

IC3, PDR, and Friends^{*}

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Abstract. We describe the IC3/PDR algorithms and their various generalizations. Our goal is to give a brief overview of the algorithms and describe them using unified notation. Many crucial optimizations and implementation details are omitted.

1 Constrained Horn Clauses

Given the sets $\mathcal{F}$ of function symbols, $\mathcal{P}$ of predicate symbols, and $\mathcal{V}$ of variables, a *Constrained Horn Clause (CHC)* is a First Order Logic (FOL) formula of the form:

$$\forall \mathcal{V} \cdot (\phi \wedge p_1[X_1] \wedge \dots \wedge p_k[X_k] \rightarrow h[X]), \text{ for } k \geq 0$$

where: ϕ is a constraint over $\mathcal{F}$ and $\mathcal{V}$ with respect to some background theory $\mathcal{A}$; $X_i, X \subseteq \mathcal{V}$ are (possibly empty) vectors of variables; $p_i[X_i]$ is an application $p(t_1, \dots, t_n)$ of an n -ary predicate symbol $p \in \mathcal{P}$ for first-order terms t_i constructed from $\mathcal{F}$ and X_i ; and $h[X]$ is either defined analogously to p_i or is $\mathcal{P}$ -free (i.e., no $\mathcal{P}$ symbols occur in h). Here, h is called the *head* of the clause and $\phi \wedge p_1[X_1] \wedge \dots \wedge p_k[X_k]$ is called the *body*. A clause is called a *query* if its head is $\mathcal{P}$ -free, and otherwise, it is called a *rule*. A rule with body true is called a *fact*. We say a clause is *linear* if its body contains at most one predicate symbol, otherwise, it is called *non-linear*. In this paper, we follow the Constraint Logic Programming (CLP) convention of representing Horn clauses as $h[X] \leftarrow \phi, p_1[X_1], \dots, p_k[X_k]$.

A CHC with constraint ϕ is satisfiable if there exists an interpretation $\mathcal{I}$ of the predicate symbols $\mathcal{P}$ such that each constraint ϕ is true under $\mathcal{I}$. A set Π of CHCs is satisfiable if there exists an interpretation $\mathcal{I}$ that satisfies all clauses in Π .

Satisfiability of a set Π of linear CHC is reducible to satisfiability of 3 clauses of the form:

$$\text{Init}(X) \rightarrow P(X) \tag{1}$$

$$P(X) \rightarrow \text{Bad}(X) \tag{2}$$

$$P(X) \wedge \text{Tr}(X, X') \rightarrow P(X') \tag{3}$$

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Input: A safety problem $\langle \text{Init}(X), \text{Tr}(X, X'), \text{Bad}(X) \rangle$.
Output: *Unreachable* or *Reachable*
Data: A cex queue Q , where $c \in Q$ is a pair $\langle m, i \rangle$, m is a cube over state variables, and $i \in \mathbb{N}$. A level N . A trace $F_0, F_1, \dots$
Initially: $Q = \emptyset, N = 0, F_0 = \text{Init}, \forall i > 0 \cdot F_i = \emptyset$.
repeat
 Unreachable If there is an $i < N$ s.t. $F_{i+1} \subseteq F_i$ **return** *Unreachable*.
 Reachable If there is an m s.t. $\langle m, 0 \rangle \in Q$ **return** *Reachable*.
 Unfold If $F_N \rightarrow \neg \text{Bad}$, then set $N \leftarrow N + 1$.
 Candidate If for some $m, m \rightarrow F_N \wedge \text{Bad}$, then add $\langle m, N \rangle$ to Q .
 Decide If $\langle m, i+1 \rangle \in Q$ and there are m_0 and m_1 s.t. $m_1 \rightarrow m, m_0 \wedge m'_1$ is satisfiable, and $m_0 \wedge m'_1 \rightarrow F_i \wedge \text{Tr} \wedge m'$, then add $\langle m_0, i \rangle$ to Q .
 Conflict For $0 \leq i < N$: given a candidate model $\langle m, i+1 \rangle \in Q$ and clause φ , such that $\varphi \rightarrow \neg m$, if $\text{Init} \rightarrow \varphi$, and $\varphi \wedge F_i \wedge \text{Tr} \rightarrow \varphi'$, then add φ to F_j , for $j \leq i+1$.
 Leaf If $\langle m, i \rangle \in Q, 0 < i < N$ and $F_{i-1} \wedge \text{Tr} \wedge m'$ is unsatisfiable, then add $\langle m, i+1 \rangle$ to Q .
 Induction For $0 \leq i < N$ and a clause $(\varphi \vee \psi) \in F_i$, if $\varphi \notin F_{i+1}, \text{Init} \rightarrow \varphi$ and $\varphi \wedge F_i \wedge \text{Tr} \rightarrow \varphi'$, then add φ to F_j , for each $j \leq i+1$.
until ∞ ;

Algorithm 1: IC3/PDR.

where X is a set of variables, $X' = \{x' \mid x \in X\}$, P is a new predicate, and Init , Tr , and Bad are constraints. We call this reduced problem *Safety*, and present it as a triple $\langle \text{Init}, \text{Tr}, \text{Bad} \rangle$.

Satisfiability of a set Π of non-linear CHC is reducible to satisfiability of 3 clauses of the form:

$$\text{Init}(X) \rightarrow P(X) \quad (4)$$

$$P(X) \rightarrow \text{Bad}(X) \quad (5)$$

$$P(X) \wedge P(X^o) \wedge \text{Tr}(X, X^o, X') \rightarrow P(X') \quad (6)$$

where, $X^o = \{x^o \mid x \in X\}$ and the rest is defined as before. We call this reduced problem *Safety* as well and present it as a triple $\langle \text{Init}, \text{Tr}, \text{Bad} \rangle$. Note that the only difference between the linear and non-linear case is that Tr depends on two sets of state-variables: X and X^o .

2 IC3 and PDR

The finite state model checking algorithm IC3 was introduced in [2] and its variant PDR in [3]. It maintains sets of clauses $F_0, \dots, F_i, \dots, F_N$, called a *trace*, that are properties of states reachable in i steps from the initial states Init . Elements of F_i are called *lemmas*. In the following, we assume that F_0 is initialized to Init . After establishing that $\text{Init} \rightarrow \neg \text{Bad}$, the algorithm maintains the following invariants (for $0 \leq i < N$):

Invariant 1

$$F_i \rightarrow \neg Bad \qquad F_i \rightarrow F_{i+1} \qquad F_i \wedge Tr \rightarrow F'_{i+1}$$

That is, each F_i is safe, the trace is monotone, and F_{i+1} is inductive relative to F_i . In practice, the algorithm enforces monotonicity by maintaining $F_{i+1} \subseteq F_i$.

Alg. 1 summarizes, in a simplified form, a variant of the IC3 algorithm. The algorithm maintains a queue of counter-examples Q . Each element of Q is a tuple $\langle m, i \rangle$ where m is a monomial over $\mathbf{v}$ and $0 \leq i \leq N$. Intuitively, $\langle m, i \rangle$ means that a state m can reach a state in Bad in $N - i$ steps. Initially, Q is empty, $N = 0$ and $F_0 = \text{Init}$. Then, the rules are applied (possibly in a non-deterministic order) until either **Unreachable** or **Reachable** rule is applicable. **Unfold** rules extends the current trace and increases the level at which counterexample is searched. **Candidate** picks a set of bad states. **Decide** extends a counter-example from the queue by one step. **Conflict** blocks a counterexample and adds a new lemma. **Leaf** moves the counterexample to the next level. Finally, **Induction** generalizes a lemma inductively. A typical schedule of the rules is to first apply all applicable rules except for **Induction** and **Unfold**, followed by **Induction** at all levels, then **Unfold**, and then repeating the cycle.

Queue. The queue is ordered by the level:

$$\langle m, i \rangle < \langle n, j \rangle \iff i < j \quad (7)$$

This drives the algorithm to the shortest counterexample.

Inductive Generalization. The **Conflict** and **Induction** rules are based on the principle of inductive generalization. Let $F_0, \dots, F_i, \dots, F_N$ be a valid trace, and let φ be a clause that is *relatively inductive* to F_i :

$$\text{Init} \implies \varphi \qquad \varphi \wedge F_i \wedge Tr \implies \varphi' \quad (8)$$

Let $\mathbf{G} = G_0, \dots, G_N$ be defined as follows:

$$G_j = \begin{cases} F_j \cup \{\varphi\} & \text{if } j \leq i + 1 \\ F_j & \text{if } i + 1 < j \leq N \end{cases} \quad (9)$$

Then $\mathbf{G}$ is a valid trace. The proof is by induction on i and follows from monotonicity of the trace.

Generalizing predecessors. The **Decide** rule picks a predecessor m_0 in Tr of some (partial) state m . While it is possible to simply pick a predecessor state, the rule attempts to find a generalized predecessor instead. The conditions of the rule is sufficient to ensure that m_0 is an implicant of $\psi = (F_i \wedge \exists X' \cdot (Tr \wedge m'))$. Finding a prime implicant of ψ would have been even better, but is too expensive in practice.

Input: A safety problem $\langle \text{Init}(X), \text{Tr}(X, X'), \text{Bad}(X) \rangle$.

Output: *Unreachable* or *Reachable*

Data: A cex queue Q , where a cex $c \in Q$ is a pair $\langle m, i \rangle$, m is a conjunction of constraints over state variables, and $i \in \mathbb{N}$. A level N . A trace $F_0, F_1, \dots$

Notation: $\mathcal{F}(A) = (A(X) \wedge \text{Tr}) \vee \text{Init}(X')$.

All rules of IC3/PDR from Alg. 1, with **Decide** and **Conflict** replaced by the following:

Decide If $\langle P, i+1 \rangle \in Q$ and there is a model $m(X, X')$ s.t. $m \models \mathcal{F}(F_i) \wedge P'$, add $\langle P_{\downarrow}, i \rangle$ to Q , where $P_{\downarrow} = \text{MBP}(X', m, \mathcal{F}(F_i) \wedge P')$.

Conflict For $0 \leq i < N$, given a counterexample $\langle P, i+1 \rangle \in Q$ s.t. $\mathcal{F}(F_i) \wedge P'$ is unsatisfiable, add $P^{\uparrow} = \text{ITP}(\mathcal{F}(F_i)(X^o, X), P)$ to F_j for $j \leq i+1$.

Algorithm 2: APDR.

Propagating lemmas. The **Induction** rule propagates lemmas to higher level, optionally generalizing them as possible. This makes the trace “more” inductive, eventually leading to convergence.

Long counterexamples. The **Leaf** rule lifts blocked counterexamples to higher levels. As a side-effect, it makes it possible to discover counterexamples longer than the current exploration bound N . For example, assume that m is blocked at level i . This means that there is a path of length $N - i$ from m to Bad (but no path of length at most i from Init to m). Assume that **Leaf** lifted m to level $j > i$, and then m was reachable from Init . Then, the discovered counterexample is a concatenation of a path of length k from Init to m and a path of length $N - i$ from m to Bad . The total length of the counterexample is $(N - i + k)$ which is bigger than N .

3 Extending IC3/PDR to Theories

Extending IC3 to theories (such as Linear Arithmetic) requires changing **Decide** and **Conflict** rules to the ones shown in Alg. 2 [1]. The **Decide** rule computes a predecessor using an under-approximation of existential quantifier elimination called *Model Based Projection (MBP)*. The **Conflict** computes new lemmas using *Craig Interpolation (ITP)*. Note that **Conflict** no longer based on the principle of inductive generalization. In the following, we briefly define MBP and ITP.

Model Based Projection. Let φ be a formula, $U \subseteq \text{Vars}(\varphi)$ a subset of variables of φ , and P a model of φ . Then, $\psi = \text{MBP}(U, P, \varphi)$ is a model based projection if (a) ψ is a monomial, (b) $\text{Vars}(\psi) \subseteq \text{Vars}(\varphi) \setminus U$, (c) $P \models \psi$, (d) $\psi \rightarrow \exists V \cdot \varphi$. Furthermore, for a fixed U and a fixed φ , MBP is finite. In [5], an MBP function is defined for LRA based on Loos-Weispfenning quantifier elimination. Note that finiteness of MBP ensures that **Decide** can only be applied finitely many times for a fixed set of lemmas F_i .

Input: A safety problem $\langle \text{Init}(X), \text{Tr}(X, X^o, X'), \text{Bad}(X) \rangle$.
Output: *Unreachable* or *Reachable*
Data: A cex queue Q , where a cex $\langle c_0, \dots, c_k \rangle \in Q$ is a tuple, each $c_j = \langle m, i \rangle$,
 m is a cube over state variables, and $i \in \mathbb{N}$. A level N . A trace $F_0, F_1, \dots$.
Notation: $\mathcal{F}(A, B) = \text{Init}(X') \vee (A(X) \wedge B(X^o) \wedge \text{Tr})$, and $\mathcal{F}(A) = \mathcal{F}(A, A)$
Initially: $Q = \emptyset, N = 0, F_0 = \text{Init}, \forall i > 0 \cdot F_i = \emptyset$
Require: $\text{Init} \rightarrow \neg \text{Bad}$
repeat
 Unreachable If there is an $i < N$ s.t. $F_{i+1} \subseteq F_i$ **return** *Unreachable*.
 Reachable if exists $t \in Q$ s.t. for all $\langle c, i \rangle \in t, i = 0$, **return** *Reachable*.
 Unfold If $F_N \rightarrow \neg \text{Bad}$, then set $N \leftarrow N + 1$ and $Q \leftarrow \emptyset$.
 Candidate If for some $m, m \rightarrow F_N \wedge \text{Bad}$, then add $\langle \langle m, N \rangle \rangle$ to Q .
 Decide If there is a $t \in Q$, with $c = \langle m, i + 1 \rangle \in t, m_1 \rightarrow m, l_0 \wedge m_0^o \wedge m_1'$ is
satisfiable, and $l_0 \wedge m_0^o \wedge m_1' \rightarrow F_i \wedge F_i^o \wedge \text{Tr} \wedge m'$ then add $\hat{t}$ to Q , where $\hat{t} = t$
with c replaced by two tuples $\langle l_0, i \rangle$, and $\langle m_0, i \rangle$.
 Conflict If there is a $t \in Q$ with $c = \langle m, i + 1 \rangle \in t$, s.t. $\mathcal{F}(F_i) \wedge m'$ is unsatisfiable.
Then, add $\varphi = \text{ITP}(\mathcal{F}(F_i), m')$ to F_j , for all $0 \leq j \leq i + 1$.
 Leaf If there is $t \in Q$ with $c = \langle m, i \rangle \in t, 0 < i < N$ and $\mathcal{F}(F_{i-1}) \wedge m'$ is
unsatisfiable, then add $\hat{t}$ to Q , where $\hat{t}$ is t with c replaced by $\langle m, i + 1 \rangle$.
 Induction For $0 \leq i < N$ and a clause $(\varphi \vee \psi) \in F_i$, if $\varphi \notin F_{i+1}, \mathcal{F}(\phi \wedge F_i) \rightarrow \phi'$,
then add φ to F_j , for all $j \leq i + 1$.
until ∞ ;

Algorithm 3: GPDR.

Craig Interpolation. Given two formulas $A[\mathbf{x}, \mathbf{z}]$ and $B[\mathbf{y}, \mathbf{z}]$ such that $A \wedge B$ is unsatisfiable, a Craig interpolant $I[\mathbf{z}] = \text{ITP}(A[\mathbf{x}, \mathbf{z}], B[\mathbf{y}, \mathbf{z}])$, is a formula such that $A[\mathbf{x}, \mathbf{z}] \rightarrow I[\mathbf{z}]$ and $I[\mathbf{z}] \rightarrow \neg B[\mathbf{y}, \mathbf{z}]$. We further require that the interpolant is a clause. An algorithm for extracting LRA clause interpolants from the theory lemmas produced during DPLL(T) proof is given in [4].

4 Generalized PDR

GPDR algorithm [4] shown in Alg. 3 extends IC3/PDR to non-linear CHC and to constraints over Linear Rational Arithmetic (LRA). The main difference is that each element of the queue Q is a tuple of counterexamples. Intuitively, the tuple corresponds to leafs of a counterexample tree. Each application of the **Decide** rule expands one leaf of a counterexample. The extension to Linear Arithmetic is via the use of interpolation in the **Conflict** rule. However, since **Decide** is based on projection, GPDR is incomplete for LRA. That is, it might get stuck alternating between **Decide** and **Conflict** rules, never making progress.

This version of GPDR does not cache reachability information. Hence, it might need to expand the derivation tree completely to find a counterexample. Thus, it is worst case exponential even for CHC over propositional constraints.

Input: A safety problem $\langle \text{Init}(X), \text{Tr}(X, X^o, X'), \text{Bad}(X) \rangle$.
Output: *Unreachable* or *Reachable*
Data: A cex queue Q , where a cex $c \in Q$ is a pair $\langle m, i \rangle$, m is a cube over state variables, and $i \in \mathbb{N}$. A level N . A set of reachable states REACH . A trace $F_0, F_1, \dots$.
Notation: $\mathcal{F}(A, B) = \text{Init}(X') \vee (A(X) \wedge B(X^o) \wedge \text{Tr})$, and $\mathcal{F}(A) = \mathcal{F}(A, A)$
Initially: $Q = \emptyset$, $N = 0$, $F_0 = \text{Init}$, $\forall i > 0 \cdot F_i = \emptyset$, $\text{REACH} = \text{Init}$
Require: $\text{Init} \rightarrow \neg \text{Bad}$
repeat
 Unreachable If there is an $i < N$ s.t. $F_{i+1} \subseteq F_i$ **return** *Unreachable*.
 Reachable If $\text{REACH} \wedge \text{Bad}$ is satisfiable, **return** *Reachable*.
 Unfold If $F_N \rightarrow \neg \text{Bad}$, then set $N \leftarrow N + 1$ and $Q \leftarrow \emptyset$.
 Candidate If for some m , $m \rightarrow F_N \wedge \text{Bad}$, then add $\langle m, N \rangle$ to Q .
 Successor If there is $\langle m, i + 1 \rangle \in Q$ and a model $M \models \psi$, where $\psi = \mathcal{F}(\vee \text{REACH}) \wedge m'$. Then, add s to REACH , where $s' \in \text{MBP}(\{X, X^o\}, \psi)$.
 DecideMust If there is $\langle m, i + 1 \rangle \in Q$, and a model $M \models \psi$, where $\psi = \mathcal{F}(F_i, \vee \text{REACH}) \wedge m'$. Then, add s to Q , where $s \in \text{MBP}(\{X^o, X'\}, \psi)$.
 DecideMay If there is $\langle m, i + 1 \rangle \in Q$ and a model $M \models \psi$, where $\psi = \mathcal{F}(F_i) \wedge m'$. Then, add s to Q , where $s^o \in \text{MBP}(\{X, X'\}, \psi)$.
 Conflict If there is an $\langle m, i + 1 \rangle \in Q$, s.t. $\mathcal{F}(F_i) \wedge m'$ is unsatisfiable. Then, add $\varphi = \text{ITP}(\mathcal{F}(F_i), m')$ to F_j , for all $0 \leq j \leq i + 1$.
 Leaf If $\langle m, i \rangle \in Q$, $0 < i < N$ and $\mathcal{F}(F_{i-1}) \wedge m'$ is unsatisfiable, then add $\langle m, i + 1 \rangle$ to Q .
 Induction For $0 \leq i < N$ and a clause $(\varphi \vee \psi) \in F_i$, if $\varphi \notin F_{i+1}$, $\mathcal{F}(\phi \wedge F_i) \rightarrow \phi'$, then add φ to F_j , for all $j \leq i + 1$.
until ∞ ;

Algorithm 4: Rule-based description of SPACER.

5 Spacer

SPACER [5], shown in Alg. 4 extends $\mathcal{APDR}$ to non-linear CHC. Unlike other variants of IC3/PDR discussed so far, it maintains the set of reachable states REACH . This set is used, among other things, to cache reachability information.

We briefly outline the key difference between Spacer and $\mathcal{APDR}$. First, the **Reachable** rule checks whether a *Bad* state became reachable. This is inefficient for linear CHC since reachability is known before the REACH set is computed.

The single **Decide** rule of $\mathcal{APDR}$ is replaced by three rules: **Successor**, **DecideMust**, and **DecideMay**. **DecideMay** is most similar to **Decide**. **DecideMust** uses reachability cache to skip over right-most predicate application. **Successor** uses reachability cache to compute a new reachable state.

For linear CHC, SPACER is equivalent to $\mathcal{APDR}$.

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