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MINISTRY OF HOME SECURITY.

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Notes on H. A. Bethe's "Theory of armor penetration".

II. Enlargement of a hole in a flat sheet at high speeds.

By Professor G. I. Taylor, F. R. S.

When the velocity of enlargement of the hole is such that the terms involving acceleration are appreciable, the effect on the plastic flow is very different in the case of a thin sheet from what it is in a long cylinder. Bethe has calculated this effect in the latter case and obtains the formula

$$W = \pi R^2 h_0 \left[\frac{1}{2} Y \left(\ln \frac{E}{1.9Y} + 1 \right) + 0.455 \rho \left(\frac{YR}{L} \right)^2 \right] \dots (1)$$

where W is the total work done in expanding the hole to radius R , Y is the strength constant which enters into Mohr's criterion and E is elastic modulus. It will be seen that the effect of inertia is merely to add a term which is independent of E and of Y to the work done statically. The extra work is in fact largely utilized in setting up compressional waves, the constant 0.455 depends partly on the form which is assumed for the nose of the bullet which makes the hole, i.e. for the velocity-time relationship at the edge of the hole.

The object of the present note is to show that when the thin sheet problem is considered rather than the two dimensional one or infinitely long expanding cylindrical hole, the effect of inertia is very much more important. The difference between the static and dynamic problems is now not independent of the strength but depends on the ratio $\gamma = \rho U^2/Y$, U being the radial velocity at the edge of the hole, ρ the density and Y the yield strength. In fact the dependence of the plastic flow round the hole on γ is so important that for values of γ greater than 0.39 the whole character of the configuration of the displaced metal round the hole changes. The strength in fact is not great enough to allow the plastic wave to be propagated fast enough to maintain a "crater" of the type contemplated by Bethe, and a discontinuity is formed at this value.

When a hole in a sheet is enlarged at a very slow rate from a pinhole, the law of expansion with time is immaterial. The configuration is similar at all times, the radial coordinate of a point where the thickness has a given value being proportional to the radius of the hole. In general this simplifying circumstance is absent when the hole is enlarged at such a rate that the inertia of the metal gives rise to a change in the stress distribution. In one case, however, this geometrical similarity of configuration at all stages of the enlargement is preserved even at high speeds of enlargement. This is the case when the radial velocity of the hole is constant. Such a state might be realized by perforating the sheet by a circular cone moving with constant speed along its axis. It would appear to be possible to work out the distribution of velocity and thickness at any given constant rate of expansion of the hole using the same stress criterion (Mohr's) and the same strain hypothesis (expressed in Lode's variables by $\mu = \nu$) as was used in the static case already treated. It is clear, however, that numerical solutions would have to be worked out for a range of rates of expansion, and this would entail great labour. On the other hand the great simplicity of the statical problem which results from making Bethe's assumption that $\sigma_\theta = 0$ in the statical case encourages the use of the same assumption when the acceleration forces cannot be neglected. The fact that the distribution of metal round the hole is not very different when Bethe's assumption is made from what it is when the physically better

assumption, $\mu = \nu$, is used makes it seem probable that the effect of the inertia stresses may be correctly predicted by introducing them into Bethe's simple stress equation.

If Q is the radial velocity, the condition that the radial velocity of the edge of the hole is constant ensures that Q is a function of r/b only where r is the radial coordinate and b is the radius of the hole.

Writing $r/b = x$, $Q/U = v$, the condition that Q is constant at a point which moves with uniform radial velocity Ur/b or xU gives

$$\frac{\partial Q}{\partial t} + xU \frac{\partial Q}{\partial r} = 0 \quad \dots (2)$$

The acceleration of a particle is

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + Q \frac{\partial Q}{\partial r} = \frac{\partial Q}{\partial t} + U^2 v \frac{dv}{dr} \quad \dots (3)$$

so that from (2)

$$\frac{DQ}{Dt} = (v-x) \frac{U^2}{b} \frac{dv}{dx} \quad \dots (4)$$

If h is the thickness the equation of stress equilibrium is

$$\frac{d}{dr} (h \sigma_r) + \frac{h}{r} (\sigma_r - \sigma_\theta) = \rho h \frac{DQ}{Dt} \quad \dots (5)$$

and Mohr's criterion when σ_θ is assumed to be zero is $\sigma_r = -Y$, so that (5) may be written

$$-Y \left\{ \frac{dh}{dx} + \frac{h}{x} \right\} = \rho U^2 h (v-x) \frac{dv}{dx} \quad \dots (6)$$

or
$$\frac{d}{dx} (\ln h/h_0) + \frac{1}{x} = -\gamma (v-x) \frac{dv}{dx} \quad \dots (7)$$

The continuity equation may be written in the form

$$\frac{\partial Q}{\partial r} + \frac{Q}{r} = -\frac{1}{h} \frac{Dh}{Dt} = -\frac{D}{Dt} (\ln h/h_0) \quad \dots (8)$$

and the condition that $\ln h/h_0$ depends on x only gives

$$\frac{D}{Dt} (\ln h/h_0) = (v-x) \frac{U}{b} \frac{d}{dx} (\ln h/h_0) \quad \dots (9)$$

The equation of continuity is therefore

$$\frac{dv}{dx} + \frac{v}{x} = -(v-x) \frac{d}{dx} (\ln h/h_0) \quad \dots (10)$$

Eliminating $\ln h/h_0$ from (7) and (10) gives

$$\frac{dv}{dx} \left\{ 1 - \gamma(v-x)^2 \right\} = 1 \quad \dots (11)$$

Writing $x-v = y$, the variables v and x are separated and (11) may be integrated yielding

$$v = \frac{1}{2\sqrt{2\gamma}} \ln \frac{\sqrt{2} + y\sqrt{\gamma}}{\sqrt{2} - y\sqrt{\gamma}} + c \quad \dots (12)$$

where c is the constant of integration. The boundary condition at the edge of the hole is $v = 1$ at $x = 1$, so that $c = 1$. (12) may therefore be written in the form

$$x = v + \sqrt{\frac{2}{\gamma}} \tanh \left\{ \sqrt{2\gamma}(1-v) \right\} \quad \dots (13)$$

or in terms of the original variables r and Q

$$\frac{r}{b} = \frac{Q}{U} + \sqrt{\frac{2Y}{\rho U^2}} \tanh \left\{ \sqrt{\frac{2\rho U^2}{Y}} \left(1 - \frac{Q}{U} \right) \right\} \quad \dots (14)$$

At the outer boundary of the plastic region $Q = 0$ so that the radius r_2 of the outer boundary is determined by

$$x_2 = \frac{r_2}{b} = \sqrt{\frac{2Y}{\rho U^2}} \tanh \sqrt{\frac{2\rho U^2}{Y}} \quad \dots (15)$$

To find the thickness the expression (13) for x may be substituted in (7) giving

$$\frac{d}{dx} (\ln h/h_0) = -\frac{1}{x} + \gamma(x-v) \frac{dv}{dx} = -\frac{1}{x} + \sqrt{2\gamma} \frac{dv}{dx} \tanh \left\{ \sqrt{2\gamma}(1-v) \right\} \quad \dots (16)$$

Integrating (16)

$$\ln(h/h_0) = -\ln x - \ln \cosh \left\{ \sqrt{2\gamma}(v-1) \right\} + \text{constant} \quad \dots (17)$$

This constant must be determined by the boundary condition at the outer edge of the region of finite plastic deformation where $h/h_0 = 1$. Here $v = 0$ and $x = x_2$ so that (17) becomes

$$\frac{h}{h_0} = \frac{x_2 \cosh \sqrt{2\gamma}}{x \cosh \left\{ \sqrt{2\gamma}(1-v) \right\}} = \frac{2 \sinh \sqrt{2\gamma}}{x \sqrt{2\gamma} \cosh \left\{ \sqrt{2\gamma}(1-v) \right\}} \quad \dots (18)$$

The thickness h_1 at the edge of the hole is given by putting $x = 1$, $v = 1$, so that

$$\frac{h_1}{h_0} = x_2 \cosh \sqrt{2\gamma} = \sqrt{\frac{2}{\gamma}} \sinh \sqrt{2\gamma} \quad \dots (19)$$

Equations (13) and (18) are the complete solution of the problem for any velocity of enlargement when Bethe's strain assumption is made.

Comparison with Bethe's static solution.

When U is small so that γ' is small (13), (15) and (18), take the forms

$$x = v + 2(1-v) = 2-v \quad \dots (20)$$

$$x_2 = 2 \quad \text{or} \quad r_2 = 2b \quad \dots (21)$$

$$\frac{h}{h_0} = \frac{x_2}{x} = \frac{2b}{r} \quad \dots (22)$$

(20) and (22) are identical with Bethe's solution in which the maximum thickness of the sheet is twice that of the undistorted sheet and the radius of the region where a finite increase in thickness occurs is 2b.

Numerical values.

Values of h/h_0 as a function of x have been obtained for $\gamma' = 0.125, 0.245, 0.32$ and 0.39 by calculating x from (13) for a sequence of values of v covering the range $0 < v < 1$; corresponding values of h/h_0 were then calculated from (18). The results are shown in Fig. 1, and the values for $\gamma' = 0$, namely $h/h_0 = 2b/r$ are shown for comparison.

Limiting values of γ' .

It will be seen in Fig. 1 that as γ' increases the maximum value of h/h_0 (i.e. the value at the hole) increases from 2.0 and r_2/b decreases from 2.0. As γ' increases the slope of the h/h_0 curve increases at the outer edge (i.e. at $r = r_2$) till when $\gamma' = 0.39$ it becomes infinite. That this must be the case can be seen from (11), dv/dx in fact becomes infinite when $x - v = 1/\sqrt{2\gamma'}$ i.e. when $\tanh\{\sqrt{2\gamma'}(1-v)\} = \frac{1}{2}\sqrt{2}$, which may be changed into

$$\sqrt{2\gamma'}(1-v) = \sinh^{-1} 1 = 0.881 \quad \dots (23)$$

If therefore $\sqrt{2\gamma'}$ is greater than 0.881, i.e. if γ' is greater than 0.39, (23) can be satisfied when v lies in the range $1 > v > 0$. In fact values of γ' greater than 0.39 lead analytically to two values of h/h_0 and of v for one value of x , which has no physical meaning.

It will be seen, therefore, that when the radial velocity at the hole exceeds $U = \sqrt{0.39Y/\rho}$ a material which obeys Mohr's criterion and Bethe's hypothesis that $\sigma_\theta = 0$ cannot expand without the formation of a discontinuity at the edge of the deformed region. It seems possible that this is a consequence of the hypothesis that $\sigma_\theta = 0$, and that if the problem were solved using the strain relations represented by $\mu = \nu$ in Lode's variables, this limitation might not be found, or at any rate might be found only for higher values of U. It will be noticed in the static case* that the slope of the h/h_0 curve at the outer boundary of the plastic region is much less for the $\mu = \nu$ relation than it is in Bethe's case. The limiting value of U, namely $\sqrt{0.39Y/U}$, for yield stress 20 tons/sq. inch is $U = 400$ ft. per second, a velocity which would be produced by a bullet with a conical nose of semi-vertical angle 22° moving at 1,000 ft. per second. The limiting value of r_2/b is 1.63 while the limiting value of h/h_0 is 2.27.

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* See Fig. 4 of "Notes on H. A. Bethe's Theory of armor penetration. I."

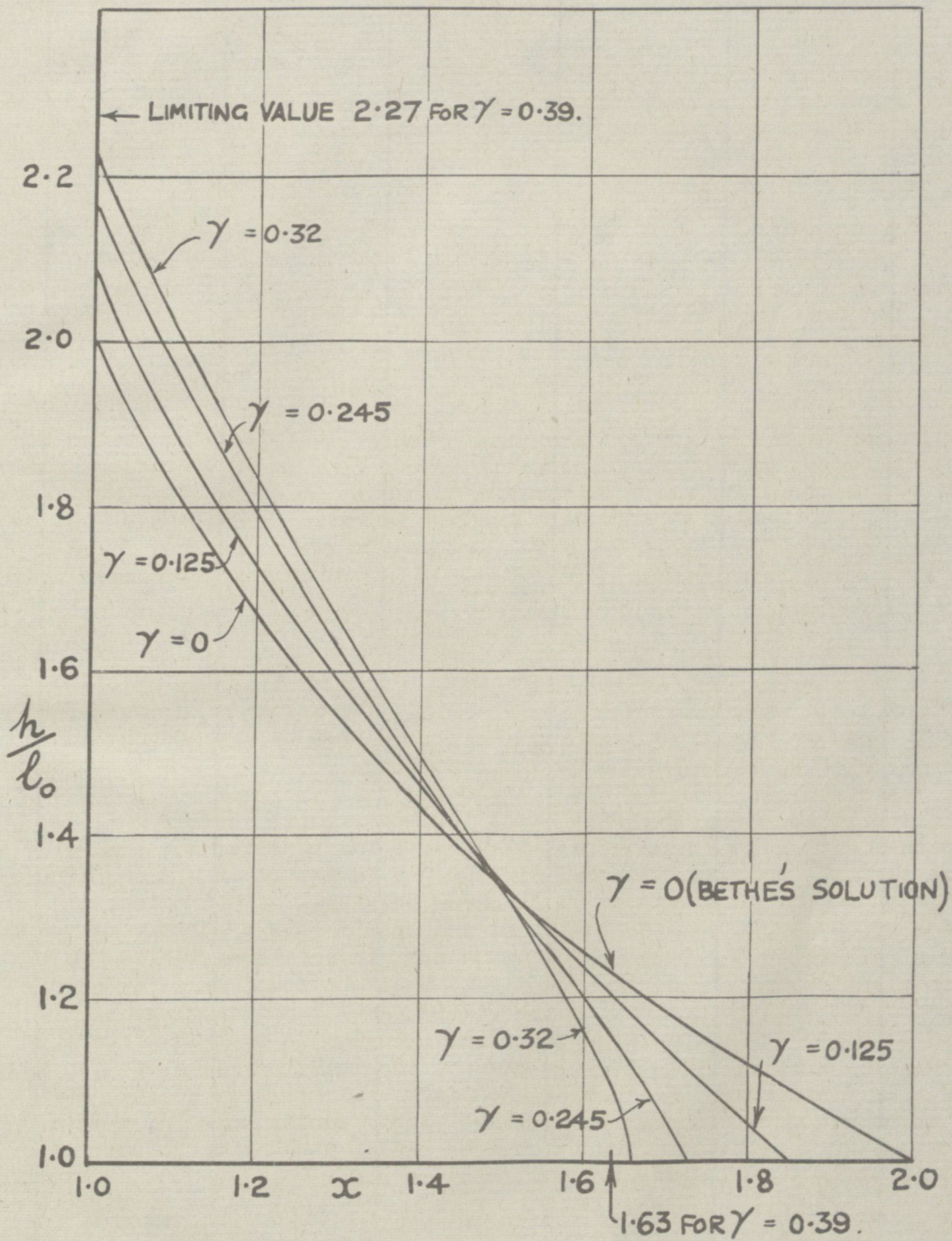


FIG. 1.