components using eqs. (3.17) and (3.32).

The criterion used to determine the convergence of the series was to keep on adding terms until their sum does not change more than 0.1%. Accordingly it was sufficient to add the first 10 terms of each series.

Fig. 3.9 shows the total magnetic field $|H_\varphi|$ computed inside the cavity and outside the cavity near the aperture, when the source is an electric dipole and region 1 is DNG and region 2 is DPS and viceversa. Looking at these figures, one observes that there is a more complex field structure when Region 1 is DNG and Region 2 is DPS (Fig. 3.9a). This more complex behavior may be explained observing that there are more material variations going from the source towards the cavity. In fact, the material changes from DPS to DNG across the aperture and then from DNG to DPS across the boundary ξ_2 . On the contrary, when Region 1 is DPS and Region 2 is DNG there is no change of material across the aperture and a transition from DPS to DNG across the boundary ξ_2 .

Field magnitude values are better appreciated using the 1D plots of Fig. 3.10, which represents quantities evaluated along the coordinate line $|\eta| = 0.7$ shown as dotted in Fig. 3.8. One should note that in Fig. 3.10 negative values of ξ correspond to locations within the cavity. Fig. 3.10a shows the behavior of $|H_\varphi|$ for an electric dipole source. The same figure contains two additional results that help understanding the effect of the presence of metamaterial. The simplest situation corresponds to the whole cavity filled with DPS material, which is accomplished by using the exact expressions of this work by setting $\xi_1 = \xi_2 = 2$. This situation was already investigated in [111] and in fact our results match with Fig. 4 of [111] with $\zeta = 1$. Then, the whole cavity filled with DNG metamaterial was investigated in [113] and our results reproduce the ones of Fig. 3(b) of [113] with $\zeta = 0.5$. A more complex situation is considered in this manuscript, where the cavity contains two layered regions. It is apparent that the strongest value of the total magnetic field inside the cavity is achieved when Region 1 is DNG and Region 2 is DPS. Fig. 3.10b shows the behavior of $|E_\varphi|$ for the same geometry and materials already considered in Fig. 3.10a, but for the case of a magnetic dipole source. Similar to the results of Fig. 3.10a, it is apparent that the strongest field is achieved inside the cavity when Region 1 is DNG and Region 2 is DPS.

The third results are the radiation intensity plots which describe the behavior of the fields in the far-region and are given by

$$U^{(e)}(\theta) = \frac{8k^2}{Z_1(\xi_0^2 + 1)} \times \left| \sum_{l=0}^{\infty} \frac{R_{1,2\ell+1}^{(1)}(-ic, i\xi_0) + a_l^{(e)}}{\tilde{\rho}_{1,2\ell+1} \tilde{N}_{1,2+1}} S_{1,2\ell+1}(-ic, \cos \theta) \right|^2, \quad (3.56)$$

$$U^{(m)}(\theta) = \frac{8k^2 Z_1}{\xi_0^2 + 1} \times \left| \sum_{l=1}^{\infty} \frac{R_{1,2\ell}^{(1)}(-ic, i\xi_0) + a_l^{(m)}}{\tilde{\rho}_{1,2\ell} \tilde{N}_{1,2\ell}} S_{1,2\ell}(-ic, \cos \theta) \right|^2. \quad (3.57)$$

The radiation intensity for $d/\lambda = 0.3183$, $\xi_1 = 1.205$ and $\xi_2 = 0.50$ is computed using eqs. (3.56-3.57), when the dipole sources are located very close to the ground plane ($z_0/\lambda = 0.01$, $\xi_0 = 0.0628$). Fig. 3.11 represents the results of the radiation intensity for the four cases studied in this work as well as with the

limiting cases of the whole cavity filled with DPS and DNG material. The strongest peak occurs when Region 1 is DNG and Region 2 is DPS thus providing a more directive pattern. In other words, the layered combination of DNG and DPS produces a total field that is stronger than DNG and DPS standalone.

Finally, we show that with an appropriate combination of the dimensions of the DNG Region 1 and DPS Region 2 one could design an antenna that maximizes the radiation intensity. In fact, Fig. 3.11 suggests an investigation to determine if the larger values of radiation intensity already obtained for the combination of DNG for Region 1 and DPS for Region 2, could be further improved. Therefore, we seek to optimize the geometrical parameters ξ_1 and ξ_2 to maximize the radiation intensity $U(\theta_{max})^{(m)}$ in the case of a magnetic dipole source. In Fig. 3.12a, ξ_1 is varied and ξ_2 fixed, while in Fig. 3.12b ξ_2 is varied and ξ_1 fixed. One obtains that the radiation intensity is maximized when $\xi_1 = 1.205$ and $\xi_2 = 0.50$. In conclusion, a DNG metamaterial allows to obtain a maximum radiation intensity that significantly exceeds the one achievable with DPS material.

3.3 Algorithm 934: Mathieu function for complex values of the parameter

The validation process may take advantage of several exact solutions of canonical electromagnetic scattering problems that were recently developed, such as, for example, [116, 117, 118, 119, 120, 121, 122, 123]. These solutions involve shapes that are coordinate surfaces in the elliptic cylinder coordinate system and require the evaluation of angular and radial Mathieu functions. When materials are lossy, the Mathieu functions must be evaluated for complex values of a parameter. Therefore, a new software was developed for the computation of both angular and radial Mathieu functions, since no other computational software was available. The new software, known as algorithm 934, was published in [29] and its main results are summarized in the following.

Introduction

Mathieu functions were introduced by Émile Léonard Mathieu in 1868 to study the vibration modes of an elliptical membrane [124]. Since then, many other authors have investigated their properties and the most important results are included in [125], [126], [127]. Mathieu functions are the solutions of Mathieu's differential equation

$$\frac{d^2 y}{dx^2} + (a - 2q \cos^2 x)y = 0, \quad (3.58)$$

which are also called angular Mathieu functions. Replacing x with ix in the previous equation yields the modified Mathieu differential equation

$$\frac{d^2 y}{dx^2} - (a - 2q \cosh^2 x)y = 0, \quad (3.59)$$

whose solutions are called radial Mathieu functions.

Solutions of both differential equations depend upon the parameter q , which could take complex values.

The computation of Mathieu functions is not a trivial problem and software packages that compute them when q is real have been developed by, among others, Clemm [128, 129], Hodge [130], Rengarajan [131], Frisch [132], Baker [133], Shirts [134], [135], IMSL [136], Zhang and Jin [137], Alhargan [138], Erricolo [139] and [140]. When q is complex, Mathematica® provides solutions, but only for the angular functions. The novelty of this software is to extend to the case of complex values of q the results presented in [139].

Angular and radial of Mathieu functions

In the following, we will use the notation of Blanch and Rhodes [141] (which is also available in [142], [127]) and Stratton [143].

Equation (3.58) is periodic and in many physical applications only periodic solutions are of interest. For a given q there exist two countable sets of values of a for which equation (3.58) admits periodic solutions. These values of a are called characteristic values and, depending upon the set, the period of the solution is either π or 2π . There are four kinds of periodic solutions of (3.58) associated with the characteristic values a :

$$\text{Se}_{2r}(q, x) = \sum_{k=0}^{\infty} \text{De}_{2k}^{(2r)} \cos 2kx \quad (\text{of period } \pi) \quad (3.60)$$

$$\text{Se}_{2r+1}(q, x) = \sum_{k=0}^{\infty} \text{De}_{2k+1}^{(2r+1)} \cos(2k+1)x \quad (\text{of period } 2\pi) \quad (3.61)$$

$$\text{So}_{2r}(q, x) = \sum_{k=1}^{\infty} \text{Do}_{2k}^{(2r)} \sin 2kx \quad (\text{of period } \pi) \quad (3.62)$$

$$\text{So}_{2r+1}(q, x) = \sum_{k=0}^{\infty} \text{Do}_{2k+1}^{(2r+1)} \sin(2k+1)x \quad (\text{of period } 2\pi) \quad (3.63)$$

The functions $\text{Se}_n(q, ix)$ and $\text{So}_n(q, ix)$ clearly satisfy (3.59) for the same characteristic values a , but (3.60)-(3.63) converge slowly. Therefore the solutions of (3.59) are written in terms of rapidly converging series of products of Bessel functions associated with the same coefficients De_m, Do_m of the angular functions. Their expressions are:

$$\text{Re}_{2r}^{(1)}(q, x) = \frac{(-1)^r}{\text{De}_0^{(2r)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{De}_{2k}^{(2r)} J_k(s) J_k(t), \quad (3.64)$$

$$\text{Re}_{2r+1}^{(1)}(q, x) = \frac{(-1)^r}{\text{De}_1^{(2r+1)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{De}_{2k+1}^{(2r+1)} [J_{k+1}(s) J_k(t) + J_k(s) J_{k+1}(t)], \quad (3.65)$$

$$\text{Ro}_{2r}^{(1)}(q, x) = \frac{(-1)^r}{\text{Do}_2^{(2r)}} \sqrt{\frac{\pi}{2}} \sum_{k=1}^{\infty} (-1)^k \text{Do}_{2k}^{(2r)} [J_{k+1}(s) J_{k-1}(t) - J_{k+1}(t) J_{k-1}(s)], \quad (3.66)$$

$$\text{Ro}_{2r+1}^{(1)}(q, x) = \frac{(-1)^r}{\text{Do}_1^{(2r+1)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{Do}_{2k+1}^{(2r+1)} [J_{k+1}(s) J_k(t) - J_k(s) J_{k+1}(t)], \quad (3.67)$$

where

$$s = \sqrt{q}e^x, \quad t = \sqrt{q}e^{-x}. \quad (3.68)$$

The radial functions of the first kind have parity either even, $\text{Re}_n^{(1)}$, or odd, $\text{Ro}_n^{(1)}$. A second set of solutions for the modified Mathieu's equation is obtained by replacing the Bessel functions $J_m(s)$ in the previous equations with the Bessel functions $Y_m(s)$. This substitution yields the modified functions of the second kind. In many physical applications they are referred to as radial functions of the second kind. They have parity either even, $\text{Re}_n^{(2)}$, or odd, $\text{Ro}_n^{(2)}$:

$$\text{Re}_{2r}^{(2)}(q, x) = \frac{(-1)^r}{\text{De}_0^{(2r)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{De}_{2k}^{(2r)} Y_k(s) J_k(t), \quad (3.69)$$

$$\text{Re}_{2r+1}^{(2)}(q, x) = \frac{(-1)^r}{\text{De}_1^{(2r+1)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{De}_{2k+1}^{(2r+1)} [Y_{k+1}(s) J_k(t) + Y_k(s) J_{k+1}(t)], \quad (3.70)$$

$$\text{Ro}_{2r}^{(2)}(q, x) = \frac{(-1)^r}{\text{Do}_2^{(2r)}} \sqrt{\frac{\pi}{2}} \sum_{k=1}^{\infty} (-1)^k \text{Do}_{2k}^{(2r)} [Y_{k+1}(s) J_{k-1}(t) - Y_{k-1}(s) J_{k+1}(t)], \quad (3.71)$$

$$\text{Ro}_{2r+1}^{(2)}(q, x) = \frac{(-1)^r}{\text{Do}_1^{(2r+1)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{Do}_{2k+1}^{(2r+1)} [Y_{k+1}(s) J_k(t) - Y_k(s) J_{k+1}(t)]. \quad (3.72)$$

Similar to Hankel functions, one defines modified functions of the third and fourth kinds. They are also referred to as radial functions and they have even parity

$$\text{Re}_n^{(3)} = \text{Re}_n^{(1)} + i\text{Re}_n^{(2)}, \quad (3.73)$$

$$\text{Re}_n^{(4)} = \text{Re}_n^{(1)} - i\text{Re}_n^{(2)}, \quad (3.74)$$

and odd parity

$$\text{Ro}_n^{(3)} = \text{Ro}_n^{(1)} + i\text{Ro}_n^{(2)}, \quad (3.75)$$

$$\text{Ro}_n^{(4)} = \text{Ro}_n^{(1)} - i\text{Ro}_n^{(2)}. \quad (3.76)$$

Let $a_r(q)$ be the characteristic values associated with even solutions and $b_r(q)$ those associated with odd solutions. Notice that $r = 0, 1, 2, \dots$ and $r = 0$ applies only to a_r . The following four properties hold for complex values of q .

Property 1 When $q \rightarrow 0$, $a_r(q) \rightarrow r^2$ and $b_r(q) \rightarrow r^2$.

Property 2 $a_{2r}(-q) = a_{2r}(q)$, $b_{2r}(-q) = b_{2r}(q)$, $a_{2r+1}(-q) = b_{2r+1}(q)$.

Property 3 When q is real and not zero, the characteristic values can be ordered as $a_0 < b_1 < a_1 < b_2 < \dots < b_r < a_r < \dots$, $q > 0$
 $a_0 < a_1 < b_1 < b_2 < a_2 < a_3 < b_3 < \dots a_{2r-1} < b_{2r-1} < b_{2r} < a_{2r} < \dots$, $q < 0$

Property 4 When q is real, the solutions associated with $a_r(q)$ and $b_r(q)$ have r zeros in $[0, \pi)$.

Properties 1 and 3 establish the ordering principle for the solutions of Mathieu equation. Specifically, the order r of a Mathieu function is given by the position r of the characteristic value a_r (or b_r) in the sequence of characteristic values. Mathieu functions of different order are associated with different expansion coefficients De^r (Do^r), which depend on a_r (b_r), as implied by the definitions (3.60)-(3.67), (3.69)-(3.76). When coupled with the continuity of $a_r(q)$ and $b_r(q)$, Properties 1 and 3 are the only criterion available to order the characteristic values $a_r(q)$ and $b_r(q)$ in the complex plane [144].

It is sufficient to know the solutions of Mathieu equation when q belongs to the first quadrant. In fact, all values of q may be reduced to the first quadrant because of Property 2 and also because if $y(z)$ is a solution associated with a and q then y^* is a solution associated with q^* and a^* .

Normalization, Properties, Orthogonality, Wronskian

There exist at least three different normalizations for the Mathieu functions. In this work, we use the normalization of Stratton [143] and, accordingly, the Mathieu angular functions are normalized so that

$$\text{Se}_r(q, 0) = 1, \left. \frac{d\text{So}_r(q, x)}{dx} \right|_{x=0} = 1. \quad (3.77)$$

Therefore the expansion coefficients must satisfy

$$\sum_{k=0}^{\infty} \text{De}_{2k+p}^{(r)} = 1, \sum_{k=0}^{\infty} (2k+p) \text{Do}_{2k+p}^{(r)} = 1 \quad (3.78)$$

with $p = 0, 1$ depending on the order r . Mathieu angular functions satisfy the following orthogonality relations

$$\int_0^{2\pi} \text{Se}_m(q, x) \text{Se}_n(q, x) dx = \begin{cases} 0 & m \neq n \\ N_m^e & m = n \end{cases} \quad (3.79)$$

$$\int_0^{2\pi} \text{So}_m(q, x) \text{So}_n(q, x) dx = \begin{cases} 0 & m \neq n \\ N_m^o & m = n \end{cases} \quad (3.80)$$

$$\int_0^{2\pi} \text{Se}_m(q_1, x) \text{So}_n(q_2, x) dx = 0, \forall m, n, q_1, q_2 \quad (3.81)$$

In the previous relations appear the normalization coefficients

$$N_{2r}^e = \pi \left[2 \left(\text{De}_0^{(2r)} \right)^2 + \left(\text{De}_2^{(2r)} \right)^2 + \left(\text{De}_4^{(2r)} \right)^2 + \dots \right], \quad (3.82)$$

$$N_{2r+1}^e = \pi \left[\left(\text{De}_1^{(2r+1)} \right)^2 + \left(\text{De}_3^{(2r+1)} \right)^2 + \dots \right], \quad (3.83)$$

$$N_{2r}^o = \pi \left[\left(\text{Do}_2^{(2r)} \right)^2 + \left(\text{Do}_4^{(2r)} \right)^2 + \dots \right], \quad (3.84)$$

$$N_{2r+1}^o = \pi \left[\left(\text{Do}_1^{(2r+1)} \right)^2 + \left(\text{Do}_3^{(2r+1)} \right)^2 + \dots \right]. \quad (3.85)$$

Relations between solutions for the parameter q and $-q$ The following relations hold for the angular functions

$$\text{Se}_{2r}(-q, x) = \frac{\text{Se}_{2r}(q, \pi/2 - x)}{\text{Se}_{2r}(q, \pi/2)} \quad \text{De}_{2k}^{(2r)}(-q) = \frac{(-1)^k \text{De}_{2k}^{(2r)}(q)}{\text{Se}_{2r}(q, \pi/2)} \quad (3.86)$$

$$\text{Se}_{2r+1}(-q, x) = \frac{\text{So}_{2r+1}(q, \pi/2 - x)}{\text{So}_{2r+1}(q, \pi/2)} \quad \text{De}_{2k+1}^{(2r+1)}(-q) = \frac{(-1)^k \text{Do}_{2k+1}^{(2r+1)}(q)}{\text{So}_{2r+1}(q, \pi/2)} \quad (3.87)$$

$$\text{So}_{2r+1}(-q, x) = -\frac{\text{Se}_{2r+1}(q, \pi/2 - x)}{\text{Se}_{2r+1}'(q, \pi/2)} \quad \text{Do}_{2k+1}^{(2r+1)}(-q) = \frac{(-1)^{k+1} \text{De}_{2k+1}^{(2r+1)}(q)}{\text{Se}_{2r+1}'(q, \pi/2)} \quad (3.88)$$

$$\text{So}_{2r}(-q, x) = -\frac{\text{So}_{2r}(q, \pi/2 - x)}{\text{So}_{2r}'(q, \pi/2)} \quad \text{Do}_{2k}^{(2r)}(-q) = \frac{(-1)^k \text{Do}_{2k}^{(2r)}(q)}{\text{So}_{2r}'(q, \pi/2)} \quad (3.89)$$

For the radial functions of even order

$$\text{Re}_{2r}^{(1)}(-q, x) = \text{Re}_{2r}^{(1)}(q, x + i\pi/2) \quad (3.90)$$

$$\text{Re}_{2r}^{(2)}(-q, x) = \text{Re}_{2r}^{(2)}(q, x + i\pi/2) \quad (3.91)$$

$$\text{Ro}_{2r}^{(1)}(-q, x) = \text{Ro}_{2r}^{(1)}(q, x + i\pi/2) \quad (3.92)$$

$$\text{Ro}_{2r}^{(2)}(-q, x) = \text{Ro}_{2r}^{(2)}(q, x + i\pi/2) \quad (3.93)$$

and for the radial functions of odd order

$$\text{Re}_{2r+1}^{(1)}(-q, x) = \text{Ro}_{2r+1}^{(1)}(q, x + i\pi/2) \quad (3.94)$$

$$\text{Ro}_{2r+1}^{(1)}(-q, x) = \text{Re}_{2r+1}^{(1)}(q, x + i\pi/2) \quad (3.95)$$

$$\text{Re}_{2r+1}^{(2)}(-q, x) = \text{Ro}_{2r+1}^{(2)}(q, x + i\pi/2) \quad (3.96)$$

$$\text{Ro}_{2r+1}^{(2)}(-q, x) = \text{Re}_{2r+1}^{(2)}(q, x + i\pi/2) \quad (3.97)$$

The Mathieu radial functions of argument $x + i\pi/2$ may be expressed in terms of the modified Bessel functions I and K using the following relations [145]

$$J_m(iw) = i^m I_m(w), \quad Y_m(iw) = i^{m+1} \left[I_m(w) + i \frac{2}{\pi} (-1)^m K_m(w) \right] \quad (3.98)$$

$$\cosh(w + i\pi/2) = i \sinh(w), \quad \sinh(w + i\pi/2) = i \cosh(w), \quad \tanh(w + i\pi/2) = \coth(w) \quad (3.99)$$

where w is a complex variable with $-\pi < \arg w \leq \pi/2$. In fact, using definition (3.68), when $x \rightarrow x + i\pi/2$, the argument of the Bessel functions become $s \rightarrow is$ and $t \rightarrow -it$, so that the radial functions of the first kind become

$$\text{Re}_{2r}^{(1)}(-q, x) = (-1)^r \sqrt{\frac{\pi}{2}} \text{Ie}_{2r}(q, x) \quad (3.100)$$

$$\text{Re}_{2r+1}^{(1)}(-q, x) = i(-1)^r \sqrt{\frac{\pi}{2}} \text{Ie}_{2r+1}(q, x) \quad (3.101)$$

$$\text{Ro}_{2r}^{(1)}(-q, x) = (-1)^r \sqrt{\frac{\pi}{2}} \text{Io}_{2r}(q, x) \quad (3.102)$$

$$\text{Ro}_{2r+1}^{(1)}(-q, x) = i(-1)^r \sqrt{\frac{\pi}{2}} \text{Io}_{2r+1}(q, x) \quad (3.103)$$

where

$$\text{Ie}_{2r}(q, x) = \sum_{k=0}^{\infty} (-1)^k \text{De}_{2k}^{(2r)}(q) I_k(s) I_k(t) / \text{De}_0^{(2r)}(q) \quad (3.104)$$

$$\text{Ie}_{2r+1}(q, x) = \sum_{k=0}^{\infty} (-1)^k \text{Do}_{2k+1}^{(2r+1)}(q) [I_{k+1}(s) I_k(t) + I_{k+1}(t) I_k(s)] / \text{Do}_1^{(2r+1)}(q) \quad (3.105)$$

$$\text{Io}_{2r}(q, x) = \sum_{k=1}^{\infty} (-1)^k \text{Do}_{2k}^{(2r)}(q) [I_{k-1}(s) I_{k+1}(t) - I_{k+1}(s) I_{k-1}(t)] / \text{Do}_2^{(2r)}(q) \quad (3.106)$$

$$\text{Io}_{2r+1}(q, x) = \sum_{k=0}^{\infty} (-1)^k \text{De}_{2k+1}^{(2r+1)}(q) [I_{k+1}(s) I_k(t) - I_k(t) I_{k+1}(s)] / \text{De}_1^{(2r+1)}(q) \quad (3.107)$$

The radial functions of the second kind are complex for real value of x , yielding

$$\text{Re}_{2r}^{(2)}(-q, x) = (-1)^r \left[i\sqrt{\pi/2} \text{Ie}_{2r}(q, x) - \sqrt{\frac{2}{\pi}} \text{Ke}_{2r}(q, x) \right] \quad (3.108)$$

$$\text{Re}_{2r+1}^{(2)}(-q, x) = (-1)^r \left[-\sqrt{\pi/2} \text{Ie}_{2r+1}(q, x) + i\sqrt{\frac{2}{\pi}} \text{Ke}_{2r+1}(q, x) \right] \quad (3.109)$$

$$\text{Ro}_{2r}^{(2)}(-q, x) = (-1)^r \left[i\sqrt{\pi/2} \text{Io}_{2r}(q, x) - \sqrt{\frac{2}{\pi}} \text{Ko}_{2r}(q, x) \right] \quad (3.110)$$

$$\text{Ro}_{2r+1}^{(2)}(-q, x) = (-1)^r \left[-\sqrt{\pi/2} \text{Io}_{2r+1}(q, x) + i\sqrt{\frac{2}{\pi}} \text{Ko}_{2r+1}(q, x) \right] \quad (3.111)$$

where

$$\text{Ke}_{2r}(q, x) = \sum_{k=0}^{\infty} \text{De}_{2k}^{(2r)}(q) I_k(t) K_k(s) / \text{De}_0^{(2r)}(q) \quad (3.112)$$

$$\text{Ke}_{2r+1}(q, x) = \sum_{k=0}^{\infty} \text{Do}_{2k+1}^{(2r+1)}(q) [I_k(t) K_{k+1}(s) - I_{k+1}(t) K_k(s)] / \text{Do}_1^{(2r+1)}(q) \quad (3.113)$$

$$\text{Ko}_{2r}(q, x) = \sum_{k=0}^{\infty} \text{Do}_{2k}^{(2r)}(q) [I_{k-1}(t) K_{k+1}(s) - I_{k+1}(t) K_{k-1}(s)] / \text{Do}_2^{(2r)}(q) \quad (3.114)$$

$$\text{Ko}_{2r+1}(q, x) = \sum_{k=0}^{\infty} \text{De}_{2k+1}^{(2r+1)}(q) [I_{k+1}(t) K_k(s) + I_k(t) K_{k+1}(s)] / \text{De}_1^{(2r+1)}(q) \quad (3.115)$$

Computation of Mathieu functions

Mathieu functions are computed according to the following three main steps: (1) computation of the characteristic values; (2) computation of the expansion coefficients De_m or Do_m ; and, (3) evaluation of the series expansions (3.60)-(3.67), (3.69)-(3.76).

Computation of the characteristic values The Mathieu characteristic values depend on the parameter q , the parity of the Mathieu function and its order n , and one approach to compute them is to solve an eigenvalue problem that leads to an infinite set of eigenvalues that approximate the characteristic values a_r (b_r) associated with the parameter q .

When q is real, the characteristic values are real and it is possible to sort them, according to Property 3. This sorting provides the rule to determine the order n of the Mathieu function. Specifically, the solution of order n is the one generated using the characteristic value at position n in the sorted sequence. However, when q is not real, the characteristic values are complex and they cannot be sorted, which prevents a direct association between a specific characteristic value and its corresponding Mathieu function.

Fortunately, the continuity of the characteristic values $a_r(q)$ and $b_r(q)$ provides a method to determine the appropriate characteristic value to generate a requested Mathieu function of order n , which is explained next.

Tracking algorithm Assume that $q = q_{\text{real}} + iq_{\text{imag}}$ and that the parity and order of the Mathieu function are given. The following algorithm determines the appropriate characteristic value, $a_r(q_{\text{temp}})$ or $b_r(q_{\text{temp}})$, when $q_{\text{temp}} = q_{\text{real}}$ and then tracks the characteristic value as q_{temp} is changed until it reaches $q_{\text{temp}} = q$.

Step 1 Let $q_0 = q_{\text{real}}$ and compute the characteristic values associated with q_0 . The characteristic values are the eigenvalues of an infinite matrix, whose entries depend on q , n , and on the parity of the Mathieu function. Since q_0 is real, the eigenvalues of the matrix are real and can be sorted. The infinite matrix is truncated to an order $N > n$, which is sufficiently large, ($N = 10 + 2n$), so that the first n eigenvalues of the truncated matrix are a good approximation of the first n characteristic values. Each characteristic value a_r^0 (b_r^0) is associated with the eigenvalue at position r in the sequence and the one at the location $r = n$ is selected. This is performed by `initial_eigenvalue_approximation`.

Step 2 Consider an increment Δq_{imag} so that $\Delta q_{\text{imag}} \leq 0.5$ and set $p = 0$.

Step 3 Let $p = p + 1$, $q_{\text{temp}}^p = q_0 + ip\Delta q_{\text{imag}}$. Then use the method described in [146] to refine the characteristic value $a_n^p(q_{\text{temp}}^p)$ using as initial approximation $a_n^{p-1}(q_{\text{temp}}^{p-1})$.

Step 4 If $q_{\text{temp}}^p \neq q$ then go back to Step 3, otherwise the last value found for a_n^p (b_n^p) is the characteristic value of order n associated with q .

Steps 3 and 4 are both performed by `Mathieu_Eigenvalue_and_Coefficients`.

This software could be extended to allow for the simultaneous computation of all Mathieu functions of order below the maximum order n . This extension requires the simultaneous computation of all the characteristic values $a_r(q)$ or $b_r(q)$, $r = 0, \dots, n$, and could be accomplished with the following variation of the tracking algorithm. At Step 3, instead of using Blanch's algorithm, one could replace it with the following.

Step 3. Let $p = p + 1$, $q_{\text{temp}}^p = q_0 + ip\Delta q_{\text{imag}}$. Create a new truncated matrix, similar to Step 1, for the parameter q_{temp}^p and compute its eigenvalues e_ℓ , $\ell = 1, \dots, N$. Let $d_{\ell s}$ be the distance between the eigenvalue e_ℓ and the previous characteristic value approximation $a_s^{p-1}(q_{\text{temp}}^{p-1})$ or $b_s^{p-1}(q_{\text{temp}}^{p-1})$. For each $s = 1, \dots, n$, compute all distances $d_{\ell s}$. The characteristic value $a_s^p(q_{\text{temp}}^p)$ or $b_s^p(q_{\text{temp}}^p)$ is the eigenvalue e_ℓ for which the distance $d_{\ell s}$ is minimal.

Computation of the expansion coefficients The expansion coefficients De_m (Do_m) are computed using four recurrence relations obtained by substituting the series expansion for the angular functions (3.60)-(3.63) into Mathieu differential equation (3.58).

Blanch [146] showed that each relation may be written as both a forward or a backward recurrence for any characteristic value a_r (b_r). In particular, she found the property that when the value of a_r (b_r) is the characteristic value then both the forward and the backward recurrence relation produce the same expansion coefficients. This property is also used at Steps 3 and 4 of the previous section to refine the value computed for a_r (b_r), by computing appropriate derivatives of a_r (b_r) and using them into a Newton scheme to improve the initial value of a_r (b_r). Once the value of a_r (b_r) has been determined, the expansion coefficients are computed using both the forward and the backward recurrence relation by the subroutine `Mathieu_Eigenvalue_and_Coefficients`.

In the computation of the expansion coefficients, it is sufficient to consider values of the parameter q such that $\text{real}(q) \geq 0$ because of the relations between Mathieu functions for the parameter q and $-q$ presented in Section 3.3.

Evaluation of the series expansions: Angular functions Mathieu angular functions are computed using the series expansions (3.60)-(3.63) by the function `MathieuAngular` for real values of the variable x and complex values of the parameter q . When x takes complex values, the appropriate function call is `MathieuAngular_cmplx`.

Evaluation of the series expansions: Radial functions Mathieu radial functions are computed using the series expansions (3.64)-(3.67), (3.69)-(3.76) by the function `MathieuRadial`, which accepts real values of the variable x and complex values of the parameter q .

Specifically the function `MathieuRadial` operates in the following way. When $\text{real}(q) \geq 0$, it applies the tracking algorithm, it determines the expansion coefficients and computes the appropriate series expansions using the definitions (3.64)-(3.67), (3.69)-(3.76). When $\text{real}(q) \leq 0$, it introduces a temporary variable $q1 = -q$ and performs the same steps as for the previous case, where $q1$ replaces q . Then, the radial functions are obtained by applying the relations given in Section 3.3.

Validation

These subroutines have been validated by making comparisons as described in the following.

- When $|q|$ is sufficiently small, comparisons can be made with the McLaurin expansions of the Mathieu functions described in [147].
- Comparison with Mathematica®.
- Another check is obtained by controlling the wronskian value.

In the following, u is the argument of the angular Mathieu function and v is the argument of the radial Mathieu function because of the physical meaning of u and v , which is related to the transformation from elliptic cylinder to rectangular Cartesian coordinates given in eq. (3.119) as part of the sample application given in Section 6.

Comparison with power series We compare our results to the values obtained by using the small argument series expansion approximation [147] of the solutions calculated by a subroutine programmed in MATLAB®. Fig. 3.13 shows the results of the comparison in the case of the even angular Mathieu function $\text{Se}_m(q, v)$ for $m = 15$ and $q = -0.9 + i0.9$. Fig. 3.14 shows the results of the comparison in the case of the

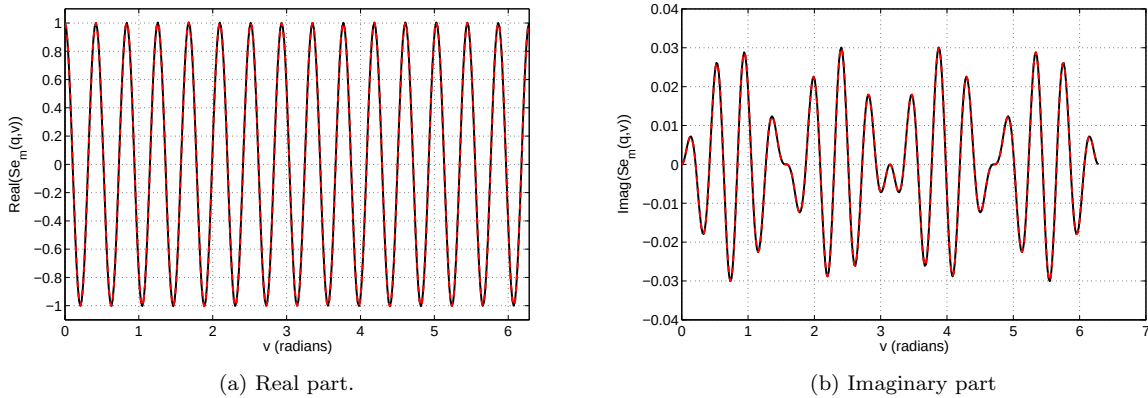


Figure 3.13: Plot of $\text{Se}_m(q, v)$ for $m = 15$ and $q = -0.9 + i0.9$. Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

odd angular Mathieu function $\text{So}_m(q, v)$ for $m = 10$ and $q = -0.9 + i0.9$.

Fig. 3.15 shows the results of the comparison in the case even radial Mathieu functions of the first kind $\text{Re}_m^{(1)}(q, u)$ for $m = 5$ and $q = -0.8 + i0.7$. Fig. 3.16 shows the results of the comparison in the case of even radial Mathieu functions of the second kind $\text{Re}_m^{(2)}(q, u)$ for $m = 5$ and $q = -0.8 + i0.7$. Fig. 3.17 shows the results of the comparison in the case of odd radial Mathieu functions of the first kind $\text{Ro}_m^{(1)}(q, u)$ for $m = 3$ and $q = 0.2025 + i0.5$. Fig. 3.18 shows the results of the comparison in the case of even radial Mathieu functions of the second kind $\text{Ro}_m^{(2)}(q, u)$ for $m = 3$ and $q = 0.2025 + i0.5$. Next we show the comparisons

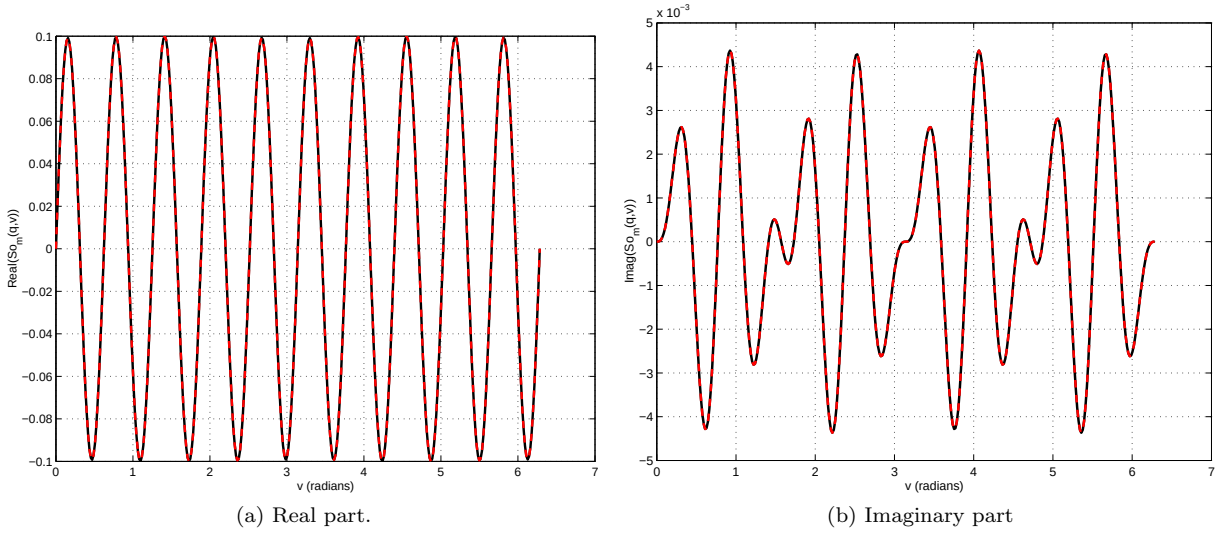


Figure 3.14: Plot of $So_m(q, v)$ for $m = 10$ and $q = -0.9 + i0.9$. Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

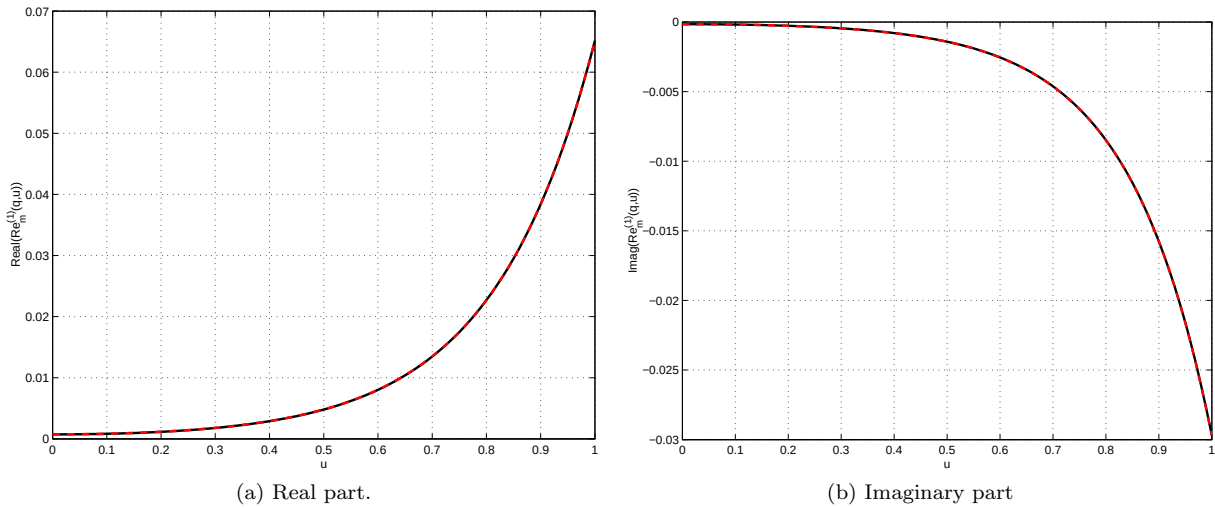


Figure 3.15: Plot of $Re_m^{(1)}$ for $m = 5$ and $q = -0.8 + i0.7$. Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

with the small argument power series as a function of q . In Fig.3.19, the angular Mathieu function $Se_m(q, v)$ is evaluated for $m = 1$, $v = 1$ and $q = q_r + iq_i$, with $q_i = 0.5$ while $-0.5 < q_r < 0.5$. In Fig. 3.20, the even radial Mathieu function of the first kind $Re_m^1(q, u)$ is evaluated for $m = 1$, $u = 1$ and $q = q_r + iq_i$, with $q_i = 0.5$ while $-0.5 < q_r < 0.5$. In both cases, there is a good agreement between the Fortran results and the small argument power expansion.

Comparison with Mathematica® For larger value of $|q|$, Mathieu angular functions are evaluated using Mathematica® and compared to the results obtained by our subroutines. The comparison with Mathematica® is limited to the angular Mathieu functions, since there are no Mathematica® functions that compute radial Mathieu functions.

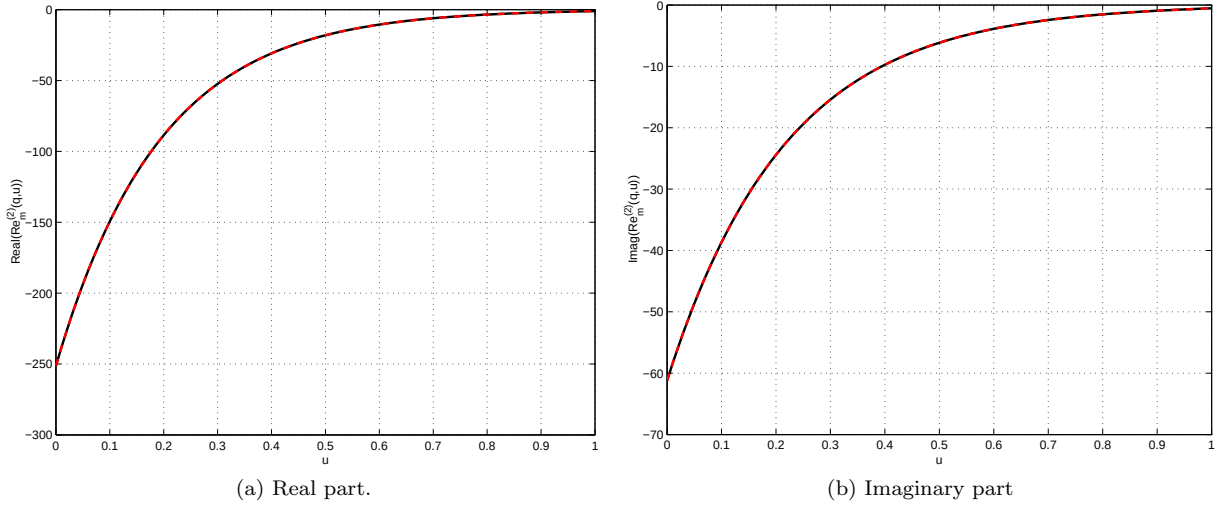


Figure 3.16: Plot of $\text{Re}_m^{(2)}(q, u)$ for $m = 5$ and $q = -0.8 + i0.7$. Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

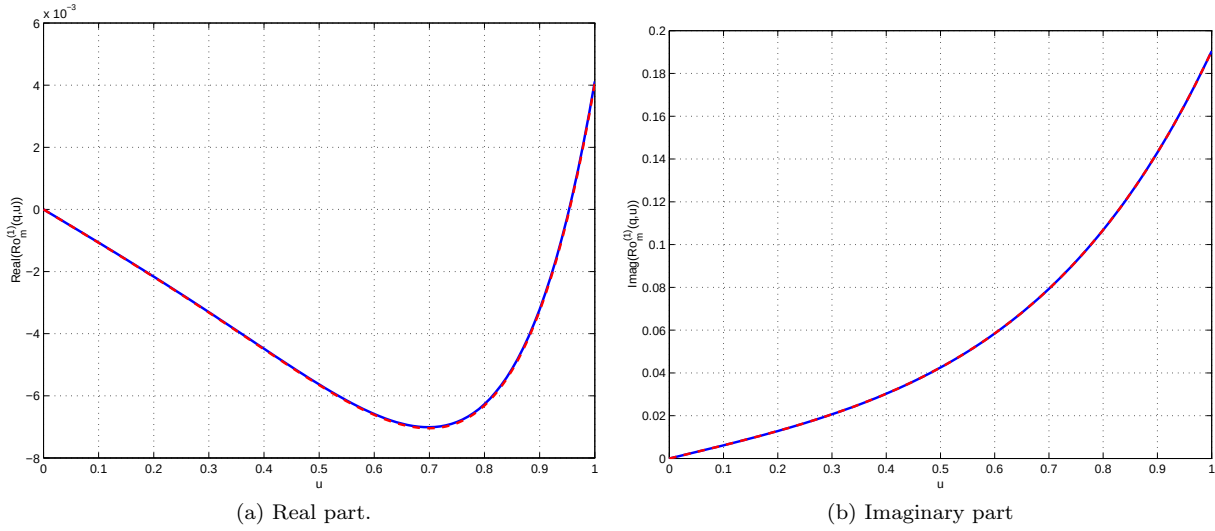


Figure 3.17: Plot of $\text{Ro}_m^{(1)}(q, u)$ for $m = 3$ and $q = 0.2025 + i0.5$. Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

Angular Mathieu functions are evaluated in three steps: (1) given a small parameter q and order n , the Mathieu characteristic values a_r and b_r are computed using the Mathematica® functions `MathieuCharacteristicA[m,q]` and `MathieuCharacteristicB[m,q]`; (2) even angular Mathieu functions are computed using the function `MathieuC[ar,q,v]` and odd angular Mathieu functions are computed using `MathieuS[br,q,v]`; (3) the results are normalized according to Stratton, *i.e.*

$$\text{Se}_m(q, v) = \frac{ce_m(q, v)}{ce_m(q, 0)}, \text{So}_m(q, v) = \frac{se_m(q, v)}{se'_m(q, 0)} \quad (3.116)$$

where $se'_m(q, 0)$ is computed using the function `MathieuSPrime[br,q,v]` [148].

Even angular functions $\text{Se}_m(q, v)$ are examined for $m = 15$, $q = 3 + 2j$ in Fig. 3.21a and Fig. 3.21b by

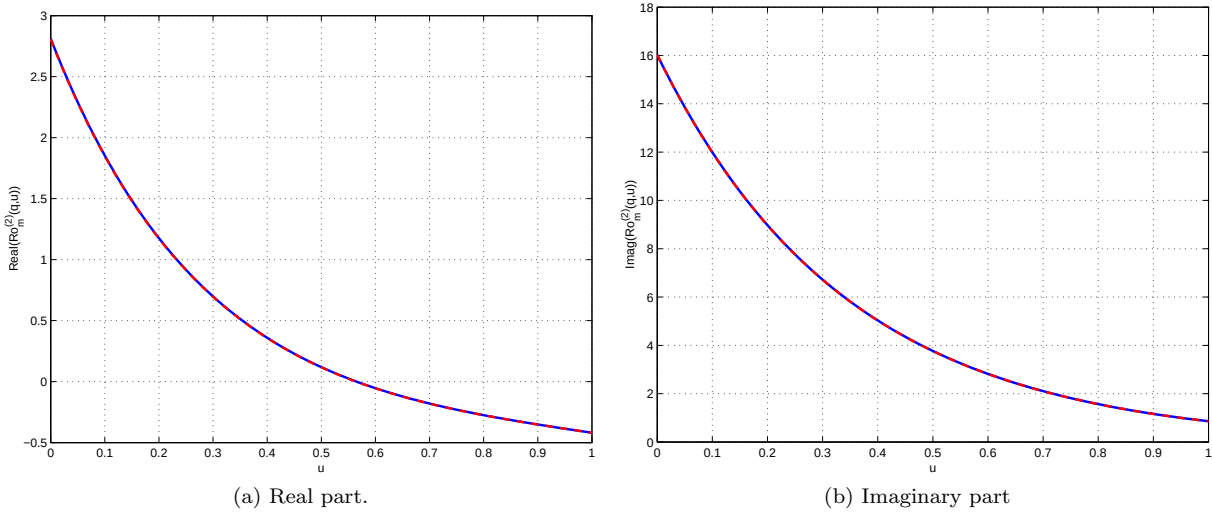


Figure 3.18: Plot of $\text{Ro}_m^{(2)}(q, u)$ for $m = 3$ and $q = 0.2025 + i0.5$. Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

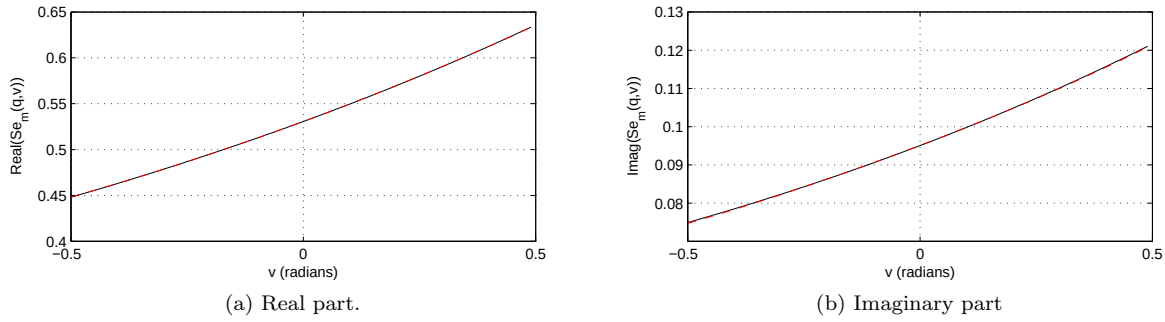


Figure 3.19: Plot of $\text{Se}_m(q, v)$ for $m = 1$, $v = 1$ while the real part of $q = q_r + iq_i$ is varied with $q_i = 0.5$. Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

comparing the real and imaginary parts, respectively.

Odd angular functions $\text{So}_m(q, v)$ are examined for $m = 10$, $q = 4 + 2j$ in Fig. 3.22a and Fig. 3.22b by comparing the real and imaginary parts, respectively.

It is clear from the plots that the Fortran subroutines are very accurate in calculating angular functions for small values of the parameter.

Wronskian check Another way to check the accuracy of the computation of radial functions is to examine their wronskian. This test is based on the wronskian property

$$W_e = \text{Re}_m^{(1)}(q, u)\text{Re}_m^{(2')}(q, u) - \text{Re}_m^{(2)}(q, u)\text{Re}_m^{(1')}(q, u) = 1 \quad (3.117)$$

$$W_o = \text{Ro}_m^{(1)}(q, u)\text{Ro}_m^{(2')}(q, u) - \text{Ro}_m^{(2)}(q, u)\text{Ro}_m^{(1')}(q, u) = 1 \quad (3.118)$$

where the prime symbol refers to the derivative with respect to argument u . Table 3.1 shows the results of this test when $q = -0.9 + j0.9$ and $u = 2$. Table 3.2 shows the results of this test when $q = 2.5 + j1.8$ and $u = 2$.

Table 3.1: Wronskian test for $q = -0.9 + j0.9$ and $u = 2$.

n	real($W_e - 1$)	imag(W_e)	real($W_o - 1$)	imag(W_o)
0	-0.497676E-07	-0.104774E-07		
1	0.582077E-10	0.000000E+00	-0.145519E-10	0.000000E+00
2	-0.203727E-09	0.232831E-09	0.000000E+00	0.000000E+00
3	0.000000E+00	0.000000E+00	-0.291038E-10	0.000000E+00
4	0.756700E-09	-0.631553E-08	0.151049E-07	0.321306E-07
5	0.538421E-09	-0.258660E-08	0.698492E-09	-0.217915E-08
6	-0.625732E-09	0.894943E-09	0.298314E-09	-0.107138E-08
7	-0.483851E-09	0.605269E-09	-0.471118E-09	0.537057E-09
8	0.647105E-09	-0.112777E-09	-0.369027E-09	0.383125E-09
9	0.696215E-09	-0.852083E-10	0.655563E-09	-0.596287E-10
10	-0.106102E-08	-0.669896E-09	0.688974E-09	-0.345644E-10
11	-0.138795E-08	-0.100460E-08	-0.129301E-08	-0.978877E-09
12	0.119574E-08	0.515956E-08	-0.162720E-08	-0.136690E-08
13	0.147028E-08	0.851756E-08	0.125216E-08	0.811601E-08
14	0.224559E-07	-0.356762E-07	0.123325E-08	0.124891E-07
15	0.441156E-07	-0.586881E-07	0.426029E-07	-0.535508E-07
16	-0.530413E-06	0.677142E-07	0.859084E-07	-0.964482E-07
17	-0.919311E-06	0.312372E-06	-0.878474E-06	0.138578E-06
18	0.875018E-05	0.841740E-05	-0.245003E-05	0.366472E-07
19	0.275663E-04	0.256637E-04	0.356364E-04	0.243828E-04
20	-0.324346E-03	-0.298998E-05	0.566832E-04	-0.444547E-04

Table 3.2: Wronskian test for $q = 2.5 + j1.8$ and $u = 2$.

n	real($W_e - 1$)	imag(W_e)	real($W_o - 1$)	imag(W_o)
0	0.153153E-08	0.375110E-09		
1	0.122702E-08	-0.992827E-09	0.483169E-11	0.964064E-10
2	-0.105049E-07	-0.793534E-09	-0.142109E-11	0.583213E-10
3	-0.110762E-07	0.100954E-09	0.852651E-13	0.227374E-12
4	0.833893E-10	-0.295699E-09	-0.238742E-11	-0.454747E-12
5	0.170530E-12	0.248690E-12	0.113687E-12	0.568434E-13
6	-0.476831E-09	-0.367226E-08	0.290066E-08	0.727772E-08
7	-0.598845E-10	-0.171605E-08	0.107889E-09	-0.176777E-08
8	-0.641023E-09	-0.612374E-09	0.253621E-09	-0.876405E-09
9	-0.464766E-09	-0.531671E-09	-0.446349E-09	-0.581082E-09
10	-0.378250E-09	0.373390E-10	-0.281403E-09	-0.498641E-09
11	-0.435564E-09	-0.457980E-10	-0.441648E-09	-0.713598E-10
12	-0.236587E-09	0.336847E-09	-0.453785E-09	-0.181238E-09
13	-0.478929E-09	0.334858E-09	-0.496685E-09	0.315323E-09
14	-0.282224E-10	0.923046E-09	-0.796111E-09	0.188488E-09
15	-0.672772E-09	0.141019E-08	-0.722514E-09	0.138122E-08
16	0.130262E-08	0.345546E-08	-0.202385E-08	0.173100E-08
17	-0.465085E-09	0.750809E-08	-0.679129E-09	0.742430E-08
18	0.160675E-07	0.199750E-07	-0.673906E-08	0.137299E-07
19	0.165958E-07	0.583738E-07	0.157591E-07	0.580741E-07
20	0.226897E-06	0.157335E-06	-0.941438E-08	0.142480E-06

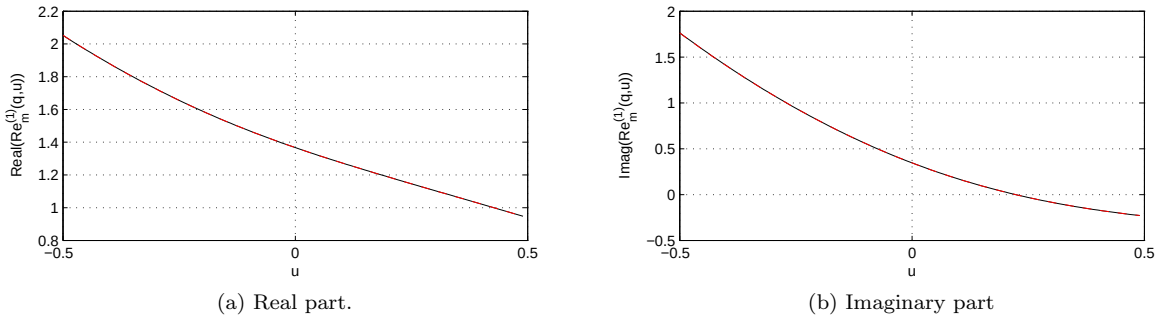


Figure 3.20: Plot of $\text{Re}_m(q, u)$ for $m = 1$, $u = 1$ while the real part of $q = q_r + iq_i$ is varied with $q_i = 0.5$. Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

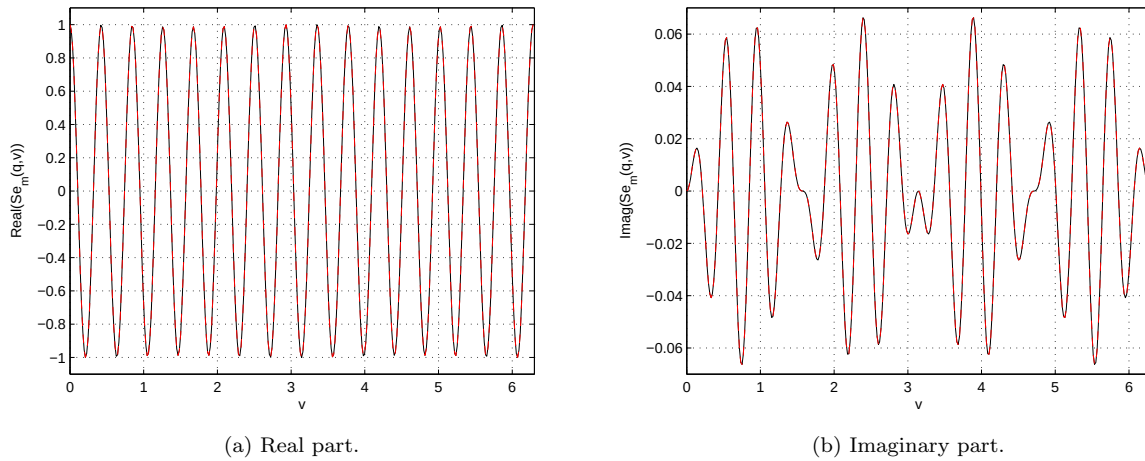


Figure 3.21: Comparisons between Fortran subroutines and Mathematica® in the case of the even angular function $\text{Se}_m(q, v)$ when $m = 15$ and $q = 3 + 2i$. Fortran results are indicated with a solid black line, while Mathematica® results are indicated with a dashed red line. The two curves overlap.

Tables 3.1 and 3.2 are computed using the version of our subroutines that employs the IMSL numerical subroutines to evaluate Bessel functions and obtain the initial characteristic values a_r^0 (b_r^0). Since IMSL numerical subroutines may not be available to everyone, the software associated with this software contains other Fortran libraries to compute the Bessel functions [149] and the eigenvalues. We would like to point out that higher accuracy is achieved when IMSL subroutines are used.

Sample application

Mathieu functions are frequently applied in problems involving elliptical cylinder coordinates [150, 151, 152, 153, 116, 117, 111, 118], [119, 120, 154, 155, 112, 156, 157], [104, 121, 122, 123, 158, 159, 160, 161]. As a sample application of the Mathieu functions, we consider the two-dimensional problem of the computation of the electromagnetic field distribution inside an infinitely long lossy elliptic cylinder made of three layers, shown in Fig. 3.23a, when the structure is illuminated by an incident time-harmonic plane wave. The time factor dependence $e^{i\omega t}$ is assumed and neglected throughout.

Without any loss of generality, we select a coordinate system where the z axis is the axis of the cylinder and the x and y axes correspond to the axes of symmetry of the elliptic cross section. For this geometry, it is possible to express the solution in terms of series expansions of Mathieu functions. Accordingly, we

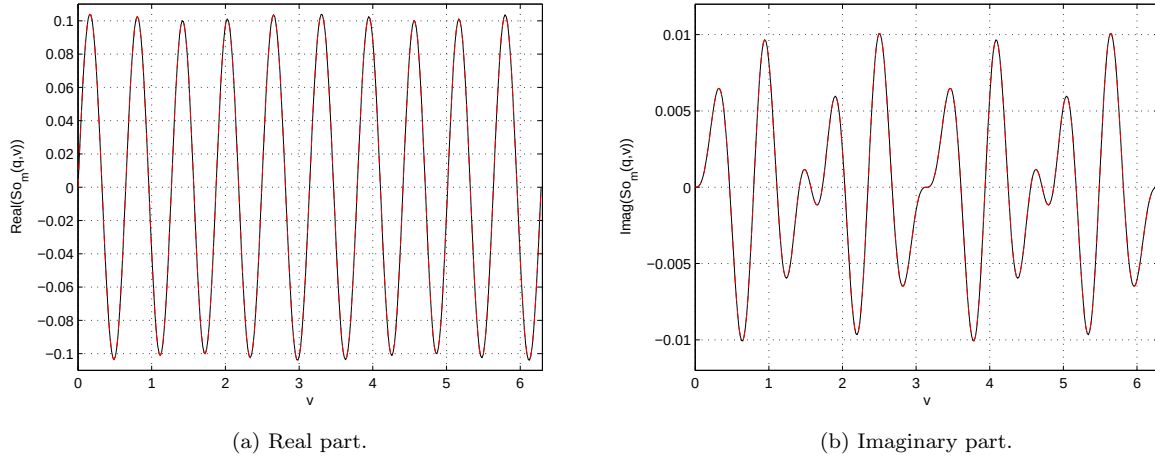


Figure 3.22: Comparisons between Fortran subroutines and Mathematica® in the case of the odd angular function $\text{So}_m(q, v)$ when $m = 10$ and $q = 4 + 2j$. Fortran results are indicated with a solid black line, while Mathematica® results are indicated with a dashed red line. The two curves overlap.

select an elliptic cylinder coordinate system [143, 162], (u, v, z) with focal distance d , having two foci located along the x axis at $\pm d/2$. The elliptic cylinder coordinates (u, v, z) and the Cartesian coordinates (x, y, z) are related by

$$\begin{aligned} x &= \frac{d}{2} \cosh u \cos v \\ y &= \frac{d}{2} \sinh u \sin v \\ z &= z \end{aligned} \tag{3.119}$$

From the previous definition, surfaces having $u = \text{constant}$ represent families of confocal elliptic cylinders, while surfaces having $v = \text{constant}$ represent families of confocal hyperbolic cylinders. There are four regions in Fig. 3.23a: region 1, located inside the surface $u = u_1$; region 2, limited between the surface $u = u_2$ and the surface $u = u_1$; region 3, limited between the surface $u = u_3$ and the surface $u = u_2$; and region 0, which is external to the surface $u = u_3$.

The source of the electromagnetic field is a plane wave, whose direction of propagation lies in the xy plane and makes the angle ϕ_0 with the negative x axis, and polarized with the electric field along the z direction, i.e.

$$E_z^i = E_0 e^{ik_0(x \cos \phi_0 + y \sin \phi_0)}. \tag{3.120}$$

The main steps that lead to the exact electromagnetic solution of this problem are briefly summarized in the following.

We first express the z component of the electric field in each region as a series expansion containing Mathieu functions [143, 125]. In doing so, we assume that each region is filled with a medium that is linear, uniform, and isotropic. The wavevector in region ℓ is k_ℓ . The z component of the electric field in regions 1,

2 and 3 are given by [114]

$$E_{1,z} = \sum_{m=0}^{+\infty} e_{1,m} \text{Re}_m^{(1)}(q_1, u) \text{Se}_m(q_1, v) + \sum_{m=1}^{+\infty} f_{1,m} \text{Ro}_m^{(1)}(q_1, u) \text{So}_m(q_1, v), \quad (3.121)$$

$$E_{2,z} = \sum_{m=0}^{+\infty} \left[e_{2,m} \text{Re}_m^{(1)}(q_2, u) \text{Se}_m(q_2, v) + f_{2,m} \text{Re}_m^{(2)}(q_2, u) \text{Se}_m(q_2, v) \right] \\ + \sum_{m=1}^{+\infty} \left[g_{2,m} \text{Ro}_m^{(1)}(q_2, u) \text{So}_m(q_2, v) + h_{2,m} \text{Ro}_m^{(2)}(q_2, u) \text{So}_m(q_2, v) \right], \quad (3.122)$$

$$E_{3,z} = \sum_{m=0}^{+\infty} \left[e_{3,m} \text{Re}_m^{(1)}(q_3, u) \text{Se}_m(q_3, v) + f_{3,m} \text{Re}_m^{(2)}(q_3, u) \text{Se}_m(q_3, v) \right] \\ + \sum_{m=1}^{+\infty} \left[g_{3,m} \text{Ro}_m^{(1)}(q_3, u) \text{So}_m(q_3, v) + h_{3,m} \text{Ro}_m^{(2)}(q_3, u) \text{So}_m(q_3, v) \right], \quad (3.123)$$

respectively. In region 0, external to the elliptic cylinder, the electric field is given by the sum of the incident field E_z^i (3.120) which can be expanded in terms of Mathieu functions

$$E_z^i = \sqrt{8\pi} \left[\sum_{m=0}^{+\infty} \frac{i^m}{N_m^e} \text{Re}_m^{(1)}(q_0, u) \text{Se}_m(q_0, v) \text{Se}_m(q_0, \phi_0) + \sum_{m=1}^{+\infty} \frac{i^m}{N_m^o} \text{Ro}_m^{(1)}(q_0, u) \text{So}_m(q_0, v) \text{So}_m(q_0, \phi_0) \right], \quad (3.124)$$

and the scattered field

$$E_{0,z} = \sum_{m=0}^{+\infty} e_{0,m} \text{Re}_m^{(4)}(q_0, u) \text{Se}_m(q_0, v) + \sum_{m=1}^{+\infty} f_{0,m} \text{Ro}_m^{(4)}(q_0, u) \text{So}_m(q_0, v), \quad (3.125)$$

where radial Mathieu functions of the fourth kind are used to satisfy the Sommerfeld radiation condition and $q_\ell = (k_\ell d)^2/16$. For a lossy cylinder, its wavenumber k_ℓ is complex, and consequently, q_ℓ is complex too. This motivates the use of the special functions presented in this work to compute the fields distribution. The magnetic field is obtained from the electric field using the relation

$$\mathbf{H}(u, v) = H_u(u, v) \hat{\mathbf{u}} + H_v(u, v) \hat{\mathbf{v}} = \frac{i}{\omega \mu d \sqrt{\cosh u^2 \cos v^2}} \left[\hat{\mathbf{u}} \frac{\partial}{\partial v} - \hat{\mathbf{v}} \frac{\partial}{\partial u} \right] E_z(u, v). \quad (3.126)$$

The expansion coefficients $e_{\ell,m}$, $f_{\ell,m}$, $g_{\ell,m}$, $h_{\ell,m}$ are obtained by imposing the electromagnetic boundary conditions on the continuity of the tangential components of the total electric and magnetic fields across the interfaces between different materials.

This approach is similar to, for example, the one of the multilayer circular cylinder. However, in this case it is not possible to write explicitly in closed form the expansion coefficients of the scattered field in terms of the expansion coefficients of the incident field because the angular Mathieu functions depend on the material properties through the parameter q , which does not happen in the circular cylinder case. It is also worth noting that the orthogonality relations (3.79)-(3.80) are not valid across an interface when the material properties change, however they may be used as done, for example in [150], [151], [153]. Therefore, the point-matching method, which enforces the boundary conditions at predetermined points [65], is used in this work. Let $m_{\max}$ be the maximum order of the expansion coefficients, then $N_t = 6m_{\max} + 3$ is the total number of uniformly distributed points where the boundary conditions are enforced, as shown in Fig. 3.23b.

The application of the boundary conditions at each point leads to a system of $12m_{\max} + 6$ equations that is solved by a matrix inversion.

Numerical results are shown in Fig. 3.24 for the electric field computed with the methods presented in [151] and with the method presented in this work. The geometry is the same in all cases, where the semimajor axes of the three layers of the elliptic cylinder have length $u_1 = 0.10\lambda_0$, $u_2 = 0.13\lambda_0$ and $u_3 = 0.16\lambda_0$ and $d = 0.16\lambda_0$. The complex relative dielectric permittivities $\epsilon_i = \epsilon'_i - i\epsilon''_i$ of the three layers are $\epsilon'_1 = 1.0$,

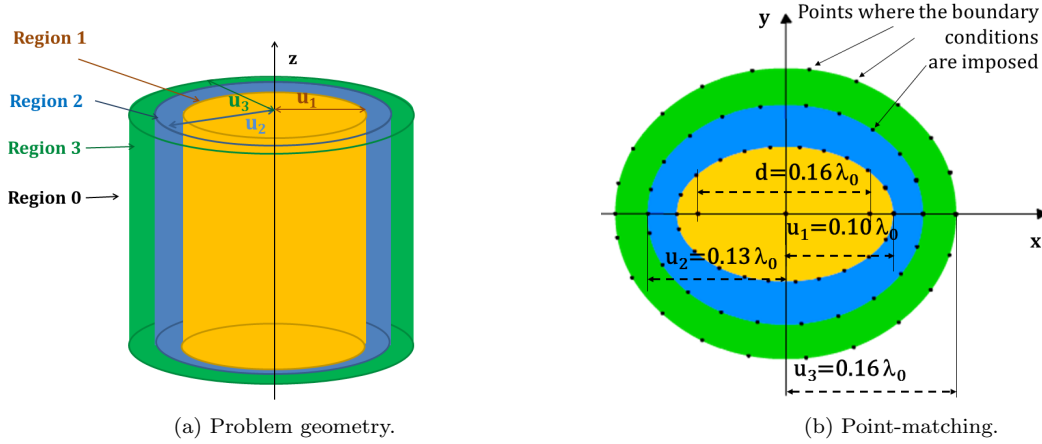


Figure 3.23: Sample problem.

$\epsilon'_2 = 4.0$, $\epsilon'_3 = 2.5$ and $\epsilon''_1 = \epsilon''_2 = \epsilon''_3 = \epsilon''$. The solution has been computed for $\epsilon'' = 0$ (pure dielectric cylinder), $\epsilon'' = 0.06$ (weakly lossy cylinder), $\epsilon'' = 0.6$ (lossy cylinder) and $\epsilon'' = 1.89$ (very lossy cylinder).

All results are computed by considering $m_{\max} = 9$. For all values of ϵ'' there is good agreement between the methods presented in [151] and the method presented here. The differences between the two results may be attributed to how the boundary conditions are applied and to the computation of the Mathieu functions. Specifically, in [151] the boundary conditions are enforced using the Galerkin method, which is different from the point matching method used in this example. In addition, in [151] the Mathieu functions for complex values of the parameter q are computed using a Taylor series expansion.

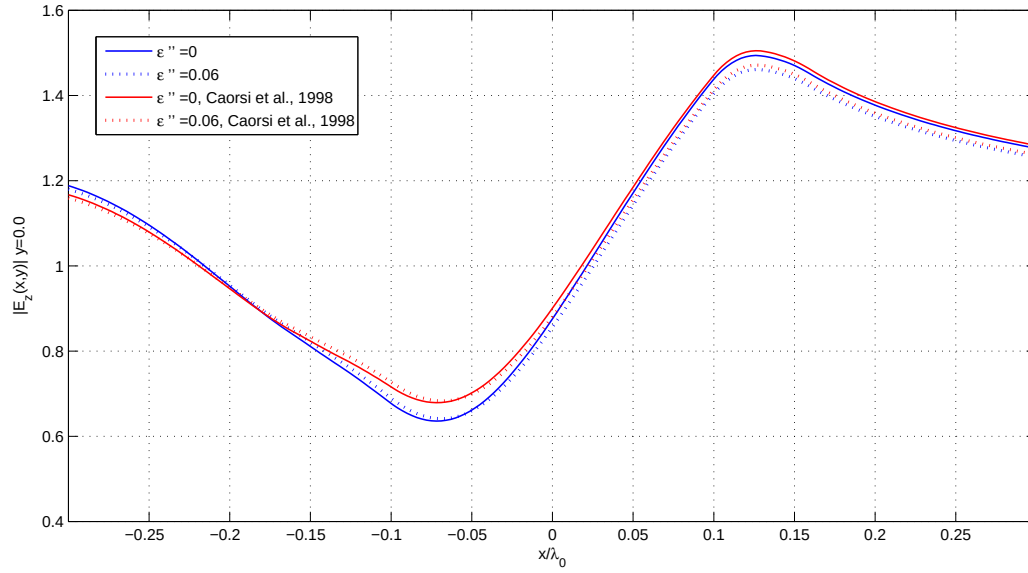
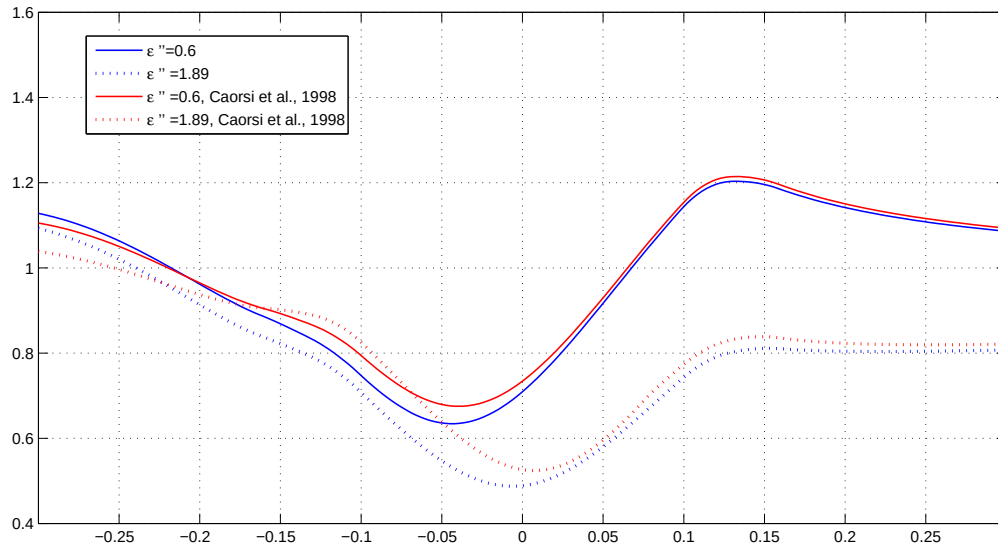

 (a) Results for $\epsilon'' = 0$ and $\epsilon'' = 0.06$.

 (b) Results for $\epsilon'' = 0.6$ and $\epsilon'' = 1.89$.

 Figure 3.24: Magnitudes $|E_z|$ in the four regions of the geometry: comparison between the computation method presented in [151] and the method presented here.

3.4 Symmetry properties of spheroidal functions with respect to their parameter

The validation process may take advantage of several exact solutions of canonical electromagnetic scattering problems that were recently developed, such as, for example, [22, 23, 24, 25, 26, 27, 28]. These solutions involve shapes that are coordinate surfaces in the spheroidal oblate or prolate coordinate system and require the evaluation of angular and radial prolate spheroidal functions or angular and radial oblate spheroidal functions. When these shapes are filled with metamaterials, knowledge of the symmetry properties of the spheroidal functions with respect to their parameter is necessary to perform the analytical derivations and computations.

Introduction

Spheroidal functions are used in problems that involve the prolate or the oblate spheroidal coordinate system. Sample application problems include, for example, the scattering of a charged particle [163], semiconductor nanodevices [164], Schrödinger's equation [165], and various acoustics [166] and electromagnetic problems whose boundaries correspond to coordinate surfaces in the prolate or oblate spheroidal coordinate system [114]. In the the solution of Helmholtz equation in electromagnetics applications, spheroidal functions depend upon a parameter $c = \beta d/2$, where β is the wavenumber and d is the focal distance. In turn, the wavenumber $\beta = \omega\sqrt{\varepsilon\mu}$ depends on the angular frequency ω , the dielectric permittivity ε , and on the magnetic permeability μ . For most materials, $\varepsilon > 0$ and $\mu > 0$ and these are called double positive or DPS. Artificial materials or metamaterials with $\varepsilon < 0$ and $\mu < 0$ are referred to as double negative or DNG and have been theoretically proposed by Veselago [167]. More recently, properties of DNG metamaterials have been investigated by many researchers, e.g. in [168], [169] because they allow applications such as perfect lenses, super-resolution and invisibility.

For a DPS material, $\beta > 0$ and $c > 0$, while for a DNG material $\beta < 0$ and $c < 0$ to satisfy causality [170]. New exact solutions of electromagnetic scattering problems were obtained when the materials involved are isorefractive [171] to each other, *i.e.* in the case of two materials when

$$\varepsilon_1\mu_1 = \varepsilon_2\mu_2 \quad (3.127)$$

A special case of the previous condition occurs when the two materials are anti-isorefractive to each other. In such a case, assuming medium 1 is DPS and medium 2 is DNG one obtains that

$$\beta_1 = \omega\sqrt{\varepsilon_1\mu_1} > 0 \rightarrow c_1 = \frac{d}{2}\beta_1 > 0 \quad (3.128)$$

$$\beta_2 = \omega\sqrt{\varepsilon_2\mu_2} < 0 \rightarrow c_2 = \frac{d}{2}\beta_2 < 0 \quad (3.129)$$

and

$$c = c_1 = -c_2 \quad (3.130)$$

where it is assumed that $c > 0$. As a result, when anti-isorefractive metamaterials are involved [172], [173], [17], it is necessary to know the behavior of spheroidal functions for $\pm c$.

The symmetry properties of the spheroidal functions with respect to the parameter c are not provided anywhere to the best of these authors knowledge, including classical reference such as [174], [126], [175], [115], [145], [176].

This article is structured as it follows. In Section 3.4 spheroidal functions are reviewed and the symmetry properties are introduced; then in Section 3.4 a sample application of these symmetry properties is provided and finally in Section 3.4 a derivation of the symmetry properties is given.

Spheroidal functions

According to the definition of Flammer [115], the prolate spheroidal coordinates are related to the cartesian coordinates by

$$x = \frac{d}{2} \sqrt{\xi^2 - 1} \sqrt{1 - \eta^2} \cos \varphi \quad (3.131)$$

$$y = \frac{d}{2} \sqrt{\xi^2 - 1} \sqrt{1 - \eta^2} \sin \varphi \quad (3.132)$$

$$z = \frac{d}{2} \xi \eta \quad (3.133)$$

where $\xi \geq 1$, $-1 \leq \eta \leq 1$, and $0 \leq \varphi \leq 2\pi$. Surfaces with $\xi = \text{constant}$ are confocal prolate spheroids, surfaces with $\eta = \text{constant}$ are confocal hyperboloids with two sheets, and surfaces with $\varphi = \text{constant}$ are planes originating in the z axis. On the other hand, the oblate spheroidal coordinate system is related to the cartesian coordinate system by

$$x = \frac{d}{2} \sqrt{\xi^2 + 1} \sqrt{1 - \eta^2} \cos \varphi \quad (3.134)$$

$$y = \frac{d}{2} \sqrt{\xi^2 + 1} \sqrt{1 - \eta^2} \sin \varphi \quad (3.135)$$

$$z = \frac{d}{2} \xi \eta \quad (3.136)$$

where $\xi \geq 0$, $-1 \leq \eta \leq 1$. Surfaces with $\xi = \text{constant}$ are confocal oblate spheroids, surfaces with $\eta = \text{constant}$ are confocal hyperboloids with two sheets, and surfaces with $\varphi = \text{constant}$ are planes originating in the z axis.

When the scalar wave equation

$$\nabla^2 \psi + \beta^2 \psi = 0, \quad (3.137)$$

is solved in the prolate spheroidal coordinate system with the method of separation of variables, the solution is written in the form

$$\psi_{mn} = S_{mn}(c, \eta) R_{mn}(c, \xi) \begin{matrix} \cos \\ \sin \end{matrix} m\varphi \quad (3.138)$$

where $S_{mn}(c, \eta)$ are prolate spheroidal angular functions and $R_{mn}(c, \xi)$ are prolate spheroidal radial functions. These functions satisfy

$$\begin{aligned} & \frac{d}{d\eta} \left[(1 - \eta^2) \frac{d}{d\eta} S_{mn}(c, \eta) \right] \\ & + \left[\lambda_{mn} - c^2 \eta^2 - \frac{m^2}{1 - \eta^2} \right] S_{mn}(c, \eta) = 0 \end{aligned} \quad (3.139)$$

$$\begin{aligned} & \frac{d}{d\xi} \left[(\xi^2 - 1) \frac{d}{d\xi} R_{mn}(c, \xi) \right] \\ & - \left[\lambda_{mn} - c^2 \xi^2 + \frac{m^2}{\xi^2 - 1} \right] R_{mn}(c, \xi) = 0 \end{aligned} \quad (3.140)$$

where m and λ_{mn} are separation constants of the original scalar Helmholtz equation (3.137). In addition, λ_{mn} is the eigenvalue of the differential equations (3.139) and (3.140). One should notice that both angular and radial functions satisfy the same type of differential equation.

The solution of the differential equation (3.137) in the oblate coordinate system may be obtained by the following transformation

$$c \rightarrow \mp ic \quad (3.141)$$

$$\xi \rightarrow \pm i\xi \quad (3.142)$$

so that it is sufficient to discuss in detail the prolate case and apply the previous transformation to address the oblate case.

Most physical problems require angular spheroidal functions that are finite for all possible values $-1 \leq \eta \leq 1$, thus limiting solutions of the differential equation (3.139) to the angular functions of the first kind $S_{mn}^{(1)}(c, \eta)$ or simply $S_{mn}(c, \eta)$, which may be written as a series expansion

$$S_{mn}(c, \eta) = \sum_{r=0,1}^{\infty} d_r^{mn}(c) P_{m+r}^m(\eta) \quad (3.143)$$

In the previous expression, $P_{m+r}^m(\eta)$ are associated Legendre functions of the first kind, $d_r^{mn}(c)$ are expansion coefficients. In this expression and later on the prime over the summation symbol means that r must take even values when $m-n$ is even and odd values when $m-n$ is odd, respectively. The normalization coefficient of the prolate angular function is

$$N_{mn}(c) = 2 \sum_{r=0,1}^{\infty} \frac{(r+2m)!(d_r^{mn}(c))^2}{(2r+2m+1)r!} \quad (3.144)$$

In addition to the angular functions, radial spheroidal functions are also needed. These are defined as

$$R_{mn}^{(1)}(c, \xi) = \left(\sum_{r=0,1}^{\infty} d_r^{mn}(c) \frac{(2m+r)!}{r!} \right)^{-1} \left(\frac{\xi^2 - 1}{\xi^2} \right)^{m/2} \times \sum_{r=0,1}^{\infty} i^{r+m-n} d_r^{mn}(c) \frac{(2m+r)!}{r!} j_{m+r}(c\xi) \quad (3.145)$$

$$R_{mn}^{(2)}(c, \xi) = \left(\sum_{r=0,1}^{\infty} d_r^{mn}(c) \frac{(2m+r)!}{r!} \right)^{-1} \left(\frac{\xi^2 - 1}{\xi^2} \right)^{m/2} \times \sum_{r=0,1}^{\infty} i^{r+m-n} d_r^{mn}(c) \frac{(2m+r)!}{r!} y_{m+r}(c\xi) \quad (3.146)$$

$$R_{mn}^{(3)}(c, \xi) = \left(\sum_{r=0,1}^{\infty} d_r^{mn}(c) \frac{(2m+r)!}{r!} \right)^{-1} \left(\frac{\xi^2 - 1}{\xi^2} \right)^{m/2} \times \sum_{r=0,1}^{\infty} i^{r+m-n} d_r^{mn}(c) \frac{(2m+r)!}{r!} h_{m+r}^{(1)}(c\xi) \quad (3.147)$$

$$R_{mn}^{(4)}(c, \xi) = \left(\sum_{r=0,1}^{\infty} d_r^{mn}(c) \frac{(2m+r)!}{r!} \right)^{-1} \left(\frac{\xi^2 - 1}{\xi^2} \right)^{m/2} \times \sum_{r=0,1}^{\infty} i^{r+m-n} d_r^{mn}(c) \frac{(2m+r)!}{r!} h_{m+r}^{(2)}(c\xi) \quad (3.148)$$

where $j_n(z)$ are the spherical Bessel functions

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z) \quad (3.149)$$

$y_n(z)$ are the spherical Neumann functions

$$y_n(z) = \sqrt{\frac{\pi}{2z}} Y_{n+1/2}(z) \quad (3.150)$$

$h_n^{(1)}$ are the spherical Hankel functions of the first kind

$$h_n^{(1)}(z) = \sqrt{\frac{\pi}{2z}} H_{n+1/2}^{(1)}(z) \quad (3.151)$$

Table 3.3: Symmetry properties for prolate spheroidal functions

	Prolate	Oblate
Eigenvalues	$\lambda_{mn}(-c) = \lambda_{mn}(c)$	$\lambda_{mn}(ic) = \lambda_{mn}(-ic)$
Expansion coefficients	$d_r^{mn}(-c) = d_r^{mn}(c)$	$d_r^{mn}(ic) = d_r^{mn}(-ic)$
Angular functions	$S_{mn}(-c) = S_{mn}(c)$	$S_{mn}(ic) = S_{mn}(-ic)$
normalization coefficient	$N_{mn}(-c) = N_{mn}(c)$	$N_{mn}(ic) = N_{mn}(-ic)$
Radial functions	$ \begin{aligned} R_{mn}^{(1)}(-c, \xi) &= (-1)^n R_{mn}^{(1)}(c, \xi) & R_{mn}^{(1)}(ic, \xi) &= (-1)^n R_{mn}^{(1)}(-ic, \xi) \\ R_{mn}^{(2)}(-c, \xi) &= -(-1)^n R_{mn}^{(2)}(c, \xi) & R_{mn}^{(2)}(ic, \xi) &= -(-1)^n R_{mn}^{(2)}(-ic, \xi) \\ R_{mn}^{(3)}(-c, \xi) &= (-1)^n R_{mn}^{(3)}(c, \xi) & R_{mn}^{(3)}(ic, \xi) &= (-1)^n R_{mn}^{(3)}(-ic, \xi) \\ R_{mn}^{(4)}(-c, \xi) &= (-1)^n R_{mn}^{(4)}(c, \xi) & R_{mn}^{(4)}(ic, \xi) &= (-1)^n R_{mn}^{(4)}(-ic, \xi) \end{aligned} $	
normalization coefficient	$\rho_{mn}(-c) = (-1)^m \rho_{mn}(c)$	$\rho_{mn}(ic) = (-1)^m \rho_{mn}(-ic)$

and $h_n^{(2)}$ are the spherical Hankel functions of the second kind

$$h_n^{(2)}(z) = \sqrt{\frac{\pi}{2z}} H_{n+1/2}^{(2)}(z). \quad (3.152)$$

The normalization coefficient for the radial functions is

$$\rho_{mn}(c) = \frac{i^{m-n} c^m}{\sum_{r=0,1}^{\infty} d_r^{mn}(c) \frac{(2m+r)!}{r!}}. \quad (3.153)$$

When c is real, the substitutions $c \rightarrow -ic$ and $\xi \rightarrow i\xi$ cause changes of sign so that the differential equations for the prolate functions are transformed into the differential equations for the oblate functions. Hence, these substitutions do not cause the spheroidal functions to produce any complex number and the arguments $-ic$ and $i\xi$ should be understood as labels to identify the oblate functions in terms of the prolate functions. In the oblate case, the eigenvalues $\lambda_{mn}(-ic)$ are all real because they are the solution of an eigenvalue problem with all real values. Consequently, the expansion coefficients $d_r^{mn}(-ic)$ are all real, the angular functions $S_{mn}(-ic, \eta)$ and the normalization coefficient $N_{mn}(-ic)$ are all real valued.

In the prolate case, the only complex quantity in addition to the radial functions $R_{mn}^{(3)}(c, \xi)$ and $R_{mn}^{(4)}(c, \xi)$ is the normalization coefficient for the radial functions, due to the term $i^{(m-n)}$ appearing in (3.153); in particular,

$$\rho_{mn}^*(c) = (-1)^{m-n} \rho_{mn}(c) \quad (3.154)$$

In the oblate case, the radial functions $R_{mn}^{(3)}(-ic, i\xi)$ and $R_{mn}^{(4)}(-ic, i\xi)$ are complex valued, as well as the radial normalization coefficient $\rho_{mn}(-ic)$, for which

$$\rho_{mn}^*(-ic) = (-1)^n \rho_{mn}(-ic). \quad (3.155)$$

The symmetry properties of spheroidal functions with respect to the parameter c are summarized in Table 3.3 and derived in Section 3.4.

Sample application

As a sample application of these symmetry properties, we provide examples in the context of electromagnetic scattering. Exact analytical solutions of some electromagnetic scattering problems exist in the prolate or

oblate spheroidal coordinate system when the source is a dipole, provided that the dipole is located along the axis of symmetry and axially oriented, as shown in Fig. 3.25 and Fig. 3.26. In the following examples, the dipole is always located at $(\xi_0, \eta = 1)$ and the material is either DPS or DNG depending upon the case considered.

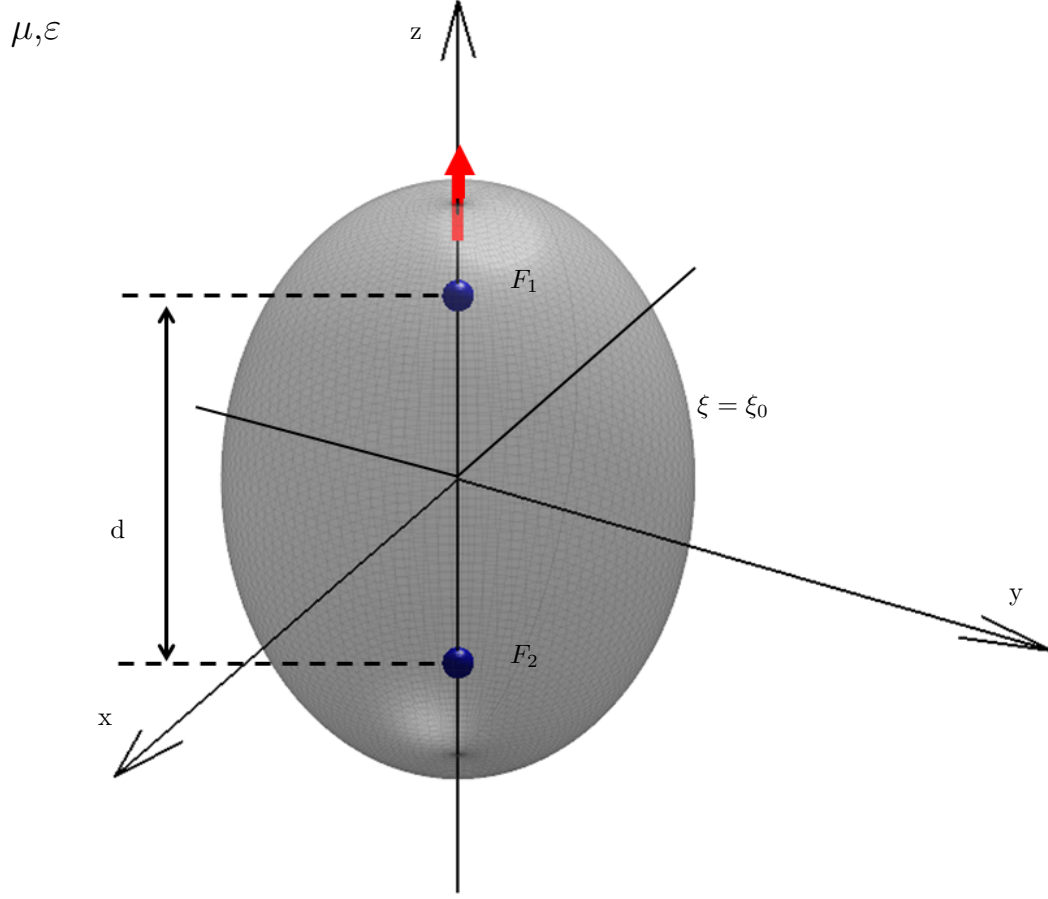


Figure 3.25: A dipole source located in an prolate spheroid coordinate system.

We consider a time-harmonic electromagnetic analysis where all quantities vary in sinusoidal fashion and are expressed in terms of their phasors. There are two conventions used in the literature to express phasors: the $\exp(-i\omega t)$ time convention used in physics and the $\exp(j\omega t)$ used by IEEE, where ω is the angular frequency. Note that the imaginary unit is indicated in physics by $i = \sqrt{-1}$, while $j = \sqrt{-1}$ is used by IEEE. In the following, both time-conventions and notations for the imaginary unit are retained to emphasize the different conventions. As a result of these two conventions, the same quantity $v(x, t)$ is represented in physics by the phasor $V_{\text{physics}}(x)$ so that

$$v(x, t) = \Re \left[V^{(\text{physics})}(x) \exp(-i\omega t) \right] \quad (3.156)$$

while the IEEE representation is

$$v(x, t) = \Re \left[V^{(\text{IEEE})}(x) \exp(j\omega t) \right]. \quad (3.157)$$

Therefore, the phasors in the two conventions are one the complex conjugate of the other, i.e.

$$V^{(\text{physics})} = \left(V^{(\text{IEEE})} \right)^*, \quad (3.158)$$

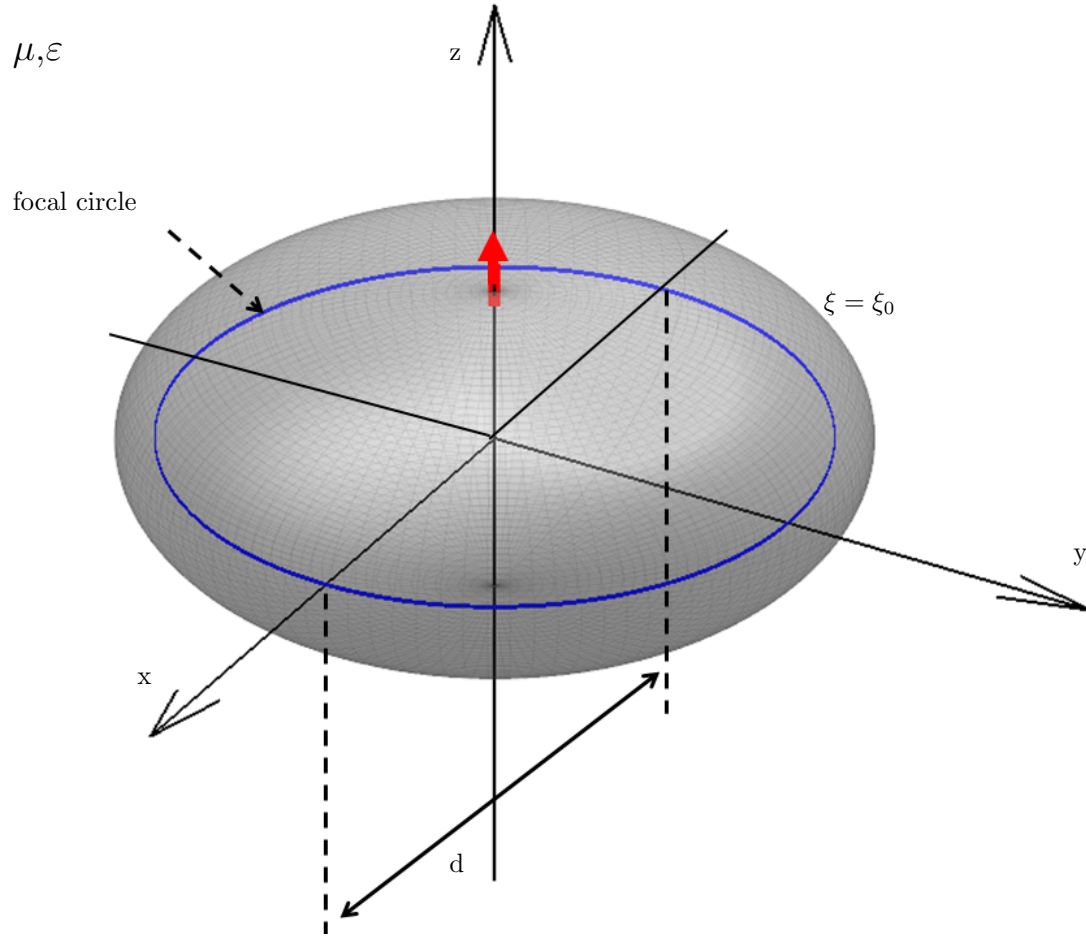


Figure 3.26: A dipole source located in an oblate spheroid coordinate system.

where the symbol $*$ indicates the complex conjugate. While it is easy to recognize that phasor quantities according to one time-convention notation are the complex conjugate of phasor quantities in the other time-convention notation, taking the complex conjugate of quantities involving spheroidal functions requires the knowledge of the symmetry properties discussed in this article, as it shown in the following examples.

Prolate spheroidal coordinates

In the following, all potentials and fields are represented by phasors according to one of the two time-conventions.

Physics time-convention $\exp(-i\omega t)$

A DPS material is characterized by the parameter c , the wavenumber $\beta = 2c/d$ and the electric dipole source has a Hertz vector potential

$$\mathbf{\Pi}^{(\text{physics})} = \hat{\mathbf{z}} \frac{e^{i\beta R}}{\beta R} \quad (3.159)$$

resulting into the magnetic field [114]

$$H_{\varphi}^{(\text{physics})}(\xi, \eta) = \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \times \sum_{n=1}^{\infty} \frac{(-i)^{n-1}}{\rho_{1,n}(c)N_{1,n}(c)} R_{1,n}^{(1)}(c, \xi_{<}) R_{1,n}^{(3)}(c, \xi_{>}) S_{1,n}(c, \eta) \quad (3.160)$$

where $Y = \sqrt{\varepsilon/\mu}$ is the admittance of the material.

A DNG material is characterized by the parameter $-c$, the wavenumber $\beta = -2c/d$ and the electric dipole source has a Hertz vector potential

$$\mathbf{\Pi}^{(\text{physics})} = -\hat{\mathbf{z}} \frac{e^{-i\beta R}}{\beta R} \quad (3.161)$$

resulting into the magnetic field

$$\begin{aligned} H_{\varphi}^{(\text{physics})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \sum_{n=1}^{\infty} \frac{(-i)^{n-1}}{\rho_{1,n}(-c)N_{1,n}(-c)} \times \\ &R_{1,n}^{(1)}(-c, \xi_{<}) R_{1,n}^{(3)}(-c, \xi_{>}) S_{1,n}(-c, \eta) \\ &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \sum_{n=1}^{\infty} \frac{(-i)^{n-1}}{[-\rho_{1,n}(c)] N_{1,n}(c)} \times \\ &\left[-(-1)^{n-1} R_{1,n}^{(1)}(c, \xi_{<}) \right] \left[-(-1)^{n-1} R_{1,n}^{(4)}(c, \xi_{>}) \right] S_{1,n}(c, \eta) \\ &= -\frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \times \\ &\sum_{n=1}^{\infty} \frac{(-i)^{n-1}}{\rho_{1,n}(c)N_{1,n}(c)} R_{1,n}^{(1)}(c, \xi_{<}) R_{1,n}^{(4)}(c, \xi_{>}) S_{1,n}(c, \eta) \end{aligned} \quad (3.162)$$

IEEE time-convention $\exp(j\omega t)$

A DPS material is characterized by the parameter c , the wavenumber $\beta = 2c/d$ and the electric dipole source located has a Hertz vector potential

$$\mathbf{\Pi}^{(\text{IEEE})} = \hat{\mathbf{z}} \frac{e^{-j\beta R}}{\beta R} \quad (3.163)$$

that is the complex conjugate of (3.159), resulting into the magnetic field

$$\begin{aligned} H_{\varphi}^{(\text{IEEE})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \times \\ &\sum_{n=1}^{\infty} \frac{j^{n-1}}{\rho_{1,n}^*(c)N_{1,n}(c)} R_{1,n}^{(1)}(c, \xi_{<}) R_{1,n}^{(4)}(c, \xi_{>}) S_{1,n}(c, \eta) \\ &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \sum_{n=1}^{\infty} \frac{j^{n-1}}{[(-1)^{n-1} \rho_{1,n}(c)] N_{1,n}(c)} \times \\ &R_{1,n}^{(1)}(c, \xi_{<}) R_{1,n}^{(4)}(c, \xi_{>}) S_{1,n}(c, \eta) \\ &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \times \\ &\sum_{n=1}^{\infty} \frac{(-j)^{n-1}}{\rho_{1,n}(c)N_{1,n}(c)} R_{1,n}^{(1)}(c, \xi_{<}) R_{1,n}^{(4)}(c, \xi_{>}) S_{1,n}(c, \eta) \end{aligned} \quad (3.164)$$

that is the complex conjugate of (3.160).

DNG material The material is characterized by the parameter $-c$, the wavenumber $\beta = -2c/d$ and the electric dipole source located at $(\xi_0, \eta = 1)$ has a Hertz vector potential

$$\mathbf{\Pi}^{(\text{IEEE})} = -\hat{\mathbf{z}} \frac{e^{j\beta R}}{\beta R} \quad (3.165)$$

and applying these substitutions into (3.164) yields

$$\begin{aligned} H_{\varphi}^{(\text{IEEE})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \times \\ &\sum_{n=1}^{\infty} \frac{(-j)^{n-1}}{\rho_{1,n}(-c) N_{1,n}(-c)} R_{1,n}^{(1)}(-c, \xi_{<}) R_{1,n}^{(4)}(-c, \xi_{>}) S_{1,n}(-c, \eta) \\ &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \sum_{n=1}^{\infty} \frac{(-j)^{n-1}}{[-\rho_{1,n}(c)] N_{1,n}(c)} \times \\ &\left[-(-1)^{n-1} R_{1,n}^{(1)}(c, \xi_{<}) \right] \left[-(-1)^{n-1} R_{1,n}^{(3)}(c, \xi_{>}) \right] S_{1,n}(c, \eta) \\ &= -\frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \times \\ &\sum_{n=1}^{\infty} \frac{(-j)^{n-1}}{\rho_{1,n}(c) N_{1,n}(c)} R_{1,n}^{(1)}(c, \xi_{<}) R_{1,n}^{(3)}(c, \xi_{>}) S_{1,n}(c, \eta) \end{aligned} \quad (3.166)$$

A second approach is to take the complex conjugate of (3.160), replace $c \Rightarrow -c$ yielding

$$\begin{aligned} H_{\varphi}^{(\text{IEEE})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \times \\ &\sum_{n=1}^{\infty} \frac{j^{n-1}}{\rho_{1,n}^*(-c) N_{1,n}(-c)} R_{1,n}^{(1)}(-c, \xi_{<}) R_{1,n}^{(4)}(-c, \xi_{>}) S_{1,n}(-c, \eta) \end{aligned} \quad (3.167)$$

A third approach is to take the complex conjugate of (3.162) yielding

$$\begin{aligned} H_{\varphi}^{(\text{IEEE})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \times \\ &\sum_{n=1}^{\infty} \frac{j^{n-1}}{\rho_{1,n}^*(-c) N_{1,n}(-c)} R_{1,n}^{(1)}(-c, \xi_{<}) R_{1,n}^{(4)}(-c, \xi_{>}) S_{1,n}(-c, \eta) \end{aligned} \quad (3.168)$$

Oblate spheroidal coordinates

Physics time-convention $\exp(-i\omega t)$

A DPS material is characterized by the parameter c , the wavenumber $\beta = 2c/d$ and the electric dipole source located at $(\xi_0, \eta = 1)$ has a Hertz vector potential

$$\mathbf{\Pi}^{(\text{physics})} = \hat{\mathbf{z}} \frac{e^{i\beta R}}{\beta R} \quad (3.169)$$

resulting into the magnetic field

$$\begin{aligned} H_{\varphi}^{(\text{physics})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-i)^n}{\rho_{1,n}(-ic) N_{1,n}(-ic)} \times \\ &R_{1,n}^{(1)}(-ic, i\xi_{<}) R_{1,n}^{(3)}(-ic, i\xi_{>}) S_{1,n}(-ic, \eta) \end{aligned} \quad (3.170)$$

A DNG material is characterized by the parameter $-c$, the wavenumber $\beta = -2c/d$ and the electric dipole source located at $(\xi_0, \eta = 1)$ has a Hertz vector potential

$$\mathbf{\Pi}^{(\text{physics})} = -\hat{\mathbf{z}} \frac{e^{-i\beta R}}{\beta R} \quad (3.171)$$

resulting into the magnetic field

$$\begin{aligned} H_{\varphi}^{(\text{physics})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \times \\ &\sum_{n=1}^{\infty} \frac{(-i)^n}{\rho_{1,n}(ic)N_{1,n}(ic)} R_{1,n}^{(1)}(ic, i\xi_{<}) R_{1,n}^{(3)}(ic, i\xi_{>}) S_{1,n}(ic, \eta) \\ &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-i)^n}{[-\rho_{1,n}(-ic)] N_{1,n}(-ic)} \times \\ &\left[-(-1)^{n-1} R_{1,n}^{(1)}(-ic, i\xi_{<}) \right] \left[-(-1)^{n-1} R_{1,n}^{(4)}(-ic, i\xi_{>}) \right] \times \\ &S_{1,n}(-ic, \eta) \\ &= -\frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-i)^n}{\rho_{1,n}(-ic)N_{1,n}(-ic)} \times \\ &R_{1,n}^{(1)}(-ic, i\xi_{<}) R_{1,n}^{(4)}(-ic, i\xi_{>}) S_{1,n}(-ic, \eta) \end{aligned} \quad (3.172)$$

IEEE time-convention $\exp(j\omega t)$

A DPS material is characterized by the parameter c , the wavenumber $\beta = 2c/d$ and the electric dipole source located at $(\xi_0, \eta = 1)$ has a Hertz vector potential

$$\mathbf{\Pi}^{(\text{IEEE})} = \hat{\mathbf{z}} \frac{e^{-j\beta R}}{\beta R} \quad (3.173)$$

that is the complex conjugate of (3.169), resulting into the magnetic field

$$\begin{aligned} H_{\varphi}^{(\text{IEEE})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{j^n}{\rho_{1,n}^*(-jc)N_{1,n}(-jc)} \times \\ &R_{1,n}^{(1)}(-jc, j\xi_{<}) R_{1,n}^{(4)}(-jc, j\xi_{>}) S_{1,n}(-jc, \eta) \\ &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{j^n}{[(-1)^n \rho_{1,n}(-jc)] N_{1,n}(-jc)} \times \\ &R_{1,n}^{(1)}(-jc, j\xi_{<}) R_{1,n}^{(4)}(-jc, j\xi_{>}) S_{1,n}(-jc, \eta) \\ &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-j)^n}{\rho_{1,n}(-jc)N_{1,n}(-jc)} \times \\ &R_{1,n}^{(1)}(-jc, j\xi_{<}) R_{1,n}^{(4)}(-jc, j\xi_{>}) S_{1,n}(-jc, \eta) \end{aligned} \quad (3.174)$$

that is the complex conjugate of (3.170).

A DNG material is characterized by the parameter $-c$, the wavenumber $\beta = -2c/d$ and the electric dipole source located at $(\xi_0, \eta = 1)$ has a Hertz vector potential

$$\mathbf{\Pi}^{(\text{IEEE})} = -\hat{\mathbf{z}} \frac{e^{j\beta R}}{\beta R} \quad (3.175)$$

and applying these substitutions into (3.174) yields

$$\begin{aligned}
H_{\varphi}^{(\text{IEEE})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \times \\
&\sum_{n=1}^{\infty} \frac{(-j)^n}{\rho_{1,n}(jc)N_{1,n}(jc)} R_{1,n}^{(1)}(jc, j\xi_{<}) R_{1,n}^{(4)}(jc, j\xi_{>}) S_{1,n}(jc, \eta) \\
&= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-j)^n}{[-\rho_{1,n}(-jc)] N_{1,n}(-jc)} \times \\
&\left[-(-1)^{n-1} R_{1,n}^{(1)}(-jc, j\xi_{<}) \right] \left[-(-1)^{n-1} R_{1,n}^{(3)}(-jc, j\xi_{>}) \right] \times \\
&S_{1,n}(-jc, \eta) \\
&= -\frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-j)^n}{\rho_{1,n}(-jc)N_{1,n}(-jc)} \times \\
&R_{1,n}^{(1)}(-jc, j\xi_{<}) R_{1,n}^{(3)}(-jc, j\xi_{>}) S_{1,n}(-jc, \eta)
\end{aligned} \tag{3.176}$$

A second approach is to start from (3.170), take its complex conjugate and replace $c \Rightarrow -c$ yielding

$$\begin{aligned}
H_{\varphi}^{(\text{IEEE})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{j^n}{\rho_{1,n}^*(jc)N_{1,n}(jc)} \times \\
&R_{1,n}^{(1)}(jc, j\xi_{<}) R_{1,n}^{(4)}(jc, j\xi_{>}) S_{1,n}(jc, \eta)
\end{aligned} \tag{3.177}$$

A third approach is to start from (3.172) and take its complex conjugate, yielding

$$\begin{aligned}
H_{\varphi}^{(\text{IEEE})}(\xi, \eta) &= \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{j^n}{\rho_{1,n}^*(jc)N_{1,n}(jc)} \times \\
&R_{1,n}^{(1)}(jc, j\xi_{<}) R_{1,n}^{(4)}(jc, j\xi_{>}) S_{1,n}(jc, \eta)
\end{aligned} \tag{3.178}$$

Derivations

The symmetry properties summarized in Table 3.3 are justified in the following.

Prolate functions

The symmetry properties of the prolate spheroidal functions require a careful review of the process that leads to their computation, since closed form expressions are not available. The starting point of this review are the differential equations (3.139), (3.140) that are of the same type and are satisfied by the angular and radial functions, respectively. The computation of the solution of the prolate spheroidal equation starts from the determination of its eigenvalue λ_{mn} . In particular, it must be noticed that the differential equation depends on the parameter c only through c^2 so that the eigenvalue λ_{mn} must satisfy

$$\lambda_{mn} = \lambda_{mn}(c^2) \iff \lambda_{mn}(-c) = \lambda_{mn}(c). \tag{3.179}$$

Next the expansion coefficients $d_r^{mn}(c)$ of the series that define the prolate functions are evaluated. The angular functions are defined by the series expansion given in eq. (3.143) and when this series expansion is

substituted into the differential equation (3.139), one obtains the recursion formula

$$\begin{aligned} & \frac{(2m+r+2)(2m+r+1)c^2}{(2m+2r+3)(2m+2r+5)} d_{r+2}^{mn}(c) + \\ & \left[(m+r)(m+r+1) - \lambda_{mn}(c) + \right. \\ & \left. \frac{2(m+r)(m+r+1) - 2m^2 - 1}{(2m+2r-1)(2m+2r+3)} c^2 \right] d_r^{mn}(c) + \\ & \frac{r(r-1)c^2}{(2m+2r-3)(2m+2r-1)} d_{r-2}^{mn}(c) = 0 \end{aligned} \quad (3.180)$$

that allows only to compute the ratios $d_r^{mn}(c)/d_{r-2}^{mn}(c)$. Unique values for the expansion coefficients are obtained by imposing a normalization condition. Using Flammer's convention

$$\begin{aligned} S_{mn}(c, 0) &= P_n^m(0) = \\ & \frac{(-1)^{\frac{n-m}{2}} (n+m)!}{2^n \left(\frac{n-m}{2}\right)! \left(\frac{n+m}{2}\right)!}, \quad (n-m) \text{ even}, \\ S'_{mn}(c, 0) &= P_n^{m'}(0) = \\ & \frac{(-1)^{\frac{n-m-1}{2}} (n+m+1)!}{2^n \left(\frac{n-m-1}{2}\right)! \left(\frac{n+m+1}{2}\right)!}, \quad (n-m) \text{ odd}. \end{aligned} \quad (3.181)$$

Since the previous normalization condition does not depend on c and the recurrence relation (3.180) depends on c directly through c^2 and indirectly through λ_{mn} , which has even symmetry, one concludes that

$$d_r^{mn}(-c) = d_r^{mn}(c). \quad (3.182)$$

Next, the evaluation of the series (3.143) yields

$$S_{mn}(-c, \eta) = S_{mn}(c, \eta) \quad (3.183)$$

because of the symmetry of the expansion coefficients (3.182). The normalization coefficient of the angular functions (3.144) satisfies

$$N_{mn}(-c) = N_{mn}(c). \quad (3.184)$$

given the symmetry of the angular functions (3.182).

The radial prolate spheroidal functions are defined using the same expansion coefficients as the angular prolate spheroidal functions. The radial functions of the first kind are defined according to (3.145) and in order to investigate their behavior with respect to changes of sign in c , we observe that the answer depends on the behavior of the spherical Bessel function $j_{m+r}(c\xi)$. Using the property

$$j_n(-z) = e^{jn\pi} j_n(z) \quad (3.185)$$

we have

$$\begin{aligned} R_{mn}^{(1)}(-c, \xi) &= \left(\sum_{r=0,1}^{\infty} d_r^{mn}(c) \frac{(2m+r)!}{r!} \right)^{-1} \left(\frac{\xi^2 - 1}{\xi^2} \right)^{m/2} \times \\ & \sum_{r=0,1}^{\infty} i^{r+m-n} d_r^{mn}(c) \frac{(2m+r)!}{r!} j_{m+r}(-c\xi) \end{aligned} \quad (3.186)$$

resulting into the following cases. When $(n - m)$ is even,

$$\begin{aligned}
R_{mn}^{(1)}(-c, \xi) &= \frac{\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}}{\sum_{\ell=0}^{\infty} d_{2\ell}^{mn}(c) \frac{(2m+2\ell)!}{(2\ell)!}} \times \\
&\sum_{\ell=0}^{\infty} i^{2\ell+m-n} d_{2\ell}^{mn}(c) \frac{(2m+2\ell)!}{(2\ell)!} (-1)^{m+2\ell} j_{m+2\ell}(c\xi) \\
&= \begin{cases} R_{mn}^{(1)}(c, \xi) & m \text{ even} \Rightarrow n \text{ even} \\ -R_{mn}^{(1)}(c, \xi) & m \text{ odd} \Rightarrow n \text{ odd} \end{cases} \quad (3.187)
\end{aligned}$$

and when $(n - m)$ is odd,

$$\begin{aligned}
R_{mn}^{(1)}(-c, \xi) &= \frac{\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}}{\sum_{\ell=0}^{\infty} d_{2\ell+1}^{mn}(c) \frac{(2m+2\ell+1)!}{(2\ell+1)!}} \sum_{\ell=0}^{\infty} i^{2\ell+m-n+1} \times \\
&d_{2\ell+1}^{mn}(c) \frac{(2m+2\ell+1)!}{(2\ell+1)!} (-1)^{m+2\ell+1} j_{m+2\ell+1}(c\xi) \\
&= \begin{cases} -R_{mn}^{(1)}(c, \xi) & m \text{ even} \Rightarrow n \text{ odd} \\ R_{mn}^{(1)}(c, \xi) & m \text{ odd} \Rightarrow n \text{ even} \end{cases} \quad (3.188)
\end{aligned}$$

Hence, by combining the results of (3.187) and (3.188) we obtain

$$R_{mn}^{(1)}(-c, \xi) = (-1)^n R_m^{(1)}(c, \xi) \quad (3.189)$$

For the radial functions of the second kind, we replace the spherical Bessel functions of the first kind $j_{m+r}(c\xi)$ with the spherical Bessel functions of the second kind $y_{m+r}(c\xi)$ in (3.145). Then using the property

$$y_n(-z) = (-1)^{n+1} y_n(z) \quad (3.190)$$

we have

$$\begin{aligned}
R_{mn}^{(2)}(-c, \xi) &= \left(\sum_{r=0,1}^{\infty} d_r^{mn}(c) \frac{(2m+r)!}{r!} \right)^{-1} \left(\frac{\xi^2-1}{\xi^2} \right)^{m/2} \times \\
&\sum_{r=0,1}^{\infty} i^{r+m-n} d_r^{mn}(c) \frac{(2m+r)!}{r!} y_{m+r}(-c\xi) \quad (3.191)
\end{aligned}$$

resulting into the following cases. When $(n - m)$ is even,

$$\begin{aligned}
R_{mn}^{(2)}(-c, \xi) &= \frac{\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}}{\sum_{\ell=0}^{\infty} d_{2\ell}^{mn}(c) \frac{(2m+2\ell)!}{(2\ell)!}} \times \\
&\sum_{\ell=0}^{\infty} i^{2\ell+m-n} d_{2\ell}^{mn}(c) \frac{(2m+2\ell)!}{(2\ell)!} (-1)^{m+2\ell+1} y_{m+2\ell}(c\xi) \\
&= \begin{cases} -R_{mn}^{(2)}(c, \xi) & m \text{ even} \Rightarrow n \text{ even} \\ R_{mn}^{(2)}(c, \xi) & m \text{ odd} \Rightarrow n \text{ odd} \end{cases} \quad (3.192)
\end{aligned}$$

and when $(n - m)$ is odd,

$$\begin{aligned}
 R_{mn}^{(2)}(-c, \xi) &= \frac{\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}}{\sum_{\ell=0}^{\infty} d_{2\ell+1}^{mn}(c) \frac{(2m+2\ell+1)!}{(2\ell+1)!}} \sum_{\ell=0}^{\infty} i^{2\ell+m-n+1} \times \\
 & d_{2\ell+1}^{mn}(c) \frac{(2m+2\ell+1)!}{(2\ell+1)!} (-1)^{m+2\ell+2} y_{m+2\ell+1}(c\xi) \\
 &= \begin{cases} R_{mn}^{(1)}(c, \xi) & m \text{ even} \Rightarrow n \text{ odd} \\ -R_{mn}^{(1)}(c, \xi) & m \text{ odd} \Rightarrow n \text{ even} \end{cases}
 \end{aligned} \tag{3.193}$$

Hence, by combining (3.192) and (3.193) we obtain

$$R_{mn}^{(2)}(-c, \xi) = -(-1)^n R_m^{(2)}(c, \xi) \tag{3.194}$$

For the radial functions of the third and fourth kind, we replace the spherical Bessel functions of the first kind $j_{m+r}(c\xi)$ with the spherical Hankel functions of the first and second kind $h_{m+r}^{(1)}(c\xi)$ and $h_{m+r}^{(2)}(c\xi)$, respectively, in (3.145). Then recalling that

$$R_{mn}^{(3)}(c, \xi) = R_{mn}^{(1)}(c, \xi) + iR_{mn}^{(2)}(c, \xi) \tag{3.195}$$

$$R_{mn}^{(4)}(c, \xi) = R_{mn}^{(1)}(c, \xi) - iR_{mn}^{(2)}(c, \xi) \tag{3.196}$$

and using the previous results (3.189), (3.194), we obtain

$$\begin{aligned}
 R_{mn}^{(3)}(-c, \xi) &= (-1)^n R_{mn}^{(1)}(c, \xi) - i(-1)^n R_{mn}^{(2)}(c, \xi) = \\
 & (-1)^n R_{mn}^{(4)}(c, \xi)
 \end{aligned} \tag{3.197}$$

$$\begin{aligned}
 R_{mn}^{(4)}(-c, \xi) &= (-1)^n R_{mn}^{(1)}(c, \xi) + i(-1)^n R_{mn}^{(2)}(c, \xi) = \\
 & (-1)^n R_{mn}^{(3)}(c, \xi)
 \end{aligned} \tag{3.198}$$

Finally, the normalization coefficient of the radial functions (3.153) behaves according to

$$\rho_{mn}(-c) = (-1)^m \rho_{mn}(c) \tag{3.199}$$

Oblate functions

The derivations carried out for the prolate functions are valid for the oblate functions provided that the substitutions given in equations (3.141) and (3.142) are made.

Chapter 4

Task 3: Physic-based inversion schemes

Alternative approaches to the classical Truncated Singular Value Decomposition (TSVD) method to invert the linear matrix equation that links the (measured) scattered field to the unknown contrast function have been investigated due to two main factors. One factor is obtaining sharper images because the transition between the background material and the object is too smooth with TSVD, which is not physical. The other factor is to introduce physical bounds to enforce that the returned values of the contrast function correspond to what is possible to find in nature. In fact, TSVD does not allow to enforce such condition. In particular, we focused on iterative inversion methods such as the conjugate gradient method (CG) and the Algebraic Reconstruction Technique which are faster to invert ill-conditioned matrix and simple to manipulate rational physical constraints. The comparison of TSVD, CG and ART has been discussed in [32, 33, 34]. TSVD, CG and ART provide equivalent results in general except that CG and ART do not require large memory usage and execution time. Furthermore, ART has resulted high resolution when physical constraints are introduced. In addition, dielectric and metallic targets can be distinguished by value of contrast function [4, 35]. Additional details are provided in this chapter.

4.1 Inversion algorithms

If the matrix $\mathbf{L}$ were not ill-conditioned, the inversion could be performed using a pseudo-inverse matrix. For matrices with more columns than rows this is normally computed as [177]

$$\mathbf{L}^\dagger = \mathbf{L}^H (\mathbf{L}\mathbf{L}^H)^{-1}, \quad (4.1)$$

where the notation ^{H} indicates the Hermitian transpose. Unfortunately the inversion performed in this way is poor, because of ill-conditioning. Since the condition number can be written as

$$\kappa(\mathbf{L}) = \|\mathbf{L}^\dagger\| \cdot \|\mathbf{L}\|, \quad (4.2)$$

a high condition number means that the pseudo-inverse is really just a numerically bad approximation of a matrix that should verify the relationship $\mathbf{L}^\dagger \cdot \mathbf{L} = \mathbf{I}$, $\mathbf{I}$ being the identity matrix. The result is that small noise in the data lead to extremely large errors in the reconstruction of $\mathbf{v}$ if (4.1) is used.

The solution to the ill-conditioning problem is to use regularization techniques. These methods attempt to mitigate the effects of small singular values by attenuating them or completely eliminating them from the computation of the pseudo-inverse.

One of the most popular methods is called the Truncated Singular Value Decomposition (TSVD) [178], which is closely related to other methods such as the Tikhonov regularization [179]. These and other methods based on similar principles are called *direct* because as a result of the computation they return directly the final solution. This is opposed to the approach taken by *iterative* methods, which start from a rough approximation of the solution and then refine it in an iterative matter [177].

4.1.1 Truncated Singular Value Decomposition

TSVD works as follows. First, the Singular Value Decomposition is performed, which decomposes the matrix $\mathbf{L}$ as

$$\mathbf{L} = \mathbf{U}\mathbf{S}\mathbf{V}^H. \quad (4.3)$$

The diagonal matrix $\mathbf{S}$ contains the singular values of the matrix $\mathbf{L}$ usually sorted in descending order of magnitude. The pseudo-inverse can be easily written in terms of its singular value decomposition terms as

$$\mathbf{L}^\dagger = \mathbf{V}\mathbf{S}^{-1}\mathbf{U}^H. \quad (4.4)$$

Since $\mathbf{S}$ is diagonal, its inverse is obtained simply by replacing every nonzero diagonal entry by its reciprocal and transposing the resulting matrix.

Since the small singular values are the ones that create reconstruction problems, they are simply cut off, and so are the corresponding generalized eigenvectors contained in the matrices $\mathbf{U}$ and $\mathbf{V}$. The regularized truncated pseudo-inverse, therefore, is found as

$$\tilde{\mathbf{L}}^\dagger = \mathbf{V}_k \mathbf{S}_k^{-1} \mathbf{U}_k^H, \quad (4.5)$$

where k indicates the number of singular values that have been retained. The dimension of the final matrix is unchanged, because the truncation of $\mathbf{V}$ is done only on its columns, and the truncation of $\mathbf{U}^H$ is done only on its rows. However, the small singular values responsible for the ill-conditioning are not part of the regularized truncated pseudo-inverse. The final solution is found by a simple matrix multiplication

$$\mathbf{v} = \tilde{\mathbf{L}}^\dagger \cdot \mathbf{E}^s. \quad (4.6)$$

The TSVD is an extremely popular method to obtain regularized solutions of ill conditioned problems. Another solution, possibly even more popular, is the Tikhonov regularization, which is similar both in terms of results and theoretical approach. Instead of simply cutting the small singular values, the Tikhonov regularization reduces their effect by scaling them with an appropriate regularizing parameter λ .

Both methods have drawbacks. TSVD is based on a very computationally heavy operation, the singular value decomposition. In addition, it is not clear how to choose the regularization parameter k . One way is simply to manually guess *a posteriori*, judging subjectively the quality of the reconstructed image; a more rigorous method is to analyze the L-curve [180, 181], which is described briefly in the following.

When finding a regularized solution of an under-determined, ill-conditioned problem, it is possible to plot two quantities as function of the regularization parameter. The first is the norm of the solution itself, $\|\mathbf{v}\|_2$, the other is the norm of the residual $r = \|\mathbf{E}^s - \mathbf{L} \cdot \mathbf{v}\|$. Plotting the first versus the second, a curve that looks like a capital L is often obtained (see Fig. 4.1).

Each point on the L-curve corresponds to a different solution. Oftentimes, the best solution to the problem is the one corresponding to the k (or λ for Tikhonov regularization) at the knee of the curve. For example, in Fig. 4.1 a $\lambda = 0.01$ will return a good solution.

This choice is somewhat subjective, because after all it relies on the quality of the image as it is perceived by the final user; for some problems it might not be the best. In other cases, the knee might not be easy to identify, or the method itself might fail in the case of solutions which are dominated by the first few SVD components [181]. In these cases the choice of k is not simple.

In addition, the solutions returned by any regularized method are often blurry and poorly resolved. This is because the regularization has the property of obtaining a faster convergence for low frequency components¹ [182]. The regularization acts as a low-pass filter. This behavior is often desirable, because it reduces high frequency noise, but it also limits the performance of the inversion, which is unable to return images with sharp transitions and which tends to overestimate the effect of small, localized artifacts. The L-curve can be seen as a graph which separates blurry images from noisy images: the solution often lies in the middle but it is just the choice between the lesser of two evils.

The drawbacks displayed by regularized direct methods motivated research into alternative approaches. In particular, iterative methods have been investigated and are described in Chapter 4.2.

¹Frequency here is intended as spatial frequency, in the context of the spatial Fourier transform of the solution image.

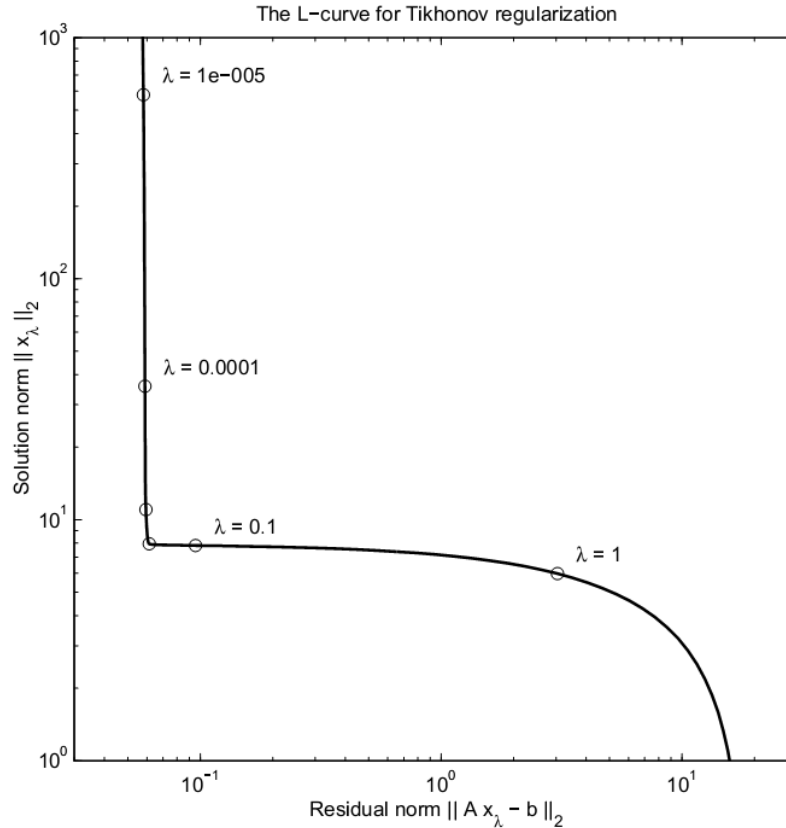


Figure 4.1: Example of an L-curve for an ill-conditioned problem. The points on the curve identified with different values of λ correspond to different values of the regularization parameter in a Tikhonov regularization. A similar behavior is obtained for TSVD for different values of k . Image obtained from [181].

4.2 Iterative inversion algorithms

This chapter is dedicated to describing two main inversion algorithms developed to invert RF Tomography data.

Section 4.1 introduced some direct methods used to solve under-determined, ill-conditioned, dense and large linear systems of equations. Those methods are all direct methods, i.e. they attempt to find a solution to the problem $\mathbf{E}^s = \mathbf{L} \cdot \mathbf{v}$ in one step, by using some regularization procedure. In particular, the most common method used in RF Tomography is the Truncated Singular Values Decomposition (TSVD) [183, 184, 185].

In this chapter an imaging scenario is chosen, which highlights both strengths and weaknesses of TSVD. Then, the new developed methods that allow for the introduction of physical constraints are presented and discussed.

4.2.1 Reference scenario

In the reference scenario a simple target is imaged. It consists of a short section of a PVC pipe: its outer diameter is 4.8 cm, its height is 5 cm, its thickness is 4 mm; the dielectric constant of PVC is estimated to be $\epsilon_r = 2.5$ with negligible conductivity ($\tan(\delta) = 0.004$ at 3 GHz). The object is illuminated from 11 positions along a circumference of radius 38.4 cm, uniformly spaced between 0° and 280° . The electric field is sampled at 20 locations along a circumference of radius 15.4 cm, uniformly spaced between 0° and 360° . A 3D model of the target, along with the location of the antennas, is shown in Fig. 4.2. All images are obtained considering a square domain of investigation of side equal to 20 cm, centered around the center of rotation of the antennas, and divided into 75 pixels per side. Each pixel is therefore 2.67 mm (or approximately $\lambda/37$).

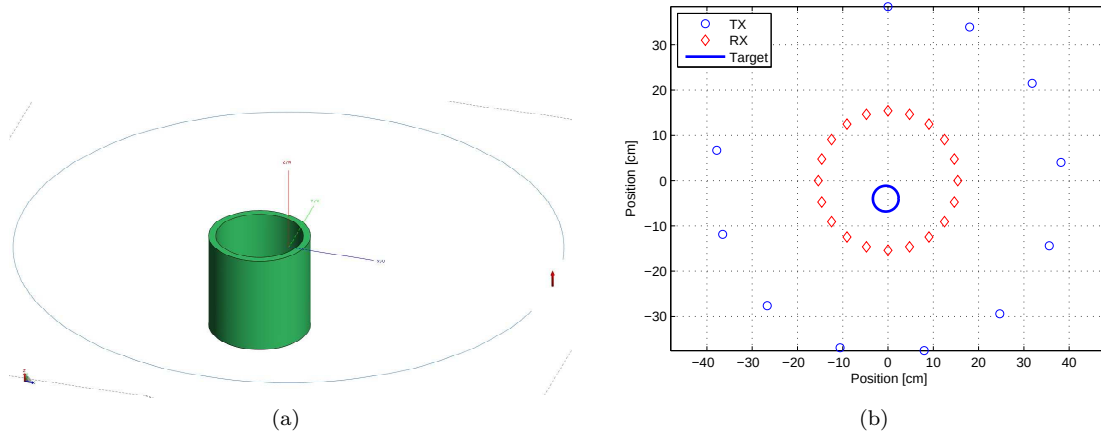


Figure 4.2: Figure (a) shows a 3D model of the PVC pipe used in the measurement. The red arrow in the right part of the image represents one of the positions of the receiving antenna. Figure (b) shows all the positions used in the measurement.

This leads to a problem with $75 \times 75 = 5625$ unknowns which must be reconstructed from 220 measurements. The conditioning number $\kappa(\mathbf{L})$ turns out to be approximately equal to 1.3×10^5 .

4.2.2 Example of TSVD reconstruction

Reconstruction using TSVD is satisfactory, and is shown in Fig. 4.3. The object is reconstructed in the correct location and its shape is preserved, although the thickness of the pipe is exaggerated. The image is affected by noise and presents a few artifacts, which are however weak when compared to the main target.

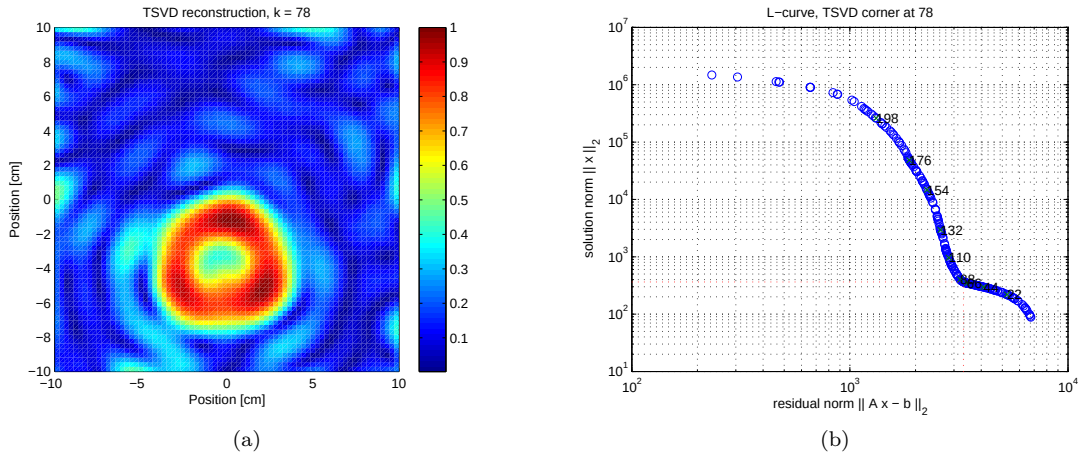


Figure 4.3: Figure (a) shows the reconstruction obtained with TSVD in terms of magnitude of the complex dielectric permittivity contrast function, $|\mathbf{v}|$. The image is obtained by retaining $k = 78$ out of 220 singular values: the number has been automatically determined by identifying the corner of the L-curve (b).

The imaging result depends greatly on the number of singular values that are retained in the inversion, as shown in Fig. 4.4. In addition, the TSVD reconstruction provides a solution that does not make physical sense (see Fig. 4.5): both real and the imaginary parts of $\mathbf{v}$ are negative, while it is clear that the real part (dielectric permittivity) must be positive.

On a recent desktop computer (quad-core CPU operating at 3.7 GHz, 8 GB of memory) the time required to generate each image is approximately 4 seconds. The result is not bad, because the code was

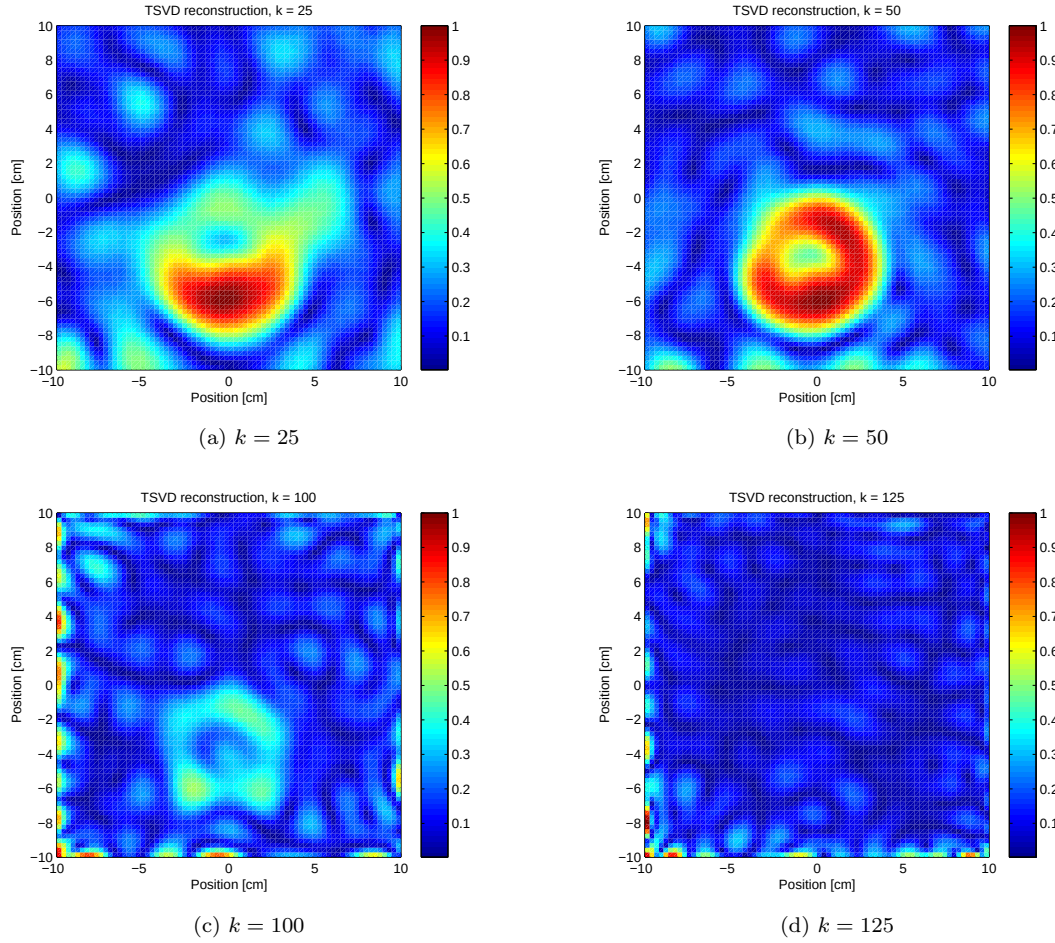


Figure 4.4: Reconstruction quality as function of the number of singular values retained. The solution rapidly goes from over smoothed (a), to meaningful (b), to useless (c-d)

not pre-compiled or optimized. However, on a less recent laptop computer (single core CPU operating at 1.9 GHz, 1.5 GB of memory) reconstruction is impossible: the algorithm crashes because the singular value decomposition exhausts the memory of the computer.

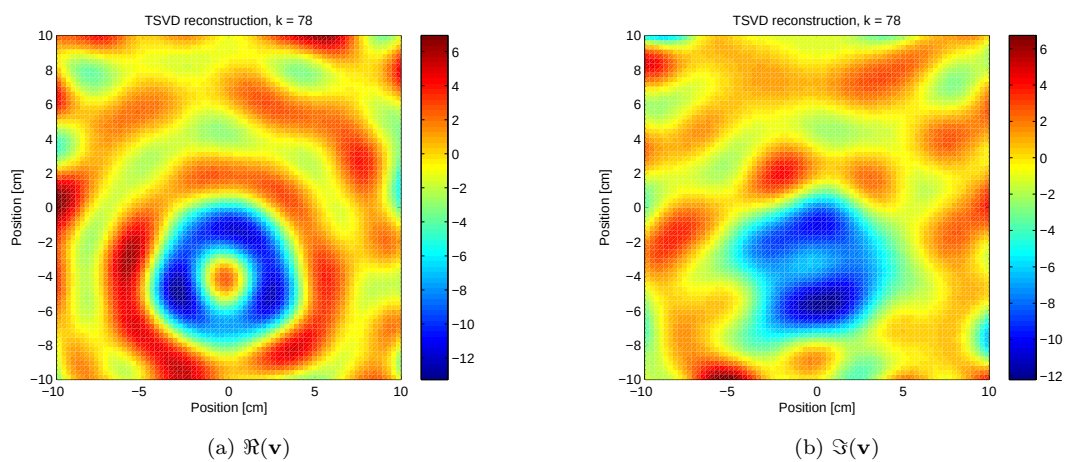


Figure 4.5: Real (a) and imaginary (b) parts of the solution retrieved by TSVD, for $k = 78$. The real part is negative although this does not make sense physically.

4.2.3 Conjugate Gradient

The Conjugate Gradient algorithm [186, 78] is a very popular method for solving linear systems of equations. It is based on the steepest descent method and it improves on it by choosing in a smarter way the direction where to look for the solution at every iteration. The difference between the two methods can be efficiently explained with a simple example.

Suppose the minimum of a convex cost function needs to be found. The method of steepest descent starts from an initial guess, computes the gradient of the function and moves along that direction (search line) until a minimum in the gradient is reached. From that point, it computes a new gradient and it follows it again, repeating this process until a minimum in the function is reached. Thus, the method follows at every iteration a *steepest descent* path. It turns out that while moving along the search line, the function is minimized when the gradient is perpendicular to it. Therefore in the steepest descent method each step will always be taken in a direction perpendicular to the direction taken at the previous step. The phenomenon is depicted in Fig. 4.6.

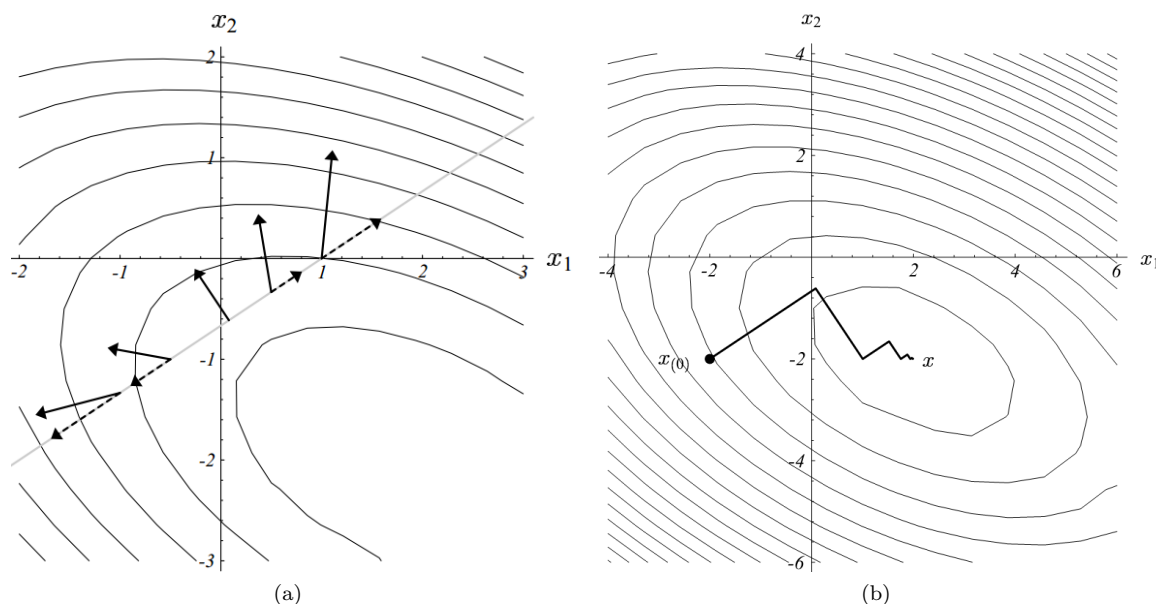


Figure 4.6: The steepest descent method at work. The function is minimized along a certain search line when the gradient is perpendicular to it (a). The path taken is therefore made of orthogonal segments (b). ©1994 by Jonathan Richard Shewchuk [187]

When dealing with particularly “oval” functions, unless the starting point is a particularly good one, the steepest descent method will need an extremely long time to find the solution because it will keep moving left and right along perpendicular search lines.

The Conjugate Gradient (CG) method addresses this problem by noticing that it would be better to move not along the direction of steepest descent, but along a direction that would minimize the number of steps necessary to find the minimum of the target function. This thesis is not concerned with the details of the CG algorithm, but there are many excellent references that discuss them exhaustively, [177] and [187] particularly. In a nutshell, by properly exploiting some properties of the search directions and of the residuals at every step, the CG method is guaranteed to find the solution of an n -dimensional problem in n steps. In the example used before, the CG method finds the minimum solution in 2 steps, as shown in Fig. 4.7.

The beauty of CG is that it is very efficient, it works naturally with complex numbers, it does not require computationally heavy operations such as the Singular Value Decomposition, and it can be more or less easily customized so as to enforce some constraints on the solution. For example, the starting point can be chosen as the origin (the null vector) and then it can be enforced that the path of convergence only includes positive numbers. This can be done to apply some physical constraints on the final solution returned. Even though from a mathematical point of view it might make sense to move in a certain direction because the

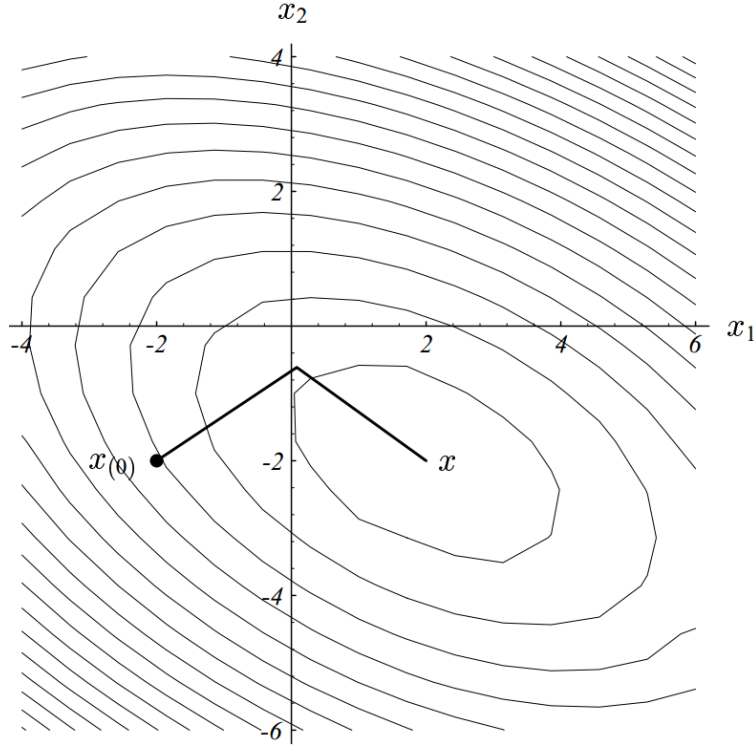


Figure 4.7: The CG method at work. Compare this convergence path, that takes 2 steps, with the one followed by steepest descent, in Fig. 4.6b. ©1994 by Jonathan Richard Shewchuk [187]

cost function is minimized, from a physics point of view the solution thus found might not make sense.

In addition, it is usually convenient to stop the iterative process before the minimum of the target function is actually found. Considering the details of RF Tomography, the minimum norm solution to is usually affected by a lot of noise, because of the ill-conditioning of the problem. Therefore it can be a good idea to stop before this point is reached, and observe the quality of the reconstruction as the CG algorithm proceeds along its path.

These considerations provide motivation for applying the Conjugate Gradient method to RF Tomography, by applying some modifications to the basic algorithm.

Physical bounds, re-orthogonalization and termination

In RF Tomography the contrast function is the difference between the permittivity of the target and the permittivity of the background. Both quantities are in general complex, with the imaginary part depending on the conductivity σ . Therefore:

$$v()^{-1} = [\varepsilon()^{-1} - \varepsilon_b] - j \left[\frac{\sigma()^{-1} - \sigma_b}{2\pi f \varepsilon_0} \right], \quad (4.7)$$

for each pixel, where $\mathbf{r}'$ identifies each pixel. In free space it is assumed that:

$$\varepsilon()^{-1} \geq \varepsilon_b \quad (4.8)$$

$$\sigma()^{-1} \geq \sigma_b, \quad (4.9)$$

i.e. the target is denser and more conductive than free space. Therefore $\mathbf{v}$ will have positive real part and negative imaginary part. Unless enforced explicitly, nothing guarantees that this simple physical rule will be respected by the CG inversion. Therefore the CG algorithm has been modified by imposing *hard* constraints

on the solution returned. At every iteration:

$$\text{if } \operatorname{Re}(\mathbf{v}) < 0 \Rightarrow \operatorname{Re}(\mathbf{v}) = |\operatorname{Re}(\mathbf{v})| \quad (4.10)$$

$$\text{if } \operatorname{Im}(\mathbf{v}) > 0 \Rightarrow \operatorname{Im}(\mathbf{v}) = -|\operatorname{Im}(\mathbf{v})|, \quad (4.11)$$

which imposes the physical bound without changing the norm of the solution (as it would setting the pixel equal to zero). Still, this hard constraint has an important effect on the algorithm. Even though the norm of the solution remains unchanged, the solution is indeed changed, and so is its corresponding residual. In practice a jump is introduced in the solution space. This is undesired because it disrupts the natural convergence path that CG would take. In particular, CG is based on the property that at each iteration the residual vector is perpendicular to *every* residual found in the previous iterations. Since the residuals are used at every iteration to determine the next search direction, this operation disrupts CG.

The solution to this problem comes by operating a re-orthogonalization procedure [188, 189], using the Gram-Schmidt procedure [190, 191]. Given a set of vectors, the Gram-Schmidt process generates a set of orthogonal vectors, in an iterative fashion. The first vector is taken as is, the second vector is made perpendicular to the first one, and so on, until all the vectors in the set have been made perpendicular to each other. In the CG inversion developed here, the quantity made orthogonal is the set of residuals $r = \|\mathbf{E}^s - \mathbf{L} \cdot \mathbf{v}\|$. Therefore, at each iteration, after the new solution has been found and the physical bounds have been applied, the Gram-Schmidt procedure is applied to re-orthogonalize the residuals.

The last step in the algorithm is its termination. The algorithm must be stopped before it completes all the iterations, otherwise there would be no regularization effect on the solution. In order to establish the point when to stop, the evolution of the residual at every iteration is analyzed. The residual at the i -th iteration is defined as:

$$r^{(i)} = \|\mathbf{E}^s - \mathbf{L} \cdot \mathbf{v}^{(i)}\|_2. \quad (4.12)$$

This residual is analyzed in more details in the next section. Generally, $r^{(i)}$ changes rapidly with the first few iterations and then tends to stabilize toward an asymptote. Therefore a stopping condition can be applied on the derivative (finite-difference) of the residual. At every iteration the difference between $r^{(i)}$ and $r^{(i-1)}$ is computed. The algorithm continues until the difference at a step i is smaller than 5% of the maximum difference observed during the process. This procedure results in a stopping point reached very quickly, normally within 10 iterations.

Comparison of CG and TSVD

Conjugate Gradient and the Truncated Singular Value Decomposition have been compared using data coming from both real measurements and noise-free simulations. The conditions used in the experiments are replicated in the commercial software FEKO, which produces as a result a vector containing the scattered field $\mathbf{E}_S(P)$. This field is in turn used to generate images.

The choice of a Method of Moments simulation provides certain advantages over other types of simulations. First, since it is a method based on the surface equivalence theorem, it allows to obtain the scattered field directly, with one simulation. A method like Finite Difference Time Domain (FDTD) would require two simulations, one with the object and one without, to obtain the scattered field by subtraction. Second, the method requires to mesh only the object that causes scattering, again thanks to the surface equivalence theorem. This represents another computational advantage against FDTD and the Finite Elements Method (FEM), because both would require to mesh the whole space where the measurement occur. Since the targets are small compared to the area where the measurement is performed, the computational advantage given by meshing only the target is very large.

Generating images using the simulated scattered electric field allows drawing conclusions on the quality of the inversion algorithm itself, without the interference of noise, positioning errors, measurement errors, etc. Nonetheless, showing the imaging results from actual measured data remains the main goal of this chapter.

First, the performance of CG in its basic formulation, without any modification is analyzed. The test conditions are the same as in Section 4.2.1.

Fig. 4.8 shows the result. TSVD and CG return almost identical images. Therefore, CG has regularizing properties identical to TSVD, which come at a much lower computational cost. On a desktop PC equipped

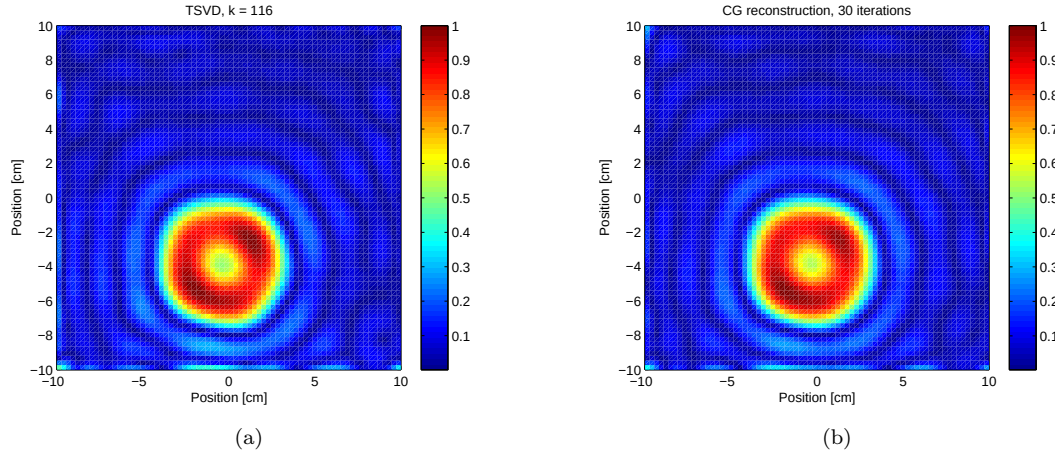


Figure 4.8: Reconstruction of an empty PVC pipe from simulated data using TSVD (a) and CG (b). For TSVD, 116 singular values have been retained. For CG, the result is obtained after 30 iterations.

with a quad-core CPU operating at 3.7 GHz, 8 GB of memory, running a MATLAB code (hence, not compiled) TSVD required 4.1 seconds to find the solution, CG required less than 0.2 seconds.

For this test, CG has been forced to run for 30 iterations, overriding the automatic termination rule. This has been done to show the behavior of the residual and of the solution norm, which are displayed in Fig. 4.9. The residual quickly decreases then it stabilizes, while the solution norm quickly increases then it stabilizes. This behavior can be observed in basically all cases, and it is what motivated the choice of the introduction of the stopping rule for the CG algorithm.

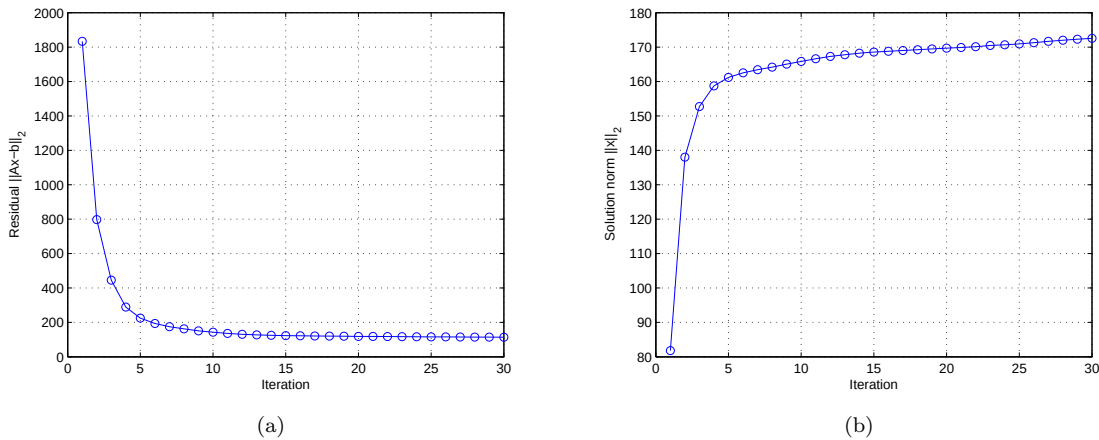


Figure 4.9: Plots of residual (a) and solution norm (b) for CG, for simulated data. The plots show that convergence is obtained very quickly: after 10 iterations the change on both quantities is small.

Similar results are obtained for measured data, in Fig. 4.10. The result for TSVD is a replica of the one shown in Fig. 4.3, while the one for CG is obtained by introducing the stopping condition. Again, the two images are extremely similar in terms of quality, with TSVD taking 4.1 seconds to find the solution and CG requiring only 0.04 seconds. CG found the solution after only 5 iterations.

Effect of the physical bound

The introduction of the physical bounds has beneficial effects on the quality of the solution returned by CG. Apart from the obvious advantage of avoiding solutions that do not make physical sense, restricting the space

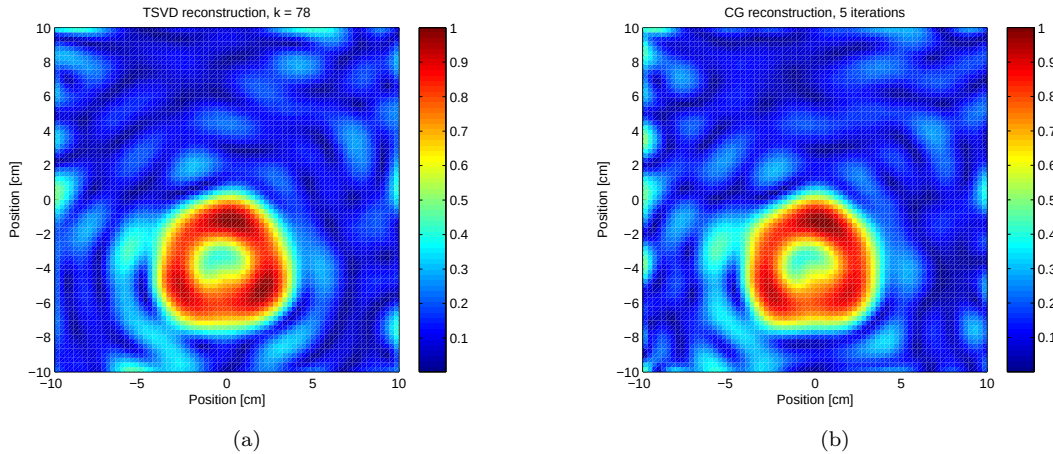


Figure 4.10: Reconstruction of an empty PVC pipe from measured data using TSVD (a) and CG (b). For TSVD, 78 singular values have been retained. For CG, the result is obtained after only 5 iterations.

where a valid solution can be searched for has the consequence of reducing the noise present in the image. To verify this, data coming from measurements only is studied, since they are more affected by noise. The effect of the introduction of the physical bound is shown in Fig. 4.11.

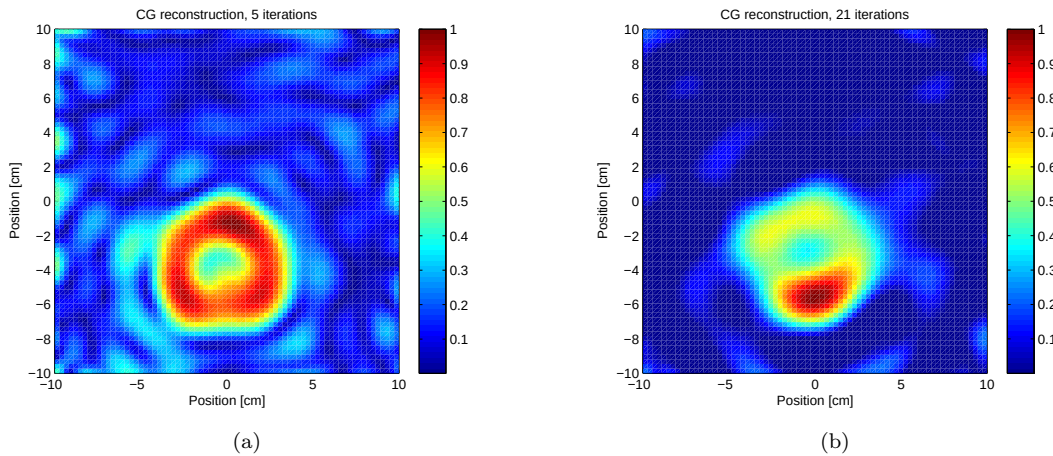


Figure 4.11: Comparison between the regular CG algorithm (a) and the same algorithm with the enforcement of physical bounds (b). The image loses quality, but it is also much less noisy. Reorthogonalization can be used to improve the reconstruction.

The enforcement of the physical bounds has two effects. On one hand, the image quality gets worse: the shape is not reconstructed correctly and it is hard to infer the shape of the target. On the other hand, the noise which affected Fig. 4.10 is greatly reduced.

As explained previously, the enforcement of the *hard* physical bounds has the undesirable effect of interfering with the progression of the algorithm, because it breaks the property of orthogonality of the residual. Evidence of this is shown in the residual and solution norm plots, which become irregular, as shown in Fig. 4.12. While the norm of the solution remains fairly stable, the norm of the residuals shows a staggering pattern. As a consequence, while the solution could be found in 5 iterations the termination condition is now met only after 21 iterations.

These issues are solved with the application of the Gram-Schmidt re-orthogonalization procedure.

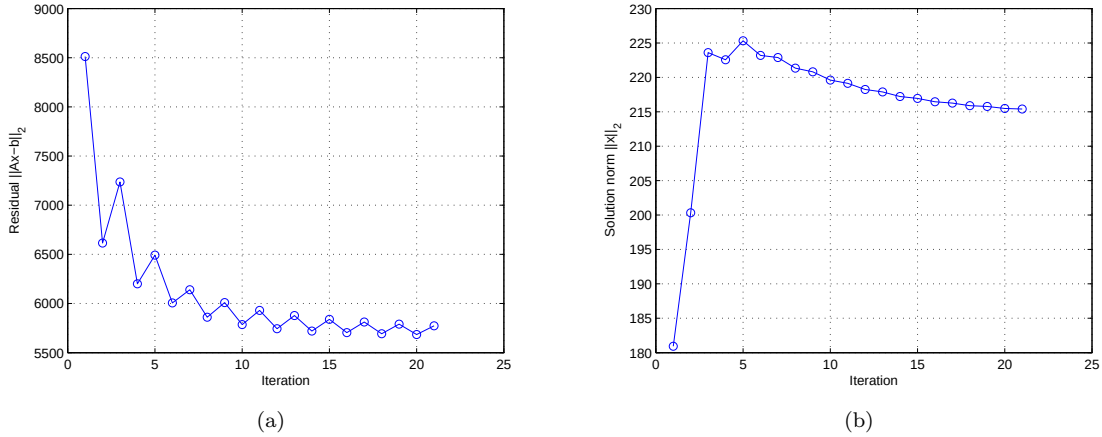


Figure 4.12: Residual (a) and solution norm (b) as functions of the iteration step. As expected, the residuals are very irregular, because of the introduction of the physical bound.

Effect of re-orthogonalization

Since the simple introduction of the physical bound has been shown to make the image quality worse (although it reduces the noise), a re-orthogonalization technique is used. Before computing the next solution, the residuals are re-orthogonalized, following the Gram-Schmidt process. As a result, the CG algorithm will return a solution which is computed on the basis of a set of orthogonal residual vectors. The benefit of this operation is easy to appreciate.

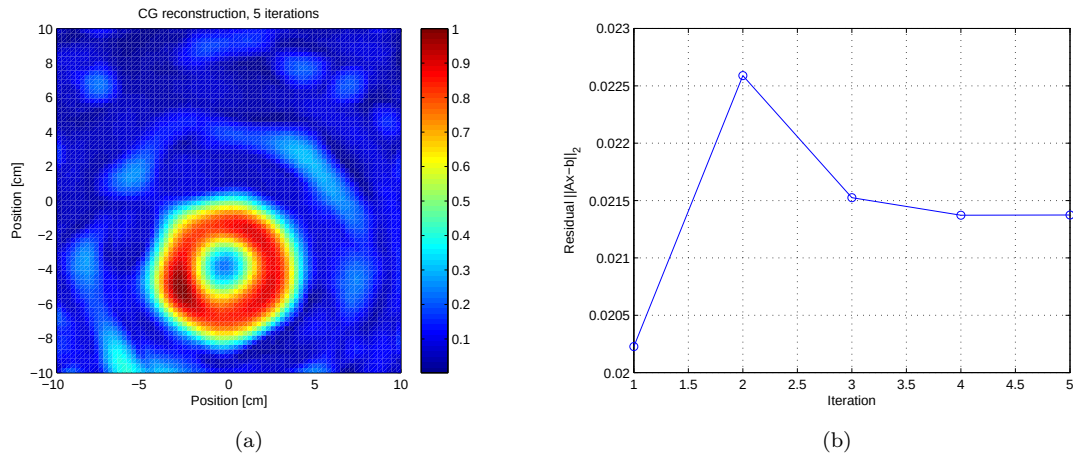


Figure 4.13: Solution obtained using CG, enforcing the physical bounds and reorthogonalizing the residuals (a). Convergence is achieved in 5 iterations. Figure (b) shows the corresponding residual at every iteration. ©2013 IEEE, figure (a) from [34]

Fig. 4.13 shows that the image returned is less affected by artifacts than the one of Fig. 4.10b, and the reconstruction of the shape is slightly more accurate. The residuals are stable again. In fact, if the iterations continued past this point, the solution would not change at all. The enforcement of the physical bound with the addition of the re-orthogonalization step effectively restricts the space where the solution can be searched for. When the CG algorithm hits the bounds, no better solution can be found and the algorithm stops. This happens after only 0.04 seconds.

As a conclusion, this algorithm is proposed as an alternative to TSVD: it shows similar regularizing properties, it can achieve similar or better quality of the reconstructed image, it limits the noise, and it greatly reduces the computational resources needed to find a solution. Results obtained using this methods

were presented in [32, 54, 34].

4.2.4 Algebraic Reconstruction Technique

The result obtained with CG is an improvement with respect to TSVD mostly in terms of computational resources needed and of noise reduction. However, the overall quality of the image is still affected by poor resolution. Two main factors cause this: first the Born approximation, which was used to derive the forward model in the first place [79], then CG, which is used to invert the data, operate like a low-pass spatial filter [73]. Rapid transitions, such as the ones presented by the edges of the PVC object analyzed here, cannot be reconstructed exactly. The limitations given by the Born approximation are described in details in Chapter 2.2 where a partial solution is also proposed. In this chapter, instead, attention is given to what can be done from the inversion point of view, remaining under the umbrella of the Born approximation. An additional inversion algorithm, alternative to CG, is therefore proposed.

The Algebraic Reconstruction Technique (ART) has been developed in 1937. The original algorithm was devised by Polish mathematician Stefan Kaczmarz [192] to solve linear systems of equations². Like most numerical methods developed before the 60s, it found little application simply because of the lack of powerful computers. In the 70s the method has been re-discovered and used in the field of x-ray tomography [193].

The reason why the ART method has been looked into is because it is employed in biomedical imaging, where it has been shown to be superior in some cases to the filtered backprojection method [194]. In biomedical imaging the variability of the properties of tissue is larger than what is expected in RF Tomography, therefore it is expected that the algorithm can produce good results. In addition, the ART method does not rely on any strong mathematical assumption, but can be applied in principle to any linear system of equations. Considerations about its performance in terms of speed of convergence or other possible computational advantages are not of immediate concern. However, it is important that the algorithm is iterative, which allows to easily introduce physical bounds on the solution returned and to also easily introduce a regularization parameter.

The ART method is based in principle on the pseudo-inverse (see section 4.1). The fundamental difference is that the principle of pseudo-inverse is applied in a double-iterative fashion. In the first, inner iteration, the rows of the matrix $\mathbf{L}$ are scanned one by one to produce an image, starting from a guess solution. In the second, outer iteration, the image is used as the starting point to repeat the inner iteration. The inner iteration is run as many times as the number of rows, the outer iteration is stopped after a certain criterion is met, so as to achieve a regularizing effect, controlled by the number of iterations.

The advantage of ART over pseudo-inverse is twofold. First and foremost, the possibility to obtain a regularized solution. Second, and not less important, is the fact that the inner iteration requires a computational effort that is much smaller than the one needed by the pseudo-inverse, although in practice it realizes the same function.

In detail, ART works as follows. The original pseudo-inverse formulation (see (4.1)) is changed so as to operate on one row of the matrix at a time:

$$\mathbf{L}_i^{\dagger, \text{ART}} = \mathbf{L}_i^H (\mathbf{L}_i \mathbf{L}_i^H)^{-1} \quad (4.13)$$

$$= \mathbf{L}_i^H (\|\mathbf{L}_i\|_2^2)^{-1} \quad (4.14)$$

$$= \frac{\mathbf{L}_i^H}{\|\mathbf{L}_i\|_2^2}, \quad (4.15)$$

where i identifies the i -th row being scanned. Notice that the inverse operation in (4.13) is not a matrix operation, hence it does not involve anymore a very large, complex-valued matrix, but it is simply the reciprocal of a real positive scalar, resulting in a computational advantage.

The row-wise inverse obtained in (4.15) is then used in the outer iteration to compute the solution to the inverse problem. Starting from a guess solution $\mathbf{v}_{old}$ (usually an empty vector), the algorithm computes:

$$\mathbf{v}_{new} = \mathbf{v}_{old} + \frac{\mathbf{L}_i^H}{\|\mathbf{L}_i\|_2^2} (E_i - \mathbf{L}_i \cdot \mathbf{v}_{old}) \quad (4.16)$$

To terminate the algorithm, the same procedure used for CG is adopted. At every iteration, the residual $r = \|\mathbf{E}^S - \mathbf{L} \cdot \mathbf{v}\|$ is computed. The evolution of the residual over time is observed, and when the variation of

²an English translation of the original German paper is available at: http://jasonstockmann.com/Jason_Stockmann/Welcome_files/kaczmarz_english_translation_1937.pdf

the residual is less than 5% of the maximum variation observed, the solution is considered sufficiently stable, and the algorithm is terminated.

The result of this simple algorithm is not very good, as shown in Fig. 4.14. In its original formulation, which is so similar to the pseudo-inverse, ART does not tackle the ill-conditioning. The reconstruction result is very similar to what is obtained in TSVD when too many singular values are retained (compare Fig. 4.14 with Fig. 4.4). In addition, convergence is very slow. After 30 iterations the residual still does not fall below the 5% threshold, so the algorithm is manually stopped.

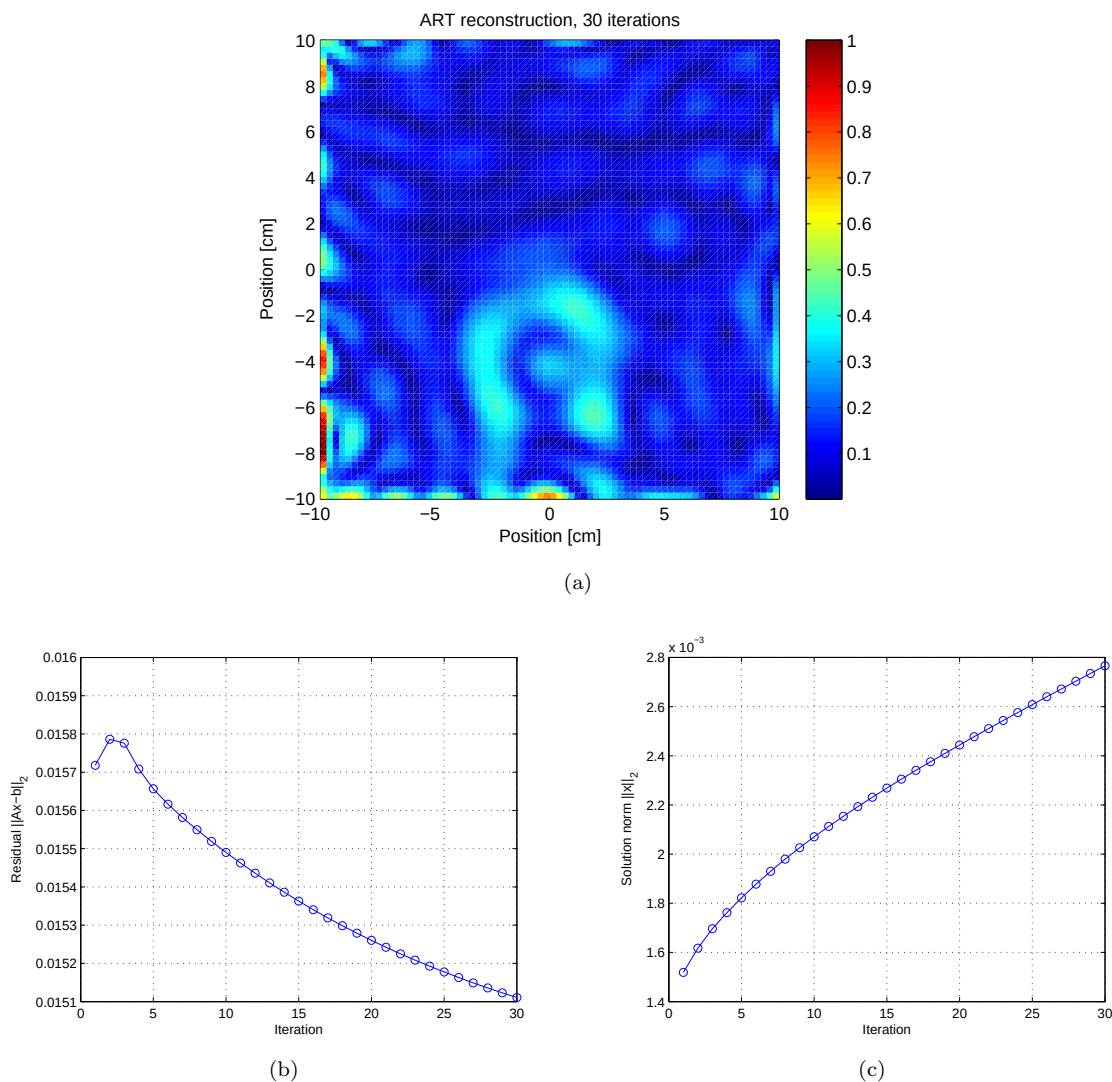


Figure 4.14: Solution obtained using the basic ART method (a), and corresponding plots of residual (b) and solution norm (c).

Two techniques can be used to solve this problem.

Physical bounds and convergence

ART can be modified like CG to include physical bounds in the solution. The procedure applied is the same, i.e. the solution computed at each iteration is forced to have positive real and negative imaginary parts. However, unlike Conjugate Gradient, ART is not based on the orthogonality of the residuals. In CG the next iteration is computed based on the assumption that the next residual must be orthogonal to all the previous residuals. ART is a simpler method: even though it uses the residual to compute the next solution, there is no explicit assumption on the nature of these residuals.

The sole application of the physical bound has incredibly positive effects on the quality of the solution. Fig. 4.15 shows the result.

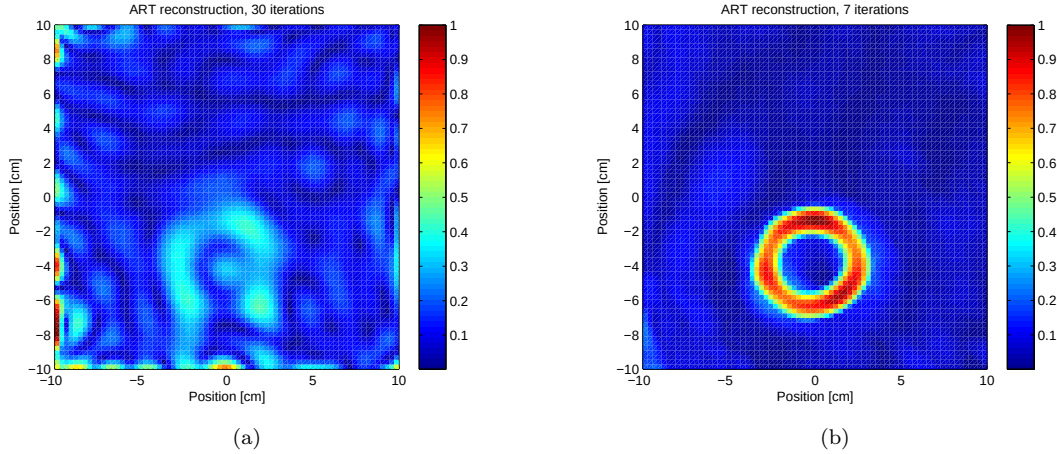


Figure 4.15: Solution obtained using ART (a), and solution obtained using ART and enforcing the physical bounds (b). ©2012 IEEE, figure (b) from [33]

The reconstructed image is sharper, more detailed, noise-free, overall more actionable, and is obtained in a shorter time. While the original ART algorithm after 30 iterations still does not converge, after applying the physical bounds the solution is found in 7 steps.

Overall, this method required 0.36 seconds to converge (compared to 4.1 for TSVD and 0.04 for CG), making it still a very fast method. From a computational point of view ART is limited by the fact that it nests two iterations: the inner iteration scans all the rows of $\mathbf{L}$ as in (4.15), while the outer iteration updates the solution based on the previous solution ((4.16)). The inner iteration always runs for as many steps as the number of rows in the matrix, i.e. the number of TX multiplied by the number of RX, in this case 220. The outer iteration is the one that is controlled and stopped upon request (convergence or maximum number of iterations reached). Therefore ART is inherently slower than CG. However, the reconstruction result is by far superior.

The second technique used to improve ART is not needed in all situations. In some cases, often when imaging metallic targets, the solution returned is greatly disturbed by noise.

Eq. (4.16) consists of the summation of two terms: the first one is the solution found at the previous iteration, and the second one is an update term which depends on the residual and on the pseudo-inverse term. Instead of summing the two terms directly, the update term is pre-multiplied by a constant μ . As a result, (4.16) is changed as:

$$\mathbf{v}_{new} = \mathbf{v}_{old} + \mu \frac{\mathbf{L}_i^H}{\|\mathbf{L}_i\|_2^2} (E_i - \mathbf{L}_i \cdot \mathbf{v}_{old}) \quad (4.17)$$

The effect of μ is to operate as a regularizing parameter. Small values of μ return smoother solutions (like small values of k do with TSVD), while larger values of μ provide sharper images, but they also introduce noise (as large values of k do with TSVD). Also, small values of μ could make the algorithm converge more slowly, since the solution would be updated by very small amounts at each step. The effect of μ on the usual empty PVC pipe image is shown in Fig. 4.16.

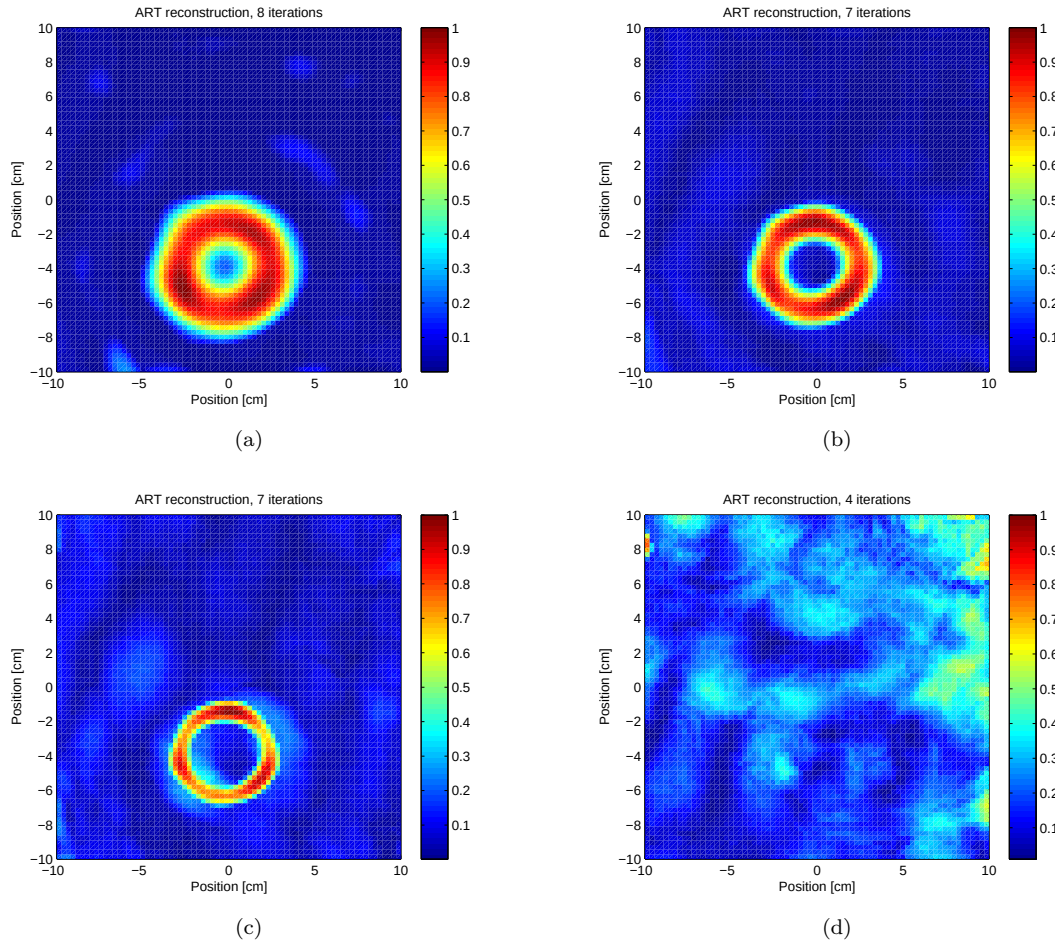


Figure 4.16: Reconstruction using ART and enforcing the physical bounds, with $\mu = 0.1$ (a), $\mu = 0.5$ (b), $\mu = 1.5$ (c), and $\mu = 2$ (d). Images are obtained after 8, 7, 7 and 4 iterations, respectively.

When μ is too large, the solution becomes noisy. However, the solution is also sharper. Therefore, μ should be chosen to be as large as possible before convergence is disrupted (as is the case in Fig. 4.16d).

4.2.5 Imaging results: detection of metallic and dielectric targets

A particular feature offered by RF Tomography is the ability to distinguish between metallic and dielectric targets. This possibility is interesting because it can provide additional information, not limited to the shape and location of the target. In many settings, the possibility to tell apart metals and non-metals could make the difference between succeeding or failing in identifying buried targets of interest.

Since the contrast vector $\mathbf{v}$ is complex-valued, separating its real and imaginary components allows to obtain independent images of conducting and insulating objects. This is also enforced by the physical bounds applied to the solution in the iterative methods.

The test used as example has been carried out using 11 transmitters rotating around a circle of radius equal to 38.4 cm and 20 receivers whose radius of rotation was 15.4 cm. Two targets are present in the area under investigation. The first target is a copper cylinder, of diameter 1.8 cm and 2.8 cm tall. The cylinder is hollow and its thickness is equal to 1 mm. The second target is a plastic cylinder (PVC, $\epsilon_r \approx 2.5$ at 3 GHz), with diameter 7 cm and height 4.3 cm. This cylinder is also hollow, with its side having a thickness of 5 mm. The copper cylinder is located inside the PVC pipe, in a non-concentric position. The geometry is shown in Fig. 4.17.

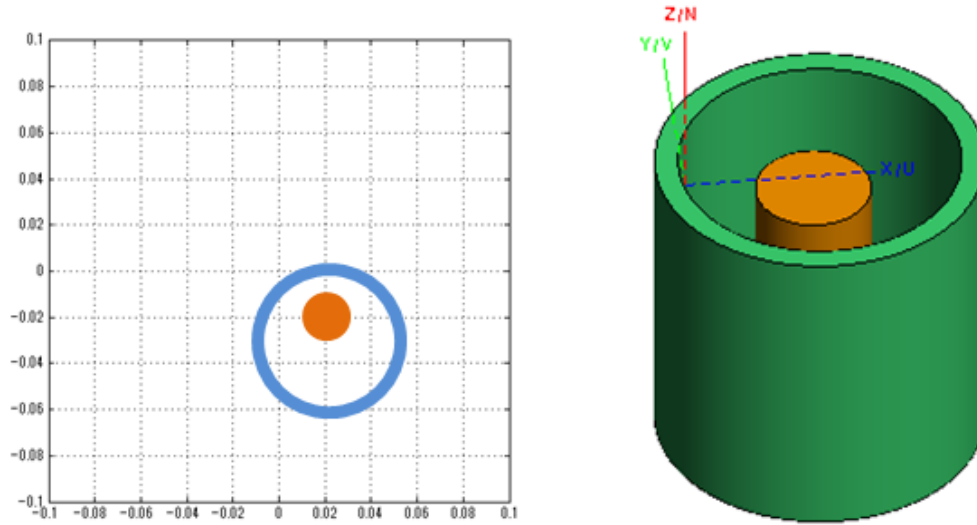


Figure 4.17: Geometry for the test case.

The first attempt of reconstruction is made with TSVD and the result is presented in Fig. 4.18. The absolute part of the image is due for the most part to the copper cylinder, which represents a stronger scatterer with respect to the PVC pipe. By separating real and imaginary parts this becomes evident. The images are fairly noisy, and, since no physical bounds have been enforced, parts of $\text{Re}(\mathbf{v})$ are negative, while parts of $\text{Im}(\mathbf{v})$ are positive. The image has been partitioned into 101×101 pixels, so that reconstruction is performed in 14.9 seconds. Inversion is performed only once, the real and imaginary parts are extracted from the complex vector resulting from the inversion. The number of singular values to be retained in the inversion is automatically determined by the location of the corner in the L-curve.

The second reconstruction attempt is made with CG without the introduction of any physical bound, and is shown in Fig. 4.19. The quality of the reconstruction is very similar to the one of TSVD, as expected. The advantage is in terms of computational burden, since the inversion requires only 5 iterations to converge, taking 0.08 seconds (about 190 times faster than TSVD).

The result of Fig. 4.19 can be improved by applying physical bounds to the CG inversion. The result is shown in Fig. 4.20. The reconstruction improves visibly. The sole application of the physical bounds improves the reconstruction of the real part in particular. The shape of the PVC pipe appears now more round and defined, and the images are overall more sharp, due to a reduction of the noise level. The time required to obtain these images is 0.09 seconds, still corresponding to 5 iterations. The application of the

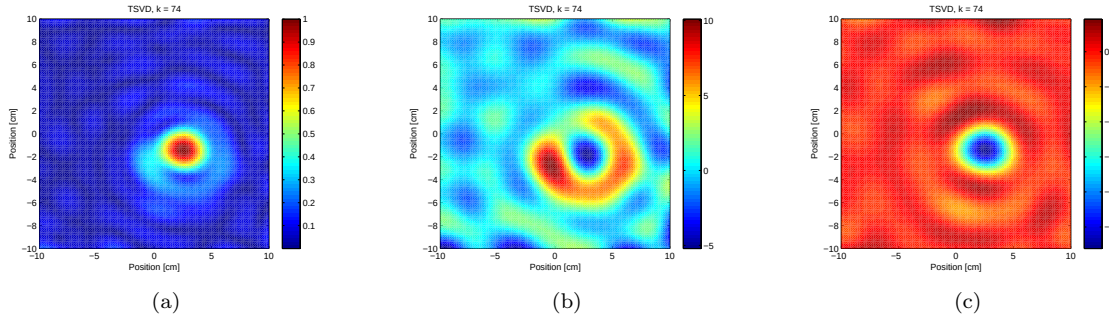


Figure 4.18: Reconstruction of a metallic and plastic objects using TSVD. Images are obtained using 74 singular values. Figure (a) shows the absolute value of the contrast vector and is normalized, (b) and (c) show the real and imaginary parts, respectively, and have not been normalized.

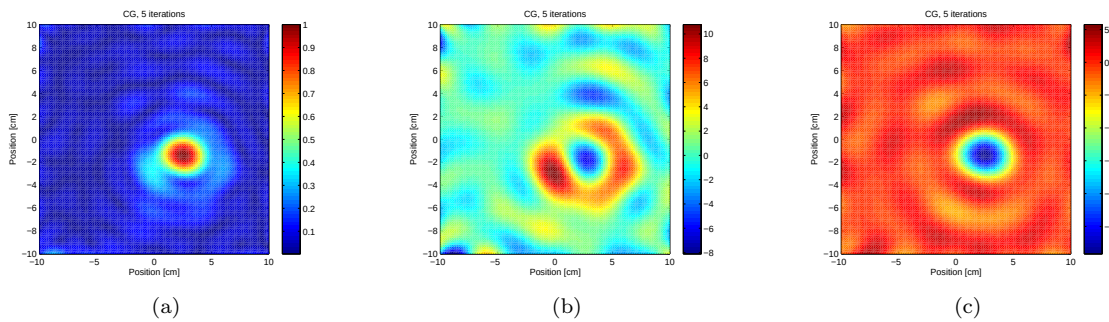


Figure 4.19: Reconstruction of a metallic and plastic objects using Conjugate Gradient without enforcing any physical bound.

physical bound provides no computational burden, and the re-orthogonalization using the Gram-Schmidt technique proves to be extremely quick.

Next, the performance of ART is analyzed. Fig. 4.21 shows the reconstruction produced by the Algebraic Reconstruction Technique without any modification. Quality is comparable to the ones of Fig. 4.18 and Fig. 4.19. The image is obtained in 0.53 seconds, or 8 iterations. The image has been obtained with $\mu = 0.5$, chosen empirically. The problems in the reconstruction are the same as CG and TSVD: the images are noisy, both real and imaginary parts show positive and negative values, and in general it is difficult to distinguish the PVC pipe.

The best result is obtained when the physical bound is applied to ART, as shown in Fig. 4.22. The absolute value is still dominated by the copper cylinder, which represents a strong scatterer. However, both real and imaginary parts of the contrast vector show very little noise, and the real part depicts more clearly the PVC pipe. The images in Fig. 4.22 have been obtained again with $\mu = 0.5$; the application of the physical bounds makes the algorithm converge after 8 iterations instead of 7, taking a total of 0.55 seconds.

Comparing Fig. 4.22 against Fig. 4.20 and Fig. 4.18 it is obvious the big advantage of ART over CG and TSVD. Even though ART is slower than CG, due to the nested iterations, it is still almost 30 times faster than TSVD.

Overall, ART, with the addition of the physical bounds and the use of the regularizing parameter μ proves to be a very good inversion algorithm that allows to overcome some of the resolution limitations given by CG; it is therefore proposed as an alternative inversion method.

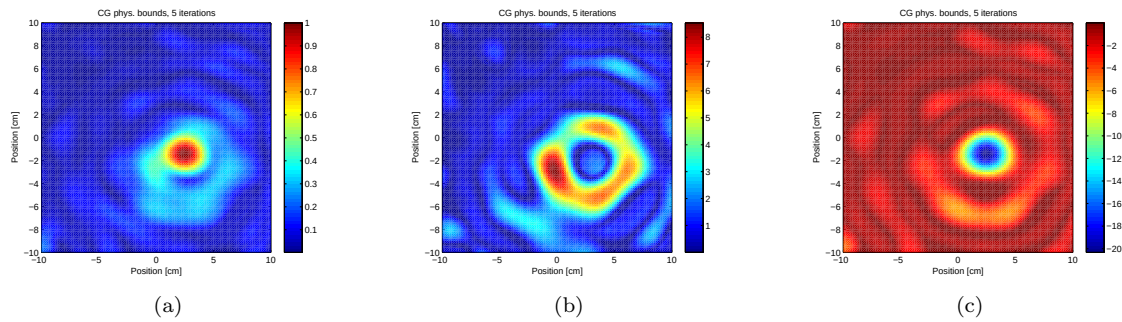


Figure 4.20: Reconstruction of a metallic and plastic objects using Conjugate Gradient modified with the introduction of the physical bounds and the reorthogonalization algorithm. Figure (a) shows the absolute value of the contrast vector and is normalized, (b) and (c) show the real and imaginary parts, respectively, and have not been normalized.

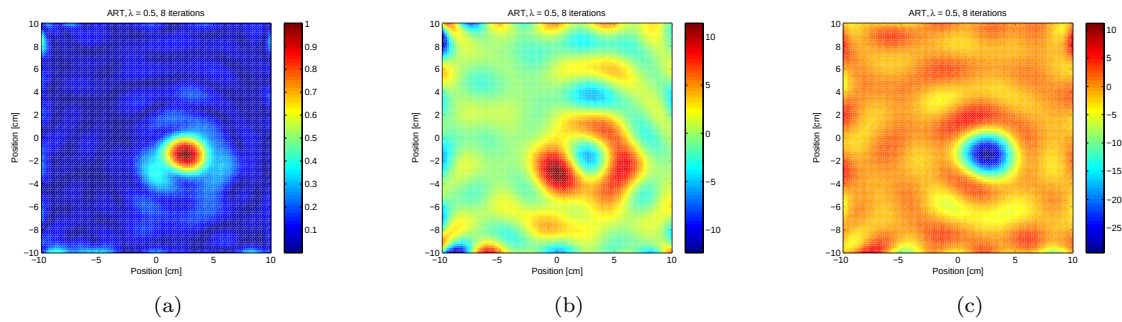


Figure 4.21: Reconstruction of a metallic and plastic objects using ART. Figure (a) shows the absolute value of the contrast vector and is normalized, (b) and (c) show the real and imaginary parts, respectively, and have not been normalized.

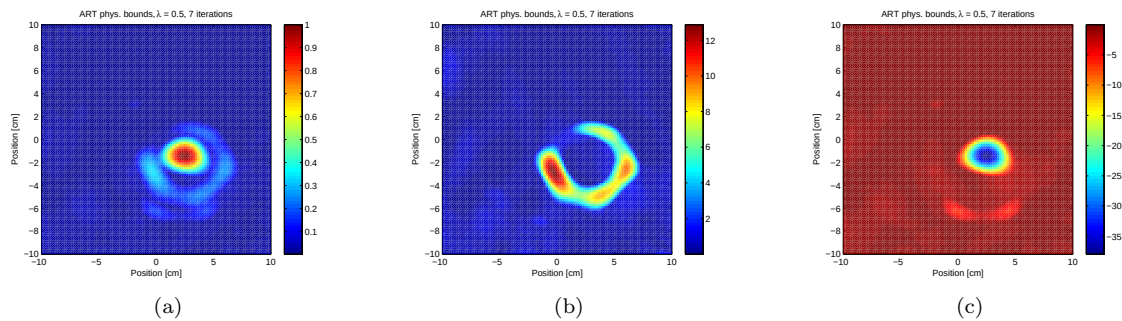


Figure 4.22: Reconstruction of a metallic and plastic objects using ART modified with the introduction of the physical bounds. Figure (a) shows the absolute value of the contrast vector and is normalized, (b) and (c) show the real and imaginary parts, respectively, and have not been normalized.

Chapter 5

Task 4: System implementation

Related to the problem of distributed sensing, we considered the exploitation of prior information about the environment with the purpose of predicting multipath and using this information to obtain better models of the received signal to improve, for example, the probability of detection of a target. Our results have been published in [42], preliminary results were presented in conferences [43, 44, 45], and additional details are given in this chapter.

5.1 Exploitation of multipath information

We considered the detection of a target by a radar in a multipath environment. We show that by taking advantage of the multipath it is possible, in general, to increase the probability of detection of the target, compared to a conventional detection problem based on a model of the return signal that accounts only for the direct signal return from the target.

Multipath is accounted for by leveraging on prior knowledge of the environment where the radar operates. Using this prior knowledge and electromagnetic high-frequency ray-tracing analysis, we can predict the time of arrival of each multipath return depending upon the assumed location of the target. In the case study considered, we show that the environment can be divided into three regions: (1) Region II where multipath components can be clearly distinguished and where the probability of detection is improved by properly accounting for the multipath; (2) Region I where multipath components cannot be distinguished and there is no possibility of improving the probability of detection; and, (3) the Transition Region where, depending upon the SNR of each individual component and the correlation coefficient of the multipath components, it is possible to improve upon a conventional detector. Thus, it is also shown here that diverse receiving strategies, which are optimum in the particular regions of the multipath environment, can be applied to exploit the best performing receivers.

We describe a method to account for multipath as well as the quantitative analysis of the performance increase due to the multipath exploitation. The method was explained by referring to a basic case study scenario, however the approach is quite general and it could be extended to more complex environments.

Radar detection problems are still challenging in all environments where multipath effects are present, notwithstanding the fact that they have been widely studied through advanced electromagnetic (EM) modeling [195], [196]. One approach to deal with multipath problems is to take the advantage of diversity [197]. In addition, with adaptive radars it has been well understood that prior knowledge of the environment and its effective parameters may be used to enhance the detection, estimation and tracking performance of radar systems, *e.g.* [198, 199, 200, 201, 202, 203], and references therein. Recently, with the development of advanced and computationally efficient EM tools, EM propagation models and simulations are being incorporated into radar and sensing problems as well, *e.g.* [204, 205]. Along these lines, this work investigates the problem of target detection in a multipath environment. Our work considers diverse receiving strategies by exploiting prior knowledge of the radar-target environment through advanced EM modeling of the corresponding multipath structure.

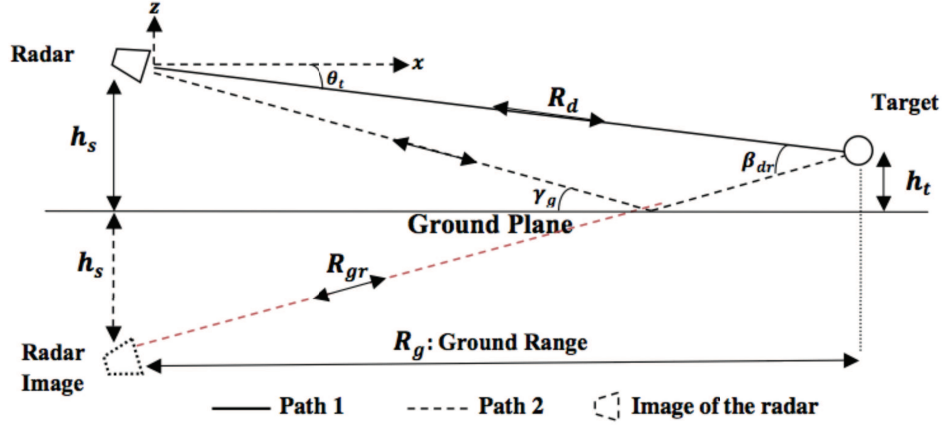


Figure 5.1: Geometry of the problem: Radar-Target over a Ground Plane.

Multipath Model and Time-Delay Analysis

We examine the case study shown in Fig. 5.1 because it contains only a few parameters and it has the advantage of allowing for an analytic discussion of this detection problem. This is a 2D geometry, where the origin of the cartesian coordinate system is the location of the radar and the target is located in a vertical plane at (x_t, z_t) .

A high frequency ray-tracing electromagnetic analysis of this geometry provides the following expression for the received signal of interest with a two-ray model

$$r_s(t) = \alpha_1(t)s(t - \tau_1) + \alpha_2(t)s(t - \tau_2), \quad (5.1)$$

where $r_s(t)$ and $s(t)$ are the baseband equivalents of the received signal of interest and transmitted signal respectively, $\alpha_1(t)$ and $\alpha_2(t)$ are complex parameters accounting for propagation and scattering effects, and τ_1 and τ_2 are time delays along the corresponding propagation paths. Although there would be three propagation paths and three time delays for the geometry depicted in Fig. 5.1, a two-ray model is sufficient to show the potential improvement of multipath exploitation. We assume a planar surface that provides only a specular reflection. For applications related to modeling the reflection from rough surfaces, one would have to add a contribution from diffuse reflection mechanisms. These mechanisms have been investigated, for example, in [206, 207].

The electromagnetic ray-tracing analysis of the environment is always capable to predict the time-delays associated with each multipath component as shown, for example, in [96, 94, 95]. The time-delays and associated time-delay differences for all possible locations of the target can be computed with the geometrical parameters, *i.e.* x_t , z_t , and h_s , as shown in [208]. Assuming a pulse duration T , we define the regions where multipath components overlap, when $|\tau_2 - \tau_1| \leq T$, and the regions where the multipath components are distinguishable, when $|\tau_2 - \tau_1| > T$. An example of partitioning the multipath environment for $T = 10$ ns can be found in [208]. This prompts us to diversify the detection strategy within each region.

5.1.1 Formulation of the Detection Problem

The signal representations and corresponding hypothesis testing problems for each region are derived. Exploiting the useful received signal model given in (5.1), we assume

$$s(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}, \quad (5.2)$$

$$\alpha_1(t) = \begin{cases} \text{unknown} & \tau_1 \leq t \leq T + \tau_1 \\ 0 & \text{elsewhere} \end{cases}, \quad (5.3)$$

$$\alpha_2(t) = \begin{cases} \text{unknown} & \tau_2 \leq t \leq T + \tau_2 \\ 0 & \text{elsewhere} \end{cases}. \quad (5.4)$$

The pulse duration T is assumed to be small compared to the coherence time of the target, so that $\alpha_1(t)$ and $\alpha_2(t)$ can be approximated with two unknown complex deterministic constants α_1 and α_2 , respectively.

Region I is defined by $|\tau_2 - \tau_1| \ll T$ and the received signal (under the hypothesis where target is present) is represented as

$$\begin{aligned} r(t) &= \alpha_1 s(t - \tau_1) + \alpha_2 s(t - \tau_2) + w(t) \\ &\simeq \alpha s(t - \tau_1) + w(t) \end{aligned} \quad (5.5)$$

without any significant loss of the received energy. α is an unknown complex deterministic parameter, $r(t)$ and $w(t)$ are the baseband equivalents of the received signal and noise, respectively. Accordingly, the detection problem can be formulated as the following hypothesis test

$$\begin{aligned} H_0: & \quad r(t) = w(t) \\ H_1: & \quad r(t) = \alpha s(t - \tau_1) + w(t) \end{aligned} \quad t \in [\tau_1, T + \tau_1] \quad (5.6)$$

where $[\tau_1, T + \tau_1]$ is the observation interval.

Region II is defined by $|\tau_2 - \tau_1| \geq T$; hence multipath components are resolvable in time-domain and exploited in the receiver. Accordingly, the detection problem can be formulated as

$$\begin{aligned} H_0: & \quad r(t) = w(t) \\ H_1: & \quad r(t) = \alpha_1 s(t - \tau_1) + \alpha_2 s(t - \tau_2) + w(t) \end{aligned} \quad (5.7)$$

where $t \in [\tau_1, T + \tau_2]$, which is the observation interval. Since $|\tau_2 - \tau_1| \geq T$, $s(t - \tau_1)$ and $s(t - \tau_2)$ are orthogonal signals, we have $\int_{-\infty}^{\infty} s(t - \tau_1) s^*(t - \tau_2) dt = 0$, where $(\cdot)^*$ is the complex conjugate operator.

The likelihood ratio test in both regions can be written as

$$\Lambda[r(t)|\Theta] = \frac{p_{r(t)|\Theta, H_1}(r(t)|\Theta, H_1)}{p_{r(t)|H_0}(r(t)|H_0)}, \quad (5.8)$$

where $p_{r(t)|\Theta, H_1}(r(t)|\Theta, H_1)$ and $p_{r(t)|H_0}(r(t)|H_0)$ are the likelihood functions, Θ is the vector of unknown parameters, *i.e.*

$$\Theta = \begin{cases} [\alpha] & \text{in Region I} \\ [\alpha_1, \alpha_2]^T & \text{in Region II.} \end{cases} \quad (5.9)$$

In the Transition Region, the time-delay difference is shorter than the transmitted pulse duration T but not as short as in Region I, *i.e.* $|\tau_2 - \tau_1| < T$. Thus multipath return signals along different paths are neither highly clumped nor resolvable in time-domain. The received signal in the Transition Region under the hypotheses H_0 and H_1 are the same as (5.7) in Region II. However, $s(t - \tau_1)$ and $s(t - \tau_2)$ are no longer orthogonal signals since $|\tau_2 - \tau_1| < T$ and we have $\int_{-\infty}^{\infty} s(t - \tau_1) s^*(t - \tau_2) dt = \rho$, which is a real number in this particular scenario.

Optimum and Sub-Optimum Detectors

The optimum and sub-optimum detectors are sought assuming that $w(t)$ is a zero-mean complex circular white Gaussian noise (CWGN), with known power spectral density (PSD) σ^2 . A conventional approach, based on projecting the received waveform along the first M functions of an orthonormal basis and letting M diverge, is adopted. Precisely, having chosen the basis $= \{\phi_i(t)\}_{i=1}^{\infty}$, the received signal is represented by components $R_i = \langle r(t), \phi_i(t) \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in the space of finite energy signals. Then the likelihood ratio (5.8) can be written as

$$\Lambda[r(t)|\Theta] = \lim_{M \rightarrow \infty} \Lambda_M[r_M(t)|\Theta] = \lim_{M \rightarrow \infty} \frac{p_{\mathbf{R}_M|\Theta, H_1}(\mathbf{R}_M|\Theta, H_1)}{p_{\mathbf{R}_M|H_0}(\mathbf{R}_M|H_0)}, \quad (5.10)$$

where $\mathbf{R}_M$ is the vector containing the first M coefficients of the received signal waveform and $r_M(t)$ is the projection of the received signal on the subspace spanned by the first M functions of the basis, [209, 210]. Since we have zero mean CWGN with PSD σ^2 , R_i is complex Gaussian with (i) μ_i mean and variance σ^2

under H_1 hypothesis; and (ii) zero mean and variance σ^2 under H_0 hypothesis. Thus, the likelihood function is readily found as

$$\lim_{M \rightarrow \infty} \Lambda_M[r_M(t)|\Theta] = \lim_{M \rightarrow \infty} \frac{\prod_{i=1}^M \frac{1}{\pi\sigma^2} \exp\left(-\frac{|R_i - \mu_i|^2}{\sigma^2}\right)}{\prod_{i=1}^M \frac{1}{\pi\sigma^2} \exp\left(-\frac{|R_i|^2}{\sigma^2}\right)}, \quad (5.11)$$

where μ_i is the expected value of R_i .

First we assume that Θ is a known set of parameters in order to deduct the optimal detector. Although this hypothesis is not realistic for real world radar and sonar problems, it provides an upper bound for any receiver operating under the same signal model. It is also important to observe whether a Uniformly Most Powerful (UMP) test exists or does not exist.

Region I: Multipath returns are highly clumped We formulate the detection problem, based on (5.6), as

$$\begin{aligned} H_1: \quad R_i &= \langle r(t), \phi_i(t) \rangle = \begin{cases} \alpha + W_1, & i = 1 \\ W_i, & i > 1 \end{cases} \\ H_0: \quad R_i &= \langle r(t), \phi_i(t) \rangle = W_i \end{aligned} \quad (5.12)$$

where $\phi_1(t) = s(t - \tau_1)$, so that $\mu_1 = \alpha$ and $\mu_{i \neq 1} = 0$. By evaluating the likelihood ratio (5.11) with the knowledge of Θ and taking the logarithm, one obtains the log-likelihood ratio, up to a constant and irrelevant factor, as

$$\ln \Lambda[r(t)|\Theta] = \frac{|R_1|^2 - |R_1 - \alpha|^2}{\sigma^2}. \quad (5.13)$$

Then the corresponding NP test, which we call NP1, is given as, [210, 211],

$$\Re \left\{ \sqrt{\frac{2}{\sigma^2}} \alpha^* R_1 \right\} \underset{H_0}{\overset{H_1}{\geq}} \gamma_1, \quad (5.14)$$

where $\Re\{\cdot\}$ is the real part operator and γ_1 is the threshold. This is a hypothetical test with the perfect measurement or the knowledge of α in the detector. Although (5.14) is not a UMP test with respect to α , it provides an upper bound to the performance of any practically implementable detector. GLRT is a common technique to devise detectors with unknown parameters, particularly when no UMP test exists. GLRT uses the maximum likelihood estimates (MLE) of unknown parameters under both hypotheses in the likelihood ratio [210], [211].

Thus, the corresponding GLRT detector with respect to α is given as

$$\frac{2}{\sigma^2} |R_1|^2 \underset{H_0}{\overset{H_1}{\geq}} \gamma'_1, \quad (5.15)$$

which we call GLRT1. It can be realized with a standard matched-filter followed by square modulus [210], [211].

Region II: Multipath returns are entirely resolvable The basis functions $= \{\phi_i(t)\}_{i=1}^M$ are selected as

$$\begin{aligned} \phi_1(t) &= s(t - \tau_1) \quad \tau_1 \leq t \leq T + \tau_1 \\ \phi_2(t) &= s(t - \tau_2) \quad \tau_2 \leq t \leq T + \tau_2 \end{aligned} \quad (5.16)$$

so that the detection problem for Region II becomes

$$\begin{aligned} H_1: \quad R_i &= \langle r(t), \phi_i(t) \rangle = \begin{cases} \alpha_1 + W_1, & i = 1 \\ \alpha_2 + W_2, & i = 2 \\ W_i, & i > 2 \end{cases} \\ H_0: \quad R_i &= \langle r(t), \phi_i(t) \rangle = W_i. \end{aligned} \quad (5.17)$$

By evaluating the likelihood ratio (5.11) in Region II with the knowledge of Θ and taking the logarithm, we obtain the corresponding log-likelihood ratio, up to a constant and irrelevant factor, as

$$\ln \Lambda[r(t)|\Theta] = -\frac{|R_1 - \alpha_1|^2 + |R_2 - \alpha_2|^2}{\sigma^2} + \frac{|R_1|^2 + |R_2|^2}{\sigma^2}. \quad (5.18)$$

After simple manipulations, the log-likelihood ratio test, which we call NP2, is derived as

$$\Re \left\{ \sqrt{\frac{2}{\sigma^2}} [\alpha_1^* R_1 + \alpha_2^* R_2] \right\} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_2, \quad (5.19)$$

where γ_2 is the threshold. This is a hypothetical test with the perfect measurement or the knowledge of α_1 and α_2 in the detector. Additionally, it is not a UMP test with respect to α_1 and α_2 , thus GLRT approach is applied next. GLRT substitutes the unknown signal parameters with $\hat{\Theta} = [\hat{\alpha}_1, \hat{\alpha}_2]^T$ in the log-likelihood ratio (5.18). $\hat{\Theta}$ is the MLE of Θ under H_1 that maximizes the corresponding likelihood function, [211, 44], *i.e.*

$$\hat{\Theta} = [\hat{\alpha}_1, \hat{\alpha}_2]^T = [R_1, R_2]^T. \quad (5.20)$$

The corresponding GLRT, which we call GLRT2, can be written as

$$\frac{2}{\sigma^2} [|R_1|^2 + |R_2|^2] \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'_2. \quad (5.21)$$

Transition Region For this region, no particular detector is devised since there is no stable structure of the multipath. Therefore, we analyze the performance of NP2 and GLRT2, which exploit multipath, in comparison with NP1 and GLRT1, which exploit the direct return path only.

Performance Assessment

In this section, we assess the performance of the proposed detectors GLRT2 (5.21) and NP2 (5.19), which exploit multipath, as well as the conventional detectors GLRT1 (5.15) and NP1 (5.14), which only account for the direct return signals.

Before discussing the performance of these detectors, we need to define the corresponding probabilities of false alarm and detection.

Probabilities of False Alarm and Detection The probability of false alarm of NP1 is

$$P_{FANP1} = Q \left(\frac{\gamma_1}{\sqrt{|\alpha|^2}} \right) \quad (5.22)$$

yielding the threshold

$$\gamma_1 = |\alpha| Q^{-1}(P_{FANP1}), \quad (5.23)$$

and the corresponding probability of detection

$$P_{DNP1} = Q \left(Q^{-1}(P_{FANP1}) - \sqrt{\frac{|\alpha|^2}{\sigma^2/2}} \right), \quad (5.24)$$

where $Q(\cdot)$ and $Q^{-1}(\cdot)$ are the Complementary Cumulative Distribution Function (CCDF), and inverse CCDF of a standard Gaussian random variable, respectively, pp. 20-28 of [211]. The probability of false alarm of GLRT1 is

$$P_{FAGLRT1} = Q_{\chi^2_2}(\gamma'_1) \quad (5.25)$$

yielding the threshold

$$\gamma'_1 = Q_{\chi_2^2}^{-1}(P_{FAGLRT1}) \quad (5.26)$$

and the corresponding probability of detection,

$$P_{DGLRT1} = Q_{\chi_2'^2(\lambda)} \left(Q_{\chi_2^2}^{-1}(P_{FAGLRT1}) \right), \quad (5.27)$$

where $Q_{\chi_2^2}(\cdot)$ and $Q_{\chi_2'^2(\lambda)}(\cdot)$ are the CCDF of a χ_2^2 , and a $\chi_2'^2(\lambda)$ where non-centrality parameter $\lambda = 2|\alpha|^2/\sigma^2$, respectively, pp. 20-28 of [211].

The test statistics of NP2 under both hypotheses are Gaussian, thus the probability of false alarm is obtained as

$$P_{FANP2} = Q \left(\frac{\gamma_2}{\sqrt{(|\alpha_1|^2 + |\alpha_2|^2)}} \right) \quad (5.28)$$

yielding the threshold,

$$\gamma_2 = Q^{-1}(P_{FANP2}) \sqrt{|\alpha_1|^2 + |\alpha_2|^2} \quad (5.29)$$

and the probability of detection,

$$P_{DNP2} = Q \left(Q^{-1}(P_{FANP2}) - \sqrt{\frac{|\alpha_1|^2 + |\alpha_2|^2}{\sigma^2/2}} \right). \quad (5.30)$$

When we consider GLRT2 under H_0 , the test statistics is the product between $2/\sigma^2$ and the sum of the square magnitudes of two complex circular Gaussian variables with zero mean and variance σ^2 . It can be shown that this test statistics has χ_4^2 distribution. Consequently, the probability of false alarm can be written as

$$P_{FAGLRT2} = Q_{\chi_4^2}(\gamma'_2) \quad (5.31)$$

yielding the threshold,

$$\gamma'_2 = Q_{\chi_4^2}^{-1}(P_{FAGLRT2}). \quad (5.32)$$

However, when we consider GLRT2 under H_1 , the test statistics is the product between $2/\sigma^2$ and the sum of the square magnitudes of two complex circular Gaussian random variables with non-zero mean and variance σ^2 . Again, it can be shown that the test statistics has a $\chi_4'^2(\lambda_{12})$ distribution, [211]. As a consequence, the probability of detection of GLRT2 can be written as

$$P_{DGLRT2} = Q_{\chi_4'^2(\lambda_{12})} \left(Q_{\chi_4^2}^{-1}(P_{FAGLRT2}) \right) \quad (5.33)$$

where

$$\lambda_{12} = \frac{2}{\sigma^2}(|\alpha_1|^2 + |\alpha_2|^2) = \lambda_1 + \lambda_2 \quad (5.34)$$

where $\lambda_1 = 2|\alpha_1|^2/\sigma^2$ and $\lambda_2 = 2|\alpha_2|^2/\sigma^2$. It is also important to note that $\lambda = \lambda_1$, *i.e.* $\alpha = \alpha_1$, in Region II.

In the Transition Region, we explore the use of two detection strategies. Accordingly, we first determine their statistical characterizations. Then the corresponding probabilities of false alarm and detection are provided.

The probabilities of false alarm of NP1 and GLRT1 in this region are the same as (5.22) and (5.25), respectively, but the probabilities of detection are different and obtained as

$$P_{DNP1T} = Q \left(Q^{-1}(P_{FANP1}) - \frac{|\alpha_1|^2 + \Re\{\rho\alpha_1^*\alpha_2\}}{\sqrt{|\alpha_1|^2\sigma^2/2}} \right) \quad (5.35)$$

and

$$P_{D_{\text{GLRT1T}}} = Q_{\chi_2'^2(\lambda_T)} \left(Q_{\chi_2^2}^{-1} (P_{FA_{\text{GLRT1}}}) \right) \quad (5.36)$$

where

$$\lambda_T = \frac{2}{\sigma^2} |\alpha_1 + \rho \alpha_2|^2. \quad (5.37)$$

The probabilities of false alarm and detection of NP2 (5.19) and GLRT2 (5.21) in the Transition Region, are not the same as (5.28), (5.30), (5.31) and (5.33), respectively. The covariance of the random variables R_1 and R_2 in detectors has to be taken into account to characterize the probability distribution functions. The covariance of R_1 and R_2 is readily found by $COV[R_1, R_2^*] = \rho \sigma^2$.

The probabilities of false alarm and detection of NP2 are obtained respectively as

$$P_{FA_{\text{NP2T}}} = Q \left(\frac{\gamma_2}{\sqrt{|\alpha_1|^2 + |\alpha_2|^2 + 2\rho \Re\{\alpha_1 \alpha_2^*\}}} \right), \quad (5.38)$$

$$P_{D_{\text{NP2T}}} = Q \left(Q^{-1}(P_{FA_{\text{NP2T}}}) - \sqrt{\frac{|\alpha_1|^2 + |\alpha_2|^2 + 2\rho \Re\{\alpha_1 \alpha_2^*\}}{\sigma^2/2}} \right). \quad (5.39)$$

The test statistics of GLRT2 is the sum of square magnitudes of two correlated complex circular Gaussian random variables R_1 and R_2 with the covariance matrix $\mathbf{C}_{\mathbf{R}}$, where

$$\mathbf{C}_{\mathbf{R}} = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad (5.40)$$

by letting $\mathbf{R} = [R_1, R_2]^T$. We represent (5.21) in the Transition Region with two statistically independent random variables, obtaining

$$[(1 + \rho)R_s + (1 - \rho)R_d] \underset{H_0}{\overset{H_1}{\geq}} \gamma'_2, \quad (5.41)$$

where $R_s = \frac{|R_1 + R_2|^2}{\sigma^2(1 + \rho)}$, and $R_d = \frac{|R_1 - R_2|^2}{\sigma^2(1 - \rho)}$. They are two independent random variables that have χ_2^2 distribution under H_0 , $\chi_2'^2(\lambda_s)$ and $\chi_2'^2(\lambda_d)$ distributions, respectively, under H_1 , where

$$\lambda_s = \frac{(1 + \rho)|\alpha_1 + \alpha_2|^2}{\sigma^2} \quad (5.42)$$

$$\lambda_d = \frac{(1 - \rho)|\alpha_1 - \alpha_2|^2}{\sigma^2} \quad (5.43)$$

so that the test statistics of (5.41) is

$$T_R = (1 + \rho)R_s + (1 - \rho)R_d. \quad (5.44)$$

Now we can compute the probabilities of false alarm and detection for (5.41). The probability of false alarm can be obtained as

$$P_{FA_{\text{GLRT2T}}} = P[T_R | H_0 > \gamma'_2] = P \left[\sum_{k=1}^2 c_k \chi_2^2 > \gamma'_2 \right], \quad (5.45)$$

where $T_R | H_0$ is the test statistics of (5.41) under H_0 , $c_1 = 1 + \rho$, and $c_2 = 1 - \rho$ are the constant coefficients of two independent χ_2^2 random variables. By the theorem for finite linear combinations of independent central χ^2 probabilities we obtain the probability of false alarm explicitly as

$$P_{FA_{\text{GLRT2T}}} = F_1(1 + \rho, \gamma'_2) + F_2(1 - \rho, \gamma'_2) = \frac{1 + \rho}{2\rho} \exp \left\{ -\frac{\gamma'_2}{2(1 + \rho)} \right\} - \frac{1 - \rho}{2\rho} \exp \left\{ -\frac{\gamma'_2}{2(1 - \rho)} \right\}. \quad (5.46)$$

In a similar manner, the probability of detection can be obtained as

$$P_{D_{\text{GLRT2T}}} = P[T_R|H_1 > \gamma'_2] = P\left[\sum_{k=1}^2 c_k \chi_2^2(\lambda'_k) > \gamma'_2\right], \quad (5.47)$$

where $T_R|H_1$ is the test statistics of (5.41) under H_1 , $c_1 = 1 + \rho$, $c_2 = 1 - \rho$ are the constant coefficients of two independent $\chi_2^2(\lambda'_k)$ distributions with non-centrality parameters $\lambda'_1 = \lambda_s$ and $\lambda'_2 = \lambda_d$, which are given by (5.42) and (5.43), respectively. As a consequence, the probability of detection is obtained in an analytic form as

$$P_{D_{\text{GLRT2T}}} = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin \theta(u)}{u \rho(u)} du, \quad (5.48)$$

where

$$\begin{aligned} \theta(u) &= \frac{1}{2} \sum_{k=1}^2 [2 \tan^{-1}(c_k u) + \lambda_k c_k u (1 + c_k^2 u^2)^{-1}] - \frac{1}{2} \gamma_2 u, \\ \rho(u) &= \prod_{k=1}^2 (1 + c_k^2 u^2)^{\frac{1}{2}} \exp \left\{ \frac{1}{2} \sum_{k=1}^2 \frac{\lambda_k (c_k u)^2}{(1 + c_k^2 u^2)} \right\}. \end{aligned}$$

Simulation Results and Discussion

Now that the appropriate probabilities for the detectors in consideration have been introduced, we are ready to compare the performance of these detectors. Since GLRT1 and GLRT2 are not optimum detectors, first it is necessary to assess the performance loss with respect to NP1 and NP2 which assume perfect knowledge of signal parameters.

This analysis was provided in [208] and the results are briefly summarized in the following paragraph.

The degradation in the detection performance of the proposed GLRT2 with respect to NP2 was found less than 2 dB in signal-to-noise ratio (SNR) for a low probability of false alarm, *i.e.* $P_{FA} = 10^{-3}$. For the convenience of making a comparison, the conventional GLRT1 and NP1 were tested under the same multipath environment as GLRT2 and NP2. The degradation in the conventional case was found to be 1 dB in SNR for the same $P_{FA} = 10^{-3}$. The fact that the degradation between NP1 and GLRT1 is smaller than the degradation between GLRT2 and NP2 is expected, since GLRT2 requires the estimation of two unknown parameters. In practice, imperfect prior knowledge on τ_1 and τ_2 can lead to a further performance loss.

The primary goal of this paper is to show that diverse receiving strategies can be utilized in challenging multipath radar-target environments for better detection performances. In a conventional approach, GLRT1 would be applied for all regions of a radar-target geometry. In the proposed approach, NP1 and GLRT1 are devised as optimum and sub-optimum detectors in Region I only, whereas NP2 and GLRT2 are devised as optimum and sub-optimum detectors in Region II. Thus it is important to assess the performance improvement of (i) NP2 relative to NP1; and, (ii) GLRT2 relative to GLRT1, particularly when multipath returns are resolvable as in Region II, and partially overlap in time-domain as in the Transition Region.

Performance Comparison in Region II The performance of the detectors in this region depends upon the ratio λ_2/λ_1 . Therefore we perform the comparison of the two detection strategies based upon the various values of the ratio λ_2/λ_1 . In Fig. 5.2 we compare NP2 and NP1 with respect to the SNR value of the direct return path, namely $\text{SNR} = 10 \log_{10} 2|\alpha_1|^2/\sigma^2$. Both receivers are tested under the same multipath environment. Despite of NP1, NP2 exploits the reflected return path which is assumed to be proportional to the direct path strength. Thus, Fig. 5.2 assesses the quantitative measure of the optimal performance improvement of the receiver that exploits multipath compared to the traditional receiver that relies on the direct return path only. We observe that (i) the performance of NP2 is always superior to the one of NP1 that the improvement amount depends upon the ratio λ_2/λ_1 between the multipath returns; and, (ii) their performance merges when the second return path is weak compared to the direct return path, *i.e.* $\lambda_2 = 0.01\lambda_1$. In Fig. 5.3 we compare sub-optimum receivers GLRT2 and GLRT1, under the same multipath

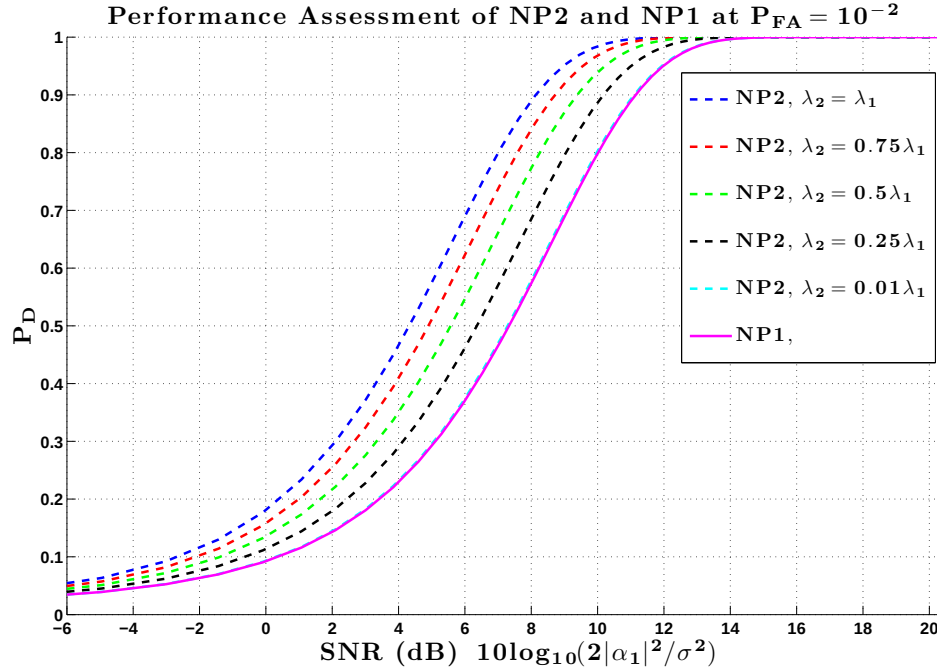


Figure 5.2: Detection performance of detectors NP2 (5.19) with $P_{D_{NP2}}$ (5.30) and NP1 (5.14) with $P_{D_{NP1}}$ (5.24) for various values of the ratio $\frac{\lambda_2}{\lambda_1} = \frac{|\alpha_2|^2}{|\alpha_1|^2}$.

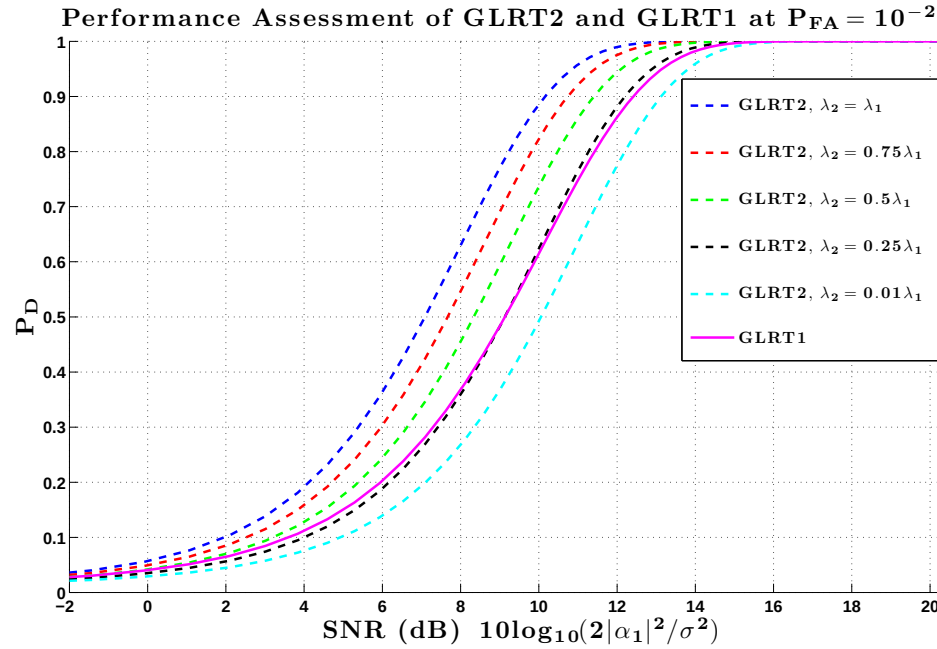


Figure 5.3: Detection performance of detectors GLRT2 (5.21) with $P_{D_{GLRT2}}$ (5.33), and GLRT1 (5.15) with $P_{D_{GLRT1}}$ (5.27) for various values of the ratio $\frac{\lambda_2}{\lambda_1} = \frac{|\alpha_2|^2}{|\alpha_1|^2}$.

environment, with respect to the SNR value of the direct return path. In general, the performance of GLRT2 is also superior to the one of GLRT1, depending upon the SNR value of the multipath returns. However, GLRT1 outperforms GLRT2 when the second return path is weak compared to the direct return path. This

is well understandable because GLRT2 has an extra cost of estimating a second unknown parameter which requires a certain level of SNR. In Fig. 5.4 we present another comparison for a lower value of the probability of false alarm, in order to emphasize that GLRT2 outperforms GLRT1 unless $\lambda_2 \ll \lambda_1$. One can observe that for lower values of the probability of false alarm it is even more evident that GLRT2 outperforms GLRT1.

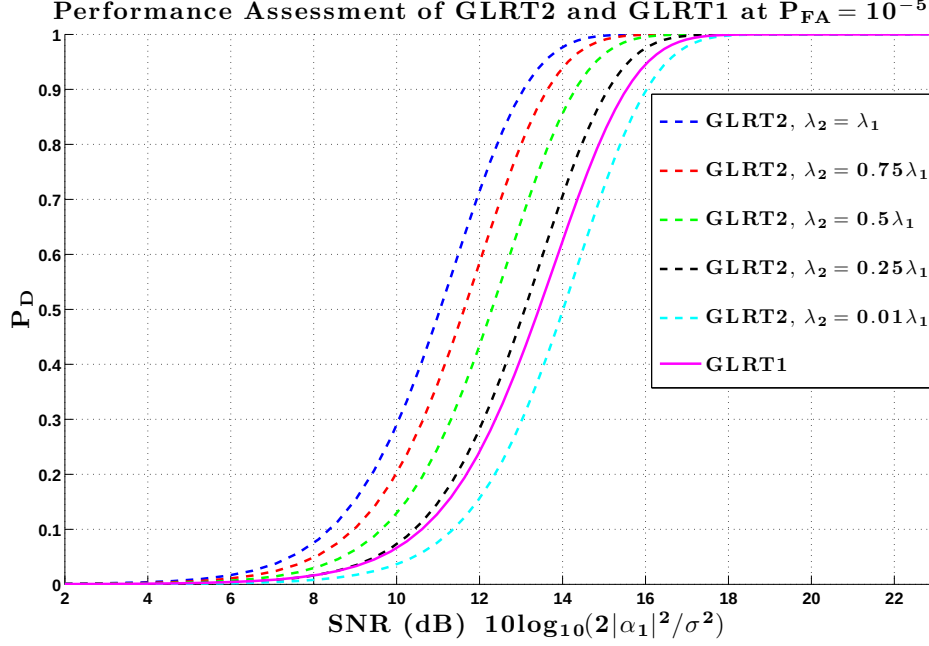


Figure 5.4: Detection performance of detectors GLRT2 (5.21) with $P_{D_{\text{GLRT2}}}$ (5.33) and GLRT1 (5.15) with $P_{D_{\text{GLRT1}}}$ (5.27), for various values of the ratio $\frac{\lambda_2}{\lambda_1} = \frac{|\alpha_2|^2}{|\alpha_1|^2}$.

Performance Comparison in the Transition Region The performance in the Transition Region depends upon the degree of overlap of the multipath returns. Therefore we perform a comparison of the two detection strategies based upon the correlation coefficient of the multipath returns. In the following analysis we assume $|\alpha_1| = |\alpha_2|$, so that direct and reflected path returns have same SNR value, *i.e.* $\lambda_1 = \lambda_2$.

First we compare the two detectors assuming very low and very high correlation of multipath signal returns: (i) $\rho = 0.01$ and (ii) $\rho = 0.99$. We observe that (1) when the correlation coefficient is very low the signals are essentially distinguishable and the probability of detection behaves similar to what was found in Region II as shown in Fig. 5.5a; and, (2) when the correlation coefficient is very high then the signals are essentially highly clumped and the probability of detection behaves similar to what was found in Region I as shown in Fig. 5.5b. In fact, $\rho = 0.01$ and $\rho = 0.99$ are not belong to Transition Region but Region II and Region I, respectively. However, it is necessary with Fig. 5.5 to validate the expressions of the probabilities of false alarm and detection in the Transition Region, *i.e.* (5.36), (5.46) and (5.48).

Second we present the performance of the detectors when $\rho = 0.5$. Our results indicate that the performance of the GLRT2 detector depends on the phase difference between the first and second returns, since the correlation of multipath signal returns are non-negligible. Thus, two extreme situations, in-phase and out-of-phase, are considered in Fig. 5.6a and Fig. 5.6b, respectively. When α_1 and α_2 are in phase, the performance of GLRT2 is always superior to the one of GLRT1, while, when α_1 and α_2 are out-of-phase, the performance of GLRT2 is only superior to GLRT1 for SNR values above 5 dB. This occurs because GLRT2 is affected by the correlation coefficient since it accounts for two signals, while GLRT1 is independent of the correlation coefficient because it exploits the direct signal only.

Finally, the improvement in the target probability of detection of GLRT2 relative to GLRT1 at any hypothetical target location in the multipath environment is presented in Fig. 5.7. It is assumed that γ'_2 , which is the threshold for GLRT2, is fixed at $P_{FA_{\text{GLRT2}}} = 10^{-5}$ by (5.32). However, $P_{FA_{\text{GLRT2T}}}$ varies

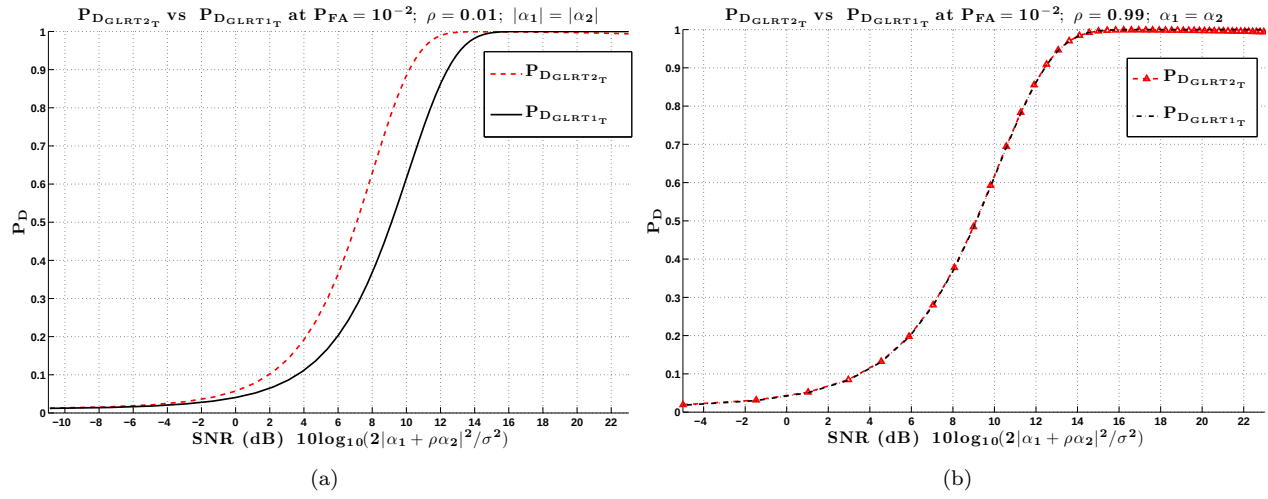


Figure 5.5: $P_{DGLRT2T}$ vs $P_{DGLRT1T}$ when (a) $\rho = 0.01$ and (b) $\rho = 0.99$.

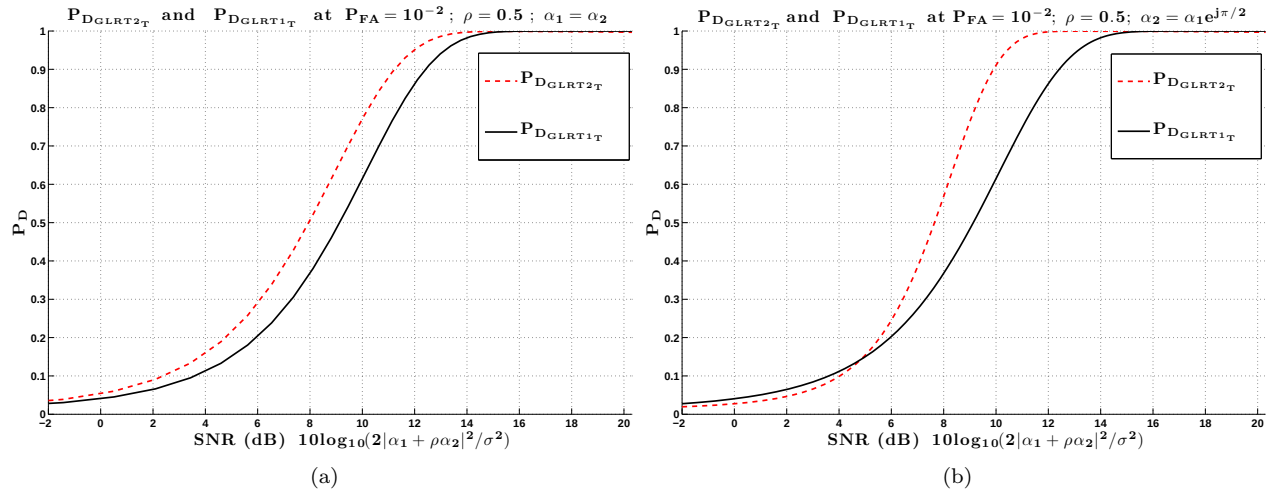


Figure 5.6: $P_{DGLRT2T}$ vs $P_{DGLRT1T}$ when $\rho = 0.5$: (a) α_1 and α_2 are in-phase; (b) α_1 and α_2 are out-of-phase.

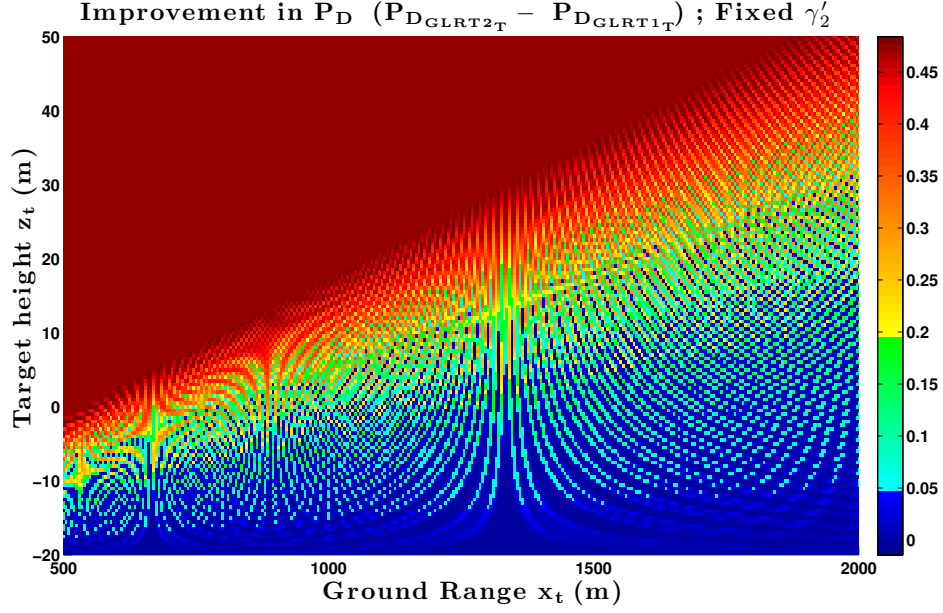


Figure 5.7: Improvement in the target probability of detection of GLRT2 relative to GLRT1 at any hypothetical target location in the multipath environment.

across the Transition Region since it depends on ρ as shown in (5.46). In order to compare two detectors at the same level of P_{FA} we make $P_{FA_{GLRT1T}} = P_{FA_{GLRT2T}}$ across all regions by changing the threshold γ'_1 of GLRT1. It is also assumed that $|\alpha_1| = |\alpha_2|$ and $\text{SNR} = 10 \log_{10}(2|\alpha_1|^2/\sigma^2) = 12$ dB. Constructive and destructive effects of multipath is also shown in Fig. 5.7 in the sense of target probability of detection at a certain SNR level of multipath returns.

Performance Comparison in Region I In Fig. 5.5b and Fig. 5.7 one can see that GLRT2 merges to GLRT1, in Region I.

Chapter 6

Task 5: Experimental validations

The budget included funds for the purchase of a vector network analyzer, which was purchased at the beginning of the funding period. As part of this task, various measurements in the anechoic room of the Andrew Electromagnetics Laboratory at the University of Illinois at Chicago were carried out. The experiments focused on the verification of various aspects of RF Tomography techniques.

First, we investigated which approaches provide better results to reconstruct dielectric and metallic targets in free space [212, 32, 5, 53, 33, 4, 54]. We investigated various reconstruction algorithms, such as truncated singular values decomposition (TSVD), conjugate gradient (CG), Algebraic Reconstruction Technique (ART) and the least squares by Lanczos bidiagonalization (LSQR). In general, iterative methods such as CG and ART, rather than classical inversion techniques such as TSVD, are better for both accuracy and speed. In addition, CG and ART are easy to introduce rational physical bounds which helps to reconstruct better contrast. We found that in CG is the most reliable, while ART provides the sharper images.

Second, we proceeded the experiments to non-trivial scenario, which involves buried targets and targets at non line of sight[55, 56, 57]. In general, the essential limitation is to find Green's function as explained in task 2. However, we could find Green's function of not large geometry with Method of Moment. We could identify the location of buried targets in sand or gravel in a box as preliminary experimental demonstration for ground penetrating radar. Details will be provided in next report.

Third, laboratory experiments were carried out to verify the recently developed triple diffraction mechanism for the Uniform Theory of Diffraction [59, 61], additional details are provided in Section 6.3.

6.1 Experimental validation of RF Tomography in free space

6.1.1 Forward model

The goal of the forward model is to link the quantity being measured (the scattered electric field) with the quantity being reconstructed. In RF Tomography we consider as contrast function the quantity ε_δ , which represents the difference between the unknown complex permittivity of the target and the known complex permittivity of the medium where the target is embedded. The formulation is considered valid inside a certain integration domain D .

As a result, the vectorial forward model for the scattered field is[184]

$$E^s(\mathbf{t}, \mathbf{r}) = Q \int_D \mathbf{a}_m^r \cdot \underline{\underline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}') \cdot \underline{\underline{\mathbf{G}}}(\mathbf{r}', \mathbf{t}) \cdot \mathbf{a}_n^t \varepsilon_\delta(\mathbf{r}') d\mathbf{r}', \quad (6.1)$$

where $\mathbf{a}_m^r$ and $\mathbf{a}_n^t$ are unit vectors representing the orientation of, respectively, receiving and transmitting ideal Hertzian dipoles (t and r are used to identify transmitting and receiving antennas more clearly). Q is a constant which depends on the characteristics of the antennas used, such as their effective height; for example, using short dipoles of length Δl , driven with a current of magnitude I at frequency f , $Q = j2\pi f \mu_0 \Delta l I$. The dyadic Green's function $\underline{\underline{\mathbf{G}}}$ embeds all the information concerning the space where the system is operating.

The frequency of operation affects the Green's function by means of the wave number k_0 . Finally, the contrast function $\varepsilon_\delta(\mathbf{r}')$ is the object of the reconstruction and it is defined as:

$$\varepsilon_\delta(\mathbf{r}') = \varepsilon_r(\mathbf{r}') - \varepsilon_D - j \frac{\sigma(\mathbf{r}') - \sigma_D}{2\pi f \varepsilon_0} \quad (6.2)$$

where ε_D and σ_D identify the dielectric permittivity and conductivity of the host medium, respectively.

The model of Equation 6.1 is linear: it is derived using the Born approximation, which is valid for weak scatterers and ignores the effect of multiple reflections.

It is possible to translate RF Tomography to the microwave domain and to a free-space scenario by choosing [74]

$$\underline{\underline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}') = \left[\underline{\underline{\mathbf{I}}} + \frac{\nabla \nabla}{k_0^2} \right] \frac{e^{jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad (6.3)$$

using the desired frequency. Equation (6.1) can then be discretized and recast in matrix form as $\mathbf{E}^S = \mathbf{L} \cdot \varepsilon_\delta$.

6.1.2 Inversion

To recover ε_δ we developed an algorithm based on the Algebraic Reconstruction Technique (ART). ART, also known as Kaczmarz method, had been originally developed in the 1930s to iteratively solve linear systems of equations, and had been re-discovered in the 1970s to create images in tomographic applications. The algorithm is loosely based on the Moore-Penrose pseudoinverse, in the sense that it follows the pseudoinverse formulation so as to operate on one row of the matrix at a time:

$$\mathbf{L}_i^{\dagger, \text{ART}} = \mathbf{L}_i^H (\mathbf{L}_i \mathbf{L}_i^H)^{-1} = \frac{\mathbf{L}_i^H}{\|\mathbf{L}_i\|_2^2}, \quad (6.4)$$

where $\mathbf{L}_i$ is a vector corresponding to the i -th row of $\mathbf{L}$. Therefore, Equation 6.4 involves the dot product of two vectors which by definition returns a scalar number. The inverse operation is thus simplified because it is not a matrix inversion, but simply the reciprocal of a real positive scalar, hence the computational advantage. The “row-wise inverse” obtained in Equation 6.4 is then used to compute the solution to the inverse problem in an iterative algorithm. Starting from a guessed solution $\varepsilon_\delta^{(0)}$ (usually the null vector), at every iteration (n) the algorithm computes:

$$\varepsilon_\delta^{(n+1)} = \varepsilon_\delta^{(n)} + \mu \frac{\mathbf{L}_i^H}{\|\mathbf{L}_i\|_2^2} \left(E_i^s - \mathbf{L}_i \cdot \varepsilon_\delta^{(n)} \right) \quad (6.5)$$

The algorithm is terminated by looking at the evolution of the residual $r^{(n)}$:

$$r^{(n)} = \|\mathbf{L} \cdot \mathbf{E}^s - \varepsilon_\delta^{(n)}\|_2. \quad (6.6)$$

For every $n > 1$, $r^{(n)}$ is compared to $r^{(n-1)}$. If the difference is less than 5% of the maximum difference ever observed, the algorithm is terminated and the corresponding solution $\varepsilon_\delta^{(n)}$ is returned.

This basic method can be further improved with the inclusion of a simple physical bound on the solution returned. Using the standard notation $\varepsilon = \varepsilon_0 \varepsilon_r (1 - j \tan \delta)$, at every ART iteration we force the solution to have positive real part and negative imaginary part, which is valid in free space or for a loss-less host medium. This physical bound is hard-imposed on the solution by flipping the sign of the elements that don't respect it. This can have detrimental effects on the solution unless the effect of this modification is not smoothed. Therefore, the multiplier μ in Equation 6.5 is empirically chosen so as to achieve meaningful reconstruction. The multiplier acts as a regularizing parameter. A large value ($\mu \approx 1$) makes the solution sharper, but could introduce noise, while a small μ tends to smooth the final image.



Figure 6.1: Experimental setup. The TX is on the left, the RX is on the right, and a target is visible in the top-center, hanging from the ceiling.

6.1.3 Experimental setup

We designed, built, and programmed an automated antenna positioning system, shown in Figure 6.1, which moves the antennas, samples the electric field, and stores the measurement data for later analysis.

Precise positioning of the antennas is paramount to achieve good quality images. The two antennas are moved along circles of different radii. Angular positioning with an error significantly smaller than 1° is achieved by using stepper motors in micro-stepping mode, and by using a pulley-belt system with a gear ratio of 4:1. The system is controlled by a LabVIEWTM application, which coordinates the movement of the antennas with the RF measurements.

Simple half-wavelength dipoles are used in the experiments instead of the electrically short dipoles of the forward model. This choice was made because half-wavelength dipoles are more readily available, provide a much higher radiation resistance, and have a lower reflection coefficient to allow for a more efficient use of the input power. The antennas were operated at 3.16 GHz ($\lambda_0 \approx 9.5$ cm), with a transmit power of 20 dBmW. The frequency has been chosen to allow a proper scaling of the problem, and has then been finely tuned to operate the antennas with minimum mismatch. The antennas were moved around the targets and the field was sampled at each position. This approach has the advantages of reducing the complexity and the cost of the measurement system, and of limiting the mutual coupling between the antennas.

A Vector Network Analyzer (HP 8753ES) measured the scattering parameter S_{21} . For each measurement 16 samples were taken and averaged to reduce additive white Gaussian noise effects. The measurements were taken in an anechoic room to further reduce undesired reflections and to mitigate external RF sources in the same frequency range. TMz polarization (i.e., vertical, perpendicular to the floor plane) was used.

6.1.4 Test on Simulated Data

To compare the performance of the ART inversion against the TSVD algorithm used in [184, 213, 212], we chose a sample test case. This test case has been simulated using the Method of Moments to obtain the scattered electric field. The data has then been used to reconstruct the shape of the target using TSVD, ART, and ART with the introduction of the physical bounds.

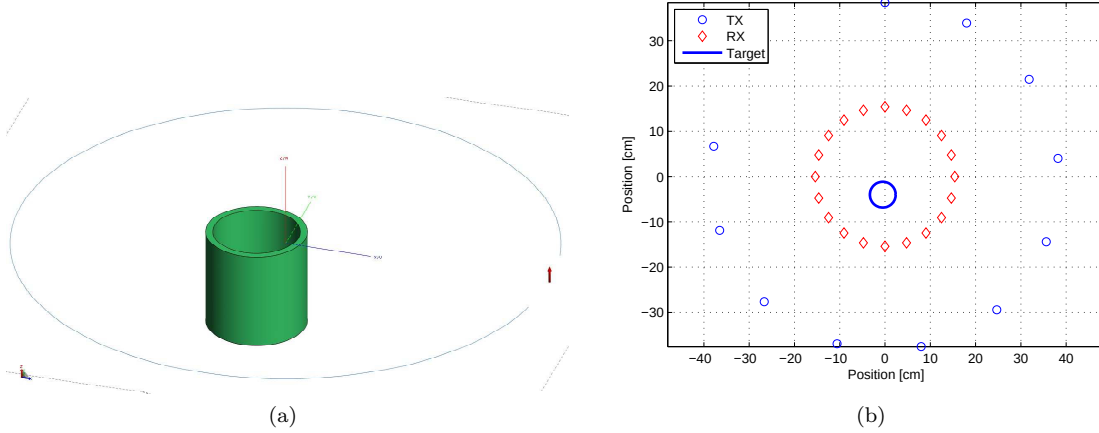


Figure 6.2: 3D model of the PVC pipe (a) and locations of the antennas used in the measurement (b).

The simulation conditions are the same that have later been tested in the laboratory. Operating at 3.16 GHz, we attempt to obtain images of a simple target, consisting of a short section of a PVC pipe: its height is 4.8 cm ($\approx 0.51\lambda_0$), its outer diameter is 4.8 cm ($\approx 0.51\lambda_0$), its inner diameter is 4.4 cm ($\approx 0.46\lambda_0$), therefore the thickness of the PVC is 4 mm ($\approx 0.04\lambda_0$); the dielectric constant of PVC is estimated to be $\epsilon_r = 2.5$ and its conductivity is negligible ($\tan(\delta) = 0.004$ at 3 GHz). The object is illuminated from 11 positions along a circumference of radius 38.4 cm, uniformly spaced between 0° and 280° . The electric field is sampled at 20 locations along a circumference of radius 15.4 cm, uniformly spaced between 0° and 360° .

All images are obtained considering a square window of sides equal to 20 cm, centered at the axis of rotation of the antennas, and divided into 101 pixels per side. Each pixel is therefore 2 mm (or approximately $\lambda_0/47$). This leads to a problem with $101 \times 101 = 10201$ unknowns which must be reconstructed from 220 measurements. The conditioning number $\kappa(\mathbf{L})$ turns out to be approximately equal to 1.7×10^5 . The problem is therefore largely under-determined and ill-conditioned.

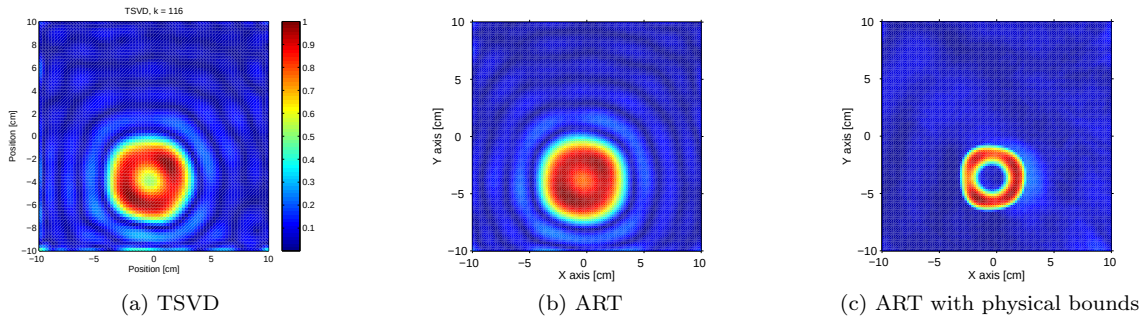


Figure 6.3: The figure shows reconstruction from simulated scattered electric fields, for an empty PVC pipe. TSVD (a) offers reconstruction very similar to ART (b), but the best image is obtained by using ART with the introduction of physical bounds (c). Both (b) and (c) have been obtained with $\mu = 1$. All images have been normalized to their maximum, since quantitative reconstruction is not of interest.

We reconstructed the absolute value of the dielectric permittivity contrast function, $|\epsilon_\delta|$. Figure 6.3a shows the reconstruction obtained using TSVD, in its MATLAB implementation, which computes all of the

singular values (even those that will be discarded later). The image is obtained by retaining 116 out of 440 singular values. This value has been automatically determined by choosing the solution representing the best compromise between the norm of the solution $\|\varepsilon_\delta\|_2$ and the norm of the residual (Equation 6.6). This procedure is commonly referred to as the analysis of the L-curve[214]. The image thus obtained barely allows to understand the shape of the object, and it is noisy and blurry.

Figure 6.3b shows the reconstruction obtained using the ART method in its original, unmodified version. Reconstruction is comparable to TSVD but the shape of the target cannot be inferred correctly. The algorithm is stopped after 4 iterations, when the residual changes by less than 5%, as explained in the previous section. The value of μ has been empirically chosen to be 1.

The best image is obtained by applying physical bounds to the ART method, as shown in Figure 6.3c. The image has been obtained with the same value of μ , and is obtained after 8 iterations (less than 5% residual change). The image is clearly superior to both TSVD and ART: the thickness of the pipe is better estimated and the image is by far less noisy. Therefore we observe that the sole introduction of the physical bound not only has the benefit of restricting the space where a solution is searched, but it also reduces noise and increases the resolution achievable by the inversion, showing sharpening properties.

Another comment can be made about the computational resources needed by TSVD and ART. TSVD is far more demanding than ART: the computation of the singular value decomposition requires time and a lot of memory. On a laptop computer equipped with 1.5 GB of memory the inversion cannot be completed because the software runs out of memory. On the other hand, ART is affected by the fact that it nests two iterations: the outer one updates the solution, the inner one generates a solution by scanning all the rows of the matrix $\mathbf{L}$. These nested loops are not efficiently executed in MATLAB, resulting in a computational time required which is comparable to TSVD. The memory requirement is however extremely smaller, and a proper efficient implementation of the code in a compiled language such as C would speed up the algorithm significantly.

In conclusion, we observe that the reconstruction obtained using ART with the introduction of the physical bound is superior to TSVD, and it is chosen as the only method used for the inversion in the following.

6.2 Results and Discussion

In this section we present reconstruction from measurements obtained using the system described above.

6.2.1 Empty PVC pipe

For the first measurement result, we replicated the conditions of the simulation shown in Figure 6.3.

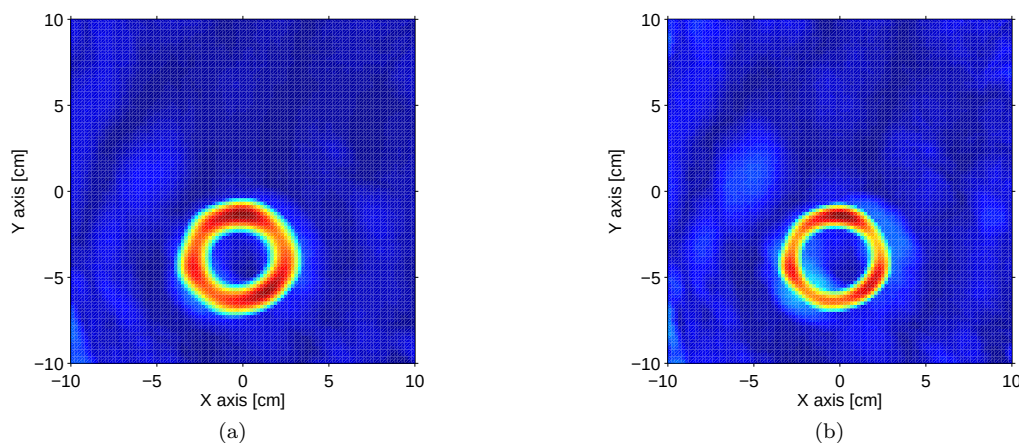


Figure 6.4: Reconstruction of PVC pipe from measurements, using ART with the enforcement of the physical bounds. The figure shows the reconstruction differences when using $\mu = 0.8$ (a) and $\mu = 1.4$ (b).

Figure 6.4 shows the reconstruction results. To show the effect of the multiplier μ on the final solution, we reported two cases, with $\mu = 0.8$ and $\mu = 1.4$, respectively. Reconstruction is extremely satisfactory, showing minimal differences when compared to the simulation result. For both values of μ , the algorithm finished in 7 iterations.

6.2.2 Two Nylon Rods

The second case study involves imaging two nylon rods (diameter equal to 2.5 cm, $\varepsilon_r \approx 3$, $\tan \delta \approx 0.3$). The centers of the two rods are separated by 3 cm ($\approx \lambda_0/3$), i.e. the sides of the rods are separated by 0.5 cm ($\approx \lambda_0/20$). The result is shown in Figure 6.5. For this measurement, due to mechanical reasons, the transmitting antenna was moving along a circumference of radius 38.2 cm and the receiving antennas along a circumference of radius 15.7 cm.

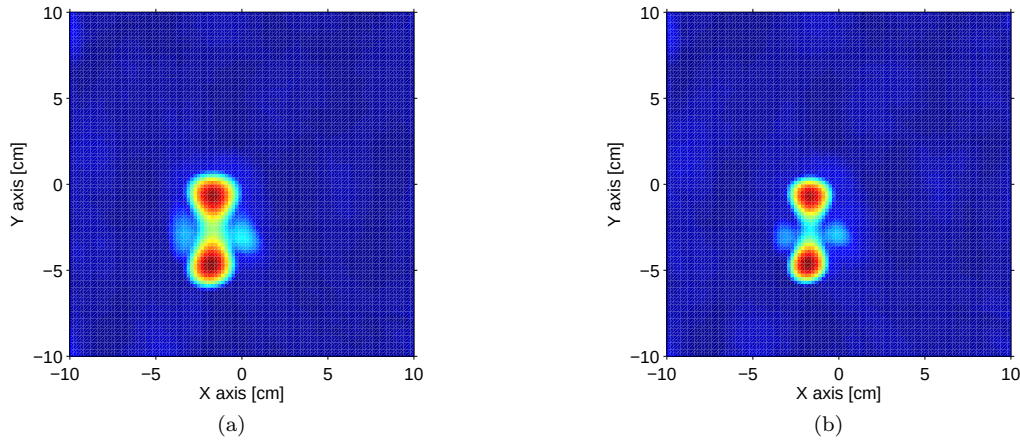


Figure 6.5: Reconstruction of two nylon rods. The figure shows the reconstruction differences when using $\mu = 1$ (a) and $\mu = 1.4$ (b).

Reconstruction is again successful, although the estimation of the size and separation of the two objects is not straightforward. To estimate size and separation of the objects we adopted a standard metric. The images are normalized to their maximum, and the objects are considered present whenever the absolute value of the reconstructed permittivity is greater than 0.5. Using this metric, the objects reconstructed in Figure 6.5a appear as having a diameter of 2.8 cm (average value, due to non perfect circular shapes), which yields an error of approximately 12%. The separation between the objects, instead, appears to be equal to 0.6 cm, an error of 20%. These estimations are also affected by the pixel size, which introduces an uncertainty of ± 2 mm.

As expected, a smaller μ makes the objects appear larger, and also closer in space. Conversely, a larger μ underestimates the size of the objects, which also appear more separated than they are in reality. The choice of μ is therefore very important and a certain strategy for choosing its value should be researched in the future.

6.2.3 Metallic and Plastic Objects

The third test case serves to validate the ability of the imaging system to distinguish between metallic and plastic cylinders. The enforcement of the physical bounds guarantees that the real part of the image is positive and the imaginary part is negative. Separating real and imaginary parts should provide a way to separate conducting versus non-conducting objects.

The following test has been carried out using 16 transmitters rotating around a circle of radius equal to 43.2 cm and 40 receivers whose radius of rotation was 22 cm.

Two targets are present in the area under investigation. The first target is a copper cylinder, of diameter 2.5 cm and 4 cm tall. The cylinder is hollow and its thickness is equal to 1 mm. The object is placed in

the lower portion of the area under reconstruction. The second target is a plastic cylinder (acrylic glass, $\epsilon_r \approx 2-3.5$ at 3 GHz), with diameter 2.5 cm and height 5.1 cm. This cylinder is also hollow, with its side having a thickness of 6 mm. This second cylinder is located in the upper left portion of the area under reconstruction.

The reconstruction obtained using ART with the enforcement of the physical bounds is shown in Figure 6.6. The absolute value of the dielectric permittivity distribution shows a prominent object at the location of the metallic target. This is normal, since metal represents a much stronger scatterer than plastic. However, when the image is split between its real and imaginary parts, the plastic and metallic objects are reconstructed correctly. The imaginary part shows a small but not zero conductivity for the plastic cylinder, while the real part is instead nonzero only for the plastic target.

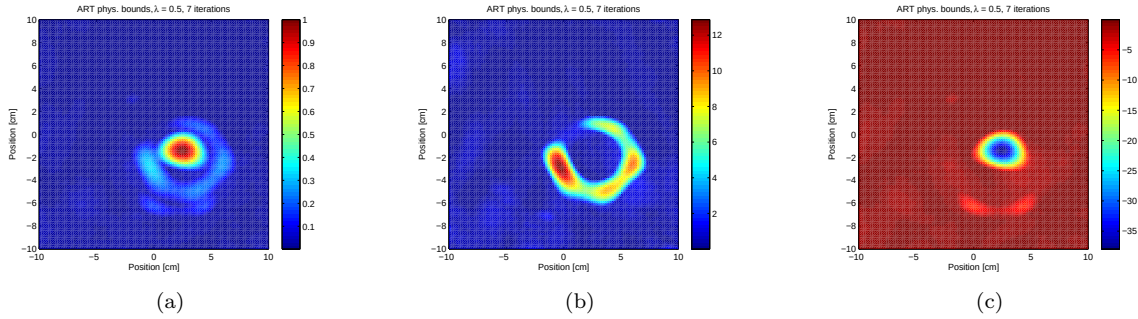


Figure 6.6: ART method with enforcement of physical bounds. Figure (a) shows $|\epsilon_\delta|$, Figure (b) shows $\text{Re}(\epsilon_\delta)$, Figure (c) shows $\text{Im}(\epsilon_\delta)$. All images are obtained with $\mu = 0.2$ and are normalized.

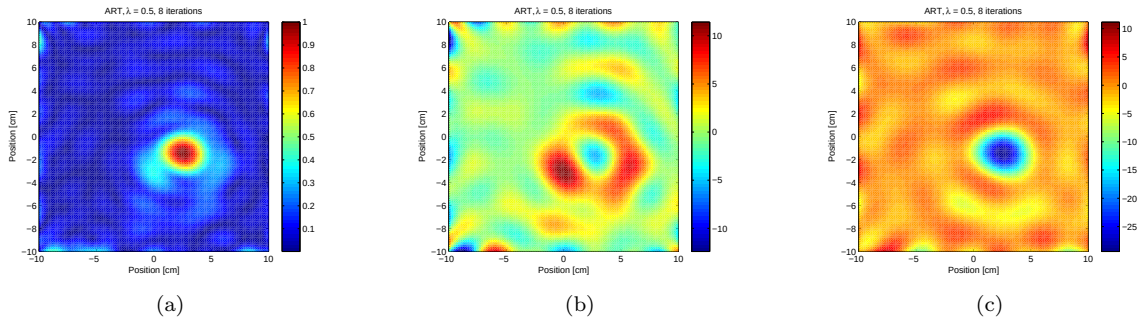


Figure 6.7: Standard ART method. Figure (a) shows $|\epsilon_\delta|$, Figure (b) shows $\text{Re}(\epsilon_\delta)$, Figure (c) shows $\text{Im}(\epsilon_\delta)$. All images are obtained with $\mu = 0.2$ and are normalized.

To evaluate the effect of the physical bound, we can compare the reconstruction shown in Figure 6.6 against the one obtained by the same ART code, with the same μ , but without the enforcement of the physical bound. The result is shown in Figure 6.7. The image is still reconstructed somewhat correctly: the real and imaginary parts show metallic and plastic objects in the right location. However, the image is much noisier, and reconstruction is not as efficient. This shows that the introduction of the forced physical bound does not necessarily correct a wrong reconstruction, but it is extremely effective in reducing the noise in the final image.

6.3 Experimental validation of the UTD triple diffraction coefficient

Carluccio *et al.* recently developed the first triple diffraction coefficient for the Uniform Theory of Diffraction (UTD) [215]. Experiments to validate this new UTD diffraction coefficient were published in [61, 59] and summarized in the following.

The UTD [216], [217], is an efficient method to compute electromagnetic fields in electrically large problems. The UTD provides solutions for the scattering from metallic single wedges. However, when more complex geometries are involved, it may happen that one edge is positioned so that it is illuminated by the transition field of another edge. Since the transition field is not ray-optical, the cascaded application of UTD single wedge diffraction coefficients provides wrong results [100].

The problem of multiple-wedge diffraction, when various transition regions overlap, has been extensively studied in the literature by using various approaches [100, 218].

Only for double and triple diffraction, a rigorous UTD uniform description valid for any arbitrary configuration of wedges is present in the literature. As for UTD double diffraction, early results on the proper transitional behavior of the double diffracted field were discussed in [219, 220, 221, 87]. Schneider and Luebbers [222] provided an expression for the double diffraction mechanism for two separate single wedges. In [223] the authors discussed the problem of double diffraction by two parallel wedges giving a result valid also in the near-region. In [101] the authors extended the previous solution to the case of two wedges sharing a common face. Capolino *et al.* [88] provided a UTD coefficient for double diffraction mechanisms by two coplanar skew edges, under oblique incidence, obtaining results for the diffracted field that are valid in the near-region and were confirmed with experimental results described in [93, 94, 95]. In [224], these double diffraction mechanisms were repeatedly applied to evaluate the field diffracted past multiple knife edges. Finally, Albani in [89] analyzed the double diffraction mechanism for the geometry of two skew edges in arbitrary configuration and provided results that are valid in the near-field.

Various experiments were designed to cause a triple diffraction mechanism in order to compare measured results with theoretically computed values of the electric field by using the formulation discussed in [215].

Experimental Setup

Experiments were designed to test the triple diffraction coefficient, under a spherical wavefront with a TEM field incidence, and according to the scenario of Fig. 6.8, which is fundamentally two-dimensional, but with a three-dimensional spreading factor. The measurements were performed in the anechoic chamber at the University of Illinois at Chicago (UIC).

Metallic obstacles of different shapes were placed between a stationary transmitting antenna (TX) and a vertically moving receiving antenna (RX). This receiving antenna was connected to a linear positioner, which moved it by small, discrete steps along the x axis. A Vector Network Analyzer (VNA, Agilent N5222A) provided the input signal for the TX antenna and measured S_{21} by sampling the electromagnetic field on the RX side at each step. The frequency of operation was 25 GHz ($\lambda = 11.9$ mm) for all measurements.

Sector Antennas The antennas were chosen to serve two goals. One goal was to provide a spherical wavefront with a local quasi-TEM incident field at a short distance from the antennas, due to the small size of the anechoic chamber. The other goal was to suppress undesired lateral waves caused by the finite transverse size of the obstacles, as shown in Fig. 6.8, in order to simulate correctly a two-dimensional geometry.

These two goals can be achieved using two sets of sector antennas (Andrew Corporation, BCAA090-250 and BCAA090-250), one set for each (soft and hard) polarization. These antennas possess a Half Power Beam Width (HPBW) of 90° in the xz plane, and a very narrow HPBW ($\approx 3^\circ$) in the yz plane. The antenna patterns were measured using an automated antenna positioner (Orbit/FR AL-360-1P15) with positioning precision of 0.05 degrees. Fig. 6.9 shows the pattern of one of the soft-polarized antennas used in the experiments. In addition, the antennas can generate a local TEM wave at a distance of less than 1 m from their mouth. These two features allow to generate an incident electric field which has spherical phase fronts and uniform amplitude in the vertical (xz) plane, while strongly attenuating lateral waves contributions in the horizontal (yz) plane, as it is desired.

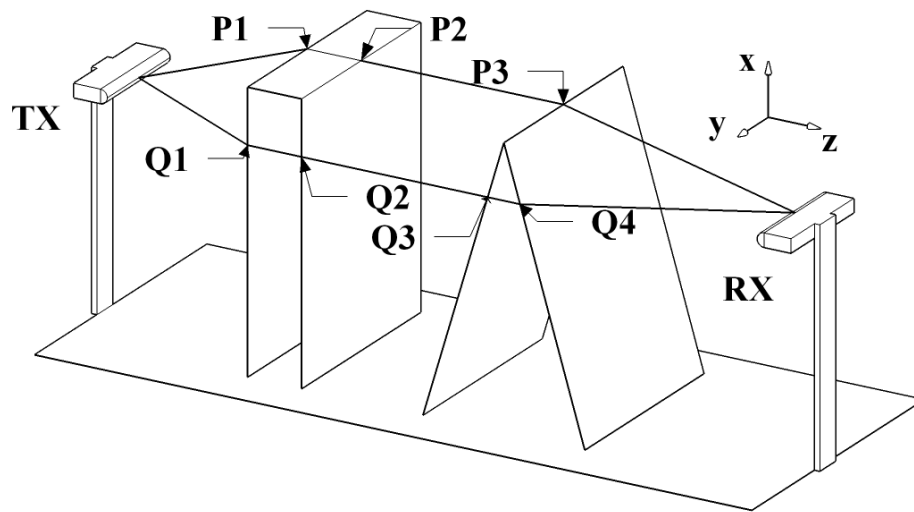


Figure 6.8: 3D sketch of the experimental setup. The trajectory $\text{TX} \rightarrow \text{P1} \rightarrow \text{P2} \rightarrow \text{P3} \rightarrow \text{RX}$ is contained in the xz plane and is of interest for the experiments. Instead, the trajectory $\text{TX} \rightarrow \text{Q1} \rightarrow \text{Q2} \rightarrow \text{Q3} \rightarrow \text{Q4} \rightarrow \text{RX}$ is an undesired lateral wave.

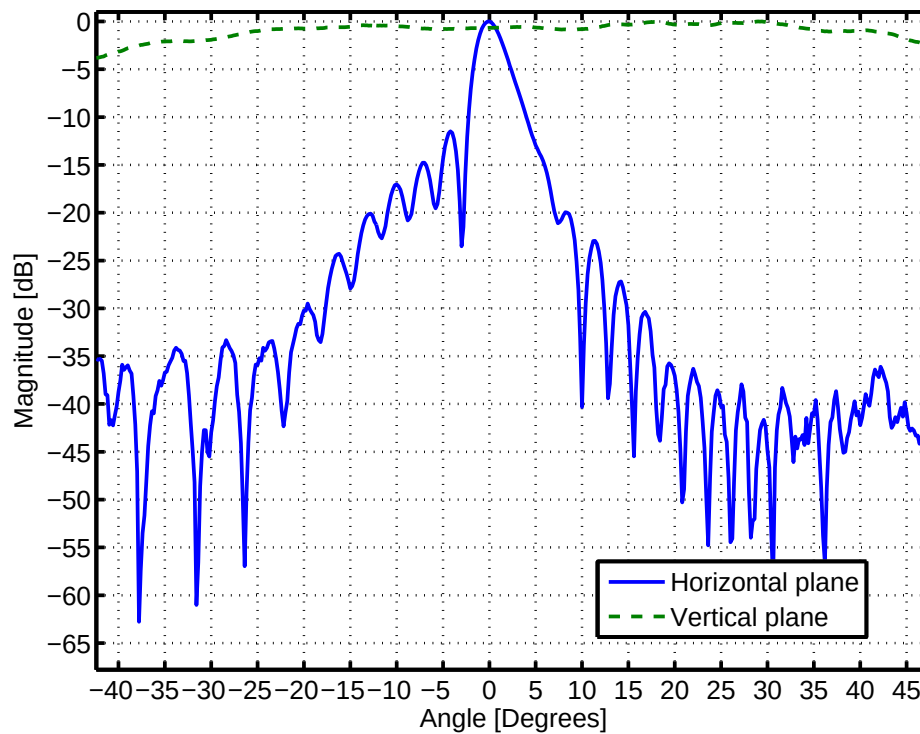


Figure 6.9: Far-field radiation pattern of the sector antenna in two orthogonal planes.

Diffracting Objects In order to create three wedges, two types of metallic obstacles were used, as shown in Fig. 6.10. The obstacles of the first type are steel objects of triangular lateral profile, 122 cm ($\approx 102\lambda$) wide and 85 cm ($\approx 71\lambda$) tall. The obstacles of the second type are copper parallelepipeds of rectangular transverse profile, 20.3 cm ($\approx 17\lambda$) deep, 76 cm ($\approx 63\lambda$) wide and 102 cm ($\approx 85\lambda$) tall. For both types of obstacles, the thickness of the metal was approximately 1 mm ($\approx 0.1\lambda$). When used in combination with

the chosen antennas, these objects are sufficiently wide to neglect undesired lateral ray contributions; the geometrical configurations have been chosen to guarantee an attenuation of at least 60 dB with respect to the main contribution.

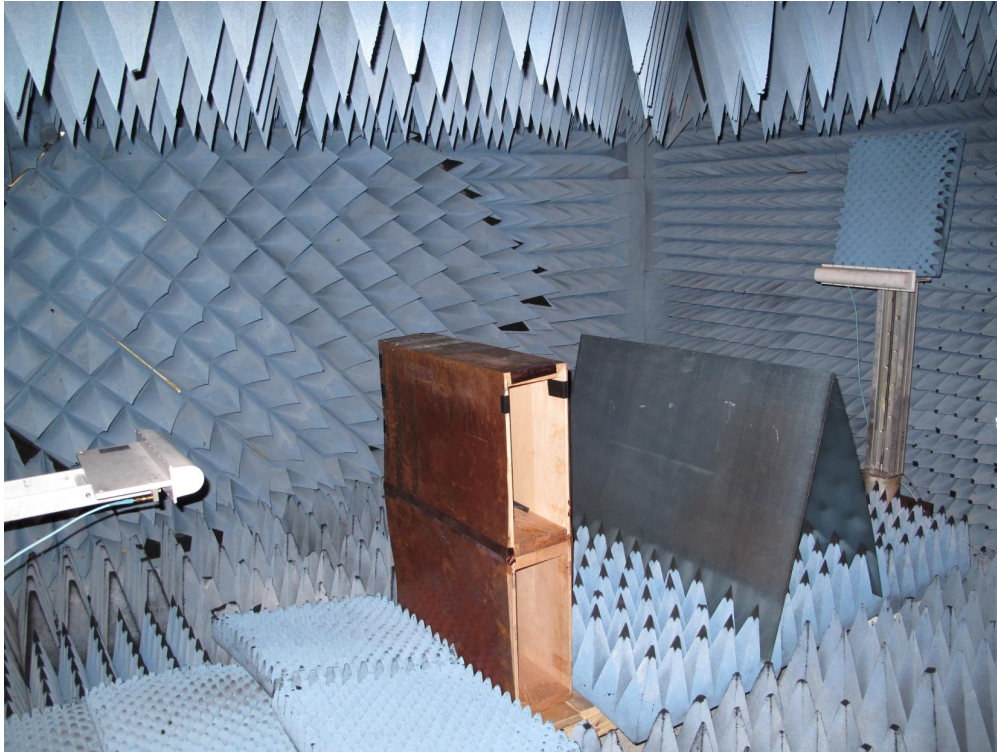


Figure 6.10: Anechoic chamber at UIC. Transmitting antenna (TX) and receiving antenna (RX) are located at each end of the room.

Triple Diffraction Measurements Diffracted fields from the three wedges were measured by reading the scattering parameter S_{21} from the VNA. All measurements were conducted above the system dynamic range (127 dB) and the receiver dynamic range (132 dB), in order to be able to measure the field even within the deep shadow region. Electromagnetic absorbers were placed everywhere around the obstacles and the antennas, in order to minimize spurious scattering and diffraction of the electromagnetic fields. Particular care was taken in minimizing the scattering originating from the metallic RX antenna positioner and from the bottom edges of the metallic obstacles.

While the TX was kept at a fixed position, the RX was moved from top to bottom by a precision linear positioner, (controller Velmex VXM-1, stepper motor American Precision 34D-9106C, lead screw UniSlide C P40), which provided a precision of $10\ \mu\text{m}$ ($\approx \lambda/1200$). The “zero” position corresponds to TX and RX aligned (same x and y position). Positive positions correspond to the RX being higher than the TX, and conversely for negative positions.

In order to obtain meaningful data, the actual positions of antennas and obstacles in the anechoic chamber must correspond as closely as possible to the one modeled in the analytic formulas. Therefore, the metallic obstacles were precisely positioned using a laser level (Craftsman Laser Trac), to make sure of the relative alignment of the obstacles with respect to both the other obstacles and the antennas. In addition, all distances were measured using a laser distance meter (Leica Disto Classic 5a), which guarantees a measurement error within 1.5 mm ($\approx \lambda/8$).

Four Case Studies Four case studies have been tested and are described in the following. For each case, the different geometries are created by arranging the two types of obstacles in different combinations and heights.

Case 1 : Three non-aligned triangular wedges. Each wedge is placed at a different height so that each Incident Shadow Boundary (ISB) is observed separately.

Case 2 : Three aligned triangular wedges. The triple diffraction UTD coefficient becomes extremely important, since the ISBs are overlapped.

Case 3 : A rectangular obstacle not aligned with a triangular obstacles. The 1st and 2nd wedge share a common face. The ray fields, that reach the third wedge, propagate at grazing of the common face between the 1st and 2nd wedge.

Case 4 : A rectangular obstacle aligned with a triangular obstacle. The 1st and 2nd wedge share a common face. The ray fields, that reach the third wedge, propagate again at grazing of the common face between the 1st and 2nd wedge. Furthermore, all the ISBs overlap.

A schematic representation of the four cases is shown in Fig. 6.11. The corresponding dimensions for all four cases have been summarized in Table 6.1.

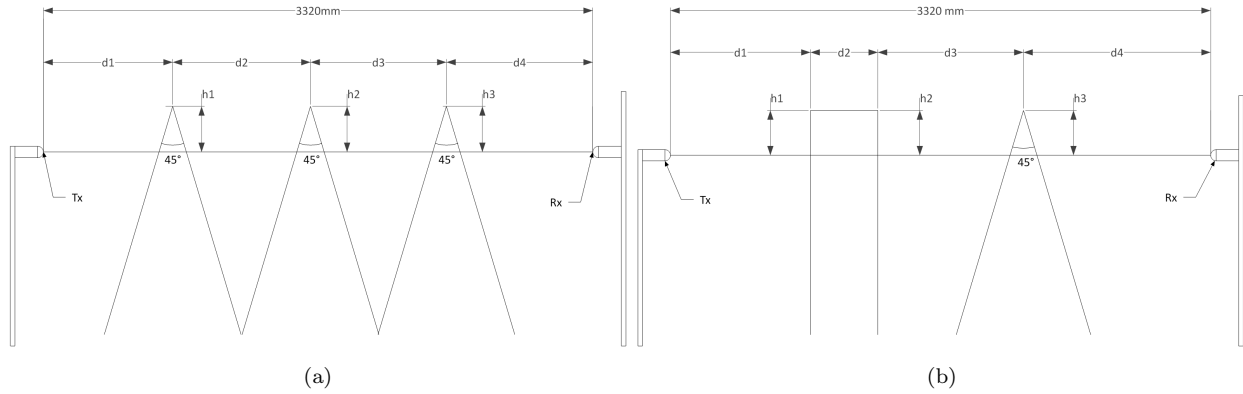


Figure 6.11: Lateral view of geometries for cases 1 and 2 (a) and cases 3 and 4 (b).

Table 6.1: Geometrical parameters and locations of the ISBs in millimeters and wavelengths. ISB2 and ISB3 in case 2 and case 4 overlap because of alignments.

Case	h1		h2		h3		d1		d2		d3		d4		ISB1		ISB2		ISB3	
	mm	λ	mm	λ	mm	λ	mm	λ	mm	λ	mm	λ	mm	λ	mm	λ	mm	λ	mm	λ
1	2	0.2	-5	-0.4	-42	-3.5	998	83	640	43	648	54	1034	86	-6.7	0.6	-23	-1.9	-101	-8.4
2	1	0.1	1	0.1	1	0.1	1000	83	670	56	570	48	1080	90	3.3	0.3	1	0.1	1	0.1
3	34	2.8	34	2.8	-16	1.3	1066	89	202	17	399	33	1653	138	106	8.8	34	2.8	-223	-19
4	2	0.2	2	0.2	2	0.2	1112	93	202	17	601	50	1405	117	6.0	0.5	2	0.2	2	0.2

Useful Definitions The measured field is plotted in terms of the normalized field $E_0(x)$, which is obtained from the appropriate component of the measured field in presence of the obstacles $\mathbf{E}(x)$, divided by the magnitude of the same component of the incident field measured for the same positions in free space $\mathbf{E}_{\text{freespace}}(x)$. This normalization to the field in free space was chosen because it has the physical meaning of additional propagation loss with respect to the free space loss. Namely,

$$E_0(x) = \frac{\mathbf{E}(x) \cdot \hat{\mathbf{n}}}{|\mathbf{E}_{\text{freespace}}(x) \cdot \hat{\mathbf{n}}|}, \hat{\mathbf{n}} = \begin{cases} \hat{\mathbf{x}} & \text{hard polarization} \\ \hat{\mathbf{y}} & \text{soft polarization} \end{cases}. \quad (6.7)$$

The field predicted using the UTD triple diffraction coefficient [215] is also normalized using Eq. (6.7).

Two polarizations are considered for each case: hard and soft. Hard polarization corresponds to the Neumann boundary condition, which occurs when the incident electric field is perpendicular to the diffracting edge and it is also known as vertical polarization. Soft polarization corresponds to the Dirichlet boundary condition, which occurs when the incident electric field is parallel to the diffracting edge and it is also known as horizontal polarization.

In all figures, the locations of ISBs are indicated with thick vertical black lines on the horizontal axis. In details, ISB1 represents the GO shadow boundary, where the singly diffracted field undergoes a transition; ISB2 represents the shadow boundary of the first order UTD solution, where doubly diffracted field undergoes a transition; and ISB3 represents the shadow boundary of the second order UTD solution, where triply diffracted field undergoes a transition.

Details of the implementation of the UTD triple diffraction coefficient can be found in [215]. In all computations it is assumed that fields propagate from the transmitter to the receiver, neglecting any backscattered contribution.

The agreement between the fields computed with the third order UTD solution and the measured fields was evaluated in terms of Mean Error (ME), Standard Deviation (SD) and Root Mean Square Error (RMSE). The ME is a good indicator for the overall behavior and cancels spurious contributions by averaging them out. The SD and RMSE require the computation of the square of the distance between the curve for the theoretical values and the curve for the measured values. The closer these curves are to each other, the smaller SD and RMSE are.

Accordingly, we first define the error between the measurement and the theoretical prediction as function of the receiver position along the x axis. For convenience, the position along the axis is identified with the auxiliary index p :

$$\Delta(p) = E_0^{\text{meas}}(p) - E_0^{\text{UTD3}}(p). \quad (6.8)$$

The subscript 0 indicates that the quantities were normalized to their respective incident field in free space values, as explained in Eq. (6.7); all values are expressed in dB. Then, the ME is computed as:

$$\text{ME} = \frac{1}{P} \sum_{x=1}^P \Delta(p), \quad (6.9)$$

where P indicates the total number of positions sampled during a linear scan.

The SD is computed as

$$\text{SD} = \left[\frac{1}{P-1} \sum_{p=1}^P (\Delta(p) - \text{ME})^2 \right]^{\frac{1}{2}}. \quad (6.10)$$

The RMSE is computed as

$$\text{RMSE} = \left[\frac{1}{P} \sum_{p=1}^P |\Delta(p)|^2 \right]^{\frac{1}{2}}. \quad (6.11)$$

The statistics defined in Eqs. (6.9)–(6.11) were calculated for all cases, and are summarized in Table 6.2, which shows excellent agreement between measurements and theoretical prediction. The ME is smaller than 1 dB in all cases except one, showing that the asymptotic behavior of the measurements follows with great precision the theoretical expectation. The SD is larger than 1 dB in only two cases, and the RMSE is always smaller than 2 dB, and smaller than 1 dB in half the cases tested.

Discussion of the experimental results

In Fig. 6.12 and in Fig. 6.13 the magnitude of the fields are shown. Thick black solid lines indicate measured fields, thick red dashed lines show the third order UTD fields computed by using the novel UTD triple diffraction coefficient [215]. Furthermore, thin blue dash lines report the field obtained by describing the double and triple diffracted rays with the simple cascading of standard (singly) diffraction UTD coefficient, as

Table 6.2: Mean Error, Standard Deviation, and Root Mean Square Error between UTD predicted field and measured field for each case.

Case	Polarization	ME [dB]	SD [dB]	RMSE [dB]
1	Soft	-0.11	1.10	1.10
	Hard	1.08	0.44	1.16
2	Soft	0.16	1.69	1.70
	Hard	0.35	0.60	0.69
3	Soft	-0.76	0.86	1.15
	Hard	0.64	0.27	0.70
4	Soft	-0.07	0.60	0.61
	Hard	0.86	0.17	0.88

it is frequently done in available codes. To emphasize the instability of the latter approach to the geometrical parameters, the thin magenta dash-dot lines represent the same formulation by cascading standard UTD solution when a negligible shift of the 2nd edge in the x direction of $+10^{-9}$ m ($\approx 10^{-7}\lambda$) slightly breaks the alignment of Case 2 and Case 4.

For all four cases, the measurements follow the theoretical expectation with very good accuracy. The measurements confirm the theoretical prediction in all regions, including the deep shadow on the left of the ISB3. The hard polarization of Case 1, Case 3, and Case 4 has a constant shift from the theoretical expectation. Authors believe this is due to errors in the normalization to the free space measurement, which affected the hard polarization more than the soft one.

Small ripples are observed for all cases, which are symptoms of multiple ray contributions caused by interference. These may be attributed to three contributions. The first contribution is due to reflections coming from an imperfect anechoic environment. Since the antenna pattern is very wide in the vertical plane, a large signal is sent towards both the floor and the ceiling. While the floor contribution is shielded by the obstacles themselves, the signal impinging on the ceiling is only attenuated by the EM absorbers, which allow some spurious reflection to reach the RX. The second contribution is due to interactions which occurred between the obstacle closest to the RX and the linear RX positioner itself. Relatively small ME and large SD in the soft polarization cases indicate larger error due to multiple ray contributions because (i) diffraction is weaker for soft polarization than for hard polarization so that the rays are comparable to the main contribution; and (ii) the exposed area of the metal frame of the receiver is quite large and cannot be covered by EM absorbers. The third contribution is due to lateral waves originating from the truncated sides of the obstacles, which are greatly attenuated by the antennas radiation pattern, but cannot be completely eliminated. The relative significance of these contributions increases with smaller measured fields, therefore we expect it to become significant only in the deeply shadowed region.

The comparison between the measurements and the cascaded first order UTD prediction shows, as expected, that only the proper UTD triple diffraction coefficient can correctly describe all the interactions among obstacles and the transitions between different shadow regions. In particular, since the direct contribution (geometrical optics) between TX and RX was blocked by the first obstacle, the cascaded first order UTD theory shows two sharp transitions at the locations of the second and third shadow boundaries. This happens for all cases and provides further motivations for the use of the UTD triple diffraction coefficient.

When all obstacles are aligned such as in Case 2 and Case 4, three ISBs are also aligned, that is, the received field crosses three incident shadow boundaries at once. While in a simulator it is trivial to set this condition, in reality it is practically impossible to guarantee perfect alignment. Very small errors in the actual alignment of the objects can lead to very large differences in the field computed by cascading the UTD single diffraction coefficients.

Fig. 6.12c and 6.12d show two different curves for the cascaded first order UTD. The first curve, Cascaded First order UTD 1, corresponds to an exactly aligned geometry. The second curve, Cascaded First order UTD 2, corresponds to the case where the middle obstacle has not been aligned properly, and it sits only

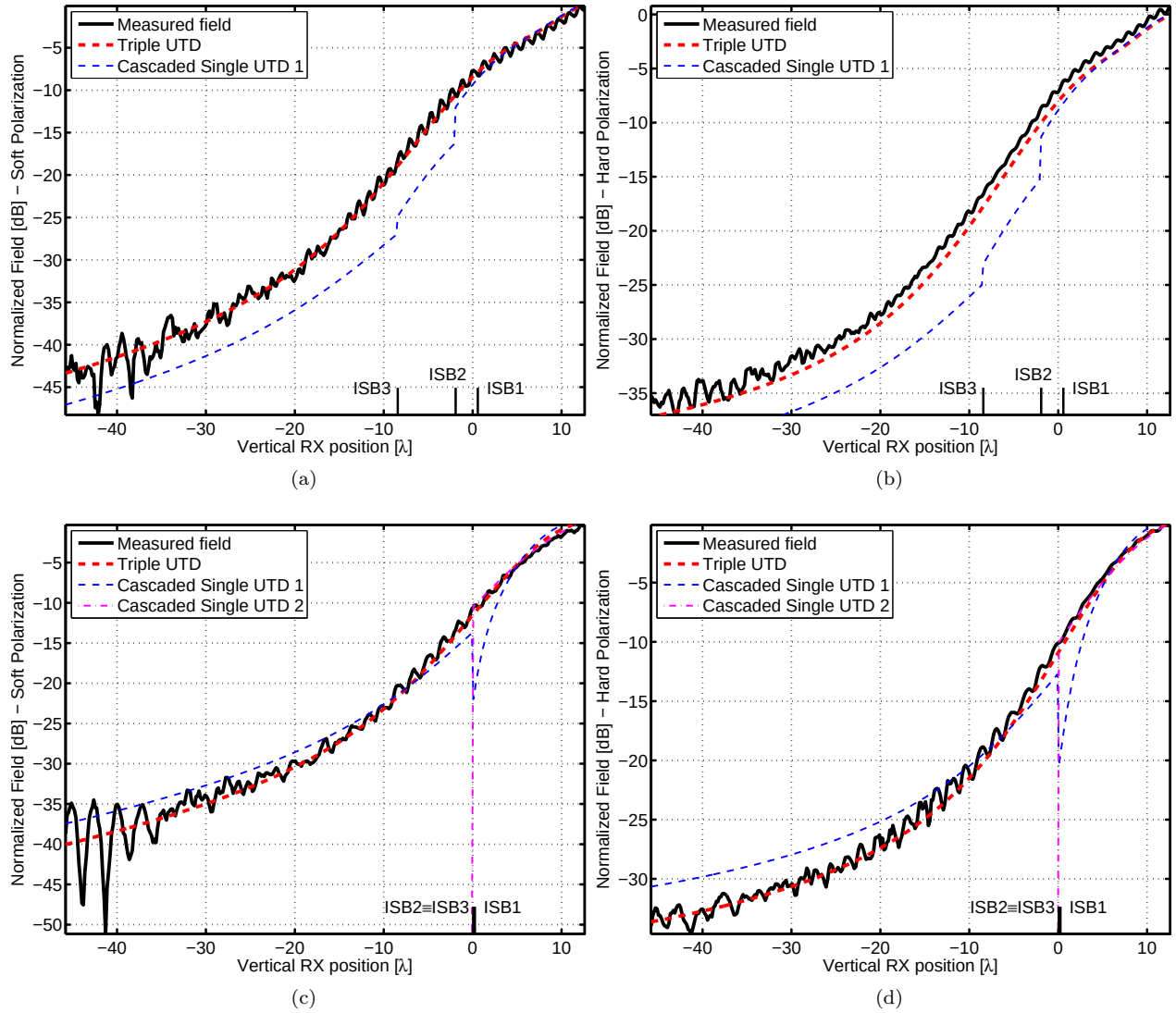


Figure 6.12: Normalized measured and calculated electric field, for the geometries involving obstacles with triangular cross-section. Case 1, (a) soft polarization and (b) hard polarization. Case 2, (c) soft polarization and (d) hard polarization; notice that the magenta dash-dot curve, corresponding to the cascaded first order UTD 2, exhibits a jump discontinuity, dropping to a very low level field, which is out of scale, when the receiver is at the ISB3.

10^{-9} m ($\approx 10^{-7}\lambda$) above the other two obstacles. In this case, the first curve indicates the field dropping before ISB3, then raising up again after the ISB3. On the other hand, because the second obstacle slightly changes the location of the second shadow boundary, the Fresnel transition function changes its behavior. As a result, the second curve indicates a rapid drop of the fields, which approaches a very low level in the dB scale of Fig. 6.12c and 6.12d. As a result, another argument in favor of the use of the UTD triple diffraction coefficient is to avoid the extreme sensitivity shown by the cascaded UTD single diffraction coefficients, for a movement as small as approximately $10^{-7}\lambda$. The reason of this not proper behavior of the multiple diffracted field by cascading UTD single wedge diffraction coefficient has been slightly discussed above in the paper. The reader, who is interested in this aspect, can find more details in [215], [100], [86]–[89].

Case 3 and Case 4 further validate the UTD triple diffraction coefficient. In such cases the prediction by using the cascaded UTD single diffraction coefficients is clearly wrong for soft polarization because, since

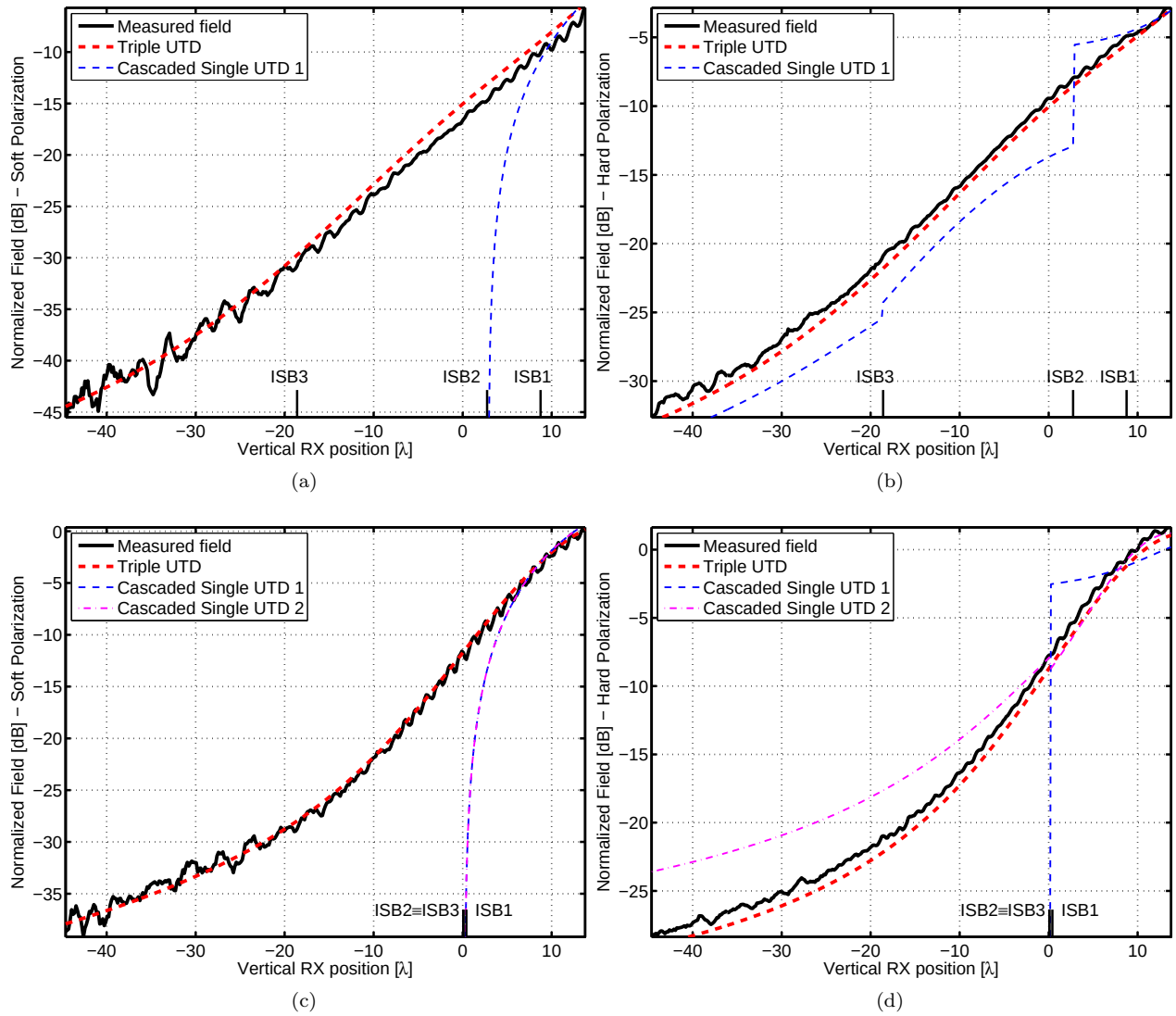


Figure 6.13: Normalized measured and calculated electric field, for the geometries involving obstacles with rectangular and triangular cross-sections. Case 3, (a) soft polarization and (b) hard polarization. Case 4, (c) soft polarization and (d) hard polarization. In Fig. 6.13c the two cascaded first order UTD results overlap.

the second edge is illuminated at grazing incidence by the field diffracted by the first edge which present a null at grazing, the double diffracted field by the first and the second edge and the triple diffracted field are zero. This also happens for the double diffracted field by the first and the third edge in Case 4, where all the edges are aligned. Also in Case 4 we reported the curve relevant to the scenario where the second edge was slightly moved of $\approx 10^{-7}\lambda$.

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1. Report Type

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Organization / Institution name

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Grant/Contract Title**The full title of the funded effort.**

Distributed RF systems for close-in sensing and imaging

Grant/Contract Number**AFOSR assigned control number. It must begin with "FA9550" or "F49620" or "FA2386".**

FA9550-12-1-0174

Principal Investigator Name**The full name of the principal investigator on the grant or contract.**

Danilo Erricolo

Program Manager**The AFOSR Program Manager currently assigned to the award**

Arje Nachman

Reporting Period Start Date

04/01/2012

Reporting Period End Date

12/31/2015

Abstract

The goal of this research is to improve close-in sensing and imaging applications through a new formulation of RF tomography, which is considered because of its flexibility and advantages when dealing with distributed sensors. This research is motivated by the removal of some limitations in the current formulation of RF tomography. One limitation is the first order Born approximation that is physically equivalent to neglecting multiple scattering phenomena. This limitation is overcome with the quadratic forward model. A second limitation is associated with the contrast function, which represents the unknown quantity to be reconstructed. In fact, while the initial formulations of RF Tomography have been based upon a scalar contrast function, it is possible to extract more information by using a dyadic contrast function, which takes advantage of the vector nature of both the incident and scattered electromagnetic field. A third limitation is the current existing inversion algorithms that do not allow for the introduction of prior knowledge to compute the solution of imaging problems, which is overcome with the introduction of physical bounds into iterative solution methods.

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Archival Publications (published) during reporting period:

1. V. Picco, G. Gennarelli, T. Negishi, F. Soldovieri, D. Erricolo, "Experimental Validation of the Quadratic Forward Model for RF Tomography," IEEE Geoscience and Remote Sensing Letters, Vol. 12, No. 7, July 2015, pp. 1461-1465.
2. T. Negishi, D. Erricolo, P.L.E. Uslenghi, "Metamaterial spheroidal cavity to enhance dipole radiation," IEEE Transactions on Antennas and Propagation, Vol. 63, No. 6, June 2015, pp. 2802-2807.
3. T. Negishi, V. Picco, D. Spitzer, D. Erricolo, G. Carluccio, F. Puggelli and M. Albani, "Measurements to Validate the UTD Triple Diffraction Coefficient," IEEE Transactions on Antennas and Propagation, Vol. 62, No. 7, July 2014, pp. 3723-3730.
4. P. Setlur, T. Negishi, N. Devroye, D. Erricolo, "Multipath Exploitation in Non-LOS Urban Synthetic Aperture Radar," IEEE Journal of Selected Topics in Signal Processing, Special Issue on Non-cooperative Localization Networks," Vol. 8, No. 1, Feb. 2014, pp. 137-152, doi 10.1109/JSTSP.2013.2287185
5. D. Erricolo and G. Carluccio, "Algorithm 934: Fortran 90 subroutines to compute Mathieu functions for complex values of the parameter," Association for Computing Machinery Transactions on Mathematical Software, Volume 40 Issue 1, Sept. 2013.
6. V. Picco, T. Negishi, S. Nishikata, D. Spitzer, and D. Erricolo, "RF Tomography in Free Space: Experimental Validation of the Forward Model and an Inversion Algorithm Based on the Algebraic Reconstruction Technique," International Journal of Antennas and Propagation, vol. 2013, Article ID 528347, 9 pages, 2013. doi:10.1155/2013/528347.
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8. H. T. Hayvaci, A. De Maio, D. Erricolo, "Improved Detection Probability of a Radar Target in the Presence of Multipath with Prior Knowledge of the Environment," IET Radar, Sonar & Navigation, Vol. 7 (1), 2013, pp. 36-46.
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Through-Wall Imaging, vol. 2012, Article ID 735414, 15 pages, 2012. doi:10.1155/2012/735414, <http://www.hindawi.com/journals/ijap/2012/735414/>.

11. L. Lo Monte, F. Soldovieri, D. Erricolo, M.C. Wicks, "Imaging Below Irregular Terrain Using RF Tomography," IEEE Trans. Geoscience and Remote Sensing, Vol. 50, No. 9, Sept. 2012, pp. 3364-3373.

Changes in research objectives (if any):

N.A.

Change in AFOSR Program Manager, if any:

There were two changes of AFOSR program manager.

(1) Dr. Tristan Ngyen who replaced the initial program manager, Dr. Jon Sjogren, at the beginning of year 2013.

(2) Dr. Arje Nachman, who replaced Dr. Tristan Nguyen on JULY 2, 2014.

Extensions granted or milestones slipped, if any:

A 9 months no-cost extension was granted in 2015 to compensate for delays in disbursement of the incremental funds.

AFOSR LRIR Number

LRIR Title

Reporting Period

Laboratory Task Manager

Program Officer

Research Objectives

Technical Summary

Funding Summary by Cost Category (by FY, \$K)

	Starting FY	FY+1	FY+2
Salary			
Equipment/Facilities			
Supplies			
Total			

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Appendix Documents

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