PROCEEDINGS *of the* FIFTH BERKELEY SYMPOSIUM ON MATHEMATICAL STATISTICS AND PROBABILITY

Held at the Statistical Laboratory
University of California
June 21–July 18, 1965
and
December 27, 1965–January 7, 1966

with the support of
University of California
National Science Foundation
National Institutes of Health
Air Force Office of Scientific Research
Army Research Office
Office of Naval Research

VOLUME II

Part 2

CONTRIBUTIONS TO PROBABILITY THEORY

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AND JERZY NEYMAN

UNIVERSITY OF CALIFORNIA PRESS
BERKELEY AND LOS ANGELES
1967

EXISTENCE OF BOUNDED INVARIANT MEASURES IN ERGODIC THEORY

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1. Introduction

We present a survey of some of the recent work done on the problem of existence of bounded invariant measure for positive contractions defined on L^1-spaces.

2. Preliminaries

1. *Positive linear forms on L^∞-spaces.* Let $(E, \mathfrak{F}, \mu)$ be a fixed measure space (with μ σ-finite). Sets in $\mathfrak{F}$ and real measurable functions defined on $(E, \mathfrak{F})$ will always be considered up to μ-equivalence; hence, all equalities or inequalities between measurable sets or functions are to be taken in the almost sure sense with respect to μ.

We will denote by f, g (with or without subscripts) elements of the Banach space $L^1(E, \mathfrak{F}, \mu)$ and by h elements of the Banach space $L^\infty = L^\infty(E, \mathfrak{F}, \mu)$. The space L^∞ is the strong dual of L^1 for the bilinear form: $\langle f, h\rangle = \int_E fh\, d\mu$. Consideration of the strong dual of L^∞, in which L^1 is isometrically imbedded, has often been helpful in analysis. We here recall the following lemma from the theory of vectorial lattices, of which we sketch a proof out of completeness.

LEMMA 1. *Let λ be a positive linear form defined on L^∞; that is, let $\lambda \in (L^\infty)'_+$. Then there exists a largest element g in L^1_+ such that the form induced by it on L^∞ verifies $g \leq \lambda$. Moreover, the complement $G = \{g = 0\}$ of the support of g is the largest set in $\mathfrak{F}$ (up to equivalence) for which there exists an $h \in L^\infty_+$ such that $h > 0$ on G and $\lambda(h) = 0$; in particular, the following equivalences hold:*

(a) $g > 0$ *a.s.* $\Rightarrow \lambda(h) > 0$ *for every* $h \in L^\infty_+$, $h \neq 0$.

(b) $g = 0$ *a.s.* $\Rightarrow \lambda(h) = 0$ *for at least one* $h \in L^\infty$ *such that* $h > 0$ *a.s.*

PROOF. The class $\Lambda = \{f\colon f \in L^1_+, f \leq \lambda \text{ on } L^\infty_+\}$ is easily seen to be closed under least upper bounds and increasing limits; hence, $g = \sup \Lambda$ belongs to Λ, and is thus the largest element of Λ.

Given two linear forms ν_1, ν_2 on L^∞, it is known and easily checked that the formula $\nu(h) = \inf\{[\nu_1(u) + \nu_2(h - u)]; 0 \leq u \leq h\}$ where $h \in L^\infty_+$, defines on L^∞_+ a linear form ν on L^∞, which is the g.l.b. of ν_1 and ν_2. Now it follows from the maximality of g that 0 is the g.l.b. of $\lambda - g$ and f_0, where f_0 is an arbitrarily fixed strictly positive element of L^1 (which is considered here as a linear form on L^∞); hence, by what precedes, one has

$$\inf_{u:0\leq u\leq h} (\lambda(u) - \langle g, u\rangle + \langle f_0, h - u\rangle) = 0 \tag{1}$$

for every h in L^∞_+.

For $h = 1_G$ where $G = \{g = 0\}$, the term $\langle g, u\rangle$ always vanishes in the last formula; we have thus shown the existence of functions u_m $(m \geq 1)$ with the following properties:

$$0 \leq u_m \leq 1_G, \qquad \lambda(u_m) + \langle f_0, 1_G - u_m\rangle \leq 2^{-m}. \tag{2}$$

Then the $v_n = \inf_{m>n} u_m$ $(n \geq 1)$ verify

$$0 \leq v_n \leq 1_G, \qquad \lambda(v_n) = 0, \langle f_0, 1_G - v_n\rangle \leq \sum_{m>n} 2^{-m} = 2^{-n} \tag{3}$$

as follows from $v_n \leq u_m$ $(m > n)$ and $1_G - v_n \leq \sum_{m>n} (1_G - u_m)$. Finally, the function $h = \sum_{n\geq 1} 2^{-n}v_n$ belongs to L^∞_+ and verifies $\lambda(h) = 0$ since

$$\lambda(h) = \sum_{n\leq p} 2^{-n}\lambda(v_n) + \lambda\left(\sum_{n>p} 2^{-n}v_n\right) \leq 2^{-p}\lambda(1_G) \to 0 \qquad \text{as} \quad p \to \infty \tag{4}$$

because $\lambda(v_n) = 0$ and $\sum_{n>p} 2^{-n}v_n \leq 2^{-p}1_G$. Moreover, one has $h > 0$ on G, because by definition $\{h = 0\} = \bigcap_n \{v_n = 0\}$, and because

$$\int_{\{v_n=0\}G} f_0\, d\mu \leq \int f_0(1_G - v_n)\, d\mu \leq 2^{-n} \to 0 \qquad \text{as} \quad n \to \infty. \tag{5}$$

We have proved the existence of h in L^∞_+ such that $\lambda(h) = 0$ and $h > 0$ on G. Conversely, if $h \in L^\infty_+$ verifies $\lambda(h) = 0$, it follows from $0 \leq \int gh \leq \lambda(h)$ that $\{h > 0\} \subset G$, and this concludes the proof of the lemma.

2. *Conservative operators on L^1-spaces.* Let T be a positive linear operator defined on L^1; we suppose that T has norm ≤ 1 (that is, a contraction) or, what is equivalent, that its dual operator T^* defined on L^∞ verifies $T^*1 \leq 1$.

If $P = \{P(x, F); x \in E, F \in \mathfrak{F}\}$ is a transition function defined on $(E, \mathfrak{F})$, the formula

$$\int_F Tf\, d\mu = \int_E fP(\cdot, F)\, d\mu, \qquad (f \in L^1, F \in \mathfrak{F}) \tag{6}$$

defines (with the aid of the Radon-Nikodym theorem) a positive linear operator T of norm 1 on L^1, provided only that the measure $\int \mu(dx)P(x, \cdot)$ is absolutely continuous with respect to μ. For the Markovian random sequence $\{X_n, n \geq 0\}$ of initial μ-density f, $(f \geq 0, \int f\, d\mu = 1)$, and transition probability P, sums of the form $\sum_{n\in M} T^n f$ where M is a subset of the set $N = \{0, 1, 2, \cdots\}$ of positive integers, can be interpreted as densities: indeed, $\int_F \sum_M T^n f$ is the expected number of times n such that $n \in M$ and $X_n \in F$. This well-known fact gives probabilistic meaning to some of the conditions of the sequel.

The operator T is said to be *conservative* if one of the following equivalent conditions is satisfied:

(a) $\sum_{n \geq 0} T^n f_0 = \infty$, a.s., where f_0 is an arbitrarily fixed element of L^1 such that $f_0 > 0$, a.s.;

(b) for any $h \in L^\infty_+$, the condition $\sum_{n \geq 0} T^{*n} h < \infty$ a.s. implies that $h = 0$;

(b′) for any $F \in \mathfrak{F}$, the condition $\sum_{n \geq 0} T^{*n}{}_{\wedge F} < \infty$ a.s. implies that $F = \phi$ a.s.

(Once it has been deduced from Hopf's maximal ergodic lemma that (a) does not depend on f_0, the equivalence of these conditions is easily proven by an argument similar to that of section 6 of the proof of theorem 1 below.)

The operator T is said to be *dissipative* if one of the following equivalent conditions is satisfied:

(a) $\sum_{n \geq 0} T^n f_0 < \infty$ a.s., with f_0 as above;

(b) $\sum_{n \geq 0} T^{*n} h \in L^\infty$ holds for at least one $h \in L^\infty_+$ such that $h > 0$ a.s.

The preceding conditions are to be compared with those of theorems 1 and 2 below.

3. *Banach limits.* A Banach limit L is by definition a positive linear form defined on $\ell^\infty(N)$, which is normalized and invariant under translation, that is, which verifies $L(\{1\}) = 1$ and $L(\{x_{n+1}, n \in N\}) = L(\{x_n, n \in N\})$. Here $\ell^\infty(N)$ denotes as usual the Banach space of bounded sequences $\{x_n, n \in N\}$ of real numbers provided with the norm $\|\{x_n\}\| = \sup_N |x_n|$. The following classical lemma proves the existence of Banach limits as a corollary and gives the value of $\sup_L L(\{x_n\})$ as found by L. Sucheston [12] by another method.

LEMMA 2. *If Λ is a subvectorial space of $\ell^\infty(N)$ containing $\{1\}$, any linear form L defined on Λ and positive (in the sense that it takes nonnegative values on $\Delta \cap \ell^\infty_+(N)$), can be extended to a linear positive form on $\ell^\infty(N)$. Moreover, for any fixed $\{x_n\} \in \ell^\infty(N)$, one has*

$$\sup_{\tilde{L}} \tilde{L}(\{x_n\}) = \inf\,[L(\{y_n\}); \qquad \{y_n\} \in \Lambda \quad \text{and} \quad y_n \geq x_n \ (n \in N)] \tag{7}$$

where $\tilde{L}$ ranges in the first member over all positive linear extensions of L to $\ell^\infty(N)$.

PROOF. The set of all linear positive forms defined on subvectorial spaces of $\ell^\infty(N)$ and extending L is provided with an order by: $L' \subset L''$, if L'' is defined and equal to L' on the domain of definition of L'; this order is clearly inductive. Let us show that any element maximal for this order is necessarily defined on the whole space $\ell^\infty(N)$.

If L' is a positive linear form defined on a vectorial subspace Λ' of $\ell^\infty(N)$ which contains $\{1\}$, and if for a given sequence $\{x_n\} \in \ell^\infty(N)$, $\{y_n'\}$ (resp. $\{y_n''\}$) is a sequence in Λ' such that $y_n' \geq x_n$ $(n \in N)$ (resp. $x_n \geq y_n''$ $(n \in N)$), then $L'(\{y_n'\}) \geq L'(\{y_n''\})$ because $\{y_n' - y_n''\} \in \Lambda' \cap \ell^\infty_+(N)$. Hence, it is possible to choose a real number c such that

$$\inf L'(\{y_n'\}) \geq c \geq \sup L'(\{y_n''\}), \tag{8}$$

where $\{y_n'\}$ (resp. $\{y_n''\}$) ranges among the sequences of Λ' such that $y_n' \geq x_n$ for all n (resp. $y_n'' \leq x_n$ for all n). The formula

$$L''(\{y_n + ax_n\}) = L'(\{y_n\}) + ac, \qquad (\{y_n\} \in \Lambda', a \in R) \tag{9}$$

then defines a positive linear extension of L' to the subspace generated by Λ'

and $\{x_n\}$. And since $\{x_n\}$ can be arbitrarily chosen in $\ell^\infty(N)$, Λ' can only be maximal if it is defined on the whole space $\ell^\infty(N)$.

This proves the first part of the lemma, and the second part is easily derived from the preceding argument.

COROLLARY. *Banach limits exist, and moreover, for every* $\{x_n\} \in \ell^\infty(N)$; *the following limit exists*

$$\lim_{p\to\infty} \sup_{n\geq 0} \frac{1}{p} \sum_{m=0}^{p-1} x_{m+n} \tag{10}$$

and is equal to $\sup_L L(\{x_n\})$ *where* L *ranges over all Banach limits.*

PROOF. Let Λ be the subvectorial space of $\ell^\infty(N)$ generated by $\{1\}$ and by $\{y_{n+1} - y_n, n \in N\}$, where $\{y_n\}$ ranges over $\ell^\infty(N)$. Define L on Λ by $L(\{c + y_{n+1} - y_n\}) = c$. Since for every $c \in R$ and every $\{y_n\} \in \ell^\infty(N)$, the inequality $c + y_{n+1} - y_n \geq 0$ $(n \in N)$ implies that $c \geq 0$ because of

$$0 \leq \frac{1}{n} \sum_{m=0}^{n-1} (c + y_{m+1} - y_m) = c + \frac{1}{n}(y_n - y_0) \to c \qquad \text{as} \quad n \to \infty, \tag{11}$$

the preceding definition of L is unambiguous (if $c + y_{n+1} - y_n = 0$ $(n \in N)$, then $c = 0$), and L is a positive linear form defined on Λ.

The lemma proves the existence of Banach limits because these are exactly the positive linear extensions of L to $\ell^\infty(N)$. It also shows that

$$\sup_L L(\{x_n\}) = \inf\,[c\colon c + y_{n+1} - y_n \geq x_n \ (n \in N)] \tag{12}$$

where c ranges over R and $\{y_n\}$ over $\ell^\infty(N)$. Let I be the infimum of the 2d member; it can be evaluated as follows.

First it follows from $x_n \leq c + y_{n+1} - y_n$ by letting $x_n^{(p)} = (1/p) \sum_{m=0}^{p-1} x_{m+n}$ that

$$x_n^{(p)} \leq c + \frac{1}{p}(y_{n+p} - y_n) \leq c + \frac{2}{p} \|\{y_n\}\|; \tag{13}$$

hence that, using the definition of I,

$$\lim_{p\to\infty} \sup \sup_n x_n^{(p)} \leq I. \tag{14}$$

On the other hand, since $x_n - x_n^{(p)}$ is of the form $\{y_{n+1} - y_n\}$ for a $\{y_n\}$ in $\ell^\infty(N)$, it follows from

$$x_n \leq \sup_\ell x_\ell^{(p)} + (x_n - x_n^{(p)}) \tag{15}$$

that the inequality $I \leq \sup_n x_n^{(p)}$ holds for every $p \geq 1$. Hence, $I = \lim_p \sup_n x_n^{(p)}$.

3. Existence of invariant measures

The main part of the following theorem was proved in [2] by Hajian and Kakutani in the particular case where the operator T is induced by a measurable and nonsingular transformation of the space $(E, \mathfrak{F}, \mu)$. It was then extended in [7] and [11], whereas its proof was at the same time simplified by the introduction of Banach limits ([12]; see also [1]).

THEOREM 1. *For any positive linear contraction T of a space $L^1(E, \mathfrak{F}, \mu)$, the following conditions are equivalent:*

(a) *there exists $g \in L^1$ such that $Tg = g$ and $g > 0$, a.s.;*

(b_n) *for any $h \in L^\infty_+$, the equality $\liminf_{n\to\infty} \langle T^n f_0, h\rangle = 0$ implies that $h = 0$ (here and in the following, f_0 denotes an arbitrary but fixed element of L^1 such that $f_0 > 0$, a.s.);*

(b_s) *for any $F \in \mathfrak{F}$, the equality $\lim_{p\to\infty} \sup_n 1/p \sum_{m=0}^{n-1} \langle T^{m+n} f_0, 1_F\rangle = 0$ implies that $F = \phi$;*

(c_n) *for any $h \in L^\infty_+$, the a.s. convergence $\sum_i T^{*n_i}h < \infty$ for an infinite sequence $0 \le n_0 < n_1 < \cdots$ of integers implies that $h = 0$;*

(c_s) *for any $F \in \mathfrak{F}$, the a.s. inequality $\sum_i T^{*n_i}1_F \le 1 + \epsilon$ for an infinite sequence $0 = n_0 < n_1 < \cdots$ of integers starting with $n_0 = 0$ implies that $F = \phi$ (here ϵ denotes an arbitrarily fixed strictly positive real number);*

(d) *$\sum_i T^{n_i} f_0 = \infty$ holds a.s. for every infinite sequence $0 \le n_0 \le n_1 < \cdots$ of integers.*

The preceding conditions imply that T is conservative. If T is conservative, then these conditions are still equivalent to the following:

(e) *for every $h \in L^\infty$ such that $h > 0$, a.s., one has $\sum_i T^{*n_i}h = \infty$, a.s. for every infinite sequence $0 \le n_0 < n_1 < \cdots$ of integers;*

(e′) *for every sequence $\{F_k, k \ge 1\}$ of measurable subsets of E such that $E = \bigcup_k F_k$, one has $\bigcup_k \{\sum_i T^{*n_i} 1_{F_k} = \infty\} = E$ for every infinite sequence $0 \le n_0 < n_1 < \cdots$ of integers;*

(f) *for any $f \in L^1_+$, the a.s. convergence $\sum_i T^{n_i} f < \infty$ for an infinite sequence $0 \le n_0 < n_1 < \cdots$ of integers implies that $f = 0$.*

REMARK. In case T is induced by a measurable non-singular transformation θ of $(E, \mathfrak{F}, \mu)$, that is, when $T^*h = h_0\theta$ $(h \in L^\infty)$, the condition (c_s) may be restated as follows (if ϵ is chosen < 1): there exists no set $F \in \mathfrak{F}$, nonnegligible, such that the $\theta^{-n_i}(F)$ are mutually disjoint for an infinite sequence $0 = n_0 < n_1 < n_2 < \cdots$ of integers (namely, there exists no weakly wandering set in the sense of [2]).

PROOF OF THEOREM 1. The proof is long and will be divided in eight parts; however, after the remark of alinea 1, only the reasoning of alinea 2 and 4 are not "immediate."

1. The following remark makes the implication $a \Rightarrow (b_n)$ obvious and will be also used in the sequel. For any fixed $h \in L^\infty_+$, the condition $\liminf \langle T^n f_0, h\rangle = 0$ where f_0 is a fixed strictly positive element of L^1, implies that

$$\liminf_{n\to\infty} \langle T^n f, h\rangle = 0 \tag{16}$$

for every $f \in L^1_+$.

Indeed, the general inequality $f \le af_0 + (f - af_0)^+$ implies that

$$\langle T^n f, h\rangle \le a\langle T^n f_0, h\rangle + \|(f - af_0)^+\|_1 \|h\|_\infty, \qquad (a \in R) \tag{17}$$

because T^n is a contraction. Letting $n \to \infty$, one gets the desired result because $(f - af_0)^+ \downarrow 0$, a.s. and in L^1, as $a \to \infty$, since f_0 is strictly positive.

From this fact follows that the validity of $\liminf \langle T^n f_0, h\rangle = 0$ for a fixed $h \in L^\infty_+$ is independent of the strictly positive f_0 chosen in L^1. Hence, condition (b_n) does not depend on the chosen f_0 and is implied by condition (a), as is readily seen by taking $f_0 = g$.

2. If L denotes a Banach limit (see preliminaries), the formula

$$\lambda(h) = L(\{\langle T^n f_0, h\rangle, n \in N\}), \qquad (h \in L^\infty) \tag{18}$$

defines a positive linear form on L^∞ such that $\lambda(T^*h) = \lambda(h)$ for every $h \in L^\infty$. This invariance indeed follows from the invariance of L under translation and the fact that $\langle T^n f_0, T^*h\rangle = \langle T^{n+1} f_0, h\rangle$. The largest element g in L^1_+ bounded above by λ (see lemma 1 of preliminaries) is then invariant under T. Indeed, on one hand,

$$\langle Tg, h\rangle = \langle g, T^*h\rangle \leq \lambda(T^*h) = \lambda(h) \tag{19}$$

holds for every $h \in L^\infty_+$ by the definitions and shows that $Tg \leq g$; on the other hand, it follows from

$$\lambda(T^*1) = \lambda(1), \qquad (\lambda - g)(T^*1) \leq (\lambda - g)(1) \tag{20}$$

(the inequality holds because $\lambda - g \geq 0$ and $T^*1 \leq 1$), that

$$\langle Tg, 1\rangle = \langle g, T^*1\rangle \geq \langle g, 1\rangle. \tag{21}$$

Hence $Tg = g$.

Suppose that (b_n) holds; then $\lambda(h) \geq \liminf_{n\to\infty} \langle T^n f_0, h\rangle > 0$ holds for every $h \in L^\infty_+$, $h \neq 0$. By lemma 1, it follows that $g > 0$ a.s. and the implication $(b_n) \Rightarrow$ (a) is so proved.

3. The use of Banach limits, as in the preceding alinea, also gives an easy proof of the implication $(b_s) \Rightarrow (c_s)$.

If $F \in \mathfrak{F}$ verifies

$$\textstyle\sum_i T^{*n_i} 1_F \in L^\infty \tag{22}$$

for an infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers, then for any form λ obtained from a Banach limit L, as in alinea 2, one has for every integer $j \geq 1$,

$$\lambda(\textstyle\sum T^{*n_i} 1_F) \geq \left(\sum_{i<j} T^{*n_i} 1_F\right) = j\lambda(1_F), \tag{23}$$

and since the first member is finite and independent of j, $\lambda(1_F) = 0$. On the other hand, one has by the preliminaries (section 3),

$$\sup_\lambda \lambda(1_F) = \sup_L L(\{\langle T^n f_0, 1_F\rangle\}) = \lim_{p\to\infty} \sup_n \frac{1}{p} \sum_{m=0}^{p-1} \langle T^{m+n} f_0, 1_F\rangle. \tag{24}$$

Thus if F verifies the hypothesis of the beginning, this last member is 0, and if (b_s) holds, F must then be a.s. equal to ϕ; that is, condition (c_s) is implied by (b_s).

4. Since the implication $(b_n) \Rightarrow (b_s)$ is clear, the proof of the implication $(c_s) \Rightarrow (b_n)$ will establish the equivalence of (b_n), (b_s), and (c_s). This proof rests on the following generalization of a lemma of [2] given in [11].

LEMMA 3. *If for an $h \in L^\infty$ such that $0 \leq h \leq 1$, one has*

$$\liminf_{n\to\infty} \langle T^n f_0, h\rangle = 0, \tag{25}$$

then there exists for each $\delta > 0$ an element $h_\delta \in L^\infty_+$ such that $h_\delta \leq h$, $\langle f_0, h - h_\delta\rangle \leq \delta$ and $\sum_i T^{*n_i} h_\delta \leq 1$ for a suitably chosen infinite sequence $0 = n_0 < n_1 < \cdots$ of integers (starting at $n_0 = 0$). Hence for every $F \in \mathfrak{F}$ such that

$$\liminf_{n\to\infty} \langle T^n f_0, 1_F\rangle = 0, \tag{26}$$

there exists for every $\epsilon, \epsilon' > 0$ a subset $F_{\epsilon,\epsilon'}$ of F such that $\langle f_0, 1_F - 1_{F_{\epsilon,\epsilon'}}\rangle \leq \epsilon'$ and $\sum_i T^{*n_i} 1_{F_{\epsilon,\epsilon'}} \leq 1 + \epsilon$ for a suitably chosen infinite sequence $0 = n_0 < n_1 < \cdots$ of integers.

PROOF OF LEMMA. Given an infinite sequence $0 = n_0 < n_1 < \cdots$ of integers we let

$$h' = \Big(h - \sum_{0 \leq i \leq j}\sum (T^*)^{n_{j+1} - n_i} h\Big)^+. \tag{27}$$

Obviously $0 \leq h' \leq h$ and $h' \in L^\infty$.

The sequence $\{n_i\}$ can be chosen so that $\langle f_0, h - h'\rangle \leq \delta$ for a given $\delta > 0$. Indeed, it follows from

$$h - h' \leq \sum_{j\geq 0}\sum_{i=0}^{j} (T^*)^{n_{j+1}-n_i} h = \sum_{j\geq 0} (T^*)^{n_{j+1}-n_j} \sum_{i=0}^{j} (T^*)^{n_j - n_i} h \tag{28}$$

that

$$\langle f_0, h - h'\rangle \leq \sum_{j\geq 0} \langle T^{n_{j+1}-n_j} f^{(j)}, h\rangle \tag{29}$$

where we have let

$$f^{(j)} = \sum_{i=0}^{j} T^{n_j - n_i} f_0 \tag{30}$$

when $j \geq 0$. Hence, the hypothesis $\liminf_{n\to\infty} \langle T^n f_0, h\rangle = 0$ made on h, where one may substitute f_0 by $f^{(j)}$ by the remark of alinea 1, makes it possible to choose the n_{j+1} by recurrence on j from $n_0 = 0$, so that

$$\langle T^{n_{j+1}-n_j} f^{(j)}, h\rangle \leq \delta 2^{-(j+1)}, \tag{31}$$

because $f^{(j)}$ only depends on $n_0, \cdots, n_j$.

The following inequality holds for every integer $i \geq 0$ and every integer $k \geq 0$, as will be proved by recurrence on k,

$$\sum_{j=i}^{i+k} (T^*)^{n_j - n_i} h' \leq 1. \tag{32}$$

Taking $i = 0$ and letting $k \to \infty$, we obtain that

$$\textstyle\sum_j (T^*)^{n_j} h' \leq 1; \tag{33}$$

namely, that h' has the properties stated for h_δ in the lemma. The above inequality is true for $k = 0$ since $h' \leq h \leq 1$ and $(T^*)^n 1 \leq 1$ for every n. Assuming that the inequality is true for every $i \geq 0$ and for the value $k - 1$ of the recurrence parameter, we deduce from

$$\sum_{j=i}^{i+k} (T^*)^{n_j - n_i} h' = h' + (T^*)^{n_{i+1} - n_i} \left(\sum_{j=i+1}^{(i+1)+k-1} (T^*)^{n_j - n_{i+1}} h' \right) \leq h' + (T^*)^{n_{i+1} - n_i} 1 \tag{34}$$

that on the set $\{h' = 0\}$, the first member is bounded above by 1. On the other hand, we have that on $\{h' > 0\}$,

$$h' = h - \sum_{0 \leq i \leq j} \sum (T^*)^{n_{j+1} - n_i} h, \tag{35}$$

and thus that

$$\sum_{j=i}^{i+k} (T^*)^{n_j - n_i} h' = h' + \sum_{j=i}^{i+k-1} (T^*)^{n_{j+1} - n_i} h' \leq h' + \sum_{j=i}^{i+k-1} (T^*)^{n_{j+1} - n_i} h \leq h \leq 1. \tag{36}$$

The recurrence is established.

Letting $h = 1_F$ in the preceding result and $\delta = \epsilon\epsilon'/1 + \epsilon$,

$$F_{\epsilon,\epsilon'} = \{h_\delta > 1/(1 + \epsilon)\} \tag{37}$$

one obtains from

$$1_{F_{\epsilon,\epsilon'}} \leq (1 + \epsilon) h_\delta \quad \text{that} \quad \sum{}_i T^{*n_i} 1_{F_{\epsilon,\epsilon'}} \leq 1 + \epsilon \tag{38}$$

and from

$$1_F - 1_{F_{\epsilon,\epsilon'}} \leq 1 + \epsilon/\epsilon(h - h_\delta) \quad \text{that} \quad \langle f_0, 1_F - 1_{F_{\epsilon,\epsilon'}} \rangle \leq \frac{1 + \epsilon}{\epsilon} \delta = \epsilon'. \tag{39}$$

This concludes the proof of the lemma.

It is easy to deduce the implication $(c_s) \Rightarrow (b_n)$ from the preceding lemma. Indeed, if $h \in L_+^\infty$ verifies $\liminf \langle T^n f_0, h \rangle = 0$, then 1_F verifies a similar relation if $F = \{h > a\}$ and a is a strictly positive real number. The sets $F_{\epsilon,\epsilon'}$ constructed from F as above are negligible if (c_s) is valid; hence, $\langle f_0, 1_F \rangle \leq \epsilon$ for every $\epsilon > 0$, and F is itself negligible. Finally, h is 0, since a was arbitrary.

5. To conclude the proof of the first part of the theorem, we show that $(b_n) \Rightarrow (d) \Rightarrow (c_n) \Rightarrow (b_n)$.

If $0 \leq n_0 < n_1 < \cdots$ is an infinite sequence of integers, we let

$$h = \xi(1 + \sum T^{n_i} f_0)^{-1} \tag{40}$$

where ξ is a fixed strictly positive element of $L^1 \cap L^\infty$ and with the convention that $(+\infty)^{-1} = 0$. Then $0 \leq h \leq \xi$ so that $h \in L_+^\infty$ and $h(\sum_i T^{n_i} f_0) \leq \xi$, a.s. (with the convention $0.\infty = 0$) so that $\sum_i \langle T^{n_i} f_0, h \rangle < \infty$; hence,

$$\liminf_{n \to \infty} \langle T^n f_0, h \rangle = 0, \tag{41}$$

and if (b_n) is satisfied, h must be 0; that is, $\sum T^{n_i} f_0 = +\infty$, a.s. This shows that $(b_n) \Rightarrow (d)$.

If $h \in L_+^\infty$ verifies $\sum_i T^{*n_i}h < \infty$, a.s. for an infinite sequence

(42) $$0 \leq n_0 < n_1 < \cdots$$

of integers, let $f = \xi(1 + \sum T^{*n_i}h)^{-1}$. Then $f > 0$, a.s. and $f \leq \xi$ so that $f \in L_+^1$; from $f(\sum_i T^{*n_i}h) \leq \xi$ follows that $\int (\sum T^{n_i}f)h\, d\mu < \infty$. But if (d) is verified, $\sum T^{n_i}f = \infty$, a.s. so that h must be 0; hence (d) $\Rightarrow$ (c_n).

Finally, if $h \in L_+^\infty$ verifies $\liminf \langle T^n f_0, h\rangle = 0$, select an infinite sequence $0 \leq n_0 < n_1 < \cdots$ such that $\langle T^{n_i}f_0, h\rangle \leq 2^{-i}$. Then

(43) $$\int f_0(\sum T^{*n_i}h)\, d\mu = \sum \langle T^{n_i}f_0, h\rangle < \infty,$$

so that

(44) $$\textstyle\sum_i T^{*n_i}h < \infty, \text{ a.s.}$$

If (c_n) is verified, it implies that $h = 0$; hence (c_n) $\Rightarrow$ (b_n).

6. The existence of a strictly positive invariant element g in L^1 immediately implies that T is conservative since $\sum_{n\geq 0}T^n g = \sum_{n\geq 0} g = \infty$; it also implies the validity of condition (e).

Indeed, the formula $T'f = g\cdot T^*(f/g)$ where $f \in L^1$ is such that $f/g \in L^\infty$, defines a positive linear contraction T' of L^1 on the dense subspace

(45) $$\{f\colon f \in L^1, f/g \in L^\infty\}$$

of L^1; T' is indeed linear and positive on this subspace, and since it verifies these

(46) $$\int T'f\, d\mu = \langle g, T^*(f/g)\rangle = \langle Tg, f/g\rangle = \int f\, d\mu,$$

it can be extended by continuity to the whole of L^1. Moreover, g is T'-invariant since $T^*1 = 1$. Hence, condition (d) of the theorem is verified by T', and this implies that condition (e) is verified by T. Indeed, if $h \in L^\infty$ is strictly positive, so is gh in L^1 and

(47) $$g\Big(\sum_i T^{*n_i}h\Big) = \sum_i T'^{n_i}(gh) = \infty$$

holds a.s. for every infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers.

7. We show next that (e) $\Rightarrow$ (c_s) if T is conservative.

If the set F is such that $\sum_i T^{*n_i}1_F \in L^\infty$ for an infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers, then $h = \sum_{n\geq 0}2^{-n}T^{*n}1_F$ is an element of L_+^∞ such that:

(48) $$\sum_i T^{*n_i}h = \sum_n 2^{-n}\, T^{*n}\Big(\sum_i T^{*n_i}1_F\Big) \in L^\infty;$$

moreover, the set

(49) $$H = \{h > 0\} = \bigcup_{n\geq 0} \{T^{*n}1_F > 0\}$$

is, that $\sum T^{*n_i}h < \infty$ on $\{f > 0\}$; hence if (e) holds, f must be 0, that is, condition (f) holds. Conversely, if (f) holds and $h \in L^\infty$ is strictly positive, then $f = \xi(1 + \sum T^{*n_i}h)^{-1}$ belongs to L_+^1 and verifies

(50) $$\int (\sum T^{n_i}f)h\, d\mu = \int f(\sum T^{*n_i}h)\, d\mu \leq \int \xi\, d\mu < \infty.$$

Therefore, $\sum_i T^{n_i}f < \infty$, a.s. and f must be 0, that is, $\sum_i T^{*n_i}h = \infty$, a.s.

4. Strong conservativeness

The following theorem is a counterpart to theorem 1.

THEOREM 2. *For any positive linear contraction T of a space $L^1(\epsilon, \mathfrak{F}, \mu)$, the following conditions are equivalent:*

(a) *the only $g \in L^1_+$ such that $Tg = g$ is 0;*

(b_n) *there exists an element $h \in L^\infty$ such that $h > 0$, a.s. and*

$$\liminf_{n\to\infty} \langle T^n f_0, h\rangle = 0 \tag{51}$$

(f_0 denotes an arbitrarily fixed element of L^1 such that $f_0 > 0$, a.s.);

(b_s) *there exists an element $h \in L^\infty$ such that $h > 0$, a.s. and*

$$\lim_{p\to\infty} \sup_n 1/p \sum_{m=0}^{p-1} \langle T^{m+n} f_0, h\rangle = 0; \tag{52}$$

(c) *there exists an element $h \in L^\infty$ such that $h > 0$, a.s. and $\sum_i T^{*n_i} h < \infty$, a.s. for a suitably chosen infinite sequence $0 \le n_0 < n_1 < \cdots$ of integers;*

(d) *$\sum_i T^{n_i} f_0 < \infty$ holds a.s. for at least an infinite sequence $0 \le n_0 < n_1 < \cdots$ of integers.*

PROOF OF THEOREM 2. (1) To prove the implication (a) $\Rightarrow$ (b_n), consider the construction in alinea 2 of the proof of theorem 1 of an invariant $g \in L^1$ starting from a Banach limit L. Since $g = 0$ by (a), lemma 1 of the preliminaries shows the existence of a strictly positive $h \in L^\infty$ such that $\lambda(h) = 0$. Then (b_n) follows from the inequality $0 \le \liminf_{n\to\infty} \langle T^n f_0, h\rangle \le \lambda(h)$.

Conversely, (b_n) $\Rightarrow$ (a). The condition $\liminf_{n\to\infty} \langle T^n f_0, h\rangle = 0$ indeed implies by a previous remark that $\liminf_{n\to\infty} \langle T^n f, h\rangle = 0$ for any $f \in L^1_+$, hence, that $\langle g, h\rangle = 0$ if g is invariant. Since $h > 0$, a.s., this shows that 0 is the only invariant element in L^1_+.

(2) To show that (b_n) implies (c) and (d), choose an infinite sequence $0 \le n_0 < n_1 < \cdots$ of integers such that $\langle T^{n_i} f_0, h\rangle \le 2^{-i}$. Then

$$\int f_0 (\sum T^{*n_i} h)\, d\mu = \int (\sum T^{n_i} f_0) h\, d\mu \le \sum 2^{-i} < \infty \tag{53}$$

implies that $\sum T^{n_i} f_0 < \infty$ a.s. since $h > 0$ a.s., resp. that $\sum T^{*n_i} h < \infty$ a.s. since $f_0 > 0$ a.s.

Conversely, (c) $\Rightarrow$ (b_n) and (d) $\Rightarrow$ (b_n), for letting, as in alinea 5,

$$f_0 = \xi (1 + \sum T^{*n_i} h)^{-1} \tag{54}$$

in the first case and $h = \xi(1 + \sum T^{n_i} f_0)$ in the second case, one obtains that

$$0 \le \liminf_{n\to\infty} \langle T^n f_0, h\rangle \le \lim_i \langle T^{n_i} f_0, h\rangle = 0 \tag{55}$$

since $\sum_i \langle T^{n_i} f_0, h\rangle < \infty$ holds in both cases. This proves the implications above, because (b_n) does not depend on the f_0 selected, as was previously noted.

(3) It is clear that (b_s) $\Rightarrow$ (b_n). Conversely, if (b_n) holds, it is possible by lemma 3 to construct for each $\delta > 0$ an element $h_\delta \in L^\infty$ such that $0 \le h_\delta \le h$, $\langle f_0, h - h_\delta\rangle \le \delta$, and that $\sum_i T^{*n_i} h_\delta \in L^\infty$ for a suitably chosen infinite sequence

$0 \leq n_0 < n_1 < \cdots$ of integers. Then $\lambda(h_\delta) = 0$ holds whatever Banach limit L has been chosen to define λ, and it follows from the corollary to lemma 2 that

$$\lim_{n\to\infty} \sup_n \frac{1}{p} \sum_{m=0}^{p-1} \langle T^{m+n} f_0, h_\delta \rangle = 0. \tag{56}$$

Letting $h' = \sum 2^{-p} h_{2-p}$, one obtains an element $h' \in L_+^\infty$ such that

$$\lim_{p\to\infty} \sup_n \frac{1}{p} \sum_{m=0}^{p-1} \langle T^{m+n} f_0, h' \rangle = 0, \tag{57}$$

which is, moreover, strictly positive since $\{h' > 0\} = \bigcup_p \{h_{2-p} > 0\}$ and

$$\int_{\{h_2-p=0\}} f_0 h \, d\mu \leq \int f_0(h - h_{2-p}) \, d\mu \leq 2^{-p} \to 0 \qquad \text{as} \quad p \uparrow \infty. \tag{58}$$

Thus h' satisfies condition (b_s).

We propose to call the set defined in the following theorem the *strongly conservative* set associated to T.

THEOREM 3. *For any positive linear contraction T of a space $L^1(E, \mathfrak{F}, \mu)$, there exists a measurable subset $\tilde{C}$ of E (defined up to an equivalence), which is characterized by each of the following properties, the third one being valid only if T is conservative.*

(a) Every T-invariant element $g \in L^1$ is carried by $\tilde{C}$, namely, $\{g \neq 0\} \subset \tilde{C}$. Conversely, there exists a T-invariant element $\tilde{g} \in L_+^1$ such that $\{\tilde{g} > 0\} = \tilde{C}$.

(b) For any infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers, one has $\sum_i T^{n_i} f_0 = \infty$ on $\tilde{C}$, and there exists, conversely, an infinite sequence

$$0 \leq \tilde{n}_0 < \tilde{n}_1 < \cdots \tag{59}$$

such that $\{\sum_i T^{\tilde{n}_i} f = \infty\} = \tilde{C}$ (f_0 denotes a strictly positive, arbitrarily fixed element of L^1).

(c) For every strictly positive $h \in L^\infty$ and every infinite sequence

$$0 \leq n_0 < n_1 < \cdots \tag{60}$$

of integers, one has $\sum T^{*n_i} h = \infty$ on $\tilde{C}$. Conversely, there exists a strictly positive $\tilde{h} \in L^\infty$ and an infinite sequence $0 < \tilde{n}_0 < \tilde{n}_1 < \cdots$ of integers such that $\{\sum T^{*\tilde{n}_i} \tilde{h} = \infty\} = \tilde{C}$.

Moreover, $\tilde{C}$ is an invariant subset of the conservative part C of T.

PROOF OF THEOREM 3. Let G denote the set of all T-invariant g in L_+^1 and consider the essential supremum of the carriers $\{g > 0\}$ $(g \in G)$. Let $\tilde{C}$ be this set. By a general property of essential suprema, there exists a sequence $\{g_n\}$ in G such that $C = \bigcup \{g_n > 0\}$. Letting $\tilde{g} = \sum_n \|g_n\|^{-1} 2^{-n} g_n$, we obtain an element of G such that $\{\tilde{g} > 0\} = \tilde{C}$. Since $Tg = g$ $(g \in L^1)$ implies $T|g| = |g|$, one has $\{g \neq 0\} = \{|g| > 0\} \subset \tilde{C}$ for every T-invariant g in L^1. The existence and uniqueness of a set $\tilde{C}$ with property (a) is thus proved.

Moreover, since $C = \{g > 0\} = \{\sum_n T^n \tilde{g} = \infty\}$, the set C is an invariant subset of C (see [10]).

Applying theorem 1 to the restriction of T to $\tilde{C}$, which is a contraction of

$L^1[\tilde{C}, \tilde{C} \cap \mathfrak{F}, \mu(\tilde{C} \cap \cdot)]$, with the restriction of $\tilde{g}$ to $\tilde{C}$ as invariant strictly positive element, we obtain that $\sum T^{n_i} f_0 = \infty$ on $\tilde{C}$ for every infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers provided that f_0 belongs to L^1_+ and is strictly positive on $\tilde{C}$ (remark that the invariance of $\tilde{C}$ implies that the powers of the restriction of T to $\tilde{C}$ are the restrictions to $\tilde{C}$ of the powers of T). When applying theorem 2 to the restriction of T to $E - \tilde{C}$, we obtain the existence of an infinite sequence $0 \leq \tilde{n}_0 < \tilde{n}_1 < \cdots$ of integers such that $\sum_i T^{\tilde{n}_i} f_0 < \infty$ holds on $E - \tilde{C}$. This suffices to establish property (b).

When T is conservative, a reasoning similar to the preceding, but using condition (e) of theorem 1 and condition (c) of theorem 2, establishes the validity of property (c) of theorem 3 and concludes its proof.

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