AC-6024

Reference:	WO 195/6020
Description:	Shaped Charges Sub-Committee: armour penetration by high velocity projectiles and Munroe jets
Date:	1944
Held by:	The National Archives, Kew
Former reference in its original department	AC 6024
Legal status:	Public Record
Closure status:	Open Document, Open Description

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SHAPED CHARGES SUB-COMMITTEE

Penetration of armour by high velocity projectiles and Munroe jets

by

R. Hill, N.F. Mott and D.C. Pack

Communicated by C.S.A.R.

Theoretical Research Report No.13/44
March 1944.
WEB.

Received March 30th, 1944.

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Armament Research Department

THEORETICAL RESEARCH REPORT NO. 13/44

Penetration of Armour by High Velocity Projectiles and Munroe Jets

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R. Hill, N.F. Mott & D.C. Pack

Branch for Theoretical Research,
Fort Halstead, Kent.

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Summary

Following a method due to Bethe, it is shown that the hydrostatic pressure p_0 necessary to enlarge a long cylindrical hole in a ductile material, which does not show strain hardening, is about 3.5Y where Y is the yield stress. A method for taking strain-hardening into account is given; for armour this increases p_0 by 8 per cent., for mild steel by about 30 per cent.. The plastic region extends in steel to a distance of about twelve times the radius of the hole.

Calculations show also that the pressure necessary to enlarge a spherical hole in steel is 4Y, which is only slightly greater than for the cylindrical case. It is suggested that the pressure on the nose of a lubricated punch deep in a semi-infinite block of material, will lie between 3.5Y and 4Y, and the resistance to penetration will thus depend little on the shape of the punch, though a sharp punch will have a slight advantage. Taking strain hardening into account, the resistance in armour should be of the order

4.5Y x area of cross section of punch

for a blunt punch, and 10 to 15 per cent. less for a pointed cone of small semi-angle.

These results are valid only after a penetration of several calibres into a semi-infinite block. Near the surfaces of a slab the resistance to penetration will be less.

The dependence of the resistance on velocity is discussed. The yield stress of many materials is known to be different in static and dynamic tests, and for the interpretation of dynamic penetration Y must be taken to be the dynamic yield. For armour there is not much difference between the static and dynamic yield. For no material will Y vary much over the range of rates of strain of interest in armour penetration.

The validity of the Poncelet formula

$a + bv^2$

for the resistance is discussed. It is shown that for shot with conical heads a formula of this type is likely to be valid, and the work done against the dynamic term bv^2 is used up in making a hole of diameter greater than the shot. The same is true of shot of the usual ogival shapes, and for spheres, if the velocity lies above a certain critical value proportional to $\sqrt{Y}$; for values of the velocity below this, there is no large dynamic term and the hole will have approximately the diameter of the shot.

Steel balls recently fired into dural by R.R.L. with velocities up to 5000 ft./sec. must have had velocities above the critical value; tapering holes were observed, and the analysis of the results indicated agreement with a Poncelet formula. On the other hand hyper-velocity tungsten carbide shot in steel give holes with no tapering, and an analysis of firing results shows that there is no large dynamic term.

A discussion of the de Marre formula is given.

Shatter is accounted for in terms of a dynamic pressure which acts only while the nose of the shot is penetrating the plate. It is shown why this dynamic pressure does not affect the total penetration.

The laws applicable to penetration by shot are modified to apply to penetration by Munroe jets. It is shown that the depth of penetration is proportional to the length of the jet and depends on the density of metal in it and on the density of the target.

On the other hand, it is almost independent of the velocity of the jet and the yield strength of the target, which are the factors determining the volume of the hole. Some calculations are given which point to the conclusion that the mean density of a jet from a steel cone is roughly one-third that of the steel lining itself.

1. Introduction

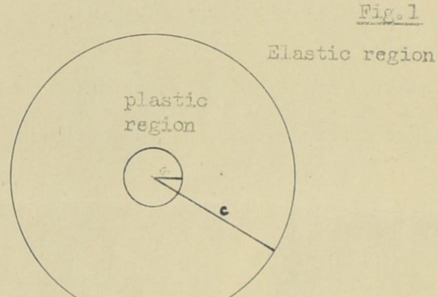
It is notoriously difficult to give any exact mathematical theory of the penetration of armour by armour-piercing shot. Attempts have been made for two limiting cases; these are, for plates of thickness t large compared with the calibre d of the shot, and the opposite case where t is small (refs.1, 2 and 3). For steel shot against armour in the cases of practical importance, t/d is from 2 to 3 for normal attack. In this report we shall attempt an analysis for the case when t/d is large (holes deep compared with their width). The analysis is applied to penetration by hyper-velocity tungsten carbide projectiles (where $t/d \sim 7$) and to Munroe jets. It also enables an improved qualitative understanding to be obtained of the penetration laws for smaller t/d.

In sections 2 to 5 we consider static punching into a semi-infinite block by a lubricated punch. The method used by Bethe (ref.1) is extended and justified, and work-hardening is taken into account. In section 6 we discuss the conditions under which there will be a dynamic term in the resistance to penetration. Section 7 discusses the experimental evidence which is available, and section 8 the phenomenon of shatter. Sections 9 - 11 deal with penetration by Munroe jets.

2. The Long Cylindrical Hole

It is not, at present, possible to give an exact solution for the flow round a punch being driven into a ductile material, and we discuss, therefore, certain simplified problems for which an exact solution can be obtained. The first of these is Bethe's solution for a long cylindrical hole.

Fig.1



Plastic and elastic regions round cylindrical and spherical holes.

A long cylindrical hole is supposed to be drilled in an infinite block of material, and a pressure p_0 applied to the surface, so that plastic flow occurs and there is a (large) increase in diameter. Details of the solution are given in Appendix I of this report. The treatment follows that given by Bethe (ref.1).

For our purpose we need only consider the radius of the hole to be initially zero and to be enlarged to a final value a. Then it is shown in Appendix I that the cylindrical hole of radius a is surrounded by a concentric cylinder of radius c within which plastic flow has occurred. Assuming the plastic volume changes to be negligible, c is given by

$$\frac{c^2}{a^2} = \frac{\sqrt{3}}{2(1+\nu)} \frac{E}{Y} \qquad (1)$$

where E is Young's Modulus, ν is Poisson's ratio and Y is the flow stress in tension. Assuming that there is no strain hardening, the pressure p_0 required to form the hole is

$$p_0 = \frac{Y}{\sqrt{3}} \left(1 + 2 \log_e \frac{c}{a}\right) \qquad (2)$$

and the energy per unit length $\pi p_0 a^2$. This energy is divided, as follows, into elastic energy and that used up in causing plastic flow:-

$\pi a^2 Y/\sqrt{3}$	Elastic
$\frac{2\pi a^2 Y}{\sqrt{3}} \log_e \left(\frac{c}{a}\right)$	Plastic

For steel the plastic energy is about five times the elastic. Half the total elastic energy is in the plastic region.

More than half the energy is actually contained outside a cylinder of radius 2a. If small plastic volume changes be allowed for, (2) remains true, and (1) becomes

$$\frac{c^2}{a^2} = \frac{\sqrt{3}}{(5 - 4\nu)} \frac{E}{Y} \qquad (3)$$

Numerical values are for steel, with $\nu = 0.3$, $E = 3 \times 10^7$ lb./sq.in., $Y = 10^5$ lb./sq.in.

$$\frac{c}{a} = 12, \qquad p_0 = 3.4\,Y$$

Since much of the energy is expended at distances from the axis equal to several multiples of a, only a fairly large specimen can be treated as infinite. The pressure required to form a cylindrical hole of radius a in a cylinder of radius b is given by

$$p_0 = \frac{Y}{\sqrt{3}}\left(1 + 2\log\frac{c}{a} - \frac{c^2}{b^2}\right), \qquad (4)$$

if $b > c$, c being given approximately by (3) as before (Appendix III). The pressure now depends on the radius of the hole. The energy required to make a hole of radius a is

$$\frac{\pi a^2 Y}{\sqrt{3}}\left(1 + 2\log\frac{c}{a} - \frac{1}{2}\frac{c^2}{b^2}\right) \qquad (5)$$

If $b \sim c$, and is thus about 12 times the diameter of the hole, the energy is decreased by about 9 per cent..

To obtain these results it has been assumed that the target material is elastic up to the yield point, and that from there on the flow stress is constant. In Appendix IV we show how to take account of strain hardening, the following assumptions being made:-

(a) There is no volume change in plastic flow,
(b) The material has a definite yield point,

above which the conventional strain s is given in terms of the true stress S as found in a tensile test by

$$S = Y + A\log_e(1 + s) \qquad (6)$$

With these assumptions the pressure p_0 needed to enlarge the hole is found to be

$$p_0 = \frac{Y}{\sqrt{3}}\left(1 + 2\log_e\frac{c}{a}\right) + \frac{\pi^2 A}{18} \qquad (7)$$

where

$$\frac{c^2}{a^2} = \frac{\sqrt{3}\,E}{2(1+\nu)Y}$$

The relation (6) is in good agreement with experiment for armour steels; Bridgman (ref. 4) gives

$$\left.\begin{array}{l} Y = 10{,}000 \\ A = 5{,}000 \end{array}\right\} \text{kg./sq.cm.}$$

the relation holding up to the very large strains achieved by Bridgman under high pressure. For these values of the constants, strain hardening increases p_0 by 5 per cent. only. Mild steel hardens more rapidly than this, and the increase in p_0 may perhaps be of the order 30 per cent..

3. Spherical Hole

A solution similar to that for the cylindrical hole can be given for the pressure required to form a spherical hole of radius a in an infinite block, starting from a hole of negligible size (Appendix II). The radius c of the plastic sphere round the hole is given by

$$\frac{c}{a} = \sqrt[3]{\frac{E}{3(1-\nu)Y}} \qquad (8)$$

and the pressure by

$$p_o = \frac{2Y}{3}\left(1 + 3\log_e \frac{c}{a}\right) \qquad (9)$$

Numerical values for steel are

$$\frac{c}{a} = 5.2, \qquad p_o = 4.0\,Y$$

It will be noticed that the pressures for the cylinder and sphere do not differ much:

$$p(\text{sphere})/p(\text{cylinder}) = 1.18$$

4. Application to Static Punching

Bethe (ref.1) in his theory of armour penetration assumed that formula (2) would give the pressure acting on the ogive of a shot if the velocity were low and friction neglected. This assumption is valid only for shot or punches with very sharp pointed heads, of semi-angle α such that

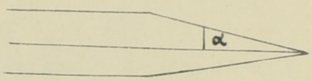


Fig.2

$$\alpha \ll a/c$$

The steel then flows radially away from the axis of the punch. For a real ogive or head of a punch, however, of whatever shape, there will be motion of the target material forward as well as laterally. If the ogive be sufficiently deep into the target for surface effects to be no longer of importance, it is to be expected that the mode of enlargement of the hole will be a compromise between the purely lateral expansion of Bethe and the radial expansion treated in the preceding section. Since the value of p_o differs but little in the two cases, we may assume that the pressure p_o is approximately uniform over the ogive and independent of its shape. This implies that the energy required to punch a narrow hole will, for sufficiently deep penetrations, tend to be proportional to the volume of the hole, provided that the punch be lubricated or treated in some way such that the friction between the surfaces of hole and punch is eliminated.

In dynamic penetrations, (firing trials), the friction is believed to be negligible owing to local melting. This point is of importance when coupled with the considerations of para.6, where comparison is made between static and dynamic penetration, and where it is suggested that p_o is effectively the same in the two cases.

Allowing for work-hardening (para.3), p_o/Y should lie between 4 and 4.5 for armour, and perhaps 5 to 5.5 for mild steel. Y is always to be identified with the yield stress. A conical head of small semi-angle should have a small advantage, of order 10 per cent., over a blunt head.

5. Effect of the Surface

Near the surface the pressure on the head of a punch will necessarily be less than when the punch has penetrated far into the body of a semi-infinite block, because the material flows parallel to the hole as well as outwards, giving a lip to the crater, as shown in Fig.3. Since much of the energy expended is used up in cold work several calibres from the hole, it may be expected that a penetration of several calibres is necessary before the effect of the surface becomes negligible.

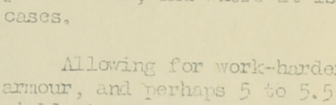


Fig.3

We do not know of any mathematical analysis of the surface effect for a semi-infinite block. G.I. Taylor (ref.3) has given an analysis of the work required to enlarge, by lateral pressure, a circular hole in a thin plate. He finds that the energy required is

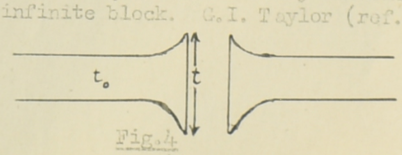


Fig.4

$$1.3Y \times \text{area of hole} \times \text{plate thickness } t_o$$

which is about one-third the equivalent quantity for a thick plate. The height of the crater is about .8 x the thickness of plate. Bethe (ref.1) makes a different stress-strain assumption and obtains a value some 50 per cent. greater than Taylor's for the work done.

It is desirable that experiments on static punching of "semi-infinite" blocks should be made to determine the effect of the surface; it would be necessary in this work to eliminate the very considerable frictional term by lubricating the punch. Such experiments should give load-penetration curves as shown in Fig. 5; the zero of the penetration is taken when the ogive of the punch is fully within the material. One expects the load to approach its ultimate value after a penetration of a few calibres.

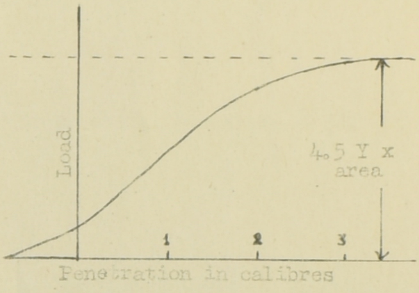


Fig. 5

6. Existence of a Dynamic Term

We have now to consider the way in which the force resisting penetration depends on the velocity V of the projectile.

The yield stress Y, of many materials, varies with the rate of strain; there are, however, many indications (refs. 5 and 6) that the relation for large rates of strain is of the type

$$Y = Y_0 + A \log (\text{rate of strain})$$

and therefore, although the flow stress may be quite different in static and dynamic punching, it will vary little over the range from 1000 to 10,000 ft./sec. of interest in armour penetration. Over this range then, we shall consider Y constant. For good armour steels there appears to be little difference between the static and dynamic yield (ref. 7). Thus the increase in resistance due to this cause will be effectively constant over a wide range of velocities.

We consider, next, under what conditions an appreciable dynamic term, varying rapidly with velocity, is likely to arise.

The formula known as Poncelet's formula

$$\text{Force} = a + bV^2$$

is often assumed for the resistance to penetration[x]. On the experimental side it has been shown by the Road Research Laboratory to give good agreement with experiment for the penetration of steel balls into dural (ref. 9); on the other hand a series of reports by Dr. Baines (ref. 10) has shown that the energy required to penetrate a steel plate statically, with lubricated shot, agrees well with the kinetic energy of the shot which will just penetrate it. Dr. Baines' work thus suggests that when the target is armour, there is no large dynamic term.

The solution to this contradiction is as follows: let us consider what happens for very high velocities ($\rho V^2 >> Y$) when the projectile is deep in the target. The resistance to flow of the material will then be negligible, and we shall treat it as a liquid. The resistance to flow is then $C \rho V^2$ x area of projectile where ρ is the density of the target material and C a drag coefficient of order about 0.1.

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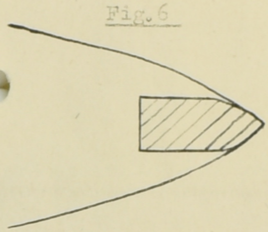
[x] cf. the recent report by Walters and Rosenhead, who show that the formula is in agreement with experiment for targets of wood or strawboard (ref. 8). As these are not ductile materials, the arguments of this report are not applicable.

Fig. 6



The energy given up by the projectile against this resistance is used in setting the fluid in motion, and cavitation occurs behind the projectile as in the sketch. The pressure at the nose of the projectile is $\frac{1}{2}\rho V^2$; the pressure drops along the surface until a value zero is reached where cavitation begins.

We now consider what happens in the case of actual interest where Y and ρV^2 are comparable. In the static case the pressure is approximately equal to p_0 all over the nose of the shot. As the velocity increases the pressure will increase near the nose and decrease near the shoulder. As long as the value of the pressure does not reach zero, the diameter of the hole made by the shot will be nowhere bigger than that of the shot; under these conditions the work done by the shot in making the hole will be almost the same as in static punching, and a large term cannot occur. It is true that the flow lines round the nose of the shot may be somewhat different in the dynamic case, but since most of the energy in making the hole is expended at some distance from it, any increase in the energy due to this will be small.

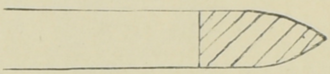


Fig. 7

Steel penetrated by A.P. or hyper-velocity shot shows holes no bigger than the shot. Thus, no dynamic term is to be expected.

At some critical value of $\rho V^2/Y$ the pressure at the shoulder of the shot will become zero, and "cavitation" will begin, a hole of greater diameter than the shot being formed. Above this velocity a dynamic term in the resistance will appear, the resistance increasing with the velocity. Examination of the holes shown in section in the R.R.L. report for steel balls in dural, show diameters which increase towards the entrance. This then, may be the explanation of the dynamic term found in these experiments; the critical value of $\rho V^2/Y$ must have been exceeded.

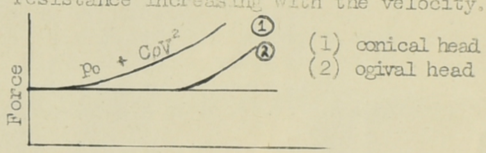


Fig. 8 →

The critical value of $\rho V^2/Y$ will depend on the shape of the head. In particular, it will be zero for a conical head. Elementary considerations of centrifugal force show that the material of the target cannot follow the surface of the punch round a sharp corner. Finally, for velocities such that a dynamic term does exist, the extra work done against it must be used up in expanding the hole against the pressure p_0. Therefore, as long as we are far from the surface, we may assume the following approximate relation

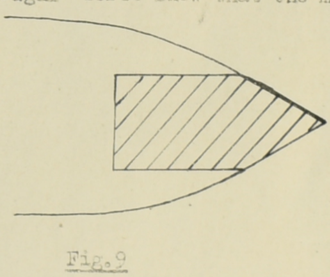


Fig. 9

$p_0 \times$ area of hole $=$ work done by projectile per unit length of hole.

The above considerations apply only to the resistance to penetration far from the surface. Near the surface the form of the lip of the crater may depend on $\rho V^2/Y$, and this may mean a rather larger dependence on V.

We may compare Taylor's solution (ref. 3) for the enlargement at high speeds of a circular hole in a thin sheet by a force applied radially. Taylor finds that the ratio h/h_0 of the thickness of the metal at the rim of the hole to its original thickness is

$$\frac{h}{h_0} = \sqrt{\frac{2}{y}}\,\sinh(\sqrt{2}\,y) \qquad y = \frac{\rho V^2}{Y}$$

$$\simeq 2\left(1 + \frac{1}{3}\,\frac{\rho V^2}{Y} + \ldots\right)$$

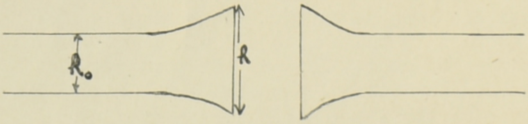


Fig. 10

Thus the shape of the rim depends on velocity, as does the energy

$$h\,Y \times \text{area}$$

required to make the hole.

An effect of this type, however, can, at most, increase the pressure near the surface to a value near to that in the body of the material. Baines' results also indicate that it cannot be very large. It would be of great interest to compare experimentally the static and dynamic energies required to penetrate a thick plate to a depth of one or two calibres.

Bethe (ref.1), in his analysis of the force on a projectile penetrating a thick plate, found a dynamic term of the Poncelet type. His analysis is, however, only applicable when the correction term is small compared with the static resistance, and when the projectile has a long, narrow head. Bethe does not give any treatment which would apply at all adequately to ogives of normal shape, nor does he consider circumstances where the hole is wider than the projectile. His results do not therefore shed any light on the ideas put forward here. The additional energy in Bethe's dynamic solution is used up in exciting elastic vibrations. In this report it will be assumed that the energy lost in this way is small.

7. Experimental Evidence

The only experimental evidence for A.P. shot known to the authors, in which the range of t/d is sufficiently large for the results to have a direct bearing on these ideas, is given by analysis of penetration of armour plate by tungsten-carbide cores (ref.11). If $MV^2 \cos\theta/d^3$ is plotted against t/d (for a given angle of incidence θ) as in the graph (Fig.15), it will be seen that the graphs are straight lines for $t/d > 2$ or 3 (to within the limits of experimental accuracy). Although the firings did not give data for low velocities, the graphs presumably extend back to the origin along a curve, representing the surface effect. The straight line part of the graphs shows that after a certain depth the resistance to penetration is constant. This is what would be expected from the observed fact that the holes made by the cores are no larger (in cross-section) than the cores themselves.

The graphs were constructed by fitting the best straight lines to firing data of English tungsten carbide. The data from firings of German tungsten carbide was plotted subsequently, and, fortuitously perhaps, fits closely to the same straight lines. Values of the constant resistance, deduced from the slopes of the lines show a remarkable consistency at about 350 tons/sq.in.. This is the reason for plotting $\frac{MV^2}{d^3}\cos\theta$, and not $\frac{MV^2}{d^3}\cos^2\theta$ as in the modified De Marre formula. Of course, the angle of entry is not strictly equal to θ but somewhat greater, and slight allowance must be made for this. Taking the static flow stress of good armour plate to be about 60 - 70 tons/sq.in., there is good agreement with theory when allowance is made for work-hardening and rate of strain.

The experiments of Baines (ref.10), in which work done in lubricated static punching is shown to be effectively equal to the energy loss in dynamic firings, support, in a general way, the ideas put forward here. The range of t/d, however, is not great enough to separate surface effects from the value of p_0 for the body of the material. The correspondence found with lubricated punches between the static and dynamic values of the energy is interesting. It may mean that the dynamic surface effect is small, or that it is compensated by a smaller expenditure of energy in the dynamic case in producing a disc or scab at the back surface. It would be interesting to see whether the static and dynamic energies required to make a hole of given depth in a thick plate are identical.

8. Shatter

The phenomenon of shatter shows that the force acting on the nose of a shot while the ogive is penetrating the surface does increase with velocity, and above some critical velocity becomes too great for the shot to stand. We have seen in the last section that near the surface some dependence of resistance to penetration on velocity is to be expected. This may be partly responsible for shatter.

While the ogive is penetrating the target, moreover, there will be an additional dynamic term due to the fact that more and more of the target material is set in motion. It is not possible, at present, to form any estimate of the magnitude of this term or of its exact dependence on the shape of ogive; we should, however, expect that it would be greater for blunt ogives since for these the material of the target has to be set more rapidly in motion than for pointed ones.

It is clear that this second dynamic term will increase indefinitely as V increases, while the first can only increase the pressure up to the value of p_0 holding far from the surface.

For this reason it may not be accurate to represent the pressure on the projectile at a given distance from the surface in the form $A + \beta\rho V^2$. However, we shall, as a first approximation, make use of this form.

The conditions for fracture of a hard steel are not known with certainty; the most natural assumption is that fracture (shatter) will occur when the maximum stress difference reaches a critical value. The maximum stress difference will be less than the pressure on the projectile, and may be written

$$q(A + \beta\rho V^2) \qquad q < 1$$

The unknowns q and β could be determined for a given head shape by determining the shatter velocity for steels of two different compressive strengths.

Finally, it must be pointed out that the dynamic term which arises because more and more matter is set into motion as the shot penetrates will probably not have any large effect on the penetration. According to the hypothesis on which this report is based, most of the kinetic energy of the shot is used up in plastic deformation of the metal; thus any energy which initially is transferred to the target material as kinetic energy must ultimately be used in the same way, that is, in making the hole. Alternatively we may consider that the shot in the material has a fictitious mass, in the same way that a shot moving in a fluid has. This fictitious mass will decrease the deceleration and thus increase the penetration, and so make up for the loss of kinetic energy due to the dynamic term at the surface, at any rate approximately.

9. Penetration by Munroe jets

In the preceding paragraphs we have discussed the penetration of ductile materials by projectiles which are assumed to be rigid; for this purpose we have had to estimate the total force on a projectile moving with a known velocity V. We have found that the pressure at the nose of the projectile always increases with increasing V; that near the shoulder it may decrease, so that the resulting force may or may not increase with V.

We have now to consider penetration by a projectile or other attacking agent that flows or breaks up on striking the target. The obvious examples are:

(a) An A.P. shot above the shatter velocity.
(b) A Munroe jet. The physical state of the metal in these jets is not yet certain and we shall consider

(i) a jet of liquid or solid metal, the flow stress being negligible in the latter case,
(ii) a jet of fragments which break up on impact.

The case of the Munroe jet is easier to treat, since the velocities are higher, and will be referred to explicitly. It provides interesting phenomena to consider in the light of the hypotheses of the previous sections. All the experimental evidence points to the penetration being achieved by lateral expansion of the target material, and the area of the hole the jet produces may be several times the area of the jet itself, suggesting that the very high velocities obtained are well in excess of the critical velocity discussed in para.6.

Following a previous report (ref.12), we treat the penetration in the following way: the jet of material, moving with velocity V, makes a hole of greater diameter than its own. The bottom of the hole moves with velocity U, which is less than the velocity V of the jet; the material of the jet, which breakes up on hitting

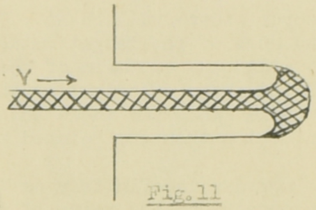


Fig. 11

the bottom of the hole, is pushed away and escapes up the side of the hole. The velocity U can be determined by equating the pressure exerted by the jet to the pressure necessary to drive the hole into the material with velocity U. It is only necessary to equate the pressures at one point, which can most conveniently be taken on the axis.

If L is the length of the jet, the time during which pressure is exerted on the bottom of the hole is $L/(V - U)$. Therefore the depth t of penetration is given by

$$t = L\,U/(V - U) \qquad (10)$$

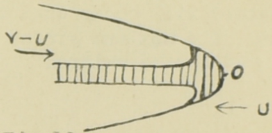


Fig. 12

To find the pressure on both sides of the hole we take axes through O, the bottom of the hole; relative to these axes the jet moves with velocity V - U, the material of the target with velocity U in the directions shown in Fig. 12. In the limiting case where U is very large, the material of the target can be treated as an inviscid fluid; the pressure that it exerts at O (the stagnation point) will then be, if ρ_T is the density of the target material,

$$\tfrac{1}{2}\,\rho_T\,U^2 \qquad (11)$$

A correcting term of some sort must be introduced to account for the strength of the material; if we write for the pressure

$$\tfrac{1}{2}\,\rho_T\,U^2 + p_0 \qquad (12)$$

where p_0 as in paras. 2 and 3 is about four times the flow stress, we have at least a formula which is correct for $U = 0$ and $U \to \infty$. For Munroe jets the correcting term is in any case small and failing more exact analysis we shall assume (12) to be correct.

A correcting term may also be required for the compressibility of the target material, assumed incompressible in the derivation of formula (11). This is mentioned in a footnote to the report quoted above (ref. 12).

The pressure exerted at O by the jet will depend on its nature. For a liquid jet or solid jet of negligible flow stress, the pressure will be

$$\tfrac{1}{2}\,\rho_j\,(V - U)^2 \qquad (13)$$

where ρ_j is the density of the metal of the jet. If the flow or shatter stress is not negligible (e.g. for A.P. shot above the shatter velocity), a term as in (12) will have to be added to take account of this, and also the velocity V will decrease during the process of penetration. We shall not attempt a quantitative treatment of this case.

For a fragment jet in which the volume of metal is a small proportion of the total volume of the jet the pressure is

$$\rho_j\,(V - U)^2$$

where ρ_j is the mass per unit volume of jet. If the fragments are pressed close together, a reasonable assumption is that the pressure should be

$$\tfrac{1}{2}\,\lambda\,\rho_j\,(V - U)^2 \qquad (14)$$

where λ is a numerical factor between 1 and 2.

Equating (12) and (14) we have

$$\rho_T\,U^2 + 2\,p_0 = \rho'_j\,(V - U)^2\,, \qquad (15)$$

where $\rho'_j = \lambda\rho_j$.

This quadratic gives U, and together with (10) determines the depth of penetration.

Three cases are of special interest:

(a) p_0 negligible compared with $\rho_j V^2$. In this case the penetration is given by

$$t = L(\rho'_j/\rho_T)^{\frac{1}{2}} \qquad (16)$$

We note that the depth of penetration is independent of the velocity of the jet.

(b) p_0 small; to the first approximation in ascending powers of p_0

$$t = L\left\{\frac{\rho'_j}{\rho_T}\right\}^{\frac{1}{2}} \left\{1 - \frac{p_0}{\rho'_j V^2}\left(1 + \sqrt{\frac{\rho'_j}{\rho_T}}\right)^2\right\} \qquad (17)$$

(c) Jets of low density, so that $U \ll V$. Then

$$U = \left\{\rho'_j V^2 - 2 p_0\right\}^{\frac{1}{2}} / \sqrt{\rho_T} \qquad (18)$$

It is clear that the approximations made in deriving these formulae break down unless $\rho'_j V^2 > 2 p_0$; if this is not the case we shall have to consider penetration by individual fragments.

As regards orders of magnitude, if ρ is the density of steel and V is 10,000 ft./sec.,

$$\tfrac{1}{2} \rho V^2 = 2400 \text{ tons/sq.in.}$$

p_0 is of the order 200 tons/sq.in. for steel.

10. The density of the jet

In order to see if some light could be shed on the density of a Munroe jet compared with the density of the lining from which it is formed, (a question on which there has been considerable speculation, but no real evidence) some calculations have been performed, based on the experimental curves reproduced in S.R.7/43/343 (ref.13). In the latter report, the velocities of emergence from plates of different thicknesses and the times of penetration have been plotted against penetrations, for steel cones fired against targets of both aluminium and copper.

We use the steady state penetration law

$$\frac{V}{U} = 1 + \sqrt{\frac{\rho_T}{\rho'_j}} \; ,$$

where $\rho'_j = \lambda \rho_j$ (equation 15), ρ_j is the density of jet and $1 < \lambda < 2$, and apply it to successive time intervals during the penetration, assuming that the velocity of emergence plotted in the graphs is the actual velocity of the jet at the corresponding depth of penetration into a semi-infinite target. This is tantamount to asserting the principle of independence of action of the different parts of the jet, of which there is experimental evidence. Then, if $\rho'_j = \lambda \mu \rho_L$, so that the actual density of the jet is $\mu \rho_L$, where ρ_L is the density of the lining material, the penetration times are given by:

$$t = \left\{1 + \sqrt{\frac{\rho_T}{\lambda \mu \rho_L}}\right\} \int_0^x \frac{dx}{V} . \qquad (19)$$

where x is the depth of penetration.

Considering $\lambda\mu$ as variable, we calculate its mean value for $\frac{1}{2}$ in. intervals of penetration, so that the corresponding increase in t computed from (19) shall be equal to the increase in t observed from the graphs. This process is carried out for penetration into both the aluminium and copper targets, and the results are shown below.

Mean Values of $\lambda\mu$.

Penetration in in.	Aluminium target	Copper target
0.0 - 0.5	0.27	0.24
0.5 - 1.0	0.35	0.31
1.0 - 1.5	0.83	0.64
1.5 - 2.0	0.65	0.84
2.0 - 2.5	0.89	0.84
2.5 - 3.0	0.76	0.77
3.0 - 3.5	0.51	0.61
3.5 - 4.0	0.58	0.36
4.0 - 4.5	0.39	
4.5 - 5.0	0.43	
Mean Value of $\lambda\mu$ over whole range	0.57	0.58

The formulae are rather susceptible to small changes, and unfortunately the curves from which the observations are made are fitted to a number of scattered experimental results. This may account for some of the erratic behaviour to be observed in the above tables. It will be noticed that the general trend of the values of $\lambda\mu$ shows marked consistency, as between the two types of target, and the mean value is practically the same in the two cases. Remembering that $1 < \lambda < 2$ (equation 14), the ratio of the density of the jet to the density of lining lies between the values in the columns and one-half these values. The density of the jet appears to increase to a maximum and then to fall away towards the rear of the jet.

Since the jet from steel cones is of the "fragment" type, $\lambda \sim 2$, and the mean density of the jet is approximately one-third that of the lining.

11. Loss of Energy to Target Material and Volume of Hole

The considerations of the earlier part of the report would lead one to suppose that for deep penetrations, it would become more and more nearly correct that p_0 x volume of hole = energy given up by jet ... (20), and that p_0 should be about 5 to 5.5 times the dynamic yield stress for penetration into mild steel (para.4). For a solid or liquid jet, relative to axes moving with velocity U (fig. 12), no energy is lost; the material of the jet flows away up the sides of the hole with its original velocity (V - U). Thus the loss of energy relative to fixed axes is

$$\tfrac{1}{2} M \left\{ V^2 - (V - 2U)^2 \right\} \qquad (21)$$

where M is the total mass of the jet. After a short calculation we find that the proportion of the total energy given up to the target material is

$$\frac{4\theta}{(\theta + 1)^2} \qquad \text{where } \theta = (\rho'_J/\rho_T)^{\frac{1}{2}} .$$

For a fragment jet, on the other hand, the kinetic energy will be partly used up in heating and deforming the fragments themselves. The amount of energy consumed in these processes will be equal to the energy loss relative to the moving axes of fig.12, i.e.

$$\tfrac{1}{2} M(V - U)^2 .$$

Thus, relative to fixed axes, the energy expended in actual penetration is

$$\tfrac{1}{2} M \left\{ V^2 - U^2 - (V - U)^2 \right\} \qquad (22)$$

which is just half as much as in formula (21); the proportion of energy given up is $2\theta/(\theta + 1)^2$.

A series of measurements of volume of holes in a mild steel target, caused by jets of steel and cadmium, is given in an early report by Evans and Ubbelohde (ref. 14). For the deepest penetrations, which are of primary interest in this discussion, the total volume of damage tended to roughly 8 c.c. for steel, and 5 c.c. for cadmium, independently of the mass of the lining, for a given charge diameter.

The density considerations of the previous section, combined with the plot of velocities in the report quoted (ref.13), lead one to suppose that the mass of the jet contributing to penetration moves with an average velocity roughly equal to two thirds of the velocity of the head of the jet. Putting the mass M of the jet equal to four-fifths of the mass of the lining, to allow for the formation of a plug, we then assume that $\frac{1}{2} M \bar{V}^2$ is the energy of that part of the jet which is effectively the penetrating agent, with $\bar{V} = 2/3\ V_h$ and V_h = velocity of the head of the jet. The graphs (ref.13) show that, for steel jets penetrating aluminium and copper U may be put equal to $\frac{1}{2}$ V, with little loss of accuracy when substituted in (22), the formula appropriate for energy given to the material by fragmenting jets. It appears very likely that the same relation between U and V for steel into steel will not lead to very large errors in assessing this energy.

Applying these assumptions in combination with equations (20) and (22) to the results of Evans and Ubbelohde (ref. 14), for thickness of lining .048D and spherical cap of radius 0.73 D where D is the charge diameter, and putting $V_h = 5 \times 10^5$ cm./sec., and Y = 50 tons/sq.in., p_0 is found to be 5.5 Y. This result can only be considered fortuitous, but it does point to p_0 being of the order of magnitude predicted.

Results will only be consistent for different masses of lining if MV_h^2 is constant, when D is constant. This is found to be true in experiments carried out by Bessent and Evans (ref. 15) for constant radius of curvature. Whether the relation is true between differing radii of curvature is not known to the authors.

The results for cadmium jets cannot be calculated without some better knowledge of the relationship between V and U during penetration.

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NOTATION

a = radius of inner hole
c = radius of plastic region
ν = Poisson's ratio
E = Young's modulus
Y = Flow stress in tensile test
p_0 = Internal pressure

APPENDIX I

Long Cylindrical Hole in Infinite Medium

The hole is supposed widened statically from zero radius by internal pressure.

For r > c the stress distribution is given by the ordinary elastic theory for plane strain.

The radial stress $\sigma_r = -Y c^2/\sqrt{3} r^2$, $r > c$

and the tangential stress $\sigma_\theta = Y c^2/\sqrt{3}\ r^2$, $r > c$ (1)

These equations satisfy the boundary condition $\sigma_r \to 0$ as $r \to \infty$, and also the Mises condition for plastic flow on r = c viz:-

$$\sigma_\theta - \sigma_r = 2Y/\sqrt{3} \qquad (2)$$

In the plastic region work-hardening is neglected and Mises' condition is taken to hold universally. The equation of equilibrium

$$\frac{d\sigma_r}{dr} = \frac{\sigma_\theta - \sigma_r}{r}$$

then gives

$$\left.\begin{aligned} \sigma_r &= -\frac{Y}{\sqrt{3}}(1 + 2\log c/r) \qquad r < c \\ \sigma_\theta &= \frac{Y}{\sqrt{3}}(1 - 2\log c/r) \qquad r < c \end{aligned}\right\} \quad (3)$$

when σ_θ and σ_r are made to vary continuously across $r = c$.

The relation determining c in terms of a follows from equating displacements in the elastic and plastic region on $r = c$. To find the displacements in the plastic region it is necessary to make some assumption about stress-strain relations there. We take the reduction in volume of the material in the plastic region, due to widening the hole from zero radius to radius a, as given by

$$-\frac{(1-2\nu)}{E}\int_a^c (\sigma_r + \sigma_\theta + \sigma_z).\ 2\pi r\,dr \qquad \text{(to order } Y/E)$$

where $\sigma_z = \frac{1}{2}(\sigma_r + \sigma_\theta)$.

If u denotes the displacement which an element finally on the plastic boundary (when the hole is of radius a) has undergone during the widening of the hole,

$$a^2 - 2uc = \frac{(1-2\nu)}{E}.\ \sqrt{3}\ Y c^2 \qquad \text{(to order } Y/E)$$

From the elastic region solution

$$u = \frac{Yc}{\sqrt{3}}.\ \frac{1+\nu}{E}$$

Eliminating u:-

$$\frac{a^2}{c^2} = \frac{Y}{E}.\ \frac{(5-4\nu)}{\sqrt{3}} \qquad (4)$$

An alternative assumption about the plastic flow is that there is no volume change, in which case

$$a^2 - 2uc = 0$$

$$\frac{a^2}{c^2} = \frac{Y}{E}.\ \frac{2(1+\nu)}{\sqrt{3}} \qquad (4a)$$

The internal pressure p_0 is given from (3) by

$$p_0 = \frac{Y}{\sqrt{3}}(1 + 2\log c/a) \qquad (5)$$

and does not vary with the hole radius. Moreover the numerical value of p_0 is but little affected by the choice of (4) or (4a).

Consider now the energy distribution in the elastic and plastic regions. For simplicity we assume there is no volume change in the plastic region, so that an element at radius r in the plastic region (when the hole is of radius a) was initially at radius s such that

$$s^2 = r^2 - a^2.$$

When the hole is of radius a', given by

$$\frac{a^2}{c^2} = \frac{a'^2}{a'^2 + s^2}$$

the element lies on the plastic boundary corresponding to radius a'. In its final position r, its energy is therefore partly elastic and partly plastic. The elastic energy per unit volume of the element is then

$$\frac{Y}{\sqrt{3}}.\ \frac{a^2}{2c^2}$$

and its plastic energy per unit volume is

$$\frac{Y}{\sqrt{3}}\left(-\frac{a^2}{c^2} + 2\log\frac{r}{s}\right) \qquad \left(\text{to order } Y.\ \frac{a^2}{c^2}\right)$$

(There is no volume change in either elastic or plastic regions).

The total energy up to radius r in the plastic region is:

$$\frac{Y}{\sqrt{3}} \int_a^r 2\pi r\, dr \left(- \frac{a^2}{2c^2} + 2 \log \frac{r}{s}\right); \qquad s^2 = r^2 - a^2$$

$$= \frac{\pi a^2 Y}{\sqrt{3}} \left\{ - \frac{s^2}{2c^2} + \frac{r^2}{a^2} \log \frac{r^2}{a^2} - \frac{s^2}{a^2} \log \frac{s^2}{a^2} \right\}. \qquad (6)$$

of which

$$\frac{\pi a^2 Y}{\sqrt{3}} \cdot \frac{s^2}{2c^2}$$

is elastic energy.

On putting r = c and neglecting Y/E compared with unity we find (remembering that $p_0 \pi a^2$ is the total work done) that:-

(i) The energy in the plastic region $= \pi a^2 \frac{Y}{\sqrt{3}} \left(\frac{1}{2} + 2 \log \frac{c}{a}\right)$

(ii) Plastic energy $= \pi a^2 \frac{Y}{\sqrt{3}} \cdot 2 \log \frac{c}{a}$

(iii) Elastic energy $= \pi a^2 \frac{Y}{\sqrt{3}}$

(iv) Half the total elastic energy is in the plastic region.

APPENDIX II

Spherical Hole in Infinite Medium

By symmetry the tangential stresses are equal. In the elastic region

$$\text{Radial stress } \sigma_r = -\frac{2Yc^3}{3r^3}$$

$$\text{Tangential stress } \sigma_\theta = \frac{Yc^3}{3r^3} \; (= \sigma_\phi) \qquad (1)$$

The Mises condition for plastic flow is now

$$\sigma_\theta - \sigma_r = Y \qquad (2)$$

In the plastic region

$$\sigma_r = -\frac{2Y}{3}\left(1 + 3 \log \frac{c}{r}\right)$$

$$\sigma_\theta = \frac{2Y}{3}\left(\frac{1}{2} - 3 \log \frac{c}{r}\right) \qquad (3)$$

These follow from the equilibrium equation

$$\frac{d\sigma_r}{dr} = \frac{2(\sigma_\theta - \sigma_r)}{r}$$

To find c/a, we proceed as in the cylindrical case, making a similar assumption, and find

$$a^3 - 3c^2 u = \frac{1 - 2\nu}{E} \cdot 2 Y c^3$$

where $u = \frac{Y}{E} \cdot (1 + \nu) \cdot \frac{c}{3}$ from the elastic region solution. Hence

$$\frac{a^3}{c^3} = \frac{Y}{E} \cdot 3(1 - \nu) \qquad (4)$$

Alternatively, supposing the plastic flow to involve no volume changes

$$\frac{a^3}{c^3} = \frac{Y}{E}(1 + \nu) \qquad (4a)$$

The internal pressure p_0 follows from (3)

$$p_0 = \frac{2Y}{3}\left(1 + 3 \log \frac{c}{a}\right) \qquad (5)$$

To find the energy distribution we proceed as in the cylindrical case. The total energy up to radius r in the plastic region is

$$\frac{4}{3} \pi a^3 Y \left\{ -\frac{1}{3}\frac{s^3}{c^3} + \frac{2}{3}\frac{r^3}{a^3} \log \frac{r^3}{a^3} - \frac{2}{3}\frac{s^3}{a^3} \log \frac{s^3}{a^3} \right\} \qquad (6)$$

where $s^3 = r^3 - a^3$. Of this, $\frac{4}{3} \pi a^3 Y \frac{s^3}{3c^3}$ is elastic energy. The total elastic energy is

$$\frac{4}{3} \pi a^3 Y. \frac{2s^3}{3c^3}.$$

APPENDIX III

Long Cylindrical Hole in Medium of Finite Cross-Section

Let b = outer radius of medium, and as usual suppose the outer boundary to be unstressed. Then in the elastic region ($b > r > c$):

$$\sigma_r = -\frac{Yc^2}{\sqrt{3}}\left(\frac{1}{r^2} - \frac{1}{b^2}\right) \qquad (1)$$
$$\sigma_\theta = \frac{Yc^2}{\sqrt{3}}\left(\frac{1}{r^2} + \frac{1}{b^2}\right)$$

In the plastic region ($c > r > a$):

$$\sigma_r = -\frac{Y}{\sqrt{3}}\left(1 + 2 \log \frac{c}{r} - \frac{c^2}{b^2}\right)$$
$$\sigma_\theta = \frac{Y}{\sqrt{3}}\left(1 - 2 \log \frac{c}{r} + \frac{c^2}{b^2}\right) \qquad (2)$$

Equating displacements on $r = c$ leads to

$$\frac{a^2}{c^2} = \frac{Y}{\sqrt{3}E}\left[5 - 4\nu - (1 - 2\nu)^2 \frac{c^2}{b^2}\right]$$

where the same assumption is made regarding plastic volume change as in equation (4) of Appendix I. The radius of plastic flow can be calculated as if the block were infinite, with an error of about 4 per cent. in the worst case when $b = c$. The internal pressure p_0 is given by

$$p_0 = \frac{Y}{\sqrt{3}}\left(1 + 2 \log \frac{c}{a} - \frac{c^2}{b^2}\right) \qquad (3)$$

and now depends on the radius of the hole. The energy required to make the hole is

$$\int_0^a 2 \pi a \, p_0 \, da$$

or, taking

$$\frac{a^2}{c^2} = \frac{Y}{\sqrt{3}E}(5 - 4\nu),$$
$$\pi a^2 = \frac{Y}{\sqrt{3}}\left(1 + 2 \log \frac{c}{a} - \frac{1}{2}\frac{c^2}{b^2}\right) \qquad (4)$$

APPENDIX IV

Effect of Work-Hardening

According to Bridgman (ref. 4) the graph of true stress against natural strain in a tensile test for steel used in armour plate can, to a close approximation, be represented by two straight lines.

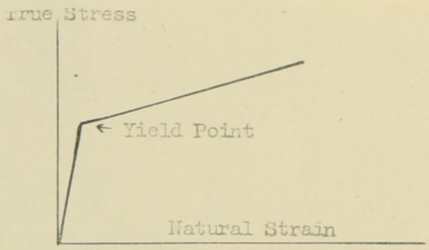

Fig. 13

For mild steel the relationship is more like that shown in Fig. (14).

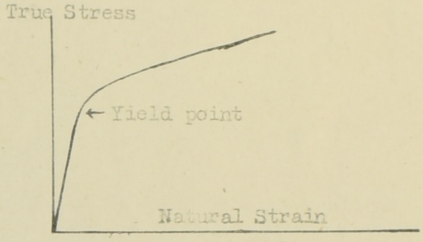

Fig. 14

the graph becoming a straight line for large strains (ref. 16).

Taking first the case of armour plate steel, it will be assumed that the true stress σ is found from a tensile test to be related to the natural strain ϵ by

$$\left.\begin{aligned} \sigma &= E\epsilon \quad (\sigma \leqslant Y) \\ \text{and} \quad \sigma &= Y + A\epsilon \quad (\sigma \geqslant Y) \end{aligned}\right\} \quad (1)$$

where Y is the yield stress. (Neglecting a term of order A/E compared with unity).

To apply this to the widening of a long cylindrical hole under internal pressure some general assumption has to be made about the stress-strain relationship in work-hardening. Following Nadai (ref. 17) it is assumed that the octahedral stress is a definite function of the unit octahedral shear strain for all strain configurations.

The octahedral stress T_n is given by

$$T_n = \frac{1}{3}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}^{\frac{1}{2}}$$

and the octahedral natural shear γ_n by the differential relation

$$d\gamma_n = \frac{2}{3}\left\{(d\epsilon_1 - d\epsilon_2)^2 + (d\epsilon_2 - d\epsilon_3)^2 + (d\epsilon_3 - d\epsilon_1)^2\right\}^{\frac{1}{2}}$$

In a tensile test

$$T_n = \frac{\sqrt{2}}{3} \cdot \sigma \qquad \text{and} \quad \gamma_n = \sqrt{2}\,\epsilon.$$

(1) then gives, as the universal relation between T_n and γ_n:-

$$T_n = \frac{\sqrt{2}}{3} Y + A\gamma_n/3 \quad \left(T_n \geqslant \frac{\sqrt{2}}{3} Y\right) \qquad (2)$$

(in the elastic region $2T_n = \frac{E}{(1+\nu)} \cdot \gamma_n$).

In the plastic region in the cylinder let ϵ_r denote the natural radial strain at radius r. Taking the plastic volume change to be negligible,

$$T_n = \frac{1}{3}\sqrt{\frac{3}{2}}\,(\sigma_\theta - \sigma_r), \quad \gamma_n \quad \text{and} \quad \gamma_n = 2\sqrt{\frac{2}{3}} \cdot |\epsilon_r|.$$

Hence

$$\sigma_\theta - \sigma_r = 2Y/\sqrt{3} + \frac{4A}{3}|\epsilon_r| \quad ; \quad (\sigma_\theta - \sigma_r \geqslant 2Y/\sqrt{3}). \qquad (3)$$

If an element initially at radius s is displaced to radius r during the widening of the hole (to radius a)

$$r^2 = a^2 + s^2 \qquad \text{and} \qquad \epsilon_r = -\log r/s \qquad (4)$$

Substituting (3) and (4) in the equation of equilibrium gives

$$d\sigma_r/dr = \frac{2Y}{\sqrt{3}} \cdot 1/r + 4A/3 \cdot 1/r \cdot \log r/s.$$

$$\therefore \sigma_r + p_0 = \frac{2Y}{\sqrt{3}} \log r/a - \frac{2A}{3} \int_a^r \log (1 - a^2/r^2) \cdot dr/r.$$

As in Appendix I,

$$a^2/c^2 = \frac{2(1+\nu)}{\sqrt{3}} \cdot \frac{Y}{E}, \text{ and } \sigma_r = -Y/\sqrt{3} \text{ on } r = c.$$

Thus

$$p_0 = Y/\sqrt{3} + 2Y/\sqrt{3} \cdot \log c/a - 2A/3 \int_1^{c/a} \log (1 - 1/t^2) \cdot dt/t.$$

Now

$$-2\int_1^{c/a} \log (1 - 1/t^2) \cdot \frac{dt}{t} = 2\int_1^{c/a} \sum_1^{\infty} 1/n \cdot 1/t^{2n} \cdot dt/t$$

$$= -\sum_1^{\infty} 1/n^2 \cdot \left[1/t^{2n} \right]_1^{c/a}$$

$$= \sum_1^{\infty} 1/n^2 \text{ (neglecting } a^2/c^2 \text{ c.f. 1)}$$

$$= \pi^2/6$$

Finally

$$p_0 = Y/\sqrt{3} + (2Y/\sqrt{3}) \log c/a + \pi^2 A/18 \qquad (5)$$

and is independent of the radius of the hole.

For mild steel, using values given in ref.(16), numerical calculation shows that the cold-work adds a term of the order 3500 kgm./sq.cm. to $Y/\sqrt{3} + 2Y/\sqrt{3}$ log c/a. Y is about 3000 kgm./sq.cm., and so the increase in p_0 is about 30 per cent..

APPENDIX V

Comparison of Static and Dynamic Works

We now attempt to make an estimate of the difference between the static and dynamic works required to enlarge a hole deep in an infinite medium. The conclusion in section (6) on general grounds was that the energy needed to widen a hole dynamically (to a final state of rest) is effectively equal to p_0 x volume of hole. The results arrived at here support this conclusion.

We consider first the dynamical widening of a long cylindrical hole in an infinite medium.

It is assumed

(a) that volume changes in the plastic region are negligible,
(b) that the Mises flow condition holds throughout the plastic region,
and (c) that rate of strain effects can be ignored.

By (a) the continuity equation in the plastic region is

$$r^2 = a^2 + s^2$$

using the notation of Appendices I and II.

The acceleration of an element at radius r is:-

$$\ddot{r} = \frac{a\ddot{a} + \dot{a}^2}{r} - \frac{a^2\dot{a}^2}{r^3}$$

and so the equation of motion in the plastic region is

$$\partial\sigma_r/\partial r = \frac{\sigma_\theta - \sigma_r}{r} + \rho\left(\frac{a\ddot{a} + \dot{a}^2}{r} - \frac{a^2\dot{a}^2}{r^3}\right)$$

By (b) $\sigma_\theta - \sigma_r = 2Y/\sqrt{3}$, and integration then gives

$$\sigma_r + p = \frac{2Y \log r/a}{\sqrt{3}} + \rho(a\ddot{a} + \dot{a}^2)\log r/a - \frac{\rho\dot{a}^2}{2}.(1 - a^2/r^2) \qquad \ldots. (1)$$

If u denotes the displacement in the elastic region, the radial strain $e_r = \partial u/\partial r$ and the tangential strain $e_\theta = u/r$. On the plastic boundary $r = c$,

$$\sigma_\theta - \sigma_r = \frac{E(e_\theta - e_r)}{1 + \nu} = 2Y/\sqrt{3} \qquad (2)$$

and

$$u = a^2/2c \qquad \text{(to order } Y/E) \qquad (3)$$

Now the elastic stress-strain relations are

$$\sigma_r = \frac{E}{1 + \nu}\left(e_r + \frac{\nu\Delta}{1 - 2\nu}\right) \text{ etc.}$$

where $\Delta = e_r + e_\theta + e_z$.

Since $e_z = 0$ here, we find for σ_r on $r = c$:-

$$\left(\frac{1 + \nu}{E}\right)(1 - 2\nu)\,\sigma_r = (1 - \nu)e_r + \nu e_\theta$$

$$= e_\theta - (1 - \nu).\,\frac{2(1 + \nu)}{\sqrt{3}}\,.\,Y/E \text{ from (2)}$$

$$= a^2/2c^2 - (1 - \nu).\,\frac{2(1 + \nu)}{\sqrt{3}}\,Y/E \quad \ldots. (4)$$

Substituting for σ_r in (1):-

$$p = \frac{2(1 - \nu)}{\sqrt{3}(1 - 2\nu)}\,Y - \frac{E}{(1 + \nu)(1 - 2\nu)}\,.\,a^2/2c^2 + \left\{\frac{2Y}{\sqrt{3}} + \rho(a\ddot{a} + \dot{a}^2)\right\}\log c/a - \rho\dot{a}^2/2$$

$$\ldots\ldots\ldots\ldots\ldots (5)$$

(neglecting a^2/c^2 c.f. 1)

a^2/c^2 is an unknown function of t (of order Y/E) which can only be determined by solving the elastic wave equation.

Denote the expression

$$\frac{2Y}{\sqrt{3}}\,.\,\frac{(1 + \nu)}{E}$$

by k, so that k is equal to the static value of a^2/c^2. From (2) and (3)

$$a^2/c^2 = 2u/c \quad \text{and} \quad u/c - (\partial u/\partial r)_{r=c} = k$$

$$\therefore \quad a^2/c^2 - k = u/c + (\partial u/\partial r)_{r=c} = \Delta_{r=c} < 0 \qquad (6)$$

since the material in the elastic region on the plastic boundary is compressed during a continual widening of the hole. This implies that the plastic region extends further in the dynamic case than in the static case (for the same instantaneous radius of hole). On the other hand since $(\sigma_\theta)_{r=c} > 0$, we have

$$(-\sigma_r)_{r=c} < \frac{2Y}{\sqrt{3}}$$

from Mises' condition. It follows from (4) that

$$a^2/c^2 > 2\nu k \qquad (7)$$

Combining (6) and (7):-

$$2\nu < a^2/kc^2 < 1 \qquad (8)$$

The work per unit length needed to widen the hole dynamically to radius r is

$$W_D = \int_0^r p \cdot 2\pi a \, da$$

where p is given by (5). The static work is

$$W_s = \int_0^r \frac{Y}{\sqrt{3}} \left\{ 1 + \log\left(\frac{1}{k}\right) \right\} \cdot 2\pi a \, da$$

$\therefore$

$$W_D - W_s = \frac{Y}{\sqrt{3}} \int_0^r \left\{ \frac{1 - a^2/kc^2}{1 - 2\nu} \right\} \cdot 2\pi a \, da + \frac{Y}{\sqrt{3}} \int_0^r \left\{ \log \frac{kc^2}{a^2} \right\} \cdot 2\pi a \, da$$

$$+ \int_0^r \rho(a\ddot{a} + \dot{a}^2) \log \frac{c}{a} \cdot 2\pi a \, da - \int_0^r \rho \frac{\dot{a}^2}{2} \, 2\pi a \, da.$$

But $\int_0^r \rho(a\ddot{a} + \dot{a}^2) \log \frac{c}{a} \cdot 2\pi a \, da = -\pi\rho \int_0^r a^2 \dot{a}^2 \, d\left(\log \frac{c}{a}\right)$

since we are taking $\dot{a} = 0$ when $a = r$.

Hence finally:-

$$W_D - W_s = \frac{Y}{\sqrt{3}} \int_0^r \left\{ \frac{1 - a^2/kc^2}{1 - 2\nu} \right\} \cdot 2\pi a \, da + \frac{Y}{\sqrt{3}} \int_0^r \left(\log \frac{kc^2}{a^2} \right) \cdot 2\pi a \, da$$

$$- \pi\rho \int_0^r a^2 \dot{a}^2 \, d(\log c) \qquad \ldots\ldots\ldots\ldots(9)$$

Using the inequalities in (8):-

$$W_D - W_s < \pi r^2 \cdot \frac{Y}{\sqrt{3}} + \pi r^2 \cdot \frac{Y}{\sqrt{3}} \log\left(\frac{1}{2\nu}\right) - \pi\rho \int_0^r \frac{a^2 \dot{a}^2}{c} \cdot \frac{dc}{da} \cdot da$$

A rough estimate can be made of the value of the last term by taking the "static" value for $a/c \cdot dc/da$ (i.e. unity) and a mean value for $\dot{a}$.

For example, for a projectile penetrating at velocity V, we can take $\frac{1}{2}$ V tan α as a mean $\dot{a}$ (α = ogive semi-angle). In this case if Y = 7500 Kgm./sq.cm. and V = 3000 ft./sec., the last two terms approximately cancel, and so the extra work done dynamically is less than about 20 per cent. of the static work. If a similar calculation is made for the enlarging of a spherical hole the upper limit to the extra work is found to be more like 10 per cent. of the static work. However, the inequalities are not very close and the actual value of $W_D - W_s$ is probably a good deal less. To obtain the precise value would re uire an explicit solution of the elastic wave equations.

APPENDIX VI

Critical Velocity of Penetration

A rough estimate of the critical velocity at which the hole becomes wider than the projectile can be made when the projectile has a long narrow head. For then the cylindrical model is appropriate and we can write approximately

$$p = \log\left(\frac{c}{a}\right) \left\{ \frac{2Y}{\sqrt{3}} + \rho(a\ddot{a} + \dot{a}^2) \right\}$$

(terms of magnitude unity or less being neglected in comparison with 2 log c/a). This is tantamount to neglecting elastic energy in comparison with plastic kinetic energy and energy of deformation.

Suppose the equation to the head is $y = r \sin \frac{\pi x}{2L}$. While the target material is still in contact with the head

$$\dot{a} = V.\ dy/dx$$

$$\ddot{a} = V^2.\ d^2y/dx^2,$$

(taking uniform velocity of penetration V).

The pressure first becomes zero on the bourrelet ($y = r$), and the critical value of V is given by

$$\frac{2Y}{\sqrt{3}} - \rho V^2. \left(\frac{\pi r}{2L}\right)^2 = 0$$

For Y = 7500 Kg./sq.cm. and L/r = 8, V is about 5000 ft./sec..

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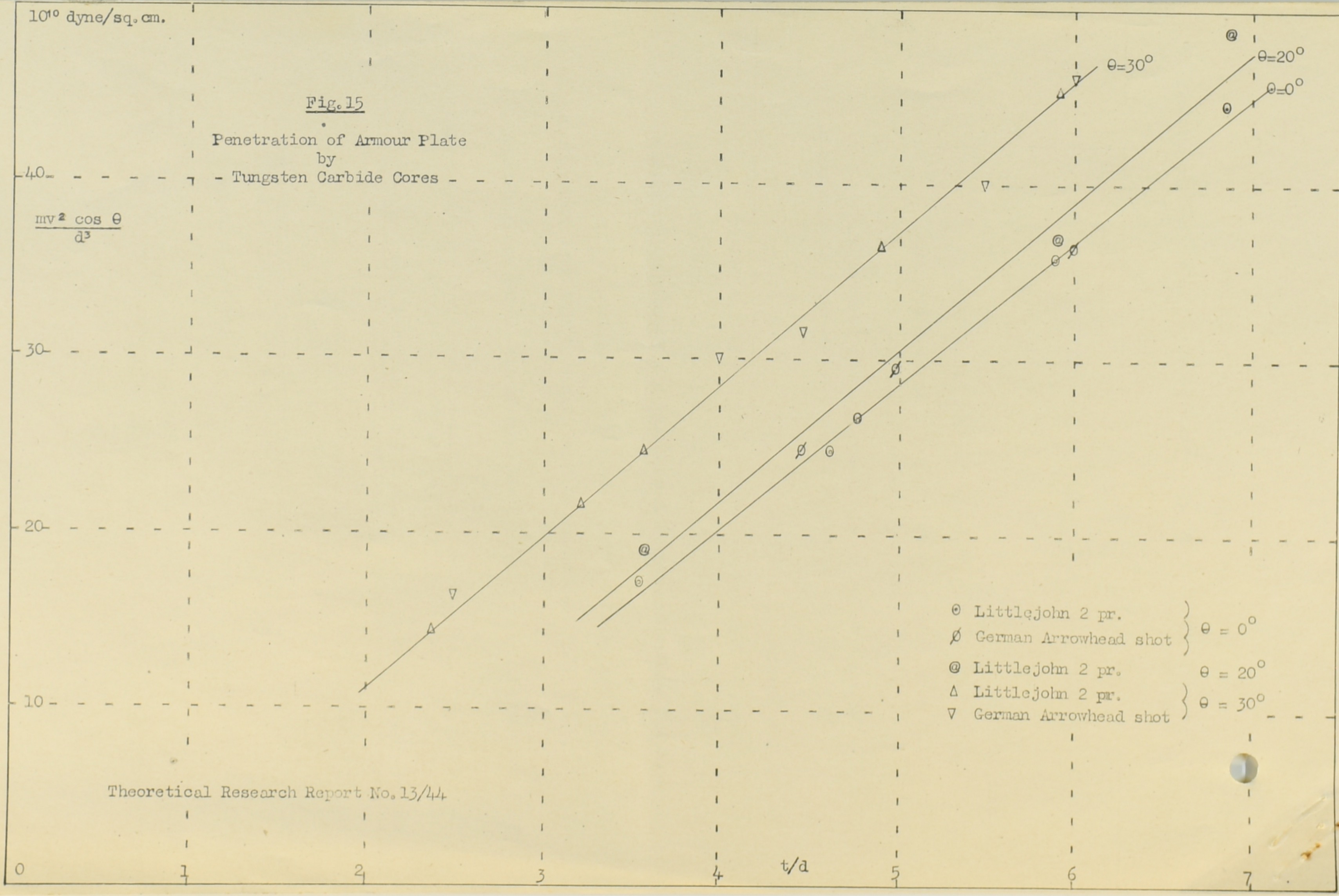

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