Stable sets, corner polyhedra and the Chvátal closure

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Abstract

We consider the edge formulation of the stable set problem. We characterize its corner polyhedron, i.e. the convex hull of the points satisfying all the constraints except the non-negativity of the basic variables. We show that the non-trivial inequalities necessary to describe this polyhedron can be derived from one row of the simplex tableau as fractional Gomory cuts. It follows that the split closure is not stronger than the Chvátal closure for the edge relaxation of the stable set problem.

Keywords: Stable set, corner polyhedron, Chvátal closure, odd cycle inequality.

1 Introduction

Consider a simple graph $G = (V, E)$, where V and E are the sets of n vertices and m edges of G, respectively. A *stable set* (independent set, vertex packing) of G is a set of pairwise non-adjacent vertices. For the sake of simplicity, we are going to assume that G has no connected component defined by a single vertex. Then, a stable set corresponds to an n-dimensional binary vector x that satisfies $x_u + x_v \leq 1$, for all $uv \in E$. The set of stable sets of G can be described by the mixed integer linear set

$$S(G) = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^m : \ Ax + y = \mathbf{1}\}, \tag{1}$$

where A is the edge-vertex incidence matrix of G, $\mathbf{1}$ is a vector of ones, x and y are vectors of variables indexed by the vertices and the edges of G, respectively. The convex hull of $S(G)$ is called the stable set polytope. The set obtained from $S(G)$ by relaxing the integrality constraints will be denoted by $R(G)$.

An important reason for studying methods to handle contraints of type (1) is that they can model restrictions appearing in many optimization problems. Indeed, notice that set packing and partitioning problems can be transformed into vertex-packing problems on the intersection graph.

A general approach to deal with mixed integer sets consists of solving the linear relaxation and adding cutting planes. Most commonly, these cutting planes are derived from integrality arguments applied to a single equation. Intersection cuts are more general [4]. Recently, cutting planes derived from two or more rows have attracted renewed interest [2, 6, 9], and it has been shown that they can better approximate the feasible set [3, 5]. Whether they are derived from one or more rows by integrality arguments, the known cuts have in common the feature that they are valid for the corner polyhedra defined by Gomory [14].

In the remainder, let $\mathcal{B}$ stand for the set of all (feasible or infeasible) bases of the constraint matrix $[A \quad I]$. The *corner polyhedron* associated with a basis $B \in \mathcal{B}$, to be denoted corner(B), is the convex hull of the feasible points of the relaxation of (1) where the non-negativity constraints on the basic variables are discarded. A valid inequality for corner(B) is valid for (1).

In this paper, we study the corner polyhedra for (1) and, consequently, investigate cuts that can be derived from a basic solution of the linear relaxation of (1). We show that all non-trivial valid inequalities necessary to the description of corner(B) can be derived from one row of (1) as a Chvátal-Gomory cut [8, 13]. In addition, we relate the intersection of the corner polyhedra associated with all bases, given by

$$\text{corner}(\mathcal{B}) = \bigcap_{B \in \mathcal{B}} \text{corner}(B),$$

to the split, Chvátal and $\{0, 1/2\}$-Chvátal closures relative to $S(G)$. The Chvátal closure is the intersection of $R(G)$ with all the Chvátal inequalities, i.e. inequalities of the form $\lfloor \lambda A \rfloor x \leq \lfloor \lambda \mathbf{1} \rfloor$ with $\lambda \in \mathbb{R}^m_+$ [8]. When λ is restricted to be in $\{0, 1/2\}^m$, the $\{0, 1/2\}$-Chvátal inequalities and closure are similarly defined [7].

It follows from [1] that the polyhedron corner$(\mathcal{B})$ is included in the split closure. In turn, the split closure is included in the Chvátal closure, which is itself contained in the $\{0, 1/2\}$-Chvátal closure. Here, we show the converse inclusions and obtain the following theorem.

Theorem 1. *For the stable set formulation* (1), *the* $\{0, 1/2\}$-*Chvátal closure, the Chvátal closure, the split closure and* corner$(\mathcal{B})$ *are all identical to*

$$\{(x, y) \geq 0 : Ax + y = \mathbf{1}, \sum_{v \in \mathcal{C}} x_v \leq \frac{|C| - 1}{2}, \forall \text{ induced odd cycle } C \text{ of } G\}.$$

In this paper, a *cycle* is a sequence $C = (v_0, e_1, v_1, \ldots, e_k, v_k = v_0)$, where $v_1, v_2, ..., v_k$ are distinct vertices and $e_i = v_{i-1} v_i$, for $i = 1, 2, \ldots, k$, are distinct edges. Such a sequence is often called a *circuit* in the literature [11]. The cycle C is *induced* if every edge of the graph connecting two vertices of C is an edge of C. It is *odd* if the number k of vertices (and edges) is odd. In the remainder, as already used in Theorem 1, $|C|$ stands for the number of vertices of C. Also, we simply write $v \in C$ or $e \in C$ to mean that a vertex v or an edge e is in C.

In order to obtain Theorem 1, we first describe a graph related to the basic matrix in the next section. Section 3 is devoted to the characterization of the corner polyhedra, whereas Section 4 relates them to the split, Chvátal and $\{0, 1/2\}$-Chvátal closures.

2   The basic graph

Consider any basis $B \in \mathcal{B}$, feasible or infeasible, and the corresponding nonbasic matrix N. Let x_B, y_B and x_N, y_N represent the basic and nonbasic variables, respectively. The vertices and edges indexing these variables will be respectively called basic and nonbasic.

Let us group the equations of (1) according to the basic and nonbasic edges to get

$$\bar{A}x + y_N = \mathbf{1}, \tag{2}$$
$$\hat{A}x + y_B = \mathbf{1}, \tag{3}$$

where $A = \begin{bmatrix} \bar{A} \\ \hat{A} \end{bmatrix}$. Since the non-negativity of the variables y_B is discarded in the definition of $\mathrm{corner}(B)$, constraints (3) become redundant. So, let us focus on the first group of constraints. Toward this end, we define the basic graph G_B, which is obtained from G by removing the basic edges. Notice that $\bar{A}$ is the edge-vertex incidence matrix of G_B. Actually, this relation can still be considered in the trivial case where all edges are basic.

Let $C_i = (V_i, E_i)$, $i = 1, 2, \ldots, k$, be the connected components of G_B that are not defined by a single vertex. If A_i is the $|E_i| \times |V_i|$ incidence matrix of C_i, then $\bar{A}$ can be organized as

$$\bar{A} = \begin{bmatrix} A_1 & & & & \\ & A_2 & & & 0 \\ & & \ddots & & \\ & & & A_k & \end{bmatrix}.$$

For every $i = 1, 2, \ldots, k$, let us partition $A_i = [B_i \ N_i]$, where B_i and N_i comprise the columns of A_i that are indexed by the basic and nonbasic vertices, respectively.

Lemma 2. *For every $i = 1, 2, \ldots, k$, the matrix B_i is square and invertible.*

Proof: We have that $B = \begin{bmatrix} \bar{B} & 0 \\ \hat{B} & I \end{bmatrix}$, where $\bar{B}$ and $\hat{B}$ are submatrices of $\bar{A}$ and $\hat{A}$, respectively. Since B is a basis, $\bar{B}$ must be invertible. Therefore,

$$\bar{B} = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{bmatrix}.$$

In addition, the rows and columns of each B_i must be linearly independent. So, each B_i is square and invertible. $\qquad\qquad\square$

Now, we show that each connected component of G_B is a tree or a 1-tree. A 1-tree is a connected graph with exactly one cycle.

Lemma 3. *For every $B \in \mathcal{B}$, each connected component of G_B is either a tree or a 1-tree with an odd cycle. Each tree has exactly one nonbasic vertex. The vertices of every 1-tree are all basic.*

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Proof: If a connected component of G_B is not a tree, it must be a component C_i, for some $i \in \{1, 2, \ldots, k\}$ such that $|E_i| \geq |V_i|$. By Lemma 2, this means that $|E_i| = |V_i|$ and B_i is the incidence matrix of C_i. Therefore, C_i is a tree together with an extra edge. Moreover, the unique cycle of C_i must be odd because B_i is invertible [15].

The remaining part of the statement follows by observing the structure of $\bar{B}$. It shows that every basic vertex is in some component C_i, which has exactly $|V_i| - |E_i|$ nonbasic vertices. $\qquad\square$

From now on, let $I_B \subseteq \{1, 2, \ldots, k\}$ be the set of indices of the 1-tree components of G_B. For $i \in I_B$, the kernel of C_i, denoted by κ_i, is its unique cycle.

3 The Corner polyhedron

The corner polyhedron associated with a basis B of the stable set formulation (1) is the convex hull of the points satisfying the relaxation where the non-negativity constraints on the basic variables are discarded, that is

$$\text{corner}(B) = \text{conv}\{(x, y) : Ax + y = \mathbf{1},\ x_N \geq 0, y_N \geq 0, x \in \mathbb{Z}^n\}. \tag{4}$$

Using (2)–(4), it follows that

$$\text{corner}(B) = \text{conv}\{(x, y) : \bar{A}x + y_N = \mathbf{1},\ x_N \geq 0, y_N \geq 0, x \in \mathbb{Z}^n, \tag{5}$$
$$\hat{A}x + y_B = \mathbf{1}\}. \tag{6}$$

Since y_B is not restricted in sign and appears only in constraints (6), we have that

$$\text{corner}(B) = \{(x, y_N, y_B) : (x, y_N) \in \text{conv}\{\bar{A}x + y_N = \mathbf{1},\ x_N \geq 0, y_N \geq 0, x \in \mathbb{Z}^n\},$$
$$\hat{A}x + y_B = \mathbf{1}\}.$$

Notice that this expression can also comprise the trivial case where all edges are basic. In this case, matrix $\bar{A}$ and vector y_N do not exist, which leads to $\text{corner}(B) = \{(x, y) : Ax + y = \mathbf{1}, x \geq 0\}$.

By defining $P(B) = \{x \in \mathbb{Z}^n : \bar{A}x \leq 1, x_N \geq 0\}$, we can simplify the expression of the corner polyhedron as

$$\text{corner}(B) = \{(x, y) : x \in \text{conv}(P(B)),\ Ax + y = \mathbf{1}\}. \tag{7}$$

Also, we can rewrite

$$P(B) = \{x \in \mathbb{Z}^n : \tilde{A}x \leq b\}, \tag{8}$$

where

$$\tilde{A} = \begin{bmatrix} \bar{B} & \bar{N} \\ \hat{0} & -2I \end{bmatrix}, \quad b = \begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix},$$

and $\bar{A} = [\bar{B}\ \bar{N}]$ is a partition of $\bar{A}$ according to (x_B, x_N).

Using expression (8), we can view $P(B)$ as the set of integral solutions of a system defined by the edge-vertex incidence matrix $\tilde{A}$ of a bidirected graph [12]. Such a graph is formed by G_B together with a loop for each nonbasic vertex.

The definition of cycle in graphs extends directly to bidirected graphs [12]. However, a cycle in a bidirected graph is called *odd* if it has an odd number of edges whose endpoints are both positive or both negative according to the incidence matrix [11, 12].

By Lemma 3, each connected component of the bidirected graph associated with $P(B)$ has exactly one odd cycle (a loop was included in each tree component of G_B). This property can be used to readily describe the convex hull of $P(B)$ by means of the following result by Gerards and Schrijver [12]. See also Schrijver [11, Section 68.6b], where an odd K_4-subdivision is defined as a subdivision of the complete graph with four vertices such that each triangle has become an odd cycle.

Theorem 4 ([12, 11]). *If M is the $p \times q$ edge-vertex incidence matrix of a bidirected graph with no odd K_4-subdivision and $b \in \mathbb{Z}^p$, then*

$$conv\{x \in \mathbb{Z}^q : Mx \le b\} =$$
$$\{x \in \mathbb{R}^q : Mx \le b, \frac{1}{2}\sum_{v \in C}\sum_{e \in C} M_{ev}x_v \le \lfloor\frac{1}{2}\sum_{e \in C} b_e\rfloor, \text{ for each odd cycle } C\}.$$

Since the bidirected graph defining $P(B)$ is such that each connected component has exactly one cycle, it does not contain a K_4-subdivision. Therefore, the above theorem gives the following statement.

Corollary 5. *For every $B \in \mathcal{B}$, $conv(P(B)) = \{x : \bar{A}x \le \mathbf{1}, x_N \ge 0, \sum_{v \in \kappa_i} x_v \le (|\kappa_i| - 1)/2, \forall i \in I_B\}$.*

Corollary 5 together with (7) imply the following characterization of the corner polyhedron.

Theorem 6. *For every $B \in \mathcal{B}$, the corner polyhedron of (1) associated with B is*

$$corner(B) = \left\{(x, y) : Ax + y = \mathbf{1}, x_N \ge 0, y_N \ge 0, \sum_{v \in \kappa_i} x_v \le \frac{|\kappa_i| - 1}{2} \le 1, \forall i \in I_B\right\}.$$

Theorem 6 can also be proved without appealing to Theorem 4 by characterizing the matrix $\bar{B}^{-1}$, which essentially gives the basis inverse. Actually, we can show that each column of the coefficient matrices of $\bar{B}^{-1}\bar{A}x + \bar{B}^{-1}y_N = \bar{B}^{-1}\mathbf{1}$ has its entries either in $\{0, \pm 1\}$ or in $\{0, \pm 1/2\}$. This special structure also leads to the characterization of conv(P(B)) and, consequently, the statement of Theorem 6.

Remark 7. *Each odd cycle inequality appearing in the description of the corner polyhedron is a $\{0, 1/2\}$-Chvátal inequality. Indeed, it can be obtained with the Chvátal procedure by taking a linear combination of the constraints $x_u + x_v \le 1$, for all $e = uv \in \kappa_i$, with multipliers $\lambda_e = 1/2$, for all $e \in \kappa_i$.*

The above results imply that each non-trivial inequality necessary to describe the corner polyhedron is a fractional Gomory cut that can be derived from a row of the simplex tableau related to a basic vertex in the cycle of a 1-tree component. Even if we consider the tighter relaxation of (1) where only constraints $y_B \ge 0$ are discarded, we cannot get stronger cuts. Indeed, if we keep the non-negativity of the (basic) variables associated with the vertices, we can still use Theorem 4 to see that only the trivial inequalities $x_B \ge 0$ can be obtained in addition to those describing corner(B).

4 The Chvátal closure

Theorem 6 and Remark 7 show that the only non-trivial inequalities needed to define corner(B) are $\{0, 1/2\}$-Chvátal inequalities. They all have the form $\sum_{v \in C} x_v \leq (|C|-1)/2$, where C is an odd cycle of G. If this cycle is not induced in G, the corresponding inequality is dominated by $\sum_{v \in C'} x_v \leq (|C'| - 1)/2$, where C' is an odd cycle of G induced by a subset of vertices of C. This is the key point to characterize the Chvátal closure of $S(G)$, as follows.

Let $\mathcal{C}$ denote the set of all the induced odd cycles of G. For every $C \in \mathcal{C}$, form the submatrix B_C of $[A \quad I]$ given by

$$B_C = \begin{bmatrix} A_C & 0 \\ 0 & I \end{bmatrix},$$

where A_C is the edge-vertex incidence matrix of C and I is related to the edges of G not in C. Notice that B_C is invertible and $B_C^{-1}\mathbf{1} \in \{1/2, 1\}^m$, which implies that B_C is a feasible basis. Let us denote by $\mathcal{B}_+ = \{B \in \mathcal{B} : B^{-1}\mathbf{1} \geq 0\}$ the set of feasible basis and by $\mathcal{B}_\mathcal{C} = \{B_C : C \in \mathcal{C}\}$ its subset corresponding to $\mathcal{C}$. Also, for any $\mathcal{B}' \subseteq \mathcal{B}$, let corner($\mathcal{B}'$) stand for the intersection of the corner polyhedra associated to all bases in $\mathcal{B}'$, that is,

$$\text{corner}(\mathcal{B}') = \bigcap_{B \in \mathcal{B}'} \text{corner}(B).$$

Lemma 8. *corner*($\mathcal{B}$) $\subseteq$ *corner*($\mathcal{B}_+$) $\subseteq \mathbb{R}_+^n \times \mathbb{R}_+^m$.

Proof: We only need to prove the second inclusion. First, note that corner($\mathcal{B}_+$) $\subseteq$ corner(I) $\subseteq \mathbb{R}_+^n \times \mathbb{R}^m$. Now, for each edge $e = uv$ of G, consider the submatrix B_e of $[A \quad I]$ defined by the column of A indexed by u and the columns of I indexed by $E \setminus \{e\}$. Notice that $B_e \in \mathcal{B}_+$. In addition, by Theorem 6, $y_e \geq 0$ is a valid inequality for corner(B_e). Since corner($\mathcal{B}_+$) $\subseteq \cap_{e \in E}$corner(B_e), the result follows. $\square$

Lemma 9. *corner*($\mathcal{B}$) = *corner*($\mathcal{B}_+$) = $\bar{S}(G)$, *where*

$$\bar{S}(G) = \left\{ (x, y) \geq 0 : Ax + y = \mathbf{1}, \sum_{v \in C} x_v \leq \frac{|C| - 1}{2}, \forall C \in \mathcal{C} \right\}.$$

Proof: If $\mathcal{C} = \emptyset$, the result trivially follows by Theorem 6 and Lemma 8. The same statements imply that

$$\text{corner}(\mathcal{B}) \subseteq \text{corner}(\mathcal{B}_+) \subseteq \text{corner}(\mathcal{B}_\mathcal{C}) \cap (\mathbb{R}_+^n \times \mathbb{R}_+^m) = \bar{S}(G).$$

To obtain equalities in the expression above, it suffices to note that there is always a subset of the vertices of any non-induced odd cycle C that defines an induced odd cycle $C' \in \mathcal{C}$. Then, the inequality $\sum_{v \in C'} x_v \leq (|C'|-1)/2$ is tighter than $\sum_{v \in C} x_v \leq (|C|-1)/2$, which shows that $\bar{S}(G) \subseteq$ corner($\mathcal{B}$). $\square$

Lemma 9 and the fact that all inequalities needed in the description of corner($\mathcal{B}$) are $\{0, 1/2\}$-Chvátal inequalities lead to the following more complete version of the theorem stated in the introduction.

Theorem 10. *For the stable set formulation* (1)*,* $\bar{S}(G)$*, the* $\{0, 1/2\}$*-Chvátal closure, the Chvátal closure, the split closure, corner$(\mathcal{B}_+)$ and corner$(\mathcal{B})$ are identical.*

We should stress that the equivalences stated in Theorem 10 may not be valid for other formulations of the stable set problem. For instance, for a quasi-line graph the split closure of the clique formulation of the stable set problem coincides with the convex hull of the integer solutions and is strictly contained in the Chvátal closure [10].

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