Elementary Closures for Integer Programs [*]

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Abstract

In integer programming, the *elementary closure* associated with a family of cuts is the convex set defined by the intersection of all the cuts in the family. In this paper, we compare the elementary closures arising from several classical families of cuts: three versions of Gomory's fractional cuts, three versions of Gomory's mixed integer cuts, two versions of intersection cuts and their strengthened forms, Chvátal cuts, MIR cuts, lift-and-project cuts without and with strengthening, two versions of disjunctive cuts, Sherali-Adams cuts and Lovász-Schrijver cuts with positive semi-definiteness constraints.

Key Words: Integer programming, cutting plane, elementary closure.

1   Introduction

Recently, the integer programming community has emphasized that many of the cuts found in the literature are essentially the same. Chvátal cuts [12] are equivalent to Gomory fractional cuts [20, 21, 23]. Lift-and-project cuts [4] are disjunctive cuts [3]. Gomory mixed integer cuts [22], disjunctive cuts [2, 9, 24] and mixed integer rounding cuts [28] are equivalent [26]. It is natural to ask which of these cuts are intrinsically different. This is the purpose of

this paper. Our approach is to compare the corresponding elementary closures. Given an integer program $P_I \equiv \{(x,y) \in Z_+^n \times R_+^p | \ Ax + Gy \leq b\}$ and a family F of inequalities $\alpha x + \gamma y \leq \beta$ generated from $P \equiv \{(x,y) \in R_+^{n+p} | \ Ax + Gy \leq b\}$ and valid for P_I, the *elementary closure P_F* is a convex set obtained as the intersection of all the inequalities in F. This concept was introduced by Chvátal [12] for a specific family of cuts. In this paper, we consider elementary closures associated with other families of cuts. Some of the cuts that we consider are designed for mixed integer programs, others just for mixed 0,1 programs or pure integer programs. They can all be applied to pure 0,1 programs. We compare eighteen families of cuts that have appeared in the literature by comparing the corresponding eighteen elementary closures. We show that ten of these sets are distinct and we establish all inclusion relationships between them.

2 Several Classical Families of Cuts

Let A and G be given rational matrices (dimensions $m \times n$ and $m \times p$ respectively) and b a given rational column vector (dimension m). Consider the polyhedron $P \equiv \{(x,y) \in R^{n+p} | \ Ax + Gy \leq b\}$. A *cut* is a valid inequality for the set $P_I \equiv \{(x,y) \in Z^n \times R^p | \ Ax + Gy \leq b\}$. We will sometimes restrict our attention to the pure case ($p{=}0$) or to the mixed 0,1 case (the constraints $Ax + Gy \leq b$ contain the inequalities $0 \leq x \leq 1$). In all cases, we assume without loss of generality that $x \geq 0$ and $y \geq 0$ are part of the constraints for P.

2.1 Chvátal cuts and P_C

In the pure case, the inequality $\lfloor uA \rfloor x \leq \lfloor ub \rfloor$ is valid for P_I for any $u \in R_+^m$. Here, $\lfloor uA \rfloor$ denotes the vector obtained from the vector uA by rounding down every component to an integer. These cuts are known as *Chvátal cuts* [12]. Let P_C denote the corresponding elementary closure.

2.2 Gomory fractional cuts and P_F, P_{FB}, P_{FBF}

In the pure case, Gomory [20, 21, 23] introduced fractional cuts when the constraints are in equality form. Assume, without loss of generality, that A and b are integral. Note that $P = \{x \in R^n | Ax \leq b\}$ can be equivalently expressed as $P' = \{(x,s) \in R^{n+m} | Ax + s = b, \ s \geq 0\}$. Let $P_I' = \{(x,s) \in Z^{n+m} | Ax + s = b, \ s \geq 0\}$. For any $u \in R^m$, the inequality $f\binom{x}{s} \geq f_0$ is

valid for P'_I, where $f = u(A, I) - \lfloor u(A, I) \rfloor$ and $f_0 = ub - \lfloor ub \rfloor$. Plugging $s = b - Ax$ into it, we get the *fractional cut* $\lfloor uA \rfloor x - \lfloor u \rfloor Ax \leq \lfloor ub \rfloor - \lfloor u \rfloor b$ in the space of x. Let P_F denote the elementary closure of the fractional cuts.

Originally, Gomory proposed to generate fractional cuts from basic feasible solutions of P'. Let $z = \binom{x}{s} \in R^{n+m}_+$. Then $P' = \{z \in R^{n+m}_+ | (A, I)z = b\}$. Consider a basic feasible solution $\bar{z}$ of P'. Let B and N be the index sets of its basic and nonbasic variables respectively. Then, for $i \in B$,

$$z_i - \sum_{j \in N} \bar{a}_{ij} z_j = \bar{a}_{i0}.$$

is obtained as a linear combination of the equations in the original system $Ax + s = b$ with some multipliers $u \in R^m$. Clearly, u depends on $\bar{z}$ and i, and there is only a finite number of such choices. Any Gomory fractional cut obtained this way is called a *fractional cut from a basic feasible solution*. Let P_{FBF} be the corresponding elementary closure. Similarly, let P_{FB} be the elementary closure of the fractional cuts generated from all the basic solutions.

2.3   Gomory mixed integer cuts and P_{MI}, P_{MIB}, P_{MIBF}

In [22], Gomory introduced cuts that are at least as strong as his fractional cuts in the pure case and, in addition, can be applied to the mixed case. Let $P' = \{(x, y, s) \in R^{n+p+m} | Ax + Gy + s = b, \; s \geq 0\}$ and $P'_I = \{x \in Z^n, \; (y, s) \in R^{p+m} | Ax + (G, I)\binom{y}{s} = b, \; s \geq 0\}$. Introduce $z = \binom{y}{s}$. For any $u \in R^m$, let $\bar{a} = uA$, $\bar{g} = u(G, I)$ and $\bar{b} = ub$. Let $\bar{a}_i = \lfloor \bar{a}_i \rfloor + f_i$ and $\bar{b} = \lfloor \bar{b} \rfloor + f_0$. The following inequality, known as *Gomory mixed integer cuts* [22], is valid for P'_I:

$$\sum_{(i:f_i \leq f_0)} f_i x_i + \frac{f_0}{1 - f_0} \sum_{(i:f_i > f_0)} (1 - f_i) x_i + \sum_{(j:\bar{g}_j \geq 0)} \bar{g}_j z_j - \frac{f_0}{1 - f_0} \sum_{(j:\bar{g}_j < 0)} \bar{g}_j z_j \geq f_0.$$

Plugging $s = b - Ax - Gy$ into it, we get a valid cut for P_I. Let P_{MI} denote the corresponding elementary closure.

Just as we defined fractional cuts from basic feasible solutions, we can also define *mixed integer cuts from basic feasible solutions*.

Let P_{MIBF} be the elementary closure of the mixed integer cuts generated from basic feasible solutions and let P_{MIB} be the elementary closure of the mixed integer cuts generated from all the basic solutions.

2.4 Mixed integer rounding cuts and P_{MIR}

Let $c^1 x + hy \leq c_0^1$ and $c^2 x + hy \leq c_0^2$ be two valid inequalities for P, and $\pi = c^2 - c^1 \in Z^n$, $\pi_0 = \lfloor c_0^2 - c_0^1 \rfloor$ and $\gamma = c_0^2 - c_0^1 - \pi_0$. Then the inequality

$$\pi x + (c^1 x + hy - c_0^1)/(1 - \gamma) \leq \pi_0$$

is valid for P_I [28]. These cuts were introduced by Nemhauser and Wolsey [28] and are called *mixed integer rounding cuts*. Let P_{MIR} be the corresponding elementary closure.

2.5 Disjunctive cuts and P_D

For any $(\pi, \pi_0) \in Z^{n+1}$, define $P(\pi, \pi_0)$ to be the convex hull of the union of the two polyhedra $P \cap \{\pi x \leq \pi_0\}$ and $P \cap \{\pi x \geq \pi_0 + 1\}$. Inequalities that are valid for some $P(\pi, \pi_0)$ are called *disjunctive cuts* or *D-cuts*. Disjunctive cuts were introduced by Balas [3]. They are also called *split cuts* [15]. D-cuts can be obtained as follows [2, 9, 24]: if $cx + hy - \alpha(\pi x - \pi_0) \leq c_0$ and $cx + hy + \beta(\pi x - \pi_0 - 1) \leq c_0$ are two valid inequalities for P, where $\alpha \geq 0$, $\beta \geq 0$, $(\pi, \pi_0) \in Z^{n+1}$, then $cx + hy \leq c_0$ is a D-cut and, conversely, all D-cuts can be obtained this way.

$P_D \equiv \cap_{(\pi, \pi_0) \in Z^{n+1}} P(\pi, \pi_0)$ is the elementary closure of the D-cuts.

2.6 Simple disjunctive cuts and P_{SD}

In the mixed $0, 1$ case, the D-cuts can be specialized to those arising from the disjunction $(x_j \leq 0)$ or $(x_j \geq 1)$, *i.e.*, valid inequalities for the union of the polyhedra $P \cap \{x_j \leq 0\}$ and $P \cap \{x_j \geq 1\}$. Let P_{SD} denote the elementary closure of these *simple D-cuts*.

2.7 Lift-and-project cuts and $P_{L\&P}$, $P_{L\&P+S}$

These cuts can be obtained in the mixed 0,1 case. For a given $j \in \{1, \ldots, n\}$, form the nonlinear system

$$(1 - x_j)(b - Ax - Gy) \geq 0$$

$$x_j(b - Ax - Gy) \geq 0$$

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and linearize it by substituting x_j for x_j^2; z_i for $x_i x_j$, $i \neq j$; and w_i for $x_j y_i$. Then project this higher dimensional polyhedron back into the (x, y) space. The resulting polyhedron is denoted by P_j. It is easy to see that $P_I \subseteq P_j$. Therefore any valid inequality for $P_{L\&P} \equiv \cap_j P_j$ is valid for P_I. Valid inequalities for $P_{L\&P}$ are called *lift-and-project cuts* [4]. See Sherali and Adams [30] and Lovász and Schrijver [25] for earlier related work.

It can be shown [3, 4] that any valid inequality $\alpha \binom{x}{y} \geq \beta$ for P_j corresponds to a feasible solution $(\alpha, \beta) \in R^{n+p+1}$ of the polyhedral cone Q

$$(*) \begin{cases} \alpha & +u(A, G) & & -u_0 e_j & & \geq 0, \\ \alpha & & +v(A, G) & & -v_0 e_j & \geq 0, \\ & -ub & & & = \beta, \\ & & -vb & & +v_0 & = \beta, \\ & u, & v \geq 0, \end{cases}$$

where e_j denotes the j-th unit vector. Furthermore the extreme rays of the cone Q correspond to the facets of P_j when $P_j \neq \emptyset$. Notice that if (α, β) is an extreme ray of Q, then $\alpha = \max\{\alpha^1, \alpha^2\}$, where

$$\alpha^1 = u_0 e_j - u(A, G),$$
$$\alpha^2 = v_0 e_j - v(A, G).$$

The lift-and-project cut $\alpha \binom{x}{y} \geq \beta$ is derived from the 0-1 condition on a single variable x_j, even though we allow the choice of x_j to be arbitrary. These cuts can be strengthened by using integrality conditions on the other variables x_i for $i \neq j$ as shown by Balas and Jeroslow [5, 7]. The *strengthened lift-and-project cut* $\gamma \binom{x}{y} \geq \beta$ is defined as follows:

$$\gamma_k = \min\{\alpha_k^1 + u_0 \lceil \bar{m}_k \rceil, \alpha_k^2 - v_0 \lfloor \bar{m}_k \rfloor\} \text{ for } k = 1, \cdots, n,$$
$$\gamma_k = \max\{\alpha_k^1, \alpha_k^2\} \text{ for } k = n+1, \cdots, n+p,$$

where α^1 and α^2 are defined above and $\bar{m}_k = \frac{\alpha_k^2 - \alpha_k^1}{u_0 + v_0}$.

The strengthened lift-and-project cuts can be viewed as valid inequalities for the union of the polyhedra $P \cap \{\pi x \leq 0\}$ and $P \cap \{\pi x \geq 1\}$ for a special choice of π (see proof of Theorem 2.2 in [5]).

The elementary closure of the strengthened lift-and-project cuts is denoted by $P_{L\&P+S}$.

2.8 Intersection cuts and P_{IBF}, P_{IB}, P_{IBF+S}, P_{IB+S}

Let $P' = \{(x, y, s) \in R^{n+p+m} | Ax + Gy + s = b, \ s \geq 0\}$ and let $(\bar{x}, \bar{y}, \bar{s})$ be a basic feasible solution. Let $(\bar{x}, \bar{y})$ be the corresponding point in $(n+p)$-space. Note that $(\bar{x}, \bar{y})$ is an extreme point of P and the $n + p$ nonbasic variables in $(\bar{x}, \bar{y}, \bar{s})$ uniquely define $n + p$ extreme rays originating at $(\bar{x}, \bar{y})$. Let C denote the cone defined by these $n + p$ rays. Valid inequalities for $C \cap \{x_k \leq \pi_0\}$ and $C \cap \{x_k \geq \pi_0 + 1\}$ where $\pi_0 \in Z$, are called *intersection cuts from basic feasible solutions*. This notion was introduced by Balas in a more general context [1]. Let P_{IBF} be the corresponding elementary closure.

If we consider the cone C defined by a basic solution $(\bar{x}, \bar{y}, \bar{s})$ which may not be feasible, then a valid inequality for $C \cap \{x_k \leq \pi_0\}$ and $C \cap \{x_k \geq \pi_0 + 1\}$, is called an *intersection cuts from a basic solution*. Denote the corresponding elementary closure by P_{IB}.

Just as lift-and-project cuts, intersection cuts can be strengthened by using integrality conditions on the other variables x_i for $i \neq k$. The elementary closures of the *strengthened intersection cuts from basic feasible solutions* and the *strengthened intersection cuts from basic solutions* are respectively denoted by P_{IBF+S} and P_{IB+S}.

2.9 Sherali-Adams cuts and P_{SA}

The Sherali-Adams cuts can be obtained in the mixed 0,1 case as follows. Form the nonlinear system

$$(1 - x_1)(b - Ax - Gy) \geq 0$$

$$x_1(b - Ax - Gy) \geq 0$$

...

$$(1 - x_n)(b - Ax - Gy) \geq 0$$

$$x_n(b - Ax - Gy) \geq 0$$

and linearize it by substituting x_j for x_j^2; z_{ij} for $x_i x_j$, $i \neq j$; and w_{ik} for $x_i y_k$. Then project this higher dimensional polyhedron M back into the (x, y) space. The resulting polyhedron is denoted by P_{SA}. Any valid inequality for P_{SA} is valid for P_I and is called a *Sherali-Adams cut* [30]. Sherali and Adams also introduced other families, obtained with more multiplication of terms, but we do not consider them here.

2.10   Lovász-Schrijver cuts and P_{LS}

The Lovász-Schrijver cuts are obtained in the mixed 0,1 case by imposing an additional constraint on the polyhedron M defined in Section 2.9. Let Z be the square matrix with $n+1$ rows and columns defined as follows. Row 0 is the vector $(1, x)$, column 0 is its transpose, and for $i, j \geq 1$, Z has entry z_{ij} if $i \neq j$ and $z_{ii} = x_i$ otherwise. Lovász and Schrijver [25] proposed to tighten the set M by imposing that the matrix Z be positive semi-definite. Let $M^* = M \cap \{Z|\ Z$ is a positive semi-definite matrix$\}$. The projection of M^* into the (x, y) space is denoted by P_{LS}. One can verify that $P_I \subseteq P_{LS} \subseteq P$. Any valid inequality for P_{LS} is a *Lovász-Schrijver cut*.

3   Comparison of the Elementary Closures

All the cuts defined in Section 2 are meaningful in the case of pure $0, 1$ programs. The purpose of this section is to compare the corresponding eighteen elementary closures in this case. When appropriate, we state the results in more general cases (for example, pure integer programs, mixed $0, 1$ programs).

The comparison between the eighteen elementary closures is illustrated in Figure 1. If A and B denote two kinds of cuts, we use the following convention in Figure 1: $A = B$ means that the elementary closures P_A and P_B are identical, $A \to B$ means that $P_A \supseteq P_B$ and the inclusion is strict for some instances, and A not related to B in the figure means that for some instances $P_A \nsubseteq P_B$ and for other instances $P_B \nsubseteq P_A$.

(1) $P_C = P_F$:

Now we are in the pure case $Ax \leq b$ with $x \in Z_+^n$. Without loss of generality, A and b are assumed to be integral. The proof of Theorem 6.34 in [13] implies that any Chvátal cut has the form $\lfloor uA \rfloor x \leq \lfloor ub \rfloor$ for some multiplier vector $0 \leq u < 1$. Since $\lfloor u \rfloor = 0$, this is the Gomory fractional cut $\lfloor uA \rfloor x - \lfloor u \rfloor Ax \leq \lfloor ub \rfloor - \lfloor u \rfloor b$. Conversely, any fractional cut $\lfloor uA \rfloor - \lfloor u \rfloor Ax \leq \lfloor ub \rfloor - \lfloor u \rfloor b$ can be written as $\lfloor uA - \lfloor u \rfloor A \rfloor x \leq \lfloor ub - \lfloor u \rfloor b \rfloor$, since A and b are integral. But this is a Chvátal cut $\lfloor \lambda A \rfloor \leq \lfloor \lambda b \rfloor$, where $\lambda = u - \lfloor u \rfloor$.

(2) $P_{SD} = P_{L\&P}$:

In the mixed 0,1 case, $(P \cap \{x_j \leq 0\}) \cup (P \cap \{x_j \geq 1\}) = P \cap \{(x, y) \in R^{n+p} : x_j \in \{0, 1\}\})$
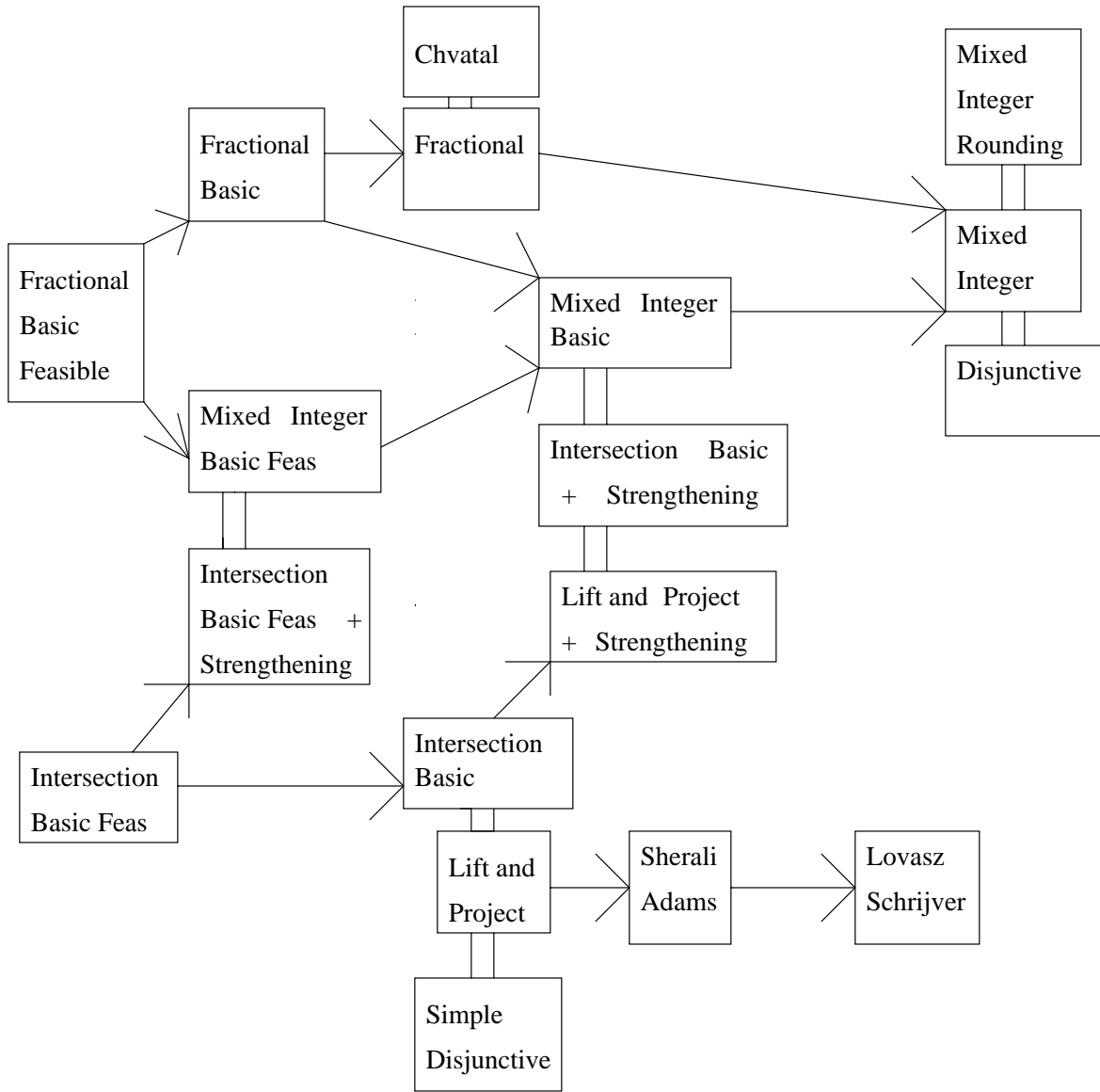
Figure 1: Comparison of Elementary Closures

Therefore $P_{SD} = \cap_{j=1}^n \text{conv}(P \cap \{(x,y) \in R^{n+p} : x_j \in \{0,1\}\})$, which is $P_{L\&P}$ by Theorem 2.1 in [4].

(3) $P_{MIR} = P_D = P_{MI}$:

Nemhauser and Wolsey [28] showed these two equalities.

To show that $P_{MIR} \subseteq P_D$, we show that any D-cut can be obtained by the MIR procedure. Let $cx + hy \leq c_0$ be a D-cut. Then there exist $(\pi, \pi_0) \in Z^{n+1}$ and $\alpha, \beta > 0$ such that $cx + hy - \alpha(\pi x - \pi_0) \leq c_0$ and $cx + hy + \beta(\pi x - \pi_0 - 1) \leq c_0$ are both valid inequalities for P. Multiplying both inequalities by $\frac{1}{\alpha+\beta}$ and applying the MIR procedure with $\gamma = \frac{\beta}{\alpha+\beta}$ gives, after simplifications, $cx + hy \leq c_0$.

$P_D \subseteq P_{MI}$ since it is well known that Gomory mixed integer cuts can be derived from a disjunction expressing integrality conditions on the x variables.

Now we show that $P_{MI} \subseteq P_{MIR}$. We show that any MIR cut can be obtained using Gomory's mixed integer procedure. Consider two valid inequalities for P, say $c^1 x + hy \leq c_0^1$ and $c^2 x + hy \leq c_0^2$ where $\pi = c^2 - c^1 \in Z^n$. After introducing two nonnegative slack variables s_1 and s_2, we get $c^1 x + hy + s_1 = c_0^1$ and $c^2 x + hy + s_2 = c_0^2$. Subtracting the first equation from the second, we get $(c^2 - c^1)x - s_1 + s_2 = c_0^2 - c_0^1$. The Gomory mixed integer cut generated from this equation is $\frac{\gamma}{1-\gamma} s_1 + s_2 \geq \gamma$, where $\gamma = c_0^2 - c_0^1 - \lfloor c_0^2 - c_0^1 \rfloor$, or equivalently $(c^2 - c^1)x - \frac{1}{1-\gamma} s_1 \leq \lfloor c_0^2 - c_0^1 \rfloor$. Plugging $s_1 = -c^1 x - hy + c_0^1$ into it, we obtain the mixed integer rounding cut $\pi x + (c^1 x + hy - c_0^1)/(1-\gamma) \leq \pi_0$, where $\pi_0 = \lfloor c_0^2 - c_0^1 \rfloor$.

(4) $P_{IB} = P_{L\&P}$, $P_{IB+S} = P_{L\&P+S} = P_{MIB}$, $P_{IBF+S} = P_{MIBF}$:

These are new results of Balas and Perregaard [8]. They are obtained by establishing the correspondence between the basic feasible solutions of $(*)$ in Section 2.7 and Gomory mixed integer cuts from basic solutions of P'.

(5) $P_{L\&P+S} \subseteq P_{L\&P}$ and the inclusion is strict for some P:

Obviously $P_{L\&P+S} \subseteq P_{L\&P}$ from their definitions. To see that the inclusion can be strict, consider the polytope $P = \{x \in R_+^2 | -1/2 \leq x_1 - x_2 \leq 0, \ x_1 \leq 1, \ x_2 \leq 1\}$. Then $P_{L\&P} = \{x \in R^2 | x_1 - x_2 \leq 0, \ -2x_1 + x_2 \leq 0, \ -x_1 + 2x_2 \leq 1\}$, while $P_{L\&P+S} = \text{conv}(P_I) = \{x \in R_+^2 | x_1 - x_2 = 0, \ x_1 \leq 1, \ x_2 \leq 1\}$.

(6) $P_F \subseteq P_{FB} \subseteq P_{FBF}$, $P_{MI} \subseteq P_{MIB} \subseteq P_{MIBF} \subseteq P_{FBF}$, $P_{MIB} \subseteq P_{FB}$, $P_{IB+S} \subseteq P_{IB} \subseteq$

P_{IBF}, $P_{IBF+S} \subseteq P_{IBF}$ and each of these inclusions is strict for some P:

These inclusions follow from the definitions. The fact that they can be strict follows from the following results, established later: The results in (15), (4) and (12) imply the strict inclusions in $P_F \subseteq P_{FB} \subseteq P_{FBF}$, and the results in (7), (15), (4), (12) and (11) imply the strict inclusions in $P_{MI} \subseteq P_{MIB} \subseteq P_{MIBF} \subseteq P_{FBF}$. The strict inclusion in $P_{MIB} \subseteq P_{FB}$ follows from (11), and the strict inclusions in $P_{IB+S} \subseteq P_{IB} \subseteq P_{IBF}$ follow from (5), (4) and (13). The strict inclusion in $P_{IBF+S} \subseteq P_{IBF}$ holds because of (10).

(7) $P_{MI} \subseteq P_C$ and the inclusion is strict for some P:

In the pure case, the Gomory mixed integer cut obtained from the multiplier vector u is a strengthening of the Gomory fractional cut obtained from the same u. The fact that the inclusion can be strict will be established in (11).

(8) $P_{SA} \subseteq P_{L\&P}$ and the inclusion is strict for some P:

The inclusion $P_{SA} \subseteq P_{L\&P}$ follows from the definitions. The fact that the inclusion can be strict follows from (14).

(9) $P_{LS} \subseteq P_{SA}$ and the inclusion is strict for some P:

The inclusion follows from the definitions. Next we show that it can be strict. For a graph $G = (V, E)$ and $P = \{x \in R^n_+ | \ x_i + x_j \leq 1, \ \forall (i, j) \in E\}$, the set P_I is the stable set polytope conv($\{x \in \{0, 1\}^n | \ x_i + x_j \leq 1, \ \forall (i, j) \in E\}$). Lovász and Schrijver [25] characterized P_{SA} as the polytope defined exactly by the constraints of P and the odd hole constraints. They showed that there are other constraints, like clique, odd wheel and odd antihole constraints, which are valid for P_{LS} but not for P_{SA}.

(10) $P_{LS} \not\subseteq P_{FBF}$ for some P:

Let $P = \{x \in R^2 | -x_1 + x_2 \leq 1/2, \ 0 \leq x_1 \leq 1, \ 0 \leq x_2 \leq 1\}$. Then $P_{FBF} = \{x \in R^2 | \ x_1 - x_2 \geq 0, \ 0 \leq x_1 \leq 1, \ 0 \leq x_2 \leq 1\} = $ conv(P_I), since the Gomory fractional cut generated from the basic feasible solution $(0, 1/2)$ is $x_1 - x_2 \geq 0$. However, the vector $(x_1, x_2)^T = (1/2 - \epsilon, 1/2 + \epsilon)^T \notin$ conv(P_I) is in P_{LS}, where ϵ is a sufficiently small positive number, because $(x_1, x_2, y_{12}) = (1/2 - \epsilon, 1/2 + \epsilon, 1/2 - \epsilon)$ satisfies both the inequalities in the higher dimensional polyhedron M and the positive semi-definiteness constraint. See also [14].

(11) $P_C \not\subseteq P_{IBF}$ for some P:

For $P = \{(x_1, x_2) | -2x_1 + x_2 \leq 0,\ 2x_1 + x_2 \leq 2,\ 0 \leq x_1,\ 0 \leq x_2 \leq 1\}$, we have $P_{IBF} = \text{conv}(P_I) = \{(x_1, x_2) | 0 \leq x_1 \leq 1,\ x_2 = 0\}$.

In order to show $\text{conv}(P_I) \subset P_C$ for this P, let us consider an arbitrary valid inequality for P and justify that $(1/2, 1/2)$ satisfies the corresponding Chvátal cut. Any valid inequality for P must either be valid for both $(0, 0)$ and $(1, 1)$ or be valid for both $(1, 0)$ and $(0, 1)$. Hence the Chvátal rounding procedure will keep either both $(0, 0)$ and $(1, 1)$ or both $(1, 0)$ and $(0, 1)$ valid for the corresponding Chvátal cut. Therefore, the center point $(1/2, 1/2)$ is valid for the cut.

(12) $P_{MIBF} \not\subseteq P_{FB}$ for some P:

Let $P = \{x \in R_+^2 |\ 10x_1 + 2x_2 \leq 11, 2x_1 + 10x_2 \leq 11,\ 10x_1 + 10x_2 \leq 19, x_1 \leq 1,\ x_2 \leq 1\}$. The only 6 basic feasible solutions are $(11/12, 11/12)$, $(1/2, 1)$, $(1, 1/2)$, $(1, 0)$, $(0, 1)$ and $(0, 0)$. The Gomory mixed integer cuts generated from basic feasible solutions do not yield $x_1 + x_2 \leq 1$, whereas $P_{FB} = \text{conv}(P_I) = \{x \in R_+^2 |\ x_1 + x_2 \leq 1\}$, since the Gomory fractional cut generated from the basic infeasible solution $(9/10, 1)$ is $x_1 + x_2 \leq 1$.

(13) $P_{MIBF} \not\subseteq P_{L\&P}$ for some P:

Consider the polytope $P = \{x \in R_+^2 |\ 4x_1 + x_2 \leq 4,\ -4x_1 + x_2 \leq 0,\ x_2 \leq 1\}$. The basic feasible solutions are $(1/4, 1)$, $(3/4, 1)$, $(0, 0)$ and $(1, 0)$. We obtain $P_{MIBF} = \{x \in R_+^2 |\ -x_1 + x_2 \leq 0,\ x_1 + x_2 \leq 1\}$, while $P_{L\&P} = \text{conv}(P_I) = \{x \in R^2 |\ 0 \leq x_1 \leq 1,\ x_2 = 0\}$.

(14) $P_D \not\subseteq P_{SA}$ for some P:

Let P be the polytope of the Steiner triple system A_9 [11], [18]. Theorem 4.4 of [11] tells us that the inequality $\mathbf{1}x \geq 27/7$ is a Sherali-Adams cut.

The point $x^* = (x_1^*, x_2^*, \cdots, x_9^*) = (3/7, 3/7, \cdots, 3/7)$ satisfies $\mathbf{1}x \geq 27/7$ as equality and satisfies all the constraints of P as strict inequalities. If we could show that x^* satisfies any D-cut as a strict inequality, then x^* is in the interior of P_D, which means that the Sherali-Adams cut $\mathbf{1}x \geq 27/7$ is not valid for P_D. Hence we have $P_D \not\subseteq P_{SA}$ for this P.

It is shown in [3] that an inequality $\alpha x \leq \alpha_0$ is a D-cut if and only if

$$
(**) \begin{cases} \alpha \leq \gamma_1 \pi + \theta_1 A, \\ \alpha \leq -\gamma_2 \pi + \theta_2 A, \\ \alpha_0 \geq \gamma_1 \pi_0 + \theta_1 b, \\ \alpha_0 \geq -\gamma_2 (\pi_0 + 1) + \theta_2 b, \\ (\alpha, \ \alpha_0) \in R^{p+1}, \\ (\gamma_1, \ \theta_1), \ (\gamma_2, \ \theta_2) \in R_+^{m+1}, \\ (\pi, \ \pi_0) \in Z^{p+1}. \end{cases}
$$

Since we assume that $\alpha x \leq \alpha_0$ is not valid for P, we have $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$. After dividing γ_1 in the both sides of all the above inequalities and resetting the variables, we see that $\alpha x \leq \alpha_0$ is a D-cut if and only if

$$
(***) \begin{cases} \alpha \leq \pi + \theta_1 A, \\ \alpha \leq -\gamma \pi + \theta_2 A, \\ \alpha_0 \geq \pi_0 + \theta_1 b, \\ \alpha_0 \geq -\gamma(\pi_0 + 1) + \theta_2 b, \\ \gamma > 0, \ \theta_1, \ \theta_2 \in R_+^m, \\ (\alpha, \ \alpha_0) \in R^{p+1}, \\ (\pi, \ \pi_0) \in Z^{p+1}. \end{cases}
$$

GAMS was used to verify the infeasibility of the nonlinear mixed integer system consisting of $(***)$ for A_9 plus the inequality constraint $\alpha x^* \geq \alpha_0$ for $x^* = (3/7, 3/7, \cdots, 3/7)$. We conclude that x^* is in the interior of P_D, which leads to $P_D \not\subseteq P_{SA}$ for this polytope P.

(15) $P_{L\&P+S} \not\subseteq P_C$ for some P:

Consider the polytope $P = \{x \in R_+^{10} | \sum_{j \neq i} x_{ij} \leq 1, \text{ for } i = 1, 2, \cdots, 5\}$ for the matching problem on the complete graph with five nodes. The facet-defining inequality $\sum_{ij \in E} x_{ij} \leq 2$ of conv(P_I) is a Chvátal cut of rank one. We will show that this facet-defining inequality cannot be generated from any disjunction of $P \cap \{\pi x \leq 0\}$ and $P \cap \{\pi x \geq 1\}$, where π is integral. By the property of strengthened lift-and-project cuts shown in 2.7, $\sum_{ij \in E} x_{ij} \leq 2$ is not valid for $P_{L\&P+S}$.

By the argument and $(***)$ in (14), an inequality $\alpha x \leq \alpha_0$ can be generated from a disjunction of $P \cap \{\pi x \leq 0\}$ and $P \cap \{\pi x \geq 1\}$, where $P = \{x \in R^n | Ax \leq b\}$, if and only if the following linear integer system is feasible:

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$$\begin{cases} \alpha \leq \pi + \theta_1 A, \\ \alpha\gamma' \leq -\pi + \theta_2' A, \\ \alpha_0 \geq \theta_1 b, \\ \alpha_0 \gamma' \geq \theta_2' b - 1, \\ \gamma' > 0, \ \theta_1, \ \theta_2' \in R_+^m, \\ (\pi, \ \pi_0) \in Z^{n+1}. \end{cases}$$

For $\sum_{ij \in E} x_{ij} \leq 2$ and $P = \{x \in R_+^{10} | \sum_{j \neq i} x_{ij} \leq 1, \ \text{for } i = 1, 2, \cdots, 5\}$, we used LINDO to check numerically that the corresponding linear integer system is infeasible.

4 Properties of Elementary Closures

In this section we address two questions about the elementary closures introduced above: Are they polyhedra? Can linear functions be optimized in polynomial time over these elementary closures?

P_{IBF}, P_{FBF}, P_{FB}, P_{MIBF} (or P_{IBF+S}) and P_{MIB} (or P_{IB+S}, $P_{L\&P+S}$) are polyhedra, since there are only finitely many constraints for them. But it is unknown whether one can optimize linear functions over them in polynomial time.

P_C (or P_F) is a polyhedron [12, 29], but it is NP-hard to optimize linear functions over it [16]. Recently, Bockmayr and Eisenbrand [10] showed that, in fixed dimension, P_C has a description with polynomially many inequalities.

$P_{L\&P}$ (or P_{SD}, P_{IB}) is obtained by projecting a polyhedron and therefore is a polyhedron. Furthermore, optimizing a linear function over it can be done by solving a polynomial size linear program in the higher dimensional space. Hence this optimization problem is solvable in polynomial time. So is P_{SA} for the same reason.

P_{LS} is a convex set. It is not a polyhedron in general, but one can optimize linear functions over P_{LS} in polynomial time [25]. Recently, Goemans and Tuncel [19] gave some conditions under which $P_{LS} = P_{SA}$. Cook and Dash [14] considered combining Lovász-Schrijver cuts with Chvátal cuts.

P_{MI} (or P_{MIR}, P_D) is a polyhedron [15]. However, it is still an open problem whether the optimization of linear functions over it can be done in polynomial time.

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