Analysis of the Robustness Dynamics of Wireless Mobile Ad Hoc Networks via Time Varying Dual Basis Representation

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Abstract

Many network models are too complex to readily identify which structural aspects of a network are most influential on robustness. To analyze the dynamics and robustness of a network, many of the protocol details can be reduced to a graph theory representation of nodes, links and link weights. Our method uses the spectral analysis of the Laplacian matrix to decouple the interactions between nodes to analyze the robustness of a wireless mobile ad hoc network (MANET) with a time-varying wireless channel. This spectral analysis and the resulting algebraic connectivity can be used to determine how robust a network is, where the weak links are, and how to best increase overall performance of a network. Using a simulation of wireless devices in a MANET with a time-varying channel, we show that robustness is a function of time, that nodes become coupled and decoupled as the structure of the network changes and that any robustness analysis is more complete when more than a single eigenvalue is evaluated.

1. Introduction

Mobile ad hoc networks (MANETs) are complex systems that can be foreseen supporting multiple applications to include vehicle-to-vehicle communications (V2V), which the United States' Department of Transportation is considering mandating to improve highway safety [1]. V2V communications and more generally, MANETs can be implemented using any one of a large number of protocols at all layers of the open system interconnection (OSI) stack. To add to the complexity of the analysis of these networks, there are typically large numbers of nodes that can enter and leave the network. For instance, one should consider the number of cars that would need to communicate with a wireless infrastructure at any major intersection in New York City. A MANET is in constant flux due to these changes. Additionally, the links between nodes are wireless, which means the quality of each link changes with time which impacts data throughput and transmission errors.

All of these factors lead to difficulty when analyzing the robustness of a mobile network. There are many modeling resources available to researchers, such as Network Simulator (NS-3) [2]. Many of these resources allow the user to select from a myriad of options at each layer of the OSI stack. The overall topology, structure and robustness of the network can be lost in the details of protocols. Many research cases do not require this level of detail. A simpler model of nodes, links and link weights can be used to analyze the structure of the network to determine a node and link's importance to the overall robustness of a network. Graph theory has been used to provide analytic foundations for this robustness analysis.

1.1 Graph Theory Modeling for Robustness

Graph theory is a tool to model networks of interconnected devices with the goal of providing the analytical groundwork for developing algorithms that maximize performance. This goal is achieved through matrix representations of the graphs and developing graph metrics to better understand the state of the modeled network. Spectral graph theory uses the eigendecomposition of these matrices to gain a better understanding of the robustness of the network. The primary two matrices used in spectral graph theory are the adjacency matrix and the Laplacian matrix [3]. The adjacency matrix is a representation of the network focused on the links between nodes. The Laplacian matrix goes a step further and incorporates the degree matrix, which is a matrix solely focused on nodal connectivity. In our application, the Laplacian matrix is used because it combines both a nodal matrix and a link matrix. This allows us to define eigenvectors and eigenvalues in terms of nodes and couplings between nodes.

The main application of spectral graph theory has been to maximize the algebraic connectivity, which is defined as the smallest, non-zero eigenvalue of the Laplacian matrix [3]. It has been shown that algebraic connectivity is well correlated with robustness and performance of various networks to include wireless mobile ad hoc networks [4] [5] [6]. Graphs with weighted links have been used to develop formations of mobile nodes that maximize algebraic connectivity [7]. Methods have been developed that calculate algebraic connectivity in a distributed manner as an input to the control law to maintain formations that maximize algebraic connectivity [8] [9]. All of these examples only use the algebraic connectivity and the associated eigenvector. Much more information is contained in the remaining eigenvalues and eigenvectors that has not been exploited in modeling and analysis of MANETs.

We propose a novel method that first simplifies the network to a set of links, nodes, and weights. Second, we use the entire eigenspace of the Laplacian matrix to decouple the interactions between nodes and analyze the robustness of a wireless MANETs with a time-varying wireless channel. Once analyzed, we are able to use spectral graph theory techniques to propose methods to maximize the robustness of the network.

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This paper is organized as follows. In section 2, we describe how the MANET is modeled in a graph theory matrix, propose a method to weight each link and then describe the eigendecomposition of the weighted graph. Section 3 describes the dual basis, eigencentrality basis, and nodal basis. Section 4 shows the simulations of the modeled network and analysis of the results. Section 5 provides our conclusions and future work.

2. Eigendecomposition of a Wireless Ad Hoc Network

The Laplacian matrix is a positive, semi-definite matrix which means it will always have at least one zero eigenvalue and will always create a dual basis [10]. A dual basis is defined as a matrix in which all rows are orthogonal to all columns. To exploit this double orthogonality of the Laplacian matrix, Q, the adjacency matrix, A, must be defined, which requires a definition of the link weights, w_{ij}. Our approach uses the empirical path loss model with shadowing to weight each link in the graph. The objective is to build the Laplacian matrix based on a model that can be used in multiple applications and is firmly rooted in the dynamics of a wireless channel. The dual basis is only as meaningful as the graph theory matrix on which it is based.

2.1 Graph Representation of Wireless Ad Hoc Network

Figure 1 shows an example mesh network with six nodes and one mobile node. One of the nodes, node seven, is too far away to communicate reliably with the other nodes in the network. This simple graph can be described using various graph theory matrices. The adjacency matrix describes the links between nodes, and the degree matrix, D, is a diagonal matrix that describes how many connections each node has [3].
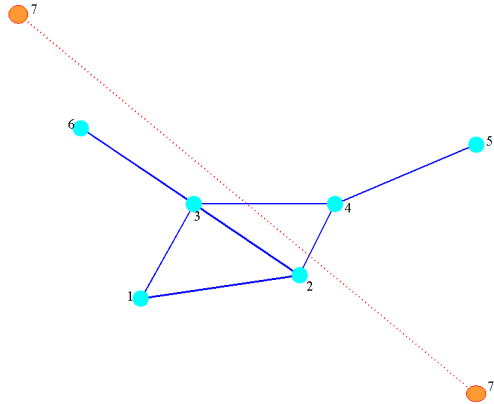

Figure 1.   A six node mesh network with one mobile node

An unweighted adjacency matrix is a binary matrix of ones and zeroes. A weighted graph is able to describe more than just the existence of a link; it can take into account information about the link quality at the physical layer and utilization of the available capacity at the network layer. This multi-layered approach allows a single network to be modeled across multiple functional layers to achieve a better understanding of the performance of each layer in context of the whole network.

2.2 Link Weights Based on Wireless Path Loss Model

Because of the time-varying nature of the wireless links between ad hoc nodes, the effects of the physical channel and a node's mobility, the MANET cannot be modeled using an unweighted graph. The model must take into account the time varying nature of the channel. We propose using energy per bit to noise power spectral density ratio, E_b/N_0, which is a widely used metric for the bit error rate between nodes. By using E_b/N_0 as a link weight, the eigendecomposition of the network model is time-varying as well.

Using the empirical path loss model with shadowing, the weight of each link is defined as [11]

$$w_{ij} = \frac{E_b}{N_0} = \left(\frac{P_s}{P_N}\right)\left(\frac{B}{R}\right) = \left(\frac{P_{Tx}L_oX\left(\frac{d_o}{d}\right)^\alpha}{P_{AWGN}+P_{\text{Int}}}\right)\left(\frac{B}{R}\right) \quad (1)$$

If bandwidth, B, data rate, R, loss at reference distance, L_o, and the reference distance, d_o, are held constant, the link weights are a function of transmit power, P_{Tx}, the noise power, the distance between the nodes, d, and the random variable, X. The noise power, P_N, is a combination of both the additive white Gaussian noise, P_{AWGN}, and the interference from other in-band transmitters, P_{Int}. The probability density function of the Gaussian random variable X in decibels is

$$f\left(x_{dB}\right) = \frac{1}{\sqrt{2\pi}\sigma_x}e^{\left(-\frac{x_{dB}^2}{2\sigma_x^2}\right)} \quad (2)$$

The variation in the received signal power, P_s, is accounted for by the random variable. By varying the standard deviation, σ, we were able to model variations of received signal strength from the mean [11].
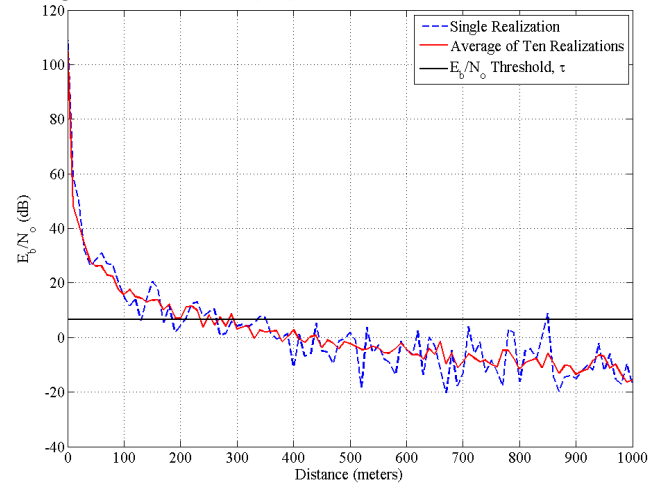
Figure 2. Simulated results of eqn. 2

Using (1) to define the link weight between node i and node j, the weighted Laplacian matrix, Q, is defined as

$$Q(i,j) = \begin{cases} -w_{ij} & \text{if } e_{ij} \in L \\ \sum_{i=1}^{n} w_{ij} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

To construct the Laplacian matrix, there exists a link, e_{ij}, in the set of all links, L, if and only if the E_b/N_0 of the link is sufficient to meet the bit error rate (BER) threshold (τ) required for the modulation scheme in use. If the link does not support the required BER, the link is set to zero. A zero indicates that node i and node j are unable to reliably communicate.

2.3 Eigendecomposition of Weighted Graph

The Laplacian matrix is positive semi-definite; therefore, all eigenvalues are real, and there will always be at least one zero eigenvalue [10]. The eigenvalues are typically ordered from smallest to largest by

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n \qquad (4)$$

Additional zero eigenvalues are interpreted as additional disconnected nodes. For each connected group within the graph model, there will be a zero eigenvalue [3].

Maximizing the smallest non-zero eigenvalue, only minimizes the difference between the largest and smallest eigenvalues. In an unweighted graph, the largest eigenvalue is bound by [3]

$$\lambda_n \leq n \qquad (5)$$

Using only the smallest as a metric only provides insight into the least connected area of the network. It provides no information about any other portion of the network. All the eigenvalues must be considered.

3. Dual Basis of a Wireless Ad Hoc Network

Spectral graph theory uses the eigendecomposition of the Laplacian matrix to describe characteristics of the network. The characteristics that are described are dependent upon how the links are modeled. In this case, we are modeling the wireless channel and the mobility of the nodes. The characteristics that are described by the eigenspace are the robustness of the network and the reachability within the network. These characteristics are fully described by the dual basis, the reachability space and null space.

3.1 Double Orthogonality

The eigenvector associated with each eigenvalue creates an orthogonal basis for the network model. In the case of a

symmetric matrix, a dual basis is always created [10]. A dual basis is defined as

$$V^T V = I \qquad (6)$$

where V is an orthonormal matrix of eigenvectors, and I is the identity matrix.

The dual basis in this case is a result of solving for the eigenvector, v, in the equation

$$(Q - \lambda I)v = 0 \qquad (7)$$

By solving eqn. (7) for all n eigenvalues and eigenvectors, the matrices D and V can be defined and used to recreate Q. In most applications, only one eigenvector matrix is required, the right eigenvector. Transformations using the right eigenvector can be demonstrated by rearranging

$$Q = VDV^T \qquad (8)$$

$$QV = VD(V^T V) = VD \qquad (9)$$

$$QV = z_1 \qquad (10)$$

The right eigenvector's dual is the left eigenvector which can be demonstrated by rearranging eqn. (8) to result in

$$V^T Q = (V^T V)DV^T = DV^T \qquad (11)$$

$$V^T Q = z_2 \qquad (12)$$

Together the left and right eigenvectors create a dual basis for the network. The dual basis provides two orthogonal bases in which the network can be defined. V transforms Q into the eigencentrality space [12], which is the space where each column vector of V is associated with one eigenvalue. V^T transforms Q into the nodal space, which is the space where each column vector of V^T is associated with a specific node in the network. These two transformations allow an analysis of the network from an eigencentrality perspective or a nodal perspective. Both of which are useful depending on the specific application of the model.

The eigencentrality basis is the basis that defines how influential a specific node is at a given eigenvalue. The eigencentrality metric is a measure of how important a node is to the network and what the impact is of its removal. The eigencentrality metric is defined as

$$E_j^k = (v_k)_j^2 \qquad (13)$$

or as the square of the kth eigenvector's j^{th} value, [12].

3.2 Eigencentrality Basis

Figure 3 shows a two-dimensional representation of the eigencentrality basis; each node is represented as a light blue circle, and each link is represented as a blue line. The orange circle in the middle of the graph represents the disconnected node. Each eigencentrality vector will be an n-dimensional vector. In this example, the eigencentrality vectors are 7-dimensional vectors; one value for each node. Plotting the first two eigencentrality vectors typically produce visually pleasing representation of the network because they place the least connected nodes at the edge and the most connected nodes at

the center [13]. Networks are typically drawn this way; the core of the network is in the center of the diagram, and the access network is at the edge.

The network could be as easily plotted using the two eigenvectors associated with the two largest eigenvalues. This representation would place the most connected nodes at the edge of the graph and the least connected in the center as shown in Figure 4. This is not an intuitive way of visualizing the network, but when analyzing network structure it is important to understand the two extremes.
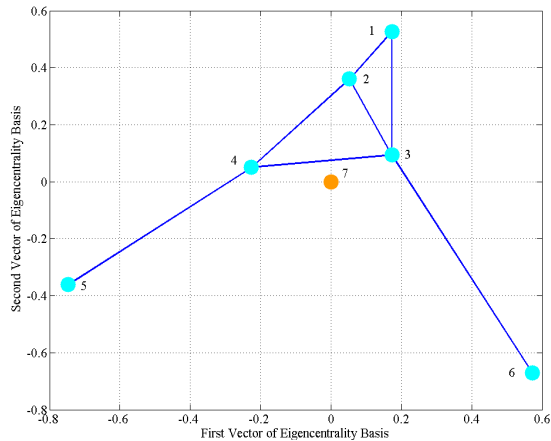
Figure 3. Plot of the first two components of the eigencentrality basis

3.3 Nodal Basis

The nodal basis is the basis that defines how influential a specific node is across a range of eigenvalues. This space defines the magnitude spectrum of the node across the full range of eigenvalues or eigenspectrum [14]. From this space, a node's importance to the rest of the graph can be determined.
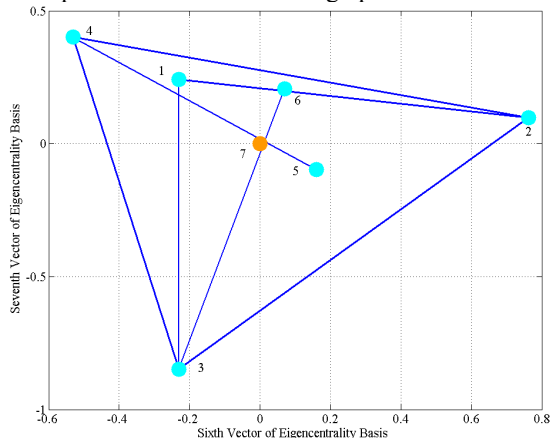

Figure 4. Plot of the sixth and seventh components of the eigencentrality basis

The nodal basis of node six only has five values because there are only five non-zero eigenvalues: [0.588, 0.130, 0.110, 0.026, 0.009]. The magnitude response across the eigenspectrum demonstrates a node's importance at each eigenvalue. Node six has its greatest magnitude at eigenvalue three, the first non-zero eigenvalue. This peak corresponds with the fact that it is not well connected and is the most important node when determining the algebraic connectivity of this network. Each node in Figure 1 has similar responses with a peak at a distinct eigenvalue. Using this information, the nodes can be ranked from least connected to most connected or least important to most important to the robustness of the network.

3.4 Null and Reachability Space

There will always be one or more zero eigenvalues of the Laplacian. The zero eigenvalues define the null space of the Laplacian. The null space is mathematically and physically interpreted as a part of the solution space or network that is unreachable. The reachability space is defined as the space in which there is guaranteed to be a route from all nodes to all other nodes. The reachability space is fully defined by the dual basis of V and V^T after the null space has been removed.

The number of non-zero eigenvalues and the size of the reachability space is equal to the $\text{rank}(Q)$. The size of the null space is equal to the $n - \text{rank}(Q) > 0$. The size of the null space determines the length of the vectors in the nodal space; the nodal space eigenvectors will have dimensionality equal to the $\text{rank}(Q)$. The eigencentrality basis vectors will always have a length equal to n [12].

As the bit error rate increases to a threshold, nodes will become disconnected from the larger ad hoc network and will enter the null space. As a function of time, the null space will grow and shrink as nodes leave and enter the network. Nodes within the null space may be able to reach each other, but they are not useful because they are not able to connect to the larger ad hoc network that is performing the required task.

4. Modeling and Analysis Simulations

To demonstrate the usefulness of the dual basis, reachability space and null space in practice, we simulated a mobile node moving through a mesh network. As the mobile node moves through the network, different nodes in the network are reachable within one hop. The E_b/N_0 between nodes determines its eigencentrality within the network.

4.1 Simulation Description

Figure 1 shows the path node seven will take through the network. Two sets of simulations were run; the first set did not include shadowing and the second did. The reason for this choice was to first show the mean behavior of the network in the dual basis, and then show an example of decoupled nodal behavior when shadowing is added to the simulation. The threshold for a connection was selected to be 6.5 dB because

binary phase shift keying (BPSK) requires a E_b/N_0 greater than that value to ensure a probability of bit error of 10^{-3}[11]. The transmit power was held fixed at 1 W. The AWGN and interference were set constant at -75 dB. The node moves in a straight line at 1.3 m/s (2.9 mph). These selections were arbitrary and could be changed in order to better model a specific mobile device and wireless environment.

4.2 Simulation Results without Shadowing

The eigenvalues of the network are shown in Figure 5. The left hand axis shows the magnitude of the eigenvalues, and the right hand axis shows the number of nodes that node seven is able to reach in one hop. The discontinuities in the figure are due to the use of a threshold, τ. As soon as E_b/N_0 increases above the threshold, a new link is added to the set of all links in the graph. This discrete change in the integer number of links is reflected in a discrete change in the eigenvalues. The continuous behavior of the eigenvalues is due to the continuous nature of E_b/N_0 as a function of distance.

Between 90 and 600 seconds, the algebraic connectivity of the network does not change significantly until node seven is only connected to node six and is exiting the network. The result of this topology is the decreased algebraic connectivity after 600 seconds. In this simulation, the algebraic connectivity provides little information about the overall dynamics of the network. The remaining eigenvalues provide the majority of the information, specifically, λ_5, λ_6, and λ_7.
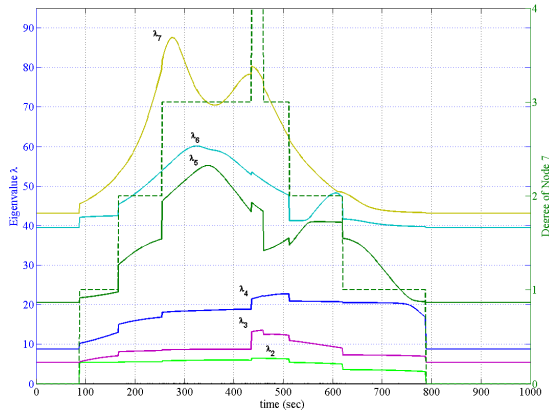

Figure 5. Eigenvalues without shadowing and the degree of node seven (dashed)

As noted above, the discontinuities are due to the use of the threshold to determine inclusion into the set of all links. The difference in the sum of the magnitudes of any set of eigenvalues at time $t - \Delta t$, just before a discontinuity, and time t is

$$2\tau = \sum_{k=1}^{n} \lambda_k(t) - \sum_{k=1}^{n} \lambda_k(t - \Delta t) \qquad (14)$$

Eqn. 14 is derived from the fact that [3]

$$\sum_{k=1}^{n} \lambda_k = \sum_{k=1}^{n} Q(k,k) \qquad (15)$$

Since the Laplacian matrix was not normalized, when a new link is added to or removed from the set of all links there is a discrete change of 13 dB in the sum of the eigenvalues. The reason the difference is 2τ is because each link is counted twice; it is counted once from node i to node j and counted again from node j to node i. The proportion of the 2τ that shows up in each eigenvalue is determined by where the node connects to the network, which is a benefit of our approach. The eigendecomposition determines how the additional link weights should be distributed among each basis.

Figure 6 shows the eigencentrality basis related to the largest eigenvalue. This basis shows the relative centrality of the most central nodes. In this simulation, there are three nodes that become the most central node at different times. When node seven connects to the network, node two (green, dashed line) is the most central node. Then as node seven approaches the center of the network, it takes over as the most central. Then as node seven leaves the network, node three, and briefly node six, become the most central nodes. At the crossover points, two nodes share the same centrality metric, which implies that these two nodes will have equal impact if removed from the network. Since they are associated with the largest eigenvalue, these most central nodes will have the largest negative impact on performance if removed [12].
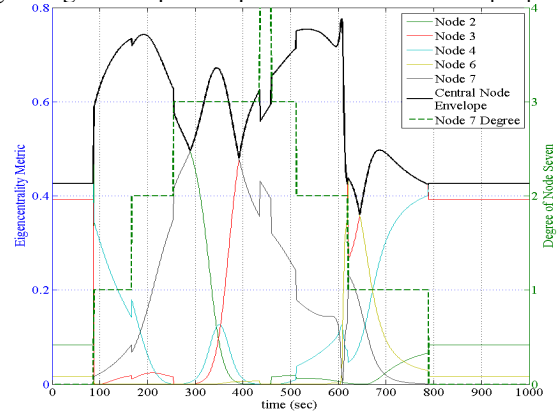

Figure 6. Eigencentrality metrics for the maximum eigenvalue

Figure 7 shows the first crossover from Figure 6 in the two-dimensional eigencentrality space. The crossover at 290 seconds is clearly depicted in Figure 7 since nodes two and seven are equal distance from the center of the graph and have opposite signs. Node four has a small value along the seventh vector axis, but has a large value along the sixth vector axis. This representation provides a plot of the graph that is visually appealing and clearly distinguishes the most critical nodes in the network. As compared to Figure 4, Figure 7 is similar, but now one can see that node seven has taken the place of node three as more critical to the performance of the network. Figures 6 and 7 are presenting similar information, but because the eigencentrality basis has more dimensions than can be visualized, it is difficult to present a single figure that captures all of the relevant information. Figure 6 shows the changes over time, but for only one eigenvector. Figure 7

shows an instance in time, but is able to show multiple vectors. Adding animation to Figure 7 provides a way to visualize the dynamics of the network in two or three dimension as a function of time.

The detail shown in Figure 8 shows the crossover points of the various nodes when node seven is only connected to node six and is leaving the network. Because node six is strongly connected to node seven and node three, it is the most central after node seven disconnects from node three. This changes as node seven leaves the network. The nodes one would expect to be most critical, nodes three and four, are increasing in value as node seven departs the network. In the context of decoupling, the crossover points are times when two nodes share equal influence over an eigenvalue. This situation does not allow the criticality to the structure of the two nodes to be decoupled.
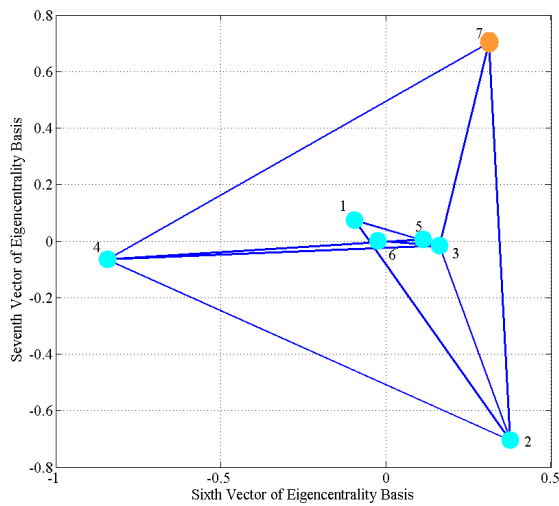

Figure 7. Plot of the sixth and seventh components of the eigencentrality basis at 290 seconds

Figure 9 shows the eigencentrality basis associated with the smallest eigenvalue. This basis describes the least central nodes. The nodes with only one link are the most influential in this basis throughout the simulation. Using only λ_2 in this simulation would provide little information about the overall health and robustness of the network. The core of the network is becoming more robust with increased E_b/N_0 and increased path diversity. While the nodes with only one link can be easily separated from the network. Therefore, the overall network is becoming more robust, but the areas of the network that affect algebraic connectivity have not changed.

4.3 Simulation Results with Shadowing

The addition of shadowing adds a new dimension to the analysis via dual basis. Because of the way we modeled the presence of a link in the set, a link is added to the set and removed from the set multiple times if node seven is near the mean distance for connectivity. This is due to the changes in

E_b/N_0 as a function of time. The key insight to notice in Figure 10 is that node seven is rapidly adding and removing a link to node one between 300 and 400 seconds. The result of this additional 13 dB is almost entirely accounted for in λ_4 and λ_7.
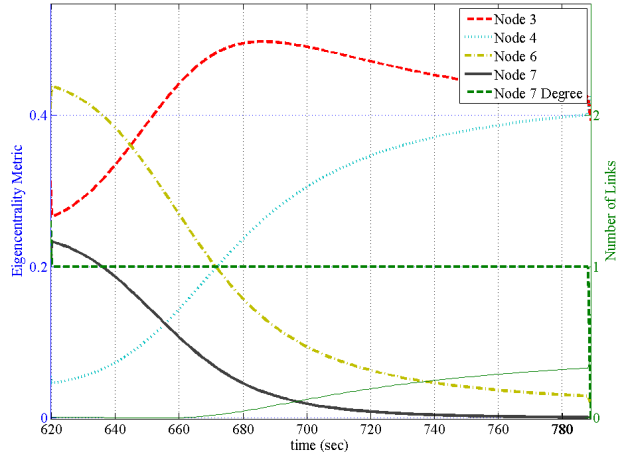

Figure 8. Detail of the eigencentrality metric when node seven is only connected to node six
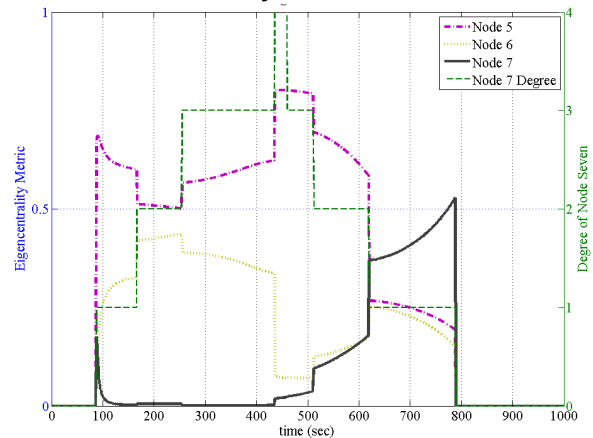

Figure 9. Eigencentrality metrics for the minimum eigenvalue as node seven traverses the network

Figure 11 shows the reason for this distribution of the 13 dB. The nodal basis of node one shows that it has a large eigencentrality metric in λ_4, which implies that it has a large influence over λ_4 [12]. The spectrum of node seven is similar to that shown in Figure 11, but it reveals that node seven has a large influence over λ_7. The addition of this link has been decoupled from the adjacent nodes, and its impact has been limited to two eigenvalues.

The analysis of the network with shadowing shows the benefit of our approach. The eigendecomposition of the network presents a method to simplify a complex network to understand the decoupled interactions between nodes in the graph. Figures 10 and 11 show that the interaction between nodes one and seven is almost completely contained in

eigenvalues four and seven. There is a clear decoupling of this interaction from the interaction with the other nodes. This decoupling is not unique to this simulation because the dual representation creates two orthonormal bases that decouple nodes in a network to the extent possible [15].
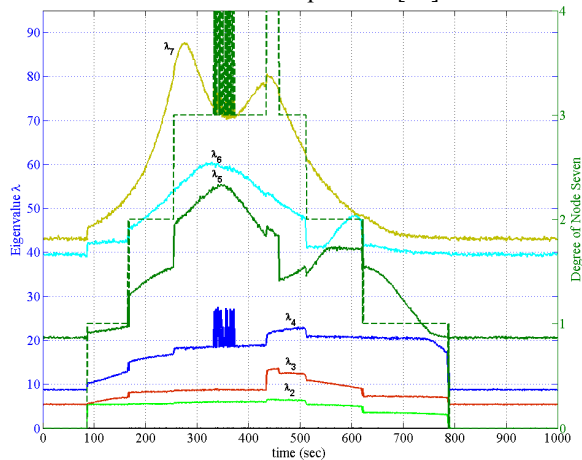

Figure 10. Eigenvalues with shadowing and the degree of node seven (dashed)

Repeated eigenvalues and crossover points are two examples when the interactions cannot be decoupled. In these two cases, not being able to decouple the two is an important characteristic of the network. Repeated eigenvalues indicate that the network has some duplicate connections that may or may not be desired. The crossover points indicate that two or more nodes have equal influence over a particular eigenvalue or they have equal impact when removed from the network.
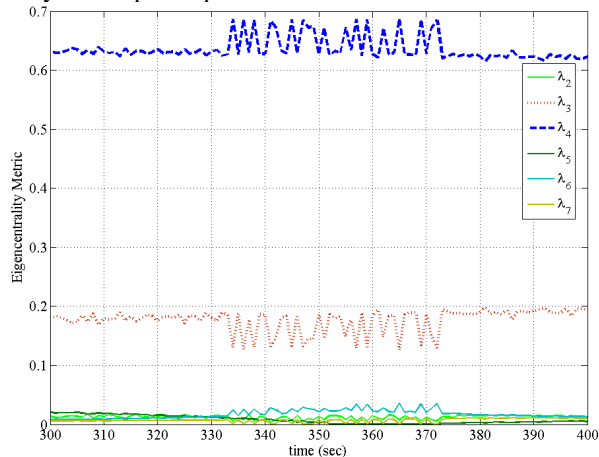

Figure 11. Spectrum of node one as node seven nears the mean distance for connectivity

5. Conclusions

In this paper, we have presented the analytical basis for a novel way to model and analyze the performance of wireless mobile ad hoc networks. We have also presented an analysis of a simulation to show the efficacy of our approach. Our approach's use of the dual basis representation allows researchers to better understand the dynamics of their networks via a decoupling of the behavior of independent nodes. The eigencentrality basis and nodal basis provide information about which nodes are most influential. We have presented a small number of cases where one can and cannot decouple the behavior of various nodes. These few cases show that our analysis effectively locates coupled nodes, the most central node in the network and weakly connected nodes. All of these are required when using the analysis to improve the performance and robustness of a network. Once these have been identified, known techniques can be used to maximize algebraic connectivity, path diversity, and minimize average path length [6][7][8].

We have modeled the link weights using one particular method. More detailed models of the wireless channel and the mobility of nodes could be used. More detailed models provide a better foundation to understand the performance of the network. The models that are chosen by researchers should be focused on their needs and requirements.

In future work, we will explore the meaning of repeated eigenvalues in the context of redundant connectivity in a network and how to interpret the meaning of the eigencentrality metric for these repeated eigenvalues. The link weight can also be used to model the capacity of a link and the amount of capacity that is used on that link. Greater link weights would be given to links that have more capacity and less utilization. Modeling a network in this fashion provides information about which areas of the network are most utilized and underutilized.

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