



FORMATION FLIGHT OF EARTH SATELLITES ON LOW-ECCENTRICITY

KAM TORI

DISSERTATION

Christopher T. Craft, CTR

AFIT-ENY-DS-16-M-201

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DISSERTATION

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Abstract

The problem of Earth satellite constellation and formation flight is investigated in the context of Kolmogorov-Arnold-Moser (KAM) theory. KAM tori are constructed utilizing Wiesel's Low-Eccentricity Earth Satellite Theory, allowing numerical representation of the perturbed tori describing Earth orbits acted upon by geopotential perturbations as sets of Fourier series. A maneuvering strategy using the local linearization of the KAM tangent space is developed and applied, demonstrating the ability to maneuver onto and within desired torus surfaces. Constellation and formation design and maintenance on KAM tori are discussed, along with stability and maneuver error concerns. It is shown that placement of satellites on KAM tori results in virtually no secular relative motion in the full geopotential to within computational precision. The effects of maneuver magnitude errors are quantified in terms of a singular value decomposition of the modal system for several orbits of interest, introducing a statistical distribution in terms of torus angle drift rates due to mismatched energies. This distribution is then used to create expectations of the steady-state station-keeping costs, showing that these costs are driven by operational and spacecraft limitations, and not by limitations of the dynamics formulation. A non-optimal continuous control strategy for formations based on Control Lyapunov Functions is also outlined and demonstrated in the context of formation reconfiguration.

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Christopher T. Craft

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List of Abbreviations

Abbreviation	Page
ISR	Intelligence/Surveillance/Reconnaissance 1
HCW	Hill-Clohessy-Wiltshire 1
TH	Tschauner-Hempel 1
KAM	Kolmogorov, Arnold, Moser 2
LVLH	Local Vertical/Local Horizontal 7
MEO	Medium Earth Orbit 11
LEO	Low Earth Orbit 12
ECI	Earth-centered Inertial 18
LEST	Low- Eccentricity Earth Satellite Theory 24
ECR	Earth-centered rotating 26
LE	Low-Eccentricity 33
CLF	Control Lyapunov Function 108

FORMATION FLIGHT OF EARTH SATELLITES ON LOW-ECCENTRICITY KAM TORI

I. Introduction

1.1 Motivation

Since the dawn of the era of spaceflight, dynamicists and engineers have been interested in the problem of spacecraft relative motion. The application of relative motion theories include rendezvous between two bodies, the design of satellite formations, and the design of satellite constellations. This research area is clearly of profound importance to applications which require precise positioning of space assets relative to other items of interest. Systems such as space-based navigation services, communications constellations, and Intelligence/Surveillance/Reconnaissance (ISR) systems all require some level of consideration for the relative motion solution.

Historically, most relative motion solutions to are based upon either the Hill-Clohessy-Wiltshire (HCW) equations or the Tschauner-Hempel (TH) equations [42, 58]. The HCW equations are linearized equations based upon the two-body problem (or some variant thereof which may include the main J_2 geopotential term as a perturbation) in which the leader or “chief” satellite follows a circular orbit. The TH equations are similar, save that the chief satellite follows an elliptical orbit. Since these solutions contain, at best, only a small portion of the effects of the Earth’s full gravitational potential, there is inevitably a significant error between the computed relative motion solution and reality, which manifests as a physical separation of unknown or undesired magnitude and a secular drift for which the satellite must correct, consuming precious maneuvering fuel [9].

To accomplish the task of reducing the residual drift between satellites in a formation, the key task is to reformulate the problem in such a way that the maximal effects of the true dynamics are included. To this end, recent work by Wiesel [60] on periodic orbits in Earth's zonal potential may be considered alongside work by Wiesel and Craft [21] on the subject of Kolmogorov, Arnold, Moser (KAM) tori for earth-orbiting satellites. This provides a framework in which the dynamics of satellites in low-eccentricity orbits can be studied which includes dynamical effects resulting from the gravitational potential to arbitrary order and degree. Specifically, after finding the appropriate periodic orbit with the low-eccentricity theory, a KAM torus for a particular satellite, represented in modal coordinates, may be found by introducing non-zonal potential terms. The resulting torus contains the perturbation solution due to non-zonal geopotential terms and non-zero eccentricity. A control methodology can then be designed which forces the satellite onto a desired torus, or relocates to a different desired location on the same torus. The primary result of KAM theory dictates that the satellite will remain on that phase-space torus for all time; this is supported by findings from Craft (as discussed in [21]). By specifying multiple satellites possessing modal/torus coordinates with desired separations, formations and constellations can be designed in which the member satellites would theoretically remain fixed in modal space (with the exception of small oscillations due to the periodic perturbing terms), resulting in relative positions with little or no secular drift. Because the formulation includes the maximal Earth gravitational dynamics, any maneuvers should be significantly reduced in magnitude and, by design, do waste fuel on purely oscillatory behaviors whose effects are unavoidable and non-deleterious for most formations. In addition, nonconservative effects like atmospheric drag, which don't directly lend themselves to the KAM approach, still do not cause secular drift between satellites, as long as the satellites are nearly identical in ballistic coefficient, while third-body conservative effects not included in the formulation (such as those from the moon and

sun) should average out over the satellite formation and can be included in the theory, if desired.

1.2 Expected Contributions and Organization

The aim of the current work is to assemble the framework of control on and in the vicinity of KAM tori; we begin from periodic orbit theory and utilizing the insights of KAM theory to create the underlying dynamical fabric for the control problem. Methods are developed to accomplish impulsive control onto tori, and the applicability of these methods to both formations and constellations are discussed. Stability and error analyses are completed to determine the potential for real-world application of the outlined method, and the feasibility of the method is demonstrated through numerical implementation. In addition, an approach to the problem of continuous (low-thrust) control for formation maintenance is developed and discussed. To the author's knowledge, no previous work has focused on the use of KAM theory as the basis for a control framework.

The expected result of this research is a novel, robust paradigm for the engineering of satellite relative motion in the low-eccentricity orbit regime. It is expected that the numerical methods and procedures developed in this work will allow efficient design of satellite formations with some desired relative orbital geometry, along with a method for controlling the individual satellites onto those orbits and keeping station. The inclusion of the Low-Eccentricity KAM theory as the underlying dynamical fabric should result in formations with arbitrarily small inter-body secular drift, driven only by inaccuracies in maneuver thrust magnitudes and attitude control, having avoided gross simplifications of the dynamics.

The remainder of this document is organized in the following manner: Chapter 2 introduces the problem of satellite relative motion, along with a historical perspective. Chapter 3 gives an introduction to KAM theory as applied to Earth satellites. Chapter 4 describes Low-Eccentricity Earth Satellite Theory and discusses the connections with

KAM theory. Chapter 5 is designed to provide familiarity with the features of low-eccentricity KAM tori, as well as to provide techniques for using them in a design sense. Chapter 6 contains the derivation of an impulsive strategy for maneuvering onto KAM tori. Chapter 7 proceeds to discuss the applicability of the theory to constellations and formations. Finally, Chapter 8 expounds upon the stability and behavior of dynamics on KAM tori, as well as providing a method for estimating maneuvering cost requirements given spacecraft hardware limitations. To close Chapter 8, a continuous control law derived from Lyapunov stability theory is presented and results of the implementation discussed. Chapter 9 contains a discussion of the overall implications of the results of this work and gives recommendations for future research related to it.

Note: The terms *formation* and *constellation* have no universal, concrete definitions when applied to spacecraft, but there are commonly accepted meanings which we will use. According to the NASA Goddard Space Flight Center (from [9]), formation flying is

the tracking or maintenance of a desired relative separation, orientation, or position between or among spacecraft.

In this work, the concepts of *formation* and *constellation* will usually be distinguished to mean the following. The term *formation* will be used to describe groups of two or more satellites in which the inter-satellite distances are on the order of tens of kilometers or less (that is, much less than the orbital radius), and usually monitored and controlled in such a way that ancillary, or “follower” satellites have some desired configuration with respect to a reference, or “chief”, satellite. The term *constellation* will be used to refer to a group of satellites orbiting at widely-spaced intervals (at distances on the order of the orbital radius) and operating in some interconnected capacity in terms of signal or optical coverage geometry.

II. Background: Relative Motion

2.1 Satellite Relative Motion Theory

In 1960, a seminal work by Clohessy and Wiltshire [19] described the problem of terminal guidance of an Earth satellite in the process of rendezvous with another satellite, with specific application to the Gemini program [14]. Their paper built upon the theory developed by Hill in 1878 [27] on lunar dynamics and so the resulting equations are termed the “Hill-Clohessy-Wiltshire” (HCW) equations. The HCW equations are constructed with the assumptions of a spherical Earth and a circular reference orbit (i.e., that which is followed by the chief satellite) [51] by linearizing in a Taylor series expansion about the Keplerian reference orbit; this implicitly assumes small separations between the chief and follower, or “deputy”, satellites as compared to their respective distances from the gravitational primary. The HCW equations of relative motion are given by

$$\ddot{x} - 2\omega\dot{y} - 3\omega^2x = f_x \quad (2.1)$$

$$\ddot{y} + 2\omega\dot{x} = f_y \quad (2.2)$$

$$\ddot{z} + \omega^2z = f_z \quad (2.3)$$

where x , y and z are the follower’s coordinates in the target-centric frame, ω is the orbital angular velocity of the chief satellite (which is equivalent to the mean motion n in this circular case) and the f_n are external forces or control inputs [51]. As should be expected, the significant simplifications of the dynamics committed to obtain the HCW equations (i.e., the treatment of the primary’s gravitational field in a point-mass sense and the linearization about the two-body trajectory) lead to large errors in the modeled vs. true relative motion, having discarded large effects such as differential acceleration in the Earth’s potential field. For the purposes of powered spacecraft rendezvous, which take place over relatively short periods of time, these equations are of sufficient accuracy

to accomplish successful control. However, when one considers satellite formations or constellations with large inter-satellite displacements (on the order of the orbital radius) and/or missions lasting many tens, hundreds, or even thousands of orbits, the performance of the HCW equations obviously falls short as a framework for relative motion.

Other work has included efforts to include first-order nonlinear terms, such as in [34], alongside efforts (most notably by Tschauner and Hempel [50]) to extend the linearized rendezvous equations to the case in which the chief satellite follows an elliptical orbit. The latter equations are termed the Tschauner-Hempel (TH) equations, given as [44]

$$\ddot{x} - 2\dot{\nu}_c\dot{y} - (2k + \dot{\nu}_c^2)x - \ddot{\nu}_cy = f_x \quad (2.4)$$

$$\ddot{y} + 2\dot{\nu}_c\dot{x} + \ddot{\nu}_cx + (k - \dot{\nu}_c^2)y = f_y \quad (2.5)$$

$$\ddot{z} + kz = f_z \quad (2.6)$$

where ν_c is the true anomaly of the chief and k is a mean motion analog defined as:

$$k = \frac{n_c^2(1 + e_c \cos \nu_c)^3}{(1 - e_c^2)^3} \quad (2.7)$$

The quantities $\dot{\nu}_c$ and $\ddot{\nu}_c$ can be calculated in terms of the chief satellite's orbital parameters as

$$\dot{\nu}_c = \frac{n_c(1 + e_c \cos \nu_c)^2}{(1 - e_c^2)^{3/2}} \quad (2.8)$$

$$\ddot{\nu}_c = -\frac{2n_c^2 e_c \sin \nu_c (1 + e_c \cos \nu_c)^3}{(1 - e_c^2)^3} \quad (2.9)$$

where $n_c = \omega$ is the mean motion of the chief, and e_c is the chief's eccentricity.

While the HCW and TH equations above and their extensions to first-order nonlinearity terms provide adequate error performance for short-term rendezvous engagements, the resurgence of interest in the relative motion problem (*vis à vis* formation flight) in the 90s and early 2000s sparked efforts to include more of the Earth's gravitational effects in the relative motion solution. Alfrend et al. [8] and Gim and Alfrend [25] developed a

method to account for the Earth’s primary oblateness effect (J_2) and slight eccentricities using a state transition matrix which relates the changes in orbital elements to the changes in the local frame. This was shown to be an improvement over the HCW equations, but was deemed unwieldy due to the complexity of the terms in the matrix. Some researchers, such as Sedwick et al. in [46], modified the right-hand side of the HCW equations to include forcing functions in terms of J_2 , with discussions of how to use control to counter the J_2 effects. Schweighart and Sedwick [45] later analyzed in more detail the effects of the J_2 perturbations on the relative orbits, describing the phenomenon of tumbling, in which the cluster rotates around the orbit normal vector. They also discussed the orbit configurations that could minimize these effects.

Schaub and Alfriend [43], Alfriend et al. [11] and Alfriend and Yan [10] then developed and demonstrated a theory which departed from the aforementioned strategies of modifying the HCW or TH equations to include higher-order geopotential terms. Instead, these authors used the mean Delaunay orbital elements, which they transformed to and from the osculating elements and physical coordinates in the chief-centered Local Vertical/Local Horizontal (LVLH) frame, as necessary. Their work still focused on the first-order terms in J_2 and developed constraints which, when satisfied, resulted in orbits which were “ J_2 -invariant”, which implies that the drift rates of the right ascension of the ascending node (Ω) and the mean latitude angle are equal on average [43]. The method focused on developing period-matching constraints on three differential mean orbital elements (δa , δe and δi) which left four degrees of freedom in designing a desired relative orbit. The authors reported that this method experienced difficulty in the case of perfectly circular orbits, in which case the mapping between the mean and osculating elements is singular, which could be alleviated by using non-singular elements. The authors proceeded to demonstrate an impulsive control scheme in this differential element formulation, which showed fair results when compared against numerical integrations including only the J_2 through J_5

zonal harmonics in the geopotential, for moderate Δv costs. With a satellite flying at $a = 7555$ km, $e = 0.05$, and $i = 48^\circ$, corrections of approximately 100 km in semi-major axis, 0.05° in inclination, and 0.01° in Ω over 4 orbits resulted in a cost of $\Delta v = 6 - 8$ m/s.

The above work using mean orbital elements is based upon expansion about an unperturbed Keplerian reference orbit, as discussed in [9]. This is approached using analytic methods for propagation of relative motion variables through differential orbital elements and Euler parameters (between entities in formation). First-order methods, such as the Gim-Alfriend State Transition Matrix approach and linear differential equation models, lead to the requirement to average the relative motion (through mean orbital elements) so that short-period motion is discounted in the relative motion solution. A second-order approach, involving a differential equation containing a hybrid combination of both physical coordinates and orbital elements, leads to a more accurate approach for relative motion modeling. The most successful approach, developed by Yan/Alfriend, develops a somewhat unwieldy traditional perturbation solution (via expansion in Taylor series in Delaunay variables) in the mean Hamiltonian, which is closest in spirit to the current research thrust. However, since these approaches involve the J_2 effects (or up to J_2^2 in the Yan-Alfriend case) only and rely on some level of linearization about a Keplerian orbit as reference, the accuracies of their models are diminished when considering a larger cross-section of the true forcing dynamics, or applications involving large time scales. In addition, the differential element theory implicitly assumes small (with respect to orbital radius) distances between the relative bodies with error growing as a function of separation distance, and thus provides a framework for analyzing motion only for satellite formations, with only limited applicability to constellations. The current work seeks to bridge the gap between formation and constellation design by providing a global theory for low-eccentricity satellite dynamics.

2.2 Current Space Systems

2.2.1 Formations.

Although many have been conceived and proposed, very few formation-flying missions have actually been realized, due to the complexity of the dynamics involved and the required rigor of an associated control law to avoid collisions. Recent notable satellite formations include:

- PRISMA: a mission led by Sweden to develop and demonstrate control strategy and technologies (both hardware and software) needed for successful formation flight [6]. PRISMA is primarily intended to progress the technology associated with autonomous formations (i.e., on-board navigation (via weak GPS) and calculation of maneuvers) and is composed of two satellites, named Mango and Tango. This mission was successfully launched in 2010 and has been completing on-orbit experiments successfully.
- Magnetospheric Multiscale Mission (MMS): a NASA GSFC mission to investigate the interactions of the Sun's and Earth's magnetic fields [5]. It is composed of four identical satellites flying in a tetrahedral formation with separations ranging from kilometer-level to several Earth radii. The spacecrafts' unique formation will allow 3-D "imaging" of the combined magnetic field and requires precise inter-spacecraft ranging and communication.
- Gravity Recovery and Climate Experiment (GRACE): a joint NASA-DLR (German Aerospace Center) mission to map variations in the Earth's gravity field [3]. While not technically a formation flying mission, it involves two satellites separated in-track by about 220 kilometers.

There are a number of proposed missions which either have not yet been finalized or were canceled, such as the U.S. Air Force Research Laboratory's TechSat-21 mission [35],

which was intended to conduct flight experiments with three microsatellites flying in close formation to act as a single virtual X-band transceiver. While this program was canceled, the concept has merit for future radio technology applications requiring synthetic apertures or large virtual antenna. Any such mission would clearly require highly accurate control and appropriate orbit design. Another planned mission involving strict inter-spacecraft position requirements, currently projected for a late-2018 launch, is the ESA PROBA-3 mission. PROBA-3 will consist of two satellites in High Earth Orbit (HEO) forming a 150-meter-long solar coronagraph to study the Sun's corona.

2.2.2 Constellations.

Satellite constellations are far more common in operational use than formations, largely because the individual entities in the constellation typically act as a loose confederation of members with more loosely-coupled orbital requirements. That is, while formations require precise navigation and control of satellites relative to other satellites in close proximity, constellations may possess less stringent relative positioning requirements and therefore require a lower duty cycle of control. Design strategies for constellations vary depending on the specific application; however, a common class of constellations is the so-called Walker constellation [52]. These constellations are popular because of their effectiveness and simplicity: the satellites in the constellation are specified by the nomenclature $i : t/p/f$, where i is the orbital inclination (shared by all satellites in the constellation), t is the total number of satellites in the constellation, p is the number of orbital planes (such that the lines of nodes are spaced evenly in an angular sense in the equatorial plane), and f is the phase spacing between satellites in adjacent planes. All Walker orbits are assumed to be nearly circular.

There are a number of satellite constellations currently in operation; notable ones include the well-known Global Positioning System (GPS) constellation, the under-validation Galileo navigation constellation, the Iridium communications constellation,

the Russian GLONASS (*Globalnaya navigatsionnaya sputnikovaya sistema*) navigation system constellation, and the Tracking and Data Relay Satellite System (TDRSS) communications constellation. The current research, being concerned in general with the problem of satellite motion and station-keeping, is applicable to some degree to all constellation systems (with the exception of those near resonance, such as the GPS constellation, as discussed by Bordner [15]). For the purposes of demonstration of applicability, the current work utilizes representative low-Earth-orbit satellites and HEO satellites in orbits similar to Galileo.

Of particular note when discussing navigation constellations is this fact: every time a navigation satellite is maneuvered, the satellite must be taken out of operation until its orbit can be redetermined. As a result, decreasing maneuver frequency maximizes satellite uptime and provides more reliable and sustainable coverage. The current research would thus provide the framework to reduce downtime of orbital assets through inclusion of a more accurate dynamical model. This was also the intent of Bordner [15] with specific application to the GPS constellation.

According to the European Space Agency (ESA) program website [1], the Galileo satellite constellation will be composed of 30 MEO satellites spread across three orbital planes, each at an inclination of 56° . Ten satellites will occupy each orbital plane, with nine craft evenly spaced (presumably in true anomaly) and one functioning as a spare. The Walker delta notation for the operational constellation is $56^\circ:27/3/1$. The satellites will be in circular orbits at an altitude of 23,222 km, or a distance of 24,000 km from the Earth's center. The constellation is expected to function in a similar manner to (and, when necessary, be augmented by) the U.S.-owned GPS satellite constellation. The first two satellites were launched into the first orbital plane in October 2011 and the next two into the next orbital plane in October 2012. Two more satellites, carried by a Soyuz ST rocket, were launched towards the third orbital plane in August 2014; however, an anomaly at orbital

injection caused the satellites to be delivered into the incorrect orbit [2]. The resulting orbit possessed an inclination of 49.8° , an eccentricity of 0.23, and a semi-major axis of 26,200 km. A series of fourteen maneuvers over several months brought the anomalous satellites back into acceptable orbits in early 2015.

It is unclear, due to a dearth of publicly-available information, what the ESA and its supporting organizations use as strategy for orbital planning, and whether or not there exists a cohesive method for elimination of secular drift between satellites (both within an orbital plane and between orbital planes). The assorted publicly-available documentation leads one to believe that the strategy involves cycles of long-term observation (over weeks or months) and minor corrective impulses to correct satellite drift. While relatively effective in the limit, this author believes that a more robust approach is provided in the current research.

The Iridium Satellite constellation is an example of a LEO system composed of 66 active satellites in circular, nearly polar orbits ($i = 86.4^\circ$) at an altitude of about 781 km [4], with Walker notation $86.4^\circ:66/6/2$. The satellites are divided among six orbital planes spaced 30 degrees apart and communicate with other coplanar satellites and “cross-seam” satellites (those in neighboring orbits) through K_a band links. The configuration allows near-constant coverage of the Earth’s surface. As is the case concerning the Galileo constellation, albeit likely for different reasons, details on the station-keeping strategy for the Iridium constellation are not publicly available. However, as with all Walker constellations, it falls firmly within the realm of applicability of the current research.

2.2.3 Applicability of the Current Theory.

The ultimate goal in satellite formation and constellation flight is to achieve an “optimal” orbital control solution; optimal, in this case, is unavoidably some balance of short-term mission requirements (such as maintaining strict formation separations) and long-term mission viability (through minimization of satellite fuel usage). The

current research focuses primarily on the second consideration by providing a convenient dynamical representation which allows satellites to maintain desired relative torus-surface separations (as discussed in the sequel). Application of this method to current or future systems would allow longer satellite lifetimes as a result of less-frequent station keeping maneuvers. By its nature, it is better suited to applicability in constellations in which secular drift is undesired; however, it still has applicability to certain formation requirements, as discussed in subsequent chapters.

III. Kolmogorov-Arnold-Moser Theory

3.1 KAM Theory Foundations

An important result in the field of mathematics in the last century is that of KAM theory (named for its developers– Kolmogorov [29], Arnold [12] and Moser [38]) which concerns itself with lightly perturbed dynamical systems. KAM theory begins with the statement that, given an unperturbed, integrable Hamiltonian $H_0(I)$, the system phase space is foliated into invariant tori associated with actions $I = \text{constant}$. By usual Hamiltonian transform theory, the torus angle coordinates φ propagate with frequencies

$$\omega = \frac{\partial H_0}{\partial I} \quad (3.1)$$

Such a torus is called “non-resonant” if the frequencies are rationally independent (and “resonant” if they are not); that is, non-resonant tori satisfy the Diophantine condition:

$$\omega \in \mathbb{R}, \langle k, \omega \rangle \geq \gamma |k|^{-1} \quad \forall k \in \mathbb{Z} \setminus \{0\}, \quad \gamma \geq 0 \quad (3.2)$$

In addition, the tori are called “non-degenerate” if the frequencies are functionally independent (i.e., the matrix of their gradients $\partial\omega/\partial I$ has full rank) and “degenerate” if they are not. A related but distinct descriptor is “isoenergetic non-degeneracy,” in which case one of the frequencies does not vanish and the ratios of the other frequencies to the non-vanishing frequency are functionally independent on some constant energy level $H_0 = E$. With these preliminaries, we turn to consideration of the system with a slight perturbation:

$$H_\epsilon(I, \varphi) = H_0(I) + \epsilon H_1(I, \varphi) \quad (3.3)$$

where H_ϵ is the perturbed Hamiltonian, I and φ are the system coordinates in an action-angle representation, H_0 is the integrable Hamiltonian, H_1 is the perturbing Hamiltonian and ϵ is some small, real value. The KAM theorem can then be stated as follows. If the perturbation ϵ is sufficiently small, most of the non-resonant invariant tori in the phase

space persist as slightly-deformed tori, which possess phase space curves winding around them with the number of frequencies equal to the number of degrees of freedom of the system.

Upon analysis, it is seen that the theory posits that a trajectory associated with $I = \text{constant}$ lies on an N -torus in a $2N$ -dimensional phase space (where N is the number of system degrees of freedom) and that it remains on that torus for all time. That is, for the Hamiltonian $H_\epsilon(I, \varphi)$, the flow is conjugated to the translation

$$\varphi \rightarrow \varphi + \omega t \quad (3.4)$$

This theory has been shown to have physical applications in fields including celestial mechanics [17, 18, 28, 36], Earth satellite motion [33, 56, 59] and particle physics [24, 53–55].

3.2 The Problem with the Two-body Problem

There is a nuance in the KAM theorem, however, which prevents direct application to artificial satellite theory; namely, the standard integrable system in the problem of Earth-orbiting satellites is the two-body problem. Unfortunately, the two-body problem does not satisfy the resonance and non-degeneracy conditions of the KAM theorem; that is, there exists only one frequency (the Keplerian frequency), so it is (by nature) a resonant system. It is also degenerate since the number of non-zero frequencies (one) is less than the number of physical degrees of freedom of the system (three). This is called “proper degeneracy”, which occurs when the Hamiltonian does not contain one or more of the action variables. Fortunately, however, a properly degenerate system can sometimes be made to be non-degenerate by the introduction of a perturbation. Following [13], we may rewrite the perturbed Hamiltonian as

$$H_\epsilon = H_{00}(I) + \epsilon H_{01}(I) + \epsilon^2 H_{11}(I, \varphi, \epsilon) \quad (3.5)$$

where the system $H_{00} + \epsilon H_{01}$ is referred to as the intermediate system. H_{00} depends only on the first k elements of I and fulfills the KAM non-degeneracy and resonance conditions with respect to $\omega(I_{1\dots k})$, and H_{01} depends on I entirely, but is non-degenerate with respect to $\omega(I_{k+1\dots m})$, where m is the dimensionality of I . Then, an extension to the KAM theory states that the phase space of the perturbed Hamiltonian is filled with invariant tori close to the $I = \text{const}$ tori of the intermediate system. Additionally, the first k frequencies are the “fast” phases, and the last $m - k$ frequencies are the “slow” phases. This brings the theory of KAM tori squarely into the realm of utility in artificial satellite theory, as we will see shortly.

3.3 Application of KAM Theory to Artificial Satellites

KAM theory has recently been applied to Earth-orbiting satellites by Wiesel in [56] and [59] and Little in [33]. Wiesel demonstrated a least-squares method for obtaining KAM tori from numerically integrated data in [56], where he showed the torus construction for an Earth satellite, and later a refined method using Fourier analysis in [59], where he showed the construction of a torus for a restricted three-body problem resembling the Earth-Moon system. Specifically, in the Earth-satellite scenario, Wiesel found that orbits subject to the perturbing geopotential could be described in terms of three angles, each propagating with an independent frequency; this is the physical embodiment of the angles φ and the frequencies ω mentioned in the previous section. These frequencies are the well-known Keplerian frequency, the nodal precession rate, and the precession of the line of apsides. The dominant cause of the latter two frequencies is the J_2 (oblateness) term in the gravitational expansion, and so these frequencies are often approximated by

$$\omega_N \approx -\omega_\oplus - \frac{3\sqrt{\mu}J_2R_\oplus^2}{2a^{7/2}(1-e^2)^2} \cos(i) \quad (3.6)$$

$$\omega_p \approx -\frac{3\sqrt{\mu}J_2R_\oplus^2}{2a^{7/2}(1-e^2)^2} \left(\frac{5}{2} \sin^2 i - 2 \right) \quad (3.7)$$

where a is the semi-major axis, e is the orbital eccentricity, i is the orbital inclination and $R_\oplus$, $\omega_\oplus$ and μ are the Earth's equatorial radius, rotational frequency, and gravitational parameter, respectively. Little [33] provided similar evidence for Earth satellites being restricted to KAM tori by employing observed orbital data from the *GRACE* and *Jason-1* satellites to determine basis frequencies and reduce the coordinates to series representation [33].

To apply KAM theorem to a problem of orbital mechanics, one would seek the ideal map

$$\mathcal{B} : (q, p) \rightarrow (I, \varphi) \quad (3.8)$$

so that one physical state $X_0 = [\mathbf{q}, \mathbf{p}]^T$ made up of the physical canonical coordinates $\mathbf{q}$ and momenta $\mathbf{p}$, respectively, would yield directly the associated torus in action-angle representation (I, φ) . Unfortunately, such a direct map $\mathcal{B}$ doesn't appear to be readily available for general orbits, and so the strategy of Wiesel et al. was to find an indirect map using multiple states along the body's physical trajectory (determined either through numerical integration of the system Hamiltonian or measurements and observations of the orbiting body, if these are available and of sufficient accuracy), along with some spectral method of translating the trajectory into a Fourier series:

$$\mathbf{q}(t) = \sum_{\mathbf{j}} \left[\mathbf{C}_{\mathbf{j}} \cos(\mathbf{j} \cdot \boldsymbol{\varphi}) + \mathbf{S}_{\mathbf{j}} \sin(\mathbf{j} \cdot \boldsymbol{\varphi}) \right] \quad (3.9)$$

where $\mathbf{C}_{\mathbf{j}}$ and $\mathbf{S}_{\mathbf{j}}$ are coefficients determined numerically and each $\mathbf{j}$ is an index vector, used in the dot products to specify linear combinations of the torus angles $\boldsymbol{\varphi}$. The number of index combinations $\mathbf{j}$, along with their associated Fourier coefficients $\mathbf{C}_{\mathbf{j}}$ and $\mathbf{S}_{\mathbf{j}}$, necessary to represent the trajectory accurately depends upon the orbital parameters; as a benchmark, centimeter-level accuracy over a one-year period in the coordinates of a numerically-integrated 630km altitude orbit was achieved with fewer than 400 $\mathbf{j}$ vectors [21].

Craft/Wiesel [21] and Bordner [15] have developed KAM tori in the series expansion form above for assorted satellite orbits acted upon by the geopotential (including all zonal,

sectoral and tesseral terms up to degree and order 20) in the Earth-centered rotating frame. The tori are constructed by numerically integrating the satellite dynamics for a duration of one year and performing a frequency decomposition method of Fourier analysis (as introduced by Laskar [32] and adapted by Wiesel [56, 59]) to determine the system’s so-called “fundamental frequencies” ω_k , $k = 1, 2, 3$. After determining the fundamental frequencies, the torus coordinates φ_k evolve in a straightforward manner as a function of time:

$$\varphi_k = \omega_k t + \varphi_{k_0} \quad (3.10)$$

where φ_{k_0} are the initial torus coordinates; therefore, the angle coordinates are seen to be KAM torus coordinates. Craft and Wiesel then showed that one could “place” satellites at different locations on the torus by defining angular separations in the torus coordinates $\varphi_1, \varphi_2, \varphi_3$, which are the analogues of the in-track (in-orbital-plane) angular displacement, nodal displacement (rotations about the ECI $\hat{z}$ vector), and the in-plane, apsidal oscillation due to eccentricity, respectively. These angular separations were then substituted into Eq. (3.9) to obtain the requisite physical configuration coordinates, and the momenta were reconstructed using

$$\mathbf{p}(t) = \dot{\mathbf{q}}(t) + \mathcal{R}_1 \mathbf{q}(t) \quad (3.11)$$

where $\mathbf{p}$ is the vector of momenta, $\dot{\mathbf{q}}$ is the time derivative of the KAM series (3.9), and $\mathcal{R}_1$ is an ECEF-to-inertial transformation matrix in terms of the rotation rate of the Earth $\omega_\oplus$:

$$\mathcal{R}_1 = \begin{bmatrix} 0 & -\omega_\oplus & 0 \\ \omega_\oplus & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.12)$$

Craft and Wiesel used these physical coordinates (calculated by transforming from fixed torus coordinate offsets) as initial conditions for numerical integrations in the geopotential with $m, n = 20$. Sample results are shown in the following figures. For these cases, a nearly circular orbit with an inclination of 30° and an altitude of 630 km was used as the basis

for creating the KAM torus via the above method. Four simulated satellites were placed at displacements of $\delta\varphi_1 = \pm 0.001^\circ$ and $\delta\varphi_2 = \pm 0.001^\circ$ from the chief, resulting in initial physical displacements of approximately 122 m, creating a tight cluster formation. The plots in Figure 3.1 show the initial positions in both the torus configuration space and the physical Cartesian configuration space. Satellites 1 and 3 correspond to the $\delta\varphi_1 = \pm 0.001^\circ$ offsets, and satellites 2 and 4 were the equivalent offsets in $\delta\varphi_2$. Figure 3.2 displays the numerically integrated behavior of each of the satellites as a function of time over two months. As expected, the two satellites which are offset in $\delta\varphi_1$ (which is the analog of the in-track displacement) display very little oscillation, while the two satellites whose offsets were in the “nodal” direction $\delta\varphi_2$ experience a higher amplitude oscillation. For this scenario, ignoring the periodic component of the separation from the chief, each satellite drifted approximately 6-12 meters over the course of two months, which is a drift rate of approximately 200-300 nm/s. Order-of-magnitude analysis suggests that this minuscule drift rate is the result of numerical error in the torus fitting process and computational precision limitation in the integration of the dynamics. Results from the above and other cases, some of whose secular drift rates are even smaller, are summarized in Table 3.1.

An additional verification of the existence of KAM tori for Earth satellites was performed in the aforementioned work. Having in hand the numerically-constructed tori in the Action-angle space (I, φ) , the torus actions I could be calculated and examined. If the system’s phase space trajectory indeed lies on a KAM torus, the torus actions should be constant. Using the torus parameters, the action (for each dimension i) may be found using the Hamilton-Jacobi theorem (see e.g. [59], [37]) by the contour integral:

$$I_i = \frac{1}{2\pi} \oint_{\Gamma_i} p \cdot dq \quad (3.13)$$

In fact, Craft [21] found that the contour integrals, performed numerically, did indeed show that the actions were constant; these results are reproduced in Figure 3.3.

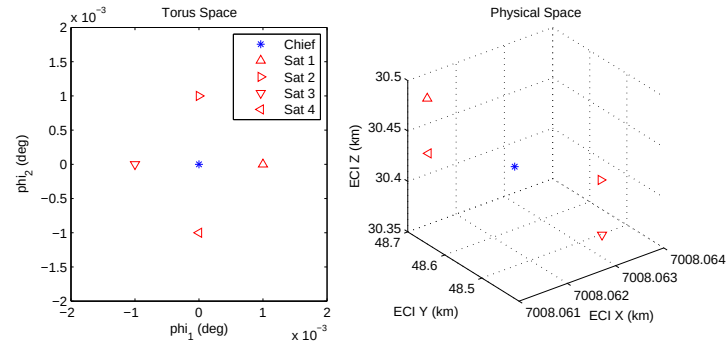


Figure 3.1: Initial position in torus space and inertial cartesian space of satellite cluster for tight formation analysis in 630km, 30° orbit, $\delta\varphi = 0.001^\circ$

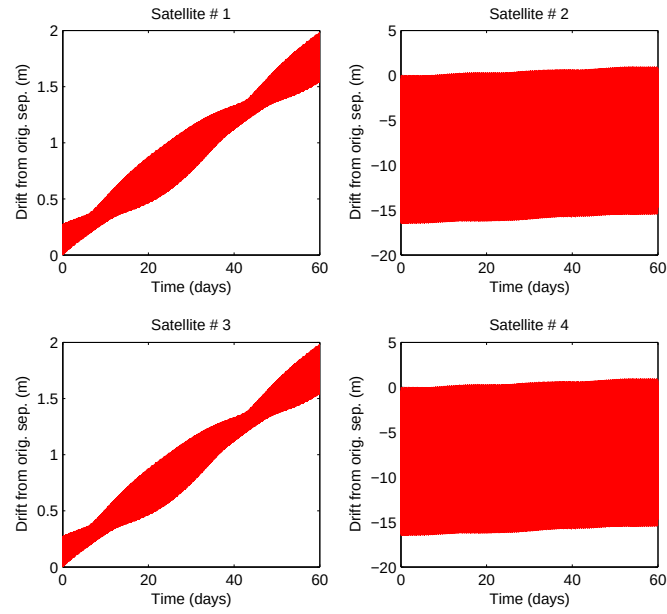


Figure 3.2: Cluster drift from initial separations for tight formation analysis in 630km, 30° orbit, $\delta\varphi_0 = 0.001^\circ$

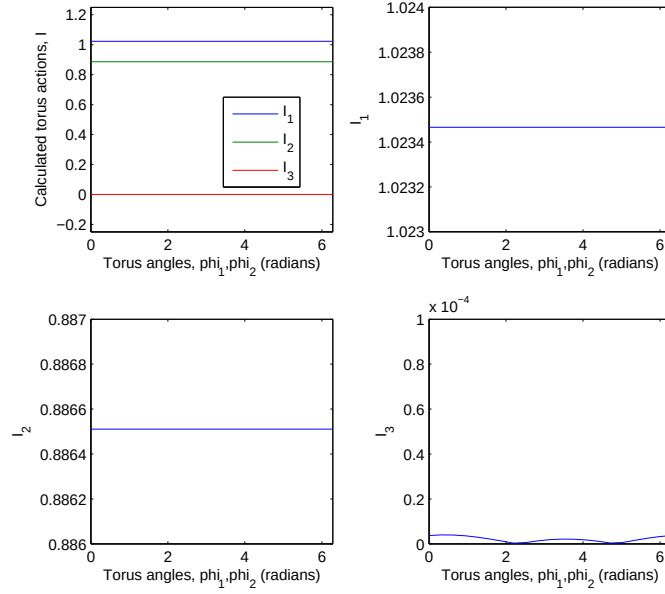


Figure 3.3: Calculated torus actions at varying locations in φ_1 and φ_2 for 320km, 30° orbit

The results summarized above from the work of Craft and Wiesel [21] demonstrated to within numerical computational accuracy that a phase trajectory lying on a KAM torus, which is a geometric object in the system's phase space, will stay on that torus. In addition, satellites whose initial conditions are close to each other on a torus surface will exhibit very small or zero drift rates relative to each other.

3.4 Tangent Space and KAM Summary

KAM theory provides a powerful theoretical framework in which we may consider perturbed systems such as the motion of an Earth-orbiting satellite acted upon by the Earth's geopotential. Recent work has shown that Earth satellite trajectories are indeed constrained to these phase space tori. With respect to formation and constellation flight, it was shown that satellites placed on the torus surface at some angular separation stay at that angular separation as the system evolves. However, while the above work provides a demonstration of the theoretical efficacy of KAM theory in formation flight, the method

used to construct the tori is largely impractical as given. The frequency decomposition framework requires an infeasibly long (in an operational sense) numerical integration to produce the configuration space data necessary for extraction of the fundamental torus frequencies and the series coefficients in (3.9). Also, in order to use the KAM construct as a practical method for orbit design, we must have a way to station-keep on a torus and maneuver onto a torus from nearby tori, which means there exists a need for at least a local description of the tangent space of the torus as it is embedded in the phase space. To see this, consider that for a Hamiltonian $\mathcal{K}$ describing a KAM system, we have as Hamilton's equations:

$$\begin{aligned}\frac{\partial \mathcal{K}}{\partial \mathbf{I}} &= \boldsymbol{\omega} \\ \frac{\partial \mathcal{K}}{\partial \boldsymbol{\varphi}} &= 0\end{aligned}\tag{3.14}$$

This implies that the Hamiltonian is a near-Hamilton-Jacobi form which is only a function of the momenta; $\mathcal{K} = \mathcal{K}(\mathbf{I})$. The simplest form of the Hamiltonian is then:

$$\mathcal{K}(\mathbf{I}) = \mathcal{K}_0 + \sum_i \omega_i I_i\tag{3.15}$$

If we are operating in a region of phase space in which KAM tori are dense, we can find the behavior of solutions on KAM tori near the current torus by considering small changes in the actions/momenta $\mathbf{I}$. Expanding the Hamiltonian to first order yields:

$$\begin{aligned}\mathcal{K}(\mathbf{I}) &= \mathcal{K}_0 + \sum_i \omega_i \delta I_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \mathcal{K}}{\partial I_i \partial I_j} \delta I_i \delta I_j + \dots \\ \mathcal{K}(\mathbf{I}) &= \mathcal{K}_0 + \sum_i \omega_i \delta I_i + \frac{1}{2} \sum_{i,j} \frac{\partial \omega_i}{\partial I_j} \delta I_i \delta I_j + \dots\end{aligned}\tag{3.16}$$

Any expression for the locally off-torus coordinates would then, by Hamilton's equations, require the gradient of the torus frequencies with respect to the momenta, $\partial \omega / \partial \mathbf{I}$. Such a tangent space solution is not readily admitted by the torus construction methods discussed in this chapter. To address this shortcoming, we turn to the Low-eccentricity Earth Satellite Theory, introduced in the next chapter.

Table 3.1: Results of tight formation analysis for various orbits and separations

Alt = 320km, $i = 15^\circ$, $\delta\varphi_0 = 0.0001^\circ$				
Sat. No.	Initial sep. from chief (m)	Avg. Oscillation amplitude (m)	Drift over 60 days (m)	Drift rate (m/s)
1	11.66	0.03379	0.01425	2.75e-009
2	11.69	0.3827	-0.01179	-2.275e-009
3	11.66	0.03378	0.01035	1.996e-009
4	11.69	0.3827	-0.01087	-2.097e-009
Alt = 320km, $i = 15^\circ$, $\delta\varphi_0 = 0.001^\circ$				
Sat. No.	Initial sep. from chief (m)	Avg. Oscillation amplitude (m)	Drift over 60 days (m)	Drift rate (m/s)
1	116.6	0.3378	0.1199	2.312e-008
2	116.9	3.827	-0.1161	-2.24e-008
3	116.6	0.3378	0.1164	2.246e-008
4	116.9	3.827	-0.1159	-2.235e-008
Alt = 320km, $i = 30^\circ$, $\delta\varphi_0 = 0.0001^\circ$				
Sat. No.	Initial sep. from chief (m)	Avg. Oscillation amplitude (m)	Drift over 60 days (m)	Drift rate (m/s)
1	11.66	0.03303	0.2361	4.555e-008
2	11.69	1.499	0.1301	2.509e-008
3	11.66	0.03303	0.2378	4.587e-008
4	11.69	1.499	0.128	2.47e-008
Alt = 630km, $i = 15^\circ$, $\delta\varphi_0 = 0.0001^\circ$				
Sat. No.	Initial sep. from chief (m)	Avg. Oscillation amplitude (m)	Drift over 60 days (m)	Drift rate (m/s)
1	12.2	0.03303	0.2361	4.555e-008
2	12.23	1.499	0.1301	2.509e-008
3	12.2	0.03303	0.2378	4.587e-008
4	12.23	1.499	0.128	2.47e-008
Alt = 630km, $i = 30^\circ$, $\delta\varphi_0 = 0.001^\circ$				
Sat. No.	Initial sep. from chief (m)	Avg. Oscillation amplitude (m)	Drift over 60 days (m)	Drift rate (m/s)
1	122	0.3048	1.626	3.137e-007
2	122.3	15.75	1.029	1.985e-007
3	122	0.3048	1.626	3.136e-007
4	122.3	15.75	1.028	1.984e-007

IV. Low-Eccentricity Earth Satellite Theory

4.1 Introduction

The primary advantage implicit in the KAM techniques outlined in the previous chapter is that, if obtainable, a KAM torus contains in its geometric structure almost all of the information of the dynamics of its associated system (this, of course, assumes conservative systems in non-chaotic regimes). Any theory which is to be usable for the satellite formation design and maintenance problem should then meet several criteria: it should contain as much of the dynamical information as possible, should reduce this information to an invariant form in a space of sufficient dimension, and should be of a form which admits a tangent space solution. Recent work by Wiesel [60] has outlined a theory of low-eccentricity Earth satellite motion which builds from periodic orbit theory in the zonal potential and includes tesseral and sectoral potential terms, third body effects, and air drag as perturbations. This theory serves as a starting point for building KAM tori for low-eccentricity systems in a general perturbation sense, which allows direct numerical calculation of coordinates at any time of interest, without long and unwieldy numerical integration and spectral analysis. This chapter gives a brief description and extension of Low-Eccentricity Earth Satellite theory as the foundation for the current research. For further details regarding this theory, the reader is referred to Appendix A and the original paper.

The core idea of the Low-Eccentricity Earth Satellite Theory (LEST) is the use of periodic, low-eccentricity orbits in the Earth's zonal potential as a base for perturbation theory, rather than beginning with the two-body problem. As mentioned in the KAM Theory chapter, when one uses the two-body problem as the "solvable" part of the dynamics, the system is degenerate, meaning that there is only one fundamental frequency of the 3-DOF system: the Keplerian frequency. Also, the perturbations to this solution

begin as disturbances on the order of about one part in 10^3 (beginning with the J_2 potential term). Using the zonal periodic orbit as the starting point eliminates the degeneracy issue and mitigates the perturbation issue; in essence, the zonal periodic orbit problem acts as the intermediate system $H_{00} + \epsilon H_{01}$ discussed previously, and serves to introduce the perturbed Hamiltonian's dependence on the remaining two actions. That is, the perturbations by the zonal geopotential terms induce two more fundamental frequencies in an orbit: the precession of the ascending node and the precession of the line of apsides (the “slow” angles). In addition, having included the largest perturbation terms in the solvable system, the perturbations to the dynamics begin at approximately one part in 10^5 , increasing the validity of rejection of higher order terms (in small quantities) in what is, at its core, a general perturbations solution.

Note: For a description of the orbital dynamics and the associated numerical implementation strategy, the current work utilizes a Hamiltonian formulation similar to Wiesel; the specific details are given in Appendix B. Also given therein is a description of the units used throughout the remainder of the work.

4.2 Periodic Orbit Construction

As discussed by Stellmacher [49] and Wiesel [57], nearly circular periodic orbits exist from equatorial through polar inclinations (excepting the region around the so-called “critical inclination”, $i^* \approx 63.4^\circ$, used for Molniya orbits to minimize apsidal regression [51]). These periodic orbits, referred to as “frozen” orbits, were also studied by Lara, Deprit and Elife [31] and Lara [30]. Additional work by Broucke [16] and Coffey et al. [20] showed that periodic orbits exist for high eccentricity systems in the neighborhood of the critical inclination, although we will not be concerned with such inclinations in this work.

Since an orbit in the zonal potential precesses a rate specific to that orbit's parameters (except at the critical inclination i^*), Wiesel's method constructs the nearly circular, zonal

periodic orbit in a frame of reference that rotates about the Earth's pole at the orbit's nodal precession rate. Given starting conditions at the ascending node in the [geometric] equatorial plane, $\mathbf{X}(0) = [x_0, 0, 0, \dot{x}_0, v_0 \cos i_0, v_0 \sin i_0]^T$, and a vector of unknowns $\Xi = [x_0, \dot{x}_0, v_0]^T$, a periodic orbit is achieved if

$$\mathbf{G} = \begin{bmatrix} z(\tau) \\ \mathbf{r}(\tau) - x_0 \dot{x}_0 \\ \mathbf{r}(\tau) \cdot \mathbf{r}(\tau) - x_0^2 \end{bmatrix} = \mathbf{0} \quad (4.1)$$

where $\mathbf{r}$ is the position vector of the satellite in the nodal rotating frame, z is the component of $\mathbf{r}$ along the positive polar axis, and $t = \tau$ is the period of the satellite. This condition states that, at one period, the satellite must intersect the equatorial plane and must have the same radial velocity and distance from the origin as at the initial time. A Newton-Raphson method is used to correct the initial condition parameters iteratively to achieve a periodic orbit (i.e., to force $\mathbf{G} = \mathbf{0}$ to within numerical precision). The nodal regression rate is then easily calculated as:

$$\dot{\Omega} = \frac{1}{\tau} \arccos \left(\frac{\mathbf{r}(\tau) \cdot \mathbf{r}(0)}{|\mathbf{r}(\tau)| |\mathbf{r}(0)|} \right) = \frac{1}{\tau} \arccos \frac{x(\tau)}{x_0} \quad (4.2)$$

The periodic orbit is then stored as a Fourier series in the frame of reference that rotates with the node, which causes it to be a function of only the angle Q_1 , which is the “mean argument of latitude” analog (essentially the angle φ_1 discussed in the KAM Theory chapter). From this periodic orbit, the orbital coordinates and momenta (forced by only the zonal potential) can be reconstructed in view of the kinematic relationships with the nodal rate. The results can easily be transformed to the Earth-centered rotating (ECR) frame by rotations through the nodal angle Q_2 about the Earth's polar axis, since the zonal potential is axisymmetric about this vector.

Motion near the periodic orbit can then be investigated by means of linearizing about the periodic trajectory $\mathbf{x}$, so that

$$\delta \dot{\mathbf{x}} = \mathbf{A}(t) \delta \mathbf{x} \quad (4.3)$$

where $A(t)$ is a periodic matrix by virtue of the underlying orbit being periodic. This leads to a problem of the form whose solution was presented by Floquet [22]. With a change of variables

$$\delta \mathbf{x} = E(t) \mathbf{y} \quad (4.4)$$

where $E(t)$ is periodic with the same period as $A(t)$, and the assumption that there is a constant matrix of Poincaré exponents in Jordan form $\mathcal{J}$ such that

$$\dot{E} = AE - E\mathcal{J} \quad (4.5)$$

the system leads to a so-called “modal form”

$$\dot{\mathbf{y}} = \mathcal{J} \mathbf{y} \quad (4.6)$$

where $\mathbf{y}$ are termed the modal variables. Upon calculation of $\mathcal{J}$ and of the modal matrix $E(t)$ over one period, information is gained about nearby solutions. The modal matrix $E(t)$ can be found by numerically integrating (4.5) over one period along with the physical states, where $A(t)$ is found from Hamilton’s equations using the Hessian of the geopotential function with respect to physical coordinates.

4.3 Structure of the Modal Solution

At this point, it is worthwhile to summarize briefly the structure of the low-eccentricity theory in the sense of KAM torus angles and the modal variables $\mathbf{y}$. In the low-eccentricity solution, there are three global angle variables, denoted Q_1 , Q_2 , and Q_3 . These angles represent, respectively, the mean argument of latitude (i.e., approximately the angle in the orbital plane from the ascending node to the current satellite position), the nodal displacement from Greenwich (which is essentially the Earth-fixed version of commonly-known Right Ascension of the Ascending Node from the ECI frame), and the argument of perigee analog. Figure 4.1 gives a graphical depiction of the angles Q_1 and Q_2 . In the

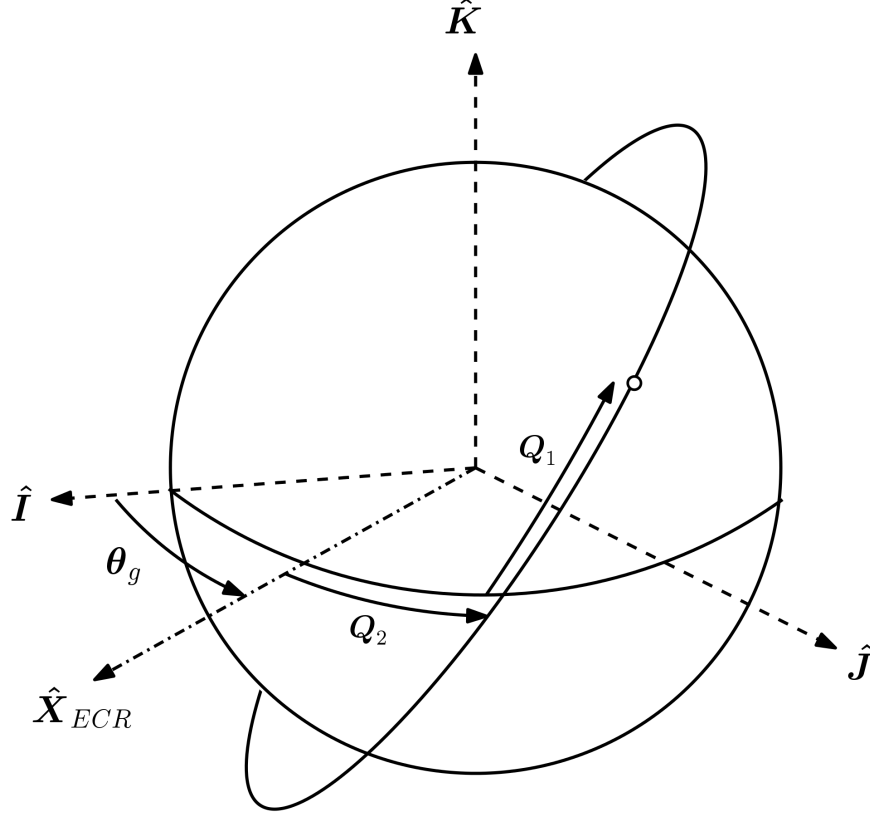


Figure 4.1: Torus Angles Q_1 and Q_2 in Periodic Orbit Solution

figure, the quantity θ_g is the angle of Greenwich's longitude (through which the ECR 1-axis $\hat{X}_{ECR}$ passes) with respect to the Earth-centered inertial frame's 1-axis ($\hat{I}$).

For low-eccentricity satellites studied in the current theory, these angles Q are equivalent to the angles φ discussed in the previous chapter (and studied in previous application of KAM theory to artificial satellites by the current author and others). We may see the parallel with the KAM theory discussed in the previous chapter in the following ways. First, these three angles increment at constant frequencies ω . Secondly, the first angle, Q_1 (the argument of latitude analog) is the so-called “fast” angle, evolving at frequency ω_1 which is close to the Keplerian frequency from the two-body problem, while the angles Q_2 and Q_3 possess much slower frequencies (by several orders of magnitude) and are results of the perturbations to the Keplerian system by the zonal gravitational effects.

The 3-torus described by the flow of the angles $\mathbf{Q}(t)$ is, then, a geometric structure in the phase space describing the dynamics of the physical system.

The modal system $\mathbf{y}$ possesses two so-called “degenerate modes” (corresponding to the two integrals of the motion for the zonal periodic orbit). Therefore, two of the y_i are local embodiments of displacements in the global angles Q_1 and Q_2 and two more of the y_i are the associated energy modes, which are local embodiments of the angular rates ω_1 and ω_2 . Specifically, in Wiesel’s work and in the current work, the modal variable y_1 is the local expression of the global angle Q_1 and the variable y_2 is its associated energy mode ω_1 ; similarly, y_3 corresponds to Q_2 and y_4 corresponds to ω_2 . Displacements in these four y_i coordinates can, to a point, be absorbed into their global counterparts, which maintains the “small displacements” necessary for applicability of the Floquet-like solution; however, the remaining two [non-degenerate] modes in $\mathbf{y}$, y_5 and y_6 , represent combinations of eccentricity and argument of perigee modes and therefore have no direct global analogs. Wiesel expands the torus further along these variables to include second order eccentricity terms in the perturbations, which extends the validity of the theory to orbits with slightly larger eccentricity, as discussed in the next section.

4.4 Perturbations

Having the periodic orbit and Floquet solution in hand, Wiesel [60] proceeds to incorporate perturbations into the solution. A perturbing acceleration is appended to the linear solution above as

$$\delta\dot{\mathbf{x}} = A(t)\delta\mathbf{x} + \dot{\mathbf{X}}_{pert} \quad (4.7)$$

and expanded about the periodic orbit as

$$\dot{\mathbf{X}}_{pert} = \dot{\mathbf{X}}_{pert}\Big|_{\mathbf{x}_{PO}} + \frac{\partial \dot{\mathbf{X}}_{pert}}{\partial \mathbf{X}}\Big|_{\mathbf{x}_{PO}} \delta\mathbf{x} + \dots \quad (4.8)$$

$$= \dot{\mathbf{X}}_{pert}\Big|_{\mathbf{x}_{PO}} + \frac{\partial \dot{\mathbf{X}}_{pert}}{\partial \mathbf{X}}\Big|_{\mathbf{x}_{PO}} E\mathbf{y} + \dots \quad (4.9)$$

where $\mathbf{x}_{PO}$ denotes evaluation on the periodic orbit. Keeping the perturbing terms to first order, the forcing term for the sectoral and tesseral geopotential effects can then be developed numerically as a double Fourier series in the angles Q_1 and Q_2 (the angle of the node from Greenwich in the equatorial plan; analog of φ_2 in the KAM section above). At first order, the perturbations to the modal variables $\mathbf{y}$ are decoupled.

As mentioned in the previous section, the fifth and sixth modal variables represent the argument of perigee/eccentricity mode, which means that the behavior of y_5 and y_6 are analogous to (but not equivalent to) the coordinates $e \sin \omega$, $e \cos \omega$ in two-body perturbations, and conveniently remain finite near zero eccentricity. Operating under the assumption of analyticity in the eccentricity, Wiesel expands the torus further along these variables to include second order eccentricity terms in the perturbations, which extends the validity of the theory to orbits with slightly larger eccentricity. The reader is referred to the Appendix and to Wiesel’s paper [60] for a full treatment and note here only that the result of the second-order eccentricity perturbations’ effect on the periodic orbit are captured in an additional term to the modal differential equation of the form

$$\dot{y}_{ecc_i} = \frac{1}{2} B''_{i\alpha\beta} y_\alpha(t_E) y_\beta(t_E) \quad (4.10)$$

where the tensor B'' is dependent upon the Keplerian-analog torus angle Q_1 and the change in apsidal torus angle Q_3 since epoch (t_E). Each term in the three-dimensional tensor B'' is a Fourier series in these two angles. The ramifications for the current research will be discussed in Chapter 6.

The low-eccentricity theory as developed by Wiesel also contains provisions for atmospheric drag as a perturbation; this is treated similarly to the conservative perturbations already discussed. The “specific” perturbing acceleration (i.e., the acceleration with the ballistic coefficient factored out) is developed as a Fourier series over one period. The resulting series solution can then be modulated by the ballistic coefficient for a given satellite upon solution summation and the drag effects can be determined. The drag force

predictably causes, due to assumptions of spherical symmetry of the atmosphere, primarily in a secular change in the energy mode y_2 and thereby a quadratic growth term in the fast angle y_1 (to be cast into Q_1). While this remains a perfectly valid drag perturbation method (to within the assumptions made of a non-stochastic, spherical atmosphere), it is not utilized in this work, for two reasons. The primary reason for exclusion is that the drag force acting on a constellation or formation of satellites will, under assumptions of geometric similarity between the individual satellites and their similarity in their attitudes, result in a net change of the overall constellation parameters, but very little change in the relative positioning of the satellites. That is, if we consider (for example) two identical satellites orbiting in a single orbital plane (only separated by a displacement in Q_1 , the relative velocity component of each satellite with the atmosphere at any specific location over the planet will be equal, and so the drag force will perturb them in the same fashion, acting to decrease the semi-major axes of the satellites identically. The second reason for exclusion of the drag force is that the current work is focused on feasibility and accuracy of maneuvering onto and between tori, and the drag force tends to change the fundamental descriptors of the tori. Specifically, the drag force steadily decreases the orbital energy and, as a result, after some amount of time, the core of the torus's development (the periodic orbit) is no longer a good approximation of the actual orbit. That is, if the semi-major axis has decreased significantly, there is a new and distinct periodic orbit describing the KAM torus core at that altitude. A full solution of the drag-perturbed problem requires re-evaluation of the periodic orbit (based on new observation data) at a frequency dependent on the rate of decay and the required state accuracy. While this continuous re-evaluation of drag-perturbed periodic orbits would be required for operational uses, it was seen by the author to obfuscate the true aim of the current work, for the reasons outlined above. As a result, the drag perturbation is not treated in detail here; for additional information, the reader is referred to [60].

4.5 LEST Summary

The end result of the application of LEST is a numerical package including sets of Fourier series which represent the reference zonal periodic orbit $\mathbf{X}_{PO}$, the modal matrix $E(t)$, and the various perturbations. The periodic solution itself forms a static (in the nodal frame) structure that is a function only of Q_1 . The forced solutions from the non-zonal geopotential perturbations then create a static geometric structure in the Earth-centered rotating frame, which is a function of torus coordinates (angles Q_1 and Q_2), not explicitly a function of time, and wraps around the Earth. The effects of second-order eccentricity depend upon the torus angles Q_1 and Q_3 , and are driven by the combined eccentricity/argument-of-perigee modes, y_5 and y_6 . These perturbations can cause oscillations and secular change in the modal state; the secular effects in y_1, y_2 can be included in the global torus frequencies.

The modal states $\mathbf{y}$ represent small displacements away from the perturbed periodic orbit, embodying small offsets in angles and energy, which manifest as motion on “nearby” tori. As a result, it is expected that, if constellations of satellites could be configured by displacements on the KAM surface (i.e., in modal space), the costs of station keeping to maintain those specific modal displacements would be driven not by inaccuracies in dynamical modeling, but rather by the limitations in spacecraft maneuvering capabilities and orbital determination.

V. Working with Low-Eccentricity Tori

5.1 Behavior of the Low-eccentricity Solution and the Effects of Conservative Perturbations

It is helpful in the application of LEST tori to satellite control problems to examine the behavior of the orbit and thereby obtain an intuition for the effects of the assorted perturbations as they vary by orbit type. This section aims to provide some qualitative examples of tori for several different orbits, along with discussion of the important features. Along the way, the accuracies of the ephemerides obtained from numerical evaluation of the LE torus description will be demonstrated, along with the associated implications.

We begin by discussing an example of a LEO satellite which possesses an orbital altitude of 630km (semi-major axis $a \approx 7008\text{km}$) and an inclination $i = 40^\circ$. The eccentricity is, of course, nearly zero due to the nature of the theory from which we are starting. This satellite has an orbital period of approximately 7.24 TU (≈ 97.3 minutes). Using the Low-Eccentricity (LE) LE torus method, we construct the periodic orbit and find that the core periodic orbit torus has the frequencies

$$\begin{aligned}\omega_1 &\approx 0.8682 \text{ rad/TU} \\ \omega_2 &\approx -0.05973 \text{ rad/TU} \\ \omega_3 &\approx -0.001126 \text{ rad/TU}\end{aligned}\tag{5.1}$$

and we notice that the nodal rate ω_2 and the apsidal rate ω_3 are, as expected, much smaller in magnitude than the Keplerian frequency analog, ω_1 . Also of note is that ω_2 given above is the rate of nodal precession combined with the rotation rate of the Earth, since the torus is constructed in the ECR frame. Even after removing the Earth's rotation rate (by the addition $\omega_2 + \omega_\oplus$), the resulting nodal rate is negative, as expected based on the satellite's inclination (recall that the node precesses westward for a prograde orbit).

If we define a simple, identically-zero modal state on the torus

$$y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (5.2)$$

we can analyze the effects of the different perturbations by separating the solution into its constituent parts. This is accomplished by summing the LEST solution series separately and examining their contributions to the final solution; it can be viewed as a sort of decomposition using the numerically-constructed Fourier series. For this particular torus, the so-called “free” state is found by summing the periodic orbit series itself. This trajectory in ECI coordinates is shown in Figure 5.1 as a function of time for one day. The plot shows the expected oscillatory behavior in all coordinates and momenta, with coordinate signals having an amplitude of approximately 1.1 Earth radii.

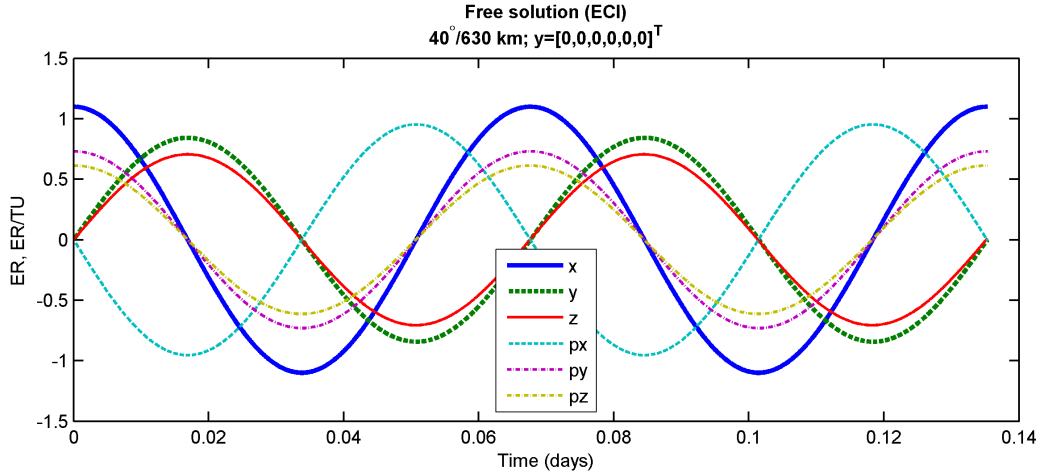


Figure 5.1: Free Solution of Physical Coordinates in ECI Frame;

630km, 40° orbit; $y = [0, 0, 0, 0, 0, 0]^T$

The sectoral and tesseral terms in the gravity field expansion obviously occupy an important place in the perturbation theory under discussion. The sectoral/tesseral effects on the modal coordinates can be found by performing the summation of the sectoral/tesseral perturbation

series for the periodic orbit as found in the LEST process. The modal coordinate variations due to these gravitational terms are shown in Figure 5.3 for a duration of one day. Figure 5.2 shows a closer view of the same perturbations (over the first two orbital periods). These sectoral/tesseral perturbations show similar behavior to that reported by Wiesel [60]. At this juncture, an important note about units is in order. The modal state y possesses mixed units: the states y_1 and y_3 are angle variables, and thus are in units of radians. Their associated energy modes y_2 and y_4 are in units of radians per time unit. The eccentricity modes y_5 and y_6 do not possess meaningful units, and their scaling is set by the periodic orbit construction. Throughout this work, when modal state results are displayed (as in, e.g., Figure 5.3), these mixed units are implied.

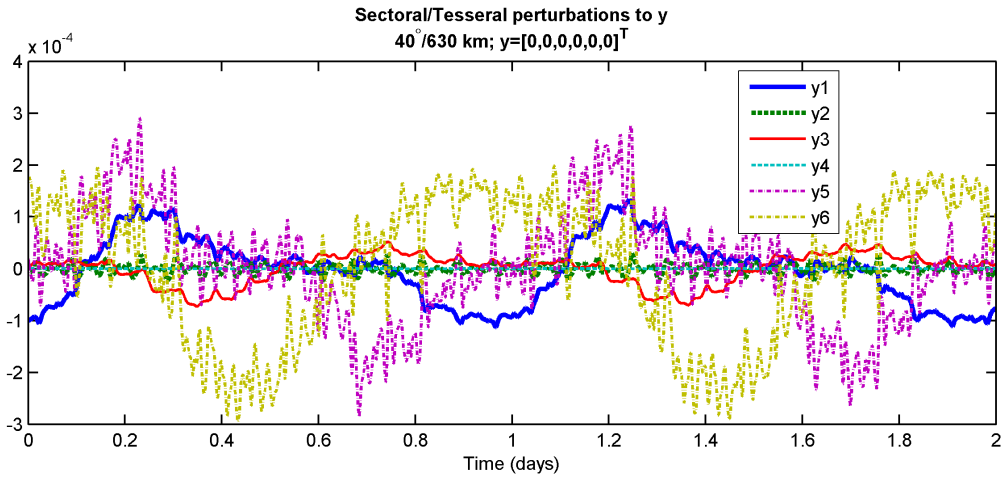


Figure 5.2: Sectoral/Tesseral Perturbations to Modal Coordinates (One Day);

$$630\text{km}, 40^\circ \text{ orbit}; y = [0, 0, 0, 0, 0, 0]^T$$

We note that, in Figure 5.3, it is evident that the sectoral/tesseral effects on the modal states are nearly periodic with period of one day. The fact that the period is nearly one day is a result of the Earth rotating beneath the satellite with a period of one day; the fact that the period is not exactly one day is a result of the precession of the node during that period (of magnitude approximately one-tenth of one radian). We can see the magnitude these effects have on the periodic orbit by transforming, at each point in time, the vector of perturbations

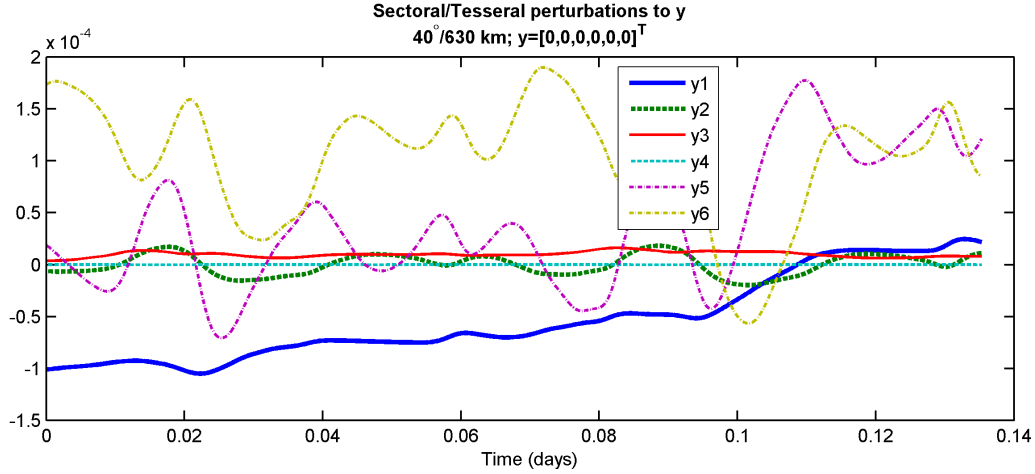


Figure 5.3: Sectoral/Tesseral Perturbations to Modal Coordinates (Two Orbits);

630km, 40° orbit; $y = [0, 0, 0, 0, 0, 0]^T$

in the modal coordinates into physical units. To do this, we utilize the modal matrix $E(t)$, so

$$\delta \mathbf{x}_{ST} = R_{2Z}(Q_2(t))E(Q_1(t))\delta \mathbf{y}_{ST} \quad (5.3)$$

where R_{2Z} is a 6×6 matrix composed of two Euler 3-axis rotations on the diagonals and zeros elsewhere, since the coordinates and momenta transform similarly. The resulting vector $\delta \mathbf{x}_{ST}$ is the representation of the change in the coordinates due to the perturbative $\delta \mathbf{y}$ represented in the ECR frame (recalling that Q_2 is the torus angle describing the node with respect to Greenwich, or any other reference we may choose). An additional 3-axis rotation (through the angle describing Greenwich with respect to the first point of Ares) will provide us the ECI representation of $\delta \mathbf{x}_{ST}$, which is shown in Figure 5.4.

One immediate conclusion that can be drawn from comparing the modal and inertial plots given above (in Figures 5.3 and 5.4) is the degree of decoupling of the perturbations that is afforded by representing them in modal coordinates. That is, we can see that, in physical coordinates, the perturbations are highly coupled; however, in the modal coordinates, the perturbations manifest largely in changes to the eccentricity modes y_5 , y_6

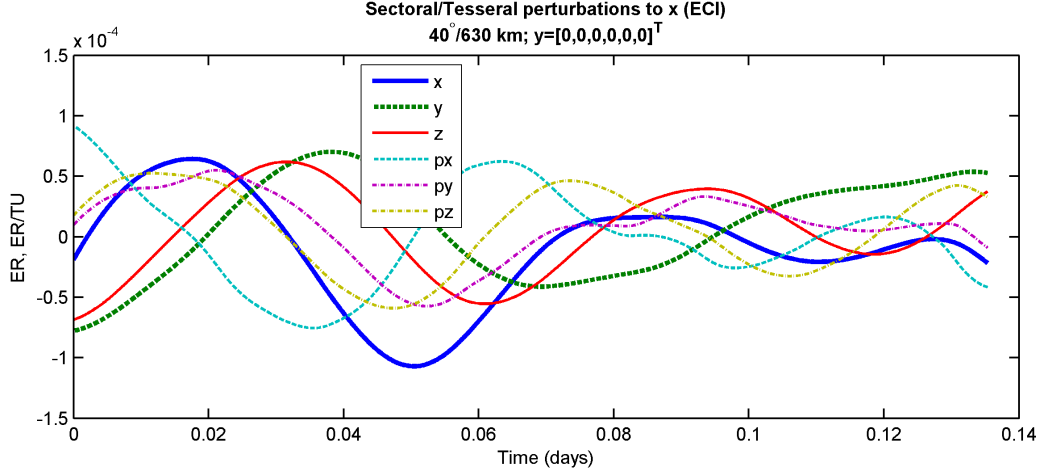


Figure 5.4: Sectoral/Tesseral Perturbations to x (ECI);

630km, 40° orbit; $y = [0, 0, 0, 0, 0, 0]^T$

and angle modes y_1, y_3 , with very little effect on the energy modes y_2 and y_4 . Also apparent from the figures is that the magnitudes of the perturbations in physical coordinates due to sectoral and tesseral harmonics are, as expected, concentrated at a few parts in 10^5 , one to two orders of magnitude lower than the perturbations to the two-body problem from the zonal harmonics. This clearly displays an advantage of beginning with the zonal periodic orbit as the reference solution.

We notice that, in the modal state (5.2), the eccentricity-related states (y_5, y_6) are zero, which implies that the perturbations due to second-order eccentricity terms should also be identically zero. Likewise, through the transformation relationship similar to (5.3), the second-order eccentricity perturbations to the physical coordinates are zero. Of course, if we increase one eccentricity state y_5 , giving a modal state to $y = [0, 0, 0, 0, 0.1, 0]^T$, we find that the second-order eccentricity perturbations appear in all modal coordinates, as shown in Figure 5.5, with associated effects in physical states shown in Figure 5.6.

Several important notes are in order regarding the above results:

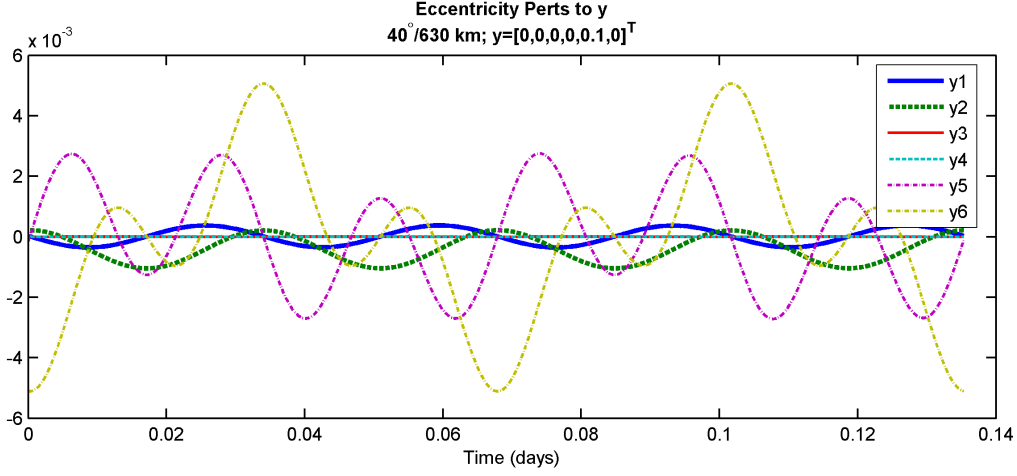


Figure 5.5: Second-order Eccentricity Perturbations to Modal Coordinates;

630km, 40° orbit; $y = [0, 0, 0, 0, 0.1, 0]^T$

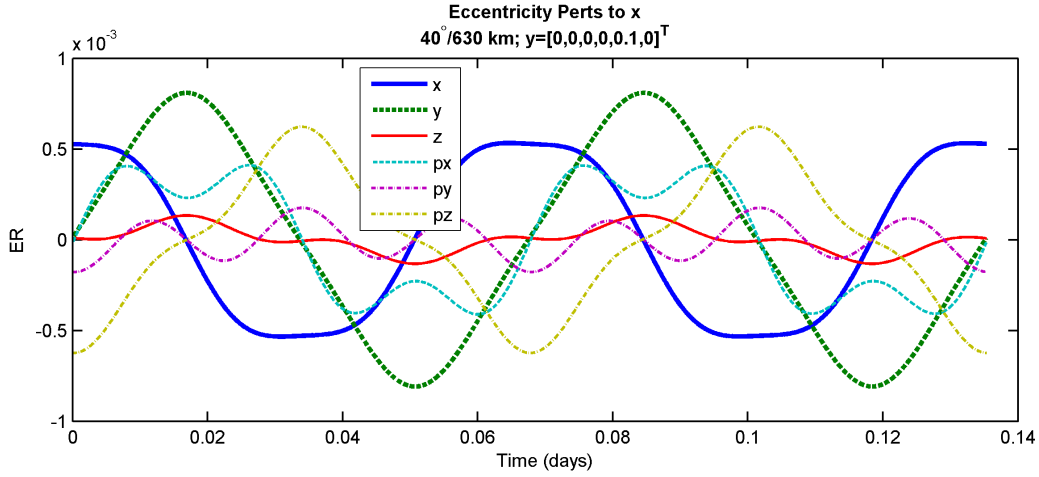


Figure 5.6: Sectoral/Tesseral Perturbations to x (ECI);

630km, 40° orbit; $y = [0, 0, 0, 0, 0.1, 0]^T$

- As cautioned by Wiesel, the eccentricity modes y_5 and y_6 , while analogous to the terms $e \cos \omega$, $e \sin \omega$ in classical orbital theory, are not exactly the same. Also, the scaling on the terms is determined by the perturbation series determination process and is somewhat arbitrary; this means that the sum of the squares of the terms y_5

and y_6 is not, in general, equal to the eccentricity. In the case given above, for $y_5 = 0.1, y_6 = 0$, the eccentricity at the start of the orbit is $e \approx 0.012$.

- The eccentricity perturbations discussed and shown in the plots are the *second-order* perturbations in eccentricity only. The first-order eccentricity effects are included in the formulation of the periodic orbit itself, as discussed previously.
- Although overshadowed by the periodic terms, both the eccentricity perturbations and sectoral/tesseral perturbations may have small constant terms in their Fourier series representations, which embody secular terms in the modal coordinates. For the orbit shown, the secular rate induced in the angle Q_1 (which manifests as an addition to y_2) by the second-order eccentricity series is approximately -4.2×10^{-4} rad/TU. The rates induced by the sectoral/tesseral perturbations are zero to double precision. The total influence of these perturbations on the nodal rate y_4 is several orders of magnitude lower, at about 7.2×10^{-7} rad/TU. As we will see, these secular rates must be handled properly during development of the maneuvering strategy.

5.2 Torus Fidelity

After the brief introduction to the effects that make up a low-eccentricity KAM torus, a natural question that arises is “how accurately does the numerically-computed torus solution reproduce the real state?” The answer to this question varies, of course, depending on the parameters of the orbit: for orbits with very low eccentricities and with inclinations and semi-major axes similar to the reference periodic orbit, the reproduction is quite accurate. (Note that, recalling the discussion in §4.3, these conditions are met as $\{y_2, y_4, y_5, y_6\} \rightarrow 0$; that is, the eccentricity is small through small y_5, y_6 and the orbital energy is the same as on the periodic orbit since the degenerate mode energy displacements y_2, y_4 are small.) Figure 5.7 shows the residuals between a numerical integration (in the geopotential up to $m, n = 20$) of our 630km/40° orbit and the associated calculated

coordinates from the LE torus for a period of one day. This torus has a modal state of identically zero. Figure 5.8 shows the residuals in the momenta. In this case, the agreement is very good, achieving sub-meter RMS position error and velocity errors less than 1 mm/s.

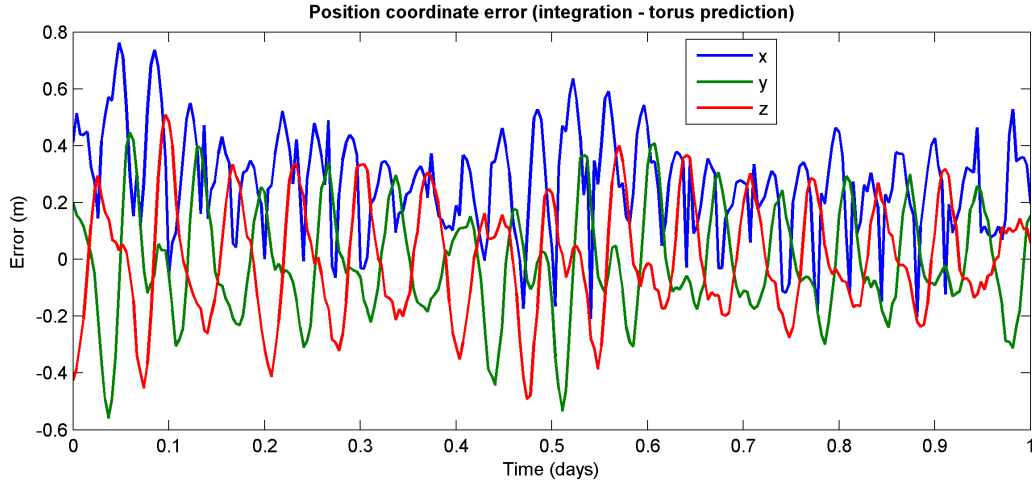


Figure 5.7: ECI Position Residuals (Numerical Integration - Torus Prediction);

$$630\text{km}, 40^\circ \text{ orbit}; y = [0, 0, 0, 0, 0, 0]^T$$

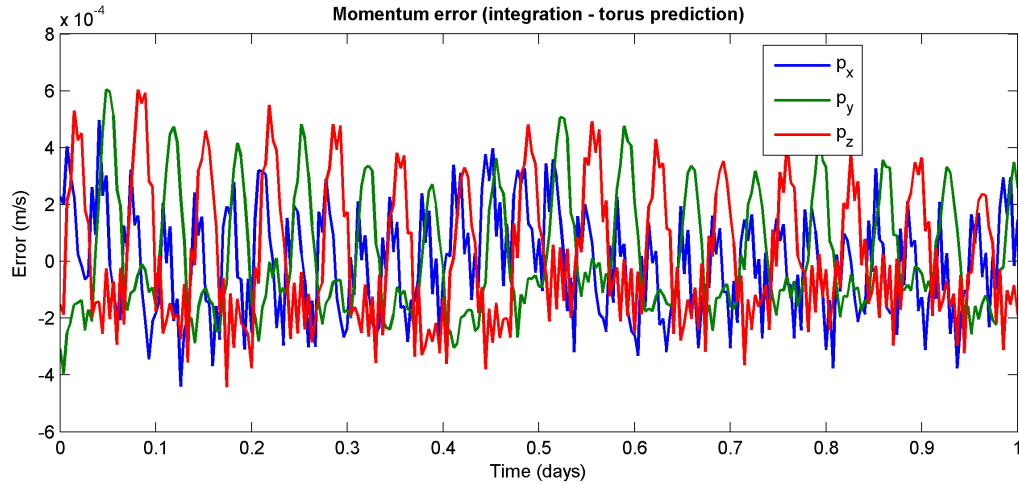


Figure 5.8: ECI Momentum Residuals (Numerical Integration - Torus Prediction);

$$630\text{km}, 40^\circ \text{ orbit}; y = [0, 0, 0, 0, 0, 0]^T$$

With the introduction of a non-zero eccentricity value in the modal state, the position residuals increase to a few tens of meters and the specific momentum residuals increase to centimeters per second, exposing the weakness of the solution to higher eccentricities; this lies in the fact that the eccentricity terms are only included to second order in the perturbation solution. However, the fit is still reasonable and, importantly for our purposes, there is no secular growth in the residuals. These plots are shown in Figures 5.9 and 5.10.

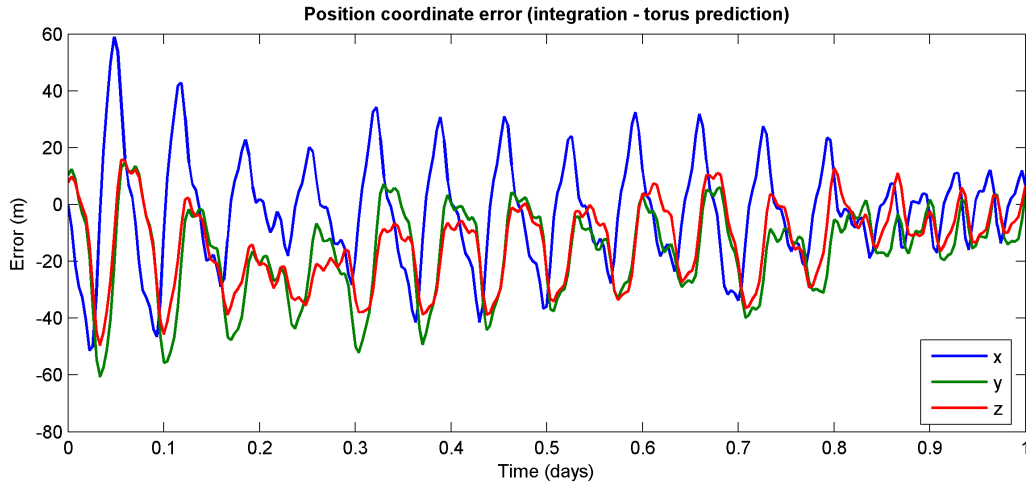


Figure 5.9: ECI Position Residuals (Numerical Integration - Torus Prediction);

$$630\text{km}, 40^\circ \text{ orbit}; y = [0, 0, 0, 0, 0, 0.1]^T$$

Finally, we present the residuals from a high-altitude orbit; this orbit is similar in parameters to those used by the Galileo system: 23222km altitude with $i = 56^\circ$. The position and momentum residuals for this orbit with zero modal state are displayed in Figures 5.11 and 5.12, respectively. They demonstrate much larger amplitude oscillations, with position residuals reaching several hundred meters and velocity residuals in the 1-10 centimeter per second regime. Adding further small displacements to the modal state yields similar results. These observations of “large” residuals must be tempered, however, with the additional observation that the residuals as a fraction of the higher orbital radius are still quite low. The position residuals shown in these plots correspond to maximum oscillations

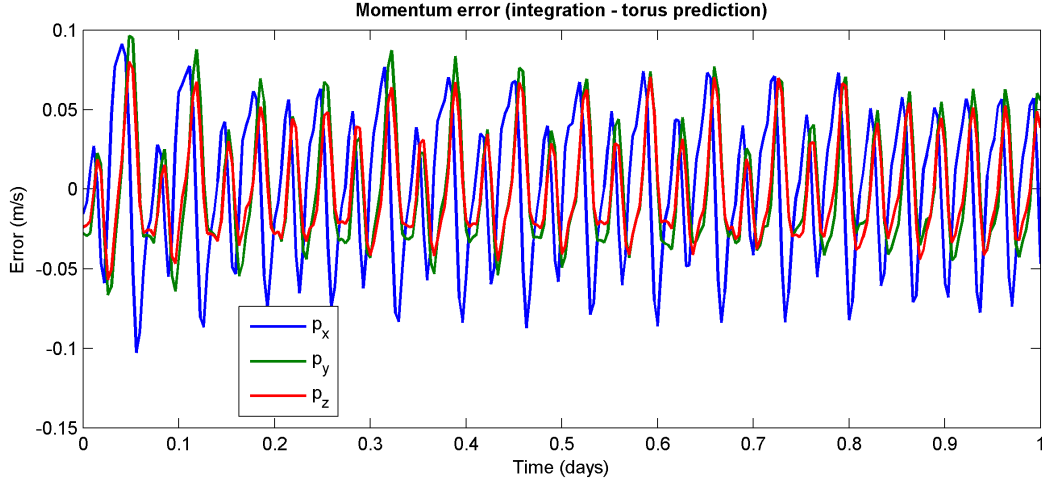


Figure 5.10: ECI Momentum Residuals (Numerical Integration - Torus Prediction);

$$630\text{km}, 40^\circ \text{ orbit; } y = [0, 0, 0, 0, 0, 0.1]^T$$

in observations from the Earth on the order of 0.0004 degrees. Most importantly, since the residuals still show no apparent secular growth, they provide further evidence that the low-eccentricity solution indeed describes the underlying KAM torus.

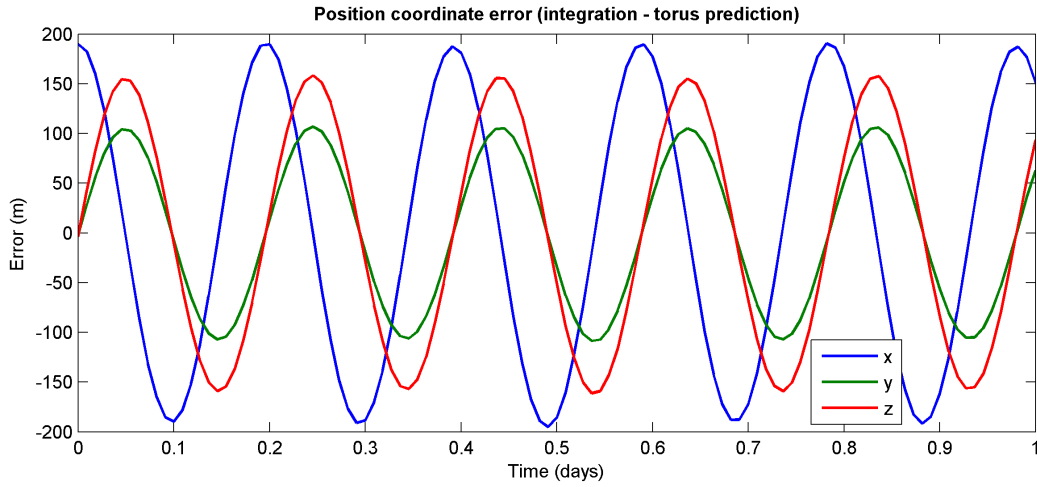


Figure 5.11: ECI Position Residuals (Numerical Integration - Torus Prediction);

$$23222\text{km}, 56^\circ \text{ orbit; } y = [0, 0, 0, 0, 0, 0]^T$$

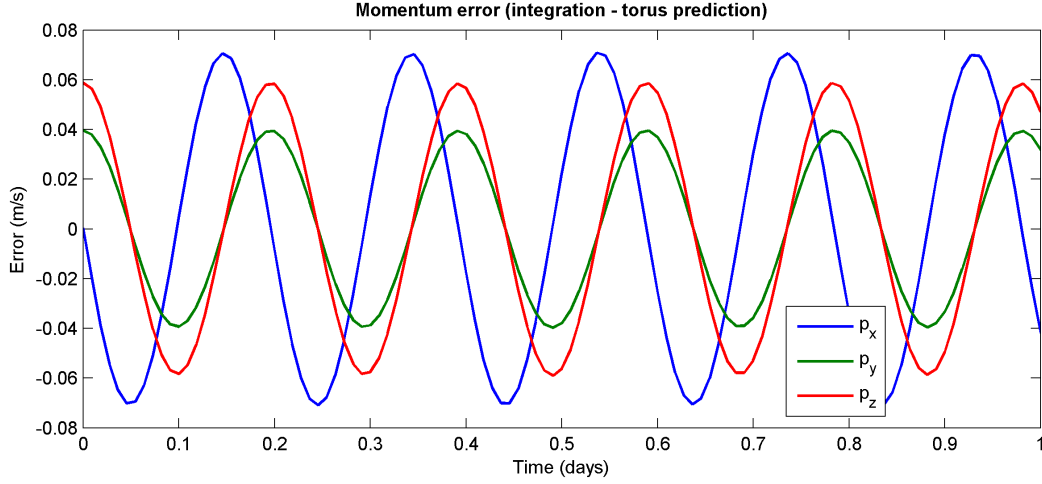


Figure 5.12: ECI Momentum Residuals (Numerical Integration - Torus Prediction);

$$23222\text{km}, 56^\circ \text{ orbit; } \mathbf{y} = [0, 0, 0, 0, 0, 0]^T$$

5.3 Bayes Estimation of Torus States

Any realistic application of a solution to a control problem must include a method to obtain an estimate of the state of the system. In the case of low-eccentricity tori, a solution has already been implied in the above discussions: due to small (but non-zero) residuals in position and momentum between the torus-predicted ECR states and the “true” (numerically integrated) states, we should, at minimum, implement some method of batch or sequential estimation strategy. The primary strategy implemented for the current work is a Bayes filter estimation method which can function in either a batch or sequential mode, although we use it in the former. This section describes the structure of the Bayes estimator used for the current work.

The general Bayes estimator is a standard least-squares scheme in which a state correction is calculated from observations and a covariance matrix. Specifically, given a state at time t_i , we have, from a previous time t_0 , a reference torus state estimate $\mathbf{y}_{ref}$ and an associated covariance matrix $P(t_0)$ which encompasses the uncertainties in the state estimate. These may be propagated to the current time t_i using the assumed or known

dynamics of the system, yielding $\hat{\mathbf{y}}(t_i)$ and $P(t_i)^-$. We assume we obtain a measurement at the current time, $\mathbf{z}_i$, which is ideally related to the state through the observation equation:

$$\mathbf{z}_i = \mathbf{g}(\hat{\mathbf{y}}(t_i), t_i) \quad (5.4)$$

At time t_i , it is necessary to obtain a linearization of the function g as

$$\tilde{G} = \frac{\partial \mathbf{g}(\hat{\mathbf{y}}, t)}{\partial \mathbf{y}} \quad (5.5)$$

from which we can calculate the so-called observation matrix:

$$T_i = \tilde{G}\Phi(t_i, t_0) \quad (5.6)$$

where Φ is the state transition matrix for the state from the reference time. We can then form the residual vector by:

$$\mathbf{r}_i = \mathbf{z}_i - \mathbf{g}(\hat{\mathbf{y}}(t_i), t_i) \quad (5.7)$$

The inverse covariance is then updated as

$$P^{-1}(t_i)^+ = P^{-1}(t_i)^- + T_i^T Q_i^{-1} T_i \quad (5.8)$$

where Q_i is the covariance of the new measurement $\mathbf{z}_i$. The estimate of the new state correction is given by

$$\Delta \mathbf{y}(t_i) = P^{-1}(t_i)^+ \left(P^{-1}(t_i)^- (\hat{\mathbf{y}} - \mathbf{y}_{ref}) + T_i^T Q_i^{-1} \mathbf{r}_i \right) \quad (5.9)$$

from which we can calculate the update to the reference solution for this iteration

$$\mathbf{y}_{ref_k} = \mathbf{y}_{ref} + \Delta \mathbf{y}(t_i) \quad (5.10)$$

The entire process is iterated until convergence; that is, until $\Delta \mathbf{y}(t_i)$ becomes sufficiently small by some appropriate measure. For batch processing, the covariance and state update

equations can be replaced with a sum over all measurements:

$$\begin{aligned}
P^{-1}(t_i)^+ &= P^{-1}(t_i)^- + \sum_{i=1}^N T_i^T Q_i^{-1} T_i \\
\Delta y(t_i) &= P^{-1}(t_i)^+ \left(P^{-1}(t_i)^- (\hat{\mathbf{y}} - \mathbf{y}_{ref}) + \sum_{i=1}^N T_i^T Q_i^{-1} \mathbf{r}_i \right)
\end{aligned} \tag{5.11}$$

and the process remains unchanged.

The application of the above process to the problem of torus estimation requires determination of the measured [physical] state at time t_i given some reference modal state at epoch. This is accomplished by simply assigning the reference modal state to the torus and performing the appropriate numerical sums in the periodic orbit and perturbations series, and propagating the results to t_i by the state transition matrix:

$$\begin{aligned}
\Phi(t, t_0) &= e^{\mathcal{J}(t-t_0)} \\
&= \begin{bmatrix} 1 & t-t_0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t-t_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(Q_3) & \sin(Q_3) \\ 0 & 0 & 0 & 0 & -\sin(Q_3) & \cos(Q_3) \end{bmatrix}
\end{aligned} \tag{5.12}$$

where $Q_3 = \omega_3(t - t_0)$ is the change in the third torus coordinate (the analog of apsidal regression). The linearized observation function necessary to convert the observed physical states to modal states is simply the modal matrix transformed by the state transition matrix:

$$\begin{aligned}
\frac{\partial \mathbf{g}(\hat{\mathbf{y}}, t)}{\partial \mathbf{y}} &= \frac{\partial \mathbf{x}(t)}{\partial \mathbf{y}(t_0)} \\
&= \Phi(t, t_0) E(Q_1(t))
\end{aligned} \tag{5.13}$$

By using the method outlined in this section, it is possible to fit a torus to a series of observed physical data. We demonstrate this by performing estimation on the 630km, 40° orbit whose modal coordinates are defined by $\mathbf{y} = [0, 1\text{E-}5, 0, -1\text{E-}7, 1\text{E-}6, 1\text{E-}3]^T$, with torus coordinates $\mathbf{Q} = [0.01, 0.5]^T$. The error in the modal state as a function of time is

shown in Figure 5.13, and the associated errors in physical position are shown in Figure 5.14. For this demonstration, the initial modal state y_{ref} was chosen to be identically zero, which led to large errors at the beginning as the estimator had difficulty converging with few input data points. However, within a reasonably short period of time, the estimator converged to very near the actual state, as reflected in the modal plot, and gave sub-meter position residuals in physical coordinates. It is noteworthy that, for estimation purposes, we transfer all of the local displacement y_1 and y_3 into the torus coordinates Q ; this allows for greater accuracy, as it amounts to choosing a point on the torus surface closest to the point we are estimating, as suggested by Wiesel [60].

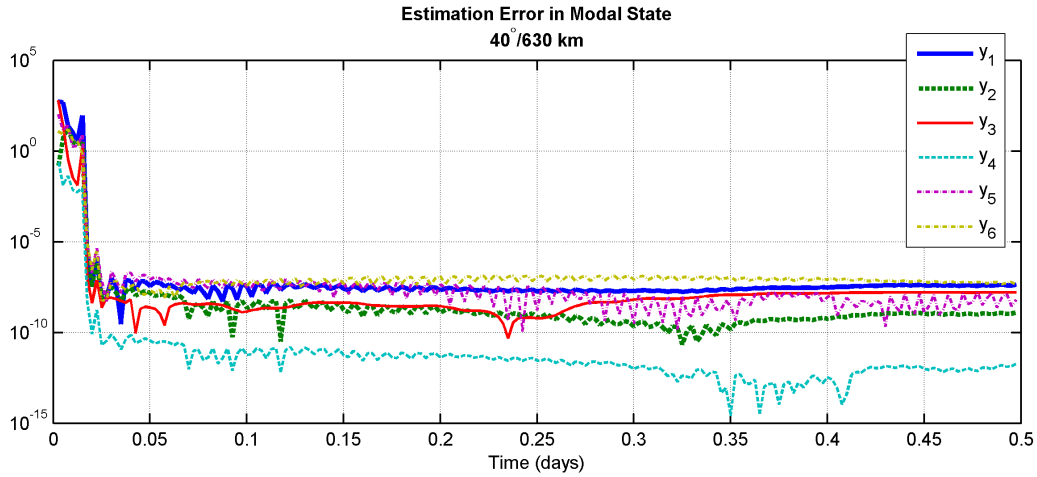


Figure 5.13: Estimation error in Modal States;

630km, 40° orbit; $y = [0, 1E-5, 0, -1E-7, 1E-6, 1E-3]^T$, $Q = [0.01, 0.5]^T$

5.4 The Inverse Transform: Torus States to Physical

In the course of numerical simulation and analysis, it is often necessary to transform from torus/modal states to physical states. This must be performed, for example, to obtain initial conditions for a numerical integration. This is somewhat more complex than the estimation problem. While it is quite straightforward to obtain a physical state

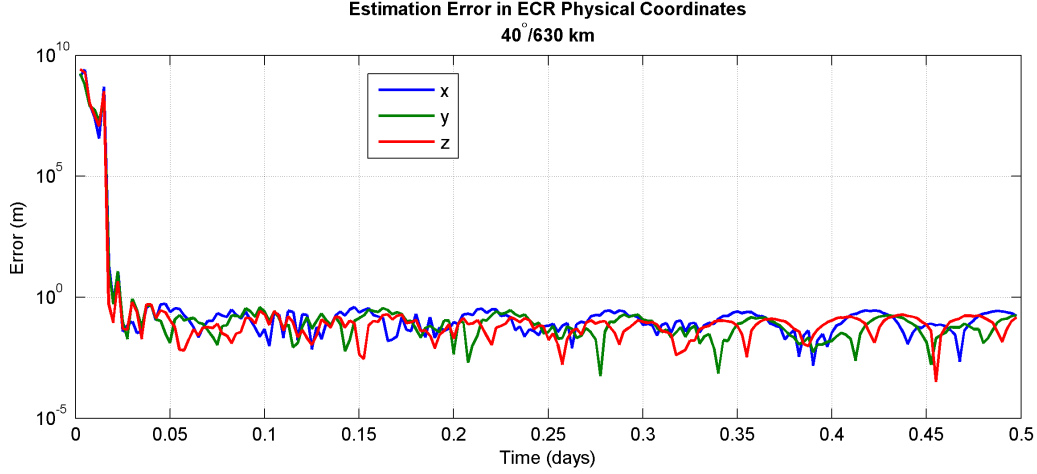


Figure 5.14: Estimation error in Physical Coordinates;

$$630\text{km}, 40^\circ \text{ orbit}; y = [0, 1\text{E-}5, 0, -1\text{E-}7, 1\text{E-}6, 1\text{E-}3]^T, Q = [0.01, 0.5]^T$$

from the torus numerical package by summing the assorted series and applying the modal perturbations to the reference periodic solution, there are small errors in the position and velocity states. We have seen these small errors in the plots given thus far; for even the best-fit tori, the discrepancies between the torus-predicted and actual physical states due to numerical precision limits and truncations in the torus construction can be on the order of a few meters for position (which is not particularly deleterious). However, the velocity errors, while relatively small (on the order of centimeters per second), would still contribute to growth in position residuals of several kilometers over a few days in most cases.

To combat this error growth and obtain the most accurate physical states from the torus package, we use an iterative fitting technique. The method begins with a best guess of the state at epoch, x_0 , derived from direct summation of the torus series package. Then, this state is numerically propagated forward in the standard geopotential until some estimation time boundary, t_e . These integrated data are passed into the torus-fitting routine outlined above, which estimates the torus states (at epoch); we call this estimate y_{est_i} . We may then

define an error vector

$$\mathbf{e}_i = \mathbf{y}_d - \mathbf{y}_{est_i} \quad (5.14)$$

where $\mathbf{y}_d$ is the desired torus state at epoch. We can then update the physical state at epoch by utilizing the tangent space linearization at epoch:

$$\begin{aligned} \mathbf{x}_{0_{i+1}} &= \mathbf{x}_{0_i} + \frac{\partial \mathbf{x}(t_0)}{\partial \mathbf{y}(t_0)} \mathbf{e}_i \\ &= \mathbf{x}_{0_i} + E(\mathbf{y}_i(t_0)) \mathbf{e}_i \end{aligned} \quad (5.15)$$

This process is then repeated until the correction to the physical state (i.e., $E(\mathbf{y}_i(t_0))\mathbf{e}_i$) has converged in the absolute value of each element to some tolerance; we typically use a value between 1E-12 and 1E-11, translating to a maximum final differential correction on the order of tens of microns (in physical coordinates) and tens of nanometers per second (in momentum variables). With an initial guess of $\mathbf{x}_0$ sufficiently close to its true value (which is almost always provided by the torus series), the process usually converges within fewer than five iterations.

VI. Low-Eccentricity KAM Torus Impulsive Maneuver Theory

6.1 Overview

Strategies for spacecraft maneuvering typically fall into one of two categories: impulsive and continuous. Continuous maneuvers are usually characterized by the use of low-thrust electric, cold-gas or chemical engines or thrusters, with the magnitude of force produced less than a few Newtons, with non-stop operation for long periods of time. In contrast, impulsive maneuvers are maneuvers with relatively large amounts of thrust operating for very short periods of time. To be precise, the term “impulsive” is somewhat abusive, since thrust occurs for some finite amount of time with some finite magnitude. However, since the maneuver “burns” are on significantly smaller time scales than the orbital period, it is common practice to approximate them as impulsive for the purposes of modeling and simulation.

This chapter is concerned solely with impulsive control; specifically, a two-maneuver impulsive control strategy for low-eccentricity KAM tori is developed. In the general case, two separate maneuvers are required to move from one torus to another, or from specific torus coordinates to different coordinates on the same torus. This is conceptually verified in the context of KAM theory by consideration of the tangent space approximation introduced in Chapter 3 and developed in Chapter 4. Recall that for some general KAM torus defined (in a near-Hamilton-Jacobi sense) by actions $\mathbf{I}$, the coordinates $\mathbf{Q}$ flow as

$$\mathbf{Q} = \mathbf{Q}_0 + \boldsymbol{\omega}(\mathbf{I})\Delta t \quad (6.1)$$

and thus trajectories are constrained for all time to an n -torus embedded in the $2n$ -dimensional phase space (where n is the dimension of the physical system configuration space). To relocate to another position $\hat{\mathbf{Q}}$ on the same torus at some future time, it would be necessary to change the propagation frequency of the torus coordinates, which amounts

to a shift onto a different, “transfer” torus with coordinate flow

$$\mathbf{Q} = \mathbf{Q}_0 + \tilde{\omega}(\tilde{\mathbf{I}})\Delta t \quad (6.2)$$

where $\mathbf{I} \neq \tilde{\mathbf{I}}$ and $\omega \neq \tilde{\omega}$. This is accomplished by a controlled change in the momenta, $\delta\mathbf{I}$, which causes an approximate torus frequency change according to:

$$\delta\omega \approx \frac{\partial\omega(\mathbf{I})}{\partial\mathbf{I}}\delta\mathbf{I} \quad (6.3)$$

This would yield new torus frequencies according to:

$$\tilde{\omega} = \omega + \delta\omega \quad (6.4)$$

Once the fundamental frequencies change (by transferring to another torus), they remain constant on that torus; therefore, to revert to the original frequencies (or some other appropriate frequencies associated with a final desired torus), a second burn is required. The entire two-burn process can be imagined as “drifting” between points on a torus by temporarily changing to an intermediate torus.

Unfortunately, because the general, literal formulation of the KAM torus for an arbitrary orbit has yet to be achieved (as discussed in §3.3), the frequency change matrix $\partial(\omega(\mathbf{I}))/\partial\mathbf{I}$ has been unavailable for use in control theory. In the current work, however, we derive a control strategy utilizing the low-eccentricity KAM tori developed in earlier chapters. This allows an application of the theoretical process above to real-world systems, limited, of course, to satellites with low eccentricity. In practice, we do not see this limitation as a major hindrance, since the vast majority of operational and proposed satellites have low eccentricities. In fact, according to the UCS catalog of operational satellites (including launches through 8/31/15), over 96% of operational satellites have orbits with eccentricities below 0.1, and over 90% have orbits with eccentricities below 0.01 [7]. Therefore, we expect the method develop subsequently to have the potential for wide application in both operational and proposed systems.

6.2 Two-impulse Maneuvers

To begin, we define a modified modal state, which we will term the “combined modal state” at time t , as $\tilde{\mathbf{y}}(t)$. This combined modal state is formed by casting the torus angle variables and frequencies back into the modal state; that is

$$\tilde{\mathbf{y}}(t) = \begin{bmatrix} y_1(t) + Q_1(t) \\ y_2(t) + \omega_1(t) \\ y_3(t) + Q_2(t) \\ y_4(t) + \omega_2(t) \\ y_5(t) \\ y_6(t) \end{bmatrix} \quad (6.5)$$

Since the modal states propagate through the modal state transition matrix [60]

$$\begin{aligned} \Phi(t, t_0) &= e^{\mathcal{J}(t-t_0)} \\ &= \begin{bmatrix} 1 & t-t_0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t-t_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(Q_3) & \sin(Q_3) \\ 0 & 0 & 0 & 0 & -\sin(Q_3) & \cos(Q_3) \end{bmatrix} \end{aligned} \quad (6.6)$$

as in

$$\mathbf{y}(t) = \Phi(t, t_0)\mathbf{y}_{t_0} \quad (6.7)$$

we can write the propagation of the combined modal state in a similar form if we temporarily neglect forced perturbations:

$$\tilde{\mathbf{y}}(t) = \Phi(t, t_0)\tilde{\mathbf{y}}(t_0) \quad (6.8)$$

In the above equation, it is important to note that the periodic orbit angles and frequencies have been combined with the modal variables. Since the modal angles y_1 and y_3 are local analogues of the global angles Q_1 and Q_2 and the modal variables y_2 and y_4 are local analogues of the global frequencies ω_1 and ω_2 , the propagated combined modal variables are not easily separated into the contributions from each individual motion. For

the purposes of orbit determination, in terms of the modal variables, it is important to keep the modal and periodic orbit variables separate; however, for the purposes of maneuvers, it is useful to consider the similar effects together.

We now consider the mechanics of a fixed-interval, 2-impulse maneuver sequence. For the remainder of this discussion, the instant in time before a maneuver will be represented as t_n^- , $n \in \mathbb{Z}^+$, and the instant after a maneuver is given by t_n^+ . Given a periodic orbit with its associated frequencies ω , if we know the torus states (Q_1, Q_2) at a particular initial time t_0 , we can write the free propagation of the combined modal states up to some maneuver time t_1 simply as

$$\tilde{\mathbf{y}}_{fr}(t_1^-) = \Phi(\Delta t_{1,0})\tilde{\mathbf{y}}(t_0) + \delta\mathbf{y}_{sec}^{1,0} \quad (6.9)$$

where $\tilde{\mathbf{y}}_{fr}$ is the free modal state, $\Delta t_{1,0} = t_1 - t_0$ and $\delta\mathbf{y}_{sec}^{1,0}$ represents the secular change in the $\tilde{\mathbf{y}}$ coordinates over the time interval $[t_0, t_1]$ due to sectoral and tesseral harmonics and second-order orbital eccentricity terms. The term $\delta\mathbf{y}_{sec}^{1,0}$ may be calculated numerically using the perturbation Fourier series. That is, we can consider the change over a time period due to secular perturbation terms as

$$\delta\mathbf{y}_{sec} = \delta\mathbf{y}_{ST} + \delta\mathbf{y}_{ecc} \quad (6.10)$$

where the contribution from sectoral/tesseral harmonics, $\delta\mathbf{y}_{ST}$, can be found from the constant term of each Fourier series for the sectoral/tesseral perturbations. In general, the perturbation to each coordinate y_i is represented by a Fourier series in the angles Q_1 and Q_2 ; however, any constant terms in these six series manifests as either a constant angle offset (in the case of y_1, y_3 and which we simply absorb into the global analogs Q_1, Q_2), a secular angular growth (from constant offsets in the energy modes y_2, y_4), or quadratic angular growth (which has been observed in all cases examined to be insignificantly small when atmospheric drag is excluded, and is thus hereafter neglected). The growth in the modal coordinates due to the second-order eccentricity terms, $\delta\mathbf{y}_{ecc}$, is slightly more complex; the

perturbation's contribution to the modal state differential equations has the form

$$\dot{y}_{ecc_i} = \frac{1}{2} B''_{i\alpha\beta} y_\alpha(t_E) y_\beta(t_E) \quad (6.11)$$

where the Greek indices are summed from five to six (i.e., including only the eccentricity modes y_5, y_6), $i = 1, 2, \dots, 6$, and the tensor B'' is dependent upon the Keplerian torus angle Q_1 and the change in apsidal torus angle Q_3 since epoch (t_E). Each term in B'' is a Fourier series in these two angles, and linear combinations of the series' constant terms, modulated of course by the eccentricity states y_5 and y_6 , cause secular drift in the modal coordinates y . Again, these effects manifest in y_1, y_3 as constant offsets, secular growth, or quadratic growth. The constant offsets are absorbed into the global torus angles, the secular growth contributes to the changes embodied in δy_{sec} , and the quadratic growth is negligible. The term δy_{sec} is clearly very small for orbits with eccentricities of zero or very near zero.

Now, an impulsive maneuver $\delta X(t_1)$ in ECEF coordinates at time t_1 will change the combined modal state by

$$\delta \tilde{y}(t_1) = E(Q_1(t_1^-))^{-1} R_{2Z}^T(Q_2(t_1^-)) \delta X(t_1) \quad (6.12)$$

(where $R_{2Z}(Q_2(t))$ is the rotation matrix from the nodal to the ECR frame, described in §5.1), which leads to the state after the maneuver:

$$\tilde{y}(t_1^+) = \tilde{y}_{fr}(t_1^-) + \delta \tilde{y}(t_1) \quad (6.13)$$

The combined modal state can then be propagated to the time of the second maneuver by:

$$\tilde{y}_{fr}(t_2^-) = \Phi(\Delta t_{2,1}) \tilde{y}(t_1^+) + \delta y_{sec}^{2,1} \quad (6.14)$$

where, again, $\delta y_{sec}^{2,1}$ represents the secular change in the modal state between t_1 and t_2 resulting from sectoral/tesseral and eccentricity perturbations. It should be noted that, even though the term $\delta y_{sec}^{1,0}$ may be nearly zero (depending on the starting orbit), the secular term $\delta y_{sec}^{2,1}$ may not be insignificant, since the maneuver changes the eccentricity of the orbit by

some amount proportional to $\|\delta\mathbf{X}_1\|$, and thus induces a secular drift with respect to the reference periodic orbit.

The second maneuver is handled similarly to the first:

$$\delta\tilde{\mathbf{y}}(t_2) = E(Q_1(t_2^-))^{-1}R_{2Z}^T(Q_2(t_2^-))\delta\mathbf{X}(t_2) \quad (6.15)$$

So, the combined modal state after the second maneuver is given by

$$\tilde{\mathbf{y}}(t_2^+) = \tilde{\mathbf{y}}_{fr}(t_2^-) + \delta\tilde{\mathbf{y}}(t_2) \quad (6.16)$$

This state, $\tilde{\mathbf{y}}(t_2^+)$, can then be propagated to any final time of interest, t_f , in the same manner as above:

$$\tilde{\mathbf{y}}_{fr}(t_f) = \Phi(\Delta t_{f,2})\tilde{\mathbf{y}}(t_2^+) + \delta\mathbf{y}_{sec}^{f,2} \quad (6.17)$$

The final state achieved with the set $\{t_1, t_2\}$ and $\{\delta\mathbf{X}_1, \delta\mathbf{X}_2\}$ is then

$$\begin{aligned} \tilde{\mathbf{y}}_{fr}(t_f) = & \Phi(\Delta t_{f,0})\tilde{\mathbf{y}}(t_0) + \Phi(\Delta t_{f,1})\left(E(Q_1(t_1^-))^{-1}R_{2Z}^T(Q_2(t_1^-))\delta\mathbf{X}(t_1) + \delta\mathbf{y}_{sec}^{1,0}\right) + \dots \\ & \Phi(\Delta t_{f,2})\left(E(Q_1(t_2^-))^{-1}R_{2Z}^T(Q_2(t_2^-))\delta\mathbf{X}(t_2) + \delta\mathbf{y}_{sec}^{2,1}\right) + \delta\mathbf{y}_{sec}^{f,2} \end{aligned} \quad (6.18)$$

Knowing that the ECEF maneuvers $\delta\mathbf{X}_1$ and $\delta\mathbf{X}_2$ are physically allowed to consist only of approximately-instantaneous momentum changes (i.e.,

$$\delta\mathbf{X}_n = \begin{bmatrix} \mathbf{0} \\ \delta\mathbf{p}_n \end{bmatrix} \quad (6.19)$$

where $\mathbf{p}$ is the inertial maneuver momentum vector represented in ECR coordinates), it is tempting simply to rearrange and solve (6.18) for the total momentum change vector $\delta\mathbf{p} = [\delta\mathbf{p}_1, \delta\mathbf{p}_2]^T$ required to accomplish some desired end state $\tilde{\mathbf{y}}_{fr}(t_f)$. However, the transformation matrices used to convert from ECEF state changes to modal variable changes, $\left[E(Q_1(t_2^-))^{-1}R_{2Z}^T(Q_2(t_2^-))\right]$, depend on the torus states $\mathbf{Q}$ at time t_2 (embedded in $\tilde{\mathbf{y}}(t_2^-)$). These combined modal states depend, in turn, on all of the frequency-interval pairs leading up to t_2 , including the interval which occurs after the first maneuver. In addition,

the secular terms $\delta \mathbf{y}_{sec}^{j,i}$ depend on the modal state at each interval's epoch, $\tilde{\mathbf{y}}(t_{i-1})$. This causes (6.18) to be a nonlinear equation in the maneuver momentum changes $\delta \mathbf{p}$.

One way to approach this issue is to treat the determination of $\delta \mathbf{p}$ as a root-finding problem and solve with a Newton-Raphson technique. To do this, we must first assemble the error function $\mathbf{g}$ as

$$\mathbf{g}(\delta X_1, \delta X_2) = \tilde{\mathbf{y}}_d - \tilde{\mathbf{y}}_{fr}(t_f) \quad (6.20)$$

$$\begin{aligned} &= \tilde{\mathbf{y}}_d - \Phi(\Delta t_{f,0})\tilde{\mathbf{y}}(t_0) - \Phi(\Delta t_{f,1}) \left(E(Q_1(t_1^-))^{-1} R_{2Z}^T(Q_2(t_1^-)) \delta \mathbf{X}(t_1) + \delta \mathbf{y}_{sec}^{1,0} \right) - \dots \\ &\quad \Phi(\Delta t_{f,2}) \left(E(Q_1(t_2^-))^{-1} R_{2Z}^T(Q_2(t_2^-)) \delta \mathbf{X}(t_2) + \delta \mathbf{y}_{sec}^{2,1} \right) - \delta \mathbf{y}_{sec}^{f,2} \end{aligned} \quad (6.21)$$

where $\tilde{\mathbf{y}}_d$ is the desired final state. In order to perform a proper traversal of the $\delta X_1, \delta X_2$ space to find the zeros of $\mathbf{g}$, we must find its gradient. We can find the partial derivative term with respect to the first maneuver as

$$\begin{aligned} \frac{\partial \mathbf{g}}{\partial \delta \mathbf{X}_1} &= -\Phi(\Delta t_{f,1}) E(Q_1(t_1^-))^{-1} R_{2Z}^T(Q_2(t_1^-)) - \dots \\ &\quad \Phi(\Delta t_{f,2}) \frac{\partial}{\partial \delta \mathbf{X}_1} \left(E(Q_1(t_2^-))^{-1} R_{2Z}^T(Q_2(t_2^-)) \right) \delta \mathbf{X}_2 - \dots \\ &\quad \Phi(\Delta t_{f,2}) \frac{\partial}{\partial \delta \mathbf{X}_1} \delta \mathbf{y}_{sec}^{2,1} \end{aligned} \quad (6.22)$$

and that with respect to the second maneuver is simply written as:

$$\frac{\partial \mathbf{g}}{\partial \delta \mathbf{X}_2} = -\Phi(\Delta t_{f,2}) E(Q_1(t_2^-))^{-1} R_{2Z}^T(Q_2(t_2^-)) - \frac{\partial}{\partial \delta \mathbf{X}_2} \delta \mathbf{y}_{sec}^{f,2} \quad (6.23)$$

Now, recognizing that the gradient expression in the second term of (6.22) is a 3-dimensional matrix, we resort to index notation for clarity. Making the temporary abbreviations

$$A = E(Q_1(t_2^-))^{-1} \quad (6.24)$$

$$R = R_{2Z}^T(Q_2(t_2^-)) \quad (6.25)$$

$$D = \frac{\partial}{\partial \delta \mathbf{X}_1} (AR) \quad (6.26)$$

we can write the partial derivative via the product rule:

$$D_{ijk} = \frac{\partial A_{i\alpha}}{\partial \delta X_{1k}} R_{\alpha j} + A_{i\alpha} \frac{\partial R_{\alpha j}}{\partial \delta X_{1k}} \quad (6.27)$$

where $i, j, k \in \mathbb{Z}[1, 6]$ and summation occurs over the Greek index α . Using the chain rule, we can then begin to compartmentalize the partial derivatives in (6.27) into quantities we can obtain either through knowledge of the maneuver method or through the low eccentricity theory. Using the chain rule,

$$\frac{\partial A_{ij}}{\partial \delta X_{1k}} = \frac{\partial A_{ij}}{\partial (Q(t_2^-))_\alpha} \frac{\partial (Q(t_2^-))_\alpha}{\partial (\delta y_1)_\gamma} \frac{\partial (\delta y_1)_\gamma}{\partial (\delta X_1)_k} \quad (6.28)$$

and, similarly:

$$\frac{\partial R_{ij}}{\partial \delta X_{1k}} = \frac{\partial R_{ij}}{\partial (Q(t_2^-))_\alpha} \frac{\partial (Q(t_2^-))_\alpha}{\partial (\delta y_1)_\gamma} \frac{\partial (\delta y_1)_\gamma}{\partial (\delta X_1)_k} \quad (6.29)$$

We immediately recognize that the quantity $\partial R_{ij}/\partial (Q(t_2^-))_\alpha$ can be calculated directly as

$$\frac{\partial R_{ij}}{\partial (Q_1(t_2^-))} = \mathbf{0}_{6 \times 6} \quad (6.30)$$

$$\frac{\partial R_{ij}}{\partial (Q_2(t_2^-))} = \begin{bmatrix} -\sin(Q_2(t_2^-)) & -\cos(Q_2(t_2^-)) & 0 & 0 & 0 & 0 \\ \cos(Q_2(t_2^-)) & -\sin(Q_2(t_2^-)) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin(Q_2(t_2^-)) & -\cos(Q_2(t_2^-)) & 0 \\ 0 & 0 & 0 & \cos(Q_2(t_2^-)) & -\sin(Q_2(t_2^-)) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.31)$$

since the angle Q_2 is the nodal torus angle. The quantity $\partial A_{ij}/\partial (Q(t_2^-))_\alpha$ can be determined by first using the matrix differentiation identity

$$\frac{\partial (M^{-1})}{\partial n} = -M^{-1} \frac{\partial M}{\partial n} M^{-1} \quad (6.32)$$

so that:

$$\frac{\partial A_{ij}}{\partial (Q(t_2^-))_\alpha} = - \left(E(Q(t_2^-))^{-1} \right)_{i\beta} \frac{\partial E(Q_1(t_2^-))_{\beta\gamma}}{\partial (Q(t_2^-))_\alpha} \left(E(Q(t_2^-))^{-1} \right)_{\gamma j} \quad (6.33)$$

The partial derivative $\partial E_{ij}/\partial(Q_1(t_2^-))$ may then be computed numerically by differentiating the periodic orbit's modal matrix Fourier series; That is, for each term in E , we have the form

$$E_{i,j}(t_2^-) = a_0^{(i,j)} + \sum_{k=1}^N \left[a_k^{(i,j)} \cos(kQ_1(t_2^-)) + b_k^{(i,j)} \sin(kQ_1(t_2^-)) \right] \quad (6.34)$$

and we can find the partial derivative with respect to Q_1 in the usual sense:

$$\frac{\partial E_{ij}}{\partial(Q_1(t_2^-))} = \sum_{k=1}^N \left[kb_k^{(i,j)} \cos(kQ_1(t_2^-)) - ka_k^{(i,j)} \sin(kQ_1(t_2^-)) \right] \quad (6.35)$$

In explicitly stating the form of $E_{i,j}$, it is also clear that the partial derivative with respect to the nodal angle is zero:

$$\frac{\partial E_{ij}}{\partial Q_2(t_2^-)} = 0 \quad (6.36)$$

since we have separated the nodal rotation R from the modal matrix.

The remaining terms in (6.28) and (6.29) may be found through the relationships used in considering the maneuvers. The term describing the change in the modal state at t_2 with respect to the modal maneuver at t_1 is calculated as

$$\frac{\partial \tilde{\mathbf{y}}(t_2^-)}{\partial \delta \mathbf{y}_1} = \Phi(\Delta t_{2,1}) \quad (6.37)$$

and the final term from the chain rule expansion in (6.28) and (6.29) is found by recalling (6.12), which directly gives

$$\frac{\partial \delta \mathbf{y}(t_1)}{\partial \delta \mathbf{X}_1} = E(Q_1(t_1^-))^{-1} R_{2Z}^T(Q_2(t_1^-)) \quad (6.38)$$

We can then combine these results to give

$$\frac{\partial \tilde{\mathbf{y}}(t_2^-)}{\partial \delta \mathbf{X}_1} = \Phi(\Delta t_{2,1}) E(Q_1(t_1^-))^{-1} R_{2Z}^T(Q_2(t_1^-)) \quad (6.39)$$

from which it follows that

$$\frac{\partial \tilde{\mathbf{Q}}(t_2^-)}{\partial \delta \mathbf{X}_1} = \tilde{I} \Phi(\Delta t_{2,1}) E(Q_1(t_1^-))^{-1} R_{2Z}^T(Q_2(t_1^-)) \quad (6.40)$$

where:

$$\tilde{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (6.41)$$

After having constructed the partial derivative matrices $\partial \mathbf{g} / \partial \delta \mathbf{X}_1$ and $\partial \mathbf{g} / \partial \delta \mathbf{X}_2$, we leverage the fact that the maneuvers cannot directly change instantaneous position (mentioned earlier) to write a square Jacobian matrix for the Newton-Raphson root-finding technique:

$$J_g = \frac{\partial \mathbf{g}}{\partial \mathbf{p}} \quad (6.42)$$

$$= \begin{bmatrix} \frac{\partial \mathbf{g}}{\partial \delta \mathbf{p}_1} & \frac{\partial \mathbf{g}}{\partial \delta \mathbf{p}_2} \end{bmatrix} \quad (6.43)$$

where

$$\frac{\partial \mathbf{g}}{\partial \delta \mathbf{p}_1} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix} \frac{\partial \mathbf{g}}{\partial \delta \mathbf{X}_1} \quad (6.44)$$

$$\frac{\partial \mathbf{g}}{\partial \delta \mathbf{p}_2} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix} \frac{\partial \mathbf{g}}{\partial \delta \mathbf{X}_2} \quad (6.45)$$

and $\mathbf{1}$ and $\mathbf{0}$ are the 3×3 identity matrix and the 3×3 matrix of zeros, respectively. The iterative updates to the $\delta \mathbf{p}$ can then be found by:

$$\delta \mathbf{p}_{i+1} = \delta \mathbf{p}_i + \Delta \delta \mathbf{p} \quad (6.46)$$

$$\Delta \delta \mathbf{p} = -J_g^{-1} \mathbf{g} \quad (6.47)$$

In order to converge reliably, the Newton-Raphson root-finding algorithm requires an initial guess which is sufficiently close to the actual solution. To supply this initial $\delta \mathbf{p}$, we have found that it is usually sufficient in practice to assume that 1) the first maneuver does not change the torus frequencies, and 2) there are no secular drift effects due to the geopotential and/or eccentricity, and then solve for $\delta \mathbf{p}$ directly. This can be accomplished by considering the non-combined modal state analog of (6.18)

$$\begin{aligned} \mathbf{y}_{fr}(t_f) &= \Phi(\Delta t_{f,0}) \mathbf{y}(t_0) + \Phi(\Delta t_{f,1}) E(Q_1(t_1^-))^{-1} R_{2Z}^T(Q_2(t_1^-)) \delta \mathbf{X}(t_1) + \dots \\ &\quad \Phi(\Delta t_{f,2}) E(Q_1(t_2^-))^{-1} R_{2Z}^T(Q_2(t_2^-)) \delta \mathbf{X}(t_2) \end{aligned} \quad (6.48)$$

and setting $\mathbf{y}_{fr}(t_f)$ equal to the desired final modal position. This gives

$$\begin{aligned} \mathbf{y}_d(t_f) - \Phi(\Delta t_{f,0})\mathbf{y}(t_0) &= \Phi(\Delta t_{f,1})E(Q_1(t_1^-))^{-1}R_{2Z}^T(Q_2(t_1^-))\delta\mathbf{X}(t_1) + \dots \\ &\quad \Phi(\Delta t_{f,2})E(Q_1(t_2^-))^{-1}R_{2Z}^T(Q_2(t_2^-))\delta\mathbf{X}(t_2) \end{aligned} \quad (6.49)$$

which we can solve for $\delta\mathbf{p}$ as

$$\delta\mathbf{p} = \Upsilon^{-1} \left(\mathbf{y}_d(t_f) - \Phi(\Delta t_{f,0})\mathbf{y}(t_0) \right) \quad (6.50)$$

where

$$\Upsilon = \begin{bmatrix} \Phi(\Delta t_{f,1})E(Q_1(t_1^-))^{-1}R_{2Z}^T(Q_2(t_1^-))I_2 & \Phi(\Delta t_{f,2})E(Q_1(t_2^-))^{-1}R_{2Z}^T(Q_2(t_2^-))I_2 \end{bmatrix} \quad (6.51)$$

and:

$$I_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \quad (6.52)$$

The 2-impulse maneuver technique outlined above can be applied in a straightforward way by cycling through $\{\Delta t_{1,0}, \Delta t_{2,1}\}$ pairs and calculating, via the Newton-Raphson method, the maneuvers $\delta\mathbf{p}$ required to transfer from the current state $\mathbf{y}(t_0)$ to the desired final state $\mathbf{y}_d(t_f)$. The “best” (in the sense of lowest-impulse) maneuver would then be the one which requires the smallest value of the Δv cost

$$\Delta v = \|\delta\mathbf{p}_1\| + \|\delta\mathbf{p}_2\| \quad (6.53)$$

In a sense, this amounts to finding the minimum on a Δv surface which is parameterized by the momentum changes $\delta\mathbf{p}$ through the two time intervals $\{\Delta t_{1,0}, \Delta t_{2,1}\}$. An example of such a surface for a simple in-track displacement change is shown in Figure 6.1 and an example for a small nodal displacement change is shown in Figure 6.2. It is important to note that, for a general desired modal displacement, the Δv surface will not be as regular as those shown in these figures.

The plot in Figure 6.1 shows the an example of the total Δv surface calculated with the two-impulse algorithm described in this chapter, as a function of $\Delta t_{1,0}$ and $\Delta t_{2,1}$, for a

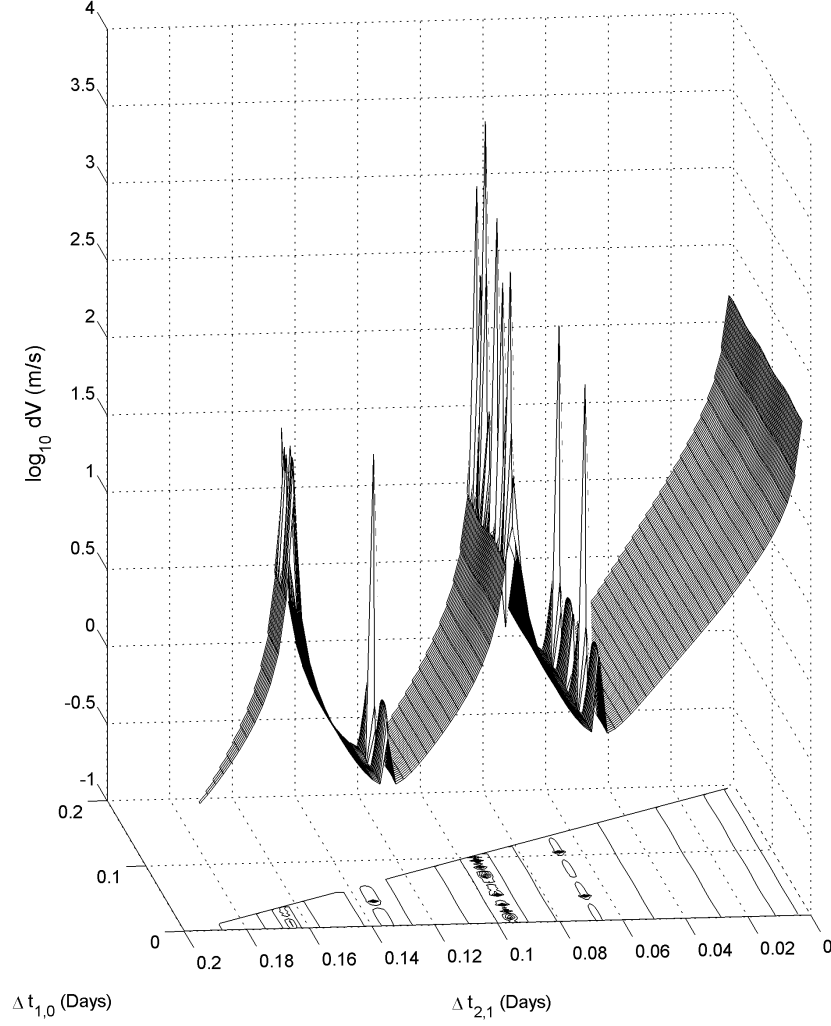


Figure 6.1: Example Δv Surface for Simple In-track Displacement ($\Delta y_1 = 0.002$ rad)

y_1 (in-track) displacement of 0.002 rad (0.115°). The orbit considered here has an altitude of 630km and an inclination of 40°, and, as such, has an orbital period of approximately 0.068 days. In Figure 6.1, we can see that the cost does not depend appreciably on the initial delay from epoch ($\Delta t_{1,0}$), since we are considering only an in-track change. The cost is decaying function in $\Delta t_{2,1}$, which is an intuitive result; this is because a small change in energy (via an in-track burn), causes a change in the Keplerian frequency analogue

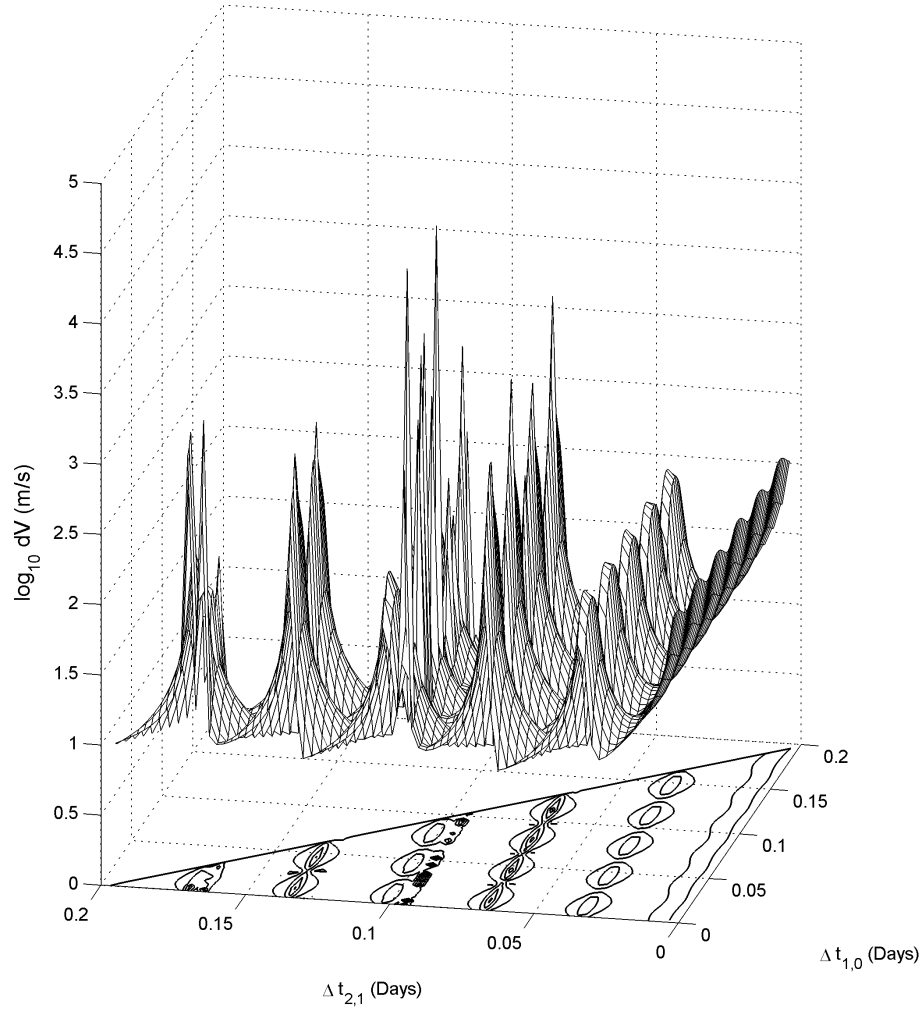


Figure 6.2: Example Δv Surface for Simple Nodal Displacement ($\Delta y_3 = 0.002$ rad)

ω_1 , which causes a secular change in the in-track displacement. Hence, for this type of maneuver, the longer one “waits” between burns, the cheaper the overall cost will be, since the required maneuver magnitudes will progressively become smaller. Also evident in the figure, though, are resonances at multiples of the orbital period τ and the semiperiod $\tau/2$. The τ resonance occurs because the modal matrix is periodic in τ and, as a result, the columns of E return to nearly their initial conditions, which would cause any calculated

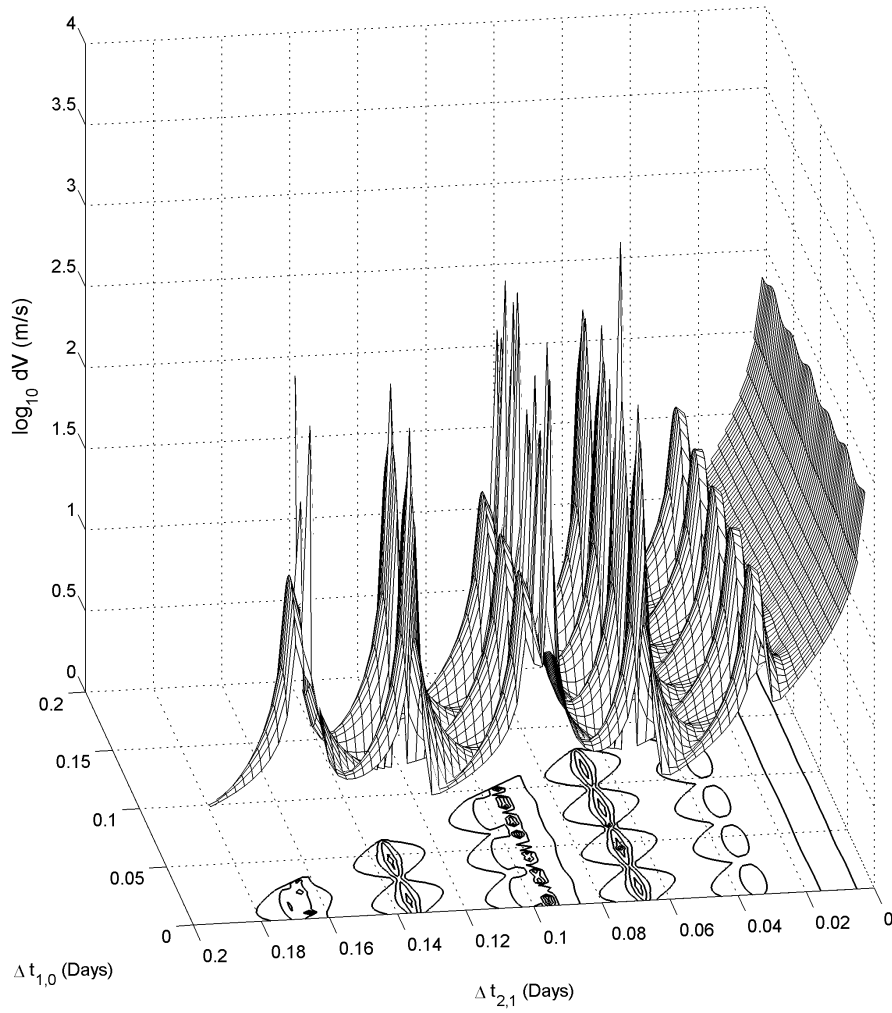


Figure 6.3: Example Δv Surface for Combined Displacement ($\Delta y_1 = 0.002$ rad; $\Delta y_3 = 0.001$ rad)

cost to spike to infinity. The $\tau/2$ ridges occur for a similar reason; at $\tau/2$, transformation embodied by E becomes practically the negative of its initial value. Significantly, the lowest costs for this case correspond to the inter-maneuver time $\Delta t_{2,1}$ very near multiples of τ , as long as careful avoidance of the resonances is exercised.

Figure 6.2 shows the cost surface for an equivalent desired angular displacement, but in the nodal angle y_3 . It is immediately evident that the total costs are much higher than the in-track displacement case, as is expected due to the nature of plane-change maneuvers. Here, again, we see resonances at multiples of the period and semiperiod, and we see a much more complex surface which decays very slowly, if at all; that is, plane change maneuvers typically do not decrease significantly in cost with increasing time (over operationally reasonable time scales).

Shown in Figure 6.2 is a cost surface for a combined displacement of $\Delta y_1 = 0.002$ rad and $\Delta y_3 = 0.001$ rad. The shape of the resulting cost surface is practically a superposition of the two scaled individual surfaces. It should be noted that, even though the combined cost surface is quite intricate, it still does not represent the most general case, which would involve general displacements in the other four modal states (the energy and eccentricity modes).

Unfortunately, the resonances discussed above mean that the cost surfaces are not everywhere differentiable and there exist many local minima; this, combined with the complexity of the maneuver calculations' dependence on the maneuver times, means that a general optimization technique is not readily available. However, for the purposes of the current work, it is sufficient to explicitly enumerate the maneuver combinations at a set of maneuver times within some allowable window. That is, we limit the total maneuver window to some time (typically several orbits) and calculate the cost surface for a mesh of $\delta t_1, \delta t_2$ combinations, as in the figures. Maneuver time combinations involving unreasonably high maneuver costs are summarily discarded, and the remaining maneuver combinations are ranked to find the lowest-cost solution within the time window.

With the information given/derived in this section, we can formally describe the process of maneuvering onto a desired torus. A graphical depiction of the process flow is shown in Figure 6.4. We begin with orbital position data, either from observations

or numerical sources, and from these data we find the average orbital radius (which, for circular orbits, is equivalent to the semi-major axis a), along with the orbital inclination i . These can be used to construct a periodic orbit, along with the associated perturbation solution (from sectoral/tesseral harmonics and second-order eccentricity effects) and the modal matrix E . The original orbital data are then used, in conjunction with the periodic orbit solution, to estimate the torus state $\mathbf{Q}$ at epoch t_0 and the modal state $\mathbf{y}$ at epoch via the process outlined in §5.3. Using these, a desired modal state $\mathbf{y}_d(t_f)$, and the techniques outlined in the current section, a cost surface is constructed, which gives the total required velocity change to accomplish the desired modal state, as a function of the two maneuver times t_1 and t_2 . From this surface, a maneuver plan is constructed by selecting a maneuver time pair which gives a sufficiently small total Δv , and the required momentum offsets (representing the impulsive burns) are calculated and stored. The maneuvers are then implemented; in a realistic application, this involves actually performing burns of magnitudes $\delta \mathbf{p}_n$ at times t_n , while in the current work, this is accomplished by high-fidelity numerical integration, as discussed in the next section. After the maneuver process, observation data (i.e, physical state history) of the new orbit is gathered. These data are then passed back into the modal state estimation routine to determine whether or not the satellite has reached the desired modal state (and therefore has maneuvered onto the desired torus). The process may then be repeated until the achieved modal state is acceptable.

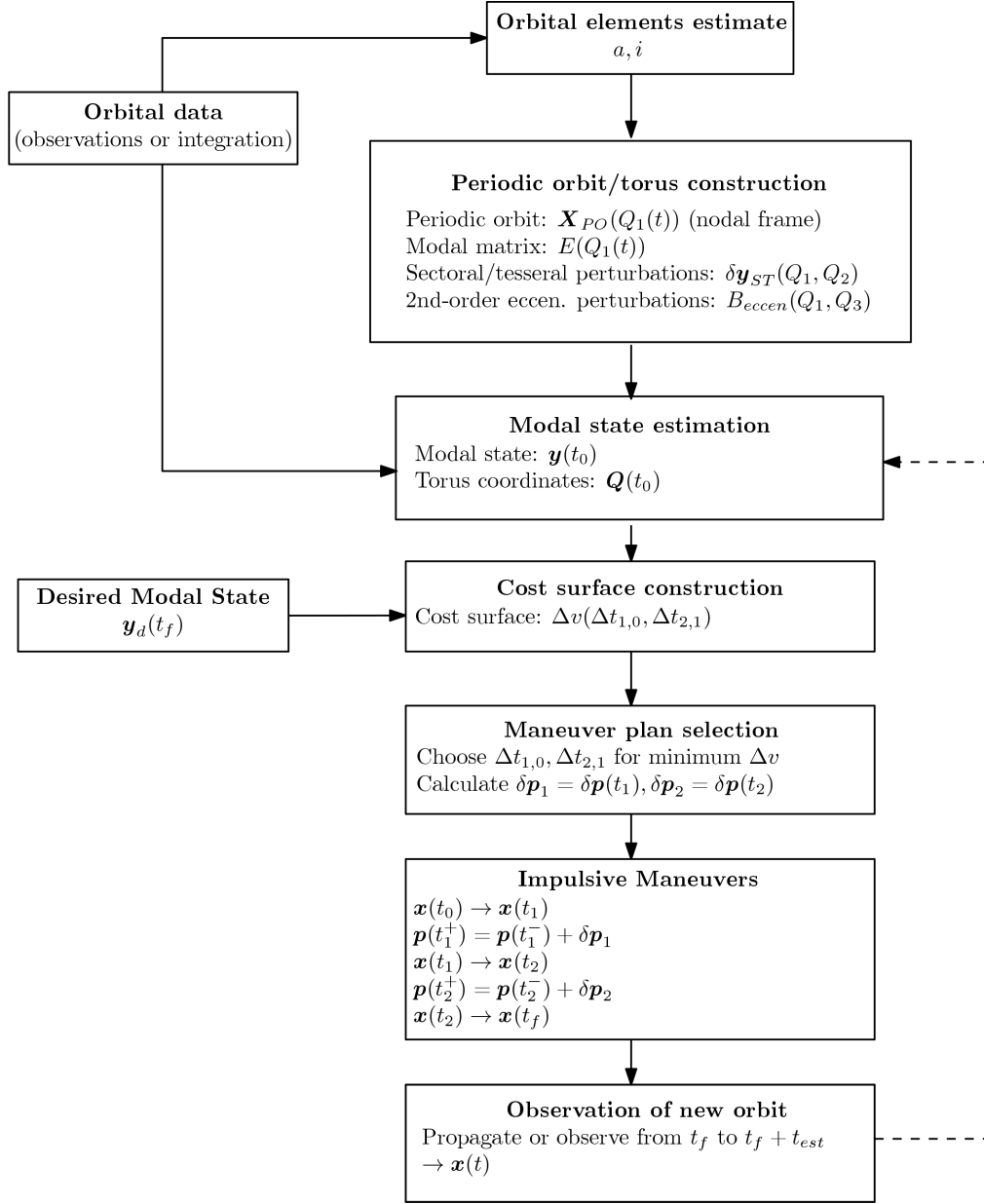


Figure 6.4: Impulsive Maneuver Process Flow

6.3 Numerical Demonstration

Representative examples of the numerical implementation of the maneuver method are shown in this section. In each of these simulations, we have two virtual satellites: the “desired” satellite (i.e., a satellite which is already in the desired orbit) and a “controlled” satellite (the satellite which starts in a different orbit and intends to reach the same position as the desired satellite). The desired satellite is numerically propagated forward for the duration of the scenario under the influence of the geopotential with $n, m = 20$, as usual, and without any reliance on the torus theory. The controlled satellite is also propagated numerically, but is assigned impulsive momentum changes at certain times, with the maneuver magnitudes, directions, and times determined from the LEST/KAM-based method derived above. Its knowledge of its current torus state comes from the torus estimation routine outlined in the previous chapter; a collection of the numerically integrated position data is used to formulate a torus/modal state estimate. The plots throughout this section examine the relative error between the desired satellite’s position/velocity and the control-achieved position/velocity.

It should be noted that, as stated, the results in this section assume “perfect” position knowledge (since numerical integration data are used in the estimator), and it also assumes “perfect” maneuver accuracy; i.e., the spacecraft is assumed to be able to change its attitude perfectly to the required orientation and create an impulsive force with the exact magnitude required by the controller. The maneuver accuracy requirement is relaxed and its ramifications examined in a later chapter.

For the general demonstration test cases we use a 630km, 40° orbit. For the first test case, we require a simple in-track displacement of 0.001 radian (7 kilometers). The initial and final torus/modal states are given as:

Case 1:

$$\mathbf{Q}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{y}_f = \begin{bmatrix} 0.001 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (6.54)$$

The plot in Figure 6.5 shows the norm of the position residuals between the numerically integrated “desired” orbit and the orbit achieved through the 2-maneuver impulsive control method. The velocity residuals are shown in Figure 6.6; the velocity residuals typically follow the behavior of the position residuals, and hence are not shown for subsequent cases.

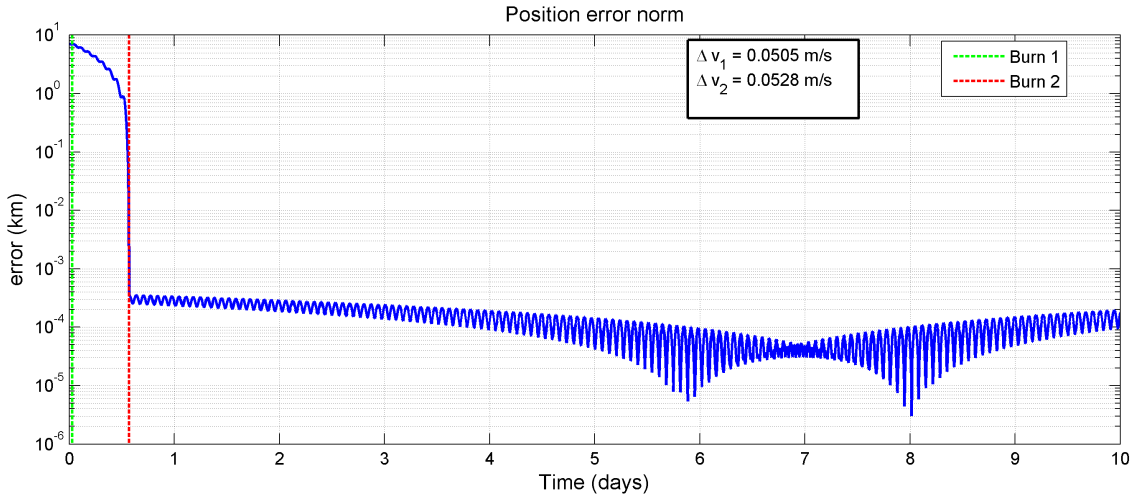


Figure 6.5: Position Error Norm, Case 1

We may also show the residuals resolved into the UVW frame (i.e., radial, along-track, and normal), sometimes referred to as the XYZ frame in relative satellite motion discussions.

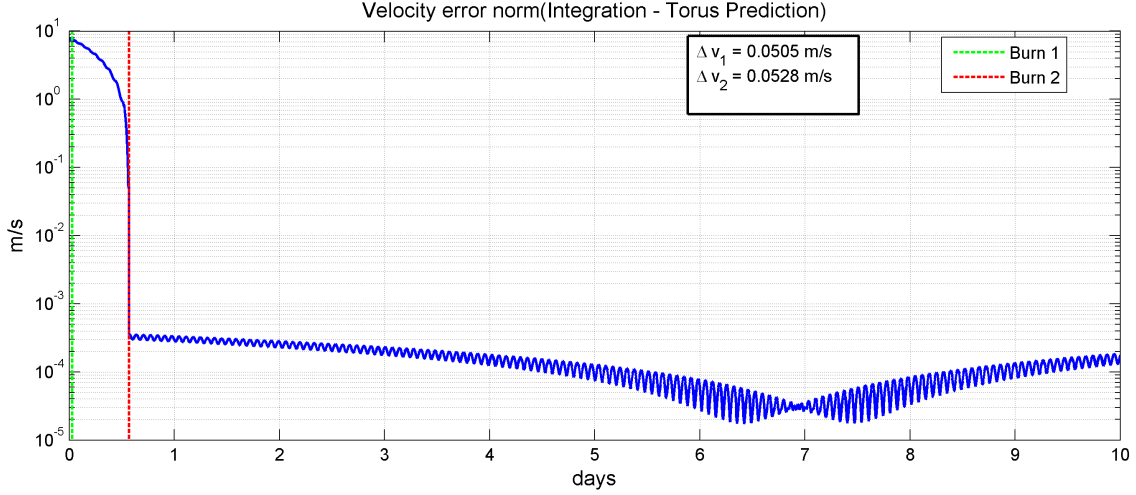


Figure 6.6: Velocity Error Norm, Case 1

To do this, we can form the basis vectors for the frame as

$$\begin{aligned}
 \hat{e}_r &= \frac{\mathbf{r}}{\|\mathbf{r}\|} \\
 \hat{e}_v &= \hat{e}_n \times \hat{e}_r \\
 \hat{e}_n &= \hat{e}_r \times \frac{\mathbf{v}}{\|\mathbf{v}\|}
 \end{aligned} \tag{6.55}$$

where $\hat{e}_r$, $\hat{e}_v$, and $\hat{e}_n$ represent the radial, velocity(along-track), and normal directions (respectively) and the vectors $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity in inertial coordinates. These basis vectors may be used to create a rotation matrix to translate between ECI and UVW/XYZ. The UVW residuals for Case 1 are shown in Figure 6.7. In Case 1, the total Δv required, as shown on the figures, is approximately 10 cm/s. For this type of maneuver, the cost would decrease if a larger time window were allowed for the maneuver to occur. The maneuver planning resulted in a highly accurate maneuver for this case, with sub-meter residuals after one maneuver.

The second case for the LEO orbit involves a slightly larger in-track displacement of 0.01 radians, or approximately 70 kilometers. The resulting position residuals are shown in Figure 6.8.

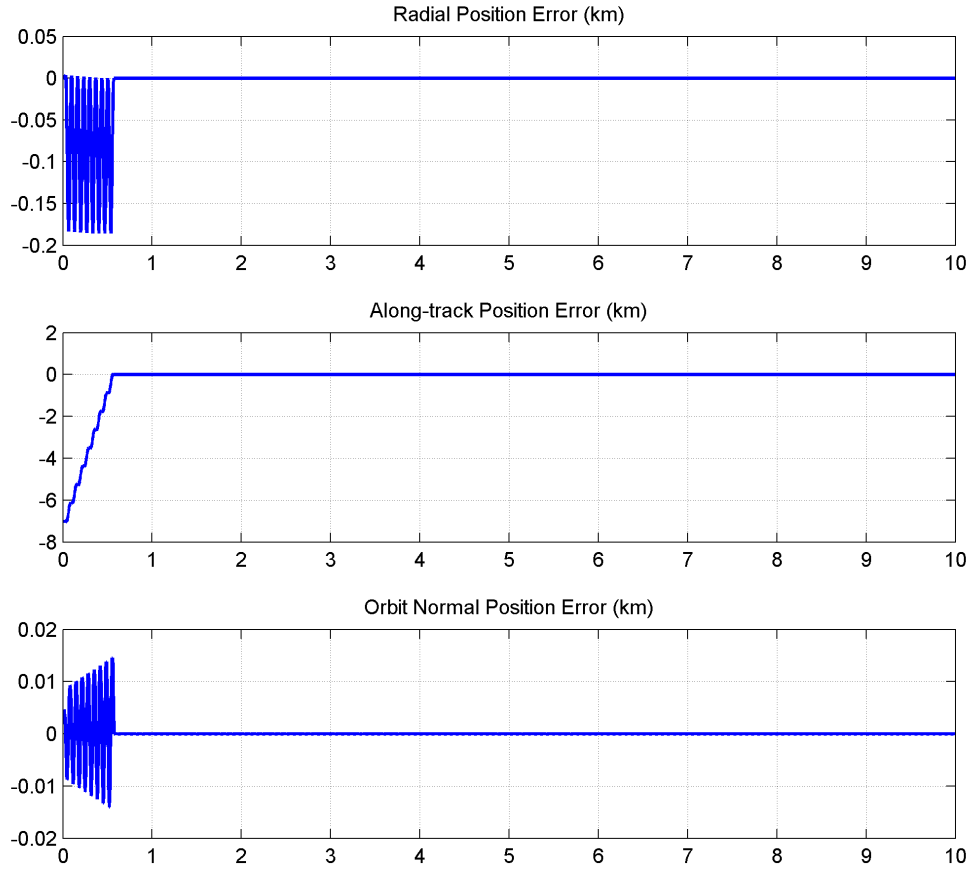


Figure 6.7: Position Residuals, UVW, Case 1

Case 2:

$$\mathbf{Q}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{y}_f = \begin{bmatrix} 0.01 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (6.56)$$

The results of Case 2 are similar in quality to those achieved before, with slightly higher residuals (11 m) after the single 2-burn maneuver. In practice, it is possible to decrease the end-state residuals to sub-meter level by performing a second, smaller maneuver after

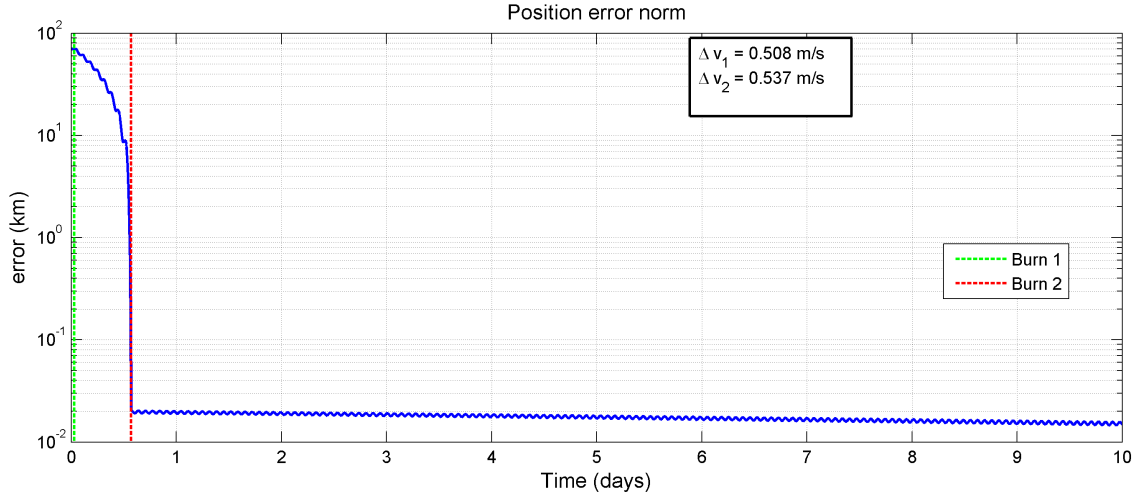


Figure 6.8: Position Error Norm, Case 2

the new torus state has been re-estimated. As shown, Case 2 requires a total cost of approximately 1 m/s, with a maximum window used for these results of approximately 0.6 days. However, as discussed previously, the Δv cost can be lowered by increasing the allowable maneuver window. As evidence of this, Figure 6.9 shows the same conditions with an extended window of 1.2 days. The maneuver cost has dropped by approximately a factor of two, at the expense of a very slightly larger error growth in the residuals.

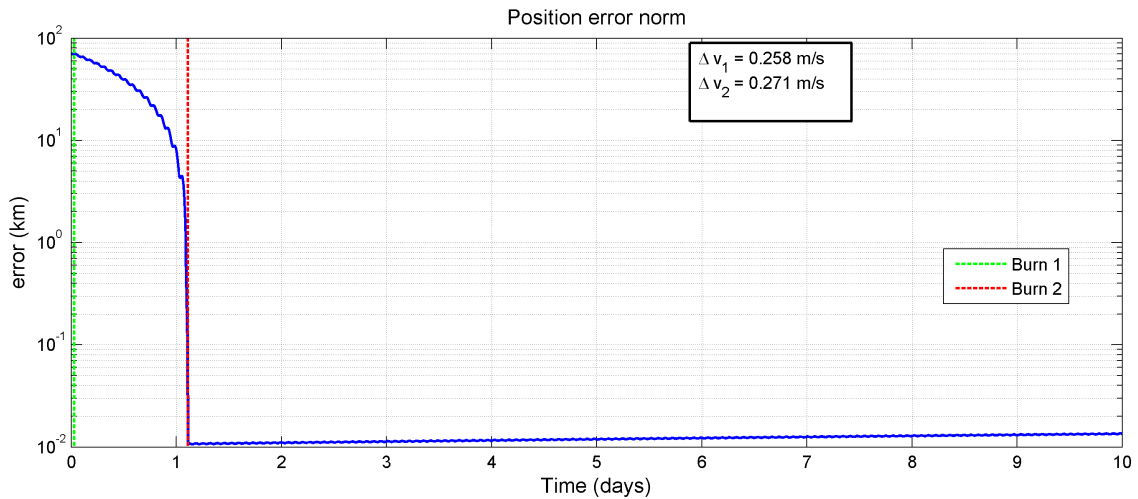


Figure 6.9: Position Error Norm, Case 2 (extended maneuver window)

The third case examines a combined change in in-track angle and orbital energy analogue y_2 . This energy change corresponds loosely to a small change in the semi-major axis of the orbit.

Case 3:

$$\mathbf{Q}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \mathbf{y}_0 = \begin{bmatrix} 0 \\ 1.0E-4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{y}_f = \begin{bmatrix} 0.01 \\ 1.0E-4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (6.57)$$

The results of a series of two maneuvers are shown in Figure 6.10. This class of maneuver, i.e., one which requires a final orbital energy much different from the reference solution, shows behavior that reinforces the previously-mentioned caveat: when one of the energy modes changes appreciably, it may in some cases be prudent to develop a new reference periodic orbit solution, as it generally requires only a few seconds on a modern computer. However, even with the current framework, after two maneuvers, the orbit is within 1-10 meters of its desired position for the remainder of the scenario.

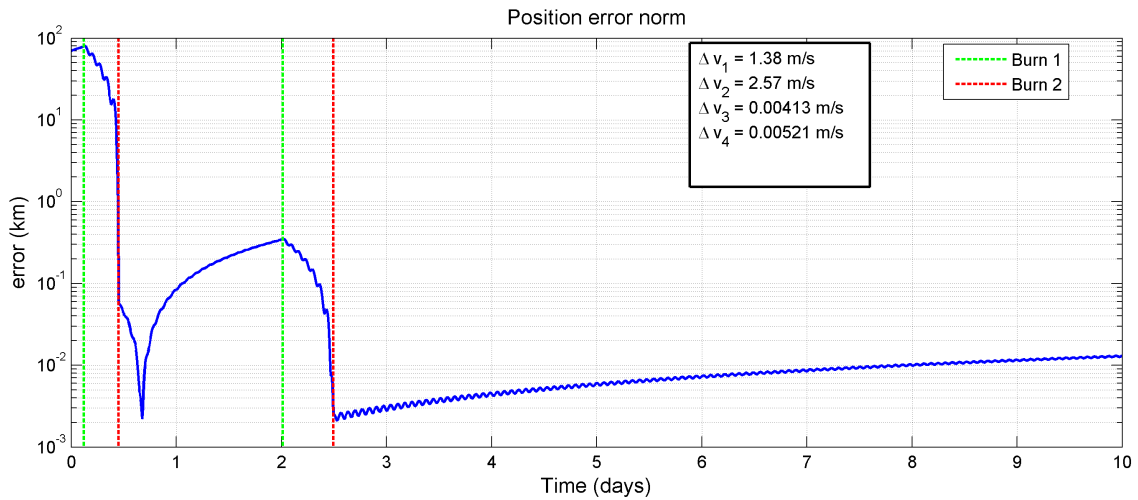


Figure 6.10: Position Error Norm, Case 3

The next case examines a pure change in the nodal state $Q_2; y_3$, and results are shown in Figure 6.11. Again, two maneuvers are allowed; the primary reason for this necessity is that nodal changes, being orbital plane changes, are relatively expensive, especially to accomplish within a short time scale. Because the magnitude of the impulses is greater, the linearizations implicit in the maneuver theory cause more error in the commanded impulse directions and magnitudes. As with previous cases, a few maneuvers is sufficient to lower the position residual magnitudes considerably.

Case 4:

$$Q_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad y_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad y_f = \begin{bmatrix} 0 \\ 0 \\ 1.0E-3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (6.58)$$

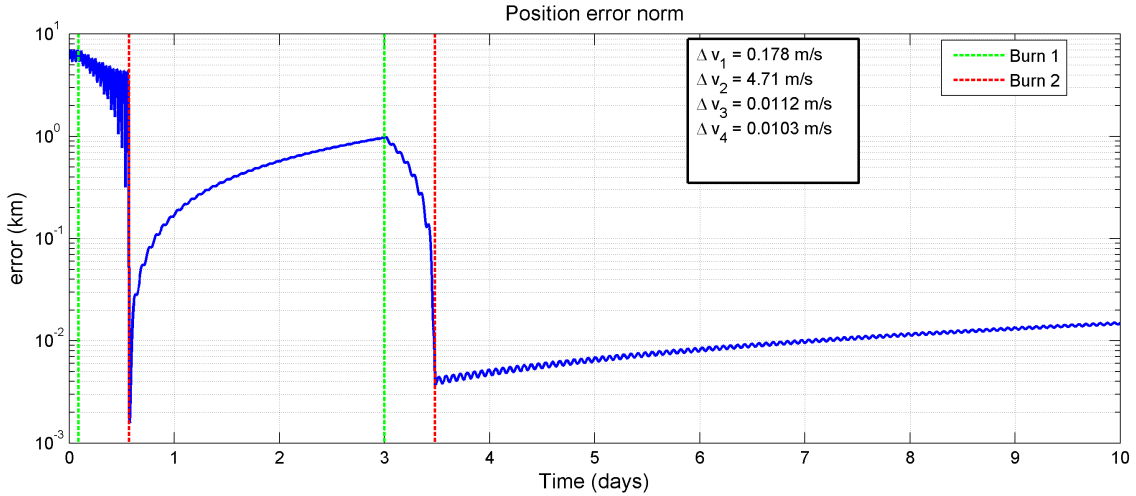


Figure 6.11: Position Error Norm, Case 4

As a final case, we examine a combined change of all modal variables, which includes slight eccentricity of the orbit. This provides evidence for the general utility of the current method for LEO satellites. The residual norms for this case are presented in Figure 6.12.

Case 5:

$$\mathbf{Q}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{y}_f = \begin{bmatrix} 0.004 \\ 1.0E-5 \\ 5.0E-4 \\ -1.0E-7 \\ 0 \\ 0.01 \end{bmatrix} \quad (6.59)$$

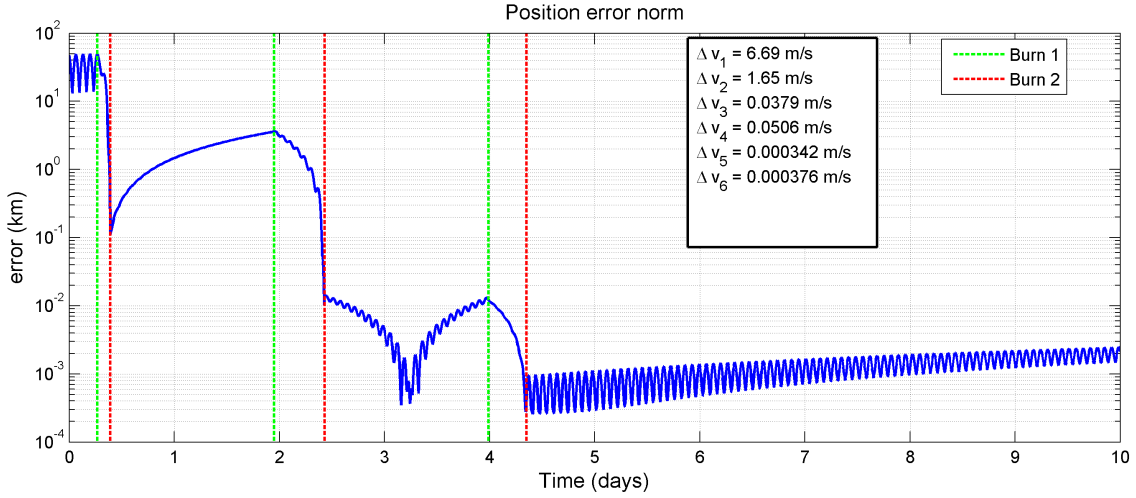


Figure 6.12: Position Error Norm, Case 5

For this case, maneuvering was allowed to be performed as often as necessary to keep the error below approximately 10 meters. This brings the residual norms down to a steady level of a few meters after three maneuver cycles. Furthermore, any of the other cases (1-4) considered in this section would demonstrate similarly small residual growth if allowed to refine the orbit with additional maneuvers.

6.4 Impulsive Maneuver Summary

The data in the previous section show that the method presented in this chapter of impulsive maneuver pairs onto a torus is quite effective. In general, larger desired displacements require larger maneuvers, especially when transitioning to tori of different

energies. This is straightforward to understand, since it is a result of the linearizations performed to achieve the control strategy. However, as with many schemes involving linearization about a reference solution, orbital reconfigurations performed with the presented method converge to a low-residual and low-drift solution within a few “iterations,” or maneuver cycles.

VII. Constellations and Formations

This chapter continues the discussion of low-eccentricity KAM tori as related to and applied to satellite constellations and formations. General design strategies for the use of KAM tori for these applications are introduced.

7.1 Constellation Design

Low-eccentricity KAM torus theory is particularly well suited to satellite constellation design. In a typical Walker constellation (as mentioned in the Background chapter), the satellites are divided onto a number of orbital planes separated by an angle $\delta\Omega$ in their right ascensions of the ascending node. Within a particular orbital plane, the satellites are further distributed by in-track angular displacements. While typical constellation design stops there and proceeds to consider how to keep the constellation's planes in the proper relative orientation based on J_2 perturbations to the osculating orbital elements, KAM theory can directly dictate these offset angles. The KAM torus coordinates Q_1 and Q_2 , are, by construction of the problem, the torus implementations of along-track and nodal coordinates. Since the time evolution of these angles are governed by constant (for a specific torus) frequencies, we see that assigning satellites to a torus and offsetting them by angles Q_1 and Q_2 is a natural way to design a low-drift constellation.

An example constellation designed by creating six equally-spaced (in Q_2) orbital planes for our 630km, 40° torus is shown in Figure 7.1. If we examine the error in the Ω locations, we see that the torus formulation is indeed a valid option for designing constellations. Figure 7.2 shows the relative error in each of the orbital planes of the constellation shown in 7.1, calculated by finding the osculating two-body elements at each time from the ECI position and velocity, and finding the differences $\Omega_i - \Omega_1$, $i \neq 1$. These differences are then compared to their “nominal” values (i.e., 60°, 120°, etc.)

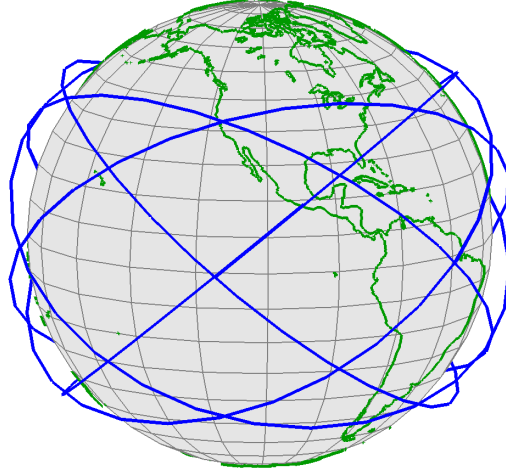


Figure 7.1: 630km, 40° Q_2 Constellation with Six Orbital Planes

Oscillations occur in the calculated Ω s, since the asymmetric gravity field acts differently on the satellites in the different planes. These are embodiments of the oscillations which the perturbation solutions of the LE KAM torus theory take into account when required (such as during state prediction and maneuver planning). However, it is of note that the errors are purely periodic to within the scale of the plot over five days, implying that there is little or zero secular drift between the constellation planes.

We can extend the same design philosophy to in-track displacements. For demonstration, four satellites were placed equally spaced in Q_1 on the $Q_2 = 0$ orbital plane discussed above. The three auxiliary satellites' positions ($Q_1 \neq 0$) were compared to the $Q_1 = 0$ satellite throughout a period of a few days. With each satellite's position at each time, an angle error was calculated through

$$\delta\theta_i(t) = \cos^{-1} \left(\frac{\mathbf{r}_i(t) \cdot \mathbf{r}_1(t)}{\|\mathbf{r}_i(t)\| \|\mathbf{r}_1(t)\|} \right) - Q_i \quad (7.1)$$

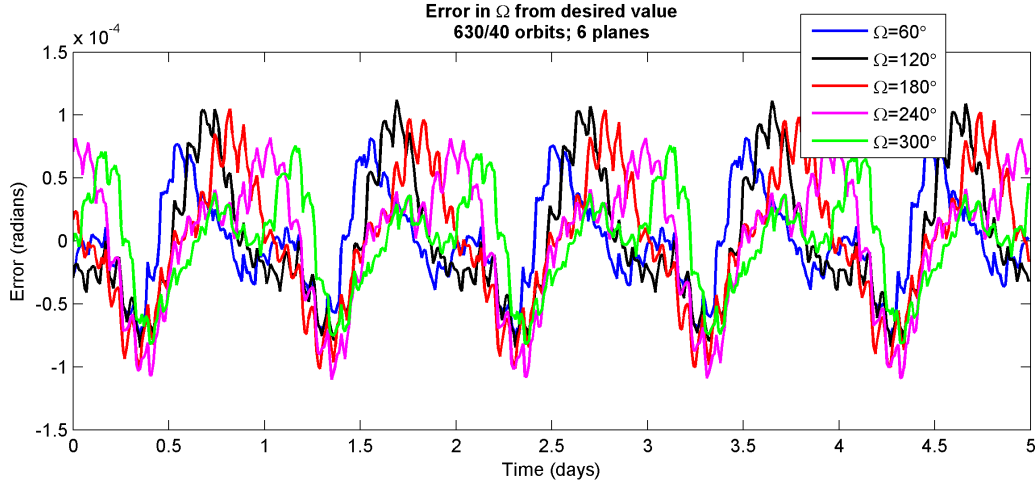


Figure 7.2: Ω Error from Torus Construction of 630km, 40° Q_2 Constellation with 6 Orbital Planes

where $\mathbf{r}_i(t)$ is the ECI position vector of satellite i . The resulting $\delta\theta$ values represent the absolute angular error from the desired initial displacement, Q_i as calculated from the position vectors of the satellites. A plot of the resulting angle errors is shown in Figure 7.3.

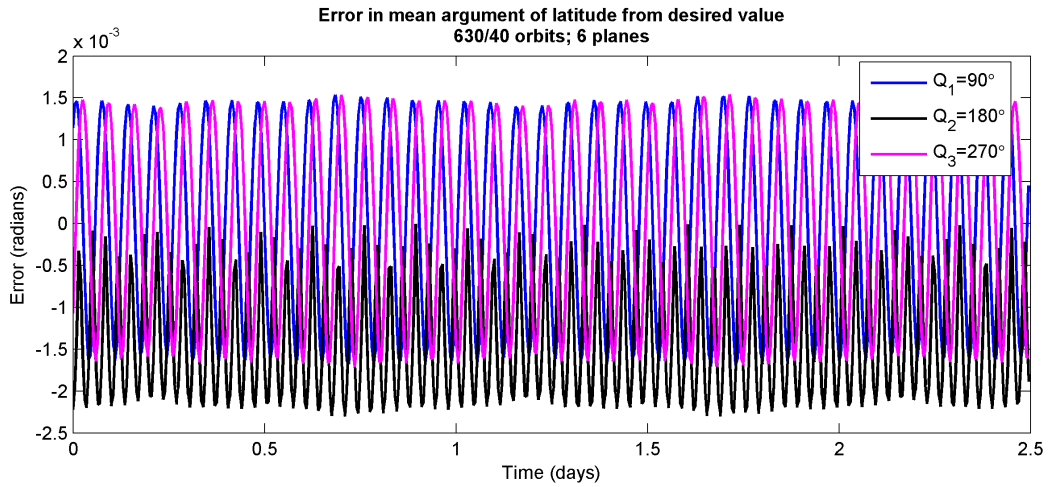


Figure 7.3: $\delta\theta$ Error of 630km, 40° Q_1 Constellation with 1 Orbital Plane

The oscillations shown in the figure, of amplitude approximately 10^{-3} radians, are due primarily to the minor eccentricity in the orbit, which is $e \approx 0.0012$, and the geopotential

perturbations. Again, these perturbations are handled by the torus theory when it is necessary to obtain the exact physical states, as mentioned above. Otherwise, there is no evident secular drift of the satellites from their stations, further reinforcing the validity of the method.

The results shown above for constellations are valid for constellations in general (away from resonance); that is, constellations of any altitude. The limiting cases are constellations with very low altitudes, in which case drag (as discussed previously) deteriorates the tori until planetary impact (while still maintaining the relative separations) and, conversely constellations with very high altitudes, in which case third-body perturbations from the Sun and Moon begin to dominate. The effects of these third-body perturbations, while not modeled in this work, may be included in the KAM method by expanding the torus definition to a larger number of basis frequencies and angles when performing numerical construction.

7.2 Formation Design

The Low-Eccentricity KAM technique can also be applied to formation design. In this section, we will investigate some strategies and geometries for design of formations. For the present discussion, we will follow convention in the field and refer to the satellite in a reference orbit as the “chief” satellite, and auxiliary satellites as “deputy” satellites. It is important to note that the “chief” satellite may be a virtual entity; that is, the chief may be considered only for reference positions insofar as the actual formation can be designed around it.

When approaching the problem of formation layout, one obvious optional case is to separate satellites by a small amount in the mean-argument of latitude angle Q_1 . This results in satellites in a roughly linear formation which stay separated in the along-track direction by a specific distance, modulated by small oscillations due to the differential gravity field (which themselves decrease as the inter-satellite distance decreases). The

geometry of this case is shown in Figure 7.4, having numerically integrated over 15 revolutions (1 day) of the reference orbit and where, again, results are presented in the chief-centric UVW/XYZ plane. (For this analysis, and for the subsequent ones in this section, the reference satellite has an identically zero modal state at epoch. This is for ease of analysis of the relative behaviors, but the methods generalize to arbitrary reference orbits within the limits of the theory. Note: ρ_d in the following figures represents the deputy's position vector in the chief's UVW frame.) In the figures throughout this chapter, the chief's position is denoted by a red asterisk, the deputy's initial position is represented by a blue asterisk, and the deputy's path over time is shown as a blue line. While potentially useful, the pure Q_1 displacement mentioned above is a trivial case to analyze and implement using tools already presented and will not be discussed further, other than to note that it is a further demonstration of the lack of secular drift in configurations designed with the torus method.

Investigating a pure change in Q_2 yields a similarly intuitive result; that is, a deputy satellite separated in node Q_2 exhibits oscillatory behavior about the chief in a direction normal to the reference orbital plane. An example is displayed in Figure 7.5. Both the magnitude of the oscillation in the z (or out-of-plane) direction and the center of the oscillation on the along-track axis y are proportional to the displacement in Q_2 , due to the inclination of the orbits.

Great flexibility is achieved when designing orbits with offsets in combinations of the modal variables. For example, the orbit shown in Figure 7.6 is created from combining displacements in the epoch angle variables Q_1 and Q_2 , as well as an eccentric displacement in y_5 . In this case, the deputy orbits the chief in what is sometimes known as a “safety orbit,” so named because the along-track drift of the satellite (seen more clearly in the 5-day simulation shown in Figure 7.7) causes the deputy to revolve around, but not collide with, the chief. This orbit could be useful for close-proximity operations before inserting

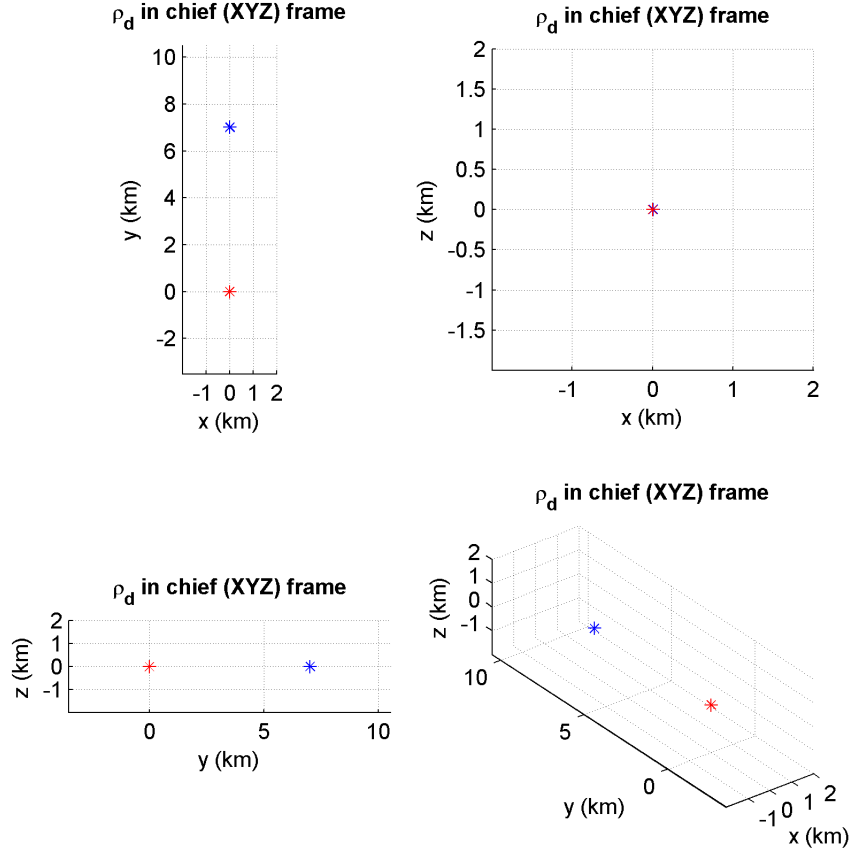


Figure 7.4: Example of Pure $Q_1 = 0.001\text{rad}$ Displacement in Formation (1 day: 15 orbits)

into a different desired orbit, as well as other proximity missions involving possibly non-cooperative chiefs. The drift seen in these relative orbits is due to the eccentricity perturbations introduced by the non-zero value of y_5 . The addition to the eccentricity modes amounts to a slight change in orbital energy (and with it a slight change in orbital period), causing relative secular drift.

While potentially useful in its own right, the secular drift involved in the “safety” orbit makes it undesirable for a formation whose purpose necessitates sustaining relative motion for a longer period. Fortunately, we can perform an energy-matching procedure to eliminate the relative drift. To do this, we calculate the angular drift rate in each of the angles Q_1 and

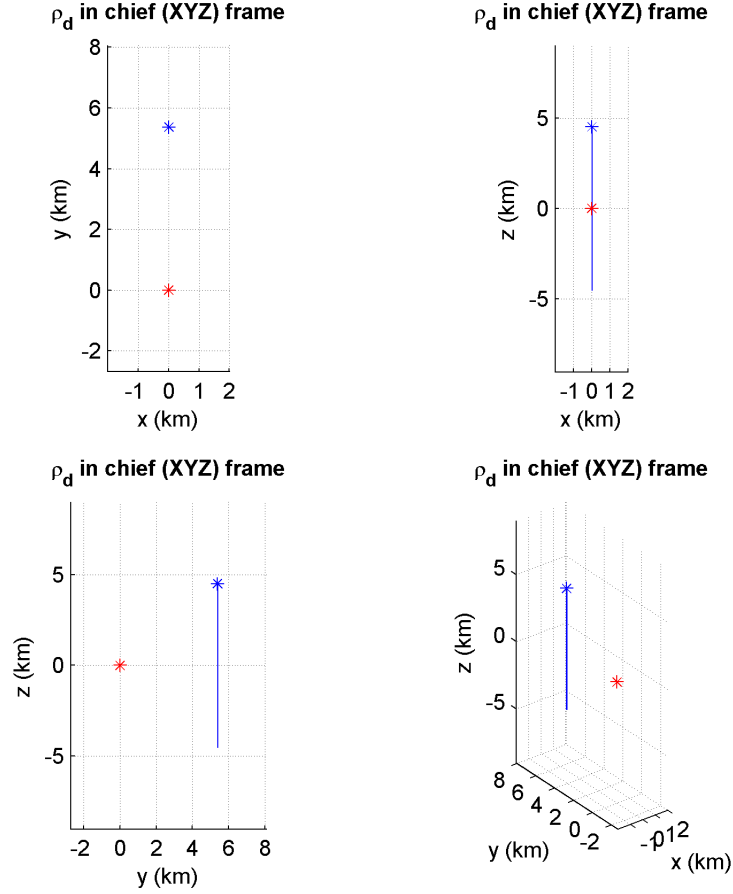


Figure 7.5: Example of Pure $Q_2 = 0.001\text{rad}$ Displacement in Formation (1 day: 15 orbits)

Q_2 due to the differential eccentricity perturbations y_4, y_5 , which can be obtained through the perturbation Fourier series, as described previously. These drift rates manifest as small changes in the frequencies y_2 and y_4 , and so we can simply subtract the calculated rates from these two modal variables to cancel the secular drift. The result of this process for otherwise the same relative modal configuration described above is shown in Figure 7.8. It is obvious (especially from the top-left plot) that the energy-matching modification causes the secular drift rate to disappear. Another effect that has become more noticeable, due to the 3-dimensional nature of the orbit and its stationarity, is the precession of the relative orbit about the chief's z axis. In fact, the deputy's relative orbit precesses at the rate of

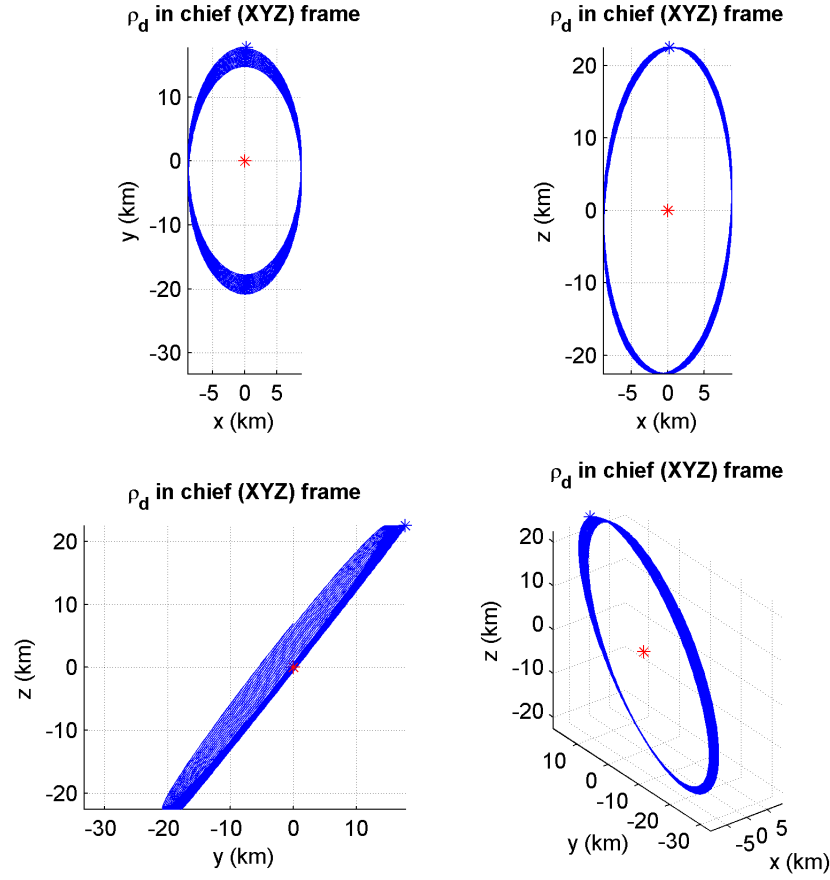


Figure 7.6: Example of “Safety” Formation from Q_1, Q_2, y_5 Displacements (1 day: 15 orbits)

apsidal regression of the reference orbit, completely filling a elliptic cylindrical surface whose projection is a 2:1 x - y ellipse (described shortly). The only reference orbit which would not engender such precession behavior is an orbit in which the apsidal regression is zero, which would occur at the critical inclination.

A configuration of possible utility (and therefore worth mention) is created by commanding an offset only in the eccentricity analog variables y_4, y_5 . An offset in some combination of these variables manifests as a relative orbit which causes the deputy to “orbit” the chief within the reference orbital plane. An example of this behavior is shown in Figure 7.9, where we have utilized a similar energy-matching modal modification as

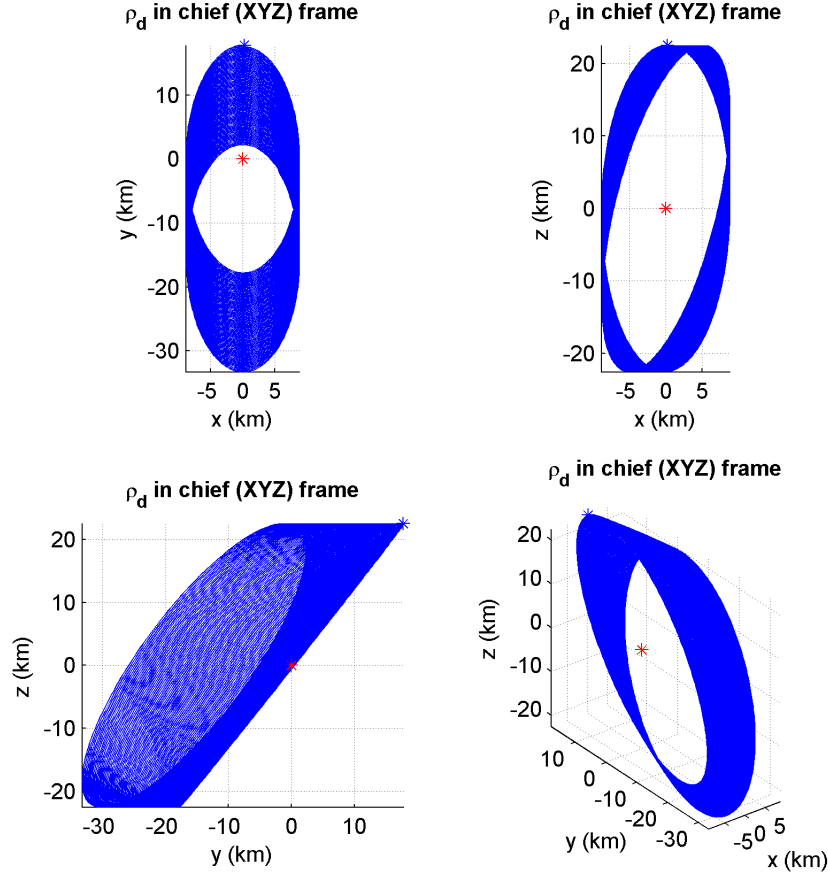


Figure 7.7: Example of “Safety” Formation from Q_1, Q_2, y_5 Displacements (5 day: 75 orbits)

above to eliminate secular drift. Several aspects of this figure are noteworthy: first, the orbit describes a 2:1 ellipse in the chief’s $x \times y$ plane, which is a common result from formation flight study (see, e.g., [9]). The ellipse’s minor axis lies in the reference orbit’s radial direction (x) and the major axis is orthogonal to it, and so is parallel to the along-track axis (y). This ellipse’s semi-minor axis has a magnitude of approximately the reference orbital radius times the difference in eccentricities between the chief and deputy orbits, which, in this example case, is approximately $\delta e \approx 1.2\text{E-}3$. This is, in fact, the same ellipse that will describe the x - y -projected motion for any configuration with non-zero relative y_5, y_6 , such as that described above, even if there is no out-of-plane oscillation. The second item of

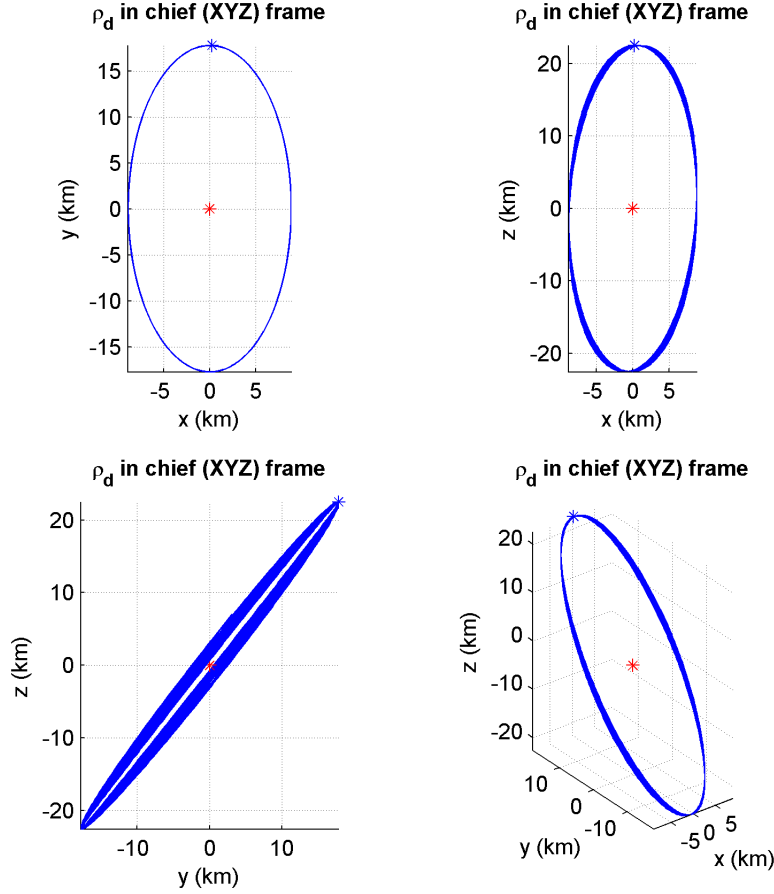


Figure 7.8: Energy-matched Formation from Q_1, y_2, Q_2, y_4, y_5 Displacements (1 day: 15 orbits)

note in the figure, and which is quite different from the results of typical formation analysis and research, is the further evidence that our definition of the satellite formation (including energy matching modifications to y_2 and y_4) leads to a practically zero relative secular drift.

Having so far delivered a somewhat qualitative description of the possible relative satellite orbits, we will now provide some specific details. As mentioned above, displacements in the nodal angle Q_2 cause along-track displacements and oscillations along the normal direction (both in the UVW reference frame of the chief). By using a combination of the Q_1 and Q_2 displacements, we can position the oscillatory center of the relative orbit at any point we desire on the chief's x axis. By using relatively straightforward

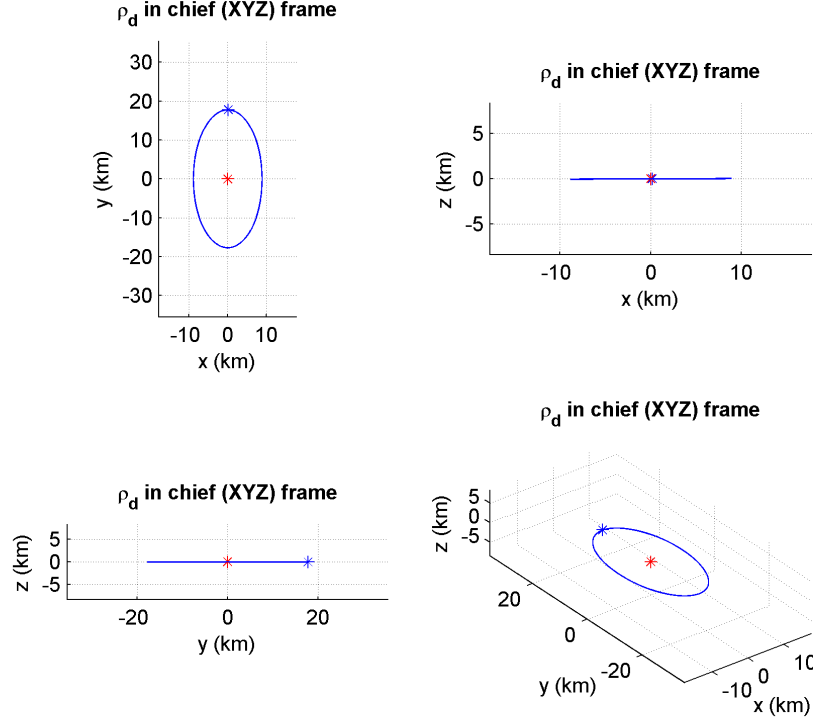


Figure 7.9: Example of $y_4 = 0.01$ Displacement in Formation (with energy matching) (1 day: 15 orbits)

geometry relationships of the chief's and deputy's ECI orbits, we realize that the conditions

$$\begin{aligned}
 Q_{1_d} &= Q_{1_c} + \delta Q_1 \\
 Q_{2_d} &= Q_{2_c} + \delta Q_2 \\
 \delta Q_1 &= -\delta Q_2 \cos i
 \end{aligned} \tag{7.2}$$

will yield an oscillatory relative orbit centered on the chief satellite's position (the origin in UVW coordinates). While this is obviously a very poor decision when dealing with pure Q_1 and Q_2 displacements (as the deputy will, twice per orbit, forcibly attempt to occupy the same physical space as the chief), it lays the foundation that applies to the general case. Specifically, when we introduce a differential eccentricity displacement, e.g. y_5 , we have removed the collision issue and created an orbit similar to that seen in Figure 7.6. The

projected x - y ellipse will have a semi-minor axis of

$$b \approx a\delta e \quad (7.3)$$

where a is the radius of the reference orbit and δe is the differential eccentricity between the chief and deputy. The projected ellipse's semi-major axis will, as mentioned above, be twice the length of b . Finally, the “height” of the relative orbit (i.e., the maximum extents along the chief's z axis) are found, through the geometric inspection of the ECI orbits, as:

$$d_z = \delta Q_2 \sin i \quad (7.4)$$

7.3 Constellation and Formation Summary

This chapter was concerned with developing an intuition about the use of LEST tori in constellation and formation design. Specifically, it was shown that Walker-type constellations may be created in a straightforward fashion by simple offsets in torus angles Q_1 and Q_2 . No-drift formations may be accomplished by determining appropriate offsets in all dimensions of the modal state $\mathbf{y}$. This may be done in such a way that the relative geometry is created with the modal/torus coordinates while secular drift is eliminated by changing the modal energy modes. While not allowing for the most general formation geometries (i.e., those which would require constant or frequent maneuvering due to dynamics in any case), the framework enables the design of a subset of formations which have low maneuver requirements.

VIII. Stability and Control

In the current chapter, an investigation of the questions of stability, and robustness to real-world maneuvering inaccuracies is performed. In addition, a continuous maneuvering strategy based on Lyapunov control theory is introduced.

8.1 LE KAM Torus Stability

The question of stability naturally arises when discussing dynamics formulations or control strategies. In the case of low-eccentricity KAM tori, the investigation is quite straightforward. Our treatment begins with a consideration of the differential equation governing the propagation of the modal states, modified to include terms from perturbations

$$\begin{aligned}\dot{\mathbf{y}} &= \dot{\mathbf{y}}_u + \dot{\mathbf{y}}_{pert} \\ &= \mathcal{J}\mathbf{y} + \dot{\mathbf{y}}_{pert}\end{aligned}\tag{8.1}$$

where $\mathcal{J}$ is, as described previously, the Jordan matrix of the modal system:

$$\mathcal{J} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_3 \\ 0 & 0 & 0 & 0 & -\omega_3 & 0 \end{bmatrix}\tag{8.2}$$

In view of this, we consider first the unperturbed system. It is clear that the rate of change in the state due to the unperturbed system ($\dot{\mathbf{y}}_u$) is weakly unstable. That is, any constant displacement in the torus angles y_1 (Q_1) or y_3 (Q_2) propagates as

$$\frac{d}{dt}(Q_{(1,2)} + \delta Q_{(1,2)}) = \omega_{(1,2)}\tag{8.3}$$

and we know from the torus theory that

$$\dot{Q}_{(1,2)} = \omega_{(1,2)}\tag{8.4}$$

and so we have $\delta\dot{Q}_{(1,2)} = 0$. This shows that any displacement in torus angles Q_1, Q_2 remains constant. Displacements in the energy modal variables y_2, y_4 for the degenerate modes also experience no rectifying force, and so will remain constant. Constant terms in these modes manifest as linear secular growth in their associated coordinates, and so, if unregulated, will cause unbounded (albeit linear) growth in the torus coordinates. This is intuitive, as a change in the energy modes amounts to a change in the torus actions, which implies that the modified torus has different frequencies as compared the unmodified torus, as this is not an iso-energetically nondegenerate system.

The eccentricity modes y_5 and y_6 cause (when nonzero) an apparent offset in the frequencies y_2 and y_4 , which causes the torus coordinates to evolve as discussed in previous chapters. Therefore, the secular effects due to small displacement in y_5 and y_6 can be treated in a similar fashion to frequency offsets. It is also of note that, when cast into the physical coordinates, an offset in y_5 and y_6 also manifests as oscillatory behavior, as discussed earlier in the section on formation design.

Real-world perturbations to the modal state typically fall into three main classes: sectoral and tesseral geopotential harmonics, atmospheric drag/solar radiation pressure, and third-body gravitational effects. Sectoral and tesseral harmonics have been discussed extensively in this document; these will cause periodic effects in the modal variables along with a possible very small linear secular growth in the angle coordinates. Third-body gravitational influence, being also a conservative effect, will cause short-term and/or long-term periodic oscillations in the modal variables, depending on the orientation of the reference orbit. As mentioned previously, third-body effects may be taken into account by expanding the KAM Fourier series definition. For example, in the case of inclusion of lunar perturbations on a HEO satellite, the perturbation series would be functions of two more angles; namely, the orientation of the Moon with respect to the spacecraft, defined by a “right-ascension/declination”-like pair of angles which are functions of time. This

expansion of the theory would increase the series construction complexity significantly, but is relatively straightforward.

Non-conservative effects such as atmospheric drag and solar radiation pressure typically present a linear secular change to the energy modes y_2, y_4 which, in turn, causes quadratic divergence of the coordinates y_1, y_3 . In practice, the atmospheric drag force acts upon the satellite via the well-known equation

$$\begin{aligned} \mathbf{a}_{drag} &= -\frac{1}{2} \frac{C_D A}{m} \rho \mathbf{v} \mathbf{v} \\ \mathbf{a}_{drag} &= -\frac{1}{2} \frac{\rho}{\beta} \mathbf{v} \mathbf{v} \end{aligned} \quad (8.5)$$

where $\mathbf{a}_{drag}$ is the acceleration due to drag, C_D is the drag coefficient of the vehicle, A is the area of the vehicle presented to the flow, m is the mass of the vehicle, $\mathbf{v}$ is the vehicle's velocity relative to the atmosphere, $v = \|\mathbf{v}\|$, and β is the ballistic coefficient, which abbreviates the three vehicle-dependent terms $m/(C_D A)$ into one quantity with units [mass/area]. This equation, in view of the torus construction, can be viewed as a continuous, small force on the satellite acting as

$$\dot{\mathbf{y}}_{drag} = -\frac{1}{2} E^{-1}(Q_1) R_{2Z}^T(Q_2) \frac{\rho}{\beta} \|\mathbf{p}\| \mathbf{p} \quad (8.6)$$

where $\mathbf{p}$ is the inertial canonical momentum in the ECR frame. As we shall see in the next section, the transformation E^{-1} causes small perturbations to act primarily on the in-track energy y_2 , which means that the maximum change in modal states due to drag may be approximated as

$$\begin{aligned} \dot{y}_{2drag} &\approx -\frac{1}{2} \xi_{1,2} \frac{\rho}{\beta} \dot{Q}_1 \\ &\approx -\frac{1}{2} \xi_{1,2} \frac{\rho}{\beta} \omega_1 \end{aligned} \quad (8.7)$$

where $\xi_{1,2}$ is the y_2 element (associated with the highest singular value) of the product of the left singular vectors of E^{-1} with its singular values; the singular value analysis of E^{-1} is discussed in the next section. Therefore, even though atmospheric drag does not generally

cause relative secular drift between satellites in formation, the global secular drift could be corrected by a constant thrust or periodic impulse to counter the secular rate (8.7).

Solar radiation pressure, over short time scales (much less than one Earth year), influences the satellite in a similar way to atmospheric drag (in that it causes a decay of energy), although the force direction becomes a function of the angle from the satellite to the Sun, rather than with respect to the atmosphere. At longer time scales, solar radiation pressure causes complex effects including variations in all of the classical osculating elements [41] with magnitude depending on the ballistic coefficient, and therefore to all of the modal states. These effects, while not discussed further in this work, could also be handled through extension of the perturbation theory; however, they are important mainly for satellites with very low ballistic coefficients (e.g., balloons or sails) operating at very high altitudes.

8.2 Impulsive Maneuvers with Imperfect Thrusters

While the results thus far in this document show that low-eccentricity KAM tori provide an accurate and effective method of representing orbital motion, and while the maneuver theory presented shows that it is possible to maneuver onto a torus, the actual applicability of the method will depend upon the effects of uncertainties in the maneuvering process on the maneuvering cost. We consider in this section errors in impulsive maneuvers caused from inaccuracies in Δv magnitude and misalignments in maneuvering attitude.

8.2.1 Error Manifestation.

Errors in impulsive maneuvers may come in the form of an inaccurate maneuver magnitude, an arbitrary attitude angular error about an arbitrary rotation axis, or some combination of the two. (We assume here, for the sake of simplicity and clarity, that any systematic or biased errors have been removed or otherwise accounted for in the spacecraft control system.) In the present discussion, we will represent the error in maneuver magnitude as $\delta\Delta v$ and an error in attitude angle as the Euler axis-angle pair $\delta\alpha, \hat{e}$. We

can represent the “true” maneuver vector (i.e., that effected by error) as

$$\delta \mathbf{p}_e = q_\alpha \delta \tilde{\mathbf{p}} q_\alpha^{-1} \quad (8.8)$$

where q is the quaternion associated with the rotation about $\hat{e}$,

$$q = \begin{bmatrix} \hat{e} \sin \frac{\alpha}{2} \\ \cos \frac{\alpha}{2} \end{bmatrix} \quad (8.9)$$

and $\delta \tilde{\mathbf{p}}$ is the quaternion with a zero scalar part formed from the planned momentum change modulated by the error $\delta \Delta v$:

$$\delta \tilde{\mathbf{p}} = \begin{bmatrix} \frac{\delta \mathbf{p}}{\|\delta \mathbf{p}\|} (\|\delta \mathbf{p}\| + \delta \Delta v) \\ 0 \end{bmatrix} \quad (8.10)$$

Note that the multiplication in (8.8) is the non-commutative Hamilton product for quaternions, and q^{-1} denotes the quaternion inverse.

We will consider in the following analysis the case in which the error magnitude is distributed according to a Gaussian distribution with zero mean; that is, $\delta \Delta v \sim \mathcal{N}(0, \sigma_{\Delta v}^2)$. While this is convenient for the current derivation, the method could be extended by substitution of this distribution with other desired distributions.

8.2.2 Singular Value Analysis.

Because the modal variables (when managing the global angles $\mathbf{Q}$ properly) constitute a very slowly-varying coordinate representation, we propose that it is instructive to analyze the singular value decomposition of the transformation between physical canonical coordinates and modal coordinates. That is, we wish to investigate the transformation embodied by the modal matrix

$$E^{-1}(t) R_{2Z}^T \quad (8.11)$$

which maps between instantaneous changes in the ECR physical variables and changes in the modal coordinates, as

$$\delta \mathbf{y} = E^{-1}(t) R_{2Z}^T \delta \mathbf{x} \quad (8.12)$$

and can be represented in a standard singular value/vector decomposition as

$$E^{-1}(t)R_{2Z}^T = U\Sigma V^* \quad (8.13)$$

where U is the matrix of so-called left-singular vectors of $E^{-1}(t)R_{2Z}^T$, Σ is the diagonal matrix of singular values, and V is the matrix of right-singular vectors. In the current application, U is a 6×6 matrix describing the transformation between what we will describe as the “scaling space” (i.e, that of the singular values) and the vector space of the modal coordinates. The matrix Σ is a diagonal 6×6 matrix, and $V^* = V^T$ is a 6×6 matrix describing the transformation between the physical coordinates and the scaling space. We can note several useful facts about the system at this point. First, since we are physically limited to instantaneous momentum changes only, we have

$$\delta \mathbf{x} = \begin{bmatrix} 0 \\ \delta \mathbf{p} \end{bmatrix} \quad (8.14)$$

and as a result, we see that, in a realistic application, the transformation is limited to $E^{-1}(t)R_{2Z}^T I_2$, where I_2 is the 6×3 matrix $I_2 = [\mathbf{0}, \mathbf{1}]^T$. Thus, the effective size of the matrix V^T is actually 6×3 , and we may not simply use matrix inversion. Secondly, we recognize that the matrix V^T is a unitary (rotation) matrix which transforms the vector $\delta \mathbf{x}$ into the scaling space, and, as such, we can append multiple rotation matrices to the quantity $E^{-1}(t)R_{2Z}^T I_2$ to transform $\delta \mathbf{p}$ into a different reference frame. In particular, a convenient representation of the change in modal coordinates due to momentum changes is

$$\delta \mathbf{y} = E^{-1}(t)R_{2Z}^T R_{UVW}^{ECEF} I_2 \delta \mathbf{p}_{UVW} \quad (8.15)$$

where R_{UVW}^{ECEF} is the rotation matrix transforming vector representations from the satellite’s UVW frame to the ECEF frame (as discussed in previous chapters) and $\delta \mathbf{p}$ is the momentum change expressed in the UVW frame.

With these preliminaries in place, the key insight gained from the singular value analysis stems from investigation of the behavior of the singular vectors and singular values

over the period of the modal matrix. That is, we can view the singular values ξ_i from the matrix Σ as measures of the “strength” of the effects of momentum changes on $\delta\mathbf{y}$; the scaled momentum effects are mapped onto the modal variables by the left singular vectors U . It also happens that, when using the UVW-ECEF form of the rotation matrix described above, we have

$$V^T \approx \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (8.16)$$

over the entire period of $E^{-1}(t)$; that is, V^T is a permutation matrix stating that, as intuition would suggest, the effects of pure radial, along-track and orbit-normal maneuvers are approximately scaled by the medium, highest, and lowest singular values, respectively. (Note that, following convention, we have ordered the columns/rows of the factorization matrices $U\Sigma V^T$ such that the diagonal [singular] values in Σ are in descending order of magnitude.)

The singular values (in order of decreasing magnitude) for the transformation $E^{-1}(t)I_2$ for the 630km, 40° orbit are shown in Figure 8.1. (For this and subsequent plots, the matrices V^T and $R_{2Z}^T R_{UVW}^{ECEF}$ have been omitted, both for clarity and because they only change the way physical changes map into the scaling space, and do not change Σ or U .) It is clear that the highest singular value is, through most of the orbital period, at least two orders of magnitude higher than the lowest singular value, with the middle singular value falling approximately halfway between the highest and lowest. In terms of the mapping just discussed, this implies that along-track maneuvers have the greatest effect on the modal variables (as a whole), while radial maneuvers have approximately half the overall effect, and normal maneuvers have a small effect. Also evident in the plot is that orbit-normal maneuvers have cyclic effectiveness; i.e., they have little effect at equatorial crossings (which was the initial condition of this orbital simulation) and greater effect at $Q_1 \approx \pi/2, 3\pi/2$, where it can be used to change the right ascension of the ascending node.

Indeed, this amounts to a confirmation of a fact that is commonly used in the astronautics community. Since orbit-normal maneuvers at equatorial crossing are typically the most effective method for pure changes in orbital inclination, these results also demonstrate, in the context of modal dynamics, the fact that orbital plane changes are relatively quite expensive in terms of Δv cost.

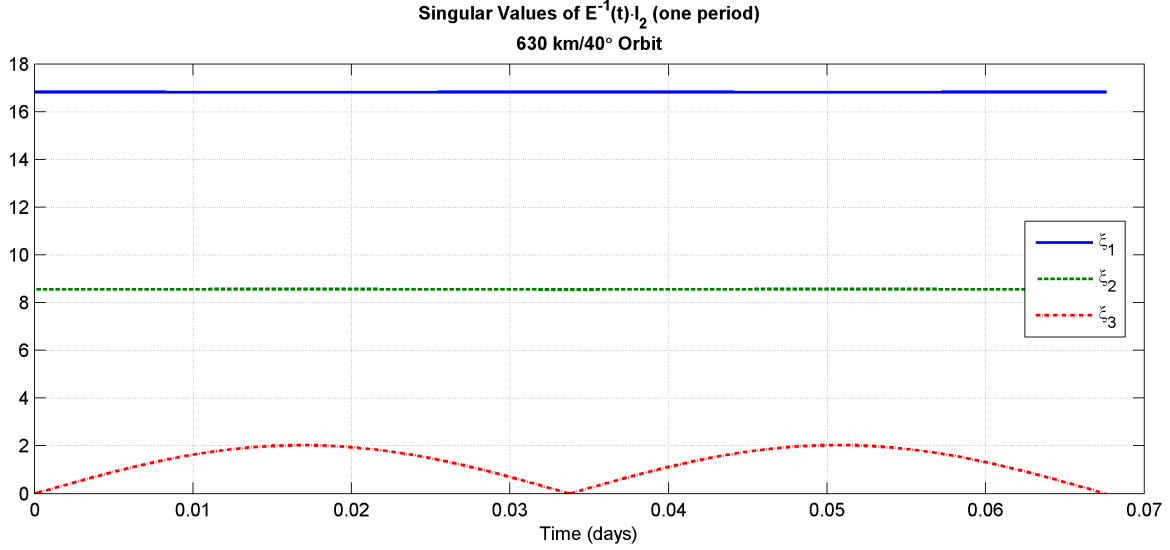


Figure 8.1: Singular Values of $E^{-1}(t)I_2$ (630km, 40°, one period)

It is also instructive for our purposes to examine the left singular vectors for the transformation $E^{-1}(t)I_2$. While the singular values described above show the strength of the effects on the modal variables as a whole, the left singular vectors determine how these effects change the modal variables. By inspecting the left singular vectors as combined with the singular values, we can determine which modal variables are affected the most, and by which types of maneuvers. We define the “left scaled singular vectors” as the columns of the partial transformation:

$$\tilde{U} = U\Sigma \quad (8.17)$$

Figures 8.2, 8.3 and 8.4 show the left scaled singular vectors for the modal transformation matrix for the same 630km, 40° orbit described above. The first figure gives the left singular vector associated with the first (and highest) singular value, ξ_1 , over one orbital period, the second figure gives the vector associated with ξ_2 , and the third figure shows the vector associated with ξ_3 . It is immediately evident in the figures that the strongest effect due to the first and second singular values is a change in the eccentricity-like variables y_5, y_6 . This effect is expected due to the fact that the reference orbit is nearly circular, which means the analogue of $e \cos \omega, e \sin \omega$ will change rapidly as the argument of perigee becomes more strongly defined. The second observation is that, for all singular values, the change in the in-track energy y_2 is greater than the change in the nodal energy y_4 , and, in the ξ_1 and ξ_2 cases, the in-track energy is 2 to 3 orders of magnitude higher. This will become important to our discussion shortly. In view of the figures, it is also noteworthy that the instantaneous changes in the coordinates y_1, y_3 do NOT represent instantaneous physical coordinate changes; rather, because the energy has changed by the maneuver, the new modal coordinates describing the same physical position (with different momentum) will possess different angles. This is essentially a further demonstration of the concept of maneuvers as a method to change to a different KAM torus.

With the observation that momentum changes in the principal directions of the transformation $E^{-1}(t)$ tend to change the y_2 energy much more than the y_4 (unless the burn direction is precisely chosen otherwise), and since a random perturbation $\delta\Delta v$ to the velocity vector will, therefore, tend to cause a y_2 change, we will make the approximation that the overriding effect of maneuver errors will be the requirement of periodic maneuvers to counter the along-track drift of the satellite. We also recognize that, with the assumption that the angle error $\delta\alpha$ is very small (on the order of one degree or less, achievable by most modern attitude determination/control systems),

$$\delta p_e \approx \delta \tilde{p} \quad (8.18)$$

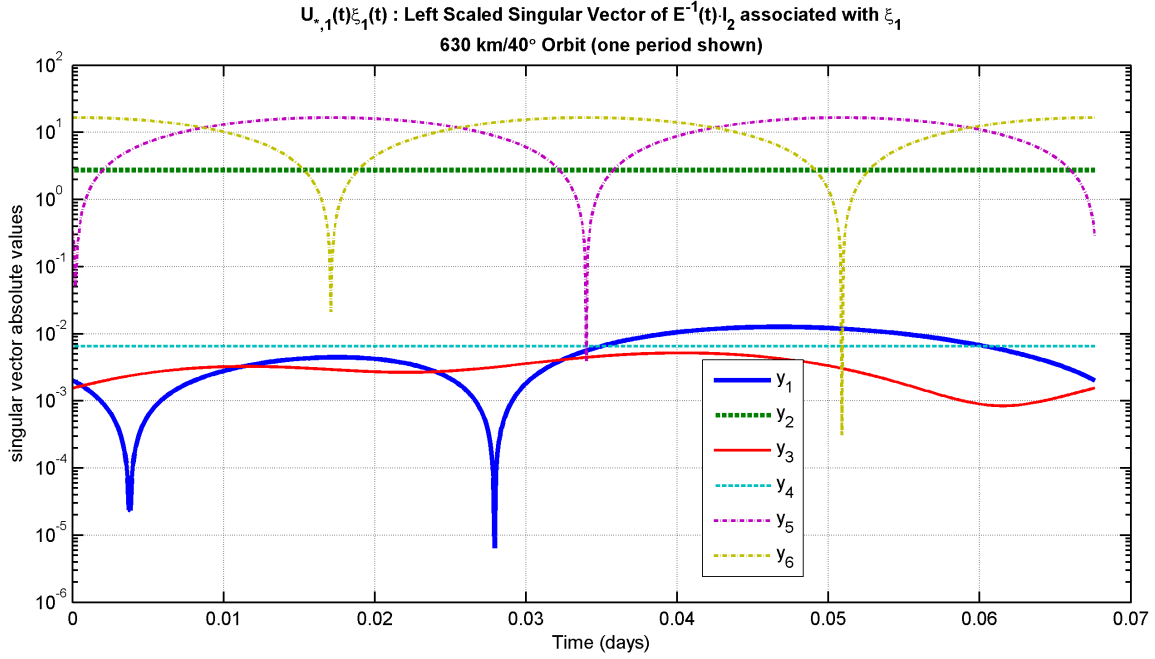


Figure 8.2: Left Scaled ξ_1 -Singular Vector of $E^{-1}(T)I_2$ (630 km, 40°, one period)

(stated in terms of the quaternions above), or, similarly, the equivalent rotation matrix embodied by the quaternion rotation through $\delta\alpha$ is very nearly identity. This implies that the error in maneuver magnitude is of much greater effect than the random error in orientation, which, incidentally, by the singular value analysis above, will still tend to effect largest change in the along-track energy mode y_2 by several orders of magnitude. Henceforth, we assume small angles $\delta\alpha$ and thereby neglect orientation errors in our current discussion, instead focusing largely on maneuver magnitude errors.

8.2.3 Station-keeping Costs.

Based on the exposition above, we may estimate the expected station-keeping costs for a satellite with non-zero errors in maneuver magnitude; we will denote the error magnitude as $\delta\Delta v$. Since the above discussion leads us to assume that, after initial orbit establishment within acceptable parameters, we will almost exclusively combat along-track drift errors, we will consider the effects of maneuver magnitude error on the terms in the

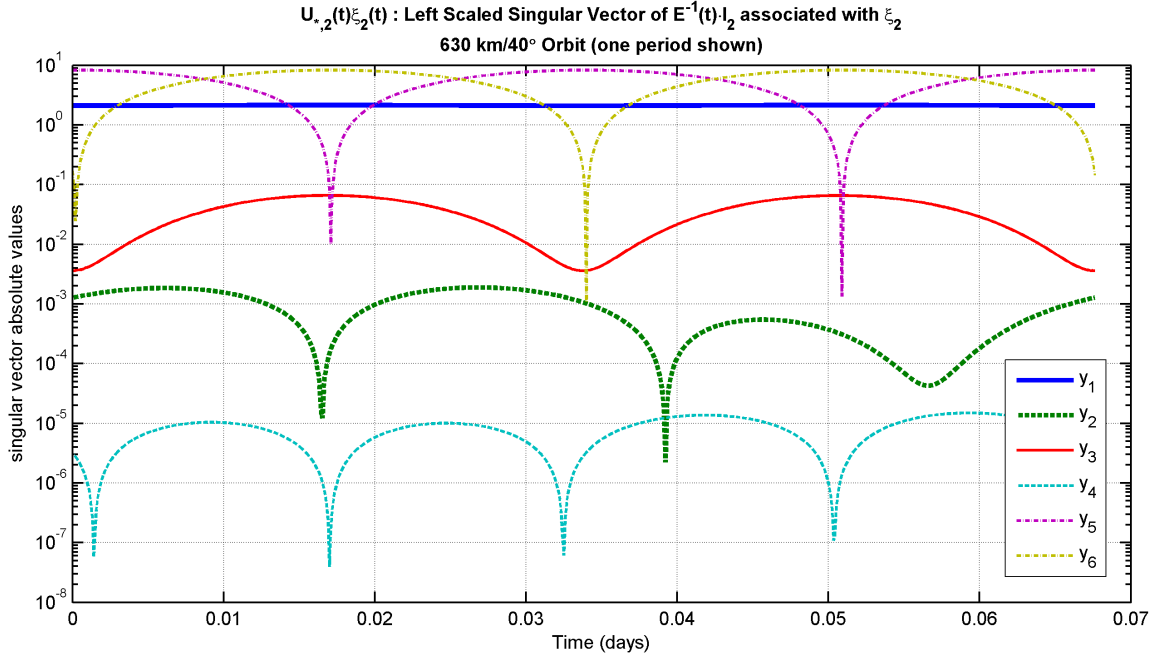


Figure 8.3: Left Scaled ξ_2 -Singular Vector of $E^{-1}(t)I_2$ (630km, 40° , one period)

left singular vectors associated with the energy mode y_2 . This amounts to projecting the three-dimensional velocity error to a one-dimensional effect. To do this, we first leverage the assumption that $\delta\Delta v \sim \mathcal{N}(0, \sigma_{\delta\Delta v}^2)$ to assert that the approximate error in energy from our desired value is

$$\delta y_2 \approx U_{2,1} \xi_1 \delta\Delta v \quad (8.19)$$

where $U_{2,1}$ denotes the y_2 element of the left singular vector associated with ξ_1 (which yields the largest y_2 effect, by three orders of magnitude). Then, since we have reduced the $\delta\Delta v$ distribution to a one-dimensional Gaussian, we may further assume that the error in each maneuver is independent and identically distributed, and posit that the distribution of the summed cost of a pair of maneuvers is $\delta\Delta v_p \sim \mathcal{N}(0, 2\sigma_{\delta\Delta v}^2)$ (i.e., a sum of two i.i.d. Gaussian distributions). Since the secular error in the y_1 (and thus the torus angle Q_1) grows

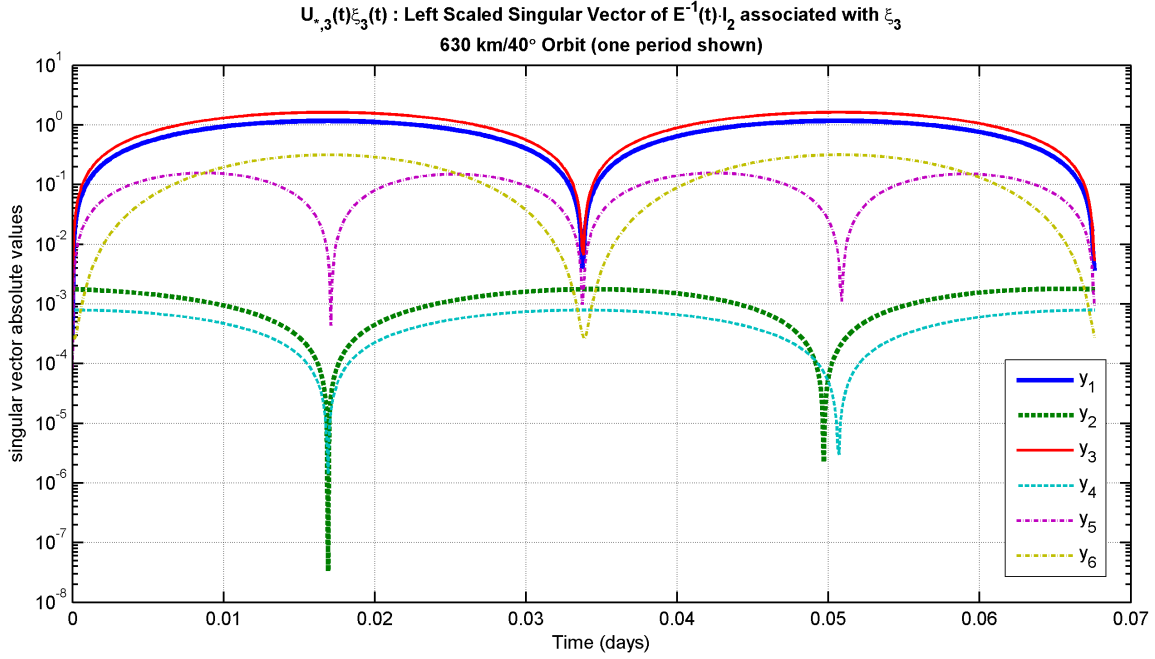


Figure 8.4: Left Scaled ξ_3 -Singular Vector of $E^{-1}(t)I_2$ (630km, 40°, one period)

as

$$\delta\dot{Q}_1 = \delta y_2 \quad (8.20)$$

we see that, for “small” displacements $e = \delta Q_1 \ll \pi$ we can approximate the linear error distance of the satellite from its desired station as:

$$e(t) = \delta y_2 \Delta t + e_0 \quad (8.21)$$

Now, since we assume that the error in each pair of maneuvers is independently realized, we can approximate the fractional number of maneuver pairs required for a specific amount of time T (in a distribution sense) to stay within a certain error bound by the equation

$$M = a \frac{U_{2,1} \xi_1 T}{e_{max}} |\delta \Delta v_p| \quad (8.22)$$

where M is the number of maneuvers, a is the orbit’s semi-major axis, and e_{max} is the maximum error (in absolute distance) allowed from the satellite’s intended station before

maneuvering. That is, for an orbit with semi-major axis a whose modal dynamics imply singular vectors and values U and ξ , a satellite whose thrusters have errors of magnitude $|\delta\Delta v_p|$ must conduct approximately M maneuvers over length of time T to maintain station within a distance e_{max} of nominal.

Now, during a maneuver pair, the satellite travels a modal in-track distance (i.e., an angle)

$$\begin{aligned}\delta y_1 &\approx \delta y_2 \Delta t_p \\ &\approx U_{2,1} \xi_1 \Delta v_p \Delta t_p\end{aligned}\tag{8.23}$$

where Δv is the magnitude of the pair of maneuvers (including base cost and errors) and Δt_p is the time between the two maneuvers comprising the pair. An additional factor of the mean orbital radius a converts this into an in-track distance traveled:

$$e \approx a U_{2,1} \xi_1 \Delta v_p \Delta t_p\tag{8.24}$$

Solving this equation for the required burn magnitude Δv_p to negate an in-track error e yields

$$\Delta v_p \approx \frac{e}{a U_{2,1} \xi_1 \Delta t}\tag{8.25}$$

Since the maneuvers for correcting in-track displacements typically occur near (but not exactly at) multiples of the orbital period τ , we may further factor the timespan into $\Delta t = n\tau$. Substituting this into the previous equation and multiplying by the fractional number of maneuvers given in (8.22), we may find an expression for the total maneuver cost:

$$\Delta v_T \approx M \Delta v_p = \frac{T}{n\tau} |\delta\Delta v_p|\tag{8.26}$$

For clarity, we note that this equation represents, in a very straightforward way, the approximate distribution of total maneuver cost over a length of time T , given that n periods are allowed for each maneuver pair duration and the satellite's thruster errors are given as

$\delta\Delta v_p \sim N(0, 2\sigma_{\delta\Delta v}^2)$. We recognize, then, that the total maneuver cost is a scaled folded normal distribution; specifically, since the cost $\delta\Delta v_p$ has zero mean, Δv_T possesses a scaled half-normal distribution. We may then write the mean of the total cost as

$$\mu_{\Delta v} = \frac{T}{n\tau} \frac{2\sigma_{\delta\Delta v}}{\sqrt{\pi}} \quad (8.27)$$

and its variance as

$$\sigma_{\Delta v}^2 = 2 \left(\frac{T\sigma_{\delta\Delta v}}{n\tau} \right)^2 \left(1 - \frac{2}{\pi} \right) \quad (8.28)$$

To investigate the validity of this approximation process, several sets of Monte Carlo simulations were conducted. Each Monte Carlo set contained at least 250 separate simulations. Each individual simulation was conducted in the following manner:

1. Create a virtual satellite and place it on an LE torus
2. Use the LE theory, along with torus-to-physical transformation principles (discussed in Chapter 5.1) to determine the satellite's physical coordinates
3. Numerically propagate a “pristine” satellite forward until the simulation end time; this satellite represents the desired station to use in position comparisons
4. Add a random initial in-track velocity error $\delta\Delta v \sim N(0, \sigma_{\delta\Delta v}^2)$
5. Numerically propagate the satellite forward until its 2-norm position error from the “pristine” satellite crosses a pre-set threshold
6. After threshold crossing, use data from the current drift to estimate the satellite's modal state
7. Use the modal state difference (estimated vs. desired) to calculate the necessary maneuvers $\delta\mathbf{p}_1, \delta\mathbf{p}_2$ and maneuver times t_1, t_2 to fly back to the desired torus (following the theory given in the current work; see Chapter 6). Allow n periods of drift between the two maneuvers in a pair.

8. Propagate to first maneuver time; add $\delta \mathbf{p}_1$ to the velocity state (to simulate an impulsive maneuver)
9. Propagate to second maneuver time; add $\delta \mathbf{p}_2$ to the velocity state (to simulate an impulsive maneuver)
10. Repeat steps 5-9 until end of simulation time

Note: all propagations performed in the current analysis include geopotential effects to order/degree 20.

Figure 8.5 shows one particular realization of the Monte Carlo experiment, following the process outlined above, for a Galileo-like satellite, in which the standard deviation of the single-impulse maneuver error was $\sigma_{\delta \Delta v} = 5\text{mm/s}$ and the station was kept to within about 5km. For this particular simulation, the resulting average total cost was approximately 0.39m/s per 30 days.

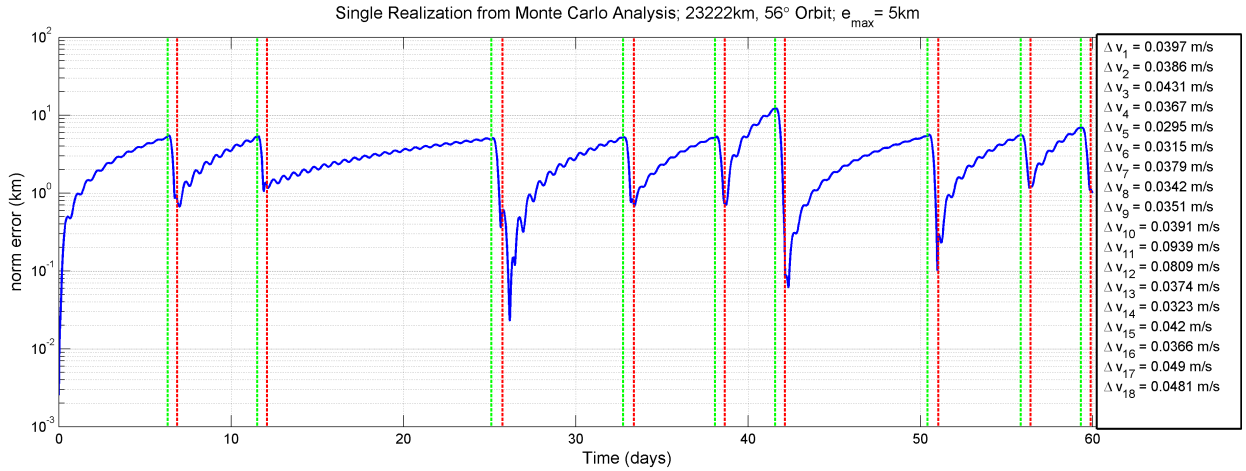


Figure 8.5: Single Realization of Station-keeping Cost over Two Months to Maintain Position within 5km (23222 km, 56° orbit; $\sigma_{\delta \Delta v} = 5\text{mm/s}$)

These Monte Carlo simulations were performed with four different sets of conditions, utilizing the two orbit classes dealt with often in this document and assigning different thrust error magnitudes:

1. 630km, 40° orbit; $\sigma_{\delta\Delta v} = 1mm/s$; $n = 8$
2. 630km, 40° orbit; $\sigma_{\delta\Delta v} = 5mm/s$; $n = 8$
3. 23222km, 56° orbit; $\sigma_{\delta\Delta v} = 1mm/s$; $n = 1$
4. 23222km, 56° orbit; $\sigma_{\delta\Delta v} = 5mm/s$; $n = 1$

As mentioned previously, over 250 simulations were performed for each of these cases, each of which in turn involved multiple propagations, maneuver planning steps, maneuver executions, etc. over 60 days. The resulting data are given in histogram form in Figures 8.6 and 8.7 (for the 630km, 40° cases) and Figures 8.8 and 8.9 (for the 23222km, 56° cases). In each of the figures, the probability density function (PDF) is also given for the half-normal distribution with the statistics $\mu_{\Delta v}, \sigma_{\Delta v}^2$ (from the process outlined in this section); this PDF is given by

$$f(x, \sigma_{\Delta v}^2) = \frac{\sqrt{2}}{\sigma_{\Delta v} \sqrt{\pi}} \exp\left(-\frac{x^2}{2\sigma_{\Delta v}^2}\right) \quad (8.29)$$

Note that the magnitude of the PDF has been scaled to allow visual comparison of the distribution shape with the histogram results of the Monte Carlo simulations.

Each plot's legend shows the mean and standard deviation of both the Monte Carlo simulation results and the theoretical distribution. In these cases, the aggregate statistics of the Monte Carlo simulations match the predicted distribution's statistics quite well, usually matching in both mean and standard deviation by less than 10 percent; in cases where mismatch is greater, the Monte Carlo results possess more favorable statistics (i.e., lower costs).

To investigate the validity of our assumption that small angle errors may be ignored, an additional two Monte Carlo cases were investigated:

5. 630km, 40° orbit; $\sigma_{\delta\Delta v} = 1mm/s$; $n = 8$; $\sigma_{\delta\alpha} = 0.5^\circ$
6. 630km, 40° orbit; $\sigma_{\delta\Delta v} = 5mm/s$; $n = 8$; $\sigma_{\delta\alpha} = 0.5^\circ$

That is, random orientation errors were added at each maneuver, with the angle error $\delta\alpha$ distributed as $\delta\alpha \sim N(0, \sigma_{\delta\alpha})$ and with each component of the rotation axis $\hat{e}$ chosen uniformly from the interval $[-1, 1]$, after which $\hat{e}$ was normalized to obtain a unit vector. The results of the simulations following this modified process are shown in Figures 8.10 and 8.11. The plotted Δv results show no appreciable difference in statistical behavior from those cases which did not include induced angular errors. This is further evidence that our approximate theory for estimating maneuver cost is valid; however, further simulation (with much higher simulation N count) would potentially provide more insight and highlight second-order differences.

The practical results of the station-keeping cost analysis can be summarized as shown in the plots in Figures 8.12 and 8.13. These plots contain isocontours of the average cost $\mu_{\Delta v}$ over a period of 30 days, for the LEO and HEO (Galileo-like) orbits, respectively. These plots show that, as expected, the station-keeping costs decrease both with increasing n (i.e., allowing more drift time between maneuvers in a pair) and with decreasing $\sigma_{\delta\Delta v}$ (i.e., more accurate thrusters). Also, it is evident by this analysis that the cost with which a satellite may station-keep on a torus is driven not by the dynamics formulation, but rather by the ability of the satellite to maneuver accurately. This further increases confidence in the validity of the maneuver theory given in this paper and in its applicability to real-world systems. However, it is noteworthy that, as mentioned previously, the current analysis does not include the dynamics effects of atmospheric drag (which would affect LEO satellites) or third-body gravitation (which would affect HEO satellites); these may introduce additional control costs to combat the associated secular drifts.

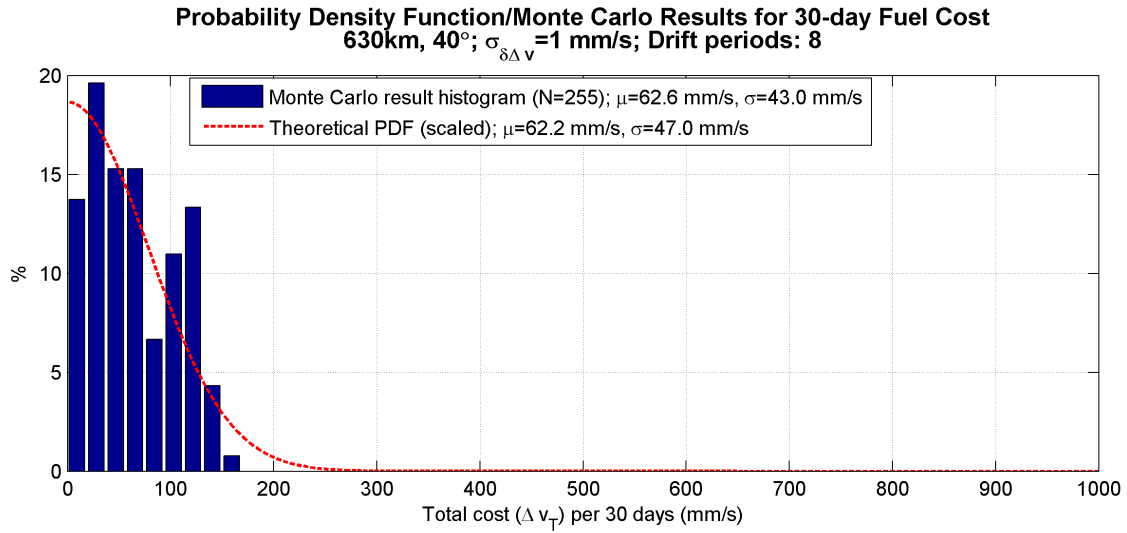


Figure 8.6: Comparison of Monte Carlo Simulations to Theoretical Cost Distribution (630km, 40° orbit; $\sigma_{\delta\Delta v} = 1\text{mm/s}$)

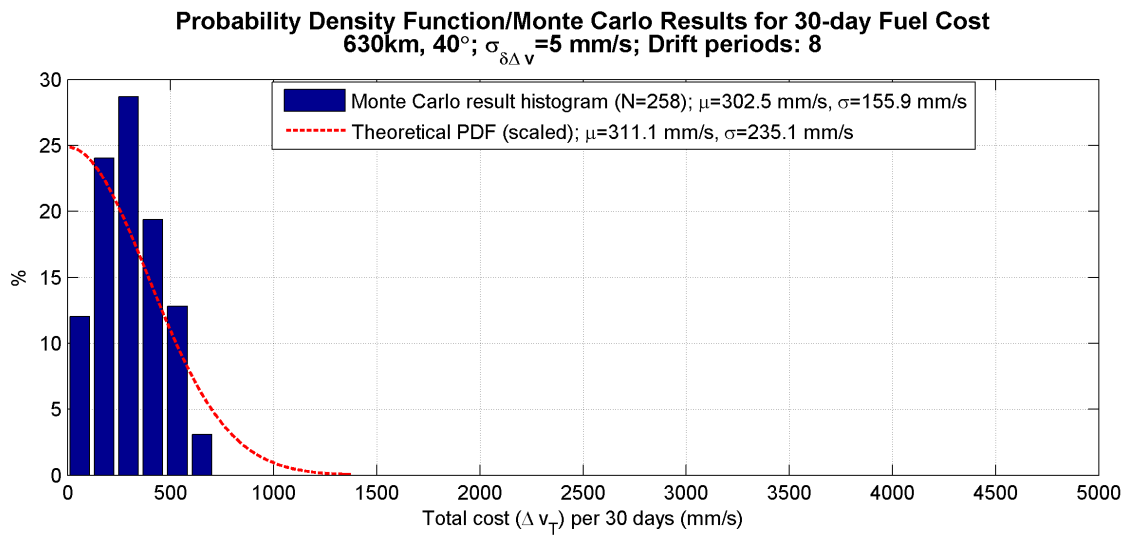


Figure 8.7: Comparison of Monte Carlo Simulations to Theoretical Cost Distribution (630km, 40° orbit; $\sigma_{\delta\Delta v} = 5\text{mm/s}$)

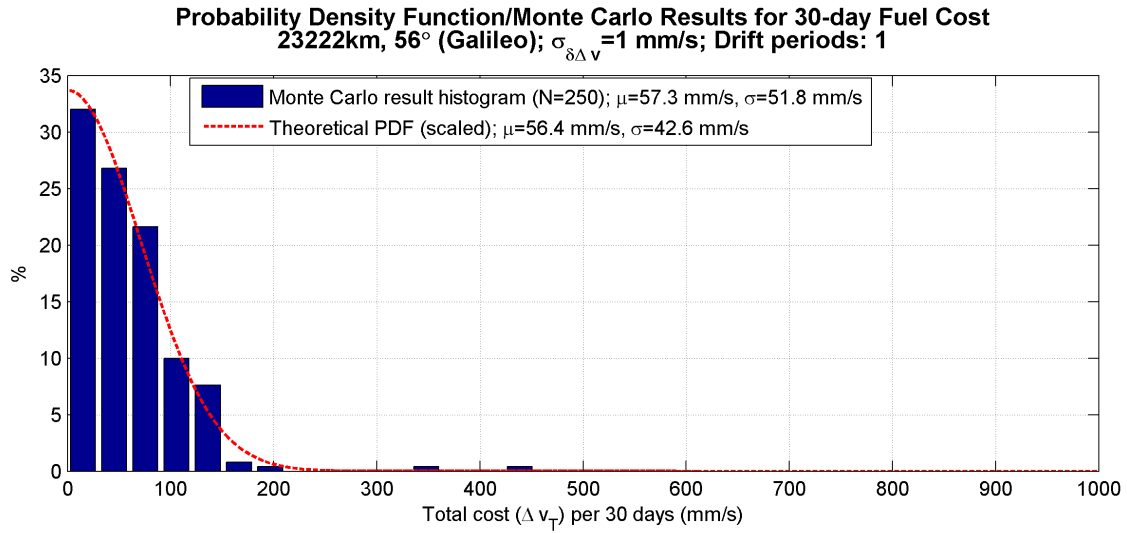


Figure 8.8: Comparison of Monte Carlo Simulations to Theoretical Cost Distribution (23222km, 56° orbit; $\sigma_{\delta\Delta v} = 1$ mm/s)

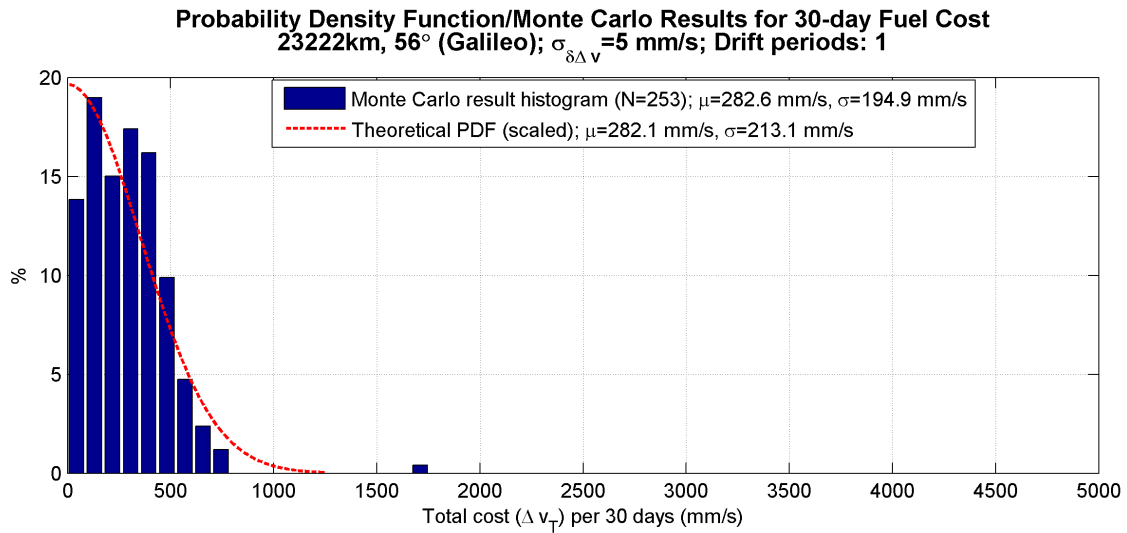


Figure 8.9: Comparison of Monte Carlo Simulations to Theoretical Cost Distribution (23222km, 56° orbit; $\sigma_{\delta\Delta v} = 5$ mm/s)

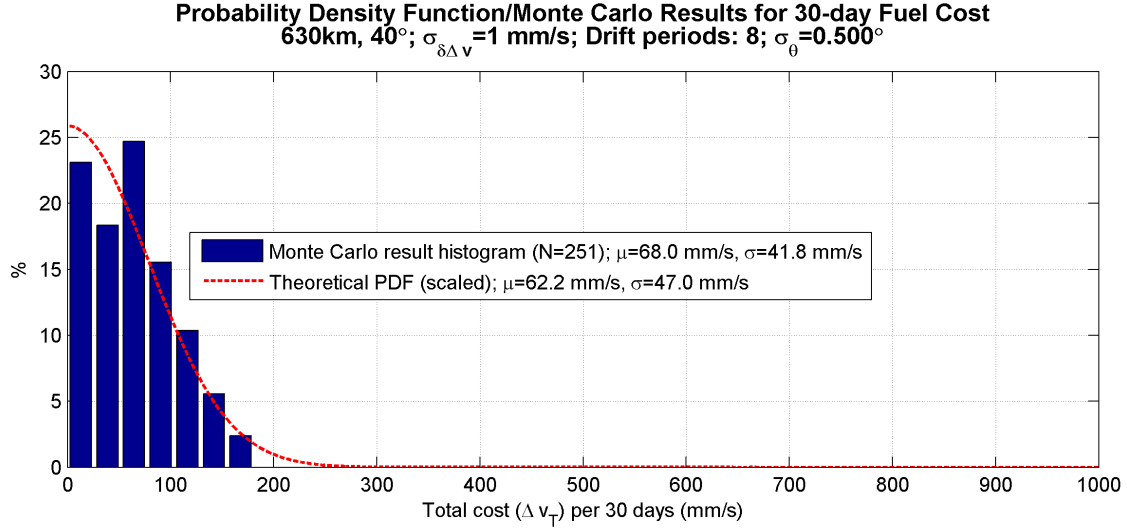


Figure 8.10: Comparison of Monte Carlo Simulations to Theoretical Cost Distribution (630km, 40° orbit; $\sigma_{\delta\Delta v} = 1 \text{ mm/s}$, $\sigma_\alpha = 0.5^\circ$)

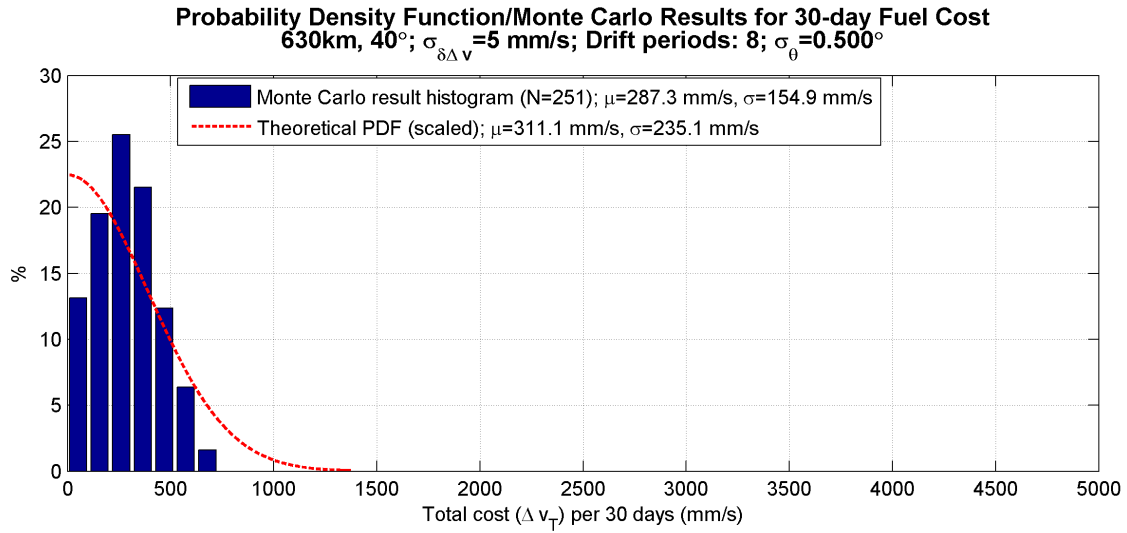


Figure 8.11: Comparison of Monte Carlo Simulations to Theoretical Cost Distribution (630km, 40° orbit; $\sigma_{\delta\Delta v} = 5 \text{ mm/s}$, $\sigma_\alpha = 0.5^\circ$)

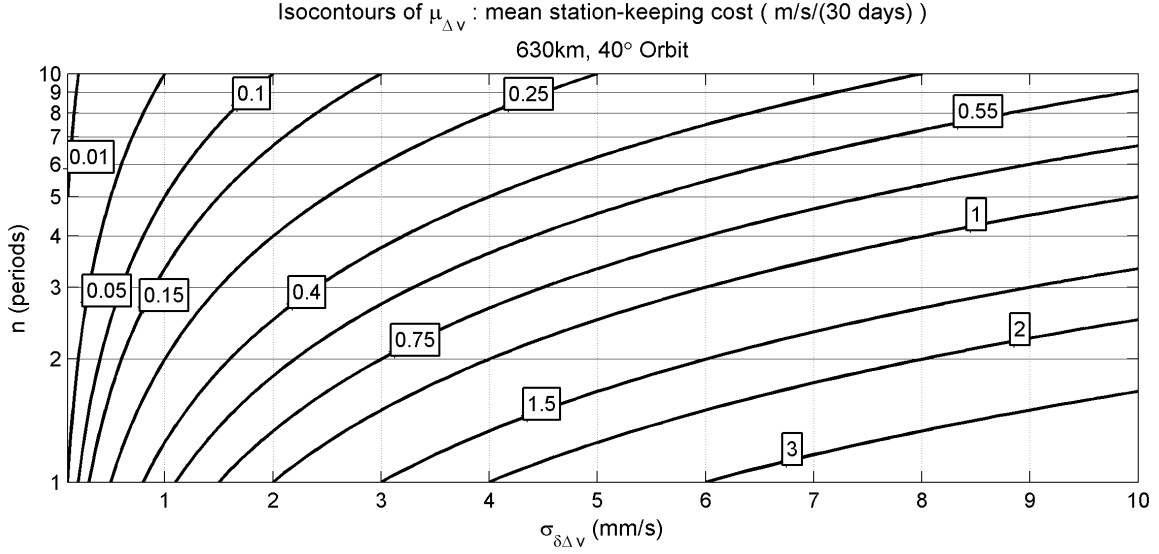


Figure 8.12: Isocontours of Mean Cost $\mu_{\Delta v}$ over 30 Days (630km, 40° orbit)

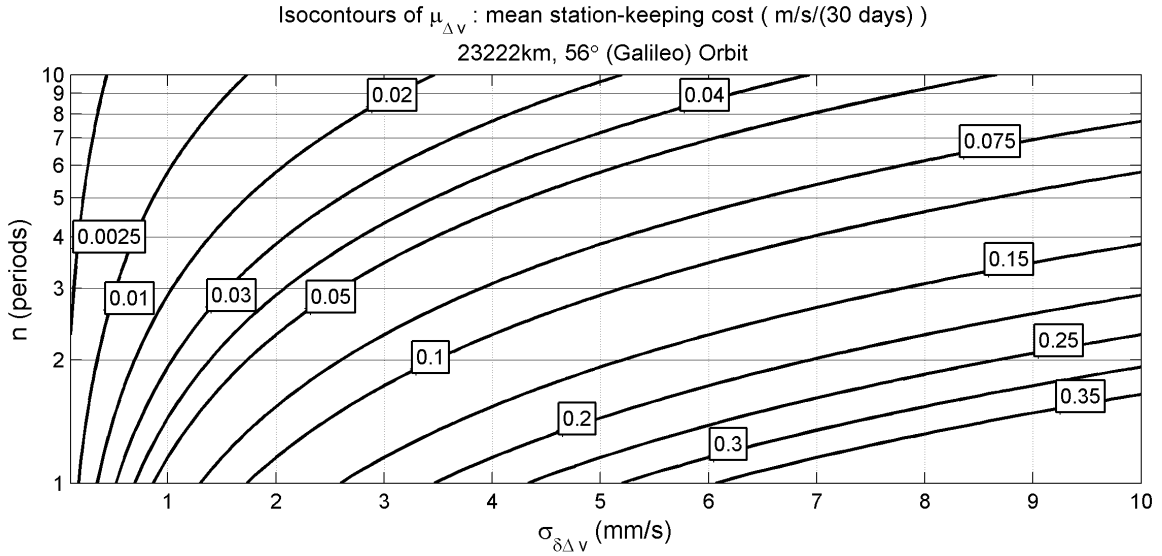


Figure 8.13: Isocontours of Mean Cost $\mu_{\Delta v}$ over 30 Days (23222km, 56° orbit)

8.2.4 A Note on Torus Estimation.

The discussions above (and the accompanying simulations) make the implicit assumption that the torus estimation process occurs without inducing any error in the overall maneuver process. This is clearly not the case in general. However, as discussed

earlier, performing a least-squares torus estimation for a sufficient length of time (fractions of a day for a LEO satellite with good coverage) will yield sufficiently accurate torus elements. In cases in which weak on-board GPS positioning can be used (which is becoming increasingly more common), state information can be known with centimeter-level precision (and cm/s level precision in the velocities) [40]. This is more than enough state accuracy to perform quality torus estimation.

8.3 Continuous Lyapunov Control

We now turn to the subject of continuous control in satellite formations. While this is not intended to be a full treatment of the general continuous control problem, it serves to demonstrate that stabilizing continuous control of a formation on a KAM torus (or set of KAM tori) is feasible. We begin with a brief discussion of Lyapunov stability; see, e.g., [37] for more detail.

According to Lyapunov's second method, for a dynamical system $\dot{x} = f(x)$, if we can find a function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, such that:

- $V(x) = 0$ if and only if $x = 0$
- $V(x) > 0$ if and only if $x \neq 0$
- $\frac{d}{dt}V(x) \leq 0 \ \forall x \in \mathbb{R}^n$

then the system $\dot{x} = f(x)$ is stable (in the Lyapunov sense), and $V(x)$ is called the Lyapunov function. Additionally, if we modify the final condition and can find a $V(x)$ such that $\frac{d}{dt}V(x) < 0 \ \forall x \in \mathbb{R}^n$ (i.e., $V(x)$ is negative definite), then the system is asymptotically stable. We will utilize the concept of a Lyapunov function to design a feedback control system for KAM stationkeeping.

8.3.1 Control Lyapunov Function and Sontag's Formula.

A common extension of Lyapunov stability concepts to control theory involves the introduction of a Control Lyapunov Function (CLF). Consider a system of the form

$\dot{x} = f(x) + g(x)u(x)$, $x \in \mathbb{R}^n$. By the converse Lyapunov theorem, there is a Lyapunov function $V(x)$ such that

$$\begin{aligned}\dot{V} &< 0 \quad \forall x \in \mathbb{D} \\ \frac{\partial V}{\partial x} \frac{dx}{dt} &< 0 \quad \forall x \in \mathbb{D}\end{aligned}\tag{8.30}$$

and thus

$$\frac{\partial V}{\partial x} [f(x) + g(x)u(x)] < 0 \quad \forall x \in \mathbb{D}\tag{8.31}$$

where $\mathbb{D}$ is some domain of control validity. The Lyapunov function is now termed the Control Lyapunov function, and the CLF is further called a Global CLF if $\mathbb{D} = \mathbb{R}^n$ [26].

We turn now to the specific case of low-eccentricity KAM torus control. We will define some reference modal state as $\mathbf{y}_r(t) \in \mathbb{R}^6$, and we will consider control onto this torus. The reference torus may itself be a displacement from a “chief” torus, in the case of formation flight. Applying the principles used elsewhere in this work, for displacements from the modal states of the reference torus, we have a dynamical system of the form

$$\delta \dot{\mathbf{y}}(t) = \mathcal{J} \delta \mathbf{y}(t) + \delta \dot{\mathbf{y}}_p(t) + G(\mathbf{y}_r(t) + \delta \mathbf{y}(t)) \mathbf{u}(t)\tag{8.32}$$

where $\mathcal{J}$ is the Jordan matrix for the modal system, $\delta \dot{\mathbf{y}}_p(t)$ is the secular perturbation term on the differential state due to sectoral/tesseral harmonics and second-order eccentricity terms, $G(\mathbf{y}_r(t) + \delta \mathbf{y}(t))$ is the matrix transforming from control input to modal states, and $\mathbf{u}(t)$ is the control input to be determined. The vector $\delta \dot{\mathbf{y}}_p(t)$ can be determined from the torus perturbation series (as mentioned previously), and the control transformation matrix follows the theme developed in the last section:

$$G = E^{-1}(Q_1(t))R_{2Z}^T(Q_2(t))\tag{8.33}$$

We can define a CLF for the system above in a common but effective manner, by considering a kinetic energy analogue of the modal state:

$$V(\delta \mathbf{y}) = \frac{1}{2} \delta \mathbf{y}^T \delta \mathbf{y}\tag{8.34}$$

where it is obvious that $V(\delta\tilde{\mathbf{y}}) > 0$ for all $\delta\tilde{\mathbf{y}} \neq 0$ and $V(\delta\tilde{\mathbf{y}}) = 0$ when $\delta\tilde{\mathbf{y}} = 0$. Now, the goal is to find $\mathbf{u}$ such that $\dot{V}(\delta\tilde{\mathbf{y}}) < 0 \forall \delta\tilde{\mathbf{y}} \neq 0$.

In his influential work [47], Sontag introduced a ‘universally’ stabilizing control law for nonlinear stabilization of a system for which a Lyapunov function of the above form could be found. The eponymous Sontag’s Universal Formula gives a stabilizing control as $\mathbf{u}(t) = k(t) \left(\nabla_{\delta\tilde{\mathbf{y}}} V G(t) \right)^T$, where $\nabla_{\delta\tilde{\mathbf{y}}} V G(t)$ is the Lie derivative of V along G , and the gain k is given by

$$k = \begin{cases} -\frac{\nabla_{\delta\tilde{\mathbf{y}}} V f + \sqrt{(\nabla_{\delta\tilde{\mathbf{y}}} V f)^2 + (\nabla_{\delta\tilde{\mathbf{y}}} V G)^4}}{\nabla_{\delta\tilde{\mathbf{y}}} V G} & \nabla_{\delta\tilde{\mathbf{y}}} V G \neq 0 \\ 0 & \nabla_{\delta\tilde{\mathbf{y}}} V G = 0 \end{cases} \quad (8.35)$$

The Lie derivatives are found as a result of differentiation of the CLF:

$$\begin{aligned} \frac{d}{dt} V &= \frac{\partial V}{\partial \delta\tilde{\mathbf{y}}} \frac{d\delta\tilde{\mathbf{y}}}{dt} \\ &= \delta\tilde{\mathbf{y}}^T \left(\mathcal{J} \delta\tilde{\mathbf{y}}(t) + \delta\dot{\tilde{\mathbf{y}}}_p(t) + G(\mathbf{y}_r(t) + \delta\tilde{\mathbf{y}}(t)) \mathbf{u}(t) \right) \\ &= \nabla_{\delta\tilde{\mathbf{y}}} V f + \nabla_{\delta\tilde{\mathbf{y}}} V G \mathbf{u}(t) \end{aligned} \quad (8.36)$$

Therefore, with the control law given above, we can regulate the modal state to a desired reference state $\mathbf{y}_r(t)$.

8.3.2 Continuous State Estimation.

To accomplish the control law given in the previous section, we need a method which attempts to provide estimates of the torus states on a continuous basis. This could be accomplished by a filtering/estimation scheme with knowledge of the control signal; however, we will utilize an approximate method described here. This method is similar in mechanism to the least-squared torus estimation procedure described earlier, but deals only with the state at a single time of interest.

Assuming accurate physical state knowledge $\mathbf{X} = [\mathbf{x}, \mathbf{p}]^T$ in the ECR frame at the current time, along with a guess of the torus states Q_1, Q_2 (which can be generated by previous states or by simple geometric relationships), we can perform the torus series

summations to obtain a predicted ECR state X_{pred} , along with the instantaneous tangent space transformation

$$\frac{\partial X}{\partial \mathbf{y}} = R_{2Z}(Q_2)E(Q_1) \quad (8.37)$$

We may then iteratively refine the estimate of the current modal state $\mathbf{y}$ by

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \left(\frac{\partial X}{\partial \mathbf{y}} \right)^{-1} (X - X_{pred}) \quad (8.38)$$

In practice, we have found that with “perfect” state knowledge (from numerical integration), this process typically converges within a few iterations and provides a modal state accurate in each element to within a few parts in 10^6 .

8.3.3 *Lyapunov Results.*

Using the control law and estimation method given above, we have implemented the continuous control of a satellite onto a reference torus which causes it to revolve around a chief satellite. This demonstration utilizes the 630km, 40° orbit, and chief satellite is defined by the identically zero modal state, $\mathbf{y}_c = [0, 0, 0, 0, 0, 0]^T$. The reference orbit to which the controller is driving the deputy satellite is defined by (at epoch) $\mathbf{y}_r = [0.00025 \cos i, 0, 0.00025, 0, -0.0055, 0.00087]^T$. For a first test case, we consider a relocation of the satellite in modal coordinate y_1 , corresponding to a regulation to the reference orbit from an initial offset in Q_1 . The controlled satellite lies at an initial modal state identical to $\mathbf{y}_r$, with the exception that y_1 has been offset by $3.83\text{E-}4$ radians, corresponding to an initial mostly-in-track displacement of approximately 2.7km. Figure 8.14 shows the relative orbit (in the chief’s frame) of the controlled satellite (in red) and the reference orbit (in blue). In these simulations, as in all other numerical studies in this work, the satellites’ dynamics were numerically integrated through the geopotential to order and degree 20. Given in Figure 8.15 are the $\delta \tilde{\mathbf{y}}$ over the control period (top plot), the 2-norm of the error in position from the reference orbit (second plot) and the radial, along-track, and orbit-normal components in the chief UVW frame of the error of the controlled

satellite from reference. The error plots show that the satellite is successfully controlled from an initial in-track displacement to maintain a near-zero error in both physical and modal coordinates. The total Δv cost, calculated by integrating the control thrust over the maneuver time, was approximately 1.72 cm/s.

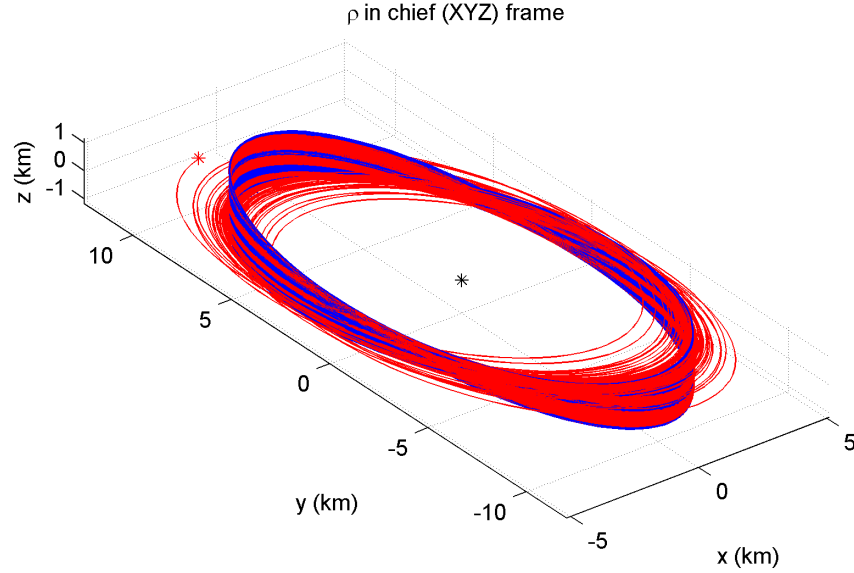


Figure 8.14: Relative Orbit during Lyapunov Control Correction of Q_1 Displacement

A more general initial displacement was also considered, in which the initial state of the controlled satellite was $y = [-0.00055 \cos i, 1.0E-5, 3E-4, 0, -3E-3, 6E-4]^T$, representing a displacement in every mode except the z-component of angular momentum (nodal rate y_4). The orbit over 20 days is shown in Figure 8.16 and the associated errors are shown in Figure 8.17. The total maneuver required approximately 13.5 cm/s of Δv . We note that the controller is again generally successful at minimizing the quadratic Lyapunov function; however, there is a residual oscillation in the out-of-plane direction of approximately 200m. This is caused by the nodal separation in the initial conditions. In fact, though

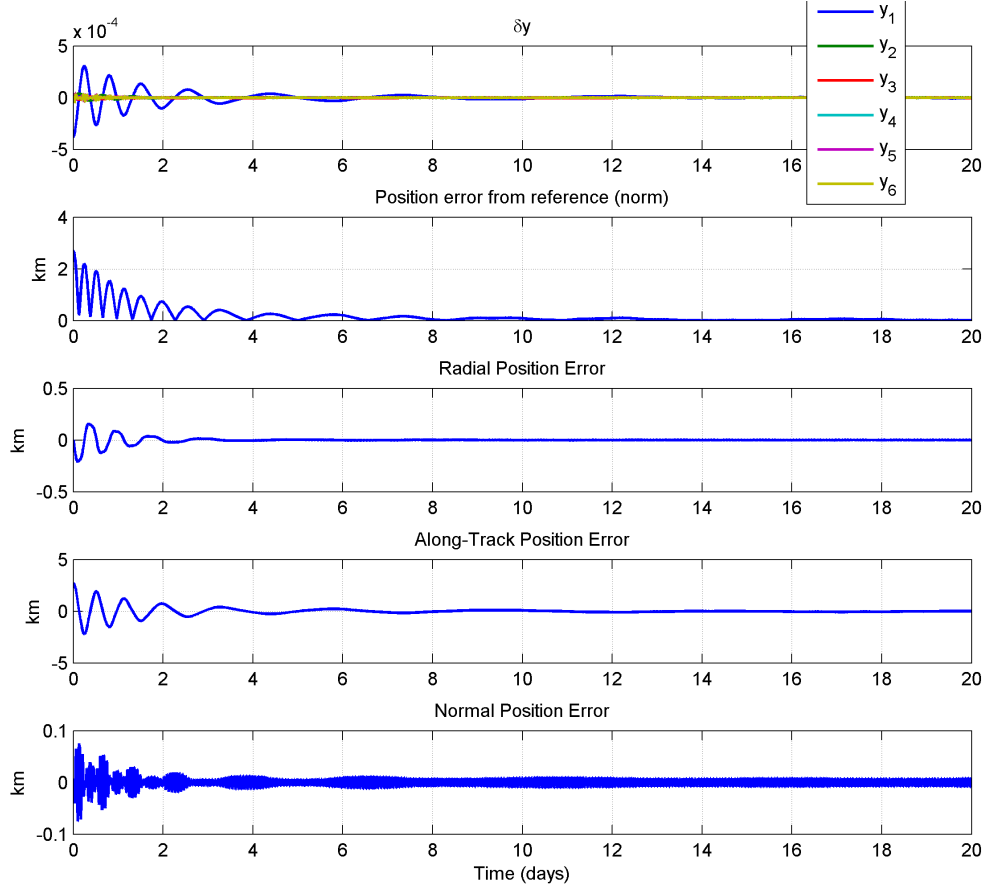


Figure 8.15: Position errors (from reference) during Lyapunov Control Correction of Q_1
Displacement

the scale is prohibitive, on the first plot of Figure 8.17 there is a final steady-state error of approximately $2.5E-5$ in the nodal angle y_3 . The plane-change operation, being very fuel-expensive because of low control authority in that modal direction, would require much longer than the time allotted for the simulation using a low-thrust controller. It is possible that a weighting matrix could be used in the Lyapunov control function in order to force faster convergence of certain states at the expense of fuel and possibly steady-state error; however, this option was not explored in this work.

It is important to note that this section, while developing a valid continuous control law, has made no claims of optimality. Rather, the CLF-derived control law is optimal,

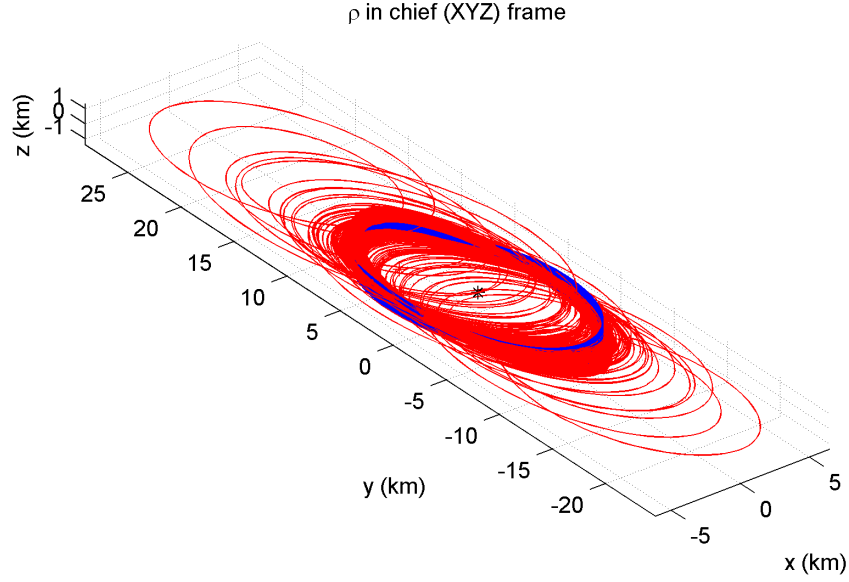


Figure 8.16: Relative Orbit during Lyapunov Control Correction of General Displacement

but not necessarily to a known or desired cost function. In fact, it has been shown (as in, for example, [23]) that control Lyapunov functions may only yield an optimal control to a specified cost function under strict conditions of the CLF. That is, the control given by a CLF is optimal for a cost functional of the form

$$J = \int_0^{\infty} [q(x) + u^2] dt \quad (8.39)$$

only if the CLF V has the same level sets as the value function associated with cost functional. This value function is the function which solves the Hamilton-Jacobi-Bellman equation for a given initial condition and essentially is the control history required to regulate the state to zero, subject to the dynamics, in an optimal fashion according to the cost functional. This implies that, in order for our CLF to yield the optimal control for the functional (8.39), the value function must have quadratic level sets; this, due the dynamical nature of the modal problem, is not generally the case, and as such, the control

law presented is not optimal, but rather a demonstration of the feasibility of control using the torus representation.

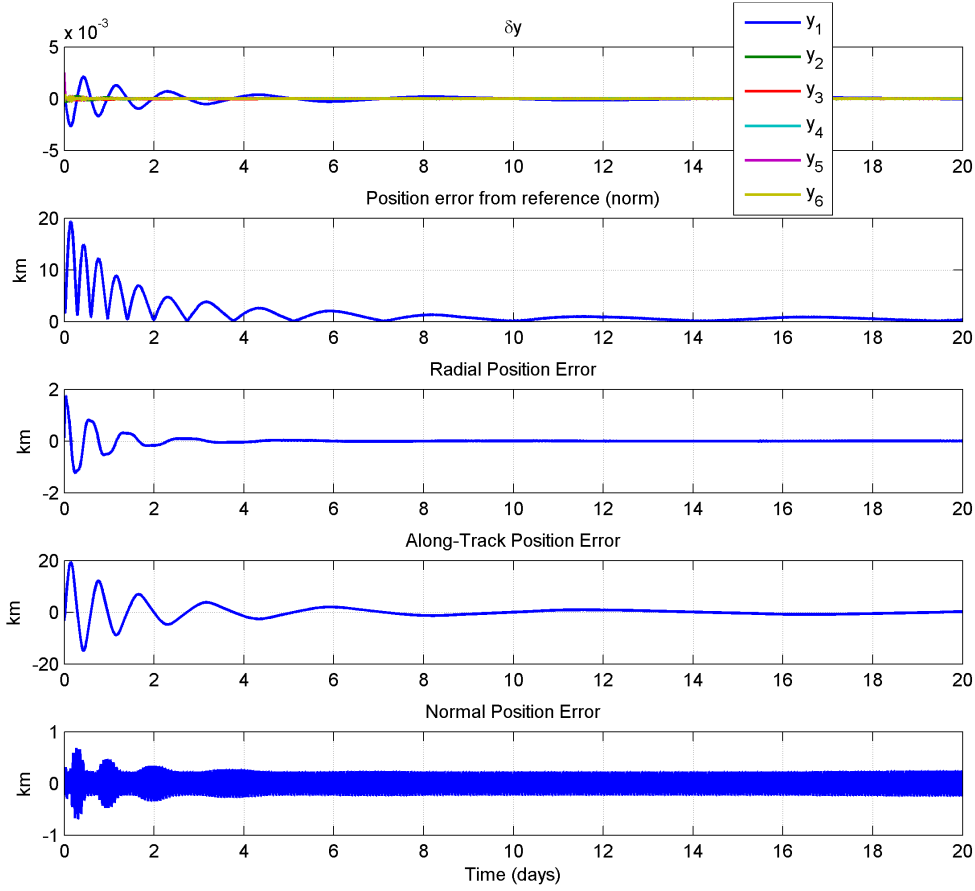


Figure 8.17: Position errors (from reference) during Lyapunov Control Correction of General Displacement

8.4 Stability and Control Summary

Some of the practical considerations for the use of LEST tori for satellite control have been addressed in this chapter. Specifically, Section 8.1 began by discussing the general behavior of satellite given small modal displacements of various types, along with the behavior of the solution under unmodeled perturbations such as non-conservative forces and third-body gravitation. Section 8.2 provided a maneuver cost exposition for the station

keeping of satellites on a torus, including a singular value analysis for the modal system and Monte Carlo simulations of maneuver sequences given a Gaussian impulse error. This analysis showed that the station-keeping cost is driven almost entirely by a satellite's thrust accuracy, and not by mismodeling of dynamics. Section 8.3 provided an overview and basic implementation of Lyapunov continuous control as applied to reconfiguration on torus. While not a cost-optimal or time-optimal formulation, it provides a demonstration of the concepts which could be extended with more specialized analyses.

IX. Conclusions and Recommendations

9.1 Summary and Conclusions

This work began with a brief survey of dynamical representations for satellite orbits, finding that, in all cases reviewed, significant simplifications to the dynamics formulations were performed in order to make the problem analytically tractable. In addition, all methods studies utilize some variant on classical orbital elements or, in the most effective cases, mean orbital elements (such as Delaunay variables). The current work utilizes recent advances in KAM theory as applied to orbital dynamics, along with Wiesel's low-eccentricity satellite theory, to provide a semi-analytic, semi-numerical approach to orbital design and control. It has been shown in this work that KAM theory can be applied through low-eccentricity theory to yield a relatively compact and intuitive geometric construction of perturbed satellite motion, allowing straightforward application to design of constellations and formations. In addition, the problem of controlling satellites onto and within KAM tori was presented, and its merits and limitations discussed.

The method of impulsive maneuvers around low-eccentricity KAM tori developed in this paper allows the placement of satellites in orbits which result in little to no secular drift from the intended station. The maneuver theory amounts to a linearized control scheme using the KAM modal system, which means that, while it may require several maneuvers to achieve station, the final drift rates will depend only upon the maneuvering accuracy of the spacecraft itself. It was shown that placement of satellites on KAM tori (that is, at specific torus angle coordinates on a torus with specific energies) results in virtually no secular motion in the full geopotential to within computational precision.

This work addressed the creation of constellations and formations on a torus and among closely-spaced tori (in an energy sense), which led to an intuitive method for choosing orbital parameters. Specifically, the theory dictates that Walker-style

constellations could be created by simple displacements in the Q_1 and Q_2 torus angles, while keeping all other modal variables the same between satellites. Satellites placed precisely on constellations formed in this way demonstrated, unsurprisingly, the same zero-secular-drift behavior discussed above, experiencing only periodic variations in relative position when numerically integrated using the full geopotential. The discussion then turned to the subject of formations, with an emphasis on the choice of torus/modal states $\mathbf{Q}$ and $\mathbf{y}$ to accomplish a select set of desired orbital geometries. It was found that certain formation configurations required slight offsets in energy modes to compensate for secular terms due to eccentricity perturbations, but that the resulting formations designed using this method show no secular drift.

In Chapter 8, the effects of perturbations and unmodeled forces on a satellite in the KAM/modal construction were discussed. It was shown that, in the case of conservative perturbations, there is at most a secular change in torus position coordinates, along with periodic behavior. Nonconservative perturbations, such as atmospheric drag, results in a secular change in energies, inducing a slow quadratic change in the torus position coordinates, which depends in magnitude upon the orbital altitude and orientation, as shown in Wiesel's formulation including drag. This chapter further discussed methods by which the theory could be extended to include these tertiary perturbations; however, being interested primarily in the relative motion of satellites in constellation or formation, we realize that effects such as drag typically cause similar forces across the constellation, changing the constellation as a whole but leaving relative displacements nearly intact.

Using a singular value analysis, the current research investigated the effects of small errors in impulsive maneuvers on the modal states, and thus on the underlying physical coordinates. It was shown that that, as could be expected, errors in the maneuver magnitudes tend to cause errors in the orbital energy mode y_2 . The effects of maneuver magnitude errors were quantified in terms of the singular value decomposition for several

orbits of interest, introducing a statistical distribution in terms of torus angle drift rates due to mismatched energies. The distribution was then used to create estimates of the steady-state station-keeping costs for several tori. This method is applicable to any valid torus (i.e., any altitude/inclination combination with frequencies away from resonance) and embodies a first-order orbital and spacecraft design requirement for fuel and maneuver accuracy. It was demonstrated that station-keeping costs are determined primarily by satellite thrust accuracy and precision, and not by limitations in the dynamics formulation.

Finally, this work demonstrated a continuous control scheme for formation station-keeping based upon Lyapunov stability theory. Using Sontag's universal formula, a Control Lyapunov Function was defined which allowed us to determine a control gain and direction which stabilized the modal state to a reference trajectory (albeit in a non-optimal manner). This was supported by a continuous modal state estimator using numerically integrated state data, representing a relatively naive estimation approach which could be replaced with other estimation techniques, such as Extended Kalman or Unscented Kalman filters. While only a brief treatment of the problem of continuous KAM control, the control solution was found to be feasible and non-cost prohibitive (in terms of required Δv), and our discussion serves as evidence that more targeted and robust continuous control solutions may be found for the KAM problem.

While the results of the methods outlined in this work are promising, they must be tempered by several caveats. First, the most obvious limitation to the theory is the requirement of low-eccentricity orbits. The underlying theory functions mostly unchanged up to orbital eccentricities past $e \approx 0.1$, as long as the second-order eccentricity perturbation terms are developed and included properly. However, the linearizations inherent in the maneuver model about the torus become less valid for orbits which are farther away from the reference low-eccentricity periodic orbit. Overall, as mentioned previously, the low-eccentricity limitation still allows applicability to a majority of

operational satellites and arguably as many future satellites, and so is not seen as a major limitation. The second caveat lies in the fact that, while the current method provides excellent results when applied in theory, the accuracy and cost of torus insertion and station-keeping will still depend on the accuracy of the controlled satellite's maneuvers and state knowledge. This concept is discussed in previous chapters, and it is found that the expected station-keeping fuel cost increases exponentially with the variance of the error in the maneuver magnitudes. The third consideration is that the theory presented does not include third-body perturbative effects and solar radiation pressure effects. While, as discussed earlier, these effects could be modeled by inclusion of additional angles in the Fourier perturbation representations, this was not attempted in the current work for conciseness and clarity. Finally, it bears repeating that the Lyapunov continuous control solution presented in this work is merely an initial proof of concept, and not intended to act as an optimal control solution.

9.2 Suggested Future Work

The theory and methods outlined in this work create the opportunity for a plethora of additional work. Following are a few examples of possible future extensions to the work. First, it would be worthwhile to perform the expansion of the theory to tertiary effects, such as third-body effects, in order to create a robust framework for HEO satellites. As mentioned previously, this would require extension of the dynamics used in numerical integration, as well as allowances for at least two additional angles (each as a function of time) for each of the extra gravitating bodies considered (e.g., the sun, moon, and Jupiter). A second extension to the current work would be the performance of an in-depth study and derivation of the continuous control solution with the goal of incorporating full state estimation, explicitly regulating uncertain/unmodelled dynamical effects, and performing fuel- or time-optimal transfers between tori or torus coordinates for the purpose of formation configuration or reconfiguration. Finally, a lofty but worthy goal (for which

work is currently underway) involves the extension of the foundations, robustness, and flexibility of KAM theory to general orbits without limitations in eccentricity. This level of extension would provide to high-eccentricity formations the same benefits presented in the current work for low-eccentricity formations; i.e, a formulation of the dynamics which virtually eliminates secular effects, and upon which the errors are a function only of the spacecraft's estimation and control abilities. Such a general extension of KAM theory would undoubtedly be a fundamental result in the field of applied perturbation theory.

Appendix A: Wiesel's Low-Eccentricity Earth Satellite Theory

Wiesel's work on Low-Eccentricity Satellite Theory, found in [60], is detailed here for the purposes of reference, as it is fundamental to the current research. The dynamics formulation used in the current research and in Wiesel's work is described in Appendix B.

Central to Wiesel's theory is creation of a periodic orbit in the zonal gravitational potential. This periodic orbit is periodic in a frame of reference rotating with the orbit's nodal frame; that is, the line of nodes rotates about the Earth's polar axis due to zonal gravitational harmonics at a rate which is unknown until the periodic orbit is constructed. This construction is accomplished by considering starting conditions at the ascending node in the [geometric] equatorial plane, $\mathbf{X}(0) = [x_0, 0, 0, \dot{x}_0, v_0 \cos i_0, v_0 \sin i_0]^T$, and a vector of unknowns $\Xi = [x_0, \dot{x}_0, v_0]^T$. The periodic orbit is achieved if

$$\mathbf{G} = \begin{bmatrix} z(\tau) \\ \mathbf{r}(\tau) - x_0 \dot{x}_0 \\ \mathbf{r}(\tau) \cdot \mathbf{r}(\tau) - x_0^2 \end{bmatrix} = \mathbf{0} \quad (\text{A.1})$$

where $\mathbf{r}$ is the position vector of the satellite in the nodal rotating frame, z is the component of $\mathbf{r}$ along the positive polar axis, and $t = \tau$ is the period of the orbit in the nodal frame. This condition states that, at one period, the satellite must intersect the equatorial plane and must have the same radial velocity and distance from the origin as at the initial time. A Newton-Raphson method is used to correct the initial condition parameters iteratively to achieve a periodic orbit (i.e., to force $\mathbf{G} = \mathbf{0}$ to within numerical precision). This is completed by finding the linearization of the cost function about the current parameter vector Ξ as

$$0 = \mathbf{G}(\Xi) + \frac{\partial \mathbf{G}}{\partial \Xi} \delta \Xi \quad (\text{A.2})$$

and expanding the gradient of G as

$$\frac{\partial \mathbf{G}}{\partial \Xi} = \left(\frac{\partial \mathbf{G}}{\partial \mathbf{X}(\tau)} \Phi(\tau, 0) + \frac{\partial \mathbf{G}}{\partial \mathbf{X}(0)} \right) \frac{\partial \mathbf{X}(0)}{\partial \Xi} \quad (\text{A.3})$$

where $\Phi(\tau, 0)$ is the state transition matrix from the initial time to the orbital period, which can be integrated with the equations of motion. The assorted gradient matrices can be calculated in a straightforward fashion from the cost function and the statement of initial conditions, and thus the relations above can be used iteratively to correct the initial conditions $\mathbf{X}(0)$ to decrease the cost function $\mathbf{G}(\Xi)$ until closure.

Once $\mathbf{G}(\Xi) = 0$ has been achieved to within the desired precision (usually near computational double precision in the current applications), harmonic analysis is used to reduce the periodic orbit to a vector of Fourier series, which, since the orbit is periodic in the nodal frame, are functions of only the angle Q_1 (the mean argument of latitude analogue). Specifically, each of the three cartesian coordinates in the nodal frame and each of the inertial cartesian momenta represented in the nodal frame are represented as Fourier series in Q_1 . Additionally, after the numerical convergence of the periodic orbit, the nodal regression rate $\dot{\Omega}$ is calculated as:

$$\dot{\Omega} = \frac{1}{\tau} \arccos \left(\frac{\mathbf{r}(\tau) \cdot \mathbf{r}(0)}{|\mathbf{r}(\tau)| |\mathbf{r}(0)|} \right) = \frac{1}{\tau} \arccos \frac{x(\tau)}{x_0} \quad (\text{A.4})$$

At any point in time, then, we may calculate a second angle Q_2 , which describes the rotation of the nodal frame about the Earth's polar axis as compared to a reference frame of our choice. In the current work and in most application in Wiesel's paper, the angle Q_2 represents the amount of rotation of the nodal frame from the Earth-centered fixed frame (i.e., the angle of the ascending node from Greenwich's longitude). In this sense, both the inertial nodal rate $\dot{\Omega}$ and the rotation rate of Earth $\omega_{\oplus}$ are included in the rate $\omega_2 = \dot{Q}_2$. Using this information, the Earth-centered Rotating (ECR) position can be found through

$$\mathbf{X}_{ECR}(Q_1, Q_2) = \begin{bmatrix} R_z(-Q_2) & 0 \\ 0 & R_z(-Q_2) \end{bmatrix} \mathbf{X}_{nodal}(Q_1) \quad (\text{A.5})$$

where R_z is a standard Euler 3-rotation about the Earth's polar axis:

$$R_z(-Q_2) = \begin{bmatrix} \cos(Q_2) & -\sin(Q_2) & 0 \\ \sin(Q_2) & \cos(Q_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{A.6})$$

Wiesel next investigates the relative motion problem by linearizing about the periodic orbit to find variational equations of the form

$$\delta\dot{\mathbf{x}} = A(t)\delta\mathbf{x} \quad (\text{A.7})$$

where $A(t)$ is a periodic matrix, since the underlying orbit is periodic. This leads to a problem of the form whose solution was presented by Floquet [22]. We can perform a linear change of variables into the so-called “modal” form,

$$\delta\mathbf{x} = E(t)\mathbf{y} \quad (\text{A.8})$$

where $E(t)$ is periodic with the same period as $A(t)$. Floquet’s method continues by assuming that there is a constant matrix of Poincaré exponents in Jordan form $\mathcal{J}$ such that

$$\dot{E} = AE - E\mathcal{J} \quad (\text{A.9})$$

and we find that the dynamical system can be written in modal form as:

$$\dot{\mathbf{y}} = \mathcal{J}\mathbf{y} \quad (\text{A.10})$$

Wiesel then gives the state transition matrix of the displacement $\delta\mathbf{x}$ as

$$\Phi(t, t_0) = E(t) \exp(\mathcal{J}(t - t_0)) E(t_0) \quad (\text{A.11})$$

and recognizes that, since the modal matrix E is periodic (with period τ), the quantity $\exp \mathcal{J}$ is the eigenvalue matrix of $\Phi(\tau, 0)$. Because the resulting eigenvalue problem is highly singular, Wiesel gives a specially-adapted method in the appendix of [60] for determining the modal matrix $E(t)$. The structure of the dynamics and the fact that there are two exact integrals of the motion (the Hamiltonian and the z -component of angular momentum) lead to two zero Poincaré exponents, and therefore two degenerate modes in the resulting local linear solution; these modes correspond to the local embodiments

of the global angles Q_1 and Q_2 , along with their associated energy modes ω_1 and ω_2 . The remaining, non-degenerate mode is an oscillatory mode resulting from the orbital eccentricity, with frequency ω_3 , the rate of regression of the argument of perigee. The 6×6 matrix $E(Q_1)$ is then reduced to a Fourier series as the next piece of the Low-eccentricity Satellite Theory package.

The reconstruction of the satellite state in the nodal frame is now given as

$$\mathbf{X}_{nodal} = \mathbf{X}_{PO}(Q_1) + E(Q_1) \exp(\mathcal{J}(t - t_0)) \mathbf{y}(t_0) \quad (\text{A.12})$$

where the modal state transition matrix is

$$\exp(\mathcal{J}(t - t_0)) = \begin{bmatrix} 1 & t - t_0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t - t_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(Q_3) & \sin(Q_3) \\ 0 & 0 & 0 & 0 & -\sin(Q_3) & \cos(Q_3) \end{bmatrix} \quad (\text{A.13})$$

where the apsidal regression since epoch is $Q_3 = \omega_3(t - t_0)$. The associated physical state represented in the ECR frame is found through the rotation through the nodal angle Q_2 :

$$\mathbf{X}_{ECR} = R_z(-Q_2) \mathbf{X}_{nodal} \quad (\text{A.14})$$

Wiesel proceeds to demonstrate excellent results for experiments in which numerically integrated data close to the periodic orbit were fit to the Low-Eccentricity theory. He shows that it is important to treat the modal variables y_1 and y_3 (the local versions of Q_1 and Q_2) properly so that the modal displacements are kept “small”; this is accomplished through absorbing static displacements in y_1 and y_2 into the global angles during the fitting process. He shows RMS residual error results of a few meters over two weeks for an orbit based on that of the Hubble telescope (practically circular at an altitude of approximately 550 km and inclination $i = 28.4^\circ$)

While the theory above includes the zonal gravity perturbations and eccentricity effects to first order, Wiesel further develops the theory to include first-order perturbation

effects in order to extend its accuracy. Considering the perturbing effects on the system

$$\delta\dot{\mathbf{x}} = A(t)\delta\mathbf{x} + \dot{\mathbf{X}}_{pert} \quad (\text{A.15})$$

he expands the perturbing acceleration about the periodic orbit as

$$\dot{\mathbf{X}}_{pert} = \dot{\mathbf{X}}_{pert}\Big|_{x_{PO}} + \frac{\partial \dot{\mathbf{X}}_{pert}}{\partial \mathbf{X}}\Big|_{x_{PO}} \delta\mathbf{x} + \dots \quad (\text{A.16})$$

$$= \dot{\mathbf{X}}_{pert}\Big|_{x_{PO}} + \frac{\partial \dot{\mathbf{X}}_{pert}}{\partial \mathbf{X}}\Big|_{x_{PO}} E\mathbf{y} + \dots \quad (\text{A.17})$$

which yields the perturbed modal equations of motion

$$\dot{\mathbf{y}} = \mathcal{J}\mathbf{y} + E^{-1}\dot{\mathbf{X}}_{pert} \quad (\text{A.18})$$

where the forcing term $E^{-1}\dot{\mathbf{X}}_{pert}$ is a function of the position on the periodic orbit. For the primary first-order perturbation effect studied by Wiesel and in this work, the sectoral and tesseral gravitational harmonics, the forcing term is a function of the two angles Q_1 and Q_2 only. A atmospheric drag perturbation demonstrates a dependence practically on Q_1 only. Further inclusion of other gravitational effects, such as third-body forces from the sun and moon, would introduce dependence on additional angles describing the bodies' orientation with respect to Earth. The perturbations are stored as Fourier series by evaluating $E^{-1}\dot{\mathbf{X}}_{pert}$ and numerically performing the Fourier integrals.

In the degenerate modes, as in (for example) y_1 , Wiesel shows that the perturbed motion can be written, using the periodic perturbing function, in the form

$$y_1(t) = \left[\left(\frac{c_2}{\omega_p} - \frac{s_2}{\omega_p^2} \right) \sin(n_1 Q_1 + n_2 Q_2) - \left(\frac{s_1}{\omega_p} + \frac{c_2}{\omega_p^2} \right) \cos(n_1 Q_1 + n_2 Q_2) \right] \Big|_{t_0}^t - \left(\frac{c_2}{\omega_p} \sin(n_1 Q_1 + n_2 Q_2) - \frac{s_2}{\omega_p} \cos(n_1 Q_1 + n_2 Q_2) \right) \Big|_{t_0}^t (t - t_0) \quad (\text{A.19})$$

where s_1 , s_2 , c_1 and c_2 are generic coefficients to be found numerically and $\omega_p = n_1\omega_1 + n_2\omega_2$. The time evolution of the corresponding energy mode y_2 is:

$$y_2(t) = \left(\frac{c_2}{\omega_p} \sin(n_1 Q_1 + n_2 Q_2) - \frac{s_2}{\omega_p} \cos(n_1 Q_1 + n_2 Q_2) \right) \Big|_{t_0}^t \quad (\text{A.20})$$

The other degenerate modal variable y_3 and its associated energy mode y_4 behave similarly (albeit obviously with distinct coefficients). The oscillatory/eccentricity mode y_5 , y_6 is found to behave according to

$$\begin{bmatrix} y_5 \\ y_6 \end{bmatrix} = \frac{1}{\omega_3} \begin{bmatrix} c_6 \\ -c_5 \end{bmatrix} + \begin{bmatrix} \alpha_5 \cos(n_1 Q_1 + n_2 Q_2) + \beta_5 \sin(n_1 Q_1 + n_2 Q_2) \\ \alpha_6 \cos(n_1 Q_1 + n_2 Q_2) + \beta_6 \sin(n_1 Q_1 + n_2 Q_2) \end{bmatrix} \quad (\text{A.21})$$

where

$$\begin{aligned} \alpha_5 &= \frac{c_6 \omega_3 + s_5 \omega_p}{\omega_3^2 - \omega_p^2} \\ \beta_5 &= \frac{s_6 \omega_3 - c_5 \omega_p}{\omega_3^2 - \omega_p^2} \\ \alpha_6 &= \frac{-c_5 \omega_3 + s_6 \omega_p}{\omega_3^2 - \omega_p^2} \\ \beta_6 &= \frac{-s_5 \omega_3 - c_6 \omega_p}{\omega_3^2 - \omega_p^2} \end{aligned} \quad (\text{A.22})$$

and we can see where small divisors may occur upon resonance between ω_3 and ω_p . After developing these first-order perturbations, Wiesel then shows the high fidelity to which the resulting model is able to match both secular and periodic changes to the physical state due to sectoral and tesseral perturbations and atmospheric drag.

Finally, Wiesel discusses the inclusion of second-order eccentricity perturbations, which is necessary in part because there is no global analogue to the y_5 and y_6 modal variables. Extension of the eccentricity solution to second order allows application of the theory to orbits in which the eccentricity is larger than the first-order solution would allow. The higher-order behavior of the modal variables is given in index summation notation as

$$\dot{y}_i = \mathcal{J}_{i\alpha} y_\alpha + \frac{1}{2} E_{i\alpha}^{-1} B_{\alpha\beta\gamma} E_{\beta\delta} E_{\gamma\epsilon} y_\delta y_\epsilon + \dots \quad (\text{A.23})$$

where the matrix B is a function of Q_1 and is given as

$$B_{\alpha\beta\gamma} = Z_{\alpha\epsilon} \frac{\partial^3 \mathcal{H}}{\partial X_\epsilon \partial X_\beta \partial X_\gamma} \quad (\text{A.24})$$

where Z is the symplectic group matrix. Abbreviating

$$B'_{i\delta\epsilon} = E_{i\alpha}^{-1} B_{\alpha\beta\gamma} E_{\beta\delta} E_{\gamma\epsilon} \quad (\text{A.25})$$

we can use the first-order solution in the eccentricity mode variables

$$\begin{aligned} \begin{bmatrix} y_5(t) \\ y_6(t) \end{bmatrix} &= \Phi(t, t_0) \begin{bmatrix} y_5(t_0) \\ y_6(t_0) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\omega_3(t - t_0)) & \sin(\omega_3(t - t_0)) \\ -\sin(\omega_3(t - t_0)) & \cos(\omega_3(t - t_0)) \end{bmatrix} \begin{bmatrix} y_5(t_0) \\ y_6(t_0) \end{bmatrix} \end{aligned} \quad (\text{A.26})$$

to evaluate the quadratic terms (with the assumption of small eccentricity), as

$$\dot{y}_i = \mathcal{J}_{i\alpha} y_\alpha + \frac{1}{2} B''_{i\alpha\beta} y_\alpha(t_0) y_\beta(t_0) + \dots \quad (\text{A.27})$$

where

$$B''_{i\alpha\beta} = B'_{i\gamma\epsilon} \Phi_{\gamma\alpha} \Phi_{\epsilon\beta} \quad (\text{A.28})$$

Assuming a general solution of the form

$$y_i(t) = \Phi_{i\alpha}^{(1)} y_\alpha(t_0) + \frac{1}{2} \Phi_{i\alpha\beta}^{(2)} y_\alpha(t_0) y_\beta(t_0) + \dots \quad (\text{A.29})$$

he imposes an equivalence requirement on the initial state and shows that the second-order state transition matrix propagates as

$$\dot{\Phi}_{ijk}^{(2)} = \mathcal{J}_{i\gamma} \Phi_{\gamma jk}^{(2)} + B''_{ijk} \quad (\text{A.30})$$

with the initial condition $\Phi_{ijk}^{(2)}(t_0, t_0) = 0$, and the entries in the matrix $\Phi^{(2)}$ are periodic functions of the angles Q_1 and Q_3 .

Appendix B: Dynamics Formulation and Numerical Methods

B.1 Dynamics

The core dynamics formulation used in the current research is the same as that used by Wiesel [56–60]. For an Earth satellite acted upon by conservative forces, the kinetic energy in a frame of reference rotating about the Earth’s polar axis is written as

$$T = \frac{1}{2} \left((\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 + \dot{z}^2 \right) \quad (\text{B.1})$$

where ω is the angular rate of the rotating reference frame with respect to the ECI basis; for example, integrating the equations of motion in the ECEF frame requires $\omega = \omega_{\oplus}$. We can find the canonical momenta as

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \quad (\text{B.2})$$

so

$$\begin{aligned} p_x &= \dot{x} - \omega y \\ p_y &= \dot{y} + \omega x \\ p_z &= \dot{z} \end{aligned} \quad (\text{B.3})$$

We may then directly write the Hamiltonian as

$$\begin{aligned} \mathcal{H} &= \sum_i p_i \dot{q}_i - T + V \\ &= \frac{1}{2} (p_x^2 + p_y^2 + p_z^2) + \omega (y p_x - x p_y) - \\ &\quad \frac{\mu}{r} \sum_{n=0}^{\infty} \sum_{m=0}^n \left(\frac{r}{R_{\oplus}} \right)^{-n} P_n^m(\sin \delta) (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \end{aligned} \quad (\text{B.4})$$

where μ is the gravitational parameter, $R_{\oplus}$ is the mean equatorial radius of Earth, P_n^m are the associated Legendre polynomials, C_{nm} and S_{nm} are the gravity fields Stokes coefficients

(with values from the Earth Gravitational Model of '96), and the radius r , latitude δ and longitude λ are given by:

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \sin \delta &= \frac{z}{\sqrt{x^2 + y^2}} \\ \tan \lambda &= \frac{y}{x} \end{aligned} \tag{B.5}$$

Now, Hamilton's equations of motions can be written in terms of the physical state vector $\mathbf{X} = [x, y, z, p_x, p_y, p_z]^T$ as

$$\dot{\mathbf{X}} = \mathbf{Z} \frac{\partial \mathcal{H}}{\partial \mathbf{X}} \tag{B.6}$$

where the symplectic group matrix $\mathbf{Z}$ is

$$\mathbf{Z} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix} \tag{B.7}$$

Where needed, specifically in periodic orbit construction and determination of the periodic modal matrix $\mathbf{E}(t)$, the linearization of Hamilton's equations is written

$$\delta \dot{\mathbf{x}} = \mathbf{Z} \frac{\partial^2 \mathcal{H}}{\partial \mathbf{X}^2} = \mathbf{A} \delta \mathbf{x} \tag{B.8}$$

and the state transition matrix may be numerically integrated alongside the physical coordinates.

It is noteworthy that the construction of the geopotential expansion given in (B.4) is mathematically correct, it experiences issues in application for satellites with high inclinations, as the latitude quantity δ becomes undefined when x and y approach zero. For this reason, in the actual implementation of the geopotential used in this work, we have leveraged the algorithm introduced by Pines [39] and clarified/extended by Spencer [48] which provides a nonsingular formulation of the potential function. This formulation is based upon the replacement of the spherical coordinates r, δ, λ with direction cosines,

which are always clearly defined and have no singularities. Position vectors are represented by four quantities; three direction cosines defining the vector orientation and a vector magnitude. Not only does this formulation eliminate singularities and there associated errors, but we find that it is, in practice, much more computationally efficient than the standard geopotential implementation.

For all numerical integrations presented in this work, with the exception of zonal limitations to construct periodic orbits, the geopotential forces are included up to and including degree and order (m, n) 20. While this provides a highly accurate representation of the dynamics, the strategies involved in the current work could be implemented with a higher-order gravitational field (to arbitrary m, n), at only the expense of higher computational time budget.

B.2 Units

The units used in the numerical integrations and manipulations in this work are the standard canonical units. We first define the distance unit to be Earth Radius (ER), with a physical value of approximately 6378.137 km, and the gravitational parameter μ to be $1 \text{ ER}^3/\text{TU}^2$. The time unit is, appropriately, called Time Unit (TU), with a physical value of approximately 13.45 minutes. The TU value is chosen as

$$\text{TU} = \sqrt{\frac{r_{\oplus}^3}{\mu}} \quad (\text{B.9})$$

such that a (hypothetical) circular orbit at 1 ER possesses a speed of 1 ER/TU (via the equation $v = \sqrt{\mu/r}$). Results in this work are sometimes presented in native canonical units, and sometimes they are given in physical units such as seconds, days, meters, and kilometers.

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14. ABSTRACT The problem of Earth satellite constellation and formation flight is investigated in the context of Kolmogorov-Arnold-Moser (KAM) theory. KAM tori are constructed utilizing Wiesel's Low-Eccentricity Earth Satellite Theory, allowing numerical representation of the perturbed tori describing Earth orbits acted upon by geopotential perturbations as sets of Fourier series. A maneuvering strategy using the local linearization of the KAM tangent space is developed and applied, demonstrating the ability to maneuver onto and within desired torus surfaces. Constellation and formation design and maintenance on KAM tori are discussed, along with stability and maneuver error concerns. It is shown that placement of satellites on KAM tori results in virtually no secular relative motion in the full geopotential to within computational precision. The effects of maneuver magnitude errors are quantified in terms of a singular value decomposition of the modal system for several orbits of interest, introducing a statistical distribution in terms of torus angle drift rates due to mismatched energies. This distribution is then used to create expectations of the steady-state station-keeping costs, showing that these costs are driven by operational and spacecraft limitations, and not by limitations of the dynamics formulation. A non-optimal continuous control strategy for formations based on Control Lyapunov Functions is also outlined and demonstrated in the context of formation reconfiguration.						
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