

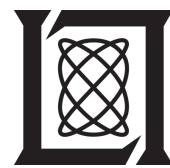
**Technical Report
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Consistent State Estimation for Very Long Range Radars

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1 December 2017

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LEXINGTON, MASSACHUSETTS



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Group 36

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ABSTRACT

Consistent state estimation for very long range radars is difficult because the true radar measurement noise covariance deviates significantly from the Gaussian ellipsoid approximation used in most track filters. Ad hoc inflation of the radar range covariance can improve inconsistency at long range, but usually at the expense of significant range accuracy. Here we present a new converted measurement extended Kalman filter algorithm (CM3EKF) for accurately estimating target states at very long ranges. It uses a third-order Taylor series approximation of the converted measurement noise covariance to calculate automatically the minimal range covariance inflation necessary to maintain track consistency without sacrificing range accuracy. In addition, the state covariance is automatically adjusted to compensate for approximation errors introduced by linearizing about a noisy track state. As the filter settles with time, the range inflation and state covariance compensation automatically decay to zero. In a systematic comparison with six other commonly used filters including the unscented Kalman filter (UKF) and regularized particle filter (RPF), we show that the CM3EKF is the most consistent and has only slightly worse convergence than the RPF, but is five orders of magnitude less computationally expensive and is less sensitive to tuning parameters.

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1. INTRODUCTION

Technological developments in the past four decades have greatly increased the sensitivity of phased array radars and have enabled target tracking at very long ranges. Long range targets (i.e., targets several thousand kilometers away) include space launch vehicles, ballistic missiles, and satellites. Apart from advances in solid-state radar electronics and signal processing, a critical technology for long range tracking is the underlying track filtering algorithm used to accurately estimate target states (e.g., position and velocity) from a time series of noisy long range position measurements.

Today, the prevailing track filter algorithm used by short- to medium-range radars is the extended Kalman filter (EKF) [2–4]. In the standard formulation of the EKF, nonlinear target state and radar measurement equations are linearly approximated using a first-order Taylor series about a nominal state trajectory. In phased array radars, the measurement equation is nonlinear because position measurements are often expressed in range direction cosine (RUV) or range-azimuth-elevation (RAE) coordinates, while target dynamics and equations of motion are expressed in Cartesian (XYZ) coordinates [2,5–7]. The different coordinates of the state and measurement necessitates a nonlinear coordinate conversion in order to calculate a filter gain.

An alternative algorithm to the EKF used commonly in practice is the converted measurement Kalman filter (CMKF) [1–3], which is linear in the measurement update step by converting the RUV measurements to XYZ coordinates. Although RUV and XYZ coordinates are related by a general closed form nonlinear function, there is no general closed form function relating RUV and XYZ covariance matrices [1]. In the CMKF, the converted (RUV to XYZ) measurement covariance is linearly approximated using a first- or second-order Taylor series [1]. This method works well when tracking targets at short to medium ranges, but at long ranges the linear approximation error leads to tracking degradation due to increased error uncertainty along the radar line-of-sight and in some cases poor covariance consistency.¹ Visually, the problem is that the transformed RUV measurement error distribution resembles a Gaussian ellipse in XYZ coordinates at short ranges, but at long ranges it resembles a very thin non-elliptical and non-Gaussian contact lens (see Figure 1). Here the thin *film* of the contact lens represents the high precision in range and the large diameter of the lens represents the very noisy angles. Since a large (possibly infinite) number of parameters are needed to correctly describe the transformed error distribution, approximations of the converted measurement covariance, decoupling of the filter states, and ad hoc range covariance inflation logic are used in practice [1,2,4,6,8].

¹Here, covariance consistency refers to the degree of statistical agreement between the filter-calculated state error covariance and the true state error covariance calculated in an off-line computer simulation (see [1]).



Figure 1. Illustration of contact lens effect at long ranges.

1.1 PROBLEM STATEMENT

Practical, accurate, and consistent target tracking for long range radars remains a long-standing problem in state estimation. Consider the nonlinear function that relates the target's position in Cartesian (XYZ) coordinates to radar measurement (RUV) coordinates. Given the target state vector in XYZ and measurement vector in RUV,

$$\mathbf{x}_{xyz} \triangleq \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{z}_{ruv} \triangleq \begin{bmatrix} r \\ u \\ v \end{bmatrix}, \quad (1)$$

the nonlinear function, $\mathbf{g}[\cdot]$, that converts from measurement to target coordinates is given by

$$\mathbf{x}_{xyz} = \mathbf{g}[\mathbf{z}_{ruv}] \triangleq \begin{bmatrix} ru \\ rv \\ r\sqrt{1-u^2-v^2} \end{bmatrix}. \quad (2)$$

The mean and error covariance of (2) can be approximated using a Taylor series expansion of $\mathbf{g}[\cdot]$ with some finite number of terms. At short ranges, a first-order approximation may suffice, however, as the range to the target increases, higher order terms are necessary to accurately describe the converted distribution.

Consider a target observed by a phased array radar directly on its boresight at a range of 5000 km ($r = 5000$ km, $u = 0$, $v = 0$). Further, consider that the radar makes one million independent measurements in RUV coordinates, each corrupted by zero mean white Gaussian noise² with statistics: $\sigma_r = 1$ m and $\sigma_u = \sigma_v = 1$ mrad. After converting each measurement from RUV to XYZ coordinates, the sample mean and covariance can be calculated. Ideally, the Taylor series approximation of the converted measurement mean and covariance is in good agreement with the sample mean and sample covariance. Figure 2a shows the one million converted measurements in the X-Z plane. Since u and v are equal to zero, the distribution is symmetric around the z -axis. The

²As will be discussed in Section 3, the noises in r , u , and v are nearly uncorrelated.

measurements near the target’s true location are colored yellow, signifying areas of high likelihood, and are colored dark blue in areas of low likelihood. The sample mean and 3σ covariance of the converted measurements are shown as a solid red square and ellipse, respectively. Notice that the sample mean falls below (in the z -direction) the area of highest likelihood, due to the large tails of the distribution. Also note the unequal plot scales: the x -axis spans 50 km, while the y -axis spans only 60 m.

Figures 2b, 2c, and 2d show results of approximating the mean and covariance of the converted measurements using first- (CM1), second- (CM2), and third-order (CM3) Taylor series expansions. The first-order approximation is the poorest in that the approximated mean does not match the sample mean. Also, the approximated covariance in the z -direction (range) is much too small. The second-order approximation is better in that it correctly matches the sample mean, however the approximated covariance in the z -direction is still too small. The third-order approximation, shown in Figure 2d, matches the sample mean and covariance extremely well. While all three Taylor series expansions approximate the cross-range covariance very well, only the third-order Taylor series correctly approximates the range covariance. Correctly approximating the converted measurement’s mean and covariance is important when calculating a filter gain, and errors in these statistics will result in tracking degradation and poor covariance consistency. Maintaining covariance consistency (both measurement and track) is extremely important for both local and remote sensors. For local sensors, measurement-to-track association may be corrupted if covariance consistency is not achieved. For remote sensors (e.g., one sensor passing its track information to another sensor or interceptor), track-to-track correlation or local acquisition based on a remote cue may either be corrupted or fail if the covariances are not consistent.

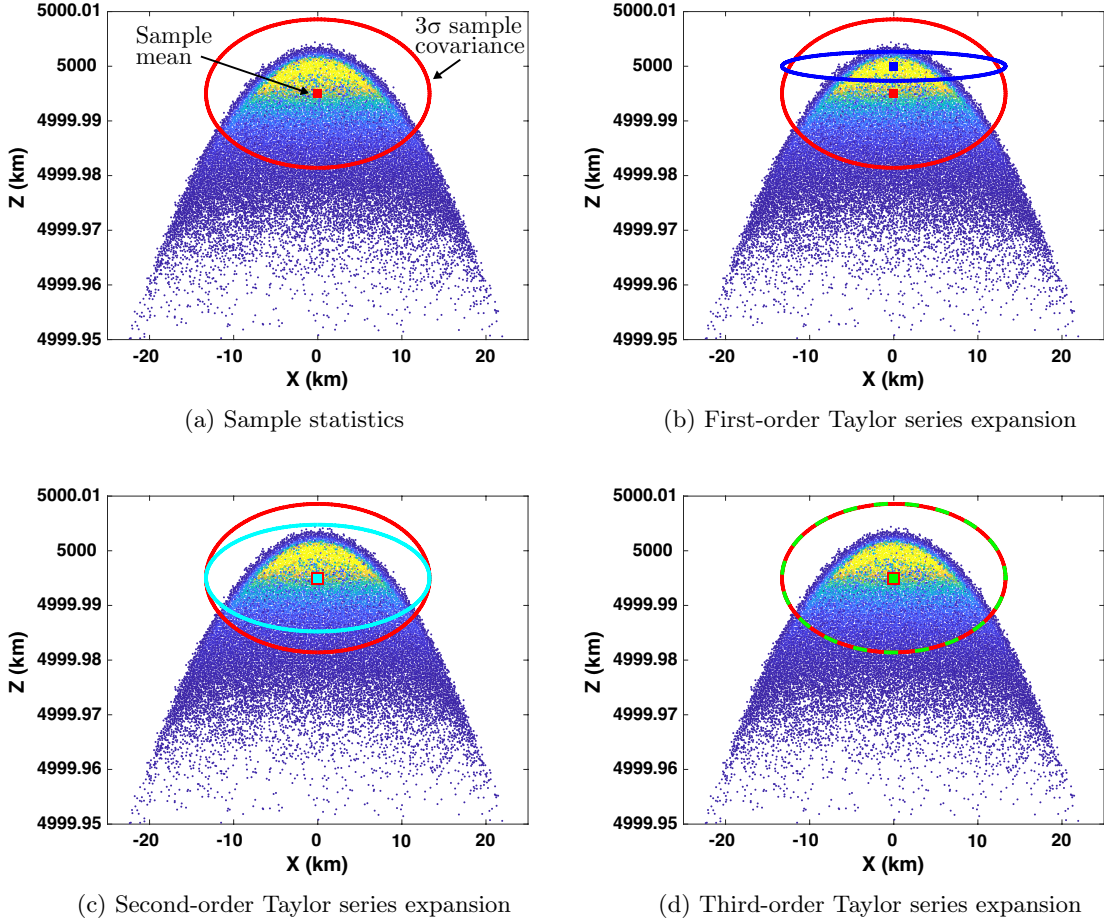


Figure 2. Approximation of mean and covariance for nonlinear conversion from RUV to XYZ coordinates. The colored dots are one million measurements converted from RUV to XYZ coordinates. The converted sample mean and 3σ covariance are denoted as a red square and red ellipse, respectively. Approximations of the converted mean and 3σ covariance are shown for the first- (blue), second- (cyan), and third-order (green) Taylor series expansions in (b), (c), and (d).

Contemporary algorithms including the unscented Kalman filter (UKF) [4, 9–11] and the particle filter (PF) [12–14] have been applied to long range tracking problems. In fact, nonlinearities associated with long range tracking motivated the development of many UKF and PF algorithms. The UKF and PF use sigma-point and Monte Carlo sampling, respectively, to approximate the converted mean and error covariance while avoiding the calculation of Jacobian matrices needed by the EKF. While previous work has compared a subset of the algorithms, none of the studies cited above have presented a systematic comparison of algorithms for the long-range phased-array radar tracking problem.

1.2 MAIN CONTRIBUTION – CM3EKF ALGORITHM

We have developed a new extended Kalman filter algorithm (CM3EKF) for accurately estimating target states for long range radars. Using a third-order Taylor series approximation of the converted (RUV to XYZ) measurement noise covariance and a novel method to handle approximation errors due to uncertainty in the linearization point, it has the advantage of superior covariance consistency without sacrificing target range estimation accuracy. The CM3EKF algorithm is unique in that it considers the two main difficulties of tracking targets at very long ranges:

1. It correctly approximates the mean and covariance of the converted (RUV to XYZ) measurements
2. It automatically compensates for linearization errors (due to using a noisy linearization point instead of truth) during transformation of the error covariance

Compensating for linearization errors is crucial during track initiation and in early portions of track, when errors are the largest. The linearization compensation in the proposed CM3EKF algorithm is computationally efficient and automatically reduces as track accuracy improves.

In this report, we compare the CM3EKF to six prevalently used algorithms using previously published scenarios to make a systematic comparison more straightforward. The six algorithms are: the first-, second-, and third-order converted measurement Kalman filters (CM1KF, CM2KF, and CM3KF) [1, 3], the measurement covariance adaptive extended Kalman filter (MCAEKF) [1], the UKF, and the regularized particle filter (RPF) which is a common variant of the PF.

1.3 PRIOR WORK

Previous work has attempted to compensate for the nonlinear relation between target state and radar measurement equations several ways. In [15], attempts were made to define the entire filter in the spherical measurement coordinate frame. While this approach leads to a linear relationship between the state and measurement equations, integrating the equations of motion in a non-Cartesian coordinate frame is generally difficult and requires either a polynomial filter or linearized dynamics. It has also been shown that this leads to filter consistency problems [15]. For these reasons, this approach is not considered further.

In [1–3, 16–18] it was suggested to convert the RUV measurements into Cartesian XYZ coordinates using either a first- or second-order Taylor series expansion and therefore realizing a linear filter during the measurement update step. The CM1KF and CM2KF algorithms presented in [1] are examples of Kalman filters using first- and second-order conversions, respectively. While this has been shown to produce consistent covariances for short to medium range cases (by inflating the measurement uncertainty in the range direction), the resulting track errors in range may actually be larger than the raw sensor measurements, providing no filter benefit in this direction.

The two prevailing algorithms for tracking long range targets are the MCAEKF and the RPF, presented in [1] and [14] respectively. The MCAEKF is a derivative of the extended Kalman filter, but inflates the measurement noise covariance (in the range direction only) in order to guarantee filter consistency. Unfortunately, the derivation of the MCAEKF in [1] has an error in the second-order Taylor series approximation (see Appendix A) that results in the algorithm producing too

large of a range uncertainty. Additionally, the need to compute the condition number ratio of the first- and second-order conversions and built-in ad hoc switching logic was added to account for this error. The RPF is a variant of the PF which performs a regularization step to reduce filter degeneracy, which is prevalent in particle filters [12, 13]. It has generally shown good performance for tracking long range targets, but is significantly more computationally demanding.

1.4 ORGANIZATION OF THE REPORT

This report is organized as follows. In Section 2 we define the mathematical state space model for the long range tracking problem and describe the most prevalently used track filter algorithms in this area. In Section 3 we describe XYZ and RUV coordinates, the nonlinear transformation relating them, and we present expressions for the converted (RUV to XYZ) measurement noise covariance using first-, second-, and third-order Taylor series approximations. In Section 4 we describe a new Kalman filter algorithm (CM3EKF) for consistent state estimation for long range radars that adaptively adjusts the range measurement covariance to maintain covariance consistency. We also describe a novel method for handling errors in the converted covariance approximation due to uncertainty in the linearization point. Section 5 compares the proposed CM3EKF algorithm to six other popular algorithms: CM1KF, CM2KF, CM3KF, MCAEKF, UKF, and RPF. Appendix A identifies errors in the derivation of the MCAEKF algorithm published in [1]. The new derivations for the first-, second-, and third-order Taylor series approximations of the converted measurements (mean and covariance) are provided in Appendices B, C, and D.

2. MATHEMATICAL MODELING AND PRIOR WORK

In this section, we present the state space model that describes the target dynamics and radar measurement equations and describe several linear and nonlinear filtering algorithms previously applied to the long range target tracking problem.

2.1 STATE SPACE MODEL FOR PHASED ARRAY RADARS

The nonlinear discrete-time equations for the target dynamics and radar measurements are given by:

$$\mathbf{x}(k+1) = \mathbf{f}[\mathbf{x}(k), k] + \boldsymbol{\nu}(k), \quad (3)$$

$$\mathbf{z}_{ruv}(k) = \mathbf{h}[\mathbf{x}(k), k] + \mathbf{w}_{ruv}(k). \quad (4)$$

At time index k , $\mathbf{x}(k)$ is the $n_x \times 1$ target state vector, and $\mathbf{z}_{ruv}(k)$ is the $n_z \times 1$ measurement vector. The state evolves according to the nonlinear time-varying equation $\mathbf{f}[\mathbf{x}(k), k]$ and is assumed to be corrupted by additive zero mean white Gaussian process noise $\boldsymbol{\nu}(k)$ with covariance $\mathbf{Q}(k)$. The measurement is given by the nonlinear time-varying equation $\mathbf{h}[\mathbf{x}(k), k]$ and is assumed to be corrupted by additive zero mean white Gaussian measurement noise $\mathbf{w}_{ruv}(k)$ with covariance $\mathbf{R}(k)$. The process and measurement noises $\boldsymbol{\nu}(k)$ and $\mathbf{w}_{ruv}(k)$ are mutually independent with

$$E[\boldsymbol{\nu}(i)\boldsymbol{\nu}^T(j)] = \mathbf{Q}(i)\delta_{ij}, \quad (5)$$

$$E[\mathbf{w}_{ruv}(i)\mathbf{w}_{ruv}^T(j)] = \mathbf{R}_{ruv}(i)\delta_{ij}, \quad \forall i, j \quad (6)$$

$$E[\boldsymbol{\nu}(i)\mathbf{w}_{ruv}^T(j)] = 0, \quad (7)$$

where $E[\cdot]$ denotes the expectation operator and δ_{ij} is the Kronecker delta function. Unless otherwise noted, the state $\mathbf{x}(k)$ is assumed to be in Cartesian XYZ coordinates.

When designing the track filter, there is flexibility in choosing the coordinate frames for the dynamics and measurement equations. The four most common choices of coordinate frames are listed in Table 1, and then described in more detail in Sections 2.2, 2.3, 2.4, and 2.5.

TABLE 1
Common Filter Coordinate Frames

Algorithm	Prediction Step	Correction Step		References
	State	State	Meas.	
1	RUV	RUV	RUV	[15]
2	XYZ	RUV ^a	RUV	[6, 19]
3	XYZ	XYZ	XYZ ^b	[1–3, 14, 16–18]
4	XYZ	XYZ	RUV	[2–4, 6, 8, 19–22]

^aDenotes nonlinear conversion of Cartesian XYZ states and covariances to RUV coordinates.

^bDenotes nonlinear conversion of RUV measurements to Cartesian XYZ coordinates.

2.2 RUV KALMAN FILTER

The first algorithm in Table 1 defines both the dynamics and measurements in RUV coordinates. This (and subsequent) estimators are referred to as *linear*, with some abuse of language, because the state and covariance correction step (measurement update) is entirely linear (i.e., the states and measurements are linearly related to each other). However, the system of equations that describe the prediction of the state and covariance (time update) for the RUV KF are likely nonlinear. Because it is more natural to model dynamics in Cartesian coordinates, this algorithm is generally a poor choice and requires using a polynomial filter or linearized dynamics equations in order to integrate the equations of motion. Since the RUV Kalman filter has been shown to cause filter consistency problems [15], it is not considered further, but we include it for completeness.

Algorithm 1 RUV Kalman Filter

Input at time $k+1$:

Updated state and covariance at time k :

$$\hat{\mathbf{x}}_{ruv}(k|k), \mathbf{P}_{ruv}(k|k)$$

Measurement and measurement covariance at time $k+1$:

$$\mathbf{z}_{ruv}(k+1), \mathbf{R}_{ruv}(k+1)$$

Time Update:

Prediction of state estimate and covariance:

$$\begin{aligned}\hat{\mathbf{x}}_{ruv}(k+1|k) &= \mathbf{f}[\hat{\mathbf{x}}_{ruv}(k|k)] \\ \mathbf{P}_{ruv}(k+1|k) &= \mathbf{J}_f \mathbf{P}_{ruv}(k|k) \mathbf{J}_f^T + \mathbf{Q}(k)\end{aligned}$$

where $\mathbf{f}[\cdot]$ is the nonlinear state transition function (may be time-varying), $\mathbf{Q}(k)$ is the process noise covariance, and $\mathbf{J}_f$ is the Jacobian of the state transition function evaluated at the current state:

$$\mathbf{J}_f = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{ruv}(k|k)}$$

Measurement Update:

Gain calculation:

$$\mathbf{K} = \mathbf{P}_{ruv}(k+1|k) \mathbf{H}^T [\mathbf{H} \mathbf{P}_{ruv}(k+1|k) \mathbf{H}^T + \mathbf{R}_{ruv}(k+1)]^{-1}$$

Update state estimate and covariance:

$$\begin{aligned}\hat{\mathbf{x}}_{ruv}(k+1|k+1) &= \hat{\mathbf{x}}_{ruv}(k+1|k) + \mathbf{K} [\mathbf{z}_{ruv}(k+1) - \mathbf{H} \hat{\mathbf{x}}_{ruv}(k+1|k)] \\ \mathbf{P}_{ruv}(k+1|k+1) &= (\mathbf{I} - \mathbf{K} \mathbf{H}) \mathbf{P}_{ruv}(k+1|k) (\mathbf{I} - \mathbf{K} \mathbf{H})^T + \mathbf{K} \mathbf{R}_{ruv}(k+1) \mathbf{K}^T\end{aligned}$$

where $\mathbf{H}$ is the measurement matrix (may be time-varying)

Output at time $k+1$:

Updated state and covariance at time $k+1$:

$$\hat{\mathbf{x}}_{ruv}(k+1|k+1), \mathbf{P}_{ruv}(k+1|k+1)$$

2.3 CONVERTED STATES LINEAR KALMAN FILTER

The second algorithm in Table 1, summarized below, is to convert the state vector and covariance to RUV coordinates and perform the measurement update in RUV. The advantage of this approach is that it converts the state covariance to measurement coordinates and produces a linear filter during the measurement update step (because measurements are linearly related to the state). However, it typically does this by using a first-order Taylor series when (i) converting the state covariance to RUV coordinates before the measurement update, and (ii) converting the state covariance back to XYZ coordinates after the update. Including higher order partial derivatives when converting from Cartesian XYZ to RUV coordinates is complicated by the fact that the Cartesian measurement noises (i.e., w_x , w_y , and w_z) are most certainly correlated, and would therefore require an extraordinarily large number of terms. As a consequence, it can be shown that this approach

(with the first-order Taylor series) suffers from consistency problems as the target's range increases and is therefore not considered further.

Algorithm 2 Converted States Linear^a Kalman Filter

Input at time $k+1$:

Updated state and covariance at time k :

$$\hat{\mathbf{x}}_{xyz}(k|k), \mathbf{P}_{xyz}(k|k)$$

Measurement and measurement covariance at time $k+1$:

$$\mathbf{z}_{ruv}(k+1), \mathbf{R}_{ruv}(k+1)$$

Time Update:

Prediction of state estimate and covariance:

$$\begin{aligned}\hat{\mathbf{x}}_{xyz}(k+1|k) &= \mathbf{f}[\hat{\mathbf{x}}_{xyz}(k|k)] \\ \mathbf{P}_{xyz}(k+1|k) &= \mathbf{J}_f \mathbf{P}_{xyz}(k|k) \mathbf{J}_f^T + \mathbf{Q}(k)\end{aligned}$$

where $\mathbf{f}[\cdot]$ is the nonlinear state transition function (may be time-varying), $\mathbf{Q}(k)$ is the process noise covariance, and $\mathbf{J}_f$ is the Jacobian of the state transition function evaluated at the current state:

$$\mathbf{J}_f = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{xyz}(k|k)}$$

Measurement Update:

Convert state estimate and covariance^b to RUV coordinates:

$$\begin{aligned}\hat{\mathbf{x}}_{ruv}(k+1|k) &= \mathbf{g}[\hat{\mathbf{x}}_{xyz}(k+1|k)] \\ \mathbf{P}_{ruv}(k+1|k) &= \mathbf{J}_{xyz}^{ruv} \mathbf{P}_{xyz}(k+1|k) (\mathbf{J}_{xyz}^{ruv})^T\end{aligned}$$

Gain calculation:

$$\mathbf{K} = \mathbf{P}_{ruv}(k+1|k) \mathbf{H}^T [\mathbf{H} \mathbf{P}_{ruv}(k+1|k) \mathbf{H}^T + \mathbf{R}_{ruv}(k+1)]^{-1}$$

Update state estimate and covariance:

$$\begin{aligned}\hat{\mathbf{x}}_{ruv}(k+1|k+1) &= \hat{\mathbf{x}}_{ruv}(k+1|k) + \mathbf{K} [\mathbf{z}_{ruv}(k+1) - \mathbf{H} \hat{\mathbf{x}}_{ruv}(k+1|k)] \\ \mathbf{P}_{ruv}(k+1|k+1) &= (\mathbf{I} - \mathbf{K} \mathbf{H}) \mathbf{P}_{ruv}(k+1|k) (\mathbf{I} - \mathbf{K} \mathbf{H})^T + \mathbf{K} \mathbf{R}_{ruv}(k+1) \mathbf{K}^T\end{aligned}$$

Convert state estimate and covariance back to Cartesian XYZ coordinates:

$$\begin{aligned}\hat{\mathbf{x}}_{xyz}(k+1|k+1) &= \mathbf{g}^{-1}[\hat{\mathbf{x}}_{ruv}(k+1|k+1)] \\ \mathbf{P}_{xyz}(k+1|k+1) &= \mathbf{J}_{ruv}^{xyz} \mathbf{P}_{ruv}(k+1|k+1) (\mathbf{J}_{ruv}^{xyz})^T\end{aligned}$$

where $\mathbf{H}$ is the measurement matrix (may be time-varying), $\mathbf{g}[\cdot]$ is the nonlinear function to convert from XYZ to RUV coordinates and $\mathbf{g}^{-1}[\cdot]^c$ is the nonlinear function from RUV back to XYZ

Output at time $k+1$:

Updated state and covariance at time $k+1$:

$$\hat{\mathbf{x}}_{xyz}(k+1|k+1), \mathbf{P}_{xyz}(k+1|k+1)$$

^aState converted to RUV coordinates such that they are linearly related to measurements.

^bConversion to/from RUV coordinates could include higher order terms; first-order linearization is assumed here.

^cBecause the conversion is nonlinear $\mathbf{J}_{xyz}^{ruv} \neq (\mathbf{J}_{ruv}^{xyz})^T$.

2.4 CONVERTED MEASUREMENTS LINEAR KALMAN FILTER

The third algorithm in Table 1 propagates the state and covariance in Cartesian XYZ coordinates and realizes a linear filter during the measurement update step by transforming the measurements into Cartesian XYZ coordinates. The use of debiased conversions for converting RAE polar/spherical to Cartesian XYZ coordinates is discussed in [2, 3, 16–18], and closed form approximations of the unbiased covariances are described that avoid linearization. In [1], Tian and Bar-Shalom approximate the converted (RUV to XYZ) measurement covariance using a first- and second-order Taylor series and describe a converted measurement Kalman filter (CM1KF and CM2KF). Unfortunately, the second-order Taylor series approximation is incorrect as published which affected their results and conclusions (see Appendix A for details). The advantage of the converted measurement approach is that the measurement noises in RUV coordinates are either nearly or completely uncorrelated and so the cross-expectations $E(w_r w_u)$, $E(w_r w_v)$, and $E(w_u w_v)$ are all equal to zero. This property enables extensions of the approach to a third-order Taylor series. As shown in Section 1.1, the addition of these higher order terms in the Taylor series expansion better characterizes the measurement error distribution and can lead to significantly improved filter consistency.

Algorithm 3 Converted Measurements Linear^a Kalman Filter

Input at time $k+1$:

Updated state and covariance at time k :

$$\hat{\mathbf{x}}_{xyz}(k|k), \mathbf{P}_{xyz}(k|k)$$

Measurement and measurement covariance at time $k+1$:

$$\mathbf{z}_{ruv}(k+1), \mathbf{R}_{ruv}(k+1)$$

Time Update:

Prediction of state estimate and covariance:

$$\begin{aligned}\hat{\mathbf{x}}_{xyz}(k+1|k) &= \mathbf{f}[\hat{\mathbf{x}}_{xyz}(k|k)] \\ \mathbf{P}_{xyz}(k+1|k) &= \mathbf{J}_f \mathbf{P}_{xyz}(k|k) \mathbf{J}_f^T + \mathbf{Q}(k)\end{aligned}$$

where $\mathbf{f}[\cdot]$ is the nonlinear state transition function (may be time-varying), $\mathbf{Q}(k)$ is the process noise covariance, and $\mathbf{J}_f$ is the Jacobian of the state transition function evaluated at the current state:

$$\mathbf{J}_f = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{xyz}(k|k)}$$

Measurement Update:

Convert measurement to Cartesian XYZ coordinates using CM1, CM2, or CM3 algorithm:

$$\left[\mathbf{z}_{xyz}(k+1), \mathbf{R}_{xyz}(k+1) \right] = \text{CM\#} \left[\mathbf{z}_{ruv}(k+1), \mathbf{R}_{ruv}(k+1) \right]$$

Gain calculation:

$$\mathbf{K} = \mathbf{P}_{xyz}(k+1|k) \mathbf{H}^T \left[\mathbf{H} \mathbf{P}_{xyz}(k+1|k) \mathbf{H}^T + \mathbf{R}_{xyz}(k+1) \right]^{-1}$$

Update state estimate and covariance:

$$\begin{aligned}\hat{\mathbf{x}}_{xyz}(k+1|k+1) &= \hat{\mathbf{x}}_{xyz}(k+1|k) + \mathbf{K} [\mathbf{z}_{xyz}(k+1) - \mathbf{H} \hat{\mathbf{x}}_{xyz}(k+1|k)] \\ \mathbf{P}_{xyz}(k+1|k+1) &= (\mathbf{I} - \mathbf{K} \mathbf{H}) \mathbf{P}_{xyz}(k+1|k) (\mathbf{I} - \mathbf{K} \mathbf{H})^T + \mathbf{K} \mathbf{R}_{xyz}(k+1) \mathbf{K}^T\end{aligned}$$

where $\mathbf{H}$ is the measurement matrix (may be time-varying)

Output at time $k+1$:

Updated state and covariance at time $k+1$:

$$\hat{\mathbf{x}}_{xyz}(k+1|k+1), \mathbf{P}_{xyz}(k+1|k+1)$$

^aMeasurements converted to Cartesian XYZ coordinates such that they are linearly related to the state.

2.5 EXTENDED KALMAN FILTER

The fourth algorithm in Table 1 is the most common [2–4, 6, 8, 19–22]. In this approach, the state and covariance are predicted in Cartesian XYZ coordinates and then related to the measurements in RUV coordinates during the correction step. This approach is the well-known extended Kalman filter (EKF) and is sometimes referred to as the mixed coordinate linearized filter. It often uses a first-order linearization, which without any modifications can produce large errors and inconsistent error covariances when applied to extremely long range tracking problems [1, 8, 14, 16]. In [1], a modification to the standard EKF was proposed that inflates the measurement noise co-

variance (`MeasNoiseCovAdj[.]` function) in order to achieve consistent results, and this algorithm is referred to as the measurement noise covariance adjusted EKF (MCAEKF). However, the inflation factor depended on the second-order Taylor series approximation of the converted (RUV to XYZ) measurement noise covariance which was derived incorrectly (see Appendix A for details). The consequence of the errors in the derivation is an unnecessarily large range uncertainty, which then requires ad hoc switching logic between the first- and second-order Taylor series conversions. In Section 4 of this report we propose modifications and corrections to the approach described in [1] so that performance is maintained while providing filter consistency.

Algorithm 4 Extended^a Kalman Filter Variant

Input at time $k+1$:

Updated state and covariance at time k :

$$\hat{\mathbf{x}}_{xyz}(k|k), \mathbf{P}_{xyz}(k|k)$$

Measurement and measurement covariance at time $k+1$:

$$\mathbf{z}_{ruv}(k+1), \mathbf{R}_{ruv}(k+1)$$

Time Update:

Prediction of state estimate and covariance:

$$\begin{aligned}\hat{\mathbf{x}}_{xyz}(k+1|k) &= \mathbf{f}[\hat{\mathbf{x}}_{xyz}(k|k)] \\ \mathbf{P}_{xyz}(k+1|k) &= \mathbf{J}_f \mathbf{P}_{xyz}(k|k) \mathbf{J}_f^T + \mathbf{Q}(k)\end{aligned}$$

where $\mathbf{f}[\cdot]$ is the nonlinear state transition function (may be time-varying), $\mathbf{Q}(k)$ is the process noise covariance, and $\mathbf{J}_f$ is the Jacobian of the state transition function evaluated at the current state:

$$\mathbf{J}_f = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{xyz}(k|k)}$$

Measurement Update:

Measurement noise covariance adjustment^b may be performed using CM2 or CM3 algorithm (e.g., Equations (42)–(55) in Section 4 for third-order conversion):

$$[\mathbf{R}_{ruv,adj}(k+1)] = \text{MeasNoiseCovAdj}[\hat{\mathbf{x}}_{xyz}(k+1|k), \mathbf{P}_{xyz}(k+1|k), \mathbf{z}_{ruv}(k+1), \mathbf{R}_{ruv}(k+1)]$$

Compute measurement Jacobian relating measurement and state:

$$\mathbf{J}_h = \left. \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{xyz}(k+1|k)}$$

Gain calculation:

$$\mathbf{K} = \mathbf{P}_{xyz}(k+1|k) \mathbf{J}_h^T [\mathbf{J}_h \mathbf{P}_{xyz}(k+1|k) \mathbf{J}_h^T + \mathbf{R}_{ruv,adj}(k+1)]^{-1}$$

Update state estimate and covariance:

$$\begin{aligned}\hat{\mathbf{x}}_{xyz}(k+1|k+1) &= \hat{\mathbf{x}}_{xyz}(k+1|k) + \mathbf{K} [\mathbf{z}_{ruv}(k+1) - \mathbf{h}[\hat{\mathbf{x}}_{xyz}(k+1|k)]] \\ \mathbf{P}_{xyz}(k+1|k+1) &= (\mathbf{I} - \mathbf{K} \mathbf{J}_h) \mathbf{P}_{xyz}(k+1|k) (\mathbf{I} - \mathbf{K} \mathbf{J}_h)^T + \mathbf{K} \mathbf{R}_{ruv,adj}(k+1) \mathbf{K}^T\end{aligned}$$

where $\mathbf{H}$ is the measurement matrix (may be time-varying)

Output at time $k+1$:

Updated state and covariance at time $k+1$:

$$\hat{\mathbf{x}}_{xyz}(k+1|k+1), \mathbf{P}_{xyz}(k+1|k+1)$$

^aMeasurements nonlinearly related to states.

^bInflates range uncertainty to correct for contact lens effect; skipping this step results in the general EKF.

2.6 OTHER ALGORITHMS FOR LONG RANGE TRACKING

Other algorithms for tracking targets at long ranges have also been suggested. In [6], Daum and Fitzgerald summarized different coordinate frame definitions that in certain applications allow one

to decouple the filter coordinates in an attempt to reduce the so-called *false cross-range observability*. In [8], Lambert and Tobias took a similar approach by suggesting a *quasi* decoupling be performed, where the predicted state covariance is decoupled during the filter gain calculation when a specified inequality is satisfied. In both of these papers, the coordinate conversions are performed assuming a first-order linearization, so important information about the true measurement uncertainty is lost.

The unscented Kalman filter [4, 9–11] and particle filter [12–14] are two common nonlinear filters that have been applied to long range tracking problems, and both of these algorithms avoid the calculation of Jacobian matrices needed by the EKF. The latter uses Monte Carlo sampling to create *particles* that describe the underlying distribution. Each particle is predicted forward using the nonlinear state transition function during the time update step. During the correction step (measurement update), the likelihood of each particle is computed using a prescribed probability density function (pdf) of the measurements. Sample statistics such as the state mean and covariance can then be computed from the probability-weighted particles. The advantage to this approach is it can describe extremely nonlinear systems and noise sources. The disadvantage is it requires a significant number of particles (typically many thousands) which can be computationally demanding, and it is sensitive to tuning parameters (e.g., it often requires a regularization step to prevent sample impoverishment). The UKF is similar to the PF in that it attempts to find a set of points referred to as sigma points whose sample pdf approximates the true pdf. However, the UKF selects the sigma points based on the state covariance matrix, and the standard formulation only requires $2n_x$ sigma points, where n_x is the dimension of the state vector. It has been shown [4, 9–11] that the approximated mean and covariance of the unscented transform matches the true distribution’s mean and covariance up to third-order when the underlying distribution is symmetric.

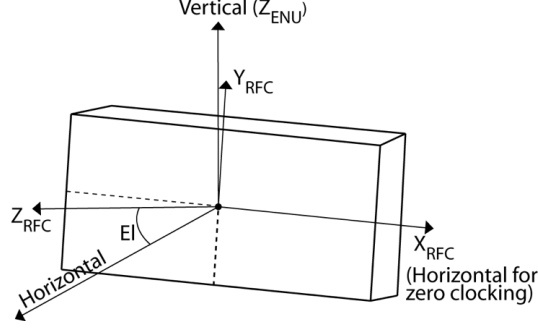
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3. NONLINEAR TRANSFORMATION FROM MEASUREMENT TO FILTER COORDINATES

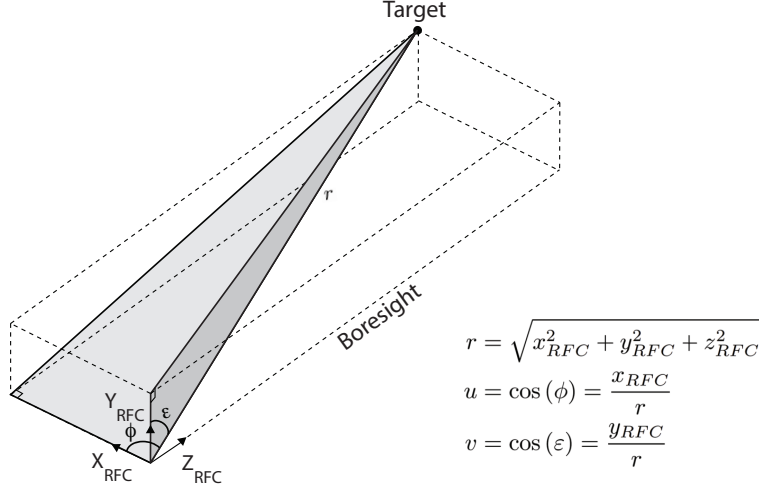
Four methods for approximating the converted (RUV to XYZ) radar measurement noise covariance are discussed in this section. Three of these methods have previously been reported: CM1, CM2, and the unscented transform (UT). The fourth method, CM3, is described here for the first time.

Phased array radars are used extensively in the areas of maritime and air surveillance, space situational awareness (SSA), and theater and ballistic missile defense (TMD and BMD). Recent advances in hardware have allowed all-digital solid-state phased array radars to become a reality, and they are increasingly being tasked to operate at longer ranges. For example, BMD and SSA targets are typically several to tens of thousands of kilometers away. Phased array radars offer several advantages over dish radars, with the most important being the ability for near-simultaneous electronically steered beams across the array face. Traditional dish arrays require large slew times in order to track multiple targets, which produces both time and mechanical stresses on the radar. Phased array radars use phase shifters to electronically steer the beam in order to track large numbers of targets dispersed across the array face. It has been shown [7] that each phase shifter's complex weights are a function of the angle off boresight in sine space, which makes the well-known range direction cosine coordinate frame a natural measurement space for phased array radars. In this coordinate system, r is the range to the target, u is the direction cosine in the array face's horizontal direction, and v is the direction cosine in the array face's vertical direction. Both u and v are defined in the interval, $u, v \in [-1, 1]$, and $\sqrt{u^2 + v^2} \leq 1$.

A Cartesian coordinate frame is natural for modeling the dynamics of a target, and in this report the radar face coordinates (RFC) frame was chosen to describe the target's state (e.g., position and velocity). In this frame, the xy -plane is coincident with the array face (x -axis points to the left when viewed outward along z -axis, y -axis points up), and the z -axis is perpendicular to the face (boresight of the radar). Note that any other Cartesian coordinate system [e.g., east-north-up (ENU), Earth centered inertial (ECI), Earth centered rotating (ECR)] is a straightforward linear rotation from the RFC coordinate frame. Illustrations of the RUV and RFC coordinate frames, and how they relate to each other, are shown in Figure 3.



(a) Radar face coordinates (RFC) is a Cartesian frame where the z -axis points outward from and normal to the radar face plane, x -axis points left when viewed outward along z -axis (horizontal if clocking angle zero), and y -axis completes right-handed coordinate frame and points up along radar face plane (vertical if clocking and elevation angles are zero).



(b) Range direction cosine is a non-Cartesian coordinate frame where r is the slant range from the array face origin to target, u is the sine of the object angular displacement from the y - z plane in x direction, and v is the sine of the object angular displacement from x - z plane in y direction.

Figure 3. Measurement and filter coordinate frames for nonlinear tracking problem.

The nonlinear transformations from RUV measurement coordinates to RFC Cartesian coordinates (and back) are defined as

$$\begin{aligned}
 x &\triangleq ru \\
 y &\triangleq rv \\
 z &\triangleq r\sqrt{1 - u^2 - v^2}
 \end{aligned}
 \iff
 \begin{aligned}
 r &\triangleq \sqrt{x^2 + y^2 + z^2} \\
 u &\triangleq \frac{x}{r} \\
 v &\triangleq \frac{y}{r}
 \end{aligned}
 \tag{8}$$

Next, define the RUV target position measurements

$$\begin{aligned} r_m &= r_t + w_r \\ u_m &= u_t + w_u \\ v_m &= v_t + w_v \end{aligned} \tag{9}$$

where r_t , u_t , and v_t are the true target range and direction cosines, and w_r , w_u , and w_v are independent Gaussian measurement noises with statistics

$$\begin{aligned} E(w_r) &= 0, & E(w_r^2) &= \sigma_r^2 \\ E(w_u) &= 0, & E(w_u^2) &= \sigma_u^2 \\ E(w_v) &= 0, & E(w_v^2) &= \sigma_v^2 \end{aligned} \tag{10}$$

Lastly, because the noise sources are independent

$$\begin{aligned} E(w_r w_u) &= E(w_r) E(w_u) = 0 \\ E(w_r w_v) &= E(w_r) E(w_v) = 0 \\ E(w_u w_v) &= E(w_u) E(w_v) = 0 \end{aligned} \tag{11}$$

These nonlinear transformations and how to properly represent the transformed random variables are the basis of this section. A brief overview of the Taylor series expansion is provided below, and then the first-, second-, and third-order approximations of the RUV to XYZ conversions are given. The derivations for each of these transformations are provided in Appendices B, C, and D on pages 63, 67, and 73.

3.1 TAYLOR SERIES REPRESENTATION OF A FUNCTION

The Taylor series expansion of an arbitrary nonlinear function $\mathbf{g}[\cdot]$ operating on a multidimensional vector $\mathbf{x} \triangleq [x_1, x_2, \dots, x_N]^T$ is given by

$$\begin{aligned} \mathbf{T}[\mathbf{x}] &= \mathbf{g}[\mathbf{x}_0] + \sum_{j=1}^N \frac{\partial \mathbf{g}[\mathbf{x}_0]}{\partial x_j} (x_j - x_{0,j}) + \frac{1}{2!} \sum_{j=1}^N \sum_{k=1}^N \frac{\partial^2 \mathbf{g}[\mathbf{x}_0]}{\partial x_j \partial x_k} (x_j - x_{0,j}) (x_k - x_{0,k}) \\ &+ \frac{1}{3!} \sum_{j=1}^N \sum_{k=1}^N \sum_{m=1}^N \frac{\partial^3 \mathbf{g}[\mathbf{x}_0]}{\partial x_j \partial x_k \partial x_m} (x_j - x_{0,j}) (x_k - x_{0,k}) (x_m - x_{0,m}) + \dots \end{aligned} \tag{12}$$

where $\mathbf{x}_0 \triangleq [x_{0,1}, x_{0,2}, \dots, x_{0,N}]^T$ is the point that we are linearizing around. In the case of converting RUV position measurements to Cartesian XYZ coordinates, $\mathbf{x} \triangleq [r_m, u_m, v_m]^T$ and so $N = 3$. Additionally, we wish to linearize around truth which reduces the perturbation $(x_j - x_{0,j})$ to the measurement noise

$$\begin{aligned} x_j - x_{0,j} &= x_{m,j} - x_{t,j} \\ &= (x_{t,j} + w_j) - x_{t,j} \\ &= w_j, \end{aligned} \tag{13}$$

where w_j is in the set $\{w_r, w_u, w_v\}$.

3.2 COVARIANCE APPROXIMATION USING FIRST-ORDER LINEARIZATION (CM1)

An approximation of the converted (RUV to XYZ) measurement noise covariance using a first-order Taylor series is described below, and the full derivation is provided in Appendix B. Given a position measurement and associated noise statistics in RUV coordinates, $\{r_m, u_m, v_m, \sigma_r^2, \sigma_u^2, \sigma_v^2\}$, the converted measurement and measurement noise covariance in Cartesian XYZ is

$$\begin{aligned} x_m &= r_m u_m \\ y_m &= r_m v_m \\ z_m &= r_m w_m \end{aligned} \quad , \quad \mathbf{R}_{xyz} = \begin{bmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{xy}^2 & \sigma_{yy}^2 & \sigma_{yz}^2 \\ \sigma_{xz}^2 & \sigma_{yz}^2 & \sigma_{zz}^2 \end{bmatrix}, \quad (14)$$

where

$$w_m = \sqrt{1 - u_m^2 - v_m^2},$$

and the covariance terms are

$$\sigma_{xx}^2 = u^2 \sigma_r^2 + r^2 \sigma_u^2 \quad (15)$$

$$\sigma_{yy}^2 = v^2 \sigma_r^2 + r^2 \sigma_v^2 \quad (16)$$

$$\sigma_{zz}^2 = w^2 \sigma_r^2 + \left(\frac{ru}{w}\right)^2 \sigma_u^2 + \left(\frac{rv}{w}\right)^2 \sigma_v^2 \quad (17)$$

$$\sigma_{xy}^2 = uv \sigma_r^2 \quad (18)$$

$$\sigma_{xz}^2 = uw \sigma_r^2 - \frac{r^2 u}{w} \sigma_u^2 \quad (19)$$

$$\sigma_{yz}^2 = vw \sigma_r^2 - \frac{r^2 v}{w} \sigma_v^2 \quad (20)$$

Here r , u , v , and w are the states to linearize around³, which are nominally truth. In practice, the track state should be used, which is discussed in more detail in Section 4.2.

3.3 COVARIANCE APPROXIMATION USING SECOND-ORDER LINEARIZATION (CM2)

An approximation of the converted (RUV to XYZ) measurement noise covariance using a second-order Taylor series is described below, and the full derivation is provided in Appendix C. Given a position measurement and associated noise statistics in RUV coordinates, $\{r_m, u_m, v_m, \sigma_r^2, \sigma_u^2, \sigma_v^2\}$, the converted measurement and measurement noise covariance in Cartesian XYZ is

$$\begin{aligned} x_m &= r_m u_m \\ y_m &= r_m v_m \\ z_m &= r_m w_m - c_z \end{aligned} \quad , \quad \mathbf{R}_{xyz} = \begin{bmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{xy}^2 & \sigma_{yy}^2 & \sigma_{yz}^2 \\ \sigma_{xz}^2 & \sigma_{yz}^2 & \sigma_{zz}^2 \end{bmatrix}, \quad (21)$$

³Care should be taken to distinguish the measurement noises w_r , w_u , and w_v from the terms $w = \sqrt{1 - u^2 - v^2}$ and $w_m = \sqrt{1 - u_m^2 - v_m^2}$.

where

$$w_m = \sqrt{1 - u_m^2 - v_m^2}, \quad (22)$$

$$c_z = -\frac{1}{2} \left(\frac{r}{w} + \frac{ru^2}{w^3} \right) \sigma_u^2 - \frac{1}{2} \left(\frac{r}{w} + \frac{rv^2}{w^3} \right) \sigma_v^2, \quad (23)$$

and the covariance terms are

$$\sigma_{xx}^2 = u^2 \sigma_r^2 + r^2 \sigma_u^2 + \sigma_r^2 \sigma_u^2 \quad (24)$$

$$\sigma_{yy}^2 = v^2 \sigma_r^2 + r^2 \sigma_v^2 + \sigma_r^2 \sigma_v^2 \quad (25)$$

$$\begin{aligned} \sigma_{zz}^2 = & w^2 \sigma_r^2 + \frac{r^2}{w^2} (u^2 \sigma_u^2 + v^2 \sigma_v^2) + \frac{u^2}{w^2} \sigma_r^2 \sigma_u^2 + \frac{v^2}{w^2} \sigma_r^2 \sigma_v^2 \\ & + \frac{r^2}{2} \left(\frac{u^2 v^2}{w^6} - \frac{u^2 + v^2}{w^4} - \frac{1}{w^2} \right) \sigma_u^2 \sigma_v^2 \\ & + \frac{r^2}{2} \left(\frac{u^4}{w^6} + 2 \frac{u^2}{w^4} + \frac{1}{w^2} \right) \sigma_u^4 + \frac{r^2}{2} \left(\frac{v^4}{w^6} + 2 \frac{v^2}{w^4} + \frac{1}{w^2} \right) \sigma_v^4 \end{aligned} \quad (26)$$

$$\sigma_{xy}^2 = uv \sigma_r^2 \quad (27)$$

$$\sigma_{xz}^2 = uw \sigma_r^2 - \frac{r^2 u}{w} \sigma_u^2 - \frac{u}{w} \sigma_r^2 \sigma_u^2 \quad (28)$$

$$\sigma_{yz}^2 = vw \sigma_r^2 - \frac{r^2 v}{w} \sigma_v^2 - \frac{v}{w} \sigma_r^2 \sigma_v^2 \quad (29)$$

Again r , u , v , and w are the states to linearize around, which would nominally be truth, but in practice the track state is used. The c_z term in the z_m -coordinate is a bias compensation term, which for small ranges and/or small u - v measurement noises goes to zero (i.e., no contact lens effect), but for large ranges and large u - v measurement noises accounts for the biases due to the severely non-elliptical, non-Gaussian uncertainty in Cartesian XYZ coordinates.

The original description of CM2 as published in [1], denoted here as CM2*, contains one sign error, two typographical errors, and two missing terms. See Appendix A for more details.

3.4 COVARIANCE APPROXIMATION USING THIRD-ORDER LINEARIZATION (CM3)

An approximation of the converted (RUV to XYZ) measurement noise covariance using a third-order Taylor series is described below for the first time, and the full derivation is provided in Appendix D. Given a position measurement and associated noise statistics in RUV coordinates, $\{r_m, u_m, v_m, \sigma_r^2, \sigma_u^2, \sigma_v^2\}$, the converted measurement and measurement noise covariance in Cartesian XYZ is

$$\begin{aligned} x_m &= r_m u_m \\ y_m &= r_m v_m \\ z_m &= r_m w_m - c_z \end{aligned} \quad , \quad \mathbf{R}_{xyz} = \begin{bmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{xy}^2 & \sigma_{yy}^2 & \sigma_{yz}^2 \\ \sigma_{xz}^2 & \sigma_{yz}^2 & \sigma_{zz}^2 \end{bmatrix}, \quad (30)$$

where

$$w_m = \sqrt{1 - u_m^2 - v_m^2}, \quad (31)$$

$$c_z = -\frac{1}{2} \left(\frac{r}{w} + \frac{ru^2}{w^3} \right) \sigma_u^2 - \frac{1}{2} \left(\frac{r}{w} + \frac{rv^2}{w^3} \right) \sigma_v^2, \quad (32)$$

and the covariance terms are

$$\sigma_{xx}^2 = u^2 \sigma_r^2 + r^2 \sigma_u^2 + \sigma_r^2 \sigma_u^2 \quad (33)$$

$$\sigma_{yy}^2 = v^2 \sigma_r^2 + r^2 \sigma_v^2 + \sigma_r^2 \sigma_v^2 \quad (34)$$

$$\begin{aligned} \sigma_{zz}^2 = & w^2 \sigma_r^2 + \frac{r^2}{w^2} (u^2 \sigma_u^2 + v^2 \sigma_v^2) - \sigma_r^2 \sigma_u^2 - \sigma_r^2 \sigma_v^2 + \left(7 \frac{r^2 u^2 v^2}{w^6} + \frac{r^2 (u^2 + v^2)}{w^4} \right) \sigma_u^2 \sigma_v^2 \\ & + \frac{1}{2} \left(7 \frac{r^2 u^4}{w^6} + 8 \frac{r^2 u^2}{w^4} + \frac{r^2}{w^2} \right) \sigma_u^4 + \frac{1}{2} \left(7 \frac{r^2 v^4}{w^6} + 8 \frac{r^2 v^2}{w^4} + \frac{r^2}{w^2} \right) \sigma_v^4 \\ & + \frac{3}{4} \left(\frac{u^4}{w^6} + 2 \frac{u^2}{w^4} + \frac{1}{w^2} \right) \sigma_r^2 \sigma_u^4 + \frac{3}{4} \left(\frac{v^4}{w^6} + 2 \frac{v^2}{w^4} + \frac{1}{w^2} \right) \sigma_r^2 \sigma_v^4 \\ & + \frac{1}{4} \left(45 \frac{r^2 u^4 v^2}{w^{10}} + \frac{36 r^2 u^2 v^2 + 6 r^2 u^4}{w^8} + \frac{6 r^2 u^2 + 3 r^2 v^2}{w^6} \right) \sigma_u^4 \sigma_v^2 \\ & + \frac{1}{4} \left(45 \frac{r^2 u^2 v^4}{w^{10}} + \frac{36 r^2 u^2 v^2 + 6 r^2 v^4}{w^8} + \frac{6 r^2 v^2 + 3 r^2 u^2}{w^6} \right) \sigma_u^2 \sigma_v^4 \\ & + \frac{15}{4} \left(\frac{r^2 u^6}{w^{10}} + 2 \frac{r^2 u^4}{w^8} + \frac{r^2 u^2}{w^6} \right) \sigma_u^6 + \frac{15}{4} \left(\frac{r^2 v^6}{w^{10}} + 2 \frac{r^2 v^4}{w^8} + \frac{r^2 v^2}{w^6} \right) \sigma_v^6 \\ & + \frac{1}{2} \left(3 \frac{u^2 v^2}{w^6} + \frac{u^2 + v^2}{w^4} + \frac{1}{w^2} \right) \sigma_r^2 \sigma_u^2 \sigma_v^2 \end{aligned} \quad (35)$$

$$\sigma_{xy}^2 = uv \sigma_r^2 \quad (36)$$

$$\begin{aligned} \sigma_{xz}^2 = & uw \sigma_r^2 - \frac{r^2 u}{w} \sigma_u^2 - \frac{1}{2} \left(\frac{u^3}{w^3} + 3 \frac{u}{w} \right) \sigma_r^2 \sigma_u^2 - \frac{1}{2} \left(\frac{uv^2}{w^3} + \frac{u}{w} \right) \sigma_r^2 \sigma_v^2 \\ & - \frac{1}{2} \left(3 \frac{r^2 uv^2}{w^5} + \frac{r^2 u}{w^3} \right) \sigma_u^2 \sigma_v^2 - \frac{3}{2} \left(\frac{r^2 u^3}{w^5} + \frac{r^2 u}{w^3} \right) \sigma_u^4 \end{aligned} \quad (37)$$

$$\begin{aligned} \sigma_{yz}^2 = & vw \sigma_r^2 - \frac{r^2 v}{w} \sigma_v^2 - \frac{1}{2} \left(\frac{u^2 v}{w^3} + \frac{v}{w} \right) \sigma_r^2 \sigma_u^2 - \frac{1}{2} \left(\frac{v^3}{w^3} + 3 \frac{v}{w} \right) \sigma_r^2 \sigma_v^2 \\ & - \frac{1}{2} \left(3 \frac{r^2 u^2 v}{w^5} + \frac{r^2 v}{w^3} \right) \sigma_u^2 \sigma_v^2 - \frac{3}{2} \left(\frac{r^2 v^3}{w^5} + \frac{r^2 v}{w^3} \right) \sigma_v^4 \end{aligned} \quad (38)$$

Again r , u , v , and w are the states to linearize around, which would nominally be truth, but in practice the track state is used. The c_z term in the z_m -coordinate is, again, a bias compensation term to handle highly non-elliptical, non-Gaussian uncertainty in Cartesian XYZ coordinates.

3.5 COVARIANCE APPROXIMATION USING THE UNSCENTED TRANSFORM

In addition to the first-, second-, and third-order Taylor series covariance approximations, we also consider the well-known unscented transform (UT) [4, 9–11]. The UT is based on two principles: (i) it is easier to perform a nonlinear transformation on a single point rather than a probability density function, and (ii) a set of points can be determined whose sample pdf approximates the true pdf. Based on these two principles, the UT determines a set of sigma points from a known distribution and transforms each of the sigma points using the nonlinear function. Using these transformed

sigma points, the sample mean and covariance can be computed. This procedure is illustrated in Figure 4. As stated earlier, it has been shown [4,9–11] that the approximated mean and covariance of the UT matches the true distribution’s mean and covariance up to third-order when the underlying distribution is symmetric. Also, because the higher order terms are not truncated, it is possible that the errors in these higher order terms will be reduced as well.

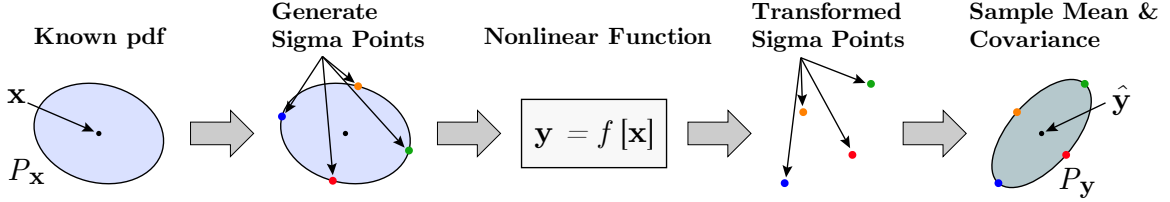


Figure 4. Illustration of the unscented transform. Given a known pdf, generate sigma points, transform each sigma point according to the nonlinear function, and compute the sample mean and error covariance from the transformed sigma points.

3.6 PERFORMANCE COMPARISON

In this section, we compare the different methods of estimating a covariance ellipsoid in Cartesian XYZ coordinates that represents the converted uncertainty volume from RUV. Because of the physical nature of the contact lens effect, the CM1, CM2, and CM3 linearization techniques all estimate the cross-range eigenvalues with high accuracy. The line-of-sight eigenvalue is the primary source of error, so this is the dimension that we will focus on. In the following examples, for each r - u - v pair, the true distributions are established by generating one million samples with additive zero mean white Gaussian noise. The samples are then converted to Cartesian XYZ coordinates using Equation (8), and the sample mean and error covariance are computed. From the error covariance, we can determine the range accuracy by taking the square-root of the smallest eigenvalue:

$$\sigma_{r,xyz} = \sqrt{\min[\text{eig}(\mathbf{R}_{xyz})]}. \quad (39)$$

where $\mathbf{R}_{xyz}$ is the sample covariance. This metric can be repeated for each covariance approximating algorithm by replacing the sample covariance with the approximated covariance. That is, $\mathbf{R}_{xyz}$ is computed using Equations (14)–(20) for CM1, Equations (21)–(29) for CM2, and Equations (30)–(38) for CM3. See [1] and Appendix A for details of $\mathbf{R}_{xyz}$ for CM2*, and [4,9–11] for details for the UT.

A percent error metric is defined as

$$\%Error = \frac{\hat{\sigma}_{r,xyz} - \sigma_{r,xyz}}{\sigma_{r,xyz}} \cdot 100, \quad (40)$$

where $\sigma_{r,xyz}$ is the range uncertainty calculated from the sample covariance (i.e., truth), and $\hat{\sigma}_{r,xyz}$ is the approximated range uncertainty for each algorithm. A particular method is more accurate to the degree of which it agrees with the sample error (truth) black lines in Figures 5–8(a), and is ideally zero everywhere on the percent error (b) figures. If the percent error metric is positive,

the approximated range uncertainty is too large; if the metric is negative, the approximated range uncertainty is too small.

Examples will be shown for the following four cases in Table 2:

TABLE 2
Scenarios Examined to Compare RUV to Cartesian XYZ Conversion Accuracy

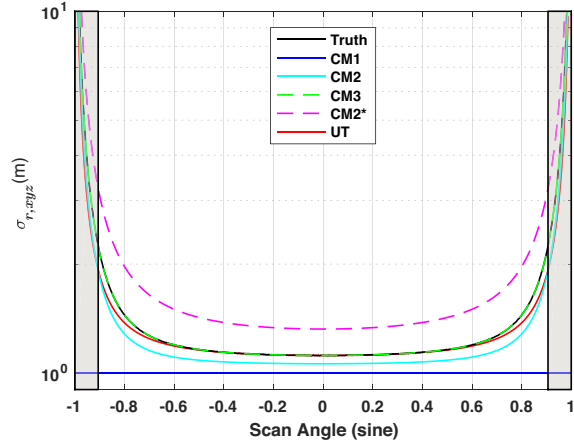
Scenario	Range (km)	σ_r (m)	σ_u (msin)	σ_v (msin)
1: Short Range	500	1.0	1.0	1.0
2: Medium Range	2000	1.0	1.0	1.0
3: Long Range	5000	1.0	1.0	1.0
4: Long Range (Unequal Noise)	5000	1.0	1.0	5.0

The first three cases use identical noise statistics in angle, but vary the range to the target. As one would expect, as the range-to-target or angle noise statistics are increased, the shape of the contact lens is accentuated. The last case examines the performance when the angle measurement noises are not equal. In this case, the angle measurement noises are varied to $\sigma_u = 1$ msin and $\sigma_v = 5$ msin. In order to perform a fair comparison between all algorithms, truth is used as the linearization point. Later, we will investigate ways to deal with uncertainty in the linearization point.

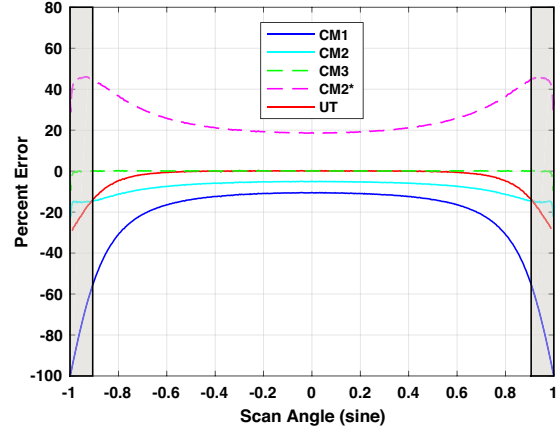
Comparison of CM1, CM2, CM2*, CM3, and UT Approximations

First we will compare the CM1, CM2, CM2*, CM3, and UT conversions, and their effectiveness along the 45° cut in u - v space (i.e., $u = v$ for all values in the unit circle) for the four cases. The independent variable will be the total scan angle, defined as $\sqrt{u^2 + v^2}$. For context, scan angles outside of $\pm 65^\circ$ are grayed out, which roughly resemble the region that a radar would operate in. Outside of this region, as one moves further off boresight, among other things, scan loss and target glint increase significantly resulting in reduced target detectability.

Care should be taken to differentiate the correctly derived second-order linearization conversion (CM2) and the incorrectly derived conversion from [1] (CM2*). Notice in Figures 5–8 that the incorrectly derived conversion artificially inflates the range uncertainty and is significantly larger than truth in all cases.

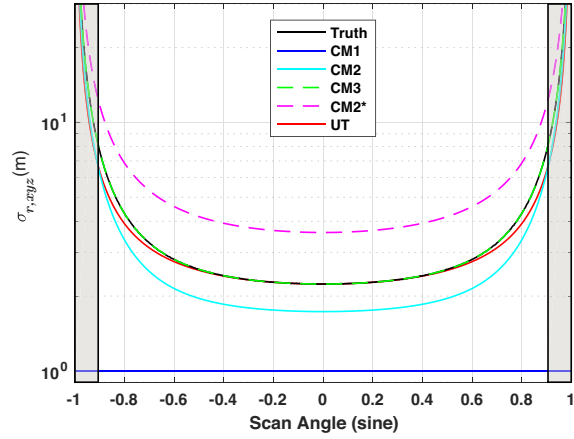


(a)

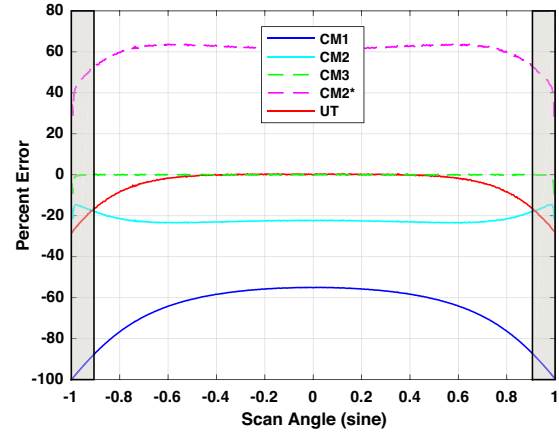


(b)

Figure 5. Case 1: Short range (500 km, $\sigma_r = 1$ m, $\sigma_u = \sigma_v = 1$ msin).



(a)



(b)

Figure 6. Case 2: Medium range (2000 km, $\sigma_r = 1$ m, $\sigma_u = \sigma_v = 1$ msin).

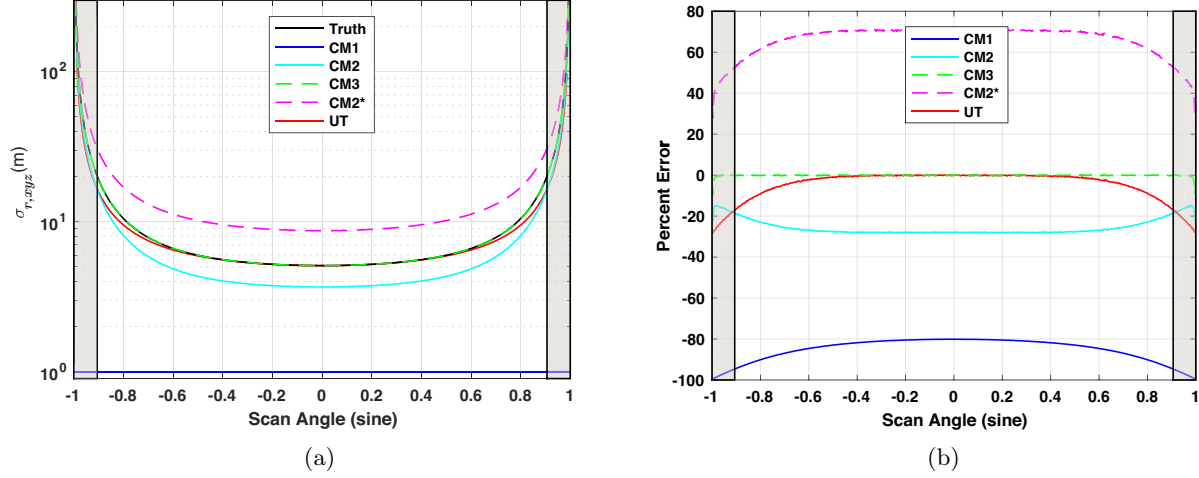


Figure 7. Case 3: Long range (5000 km, $\sigma_r = 1$ m, $\sigma_u = \sigma_v = 1$ msin).

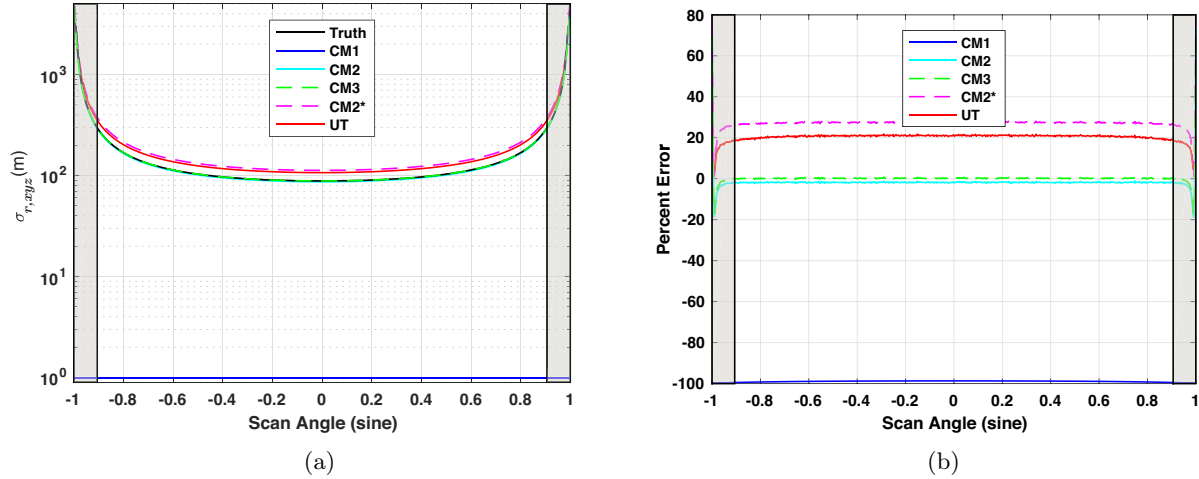


Figure 8. Case 4: Long range with unequal noise statistics (5000 km, $\sigma_r = 1$ m, $\sigma_u = 1$ msin, $\sigma_v = 5$ msin).

In all cases analyzed, the third-order Taylor series approximation is the only algorithm that accurately characterizes the true measurement's statistics across all scan angles. The first-order linearization performs the worst, as it does not attempt to account for the contact lens effect and grossly underestimates the line-of-sight noise standard deviation (it performs no inflation over the sensor's σ_r in RUV coordinates). Conversely, the CM2* algorithm as defined in [1] over-inflates the line-of-sight noise standard deviation by between 20–70% in the cases studied. This is due to errors in the derivation that leads to artificially inflated covariances. The correctly defined second-order linearization, CM2, performs better but still is between 2–28% off in the cases studied. The unscented transform performs even better but is not able to accurately estimate the correct contact

lens effect at larger scan angles ($>37^\circ$) or when the noise distributions are significantly different in u and v . For the cases shown, the tuning parameter in the unscented transform was set to $\kappa = 1$, and it was observed that the errors for the unequal noise case could be reduced when $\kappa = 0$, however it was to the detriment of the other cases. The primary cause of this is the severe non-Gaussian distribution of the converted measurements, and improved results could likely be achieved by using additional sigma points (i.e., higher order sampling). The third-order linearization is the only algorithm that accurately characterizes the effect for all values of u and v , does not require any tuning parameters, and has error less than 0.25% of truth across nearly the entire array face. Only at extreme scan angles ($>83^\circ$ – 85°) does the error approach 4%. These scan angles will almost certainly never be seen by the radar, since they tend to operate between $\pm 65^\circ$ off boresight.

Accuracy of CM3 Approximation for All Scan Angles

Next, we investigate the performance of the third-order Taylor series covariance approximation algorithm for all valid u and v that lie inside the unit circle for, (i) the medium range case with equal angle noise, and (ii) the long range case with unequal angle noise. Figure 9 shows contours of the uncertainty in the range direction for both the third-order Taylor series algorithm shown in red and truth shown in black. As you can see, both lines lie exactly on top of one another, which again shows the accuracy of this algorithm. With this in mind, there is little to gain by moving to a fourth-order Taylor series.

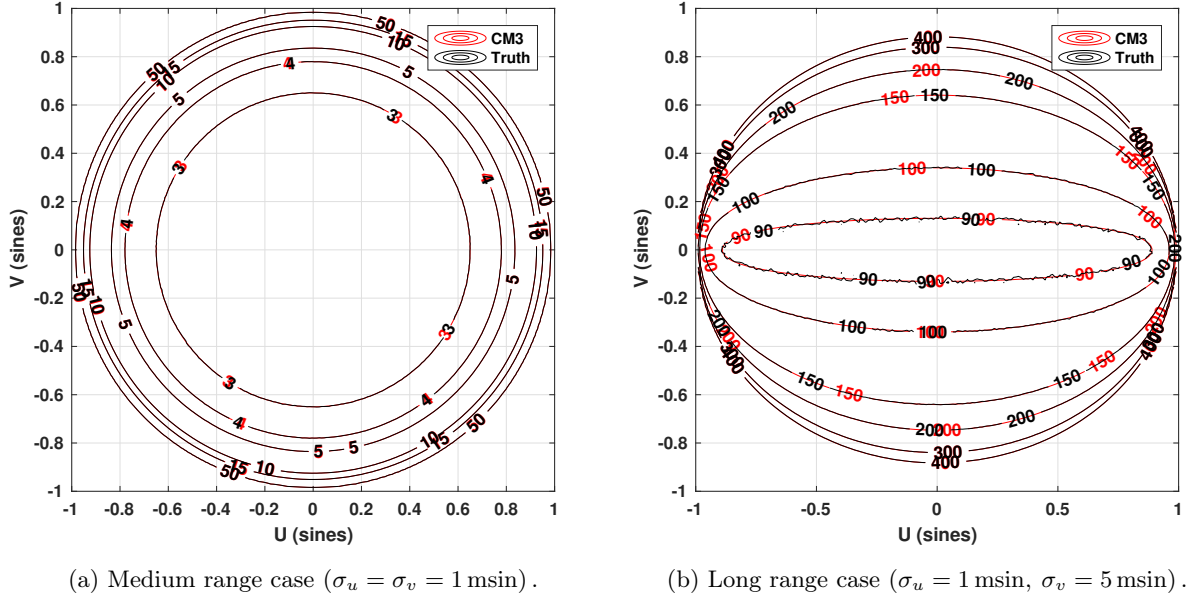


Figure 9. Comparison of σ_r for third-order Taylor series covariance approximation (red contours) and truth (black contours).

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4. CM3EKF ALGORITHM WITH LINEARIZATION POINT COMPENSATION

In this section we describe a proposed filter algorithm for long range radars based on the results from studying the accuracy of the first-, second-, and third-order conversions in the previous section. The proposed algorithm considers the two main difficulties in long range tracking for radars: (i) correctly approximating the mean and covariance of the probability density function for the nonlinear transformation of the RUV measurements to Cartesian XYZ coordinates, and (ii) effectively compensating for errors in the linearization point during the covariance transformation.

4.1 CM3 EXTENDED KALMAN FILTER

In Section 3.6, we determined that the third-order conversion CM3 most closely matched the sample converted measurement covariance. As will become more clear in Section 5, however, if one were to build a converted measurements linear⁴ Kalman filter as described in Section 2.4 based on the third-order conversion (CM3KF), it would produce consistent filter covariances but excessive errors in the range direction. Instead, we would like the consistency properties of the CM3 conversion and reduced errors in range. The third-order conversion extended Kalman filter (CM3EKF) produces this by adaptively inflating the range component of the measurement noise covariance when necessary. The CM3EKF described below is partly based on the concepts described in [1]. However, the errors in the second-order Taylor series conversion, as published in [1], led to overinflated range uncertainties (see Section 3.6) and required logic to transition between the CM2 and CM1 conversions based on the ratio of the condition numbers for the converted measurement covariances. Because the new third-order derivation corrects for these errors, we no longer need switching logic, which also requires the calculation of the condition ratio and the selection of a condition ratio switching threshold. The new filter produces consistent state errors and covariances, while ensuring small range errors.

The MCAEKF motivation and requirements, summarized in [1], are also applicable to the derivation of the CM3EKF. As we will see in the next section, the CM2KF and CM3KF are consistent for the long range tracking problem but produce larger range accuracies due to the fact that the conversions (i.e., CM2 and CM3) increase the uncertainty in the range direction such that the ellipsoids in Cartesian XYZ coordinates fully cover the measurement uncertainty. This is referred to as the *coverability requirement* in [1]. However, since the fusion of two uncertainty ellipsoids depends on their intersection, we can guarantee consistency by modifying the measurement's uncertainty so that its intersection with the track's predicted position uncertainty satisfies the *coverability requirement*. That is, we calculate the CM3 conversion range uncertainty in Cartesian XYZ coordinates for the intersection of the RUV measurement and track covariance. Figure 10 provides an illustration of this, with the blue contact lens representing the nonlinear mapping of the RUV measurement uncertainty into Cartesian XYZ coordinates, and the red ellipse representing the intersection of the measurement and predicted track covariances. From this illustration it becomes clear that as the track covariance becomes smaller, the range uncertainty inflation also becomes smaller. Note that

⁴The filter is considered linear because the converted measurements are linearly related to the state. However, the measurement conversion is nonlinear and non-Gaussian.

this process only inflates the range uncertainty of the measurement noise covariance applied during the update step of the filter.

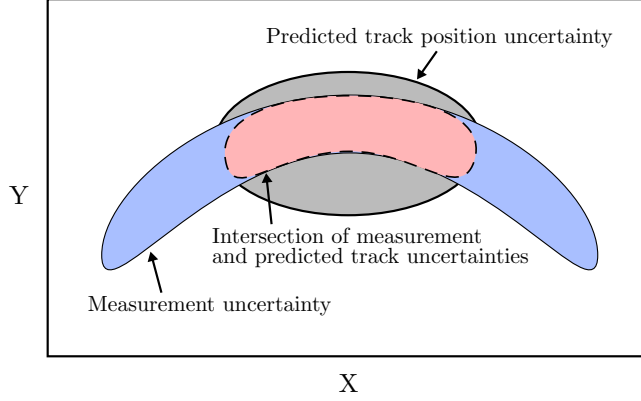


Figure 10. Illustration of the intersection between the measurement uncertainty and predicted track uncertainty satisfying the coverability requirement in two dimensions.

The range uncertainty that satisfies the coverability requirement can be found by first transforming the predicted track position uncertainty from the Cartesian XYZ filtering coordinate frame to the RUV measurement frame. Since we only require the cross-range eigenvalues, a first-order Taylor series conversion is acceptable. Given the transformation matrix

$$\mathbf{J}_{xyz}^{ruv} = \begin{bmatrix} u & v & w \\ \frac{1-u^2}{r} & \frac{-uv}{r} & \frac{-uw}{r} \\ -uv & \frac{1-v^2}{r} & \frac{-vw}{r} \end{bmatrix}, \quad (41)$$

where r , u , and v are given in Equation (8) and $w = \sqrt{1-u^2-v^2}$, the predicted track covariance is

$$\mathbf{P}_{ruv}(k+1|k) = \mathbf{J}_{xyz}^{ruv} \mathbf{P}_{xyz}(k+1|k) (\mathbf{J}_{xyz}^{ruv})^T. \quad (42)$$

The *significant region* in u - v space is then determined from $\mathbf{P}_{ruv}(k+1|k)$ and the measurement's angular noise statistics σ_u^2 and σ_v^2 :

$$\sigma_{uv}^2 \triangleq \min(c_{adj} \cdot \sigma_{uv,track}^2, \sigma_{uv,meas}^2), \quad (43)$$

$$\sigma_{uv,track}^2 \triangleq \max[\text{eig}(\mathbf{P}_{uv}(k+1|k))], \quad (44)$$

$$\sigma_{uv,meas}^2 \triangleq \max(\sigma_u^2, \sigma_v^2), \quad (45)$$

where $\sigma_{uv,track}^2$ is the largest cross-range eigenvalue of $\mathbf{P}_{uv}(k+1|k)$, the 2×2 submatrix of the predicted covariance, and $\sigma_{uv,meas}^2$ is the largest cross-range eigenvalue of the measurement uncertainty in RUV coordinates (either σ_u^2 or σ_v^2). The c_{adj} scaling factor slightly increases the cross-range intersection and was chosen through extensive simulation to be $c_{adj} = 2.5$, and is in agreement

with [1]. The *significant region* is then defined as

$$\mathbf{R}_{ruv,sig} = \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_{uv}^2 & 0 \\ 0 & 0 & \sigma_{uv}^2 \end{bmatrix}, \quad (46)$$

and the third-order conversion is calculated using this uncertainty region. From Section 3.4, the third-order Taylor series covariance in Cartesian XYZ coordinates can be simplified when $\sigma_u = \sigma_v$, which is true for our defined *significant region*.

$$\mathbf{R}_{xyz,sig} = \begin{bmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{xy}^2 & \sigma_{yy}^2 & \sigma_{yz}^2 \\ \sigma_{xz}^2 & \sigma_{yz}^2 & \sigma_{zz}^2 \end{bmatrix}, \quad (47)$$

where

$$\sigma_{xx}^2 = u^2 \sigma_r^2 + r^2 \sigma_{uv}^2 + \sigma_r^2 \sigma_{uv}^2 \quad (48)$$

$$\sigma_{yy}^2 = v^2 \sigma_r^2 + r^2 \sigma_{uv}^2 + \sigma_r^2 \sigma_{uv}^2 \quad (49)$$

$$\begin{aligned} \sigma_{zz}^2 = & w^2 \sigma_r^2 + \frac{r^2 (u^2 + v^2)}{w^2} \sigma_{uv}^2 - 2 \sigma_r^2 \sigma_{uv}^2 \\ & + \left(\frac{7}{2} \frac{r^2 (u^2 + v^2)^2}{w^6} + 5 \frac{r^2 (u^2 + v^2)}{w^4} + \frac{r^2}{w^2} \right) \sigma_{uv}^4 \\ & + \left(\frac{3}{4} \frac{(u^2 + v^2)^2}{w^6} + 2 \frac{u^2 + v^2}{w^4} + \frac{2}{w^2} \right) \sigma_r^2 \sigma_{uv}^4 \\ & + \left(\frac{15}{4} \frac{r^2 (u^2 + v^2)^3}{w^{10}} + 9 \frac{r^2 (u^2 + v^2)^2}{w^8} + 6 \frac{r^2 (u^2 + v^2)}{w^6} \right) \sigma_{uv}^6 \end{aligned} \quad (50)$$

$$\sigma_{xy}^2 = uv \sigma_r^2 \quad (51)$$

$$\begin{aligned} \sigma_{xz}^2 = & uw \sigma_r^2 - \frac{r^2 u}{w} \sigma_{uv}^2 - \left(\frac{1}{2} \frac{u (u^2 + v^2)}{w^3} + 2 \frac{u}{w} \right) \sigma_r^2 \sigma_{uv}^2 \\ & - \left(\frac{3}{2} \frac{r^2 u (u^2 + v^2)}{w^5} + 2 \frac{r^2 u}{w^3} \right) \sigma_{uv}^4 \end{aligned} \quad (52)$$

$$\begin{aligned} \sigma_{yz}^2 = & vw \sigma_r^2 - \frac{r^2 v}{w} \sigma_{uv}^2 - \left(\frac{1}{2} \frac{v (u^2 + v^2)}{w^3} + 2 \frac{v}{w} \right) \sigma_r^2 \sigma_{uv}^2 \\ & - \left(\frac{3}{2} \frac{r^2 v (u^2 + v^2)}{w^5} + 2 \frac{r^2 v}{w^3} \right) \sigma_{uv}^4 \end{aligned} \quad (53)$$

Because we do not know the true $\{r, u, v\}$ to linearize about, we estimate these values from either the track or the raw measurement, whichever has smaller corresponding uncertainty. A straightforward metric for determining which to use is to compare the square-root of the trace of each uncertainty volume, and choose the smaller of the two.

After the *significant region* uncertainty has been calculated in Cartesian XYZ coordinates, the adjusted range uncertainty is

$$\sigma_{r,adj}^2 = \min[\text{eig}(\mathbf{R}_{xyz,sig})], \quad (54)$$

and the range uncertainty is overwritten in the measurement noise covariance that is used in the track update step of the extended Kalman filter. Note that the original σ_u and σ_v values are retained; we do not inflate the measurement noise covariance in cross-range. It's also important to recognize that as the predicted position uncertainty gets smaller, the contact lens becomes more elliptical in Cartesian XYZ coordinates and the CM3-calculated range uncertainty converges to the measurement's σ_r value in RUV coordinates. The augmented measurement noise covariance is given by

$$\mathbf{R}_{ruv,adj} = \begin{bmatrix} \sigma_{r,adj}^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_v^2 \end{bmatrix}. \quad (55)$$

The algorithm to compute the adjusted measurement noise covariance using the third-order Taylor series expansion is summarized in Algorithm 5.

Algorithm 5 Adjusted Measurement Noise Covariance Using Third-Order Conversion

Input:

Predicted state and covariance at time $k + 1$:

$$\hat{\mathbf{x}}_{xyz}(k + 1|k), \mathbf{P}_{xyz}(k + 1|k)$$

Measurement and measurement covariance at time $k + 1$:

$$\mathbf{z}_{ruv}(k + 1), \mathbf{R}_{ruv}(k + 1)$$

Compute RUV covariance representing significant region:

Compute maximum cross-range uncertainty for predicted track and measurement in RUV:

$$\begin{aligned} \mathbf{P}_{ruv}(k + 1|k) &= \mathbf{J}_{xyz}^{ruv} \mathbf{P}_{xyz}(k + 1|k) (\mathbf{J}_{xyz}^{ruv})^T \\ \sigma_{uv,track}^2 &= \max[\text{eig}(\mathbf{P}_{uv}(k + 1|k))] \\ \sigma_{uv,meas}^2 &= \max(\sigma_u^2, \sigma_v^2) \end{aligned}$$

Compute intersection of track and measurement uncertainty in u - v space:

$$\sigma_{uv}^2 \triangleq \min(c_{adj} \cdot \sigma_{uv,track}^2, \sigma_{uv,meas}^2)$$

Compute RUV significant region:

$$\mathbf{R}_{ruv,sig} = \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_{uv}^2 & 0 \\ 0 & 0 & \sigma_{uv}^2 \end{bmatrix},$$

where $\mathbf{J}_{xyz}^{ruv}$ is the first-order Taylor series approximation of the XYZ to RUV conversion given in Equation (41), $\mathbf{P}_{uv}(k + 1|k)$ is the 2×2 u - v submatrix of the predicted covariance, σ_u^2 and σ_v^2 are the diagonal elements of $\mathbf{R}_{ruv}(k + 1)$ corresponding to coordinates u and v , and c_{adj} is a scaling factor that slightly increases the cross-range intersection and was chosen to be 2.5.

Approximate RUV to XYZ conversion using third-order Taylor series expansion:

$$\mathbf{R}_{xyz,sig} = \begin{bmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{xy}^2 & \sigma_{yy}^2 & \sigma_{yz}^2 \\ \sigma_{xz}^2 & \sigma_{yz}^2 & \sigma_{zz}^2 \end{bmatrix},$$

where the variances are given in Equations (48)–(53).

Compute adjusted RUV measurement noise covariance:

$$\mathbf{R}_{ruv,adj} = \begin{bmatrix} \sigma_{r,adj}^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_v^2 \end{bmatrix},$$

where $\sigma_{r,adj}^2 = \min[\text{eig}(\mathbf{R}_{xyz,sig})]$ is the adjusted range uncertainty.

Output:

Adjusted RUV measurement noise covariance: $\mathbf{R}_{ruv,adj}$

Once the measurement noise covariance has been adjusted for the contact lens effect, the extended Kalman filter update step from Algorithm 4 is performed as usual. Given that our predicted filter state vector and covariance are defined in Cartesian XYZ coordinates, we calculate the

predicted track state in the measurement frame and the Jacobian of the transformation as:

$$\mathbf{h}[\hat{\mathbf{x}}_{xyz}(k+1|k)] = \begin{bmatrix} r \\ u \\ v \end{bmatrix} = \begin{bmatrix} \sqrt{x^2 + y^2 + z^2} \\ x \\ \sqrt{x^2 + y^2 + z^2} \\ y \\ \sqrt{x^2 + y^2 + z^2} \end{bmatrix}, \quad (56)$$

$$\mathbf{J}_h = \begin{bmatrix} u & v & w & 0 & 0 & 0 \\ \frac{1-u^2}{r} & \frac{-uv}{r} & \frac{-uw}{r} & 0 & 0 & 0 \\ -uv & \frac{1-v^2}{r} & \frac{-vw}{r} & 0 & 0 & 0 \end{bmatrix}, \quad (57)$$

where $w = \sqrt{1 - u^2 - v^2}$. The filter gain is then

$$\mathbf{K} = \mathbf{P}_{xyz}(k+1|k) \mathbf{J}_h^T [\mathbf{J}_h \mathbf{P}_{xyz}(k+1|k) \mathbf{J}_h^T + \mathbf{R}_{ruv,adj}]^{-1}. \quad (58)$$

Lastly, the state and covariance are updated according to their standard equations

$$\hat{\mathbf{x}}_{xyz}(k+1|k+1) = \hat{\mathbf{x}}_{xyz}(k+1|k) + \mathbf{K} [\mathbf{z}_{ruv}(k+1) - \mathbf{h}[\hat{\mathbf{x}}_{xyz}(k+1|k)]] \quad (59)$$

$$\mathbf{P}_{xyz}(k+1|k+1) = (\mathbf{I} - \mathbf{KJ}_h) \mathbf{P}_{xyz}(k+1|k) (\mathbf{I} - \mathbf{KJ}_h)^T + \mathbf{KR}_{ruv,adj} \mathbf{K}. \quad (60)$$

4.2 COMPENSATION FOR ERRORS IN LINEARIZATION POINT

The calculation of the first-, second-, and third-order covariances during the measurement update step requires knowledge of the true state to linearize around. Clearly, this is not available in estimation applications, so in practice one typically uses either the raw measurement or a less noisy track state. Either of these options in the linearization point will create additional uncertainty, mostly in the range direction, which is more pronounced early in track while the filter converges. Compensating for the linearization point errors can be performed several ways. One such technique is to use batch processing, which stores a number of frames before performing track initialization. While this will certainly provide a more accurate initial state vector and covariance, it delays data availability for downstream functions. A much faster initialization can be performed by using the two-point differencing technique [2, 3], which amounts to a first-order polynomial fit of the measurements. This produces an initial state estimate at only the second measurement, but the state error and error covariance will be much larger. This report will use the two-point differencing approach, and a novel algorithm that adjusts the state error covariance to account for the linearization errors.

Due to the particular shape of the contact lens uncertainty volume, any errors in the linearization point during conversion of the measurements from RUV to Cartesian XYZ coordinates will lead to additional uncertainty in the range direction. This is illustrated in Figure 11, which shows the true uncertainty ellipse in blue, and an uncertainty volume calculated from a noisy linearization point in red. At long ranges, the measurements and track covariances have small errors in range and large errors in angle, which effectively rotates the ellipsoid by an angle θ and results in an increased uncertainty in range.

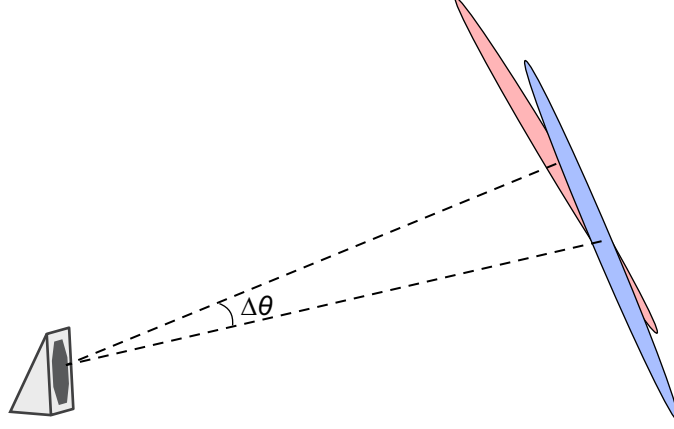


Figure 11. Illustration of errors in linearization point. Blue ellipse represents the true uncertainty volume, and red ellipse is the uncertainty volume calculated from a noisy linearization point.

In order to accurately account for this additional uncertainty, the covariance is rotated at the given range along a constant angle, which is specified by the k -th sigma of the state error covariance. In three-dimensional space, this rotation carves out an ellipse of constant range around the original position estimate. An example of this in two-dimensional space is shown in Figure 12, where a target is located 2000 km from the sensor with $\sigma_r = 1$ m and $\sigma_u = 0.5$ m/sin. Subplot 12a shows the state vector and error covariance in Cartesian XYZ coordinates

$$\hat{\mathbf{x}}_{target} = [0 \quad 2,000,000]^T, \quad \mathbf{P}_{target} = \begin{bmatrix} 1000^2 & 0 \\ 0 & 1^2 \end{bmatrix}, \quad (61)$$

relative to the target position so it's easier to read the y -axis (due to the extremely long range to target). The blue line in the figure represents the line-of-sight vector from the sensor location, located at the origin, to the estimated target position. The red and green lines are line-of-sight vectors rotated θ degrees from the target line-of-sight vector and only appear parallel because of the extremely long range to the sensor. In this example, θ is defined at the 1σ level of the state error covariance ($k=1$), which provides good coverage in practice. The angle θ is defined as

$$\theta = \arctan\left(\frac{k \cdot \sigma_{max}}{r}\right), \quad (62)$$

where σ_{max} is the largest eigenvalue of the position XYZ covariance, and r is the range to target ($\sigma_{max} = 1000$ m and $r = 2000$ km in this example).

Next, the covariance is rotated by θ in evenly spaced amounts around the line-of-sight vector to produce N transformed state vectors and error covariances. In practice, $N = 4$ is a good balance between low computational complexity and good performance. The N sample points are chosen such that they are evenly spaced at

$$\begin{bmatrix} az \\ el \end{bmatrix}_i = \begin{bmatrix} -\theta \\ 0 \end{bmatrix}, \begin{bmatrix} \theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\theta \end{bmatrix}, \begin{bmatrix} 0 \\ \theta \end{bmatrix}, i = 1, 2, 3, 4. \quad (63)$$

The rotation matrix that transforms the sample points around the estimated position of the target is given by

$$\mathbf{T}_i(az, el) = \begin{bmatrix} \cos(az) & 0 & \sin(az) \\ \sin(az)\sin(el) & \cos(el) & -\cos(az)\sin(el) \\ -\sin(az)\cos(el) & \sin(el) & \cos(az)\cos(el) \end{bmatrix}, \quad (64)$$

and the rotated sample points are

$$\hat{\mathbf{x}}_{sample,i} = \mathbf{T}_i \cdot \hat{\mathbf{x}}_{target} \quad (65)$$

$$\mathbf{P}_{sample,i} = \mathbf{T}_i \cdot \mathbf{P}_{target} \cdot \mathbf{T}_i^T. \quad (66)$$

These rotated state vectors and error covariances are shown in Figure 12b, with the red and green ellipses representing the 3σ error covariances. Notice that the range uncertainty has increased if we look at the intersection of these three ellipses.

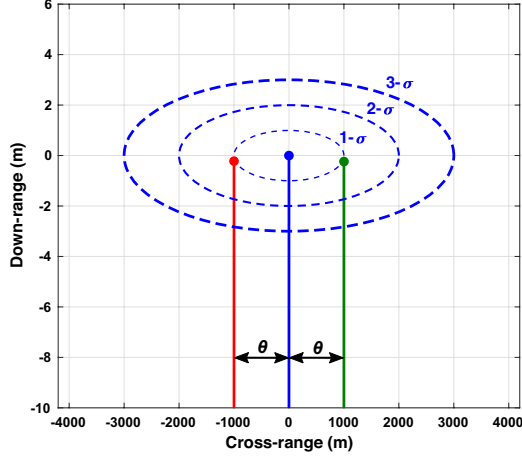
Next, we combine the N sample points and the original target state (for a total of $N + 1$ states) to produce a combined state vector and error covariance. This is accomplished assuming the final state is a Gaussian mixture of the $N + 1$ states, all with equal probability. The final state estimate and covariance in Cartesian XYZ coordinates is therefore given as

$$\hat{\mathbf{x}}_{comb} = \frac{1}{N+1} \sum_{i=0}^N \hat{\mathbf{x}}_{sample,i} \quad (67)$$

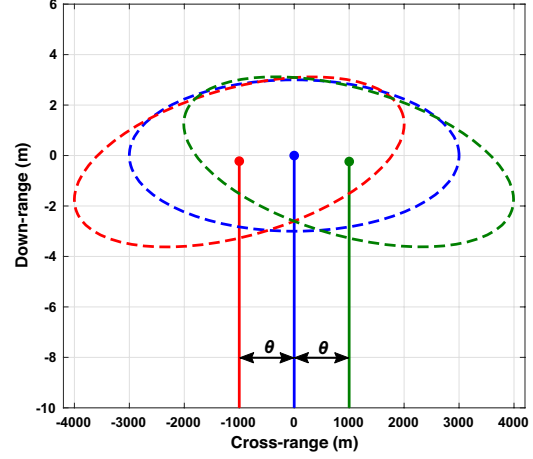
$$\mathbf{P}_{comb} = \frac{1}{N+1} \sum_{i=0}^N \left[\mathbf{P}_{sample,i} + (\hat{\mathbf{x}}_{sample,i} - \hat{\mathbf{x}}_{comb})(\hat{\mathbf{x}}_{sample,i} - \hat{\mathbf{x}}_{comb})^T \right], \quad (68)$$

where $\hat{\mathbf{x}}_{sample,0} = \hat{\mathbf{x}}_{target}$ and $\mathbf{P}_{sample,0} = \mathbf{P}_{target}$, and the second term in the covariance summation is the spread of the means term. The combined state vector and 3σ covariance is shown in Figure 12c as the orange dot and dashed ellipse, illustrating how the range uncertainty has been increased to account for errors in the linearization point.

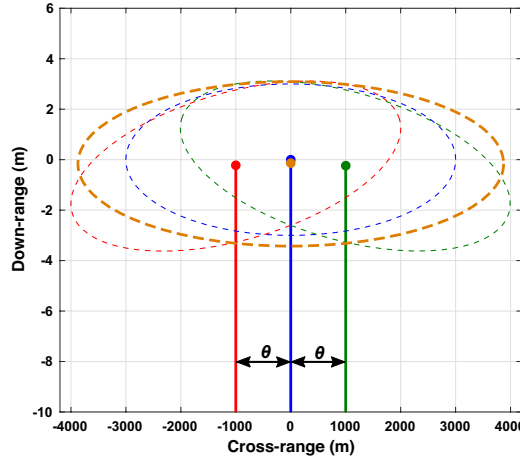
Lastly, we calculate any range uncertainty inflation that should be added, based on the combined covariance. This is used during the measurement update for this particular time step. If the measurement's range uncertainty is less than the range uncertainty of the combined covariance, $\mathbf{P}_{comb}$, then we inflate the measurement noise's range uncertainty to the combined covariance value. That is, if $\sigma_{r,meas} < \sigma_{r,comb}$, then $\sigma_{r,meas} = \sigma_{r,comb}$. An advantage to using an algorithm such as this is that as the filter converges and the track's state error covariance gets smaller, less inflation is required in the range direction until eventually no inflation is required. Using the above logic, this inflation is typically only applied for a handful to tens of seconds, as a track's covariance improves quickly with subsequent measurement updates.



(a) Estimated target position (blue dot) and 1σ , 2σ , and 3σ error covariances (blue dash). Red and green lines represent constant range line-of-sight vectors rotated θ degrees from the target.



(b) Rotated state vectors and 3σ error covariances shown in red and green dots and ellipses.



(c) Combined state vector and 3σ covariance shown in orange, which assumes a Gaussian mixture with equal weighting.

Figure 12. Two-dimensional example illustrating uncertainty compensation due to errors in the linearization point. In this example, the target is 2000 km down-range from the sensor and 0 km cross-range ($r = 2000$ km, $u = 0$ km). The noise statistics are $\sigma_r = 1$ m and $\sigma_u = 0.5$ m/sin. Note that the down-range is plotted relative to the target position so that it's easier to read the y-axis.

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5. SIMULATION RESULTS

To examine the performance of the newly defined CM3EKF, we use four trajectories at different ranges and occurring at different angular locations on the array face. Table 3 provides information for the trajectories used in this section. In order to show results that can be compared to other published work, trajectories 1 and 3 are identical to the ones used in [14], and trajectories 2 and 4 are identical to the ones used in [1]. Notice that the target speeds are much different between the two references (374 m/s for trajectories 1 and 3, and 3000 m/s for trajectories 2 and 4), which determines along with range-to-target the span of u - v space the targets will travel over the 500 s simulation time. The last column in the table shows the starting and ending scan angles on the radar’s array face in msines. Notice that the faster trajectories (2 and 4) travel significantly further in terms of scan angles than the slower ones.

TABLE 3
Initial State Information for Trajectories Used in Analysis

Traj #	Range (km)	Initial State (km, m/s)	$\sqrt{u^2 + v^2}$ (msin)
1	1090	$\mathbf{x}_0 = [750, 250, 750, -200, 300, -100]^T$	$0.73 \Rightarrow 0.74$
2	1905	$\mathbf{x}_0 = [1100, 1100, 1100, -2000, -2000, -1000]^T$	$0.82 \Rightarrow 0.23$
3	3002	$\mathbf{x}_0 = [2520, 1320, 960, -200, 300, -100]^T$	$0.95 \Rightarrow 0.95$
4	5504	$\mathbf{x}_0 = [3008, 3000, 3500, -2000, -2000, -1000]^T$	$0.77 \Rightarrow 0.69$

A graphical overview of where the trajectories lie on the radar’s array face for the 500 s simulations is shown in Figure 13. While the trajectories all lie in the upper right quadrant of the array face, the u - v space is well covered because of symmetry. Additionally, in Section 3.6 we saw that the contact lens phenomena produced smaller range uncertainties around the boresight of the radar and larger range uncertainties as $\sqrt{u^2 + v^2}$ increased, so our scenarios show good diversity there also. Notice especially trajectory 3, which is around 0.95 msines for its entire trajectory. This is extremely far off-boresight, and should be the most stressing for the filters based on earlier results.

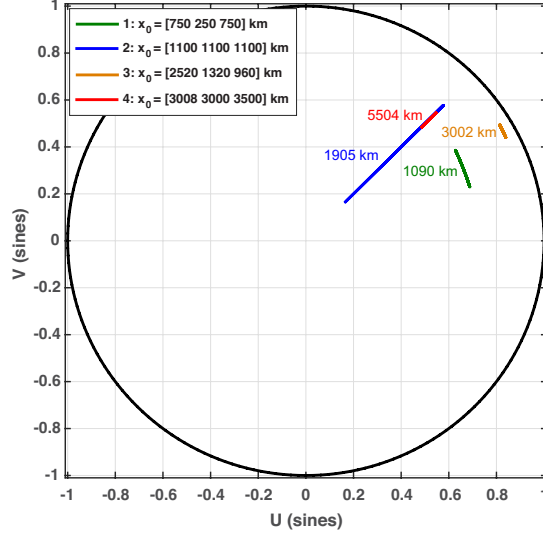


Figure 13. Overview of trajectories in $u-v$ space. The numbers next to the trajectories are the initial range-to-target. The black unit circle represents all allowable combinations of u and v .

For each scenario, 100 Monte Carlo runs were executed and several statistical measures were computed. The sensor is located at the origin of the Cartesian frame, and the measurements are obtained in the radar's natural RUV coordinate frame. Measurements are taken every one second ($T = 1$ s), and the measurement noise standard deviations are, unless otherwise noted, $\sigma_r = 1$ m and $\sigma_u = \sigma_v = 1$ msin. At these ranges this leads to measurements that are accurate in range but are very inaccurate (one to ten kilometers, 1σ) in cross-range, which results in extremely contact lens shaped uncertainty volumes. As the range uncertainty σ_r increases or the cross-range uncertainties σ_u and σ_v decrease, the contact lens effect begins to diminish and the uncertainty volume becomes more closely shaped to an ellipsoid.

5.1 FILTERS CONSIDERED

In this section we compare the CM3EKF’s performance, derived in Section 4, to several other linear and nonlinear estimators. These algorithms and a brief description of them are provided in Table 4.

TABLE 4
Filters Included in Comparison

Estimator	Description
CM1KF, CM2KF, CM3KF	Nonlinear RUV measurements are converted to Cartesian XYZ coordinates according to the first-, second-, and third-order conversions described in Section 3 in order to realize linear filters.
CM2KF*	Same as above except uses second-order conversion described in [1], which has errors in the derivation (see Appendix A).
CM3EKF	Linearized filter with measurements in RUV coordinates; uses third-order conversion to inflate the measurement’s range uncertainty in order to account for contact lens effect (see Section 4).
MCAEKF	Linearized filter with measurements in RUV coordinates; uses incorrect second-order conversion to inflate the measurement’s range uncertainty in order to account for contact lens effect (see [1]). Switching logic between CM1 and CM2 required because of errors in second-order conversion.
UKF	Nonlinear filter that uses $2n_x = 12$ sigma points to approximate true distribution, where n_x is the dimension of the state vector.
RPF	Nonlinear filter that uses brute force sampling (300,000 particles for the cases shown) to approximate the true distribution.

The two nonlinear filters, the unscented Kalman filter [4, 9–11], and the regularized particle filter [12–14, 23], were chosen because they are well known and may provide improved performance at the expense of additional computational complexity. Because of the long ranges, the RPF requires 300,000 particles in order to perform well. Regularization was also included in order to prevent sample impoverishment [12] when the effective sample size, N_{eff} , dropped below 3.5% of the total number of particles, and an Epanechnikov kernel with twice the optimal bandwidth was used. The optimal bandwidth for an Epanechnikov kernel with Gaussian density depends on both the

dimension of the state and the number of particles [4, 23], and is given by

$$h_{opt} = \left(\frac{8(n_x + 4)(2\sqrt{\pi})^{n_x}}{N_p c_{n_x}} \right)^{1/(n_x+4)}, \quad (69)$$

where n_x is the dimension of the state, N_p is the number of particles used, and c_{n_x} is the volume of the unit hypersphere in n_x -dimensional space. For our problem, the six dimensional unit hypersphere volume is

$$c_6 = \frac{\pi^3}{3!} = 5.1677. \quad (70)$$

Using twice the optimal bandwidth was found to keep the particles in areas of high likelihood while preventing sample impoverishment. Additionally, all of the filters except for the RPF require the compensation logic for errors in the linearization point which is described in Section 4.2.

5.2 SYSTEM MODEL

Because the dynamics of a non-maneuvering target are known with such high accuracy (e.g., gravity model and centrifugal and Coriolis accelerations only), we assume that $\mathbf{f}[\mathbf{x}(k), k]$ is a linear function with state transition matrix $\mathbf{F}$ and only consider the nonlinearities in the conversion from measurement to target state coordinates. As in previous analyses [1, 14], a nearly constant velocity (NCV) target was assumed with measurements received every second ($T = 1$ s) and a continuous white noise acceleration model (CWNA) was used to account for process noise. Defining our state vector as

$$\mathbf{x} \triangleq [x \ y \ z \ \dot{x} \ \dot{y} \ \dot{z}]^T, \quad (71)$$

our dynamics is given by

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{\Gamma}\boldsymbol{\nu}(k) \quad (72)$$

where $\mathbf{F}$ is the linear state transition matrix, and $\mathbf{\Gamma}$ is the gain matrix which relates how the process noise affects each state.

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & T & 0 & 0 \\ 0 & 1 & 0 & 0 & T & 0 \\ 0 & 0 & 1 & 0 & 0 & T \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (73)$$

$$\mathbf{\Gamma} = \begin{bmatrix} T^2/2 & 0 & 0 \\ 0 & T^2/2 & 0 \\ 0 & 0 & T^2/2 \\ T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} \quad (74)$$

Here, $T \triangleq t(k+1) - t(k)$ is the sensor update period, and the process noise covariance matrix $\mathbf{Q} \triangleq E(\mathbf{\Gamma}\boldsymbol{\nu}\boldsymbol{\nu}^T\mathbf{\Gamma}^T)$ is given by

$$\mathbf{Q} = \begin{bmatrix} T^3/3 & 0 & 0 & T^2/2 & 0 & 0 \\ 0 & T^3/3 & 0 & 0 & T^2/2 & 0 \\ 0 & 0 & T^3/3 & 0 & 0 & T^2/2 \\ T^2/2 & 0 & 0 & T & 0 & 0 \\ 0 & T^2/2 & 0 & 0 & T & 0 \\ 0 & 0 & T^2/2 & 0 & 0 & T \end{bmatrix} \tilde{q}, \quad (75)$$

where $\tilde{q}$ is the process noise power spectral density, which is $10^{-4} \text{m}^2/\text{s}^3$ for both the trajectory generation and used by the estimator. That is, the trajectory is perturbed by random white accelerations with the specified power spectral density, and the filters account for them using the same value. This level of process noise is a reasonable choice for ballistic objects in the exoatmosphere, where the motion is very well understood except for very small perturbations (e.g., modeling errors in the gravity field).

Depending on which filter is being used, the measurement equation is slightly different. The CM3EKF, MCAEKF, UKF, and RPF use a nonlinear measurement function that transforms the predicted state vector (or particles in the case of the UKF and RPF) from Cartesian XYZ coordinates to RUV coordinates. The measurement equation and the nonlinear function are given by

$$\mathbf{z}_{ruv}(k) = \begin{bmatrix} r_m(k) \\ u_m(k) \\ v_m(k) \end{bmatrix} = \mathbf{h}[\mathbf{x}(k)] + \mathbf{w}_{ruv}(k), \quad (76)$$

where

$$\mathbf{h}[\mathbf{x}(k)] = \begin{bmatrix} r \\ u \\ v \end{bmatrix} = \begin{bmatrix} \sqrt{x^2 + y^2 + z^2} \\ \frac{x}{\sqrt{x^2 + y^2 + z^2}} \\ \frac{y}{\sqrt{x^2 + y^2 + z^2}} \end{bmatrix}, \quad (77)$$

and the measurement noises are zero mean independent Gaussian with covariance

$$\mathbf{R}_{ruv} = E[\mathbf{w}_{ruv}(k) \mathbf{w}_{ruv}^T(k)] = \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_v^2 \end{bmatrix}. \quad (78)$$

The converted measurement linear Kalman filters (CM1KF, CM2KF, CM3KF, and CM2KF*) have a linear measurement equation

$$\mathbf{z}_{xyz}(k) = \begin{bmatrix} x_m(k) \\ y_m(k) \\ z_m(k) \end{bmatrix} = \mathbf{H}\mathbf{x}(k) + \mathbf{w}_{xyz}(k), \quad (79)$$

where

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad (80)$$

and the measurement noise covariance $\mathbf{R}_{xyz}$ uses the first-, second-, or third-order conversion from Sections 3.2, 3.3, and 3.4.

5.3 TRACK INITIALIZATION

Track initialization for all filters uses two-point differencing, where the first two measurements are converted from RUV to Cartesian XYZ coordinates and then a first-order polynomial fit is performed [2, 3]. In contrast, batch processing, which stores a number of frames before performing track initialization, could have been used to produce a more accurate initial state vector and error covariance at the expense of delayed data availability for downstream functions. The initial state and error covariance of the two-point differencing algorithm are given by

$$\hat{\mathbf{x}}(2|2) = \begin{bmatrix} \mathbf{z}_{xyz}(2) \\ (\mathbf{z}_{xyz}(2) - \mathbf{z}_{xyz}(1)) / T \end{bmatrix} \quad (81)$$

$$\mathbf{P}(2|2) = \begin{bmatrix} \mathbf{R}_{xyz} & \mathbf{R}_{xyz}/T \\ \mathbf{R}_{xyz}/T & 2\mathbf{R}_{xyz}/T^2 \end{bmatrix} \quad (82)$$

where $\mathbf{z}_{xyz}(1)$ and $\mathbf{z}_{xyz}(2)$ are the first two measurements converted to Cartesian XYZ coordinates, and $\mathbf{R}_{xyz}$ is the corresponding measurement noise covariance, which is assumed to be nearly constant between measurements.

The particle filter takes a slightly more brute force approach by sampling around the first two measurements in RUV coordinates using their measurement noise covariances [14]. For an N -sample particle filter, track initialization is performed by taking N samples around the first measurement and N samples around the second measurement. From these samples, particle-pairs are converted to Cartesian XYZ coordinates and initial state estimates are calculated using Equation (81), resulting in N position/velocity initial state vectors. The sample mean and covariance are determined from the particles for output purposes. Because of the contact-lens shape at long ranges, performing this particle-based two-point initialization tends to produce sampled estimates that more closely approximate the true measurement uncertainty distribution.

These two initialization approaches work well in most cases, but as the angular measurement noise and range-to-target increase, the variance of the velocity term in Equation (82) may become significantly larger than the maximum speed of the target [2]. This was observed when σ_u or σ_v was increased significantly (≥ 5 msines), and for these cases the target's initial velocity estimate in Equation (81) was modified such that the range-rate was estimated using the high-accuracy range measurements, and the velocity in the cross-range plane was set to zero. The velocity uncertainty was initialized to 1000 m/s isotropic, and the position/velocity cross-covariance was initialized to zero.

5.4 FILTER CONSISTENCY METRIC

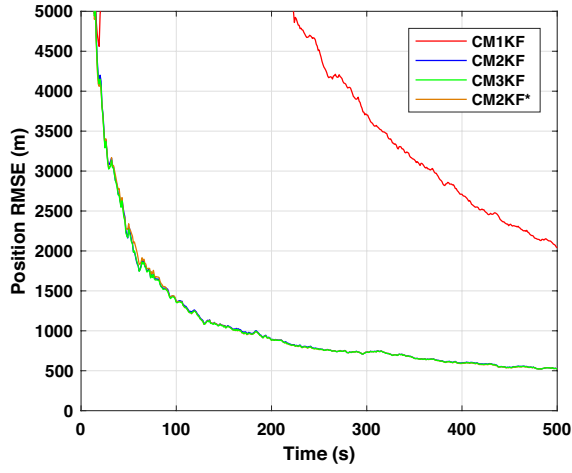
Filter consistency is an important metric for a filter to achieve. The normalized estimation error squared (NEES) metric can be used to determine if the filter computed covariance statistically represents the filter's state errors. Given N independent samples of a Gaussian random variable, $\hat{\mathbf{x}}(k)$, the average NEES is given by

$$\bar{\epsilon}(k) = \frac{1}{N} \sum_{i=1}^N [\mathbf{x}(k) - \hat{\mathbf{x}}(k|k)]^T \mathbf{P}(k|k)^{-1} [\mathbf{x}(k) - \hat{\mathbf{x}}(k|k)]. \quad (83)$$

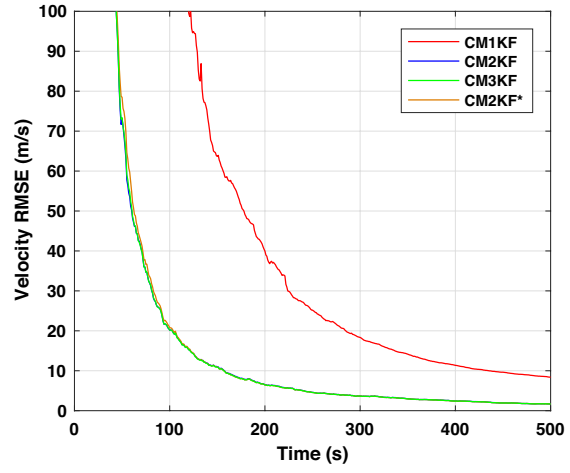
If the estimator is consistent then $N\bar{\epsilon}(k)$ will have a chi-square density with Nn_x degrees of freedom, where n_x is the dimension of the state vector. Therefore, we expect the average NEES $\bar{\epsilon}(k)$ to be close to the state dimension, which is six. The two sided 99% probability concentration region for $\bar{\epsilon}(k)$ can also be calculated and is shown with the results.

5.5 FILTER RESULTS

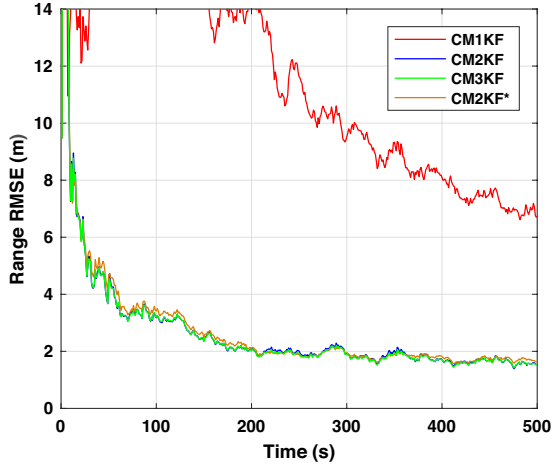
Let us first investigate the performance of the converted measurement linear filters (CM1KF, CM2KF, CM3KF, and CM2KF*) using the longest range case, trajectory 4. Figures 14a–14d show the root mean squared (RMS) position errors, velocity errors, range errors, and NEES. From the figure, the first-order conversion linear filter, CM1KF, performs very poorly since it does not model the contact lens phenomena, and therefore will not be considered further. The CM2KF, CM3KF, and CM2KF* all produce nearly identical RMS position, velocity, and range errors, however from Figure 14c it is observed that the range error never falls below the 1 m range measurement noise. This shows that in some cases there is no filter benefit in the range direction, due to the range uncertainty inflation that occurs when converting the RUV measurements to Cartesian XYZ coordinates. Finally, the NEES plot shows that the CM3KF is the most consistent filter and agrees with our expectations drawn from Section 3.6. The CM2KF filter produces a slightly smaller covariance than the sample errors, as evidenced by an average NEES value around seven, however it is still contained in the 99% probability region. The CM2KF* filter produces error covariances too large due to the errors in the derivation, as the average NEES is between 4.5–5. In light of these results, it is clear that the CM3 conversion produces the most consistent results and should be considered in the design of the extended Kalman filter described in Section 4.



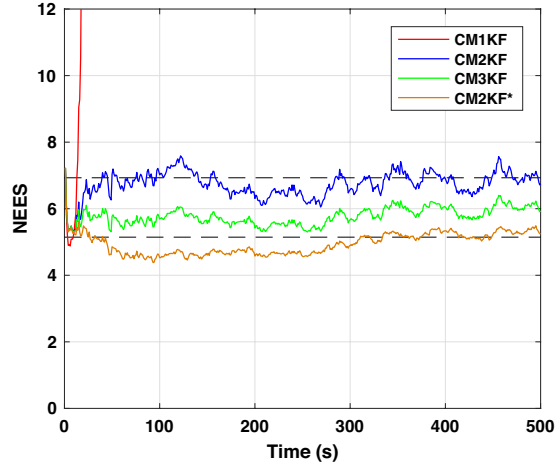
(a) RMS position errors



(b) RMS velocity errors



(c) RMS errors in range direction



(d) NEES and 99% probability region

Figure 14. RMS errors and filter consistency of the converted measurement linear Kalman filters for trajectory 4 with $\sigma_r = 1$ m, $\sigma_u = \sigma_v = 1$ msin. CM1KF, CM2KF, and CM3KF are the first-, second-, and third-order linear filters. CM2KF* is from [1] and has errors in its derivation.

In order to produce consistent error covariances *and* achieve a filter benefit in the range direction, the extended Kalman filter proposed in Section 4 was developed. In order to demonstrate the advantages of the CM3EKF, it was executed on the same trajectory as above. Figures 15a–15d show the performance of the CM3EKF compared to both the third-order converted measurement linear Kalman filter CM3KF, which achieved consistency but produced excessive range errors, and the MCAEKF from [1] which is based off the incorrect derivation of the second-order conversion. While the RMS position and velocity errors are all similar, the CM3EKF has faster convergence, especially in the range direction. Figure 15c shows the CM3EKF converges faster than the other filters and produces significant reduction in range error relative to the measurement noise. The

CM3EKF also produces consistent covariances, on the order of what was observed with the third-order converted measurement linear filter. The MCAEKF, on the other hand, converges slower in the range direction and produces covariances that are too large and inconsistent for the first few hundred seconds. Additionally, there is no switching logic required with the CM3EKF; it automatically reduces the measurement's range uncertainty as more information is acquired and the track's accuracy improves.

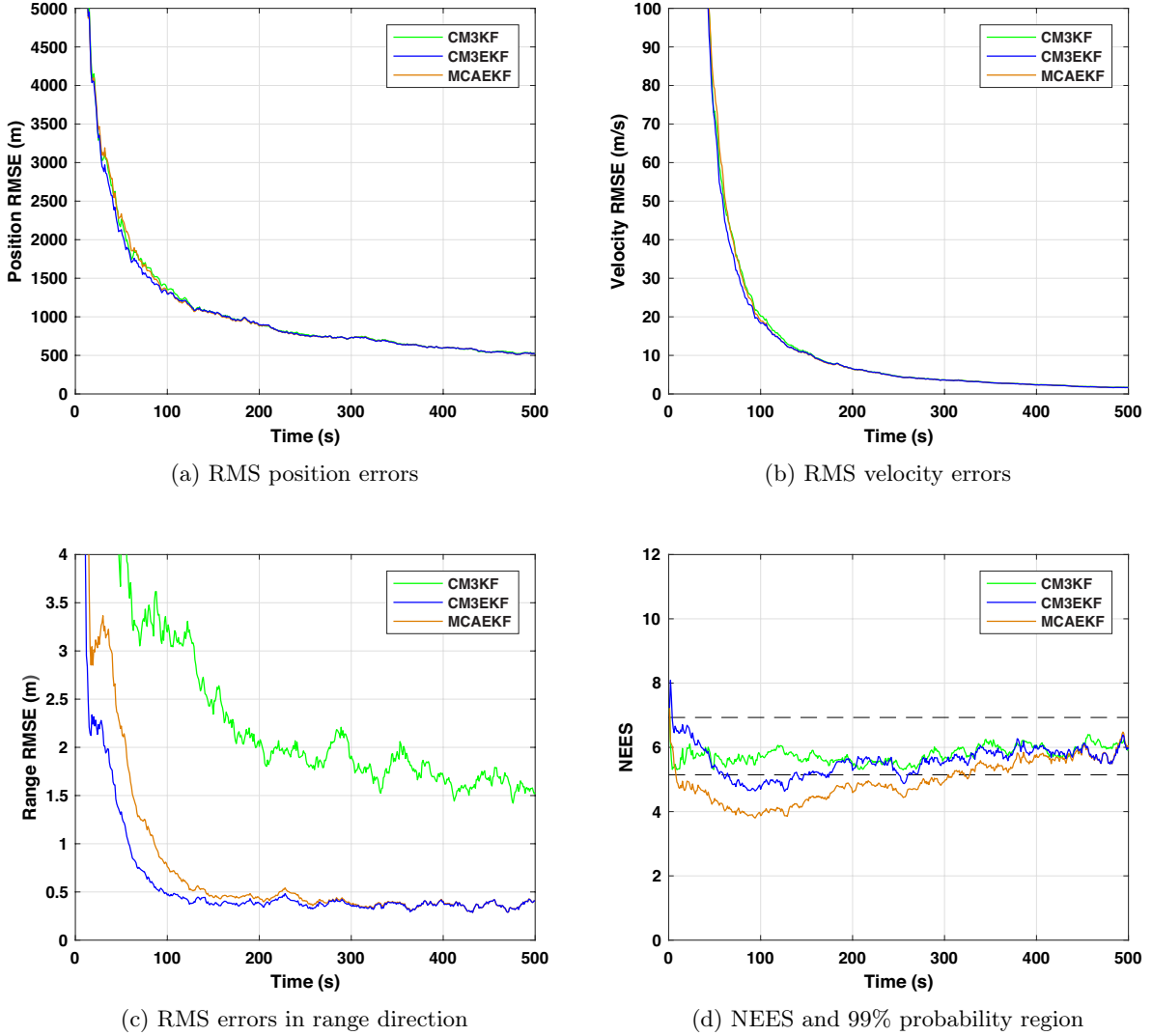
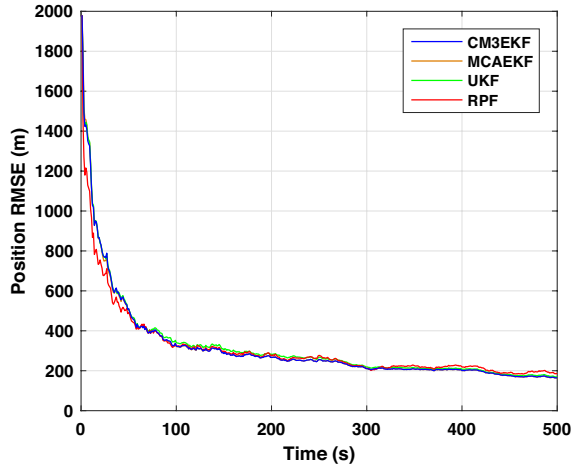


Figure 15. RMS errors and filter consistency of the extended Kalman filters (and CM3KF for comparison) for trajectory 4 with $\sigma_r = 1$ m, $\sigma_u = \sigma_v = 1$ msin. CM3EKF uses the third-order conversion to inflate the range uncertainty when required. MCAEKF is from [1] and has errors in its derivation.

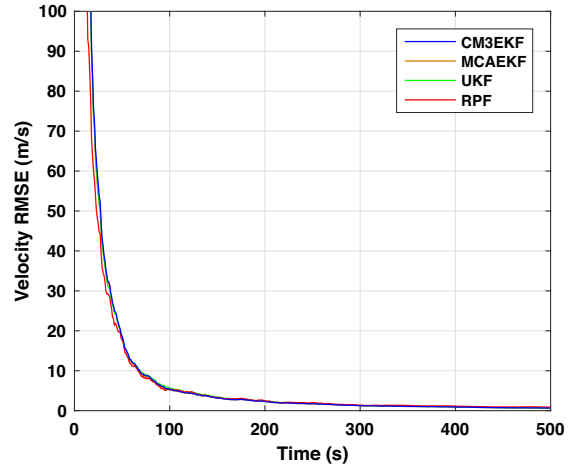
Having concluded that the CM3EKF produces the best results in terms of range errors, convergence, and consistency, we now show results for the four trajectories in Table 3 and compare

them to the UKF, RPF, and MCAEKF from [1]. From Figures 16–19, we observe that the RPF has the fastest convergence, but the covariance is slightly too large (average NEES is below 99% probability region). This is due to the fact that the bandwidth of the Epanechnikov kernel was chosen to be twice the optimal bandwidth, which effectively spreads out the particles to prevent sample impoverishment. Additional analysis showed that reducing the bandwidth caused extremely poor performance for some runs, as the particles diverged to low likelihood regions.

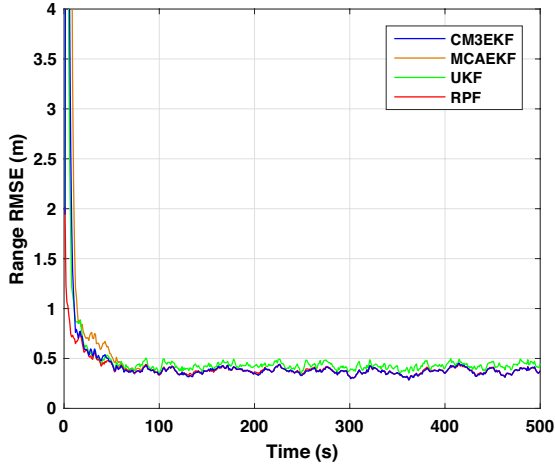
The UKF also converged quickly; however, it too had slightly larger covariances in two of the cases (Figure 16d and 17d), good filter consistency for one case (Figure 19d), and extremely poor consistency for one case (Figure 18d). In the latter case, the covariance is much too small, and can be explained by trajectory 3’s extremely large scan angle (0.95 msines), which corresponds to the region of poor performance for the UT’s estimation of the contact lens effect (see Section 3.6). The CM3EKF algorithm converged slightly slower than the RPF, but produced the smallest steady-state range errors and was the most consistent filter across all scenarios analyzed. The MCAEKF from [1] was the slowest filter to converge and produced filter covariances that were too large due to previously noted errors in its derivation.



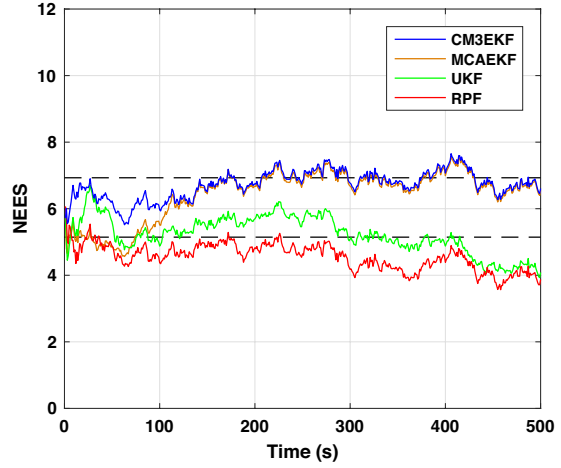
(a) RMS position errors



(b) RMS velocity errors

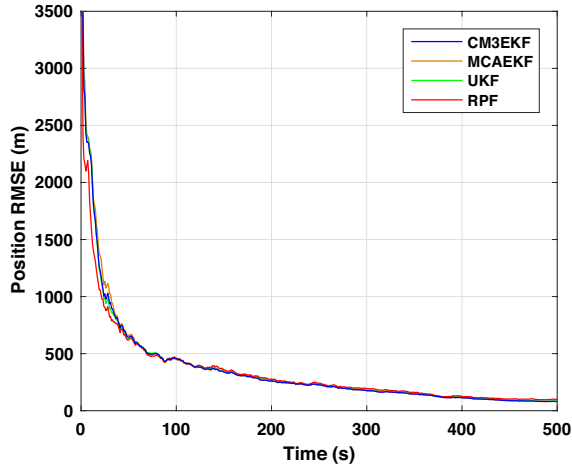


(c) RMS errors in range direction

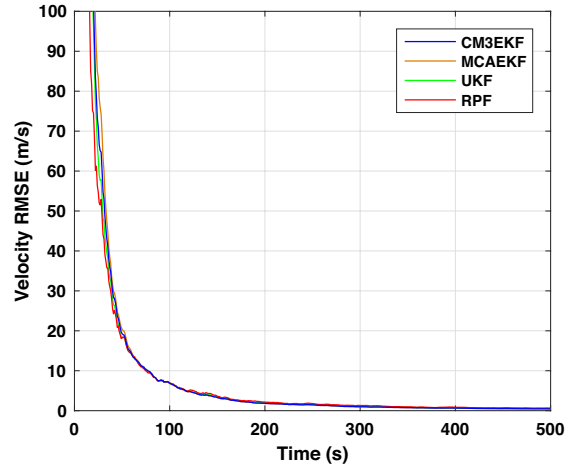


(d) NEES and 99% probability region

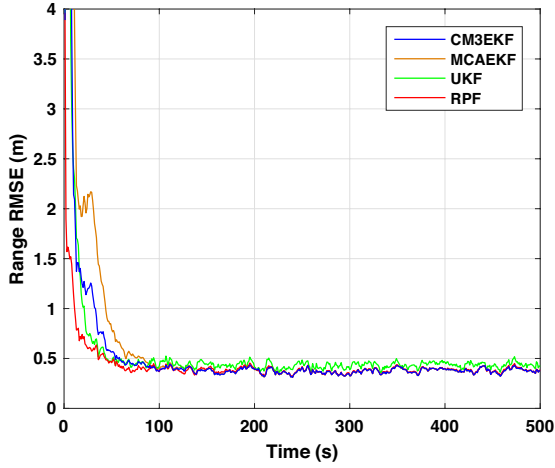
Figure 16. Filter comparison for trajectory 1 with $\sigma_r = 1$ m, $\sigma_u = \sigma_v = 1$ msin.



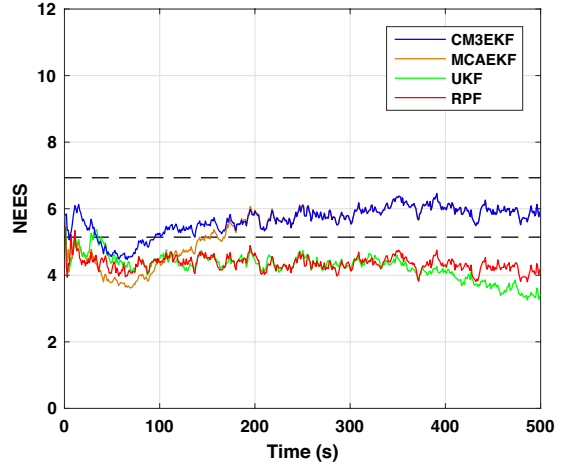
(a) RMS position errors



(b) RMS velocity errors

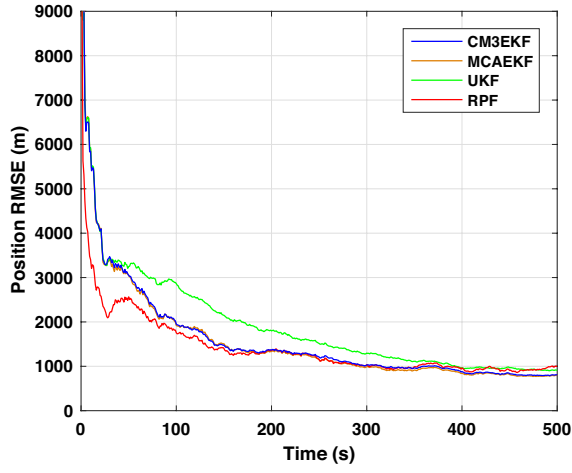


(c) RMS errors in range direction

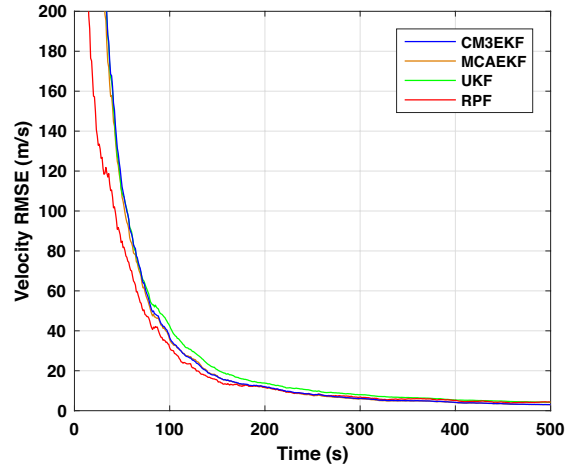


(d) NEES and 99% probability region

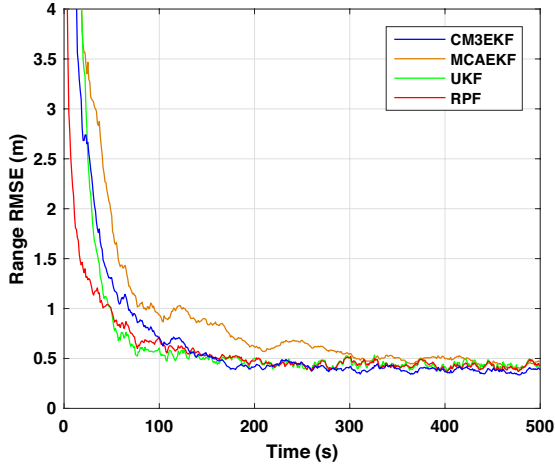
Figure 17. Filter comparison for trajectory 2 with $\sigma_r = 1$ m, $\sigma_u = \sigma_v = 1$ msin.



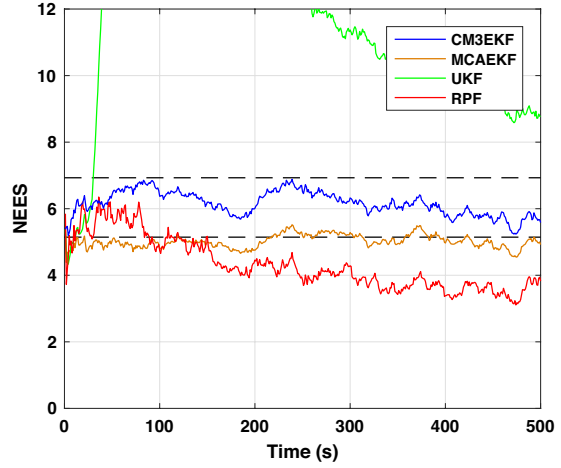
(a) RMS position errors



(b) RMS velocity errors



(c) RMS errors in range direction



(d) NEES and 99% probability region

Figure 18. Filter comparison for trajectory 3 with $\sigma_r = 1$ m, $\sigma_u = \sigma_v = 1$ msin.

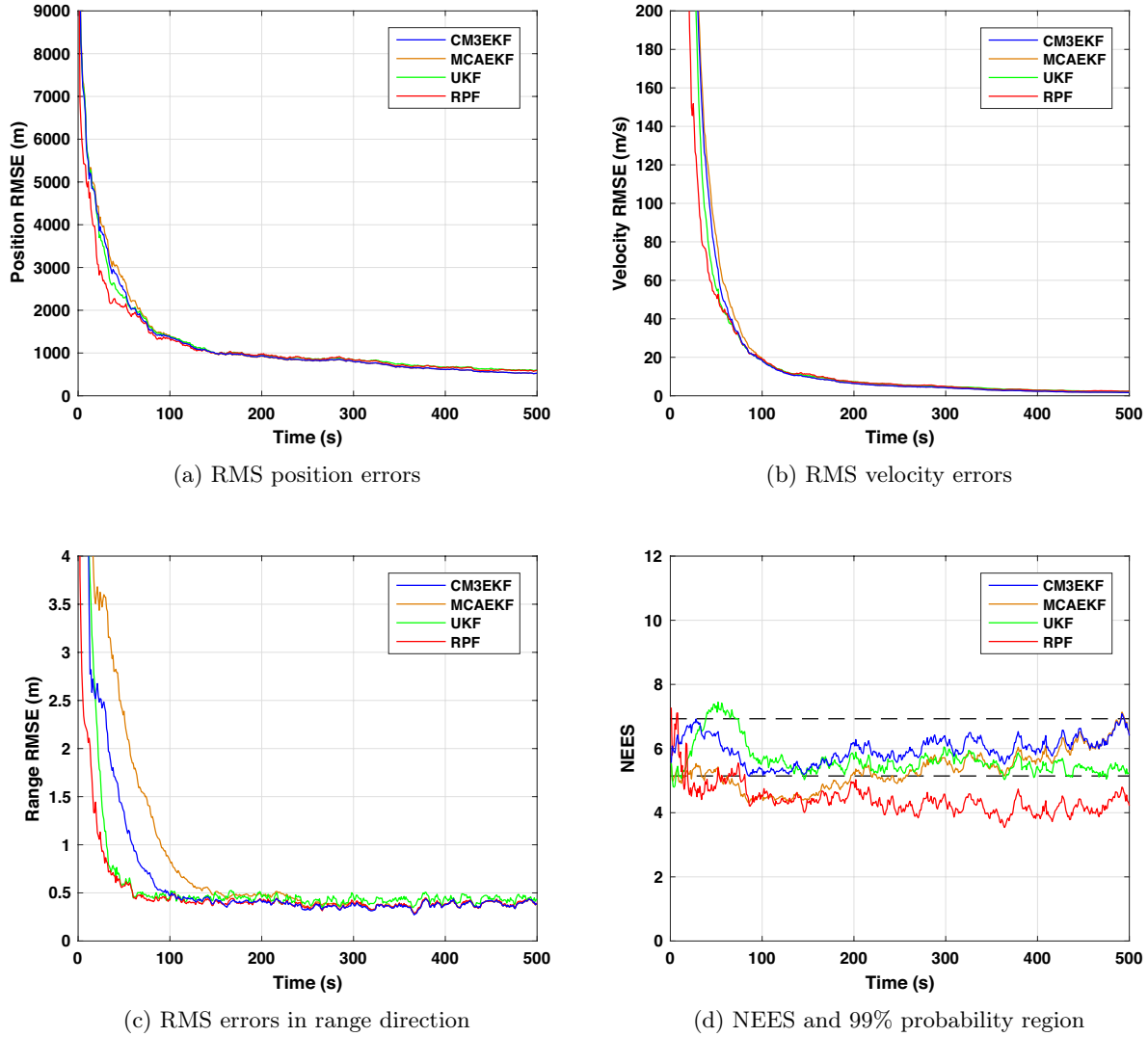


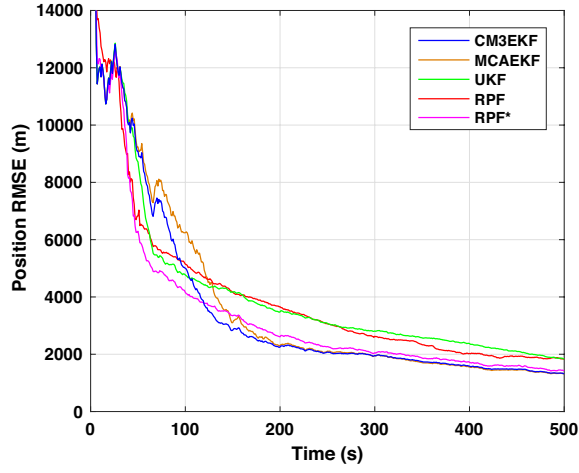
Figure 19. Filter comparison for trajectory 4 with $\sigma_r = 1$ m, $\sigma_u = \sigma_v = 1$ msin.

Lastly, we demonstrate the robustness of our algorithm by showing results for the case when the angle measurement noises are significantly different ($\sigma_u = 1$ msin, $\sigma_v = 5$ msin). For this example, the measurement noise in angle space is extremely large, specifically 6.5 km and 36.5 km, 1σ , which is an enormous volume to sample for the particle filter. From the results in Figure 20, we see that both the UKF and RPF converged quickly, but had larger steady state errors than the CM3EKF and significant consistency problems. The UKF's poor performance is in agreement with Figure 8 in Section 3.6, and can be explained by the severe non-Gaussianity of the measurements. Improved results for the UKF could likely be achieved by using more sigma points to sample the nonlinear, non-Gaussian distribution. The RPF also had very poor consistency, which was caused by sample impoverishment. For this case, because the target was at such long ranges and σ_v was so

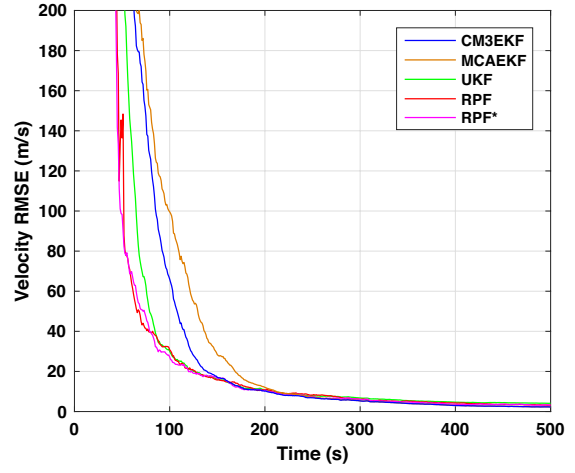
large, the probability space could not be adequately sampled, and about 10% of the Monte Carlos had divergence problems.

In general, sample impoverishment can be reduced by increasing the number of particles, which samples the likelihood space with greater density but increases the computational expense, or by increasing either the kernel bandwidth or how often regularization is performed, which spreads the particles out in likelihood space but leads to larger steady-state errors. Increasing the number of particles to two million samples, increasing the kernel bandwidth during regularization, and performing regularization at each time step were all investigated but did not show significant improvements for this particular case. Because the uncertainty in the range direction is so much smaller than in angle space (1 m compared to 36.5 km), it was found that the particles very quickly migrated into low likelihood regions. In order to mitigate this behavior, the measurement uncertainty in the range direction was initially increased to 10 m early in the track’s life when the velocity uncertainty exceeded 600 m/s, which broadened the region of high likelihood and allowed the track to converge. This was found to greatly improve the RPF performance and is shown in the magenta line titled “RPF*” in Figure 20. Although the results are greatly improved, the RPF still produces slightly smaller error covariances early in the scenario and slightly larger error covariances late in the scenario, compared to the sample errors. This example demonstrates the sensitivities of the particle filter to tuning parameters, and the difficulties with sample impoverishment.

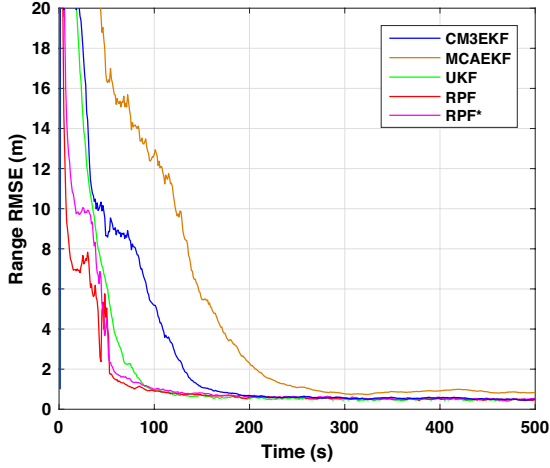
The CM3EKF, on the other hand, is again the most consistent estimator of those considered, has the smallest position and velocity steady state errors, and converges only slightly slower than the RPF and UKF.



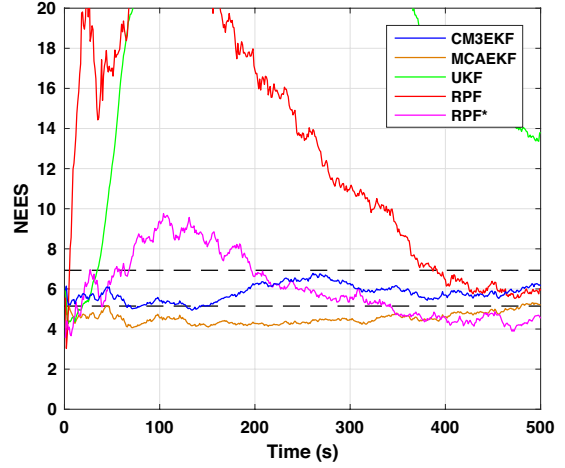
(a) RMS position errors



(b) RMS velocity errors



(c) RMS errors in range direction



(d) NEES and 99% probability region

Figure 20. Filter comparison for trajectory 4 with unequal measurement noise: $\sigma_r = 1$ m, $\sigma_u = 1$ msin, $\sigma_v = 5$ msin.

5.6 COMPUTATIONAL COMPLEXITY

Average run times for each algorithm to execute on a 500 s scenario were computed and are listed in Table 5. It should be noted that the CM1KF, CM2KF, CM3KF, CM2KF*, CM3EKF, and MCAEKF all had average run times between 0.12–0.15 s. The UKF average run time was only slightly slower, at 0.18 s. The RPF, on the other hand, had an extremely large computational demand, since it required 300,000 particles in order to sufficiently sample the broad pdf, and it averaged 915 s on a desktop computer using parallelization with 8 cores. Given that the convergence improvement of the RPF was only marginally better than the CM3EKF, that it was sensitive to tuning parameters (i.e., number of particles, regularization frequency, and kernel bandwidth selection), and that it was 6500 times slower than the other algorithms, the use of the RPF is very likely not worth it.

TABLE 5
Average Run Times to Execute 500 s of 1 Hz Data

Filter	Run Time (s)
CM3EKF	0.14
UKF	0.18
RPF	915

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6. CONCLUSIONS

In phased array radars, the measurement equation that relates the sensor measurements to the target state is nonlinear because position measurements are often expressed in RUV coordinates, while the state is expressed in Cartesian XYZ coordinates. For target tracking, the different coordinates of the state and measurement necessitate a nonlinear coordinate relationship (conversion) in order to calculate a filter gain and update the track state. In order to realize a consistent track filter (when the filter-calculated state error covariance is in agreement with the true state error covariance), this conversion must correctly approximate the non-elliptical, non-Gaussian shape of the uncertainty volume in Cartesian XYZ coordinates. This nonlinear conversion between RUV and XYZ coordinates can be approximated using a Taylor series with some finite number of terms. At short ranges, a first-order approximation may suffice; however, as the range to the target increases, higher order terms are necessary to accurately describe the converted distribution. While the topic of long range tracking has attracted significant attention over the last several decades [1–3, 6, 8, 14, 16–18], we identified errors in the most commonly used algorithm for tracking long range targets using RUV measurements [1] and reported them in Appendix A. Because this is such an important problem, this report performed a detailed analysis of the conversion problem and derived a filter that produces consistent states and covariances *and* does not sacrifice accuracy in the range direction in steady state. It accomplishes this by considering the two main difficulties of tracking targets at very long range: (i) correctly approximating the mean and covariance of the converted (RUV to XYZ) measurements, and (ii) compensating for linearization errors during transformation of the error covariance.

Section 2 of this report presented the mathematical modeling of the tracking problem for phased array radars and described several linear and nonlinear filtering algorithms previously applied to the long range target tracking problem. Section 3 was dedicated to deriving the various conversions (i.e., first-, second-, and third-order Taylor series covariance approximations), and several examples were used to compare each algorithm’s approximation of the converted mean and covariance relative to truth. From the results, the first-order conversion (CM1) was shown to be inadequate at estimating the uncertainty in range caused by the contact lens effect. The second-order conversion (CM2) approximated the true measurement uncertainty more closely, but still showed errors in the range direction between 2–28% of truth for the cases analyzed. The unscented transform performed better than both CM1 and CM2, but even it resulted in errors at large scan angles ($>37^\circ$) or when the noise distributions were significantly different in u and v . Only the third-order conversion (CM3) accurately characterized the true measurement’s uncertainty volume across all ranges and scan angles in u - v space, and resulted in errors less than 0.25% of truth.

Based on these results, a new extended Kalman filtering algorithm (CM3EKF) was developed in Section 4 for consistent state estimation for long range radars. This new algorithm adaptively adjusts the range measurement noise covariance to maintain filter consistency and appropriately accounts for errors due to the linearization point during the CM3 conversion. Unlike the MCAEKF published in [1], no switching between conversion algorithms is required and it seamlessly and accurately reduces the inflation amount as the track’s uncertainty improves. Simulation results for several long range targets at various locations on the radar’s array face were presented in Section 5 that demonstrated the CM3EKF’s superiority over the converted measurement linear Kalman filters (CM1KF, CM2KF, CM3KF, and CM2KF*), the MCAEKF, and the nonlinear UKF. It also had

only slightly worse convergence than the RPF, while producing smaller steady-state errors and more consistent error covariances. Since the RPF was shown to have tuning sensitivities (i.e., number of particles, regularization frequency, and kernel bandwidth selection) and is computationally orders of magnitude more expensive in terms of processing resources, there is little motivation to use it for such small improvements in filter convergence.

APPENDIX A: ERRORS IN MCAEKF DERIVATION

There are a number of errors in [1] that cause the MCAEKF algorithm to overinflate the range uncertainty and require ad hoc logic to transition from the CM2 to CM1 conversion. The most notable of these is in the definition of $\text{var}(z_{m_2})$ (cf. {33}⁵), where there are three errors: a sign error on the fifth term, and exponent errors on the sixth and seventh terms. The expression with corrected terms highlighted is given by

$$\begin{aligned} \text{var}(z_{m_2}) = & (\bar{c}_z)^2 + \left(\frac{\partial f_z}{\partial r}\right)^2 \sigma_r^2 + \left(\frac{\partial f_z}{\partial u}\right)^2 \sigma_u^2 + \left(\frac{\partial f_z}{\partial v}\right)^2 \sigma_v^2 \\ & \boxed{-} \bar{c}_z \left(\frac{\partial^2 f_z}{\partial u^2} \sigma_u^2 + \frac{\partial^2 f_z}{\partial v^2} \sigma_v^2 \right) + \frac{3}{4} \left(\frac{\partial^2 f_z}{\partial u^2} \right)^{\boxed{2}} \sigma_u^4 + \frac{3}{4} \left(\frac{\partial^2 f_z}{\partial v^2} \right)^{\boxed{2}} \sigma_v^4 \\ & + \left(\frac{\partial^2 f_z}{\partial r \partial u} \right)^2 \sigma_r^2 \sigma_u^2 + \left(\frac{\partial^2 f_z}{\partial r \partial v} \right)^2 \sigma_r^2 \sigma_v^2 + \left(\frac{\partial^2 f_z}{\partial u \partial v} \right)^2 \sigma_u^2 \sigma_v^2 \Big|_m. \end{aligned} \quad (\text{A.1})$$

It should be noted that the exponent errors in the definition of $\text{var}(z_{m_2})$ are likely typos in the original paper but not in the actual derivation, as the only way to match the results in their numerical example (cf. {43}–{48}) is to only include the error in the sign of the fifth term. This sign error is the most significant of the errors and results in a larger than required uncertainty in the range direction.

There are also missing terms in the definitions of $\text{cov}(x_{m_2}, z_{m_2})$ and $\text{cov}(y_{m_2}, z_{m_2})$ (cf. {37} and {38}), although their effects are small because of their relative magnitudes. They have been included here for completeness.

$$\text{cov}(x_{m_2}, z_{m_2}) = \frac{\partial f_z}{\partial u} r_m \sigma_u^2 + \frac{\partial f_z}{\partial r} u_m \sigma_r^2 + \boxed{\frac{\partial^2 f_z}{\partial r \partial u} \sigma_r^2 \sigma_u^2} \Big|_m \quad (\text{A.2})$$

$$\text{cov}(y_{m_2}, z_{m_2}) = \frac{\partial f_z}{\partial v} r_m \sigma_v^2 + \frac{\partial f_z}{\partial r} v_m \sigma_r^2 + \boxed{\frac{\partial^2 f_z}{\partial r \partial v} \sigma_r^2 \sigma_v^2} \Big|_m \quad (\text{A.3})$$

Lastly, there are typos in the definitions of $\text{var}(y_{m_2})$ and $\text{cov}(x_{m_2}, y_{m_2})$, which do not affect the final expressions, but are again corrected for completeness. Equations {35} and {36} should read:

$$\text{var}(y_{m_2}) = E \left[(\boxed{y_m} - (r_m - w_r)(v_m - w_v))^2 \right] \quad (\text{A.4})$$

$$\text{cov}(x_{m_2}, y_{m_2}) = E \left[(x_m - (r_m - w_r)(\boxed{u_m} - \boxed{w_u})) (y_m - (r_m - w_r)(v_m - w_v)) \right]. \quad (\text{A.5})$$

It is stated that the numerical example that gives the first- and second-order error covariances and corresponding eigenvalues (cf. {43}–{48}) assumes an RUV state vector with range $r = 5000$ km, angles $u = v = 0.5$ msin, and noise statistics $\sigma_r = 10$ m, $\sigma_u = \sigma_v = 2$ msin. From

⁵Equations in curly brackets in this section denote original equations found in [1].

inspection, it is clear that u and v do not equal 0.5 msin, however, since the error covariances in {43} and {44} do not equal the expected error covariances given by the first- and second-order equations. For example, $\text{cov}(x_{m_1}, y_{m_1}) = \text{cov}(x_{m_2}, y_{m_2})$ should equal $u_t v_t \sigma_r^2 = 2.5 \times 10^{-5} \text{ km}^2$, but is listed as $2.5011 \times 10^{-5} \text{ km}^2$ in {43} and {44}. Additionally, if $u = v$ and $\sigma_u = \sigma_v$, then $\text{cov}(x_{m_1}, z_{m_1}) = \text{cov}(y_{m_1}, z_{m_1})$ for CM1 and $\text{cov}(x_{m_2}, z_{m_2}) = \text{cov}(y_{m_2}, z_{m_2})$ for CM2, but they do not in {43} and {44}. Instead, it appears that the example uses a noisy measurement of u and v , and given the error covariances, range to target, and noise statistics, we can solve for the values of u and v used in their example. Solving for these two terms, we see that $u = 0.501433 \text{ msin}$ and $v = 0.498791 \text{ msin}$.

The converted covariances in km^2 (cf. {43} and {44}), the root eigenvalues (cf. {45} and {46}), the condition numbers (cf. {47} and {48}), and the condition ratios are listed in Table A.1. The left column in the table lists the numerical values taken directly from [1], and the right column contains the recomputed results for the case where the only error in the second-order conversion is the incorrect sign of the fifth term. Comparing columns, it is clear that the authors rounded the error covariances and largest root eigenvalue; in fact, their covariances are actually negative definite. The error covariances on the right side of the table are of higher numerical precision, are positive definite, and produce the corresponding root eigenvalues.

The condition numbers that are given in {47} and {48} are incorrect. Either there was a typographical error in their paper or there is an error in the condition number algorithm that they utilized. This can easily be checked by taking the ratio of the largest to smallest eigenvalues

$$\kappa(\mathbf{R}_1) = \frac{(14.145 \text{ km})^2}{(10.0 \text{ m})^2} = 2.00 \times 10^6 \quad (\text{A.6})$$

$$\kappa(\mathbf{R}_2) = \frac{(14.145 \text{ km})^2}{(65.2 \text{ m})^2} = 4.71 \times 10^6, \quad (\text{A.7})$$

which match the condition numbers listed in the right column of the table, but not the left. Conversely, to match the condition numbers from the paper, the largest root eigenvalue would have to be 17.075 km. Since the error in the condition number algorithm appears to only affect the largest eigenvalue, the eigenvalue ratios, K , are the same between the left- and right-hand columns. This is because the maximum eigenvalues of the first- and second-order conversions are the same, so the condition ratio becomes just the ratio of the minimum eigenvalues

$$K = \frac{\kappa(\mathbf{R}_2)}{\kappa(\mathbf{R}_1)} = \frac{\min[\text{eig}(\mathbf{R}_1)]}{\min[\text{eig}(\mathbf{R}_2)]}. \quad (\text{A.8})$$

TABLE A.1

Numerical Example from [1] Using Incorrect Derivation of Second-Order Conversion

Variable	Computed Values in [1]	Recomputed Values Using CM2*
$\mathbf{R}_1 :$	$\begin{bmatrix} 100 & 2.5011 \times 10^{-5} & -70.929 \\ 2.5011 \times 10^{-5} & 100 & -70.556 \\ -70.929 & -70.556 & 100.09 \end{bmatrix}$	$\begin{bmatrix} 100.000025 & 2.5011 \times 10^{-5} & -70.9294388 \\ 2.5011 \times 10^{-5} & 100.000025 & -70.5557188 \\ -70.9294388 & -70.5557188 & 100.0910975 \end{bmatrix}$
$\mathbf{R}_2 :$	$\begin{bmatrix} 100 & 2.5011 \times 10^{-5} & -70.929 \\ 2.5011 \times 10^{-5} & 100 & -70.556 \\ -70.929 & -70.556 & 100.10 \end{bmatrix}$	$\begin{bmatrix} 100.000025 & 2.5011 \times 10^{-5} & -70.9294388 \\ 2.5011 \times 10^{-5} & 100.000025 & -70.5557188 \\ -70.9294388 & -70.5557188 & 100.0994065 \end{bmatrix}$
$r_1 :$	[10 m, 10 km, 14 km]	[10.0 m, 10.0 km, 14.145 km]
$r_2 :$	[65.2 m, 10 km, 14 km]	[65.2 m, 10.0 km, 14.145 km]
$\kappa(\mathbf{R}_1) :$	2.9155×10^6	2.00091×10^6
$\kappa(\mathbf{R}_2) :$	6.856×10^4	4.70529×10^4
$K :$	0.023516	0.023516

*Assuming only error in second-order derivation is the incorrect sign of fifth term

Remembering that Table A.1 includes the sign error in the calculation of $\text{var}(z_{m_2})$, the corrected second-order conversion state error covariance, root eigenvalues, and condition number for the numerical example are calculated and provided in Table A.2. From it, we see that the range uncertainty for the second-order conversion is 25.5 m, which is about 60% smaller than the range uncertainty calculated using the incorrect second-order derivation in [1].

The second-order conversion does not always approximate the true distribution of the converted measurements accurately though. Table A.2 shows the estimated error covariances in km^2 , the root eigenvalues, and the condition numbers for (i) the correct second-order conversion, (ii) the third-order conversion, and (iii) *truth*, based on sample statistics using 100 million converted measurements. From the table, we see that the second-order linearization underestimates the range uncertainty (25.5 m as compared to the 33.2 m truth). The third-order conversion, which includes higher order terms, captures the error statistics significantly better and produces an identical range uncertainty as truth for this example.

TABLE A.2

Comparison of Second- and Third-Order Conversions Against Truth for Numerical Example in [1]

Method	Error Covariance, $\mathbf{R}$	Root Eigenvalues, r	Condition Number, κ
CM2 ^a :	$\begin{bmatrix} 100.000025 & 2.5011 \times 10^{-5} & -70.9294388 \\ 2.5011 \times 10^{-5} & 100.000025 & -70.5557188 \\ -70.9294388 & -70.5557188 & 100.0921989 \end{bmatrix}$	[25.5 m, 10 km, 14.145 km]	3.07618×10^5
CM3:	$\begin{bmatrix} 100.000025 & 2.5011 \times 10^{-5} & -70.9314265 \\ 2.5011 \times 10^{-5} & 100.000025 & -70.5576960 \\ -70.9314265 & -70.5576960 & 100.0987098 \end{bmatrix}$	[33.2 m, 10 km, 14.145 km]	1.81789×10^5
Truth ^b :	$\begin{bmatrix} 100.0012 & 3.576 \times 10^{-4} & -70.9325 \\ 3.576 \times 10^{-4} & 99.9987 & -70.5570 \\ -70.9325 & -70.5570 & 100.0990 \end{bmatrix}$	[33.2 m, 10 km, 14.145 km]	1.81710×10^5

^a Corrected second-order conversion using Equations (A.1)–(A.3).

^b Sample truth calculated using 100 million RUV measurements converted to Cartesian XYZ coordinates.

The sign error in the calculation of $\text{var}(z_{m_2})$ permeates throughout the rest of the paper, including:

1. The definition and calculation of the condition ratio (cf. {49}),

$$K \triangleq \frac{\kappa(\mathbf{R}_2)}{\kappa(\mathbf{R}_1)}, \quad (\text{A.9})$$

is incorrect since the derivation of the second-order conversion has errors. Additionally, the condition number algorithm appears to be incorrect. In this equation, $\kappa(\cdot)$ is the condition number, defined as the ratio of the largest to smallest eigenvalue.

2. The selection of the condition ratio threshold, $\tau_{KR} = 0.6$ (cf. {52}), is based on simulation results and is incorrect based on the above errors in CM2 and K .
3. The simplified definition of the “minimum value for consistency” (cf. {68}),

$$\sigma_r^* = \frac{r_m \hat{\sigma}_u^2}{\sqrt{\frac{2}{3}} (0.6^{-1} - 1)} \left(\frac{3 - 3u_1^2 + u_1^4}{1 - 2u_1^2 + u_1^4} \right)^{1/2}, \quad (\text{A.10})$$

is incorrect based on the above errors.

These errors lead to a filter that incorrectly overinflates the range uncertainty, resulting in worse filter performance (slower convergence and larger steady-state error) than achievable. Additionally, the ad hoc transition from the CM2 to CM1 conversion described in [1] (cf. {52}–{54} and {65}–{68}) is only necessary because of the errors in the second-order conversion.

APPENDIX B: DERIVATION OF FIRST-ORDER COVARIANCE APPROXIMATION

The first-order linearization from measurement-to-filter space transforms the RUV measurements to RFC Cartesian coordinates according to the nonlinear function $\mathbf{f}[\cdot]$ ⁶. The first-order Taylor series expansion assumes the derivatives of order higher than one are negligible, and is given by

$$\begin{aligned}\mathbf{T}_1[r_m, u_m, v_m] &= \mathbf{f}[r_t, u_t, v_t] + \sum_{j=1}^3 \frac{\partial \mathbf{f}[r_t, u_t, v_t]}{\partial x_j} w_{x_j} \\ &= \mathbf{f} + \frac{\partial \mathbf{f}}{\partial r} w_r + \frac{\partial \mathbf{f}}{\partial u} w_u + \frac{\partial \mathbf{f}}{\partial v} w_v \Big|_{true}\end{aligned}\quad (\text{B.1})$$

where the partial derivatives required in this section, evaluated at truth, are⁷

$$\begin{aligned}\frac{\partial f_x}{\partial r} &= u_t & \frac{\partial f_y}{\partial r} &= v_t & \frac{\partial f_z}{\partial r} &= w_t \\ \frac{\partial f_x}{\partial u} &= r_t & \frac{\partial f_y}{\partial v} &= r_t & \frac{\partial f_z}{\partial u} &= -\frac{r_t u_t}{w_t} \\ & & & & \frac{\partial f_z}{\partial v} &= -\frac{r_t v_t}{w_t}\end{aligned}\quad (\text{B.2})$$

and $w_t \triangleq \sqrt{1 - u_t^2 - v_t^2}$.

The transformation to coordinate x is approximated as⁸

$$\begin{aligned}x_m &= f_x[r_m, u_m, v_m] \triangleq r_m u_m \\ &\approx r u + \frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u \Big|_{true}\end{aligned}\quad (\text{B.3})$$

⁶In Appendix B, C, and D, $\mathbf{f}[\cdot]$ represents the nonlinear function that converts measurements from RUV to RFC Cartesian coordinates (it does not represent the nonlinear state transition function). This notation was chosen to maintain convention with previous work (i.e., [1]).

⁷Care should be taken to distinguish the measurement noises w_r , w_u , and w_v from the terms $w_t = \sqrt{1 - u_t^2 - v_t^2}$ and $w_m = \sqrt{1 - u_m^2 - v_m^2}$.

⁸While there is no closed form solution for the mean and variance of z_m (it must be approximated via Taylor series expansion), the mean and variances of x_m and y_m (and their cross covariance) found via Taylor series expansion can be compared against the squared error averaged *conditioned on the observations* technique from [2, 3, 16, 17]. For the case of the first-order linearization, the two techniques produce the same expressions for $E(x_m)$, $E(y_m)$, and $\text{cov}(x_m, y_m)$, however the expressions for $\text{var}(x_m)$ and $\text{var}(y_m)$ are not the same. That is, a higher order Taylor series expansion is required to accurately characterize the true measurement's statistics.

Using (10), (11), and (B.2), the mean and covariance of x_m is

$$\begin{aligned} E(x_m) &= ru + \frac{\partial f_x}{\partial r} E(w_r) + \frac{\partial f_x}{\partial u} E(w_u) \Big|_{true} \\ &= \boxed{r_t u_t} \end{aligned} \tag{B.4}$$

$$\begin{aligned} \text{var}(x_m) &= E[(x_m - x_t)^2] = E \left[\left(r_m u_m - \left(r_m u_m - \frac{\partial f_x}{\partial r} w_r - \frac{\partial f_x}{\partial u} w_u \right) \right)^2 \right] \Big|_{true} \\ &= E \left[\left(\frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u \right)^2 \right] \Big|_{true} \\ &= \left(\frac{\partial f_x}{\partial r} \right)^2 E(w_r^2) + \left(\frac{\partial f_x}{\partial u} \right)^2 E(w_u^2) + 2 \frac{\partial f_x}{\partial r} \frac{\partial f_x}{\partial u} E(w_r) E(w_u) \Big|_{true} \\ &= \boxed{u_t^2 \sigma_r^2 + r_t^2 \sigma_u^2} \end{aligned} \tag{B.5}$$

The transformation to coordinate y is similarly approximated as

$$\begin{aligned} y_m &= f_y[r_m, u_m, v_m] \triangleq r_m v_m \\ &\approx rv + \frac{\partial f_y}{\partial r} w_r + \frac{\partial f_y}{\partial v} w_v \Big|_{true} \end{aligned} \tag{B.6}$$

The mean and covariance of y_m is

$$\begin{aligned} E(y_m) &= rv + \frac{\partial f_y}{\partial r} E(w_r) + \frac{\partial f_y}{\partial v} E(w_v) \Big|_{true} \\ &= \boxed{r_t v_t} \end{aligned} \tag{B.7}$$

$$\begin{aligned} \text{var}(y_m) &= E[(y_m - y_t)^2] = E \left[\left(\frac{\partial f_y}{\partial r} w_r + \frac{\partial f_y}{\partial v} w_v \right)^2 \right] \Big|_{true} \\ &= \left(\frac{\partial f_y}{\partial r} \right)^2 E(w_r^2) + \left(\frac{\partial f_y}{\partial v} \right)^2 E(w_v^2) + 2 \frac{\partial f_y}{\partial r} \frac{\partial f_y}{\partial v} E(w_r) E(w_v) \Big|_{true} \\ &= \boxed{v_t^2 \sigma_r^2 + r_t^2 \sigma_v^2} \end{aligned} \tag{B.8}$$

The transformation to coordinate z is approximated as

$$\begin{aligned} z_m &= f_z[r_m, u_m, v_m] \triangleq r_m w_m \\ &\approx rw + \frac{\partial f_z}{\partial r} w_r + \frac{\partial f_z}{\partial u} w_u + \frac{\partial f_z}{\partial v} w_v \Big|_{true} \end{aligned} \tag{B.9}$$

The mean and covariance of z_m is

$$\begin{aligned} E(z_m) &= rw + \frac{\partial f_z}{\partial r} E(w_r) + \frac{\partial f_z}{\partial u} E(w_u) + \frac{\partial f_z}{\partial v} E(w_v) \Big|_{true} \\ &= \boxed{r_t w_t} \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned} \text{var}(z_m) &= E[(z_m - z_t)^2] = E \left[\left(\frac{\partial f_z}{\partial r} w_r + \frac{\partial f_z}{\partial u} w_u + \frac{\partial f_z}{\partial v} w_v \right)^2 \right] \Big|_{true} \\ &= \left(\frac{\partial f_z}{\partial r} \right)^2 E(w_r^2) + \left(\frac{\partial f_z}{\partial u} \right)^2 E(w_u^2) + \left(\frac{\partial f_z}{\partial v} \right)^2 E(w_v^2) \Big|_{true} \\ &= \boxed{w_t^2 \sigma_r^2 + \left(\frac{r_t u_t}{w_t} \right)^2 \sigma_u^2 + \left(\frac{r_t v_t}{w_t} \right)^2 \sigma_v^2} \end{aligned} \quad (\text{B.11})$$

The covariances between x_m , y_m , and z_m are

$$\begin{aligned} \text{cov}(x_m, y_m) &= E[(x_m - x_t)(y_m - y_t)] = E \left[\left(\frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u \right) \left(\frac{\partial f_y}{\partial r} w_r + \frac{\partial f_y}{\partial v} w_v \right) \right] \Big|_{true} \\ &= \frac{\partial f_x}{\partial r} \frac{\partial f_y}{\partial r} E(w_r^2) \Big|_{true} \\ &= \boxed{u_t v_t \sigma_r^2} \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} \text{cov}(x_m, z_m) &= E[(x_m - x_t)(z_m - z_t)] = E \left[\left(\frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u \right) \left(\frac{\partial f_z}{\partial r} w_r + \frac{\partial f_z}{\partial u} w_u + \frac{\partial f_z}{\partial v} w_v \right) \right] \Big|_{true} \\ &= \frac{\partial f_x}{\partial r} \frac{\partial f_z}{\partial r} E(w_r^2) + \frac{\partial f_x}{\partial u} \frac{\partial f_z}{\partial u} E(w_u^2) \Big|_{true} \\ &= \boxed{u_t w_t \sigma_r^2 - \frac{r_t^2 u_t}{w_t} \sigma_u^2} \end{aligned} \quad (\text{B.13})$$

$$\begin{aligned} \text{cov}(y_m, z_m) &= E[(y_m - y_t)(z_m - z_t)] = E \left[\left(\frac{\partial f_y}{\partial r} w_r + \frac{\partial f_y}{\partial v} w_v \right) \left(\frac{\partial f_z}{\partial r} w_r + \frac{\partial f_z}{\partial u} w_u + \frac{\partial f_z}{\partial v} w_v \right) \right] \Big|_{true} \\ &= \frac{\partial f_y}{\partial r} \frac{\partial f_z}{\partial r} E(w_r^2) + \frac{\partial f_y}{\partial v} \frac{\partial f_z}{\partial v} E(w_v^2) \Big|_{true} \\ &= \boxed{v_t w_t \sigma_r^2 - \frac{r_t^2 v_t}{w_t} \sigma_v^2} \end{aligned} \quad (\text{B.14})$$

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APPENDIX C: DERIVATION OF SECOND-ORDER COVARIANCE APPROXIMATION

The second-order Taylor series expansion assumes the derivatives of the nonlinear function $\mathbf{f}[\cdot]$ of order higher than two are negligible, and is given by

$$\begin{aligned} \mathbf{T}_2[r_m, u_m, v_m] &= \mathbf{f}[r_t, u_t, v_t] + \sum_{j=1}^3 \frac{\partial \mathbf{f}[r_t, u_t, v_t]}{\partial x_j} w_{x_j} + \frac{1}{2} \sum_{j=1}^3 \sum_{k=1}^3 \frac{\partial^2 \mathbf{f}[r_t, u_t, v_t]}{\partial x_j \partial x_k} w_{x_j} w_{x_k} \\ &= \mathbf{f} + \frac{\partial \mathbf{f}}{\partial r} w_r + \frac{\partial \mathbf{f}}{\partial u} w_u + \frac{\partial \mathbf{f}}{\partial v} w_v + \frac{1}{2} \frac{\partial^2 \mathbf{f}}{\partial r^2} w_r^2 + \frac{1}{2} \frac{\partial^2 \mathbf{f}}{\partial u^2} w_u^2 + \frac{1}{2} \frac{\partial^2 \mathbf{f}}{\partial v^2} w_v^2 \\ &\quad + \frac{\partial^2 \mathbf{f}}{\partial r \partial u} w_r w_u + \frac{\partial^2 \mathbf{f}}{\partial r \partial v} w_r w_v + \frac{\partial^2 \mathbf{f}}{\partial u \partial v} w_u w_v \Big|_{true} \end{aligned} \quad (\text{C.1})$$

where the partial derivatives required in this section, evaluated at truth, are

$$\begin{aligned} \frac{\partial f_x}{\partial r} &= u_t & \frac{\partial f_y}{\partial r} &= v_t & \frac{\partial f_z}{\partial r} &= w_t \\ \frac{\partial f_x}{\partial u} &= r_t & \frac{\partial f_y}{\partial u} &= r_t & \frac{\partial f_z}{\partial u} &= -\frac{r_t u_t}{w_t} \\ \frac{\partial^2 f_x}{\partial r \partial u} &= 1 & \frac{\partial^2 f_y}{\partial r \partial u} &= 1 & \frac{\partial f_z}{\partial v} &= -\frac{r_t v_t}{w_t} \\ & & & & \frac{\partial^2 f_z}{\partial u^2} &= -\frac{r_t}{w_t} - \frac{r_t u_t^2}{w_t^3} \\ & & & & \frac{\partial^2 f_z}{\partial v^2} &= -\frac{r_t}{w_t} - \frac{r_t v_t^2}{w_t^3} \\ & & & & \frac{\partial^2 f_z}{\partial r \partial u} &= -\frac{u_t}{w_t} \\ & & & & \frac{\partial^2 f_z}{\partial r \partial v} &= -\frac{v_t}{w_t} \\ & & & & \frac{\partial^2 f_z}{\partial u \partial v} &= -\frac{r_t u_t v_t}{w_t^3} \end{aligned} \quad (\text{C.2})$$

and $w_t \triangleq \sqrt{1 - u_t^2 - v_t^2}$.

The transformation to coordinate x is approximated as⁹

$$\begin{aligned} x_m &= f_x[r_m, u_m, v_m] \triangleq r_m u_m \\ &\approx r u + \frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u + \frac{\partial^2 f_x}{\partial r \partial u} w_r w_u \Big|_{true} \end{aligned} \quad (\text{C.3})$$

⁹While there is no closed form solution for the mean and variance of z_m (it must be approximated via Taylor series expansion), the mean and variances of x_m and y_m (and their cross covariance) found via Taylor series expansion can be compared against the squared error averaged *conditioned on the observations* technique from [2, 3, 16, 17]. For the case of the second-order linearization, the two techniques produce the same expressions for $E(x_m)$, $E(y_m)$, $\text{var}(x_m)$, $\text{var}(y_m)$, and $\text{cov}(x_m, y_m)$.

The mean and covariance of x_m is

$$E(x_m) = ru + \frac{\partial f_x}{\partial r} E(w_r) + \frac{\partial f_x}{\partial u} E(w_u) + \frac{\partial^2 f_x}{\partial r \partial u} E(w_r) E(w_u) \Big|_{true}$$

$$= \boxed{r_t u_t} \quad (C.4)$$

$$\text{var}(x_m) = E[(x_m - x_t)^2] = E \left[\left(r_m u_m - \left(r_m u_m - \frac{\partial f_x}{\partial r} w_r - \frac{\partial f_x}{\partial u} w_u - \frac{\partial^2 f_x}{\partial r \partial u} w_r w_u \right) \right)^2 \right] \Big|_{true}$$

$$= E \left[\left(\frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u + \frac{\partial^2 f_x}{\partial r \partial u} w_r w_u \right)^2 \right] \Big|_{true}$$

$$= \left(\frac{\partial f_x}{\partial r} \right)^2 E(w_r^2) + \left(\frac{\partial f_x}{\partial u} \right)^2 E(w_u^2) + \left(\frac{\partial^2 f_x}{\partial r \partial u} \right)^2 E(w_r^2) E(w_u^2) \Big|_{true}$$

$$= \boxed{u_t^2 \sigma_r^2 + r_t^2 \sigma_u^2 + \sigma_r^2 \sigma_u^2} \quad (C.5)$$

The transformation to coordinate y is similarly approximated as

$$y_m = f_y[r_m, u_m, v_m] \triangleq r_m v_m$$

$$\approx rv + \frac{\partial f_y}{\partial r} w_r + \frac{\partial f_y}{\partial v} w_v + \frac{\partial^2 f_y}{\partial r \partial v} w_r w_v \Big|_{true} \quad (C.6)$$

The mean and covariance of y_m is

$$E(y_m) = rv + \frac{\partial f_y}{\partial r} E(w_r) + \frac{\partial f_y}{\partial v} E(w_v) + \frac{\partial^2 f_y}{\partial r \partial v} E(w_r) E(w_v) \Big|_{true}$$

$$= \boxed{r_t v_t} \quad (C.7)$$

$$\text{var}(y_m) = E[(y_m - y_t)^2] = E \left[\left(\frac{\partial f_y}{\partial r} w_r + \frac{\partial f_y}{\partial v} w_v + \frac{\partial^2 f_y}{\partial r \partial v} w_r w_v \right)^2 \right] \Big|_{true}$$

$$= \left(\frac{\partial f_y}{\partial r} \right)^2 E(w_r^2) + \left(\frac{\partial f_y}{\partial v} \right)^2 E(w_v^2) + \left(\frac{\partial^2 f_y}{\partial r \partial v} \right)^2 E(w_r^2) E(w_v^2) \Big|_{true}$$

$$= \boxed{v_t^2 \sigma_r^2 + r_t^2 \sigma_v^2 + \sigma_r^2 \sigma_v^2} \quad (C.8)$$

As we will see below, the second-order linearization of the z -coordinate is biased by default, however we can debias it rather easily. Let us first calculate $\tilde{z}$ in order to determine the bias term.

$$\tilde{z}_m = f_{\tilde{z}}[r_m, u_m, v_m] \triangleq r_m w_m$$

$$\approx rw + \frac{\partial f_{\tilde{z}}}{\partial r} w_r + \frac{\partial f_{\tilde{z}}}{\partial u} w_u + \frac{\partial f_{\tilde{z}}}{\partial v} w_v + \frac{1}{2} \frac{\partial^2 f_{\tilde{z}}}{\partial u^2} w_u^2 + \frac{1}{2} \frac{\partial^2 f_{\tilde{z}}}{\partial v^2} w_v^2$$

$$+ \frac{\partial^2 f_{\tilde{z}}}{\partial r \partial u} w_r w_u + \frac{\partial^2 f_{\tilde{z}}}{\partial r \partial v} w_r w_v + \frac{\partial^2 f_{\tilde{z}}}{\partial u \partial v} w_u w_v \Big|_{true} \quad (C.9)$$

Then the mean of $\tilde{z}_m$ is

$$\begin{aligned} E(\tilde{z}_m) &= rw + \frac{1}{2} \frac{\partial^2 f_{\tilde{z}}}{\partial u^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 f_{\tilde{z}}}{\partial v^2} \sigma_v^2 \Big|_{true} \\ &= r_t w_t + \left(\frac{1}{2} \frac{\partial^2 f_{\tilde{z}}}{\partial u^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 f_{\tilde{z}}}{\partial v^2} \sigma_v^2 \right) \Big|_{true} \end{aligned} \quad (C.10)$$

and we can define the bias term as

$$\begin{aligned} c_z &\triangleq \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} \sigma_v^2 \Big|_{true} \\ &= -\frac{1}{2} \left(\frac{r_t}{w_t} + \frac{r_t u_t^2}{w_t^3} \right) \sigma_u^2 - \frac{1}{2} \left(\frac{r_t}{w_t} + \frac{r_t v_t^2}{w_t^3} \right) \sigma_v^2 \end{aligned} \quad (C.11)$$

In order to produce an unbiased conversion in z , we must subtract off the bias term. Our unbiased transformation to coordinate z is approximated as

$$\begin{aligned} z_m &= f_z[r_m, u_m, v_m] \triangleq r_m w_m - c_z, \\ &\approx rw - c_z + \frac{\partial f_z}{\partial r} w_r + \frac{\partial f_z}{\partial u} w_u + \frac{\partial f_z}{\partial v} w_v + \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} w_u^2 + \frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} w_v^2 \\ &\quad + \frac{\partial^2 f_z}{\partial r \partial u} w_r w_u + \frac{\partial^2 f_z}{\partial r \partial v} w_r w_v + \frac{\partial^2 f_z}{\partial u \partial v} w_u w_v \Big|_{true} \end{aligned} \quad (C.12)$$

Note that the bias term, c_z , is considered a constant parameter, such that its partial derivative with respect to any of the measurement variables is zero. The mean and covariance of the unbiased conversion to z_m is therefore

$$\begin{aligned} E(z_m) &= rw - c_z + \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} \sigma_v^2 \Big|_{true} \\ &= \boxed{r_t w_t} \end{aligned} \quad (C.13)$$

$$\begin{aligned} \text{var}(z_m) &= E[(z_m - z_t)^2] \\ &= E \left[\left(-c_z + \frac{\partial f_z}{\partial r} w_r + \frac{\partial f_z}{\partial u} w_u + \frac{\partial f_z}{\partial v} w_v + \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} w_u^2 + \frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} w_v^2 \right. \right. \\ &\quad \left. \left. + \frac{\partial^2 f_z}{\partial r \partial u} w_r w_u + \frac{\partial^2 f_z}{\partial r \partial v} w_r w_v + \frac{\partial^2 f_z}{\partial u \partial v} w_u w_v \right)^2 \right] \Big|_{true} \\ &= c_z^2 - c_z \frac{\partial^2 f_z}{\partial u^2} \sigma_u^2 - c_z \frac{\partial^2 f_z}{\partial v^2} \sigma_v^2 + \left(\frac{\partial f_z}{\partial r} \right)^2 \sigma_r^2 + \left(\frac{\partial f_z}{\partial u} \right)^2 \sigma_u^2 + \frac{\partial f_z}{\partial u} \frac{\partial^2 f_z}{\partial u^2} E(w_u^3) \\ &\quad + \left(\frac{\partial f_z}{\partial v} \right)^2 \sigma_v^2 + \frac{\partial f_z}{\partial v} \frac{\partial^2 f_z}{\partial v^2} E(w_v^3) + \left(\frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} \right)^2 E(w_u^4) + \left(\frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} \right)^2 E(w_v^4) \\ &\quad + \left(\frac{\partial^2 f_z}{\partial r \partial u} \right)^2 \sigma_r^2 \sigma_u^2 + \left(\frac{\partial^2 f_z}{\partial r \partial v} \right)^2 \sigma_r^2 \sigma_v^2 + \left(\frac{\partial^2 f_z}{\partial u \partial v} \right)^2 \sigma_u^2 \sigma_v^2 \Big|_{true} \end{aligned} \quad (C.14)$$

Rearranging terms and remembering that for a Gaussian random variable, a , $E(a^3) = 0$ and $E(a^4) = 3\sigma_a^4$

$$\begin{aligned} \text{var}(z_m) = & c_z^2 - c_z \left(\frac{\partial^2 f_z}{\partial u^2} \sigma_u^2 + \frac{\partial^2 f_z}{\partial v^2} \sigma_v^2 \right) + \frac{3}{4} \left(\frac{\partial^2 f_z}{\partial u^2} \right)^2 \sigma_u^4 + \frac{3}{4} \left(\frac{\partial^2 f_z}{\partial v^2} \right)^2 \sigma_v^4 \\ & + \left(\frac{\partial^2 f_z}{\partial u \partial v} \right)^2 \sigma_u^2 \sigma_v^2 + \left(\frac{\partial f_z}{\partial r} \right)^2 \sigma_r^2 + \left(\frac{\partial f_z}{\partial u} \right)^2 \sigma_u^2 + \left(\frac{\partial f_z}{\partial v} \right)^2 \sigma_v^2 \\ & + \left(\frac{\partial^2 f_z}{\partial r \partial u} \right)^2 \sigma_r^2 \sigma_u^2 + \left(\frac{\partial^2 f_z}{\partial r \partial v} \right)^2 \sigma_r^2 \sigma_v^2 \Big|_{\text{true}} \end{aligned} \quad (\text{C.15})$$

Substituting the partial derivative terms (C.2) and bias correction term (C.11) into (C.15), and simplifying gives the expression for the variance of z

$$\begin{aligned} \text{var}(z_m) = & \boxed{\begin{aligned} & w_t^2 \sigma_r^2 + \frac{r_t^2}{w_t^2} (u_t^2 \sigma_u^2 + v_t^2 \sigma_v^2) + \frac{u_t^2}{w_t^2} \sigma_r^2 \sigma_u^2 + \frac{v_t^2}{w_t^2} \sigma_r^2 \sigma_v^2 \\ & + \frac{r_t^2}{2} \left(\frac{u_t^2 v_t^2}{w_t^6} - \frac{u_t^2 + v_t^2}{w_t^4} - \frac{1}{w_t^2} \right) \sigma_u^2 \sigma_v^2 \\ & + \frac{r_t^2}{2} \left(\frac{u_t^4}{w_t^6} + 2 \frac{u_t^2}{w_t^4} + \frac{1}{w_t^2} \right) \sigma_u^4 + \frac{r_t^2}{2} \left(\frac{v_t^4}{w_t^6} + 2 \frac{v_t^2}{w_t^4} + \frac{1}{w_t^2} \right) \sigma_v^4 \end{aligned}} \end{aligned} \quad (\text{C.16})$$

The covariances between x_m , y_m , and z_m are

$$\begin{aligned} \text{cov}(x_m, y_m) = & E[(x_m - x_t)(y_m - y_t)] \\ = & E \left[\left(\frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u + \frac{\partial^2 f_x}{\partial r \partial u} w_r w_u \right) \left(\frac{\partial f_y}{\partial r} w_r + \frac{\partial f_y}{\partial v} w_v + \frac{\partial^2 f_y}{\partial r \partial v} w_r w_v \right) \right] \Big|_{\text{true}} \\ = & \frac{\partial f_x}{\partial r} \frac{\partial f_y}{\partial r} E(w_r^2) \Big|_{\text{true}} \\ = & \boxed{u_t v_t \sigma_r^2} \end{aligned} \quad (\text{C.17})$$

$$\begin{aligned} \text{cov}(x_m, z_m) = & E[(x_m - x_t)(z_m - z_t)] \\ = & E \left[\left(\frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u + \frac{\partial^2 f_x}{\partial r \partial u} w_r w_u \right) \left(\frac{\partial f_z}{\partial r} w_r + \frac{\partial f_z}{\partial u} w_u + \frac{\partial f_z}{\partial v} w_v \right. \right. \\ & \left. \left. + \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} w_u^2 + \frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} w_v^2 + \frac{\partial^2 f_z}{\partial r \partial u} w_r w_u + \frac{\partial^2 f_z}{\partial r \partial v} w_r w_v + \frac{\partial^2 f_z}{\partial u \partial v} w_u w_v \right) \right] \Big|_{\text{true}} \\ = & \frac{\partial f_x}{\partial r} \frac{\partial f_z}{\partial r} E(w_r^2) + \frac{\partial f_x}{\partial u} \frac{\partial f_z}{\partial u} E(w_u^2) + \frac{1}{2} \frac{\partial f_x}{\partial u} \frac{\partial^2 f_z}{\partial u^2} E(w_u^3) + \frac{\partial^2 f_x}{\partial r \partial u} \frac{\partial^2 f_z}{\partial r \partial u} E(w_r^2) E(w_u^2) \Big|_{\text{true}} \\ = & \boxed{u_t w_t \sigma_r^2 - \frac{r_t^2 u_t}{w_t} \sigma_u^2 - \frac{u_t}{w_t} \sigma_r^2 \sigma_u^2} \end{aligned} \quad (\text{C.18})$$

$$\begin{aligned}
\text{cov}(y_m, z_m) &= E[(y_m - y_t)(z_m - z_t)] \\
&= E \left[\left(\frac{\partial f_y}{\partial r} w_r + \frac{\partial f_y}{\partial v} w_v + \frac{\partial^2 f_y}{\partial r \partial v} w_r w_v \right) \left(\frac{\partial f_z}{\partial r} w_r + \frac{\partial f_z}{\partial u} w_u + \frac{\partial f_z}{\partial v} w_v \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} w_u^2 + \frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} w_v^2 + \frac{\partial^2 f_z}{\partial r \partial u} w_r w_u + \frac{\partial^2 f_z}{\partial r \partial v} w_r w_v + \frac{\partial^2 f_z}{\partial u \partial v} w_u w_v \right) \right] \Big|_{true} \\
&= \frac{\partial f_y}{\partial r} \frac{\partial f_z}{\partial r} E(w_r^2) + \frac{\partial f_y}{\partial v} \frac{\partial f_z}{\partial v} E(w_v^2) + \frac{1}{2} \frac{\partial f_y}{\partial v} \frac{\partial^2 f_z}{\partial v^2} E(w_v^3) + \frac{\partial^2 f_y}{\partial r \partial v} \frac{\partial^2 f_z}{\partial r \partial v} E(w_r^2) E(w_v^2) \Big|_{true} \\
&= \boxed{v_t w_t \sigma_r^2 - \frac{r_t^2 v_t}{w_t} \sigma_v^2 - \frac{v_t}{w_t} \sigma_r^2 \sigma_v^2} \tag{C.19}
\end{aligned}$$

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APPENDIX D: DERIVATION OF THIRD-ORDER COVARIANCE APPROXIMATION

The third-order Taylor series expansion assumes the derivatives of the nonlinear function $\mathbf{f}[\cdot]$ of order higher than three are negligible, and is given by

$$\begin{aligned}
\mathbf{T}_3[r_m, u_m, v_m] &= \mathbf{f}[r_t, u_t, v_t] + \sum_{j=1}^3 \frac{\partial \mathbf{f}[r_t, u_t, v_t]}{\partial x_j} w_{x_j} + \frac{1}{2} \sum_{j=1}^3 \sum_{k=1}^3 \frac{\partial^2 \mathbf{f}[r_t, u_t, v_t]}{\partial x_j \partial x_k} w_{x_j} w_{x_k} \\
&\quad + \frac{1}{6} \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \frac{\partial^3 \mathbf{f}[r_t, u_t, v_t]}{\partial x_j \partial x_k \partial x_m} w_{x_j} w_{x_k} w_{x_m} \\
&= \mathbf{f} + \frac{\partial \mathbf{f}}{\partial r} w_r + \frac{\partial \mathbf{f}}{\partial u} w_u + \frac{\partial \mathbf{f}}{\partial v} w_v + \frac{1}{2} \frac{\partial^2 \mathbf{f}}{\partial r^2} w_r^2 + \frac{1}{2} \frac{\partial^2 \mathbf{f}}{\partial u^2} w_u^2 + \frac{1}{2} \frac{\partial^2 \mathbf{f}}{\partial v^2} w_v^2 \\
&\quad + \frac{\partial^2 \mathbf{f}}{\partial r \partial u} w_r w_u + \frac{\partial^2 \mathbf{f}}{\partial r \partial v} w_r w_v + \frac{\partial^2 \mathbf{f}}{\partial u \partial v} w_u w_v \\
&\quad + \frac{1}{6} \frac{\partial^3 \mathbf{f}}{\partial r^3} w_r^3 + \frac{1}{6} \frac{\partial^3 \mathbf{f}}{\partial u^3} w_u^3 + \frac{1}{6} \frac{\partial^3 \mathbf{f}}{\partial v^3} w_v^3 \\
&\quad + \frac{1}{2} \frac{\partial^3 \mathbf{f}}{\partial r^2 \partial u} w_r^2 w_u + \frac{1}{2} \frac{\partial^3 \mathbf{f}}{\partial r^2 \partial v} w_r^2 w_v + \frac{1}{2} \frac{\partial^3 \mathbf{f}}{\partial r \partial u^2} w_r w_u^2 + \frac{1}{2} \frac{\partial^3 \mathbf{f}}{\partial r \partial v^2} w_r w_v^2 \\
&\quad + \frac{1}{2} \frac{\partial^3 \mathbf{f}}{\partial u^2 \partial v} w_u^2 w_v + \frac{1}{2} \frac{\partial^3 \mathbf{f}}{\partial u \partial v^2} w_u w_v^2 + \left. \frac{\partial^3 \mathbf{f}}{\partial r \partial u \partial v} w_r w_u w_v \right|_{true} \tag{D.1}
\end{aligned}$$

where the partial derivatives required in this section, evaluated at truth, are

$$\begin{aligned}
\frac{\partial f_x}{\partial r} &= u_t & \frac{\partial f_y}{\partial r} &= v_t & \frac{\partial f_z}{\partial r} &= w_t & \frac{\partial^3 f_z}{\partial u^3} &= -3 \frac{r_t u_t}{w_t^3} - 3 \frac{r_t u_t^3}{w_t^5} \\
\frac{\partial f_x}{\partial u} &= r_t & \frac{\partial f_y}{\partial v} &= r_t & \frac{\partial f_z}{\partial u} &= -\frac{r_t u_t}{w_t} & \frac{\partial^3 f_z}{\partial v^3} &= -3 \frac{r_t v_t}{w_t^3} - 3 \frac{r_t v_t^3}{w_t^5} \\
\frac{\partial^2 f_x}{\partial r \partial u} &= 1 & \frac{\partial^2 f_y}{\partial r \partial v} &= 1 & \frac{\partial f_z}{\partial v} &= -\frac{r_t v_t}{w_t} & \frac{\partial^3 f_z}{\partial r \partial u^2} &= -\frac{1}{w_t} - \frac{u_t^2}{w_t^3} \\
&&&&& \frac{\partial^2 f_z}{\partial u^2} &= -\frac{r_t}{w_t} - \frac{r_t u_t^2}{w_t^3} & \frac{\partial^3 f_z}{\partial r \partial v^2} &= -\frac{1}{w_t} - \frac{v_t^2}{w_t^3} \\
&&&&& \frac{\partial^2 f_z}{\partial v^2} &= -\frac{r_t}{w_t} - \frac{r_t v_t^2}{w_t^3} & \frac{\partial^3 f_z}{\partial u^2 \partial v} &= -\frac{r_t v_t}{w_t^3} - 3 \frac{r_t u_t^2 v_t}{w_t^5} \\
&&&&& \frac{\partial^2 f_z}{\partial r \partial u} &= -\frac{u_t}{w_t} & \frac{\partial^3 f_z}{\partial u \partial v^2} &= -\frac{r_t u_t}{w_t^3} - 3 \frac{r_t u_t v_t^2}{w_t^5} \\
&&&&& \frac{\partial^2 f_z}{\partial r \partial v} &= -\frac{v_t}{w_t} & \frac{\partial^3 f_z}{\partial r \partial u \partial v} &= -\frac{u_t v_t}{w_t^3} \\
&&&&& \frac{\partial^2 f_z}{\partial u \partial v} &= -\frac{r_t u_t v_t}{w_t^3}
\end{aligned} \tag{D.2}$$

and $w_t \triangleq \sqrt{1 - u_t^2 - v_t^2}$.

After canceling terms whose partial derivatives are zero, the transformations to coordinates x and y are identical to the second-order linearization, and are given by

$$\begin{aligned} x_m &= f_x[r_m, u_m, v_m] \triangleq r_m u_m \\ &\approx ru + \frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u + \frac{\partial^2 f_x}{\partial r \partial u} w_r w_u \Big|_{true} \end{aligned} \quad (D.3)$$

$$\begin{aligned} y_m &= f_y[r_m, u_m, v_m] \triangleq r_m v_m \\ &\approx rv + \frac{\partial f_y}{\partial r} w_r + \frac{\partial f_y}{\partial v} w_v + \frac{\partial^2 f_y}{\partial r \partial v} w_r w_v \Big|_{true} \end{aligned} \quad (D.4)$$

The mean and covariance of x_m and y_m were found in the previous section to be

$$E(x_m) = \boxed{r_t u_t} \quad (D.5)$$

$$\text{var}(x_m) = \boxed{u_t^2 \sigma_r^2 + r_t^2 \sigma_u^2 + \sigma_r^2 \sigma_u^2} \quad (D.6)$$

$$E(y_m) = \boxed{r_t v_t} \quad (D.7)$$

$$\text{var}(y_m) = \boxed{v_t^2 \sigma_r^2 + r_t^2 \sigma_v^2 + \sigma_r^2 \sigma_v^2} \quad (D.8)$$

The z -coordinate is again biased, so we will compensate using the same method as the second-order linearization and confirm that the resulting random variable is unbiased. The transformation to coordinate z is approximated as

$$\begin{aligned} z_m &= f_z[r_m, u_m, v_m] \triangleq r_m w_m - c_z \\ &\approx rw - c_z + \frac{\partial f_z}{\partial r} w_r + \frac{\partial f_z}{\partial u} w_u + \frac{\partial f_z}{\partial v} w_v + \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} w_u^2 + \frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} w_v^2 \\ &\quad + \frac{\partial^2 f_z}{\partial r \partial u} w_r w_u + \frac{\partial^2 f_z}{\partial r \partial v} w_r w_v + \frac{\partial^2 f_z}{\partial u \partial v} w_u w_v + \frac{1}{6} \frac{\partial^3 f_z}{\partial u^3} w_u^3 + \frac{1}{6} \frac{\partial^3 f_z}{\partial v^3} w_v^3 \\ &\quad + \frac{1}{2} \frac{\partial^3 f_z}{\partial r \partial u^2} w_r w_u^2 + \frac{1}{2} \frac{\partial^3 f_z}{\partial r \partial v^2} w_r w_v^2 + \frac{1}{2} \frac{\partial^3 f_z}{\partial u^2 \partial v} w_u^2 w_v + \frac{1}{2} \frac{\partial^3 f_z}{\partial u \partial v^2} w_u w_v^2 \\ &\quad + \frac{\partial^3 f_z}{\partial r \partial u \partial v} w_r w_u w_v \Big|_{true} \end{aligned} \quad (D.9)$$

where again we define the bias term as

$$\begin{aligned} c_z &\triangleq \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} \sigma_v^2 \Big|_{true} \\ &= -\frac{1}{2} \left(\frac{r_t}{w_t} + \frac{r_t u_t^2}{w_t^3} \right) \sigma_u^2 - \frac{1}{2} \left(\frac{r_t}{w_t} + \frac{r_t v_t^2}{w_t^3} \right) \sigma_v^2 \end{aligned} \quad (D.10)$$

Note that the bias term, c_z , is considered a constant parameter, such that its partial derivative with respect to any of the measurement variables is zero. The mean and covariance of the unbiased conversion to z_m is therefore

$$E(z_m) = rw - c_z + \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} \sigma_v^2 \Big|_{true} = \boxed{r_t w_t} \quad (D.11)$$

$$\begin{aligned} \text{var}(z_m) &= E[(z_m - z_t)^2] \\ &= c_z^2 - c_z \frac{\partial^2 f_z}{\partial u^2} \sigma_u^2 - c_z \frac{\partial^2 f_z}{\partial v^2} \sigma_v^2 + \left(\frac{\partial f_z}{\partial r} \right)^2 \sigma_r^2 + \frac{\partial f_z}{\partial r} \frac{\partial^3 f_z}{\partial r \partial u^2} \sigma_r^2 \sigma_u^2 + \frac{\partial f_z}{\partial r} \frac{\partial^3 f_z}{\partial r \partial v^2} \sigma_r^2 \sigma_v^2 \\ &\quad + \left(\frac{\partial f_z}{\partial u} \right)^2 \sigma_u^2 + \frac{1}{3} \frac{\partial f_z}{\partial u} \frac{\partial^3 f_z}{\partial u^3} E(w_u^4) + \frac{\partial f_z}{\partial u} \frac{\partial^3 f_z}{\partial u \partial v^2} \sigma_u^2 \sigma_v^2 + \left(\frac{\partial f_z}{\partial v} \right)^2 \sigma_v^2 + \frac{1}{3} \frac{\partial f_z}{\partial v} \frac{\partial^3 f_z}{\partial v^3} E(w_v^4) \\ &\quad + \frac{\partial f_z}{\partial v} \frac{\partial^3 f_z}{\partial u^2 \partial v} \sigma_u^2 \sigma_v^2 + \left(\frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} \right)^2 E(w_u^4) + \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} \frac{\partial^2 f_z}{\partial v^2} \sigma_u^2 \sigma_v^2 + \left(\frac{1}{2} \frac{\partial^2 f_z}{\partial v^2} \right)^2 E(w_v^4) \\ &\quad + \left(\frac{\partial^2 f_z}{\partial r \partial u} \right)^2 \sigma_r^2 \sigma_u^2 + \left(\frac{\partial^2 f_z}{\partial r \partial v} \right)^2 \sigma_r^2 \sigma_v^2 + \left(\frac{\partial^2 f_z}{\partial u \partial v} \right)^2 \sigma_u^2 \sigma_v^2 + \left(\frac{1}{6} \frac{\partial^3 f_z}{\partial u^3} \right)^2 E(w_u^6) \\ &\quad + \frac{1}{6} \frac{\partial^3 f_z}{\partial u^3} \frac{\partial^3 f_z}{\partial u \partial v^2} E(w_u^4) \sigma_v^2 + \left(\frac{1}{6} \frac{\partial^3 f_z}{\partial v^3} \right)^2 E(w_v^6) + \frac{1}{6} \frac{\partial^3 f_z}{\partial v^3} \frac{\partial^3 f_z}{\partial u^2 \partial v} \sigma_u^2 E(w_v^4) \\ &\quad + \left(\frac{1}{2} \frac{\partial^3 f_z}{\partial r \partial u^2} \right)^2 \sigma_r^2 E(w_u^4) + \frac{1}{2} \frac{\partial^3 f_z}{\partial r \partial u^2} \frac{\partial^3 f_z}{\partial r \partial v^2} \sigma_r^2 \sigma_u^2 \sigma_v^2 + \left(\frac{1}{2} \frac{\partial^3 f_z}{\partial r \partial v^2} \right)^2 \sigma_r^2 E(w_v^4) \\ &\quad + \left(\frac{1}{2} \frac{\partial^3 f_z}{\partial u^2 \partial v} \right)^2 E(w_u^4) \sigma_v^2 + \left(\frac{1}{2} \frac{\partial^3 f_z}{\partial u \partial v^2} \right)^2 \sigma_u^2 E(w_v^4) + \left(\frac{\partial^3 f_z}{\partial r \partial u \partial v} \right)^2 \sigma_r^2 \sigma_u^2 \sigma_v^2 \Big|_{true} \quad (D.12) \end{aligned}$$

Rearranging terms and remembering that for a Gaussian random variable, a , $E(a^4) = 3\sigma_a^4$ and $E(a^6) = 15\sigma_a^6$

$$\begin{aligned} \text{var}(z_m) &= c_z^2 + \left(\frac{\partial f_z}{\partial r} \right)^2 \sigma_r^2 + \left(\frac{\partial f_z}{\partial u} \right)^2 \sigma_u^2 + \left(\frac{\partial f_z}{\partial v} \right)^2 \sigma_v^2 - c_z \left(\frac{\partial^2 f_z}{\partial u^2} \sigma_u^2 + \frac{\partial^2 f_z}{\partial v^2} \sigma_v^2 \right) \\ &\quad + \frac{\partial f_z}{\partial r} \frac{\partial^3 f_z}{\partial r \partial u^2} \sigma_r^2 \sigma_u^2 + \frac{\partial f_z}{\partial r} \frac{\partial^3 f_z}{\partial r \partial v^2} \sigma_r^2 \sigma_v^2 + \frac{\partial f_z}{\partial u} \frac{\partial^3 f_z}{\partial u^3} \sigma_u^4 + \frac{\partial f_z}{\partial v} \frac{\partial^3 f_z}{\partial v^3} \sigma_v^4 \\ &\quad + \frac{\partial f_z}{\partial u} \frac{\partial^3 f_z}{\partial u \partial v^2} \sigma_u^2 \sigma_v^2 + \frac{\partial f_z}{\partial v} \frac{\partial^3 f_z}{\partial u^2 \partial v} \sigma_u^2 \sigma_v^2 + \left(\frac{\partial^2 f_z}{\partial r \partial u} \right)^2 \sigma_r^2 \sigma_u^2 + \left(\frac{\partial^2 f_z}{\partial r \partial v} \right)^2 \sigma_r^2 \sigma_v^2 \\ &\quad + \left(\frac{\partial^2 f_z}{\partial u \partial v} \right)^2 \sigma_u^2 \sigma_v^2 + \left(\frac{\partial^3 f_z}{\partial r \partial u \partial v} \right)^2 \sigma_r^2 \sigma_u^2 \sigma_v^2 + \frac{3}{4} \left(\frac{\partial^2 f_z}{\partial u^2} \right)^2 \sigma_u^4 + \frac{3}{4} \left(\frac{\partial^2 f_z}{\partial v^2} \right)^2 \sigma_v^4 \\ &\quad + \frac{3}{4} \left(\frac{\partial^3 f_z}{\partial r \partial u^2} \right)^2 \sigma_r^2 \sigma_u^4 + \frac{3}{4} \left(\frac{\partial^3 f_z}{\partial r \partial v^2} \right)^2 \sigma_r^2 \sigma_v^4 + \frac{3}{4} \left(\frac{\partial^3 f_z}{\partial u^2 \partial v} \right)^2 \sigma_u^4 \sigma_v^2 + \frac{3}{4} \left(\frac{\partial^3 f_z}{\partial u \partial v^2} \right)^2 \sigma_u^2 \sigma_v^4 \\ &\quad + \frac{1}{2} \frac{\partial^2 f_z}{\partial u^2} \frac{\partial^2 f_z}{\partial v^2} \sigma_u^2 \sigma_v^2 + \frac{1}{2} \frac{\partial^3 f_z}{\partial u^3} \frac{\partial^3 f_z}{\partial u \partial v^2} \sigma_u^4 \sigma_v^2 + \frac{1}{2} \frac{\partial^3 f_z}{\partial v^3} \frac{\partial^3 f_z}{\partial u^2 \partial v} \sigma_u^2 \sigma_v^4 + \frac{1}{2} \frac{\partial^3 f_z}{\partial r \partial u^2} \frac{\partial^3 f_z}{\partial r \partial v^2} \sigma_r^2 \sigma_u^2 \sigma_v^2 \\ &\quad + \frac{15}{36} \left(\frac{\partial^3 f_z}{\partial u^3} \right)^2 \sigma_u^6 + \frac{15}{36} \left(\frac{\partial^3 f_z}{\partial v^3} \right)^2 \sigma_v^6 \Big|_{true} \quad (D.13) \end{aligned}$$

Substituting the partial derivative terms (D.2) and bias correction term (D.10) into (D.13), and simplifying gives the expression for the variance of z

$$\begin{aligned}
\text{var}(z_m) = & \left[w_t^2 \sigma_r^2 + \frac{r_t^2}{w_t^2} (u_t^2 \sigma_u^2 + v_t^2 \sigma_v^2) - \sigma_r^2 \sigma_u^2 - \sigma_r^2 \sigma_v^2 + \left(7 \frac{r_t^2 u_t^2 v_t^2}{w_t^6} + \frac{r_t^2 (u_t^2 + v_t^2)}{w_t^4} \right) \sigma_u^2 \sigma_v^2 \right. \\
& + \frac{1}{2} \left(7 \frac{r_t^2 u_t^4}{w_t^6} + 8 \frac{r_t^2 u_t^2}{w_t^4} + \frac{r_t^2}{w_t^2} \right) \sigma_u^4 + \frac{1}{2} \left(7 \frac{r_t^2 v_t^4}{w_t^6} + 8 \frac{r_t^2 v_t^2}{w_t^4} + \frac{r_t^2}{w_t^2} \right) \sigma_v^4 \\
& + \frac{3}{4} \left(\frac{u_t^4}{w_t^6} + 2 \frac{u_t^2}{w_t^4} + \frac{1}{w_t^2} \right) \sigma_r^2 \sigma_u^4 + \frac{3}{4} \left(\frac{v_t^4}{w_t^6} + 2 \frac{v_t^2}{w_t^4} + \frac{1}{w_t^2} \right) \sigma_r^2 \sigma_v^4 \\
& + \frac{1}{4} \left(45 \frac{r_t^2 u_t^4 v_t^2}{w_t^{10}} + \frac{36 r_t^2 u_t^2 v_t^2}{w_t^8} + 6 \frac{r_t^2 u_t^4}{w_t^6} + \frac{6 r_t^2 u_t^2 + 3 r_t^2 v_t^2}{w_t^6} \right) \sigma_u^4 \sigma_v^2 \\
& + \frac{1}{4} \left(45 \frac{r_t^2 u_t^2 v_t^4}{w_t^{10}} + \frac{36 r_t^2 u_t^2 v_t^2}{w_t^8} + 6 \frac{r_t^2 v_t^4}{w_t^6} + \frac{6 r_t^2 v_t^2 + 3 r_t^2 u_t^2}{w_t^6} \right) \sigma_u^2 \sigma_v^4 \\
& + \frac{15}{4} \left(\frac{r_t^2 u_t^6}{w_t^{10}} + 2 \frac{r_t^2 u_t^4}{w_t^8} + \frac{r_t^2 u_t^2}{w_t^6} \right) \sigma_u^6 + \frac{15}{4} \left(\frac{r_t^2 v_t^6}{w_t^{10}} + 2 \frac{r_t^2 v_t^4}{w_t^8} + \frac{r_t^2 v_t^2}{w_t^6} \right) \sigma_v^6 \\
& \left. + \frac{1}{2} \left(3 \frac{u_t^2 v_t^2}{w_t^6} + \frac{u_t^2 + v_t^2}{w_t^4} + \frac{1}{w_t^2} \right) \sigma_r^2 \sigma_u^2 \sigma_v^2 \right]
\end{aligned}
\tag{D.14}$$

The covariances between x_m , y_m , and z_m are

$$\begin{aligned}
\text{cov}(x_m, y_m) &= E[(x_m - x_t)(y_m - y_t)] \\
&= E \left[\left(\frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u + \frac{\partial^2 f_x}{\partial r \partial u} w_r w_u \right) \left(\frac{\partial f_y}{\partial r} w_r + \frac{\partial f_y}{\partial v} w_v + \frac{\partial^2 f_y}{\partial r \partial v} w_r w_v \right) \right] \Big|_{\text{true}} \\
&= \frac{\partial f_x}{\partial r} \frac{\partial f_y}{\partial r} E(w_r^2) \Big|_{\text{true}} \\
&= \boxed{u_t v_t \sigma_r^2}
\end{aligned}
\tag{D.15}$$

$$\begin{aligned}
\text{cov}(x_m, z_m) &= E[(x_m - x_t)(z_m - z_t)] = E \left[\left(\frac{\partial f_x}{\partial r} w_r + \frac{\partial f_x}{\partial u} w_u + \frac{\partial^2 f_x}{\partial r \partial u} w_r w_u \right) \left(\dots \right) \right] \\
&= \frac{\partial f_x}{\partial r} \frac{\partial f_z}{\partial r} \sigma_r^2 + \frac{\partial f_x}{\partial u} \frac{\partial f_z}{\partial u} \sigma_u^2 + \frac{1}{2} \frac{\partial f_x}{\partial r} \frac{\partial^3 f_z}{\partial r \partial u^2} \sigma_r^2 \sigma_u^2 + \frac{\partial^2 f_x}{\partial r \partial u} \frac{\partial^2 f_z}{\partial r \partial u} \sigma_r^2 \sigma_u^2 \\
&+ \frac{1}{2} \frac{\partial f_x}{\partial r} \frac{\partial^3 f_z}{\partial r \partial v^2} \sigma_r^2 \sigma_v^2 + \frac{1}{2} \frac{\partial f_x}{\partial u} \frac{\partial^3 f_z}{\partial u \partial v^2} \sigma_u^2 \sigma_v^2 + \frac{1}{2} \frac{\partial f_x}{\partial u} \frac{\partial^3 f_z}{\partial u^3} \sigma_u^4 \Big|_{\text{true}} \\
&= \boxed{u_t w_t \sigma_r^2 - \frac{r_t^2 u_t}{w_t} \sigma_u^2 - \frac{1}{2} \left(\frac{u_t^3}{w_t^3} + 3 \frac{u_t}{w_t} \right) \sigma_r^2 \sigma_u^2 - \frac{1}{2} \left(\frac{u_t v_t^2}{w_t^3} + \frac{u_t}{w_t} \right) \sigma_r^2 \sigma_v^2} \\
&\quad - \frac{1}{2} \left(3 \frac{r_t^2 u_t v_t^2}{w_t^5} + \frac{r_t^2 u_t}{w_t^3} \right) \sigma_u^2 \sigma_v^2 - \frac{3}{2} \left(\frac{r_t^2 u_t^3}{w_t^5} + \frac{r_t^2 u_t}{w_t^3} \right) \sigma_u^4
\end{aligned}
\tag{D.16}$$

$$\begin{aligned}
\text{cov}(y_m, z_m) &= E[(y_m - y_t)(z_m - z_t)] = E\left[\left(\frac{\partial f_y}{\partial r}w_r + \frac{\partial f_y}{\partial v}w_v + \frac{\partial^2 f_y}{\partial r \partial v}w_r w_v\right)\left(\cdots\right)\right] \\
&= \frac{\partial f_y}{\partial r} \frac{\partial f_z}{\partial r} \sigma_r^2 + \frac{\partial f_y}{\partial v} \frac{\partial f_z}{\partial v} \sigma_v^2 + \frac{1}{2} \frac{\partial f_y}{\partial r} \frac{\partial^3 f_z}{\partial r \partial u^2} \sigma_r^2 \sigma_u^2 + \frac{1}{2} \frac{\partial f_y}{\partial r} \frac{\partial^3 f_z}{\partial r \partial v^2} \sigma_r^2 \sigma_v^2 \\
&\quad + \frac{\partial^2 f_y}{\partial r \partial v} \frac{\partial^2 f_z}{\partial r \partial v} \sigma_r^2 \sigma_v^2 + \frac{1}{2} \frac{\partial f_y}{\partial v} \frac{\partial^3 f_z}{\partial u^2 \partial v} \sigma_u^2 \sigma_v^2 + \frac{1}{2} \frac{\partial f_y}{\partial v} \frac{\partial^3 f_z}{\partial v^3} \sigma_v^4 \Big|_{true} \\
&= \boxed{v_t w_t \sigma_r^2 - \frac{r_t^2 v_t}{w_t} \sigma_v^2 - \frac{1}{2} \left(\frac{u_t^2 v_t}{w_t^3} + \frac{v_t}{w_t} \right) \sigma_r^2 \sigma_u^2 - \frac{1}{2} \left(\frac{v_t^3}{w_t^3} + 3 \frac{v_t}{w_t} \right) \sigma_r^2 \sigma_v^2} \\
&\quad - \frac{1}{2} \left(3 \frac{r_t^2 u_t^2 v_t}{w_t^5} + \frac{r_t^2 v_t}{w_t^3} \right) \sigma_u^2 \sigma_v^2 - \frac{3}{2} \left(\frac{r_t^2 v_t^3}{w_t^5} + \frac{r_t^2 v_t}{w_t^3} \right) \sigma_v^4} \tag{D.17}
\end{aligned}$$

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APPENDIX E: ACRONYMS

Estimation

CM1	RUV to Cartesian XYZ measurement conversion with first-order Taylor series
CM2	RUV to Cartesian XYZ measurement conversion with second-order Taylor series
CM2*	RUV to Cartesian XYZ measurement conversion with incorrect second-order Taylor series from [1]
CM3	RUV to Cartesian XYZ measurement conversion with third-order Taylor series
CM1KF	converted measurement Kalman filter with CM1
CM2KF	converted measurement Kalman filter with CM2
CM2KF*	converted measurement Kalman filter with CM2*
CM3KF	converted measurement Kalman filter with CM3
CM3EKF	extended Kalman filter with adaptive third-order measurement noise covariance compensation
CWNA	continuous white noise acceleration process noise model
EKF	extended Kalman filter
KF	Kalman filter
MCAEKF	measurement covariance adaptive extended Kalman filter with incorrect second-order Taylor expansion from [1]
NCV	nearly constant velocity
NEES	normalized estimation error squared
pdf	probability density function
PF	particle filter
RMSE	root mean squared error
RPF	regularized particle filter
UKF	unscented Kalman filter
UT	unscented transform

Coordinate Frames

ECI	Earth centered inertial (Cartesian)
ECR	Earth centered rotating (Cartesian)
ENU	east-north-up (Cartesian)
RAE	range-azimuth-elevation (non-Cartesian)
RFC	radar face coordinates (Cartesian)
RUV	range direction cosine (non-Cartesian)
XYZ	Cartesian coordinate frame (e.g., ECI, ECR, ENU, or RFC)

Miscellaneous

cf.	short for the Latin <i>confer</i> , meaning compare
BMD	ballistic missile defense
SSA	space situational awareness
TMD	theater missile defense

APPENDIX F: MATHEMATICAL NOTATION

General Notation

a	italic nonbold variables denote scalars
$\mathbf{a}$	bold lowercase variables denote column vectors
$\mathbf{A}$	bold uppercase variables denote matrices
$\mathbf{I}$	identity matrix, consisting of ones on the diagonals and zeros elsewhere
a_j	j th element of the vector $\mathbf{a}$
$\mathbf{a}^T$	vector transpose of $\mathbf{a}$
$\mathbf{A}^T$	matrix transpose of $\mathbf{A}$
$\mathbf{A}^{-1}$	matrix inverse of $\mathbf{A}$, such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$
$\text{eig}(\mathbf{A})$	eigenvalues of $\mathbf{A}$
$\kappa(\mathbf{A})$	condition number of $\mathbf{A}$, defined as the ratio of the largest to smallest eigenvalue
$\min(a, b)$	smaller element of a and b
$\max(a, b)$	larger element of a and b
$\cos(a)$	cosine of a
$\sin(a)$	sine of a
$\arctan(a)$	inverse tangent of a
$E(a), \hat{a}$	expectation (mean) of scalar a
$E(\mathbf{a}), \hat{\mathbf{a}}$	expectation (mean) of vector $\mathbf{a}$
σ_a	standard deviation of a
$\sigma_a^2, \text{var}(a)$	variance of a : $E[(a - \hat{a})^2]$
$\sigma_{ab}^2, \text{cov}(a, b)$	covariance of a and b : $E[(a - \hat{a})(b - \hat{b})]$
$\mathbf{P}_{\mathbf{a}}$	covariance of $\mathbf{a}$: $E[(\mathbf{a} - \hat{\mathbf{a}})(\mathbf{a} - \hat{\mathbf{a}})^T]$
$\forall$	symbol meaning <i>for all</i>
δ_{ij}	Kronecker delta function, which is one if $i = j$, and zero otherwise

Specific Variable Notation

$\mathbf{x}$	true state vector
$\hat{\mathbf{x}}$	estimated state vector
$\mathbf{P}$	state error covariance, $E[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T]$
$\hat{\mathbf{x}}(k k)$	estimated state vector at $t(k)$ given all measurements up through $t(k)$
$\mathbf{P}(k k)$	filter covariance at $t(k)$ given all measurements up through $t(k)$
$\hat{\mathbf{x}}(k+1 k)$	predicted state vector at $t(k+1)$ given all measurements up through $t(k)$
$\mathbf{P}(k+1 k)$	predicted filter covariance at $t(k+1)$ given all measurements up through $t(k)$
$\mathbf{F}, \mathbf{f}[\cdot]$	linear or nonlinear state transition function
$\mathbf{H}, \mathbf{h}[\cdot]$	linear or nonlinear measurement equation
$\mathbf{g}[\cdot], \mathbf{g}^{-1}[\cdot]$	nonlinear function to convert from Cartesian XYZ to RUV, and vice versa
$\mathbf{J}_f$	linearized state transition function, $\left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right _{\hat{\mathbf{x}}(k k)}$
$\mathbf{J}_h$	linearized measurement equation, $\left. \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right _{\hat{\mathbf{x}}(k+1 k)}$
$\mathbf{J}_{xyz}^{ruv}, \mathbf{J}_{ruv}^{xyz}$	Jacobian to convert error covariance from Cartesian XYZ to RUV, and vice versa
c_z	bias compensation term used in RUV to Cartesian XYZ conversion
w, w_t, w_m	variable used in RUV to Cartesian XYZ conversion, defined as $\sqrt{1 - u^2 - v^2}$
x, y, z	target position in Cartesian XYZ coordinates
$\dot{x}, \dot{y}, \dot{z}$	target velocity in Cartesian XYZ coordinates
r, u, v	target position in range direction cosine (RUV) coordinates
w_x, w_y, w_z	measurement noise in Cartesian XYZ coordinates
w_r, w_u, w_v	measurement noise in range direction cosine (RUV) coordinates
$\mathbf{w}$	measurement noise vector
$\mathbf{z}$	sensor measurement vector
$\hat{\mathbf{z}}$	estimated state vector in the measurement frame, $\mathbf{H}\hat{\mathbf{x}}$ for linear case or $\mathbf{h}[\hat{\mathbf{x}}]$ for nonlinear case
T	sensor update period, $t(k+1) - t(k)$
$\mathbf{Q}$	process noise covariance, $E(\boldsymbol{\nu}\boldsymbol{\nu}^T)$ or $E[(\mathbf{\Gamma}\boldsymbol{\nu})(\mathbf{\Gamma}\boldsymbol{\nu})^T]$
$\mathbf{R}$	measurement noise covariance, $E(\mathbf{w}\mathbf{w}^T)$
$\mathbf{K}$	filter gain

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