



Microfacet Wavelength-Scaling of the BRDF

THESIS

Samuel Edward Freda
AFIT-ENP-MS-16-J-015

**DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY**

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

DISTRIBUTION STATEMENT A
APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

The views expressed in this document are those of the author and do not reflect the official policy or position of the United States Air Force, the United States Department of Defense or the United States Government. This material is declared a work of the U.S. Government and is not subject to copyright protection in the United States.

AFIT-ENP-MS-16-J-015

MICROFACET WAVELENGTH-SCALING OF THE BRDF

THESIS

Presented to the Faculty

Department of Engineering Physics

Graduate School of Engineering and Management

Air Force Institute of Technology

Air University

Air Education and Training Command

in Partial Fulfillment of the Requirements for the

Degree of Master of Science in Optical Sciences and Engineering

Samuel Edward Freda, BS

June 1, 2016

DISTRIBUTION STATEMENT A
APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

AFIT-ENP-MS-16-J-015

MICROFACET WAVELENGTH-SCALING OF THE BRDF
THESIS

Samuel Edward Freda, BS

Committee Membership:

Maj Samuel D. Butler, PhD
Chair

Dr. Stephen E. Nauyoks
Member

Dr. Michael A. Marciniak
Member

Dr. Michael T. Eismann
Member

Abstract

The bidirectional reflectance distribution function (BRDF) describes realistic scattering of light off materials by relating incident irradiance to outbound radiance. One popular class of BRDF models assumes a surface is comprised of tiny microfacets. The drawback of microfacet BRDFs is that they often do not contain specific material parameters and neglect wavelength effects. Wave optics BRDF expressions, however, can describe wavelength effects at the expense of being more computationally cumbersome. Previous work of following a Beckmann-Kirchhoff derivation of BRDF, then relating wave optics BRDF coordinates to microfacet coordinates led to a complicated, but versatile, BRDF. In this work, the infinite summation found via this derivation is investigated. This involves algebraic simplification of the expression inside the infinite summation and curve fitting to find a functional approximation to this summation. Some methods which may accomplish this are detailed. The relationships between a wave optics and microfacet BRDF are expected to eventually lead to a simple closed-form BRDF model that more accurately describes wavelength-dependent effects and which will be fast enough to be usable in remote sensing applications.

for my babies

Acknowledgements

I'd like to thank Dr. Marciniak for bringing me to AFIT as an intern, then as a masters student. Additionally I'd like to thank Dr. Marciniak and Dr. Eismann for their collaboration and patience in getting me in as a contractor.

Once here, my progress in class has relied heavily on the collaboration of many, notably Lt Dickenson and Phil Plummer.

I'd like to thank Dr. Marciniak for guiding me through the difficult process of changing my research direction. Also, I'd like to thank Dr. Meola and Dr. Eismann for their willingness to call an audible.

I'd like to thank Maj Butler for his attention to detail, his indefatigability, and an excellent lecture on quantum computing which showed his intellect, understanding of his students, and ability to explain big things to little people. I'd like to thank Dr. Nauyoks for his objectivity and appropriate uses of the word "no" which saved much time and frustration. Also, Capt. Reeger contributed much-needed mathematical knowledge and re-direction.

Lastly, I'd like to thank the helping hands of "the clock", which taught me what it takes to do research.

Samuel Edward Freda

Table of Contents

	Page
Abstract	iv
Acknowledgements	vi
List of Figures	ix
List of Tables	xii
List of Abbreviations	xiii
I. Introduction	1
1.1 Motivation	1
1.2 Proposal	2
II. Background	4
2.1 Electromagnetics and Surfaces	4
2.2 BRDF Background	6
2.2.1 Microfacet Coordinates	7
2.2.2 The Generalized Microfacet Model	9
2.2.3 Microfacet Summary	12
2.2.4 Wave Optics Models	13
2.2.5 Coordinate Conversion	15
2.2.6 Approximate Expressions	20
III. Approximating the BRDF	24
3.1 Model Comparison	24
3.2 Linear and Exponential Combination Approximations	28
3.2.1 Linear Combination	29
3.2.2 Exponential Combination	32
3.3 Mathematical Simplification	33
3.3.1 Integration	35
3.3.2 Taylor Series Expansion	36
3.3.3 Substitution	37
3.3.4 Mathematical Approximation	37
3.4 Analysis of Variable Ranges	40
3.5 Distribution Approximation	48
3.5.1 Maximums of the Terms of the Summation	48
3.5.2 Gaussian Approximation	50
3.5.3 Limiting the Number of Terms Needed	59
3.5.4 Error Function Approximation	63

	Page
3.5.5 Gaussian Approximation Summary	68
3.6 Summation Results	69
IV. Conclusion	71
4.1 Future Work	72
4.2 Summary	73
Bibliography	75

List of Figures

Figure		Page
1	Elements of BRDF	7
2	Microfacet Coordinates	8
3	Rusinkiewicz Coordinates	9
4	Comparison Between Q and F 1	17
5	Comparison Between Q and F 2	17
6	Comparison Between Q and F 3	18
7	Comparison Between Q and F 4	18
8	Polished Approximation at Normal Incidence	21
9	Polished Approximation at 40 Degrees Incidence	21
10	Very Rough Approximation at Normal Incidence	22
11	Very Rough Approximation at 40 Degrees Incidence	22
12	MATLAB® GUI	25
13	MBK Comparison Specular 1.....	27
14	MBK Comparison Specular 2.....	27
15	MBK Comparison Specular 3.....	28
16	MBK Comparison Diffuse 1	28
17	MBK Comparison Diffuse 2	29
18	MBK Comparison Diffuse 3	29
19	Contributions of Linear Combination	30
20	Linear Combination Approximation	31
21	Large Magnitude of Approximations.....	31
22	Exponential Combination Approximation	32

Figure		Page
23	Comparison of Stirling and Gosper Approximations with the Factorial.....	39
24	Relative Error of Stirling and Gosper Approximations	40
25	Contour Error in h Cutoff	42
26	Slice of Error in h Cutoff	42
27	Behavior of g for a Specular Material	44
28	Behavior of g for a Diffuse Material	45
29	Behavior of g and BRDF a Diffuse Material	45
30	Behavior of h for a Specular Material	46
31	Behavior of h for a Specular Material Shown as Semilog	46
32	Behavior of h for a Diffuse Material	47
33	Gaussian Shape of the Terms of the Sum	50
34	Bad Gaussian with Fit	51
35	Gaussian Shape of the Terms of the Sum 2	52
36	Contour Representation of the Gaussian Mean.....	53
37	Mesh Grid Representation of the Gaussian Mean.....	53
38	Mesh Grid Representation of the Gaussian Standard Deviation.....	54
39	MATLAB® CFTool	54
40	Behavior of Gaussian Mean for a Slightly Diffuse Material	55
41	Behavior of Gaussian Mean for a Diffuse Material	55
42	Behavior of Gaussian Standard Deviation for a Diffuse Material	56
43	Behavior of Gaussian Standard Deviation for a Very Diffuse Material	56

Figure		Page
44	Gaussian Summation Error	58
45	Gaussian Summation Error Maxes	59
46	Effect of Limiting terms of Summation on Accuracy of Summation	60
47	Effect of Increasing Incident Angle on Shape of Summation	60
48	Comparison of Sums with Respect to Standard Deviation	61
49	Relative Errors of Sums with Respect to Standard Deviation	61
50	Comparison of Number of Terms Needed as a Function of g	62
51	Comparison of Number of Terms Needed as a Function of Scattered Angle	62
52	Shape of Error Function	63
53	Max RMS Error Using Error Function Approximation	65
54	First BRDF Comparison at Maximum RMS Error Using Error Function Approximation	65
55	BMBK Compared with Error Function Approximation at $\theta_i = 0^\circ$	66
56	BMBK Compared with Error Function Approximation at $\theta_i = 40^\circ$	66
57	BMBK Compared with Error Function Approximation at $\theta_i = 60^\circ$	67

List of Tables

Table		Page
1	Relative Errors in Stirling and Gosper Approximations	39
2	Error in limiting the range of h to $h \leq 10$ and $h \leq 100$	43
3	Useful ranges as determined by variable ranges	43
4	Useful ranges assuming a large g	44

List of Abbreviations

Abbreviation	Page
HSI	Hyperspectral Imaging 1
BRDF	Bidirectional Reflectance Distribution Functions 1
LWIR	Long Wave Infrared 2
DHR	Directional Hemispherical Reflectance 6
HDR	Hemispherical Directional Reflectance 6
MBK	Modified Beckmann-Kirchhoff 13
RR	Rayleigh-Rice 14
BK	Beckmann-Kirchhoff 14
KMBK	Krywonos Modified Beckmann-Kirchhoff 15
BMBK	Butler Modified Beckmann-Kirchhoff 19
RMS	Relative Root Mean Square 24
SAM	Spectral Angle Mapper 25
CDF	Cumulative Density Function 63

I. Introduction

Understanding surface reflection is challenging but rewarding. Reflectance functions describe how a material will reflect light. One application of reflectance functions to the Air Force is in Hyperspectral Imaging (HSI) models. An HSI model must contain two pieces: atmospheric modeling and material modeling. This work focuses on describing material reflectance.

1.1 Motivation

Different materials reflect light in different ways. This is how a person discerns what they see. Like an eye, hyperspectral imagers measure irradiance at multiple wavelengths in a scene. These data, however, are not sufficient to understand what is happening in the scene. First, an understanding of what materials look like must be established. A person knows what a leaf tends to look like: green and matte. Also, a person knows what metal tends to look like: gray and glossy. In this fashion, an imager must also be informed of how materials tend to look at various wavelengths. In order to fully describe this, several features of reflection must be considered. The variables that must be described are the angle at which light is coming in, the angle at which light is scattering, surface properties, and how reflectance is affected by wavelength.

Bidirectional Reflectance Distribution Functions (BRDF) describe reflectance using these variables. Fortunately, much work has already been done in developing BRDF equations of many forms. There tend to be two kinds of BRDF models, those

which are computationally difficult and applicable to a wide range of materials and wavelengths, and those which are computationally fast but describe a narrow set of materials or wavelengths. Neither of these kinds of models is sufficient for an ideal HSI model. This is because a lot of data is taken by an HSI imager, leading to a lot of processing calculations. This indicates a need for a BRDF model which can quickly process a lot of data. However, a BRDF model which is computationally fast will likely be very inaccurate outside of the wavelength band in which it was developed. Many HSI applications used by the Air Force are in the Long Wave Infrared (LWIR). Unfortunately, few BRDF models have been adapted to describe material reflectance within the LWIR wavelength range. Therefore, current BRDF models are insufficient for an effective HSI scenario in the LWIR.

1.2 Proposal

In order to produce a BRDF expression more suitable for HSI applications, one of two approaches can be taken. The first is that a simpler model, intended for a narrow wavelength range, could be generalized to work for any wavelength range. The second is that a more complex model which applies to a wide range of wavelength bands could be approximated in such a way where its calculation time could be diminished. An attempt at the first method was made by Butler [6]. It was discovered that a simple wavelength scaling factor could not be introduced to existing models. However, progress has been made on the second method by Krywonos [18] and Butler [8]. These two works follow a physical derivation which contains angular, material, and wavelength effects.

This thesis will begin by outlining some background information helpful for understanding BRDF. This includes some basic electromagnetics, a coordinate system unique to BRDF, and a brief explanation of existing models. Then there will be a dis-

cussion of various methods of approximating a complicated BRDF expression. This expression contains wavelength scaling. Therefore, if an approximation technique is found, a computationally fast and yet LWIR-applicable model will result.

In this thesis, no complete solution is given. Steps were taken to approximate the accurate form of the BRDF via mathematical and numerical approaches. The original expression proved very difficult to simplify into a convenient solution. The complexities that caused these problems are described. Although none of these techniques resulted in a full solution, some progress was made. First, a correction to an existing microfacet model is presented. Second, flaws in simple approximations using extremely polished and very rough solutions are described. Third, an alternate form of the microfacet expression is shown via mathematical approximation. Fourth, steps are taken to show what ranges of the variables are important to analyze in numerical or fitted approximations. Last, an approximation via a fitted distribution is explored. This last approach of simplifying the microfacet BRDF is most promising because it resulted in a closed-form expression for the BRDF for rough surfaces. Also, it may be even more useful if further simplification could be applied.

II. Background

BRDF models can contain electromagnetic relationships, expressions unique to BRDF, or a combination of both. This chapter will start by introducing some fundamental relationships of electromagnetics and reflectance. Then these relationships will be applied to BRDF models. This will give some understanding as to why some fundamental electromagnetic properties have been altered for use in BRDF analysis. Lastly, the conventions and equations specific to BRDF will be explained.

2.1 Electromagnetics and Surfaces

Maxwell's equations provide a succinct summary of fundamental electromagnetics [17]

$$\begin{aligned}\nabla \times \vec{\mathbf{E}} &= -\frac{\partial \vec{\mathbf{B}}}{\partial t} \\ \nabla \times \vec{\mathbf{B}} &= \mu_m (\vec{\mathbf{J}} + \epsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}) \\ \nabla \cdot \vec{\mathbf{E}} &= \frac{\rho_c}{\epsilon} \\ \nabla \cdot \vec{\mathbf{B}} &= 0\end{aligned}\tag{1}$$

where $\vec{\mathbf{E}}$ is the electric field, $\vec{\mathbf{B}}$ is the magnetic field, μ_m is the magnetic permeability, $\vec{\mathbf{J}}$ is current density, ϵ is electric permittivity, and ρ_c is charge density. These equations are simplified for a wave traveling through free space, where $\vec{\mathbf{J}} = 0$ and $\rho = 0$. BRDF, however, involves surface interaction. Therefore it is necessary to use these equations at a surface and even within a material. Geometry at a given surface is complicated, which makes Maxwell's Equations difficult to solve directly.

Two useful parameters when describing a material's electromagnetic properties of refraction and extinction are described by the equation [17]

$$n + ik = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \quad (2)$$

where ϵ_0 is the permittivity of free space, μ_0 is the permeability of free space, and n and k are the real and imaginary components of the refractive index. Generally speaking, refractive index, n , quantifies how radiation will refract in a particular medium. The extinction coefficient, k , quantifies the absorptive qualities of a material due to electron resonance [17]. Often, n and k are used in the BRDF via the Fresnel equations. These express the solution to Maxwell's equations for reflection in terms of angles and complex index of refraction $\tilde{n} = n + ik$ [17]. These Fresnel equations describe reflectance at specular angles, where the incident and reflected light are equal: [17]

$$\begin{aligned} r_p &= \frac{\tilde{n} \cos \theta - \sqrt{1 - (\frac{\sin \theta}{\tilde{n}})^2}}{\tilde{n} \cos \theta + \sqrt{1 - (\frac{\sin \theta}{\tilde{n}})^2}} \\ r_s &= \frac{\cos \theta - \sqrt{\tilde{n}^2 - \sin^2 \theta}}{\cos \theta + \sqrt{\tilde{n}^2 - \sin^2 \theta}} \end{aligned} \quad (3)$$

where r_p describes p-polarized light, r_s describes s-polarized light, and θ is the incident angle relative to the surface normal. Unpolarized effects are found simply by taking the average of these expressions

$$F = \frac{|r_p|^2 + |r_s|^2}{2} \quad (4)$$

Fresnel reflectance is used extensively in BRDF models because it is mathematically simple and contains information about reflection geometry and the material itself. This form of the Fresnel equations only holds true for an infinite, optically smooth surface. In most cases, the reflecting surface is very large relative to wavelength, so the assumption for an infinite plane is satisfied. However, very few surfaces

can be assumed to be perfectly flat. Therefore, additional terms are used to modify these Fresnel equations for surfaces with roughness. Modifications to the Fresnel equations will be developed after BRDF models are explained. It should be noted that unlike Fresnel, BRDF is dependent on both incident and scattered angles because BRDF gives a more full picture of how light reflects.

These are a few physical relationships used by BRDF models. However, there are several properties commonly used for BRDF models which were developed specifically for reflectance studies.

2.2 BRDF Background

BRDF was formally introduced by Nicodemus in order to quantify “directional variations in reflectance and emissivity of an opaque surface element” [21]. There are a few qualitative features of BRDF which will be discussed before any mathematical treatment.

When incident light irradiates a material, the light scatters in all directions. The light whose reflectant angle is directly opposite (equal in magnitude, opposite in sign) to its incident angle is called the specular peak. The light which scatters nearly equally in all directions is called the diffuse portion. A typical surface reflection will contain characteristics of these two parts, as shown in Figure 1 [19].

Another useful way to look at scatter involves adding up all the reflected light. The Directional Hemispherical Reflectance (DHR) is the normalized sum of scattered light in all directions when illuminated by one specific incident angle. Related to this is Hemispherical Directional Reflectance (HDR), which is the scattered light at one particular angle when the surface is illuminated by a full hemisphere of incident light. The HDR and DHR must be equivalent by Helmholtz reciprocity, that is, incident and reflected angles may be exchanged with no effect on magnitude [23].

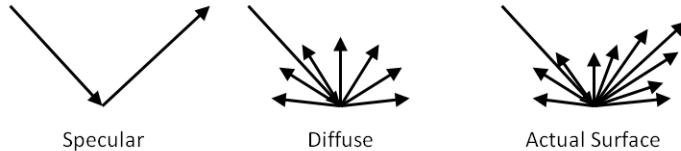


Figure 1. Specular, diffuse, and realistic surface reflections

2.2.1 Microfacet Coordinates.

Any surface, when looked at very closely, will resemble a collection of microfacets. These are tiny flat surfaces that lie at different angles on a surface. Microfacets and the coordinate system used to analyze them are shown in Figure 2 [12].

The rough surface shown is a collection of flat microfacets lying at various angles. $\vec{N}$ is the macroscopic surface normal which points in the z direction, θ_i is the incident angle measured from macroscopic normal, and θ_r , sometimes denoted θ_s , is the reflected, or scattered angle. All angles are measured from macroscopic normal.

Rusinkiewicz formally defined a microfacet coordinate system that had been used in many prior BRDF models. His coordinate system includes three dimensional vectors $\omega_i(\theta_i, \phi_i)$ and $\omega_s(\theta_s, \phi_s)$, which indicate incident and scattered angles, respectively. Rusinkiewicz's coordinate system is shown in Figure 3 [28].

In the Rusinkiewicz coordinate system, $\vec{h}$ points normally from individual microfacets.

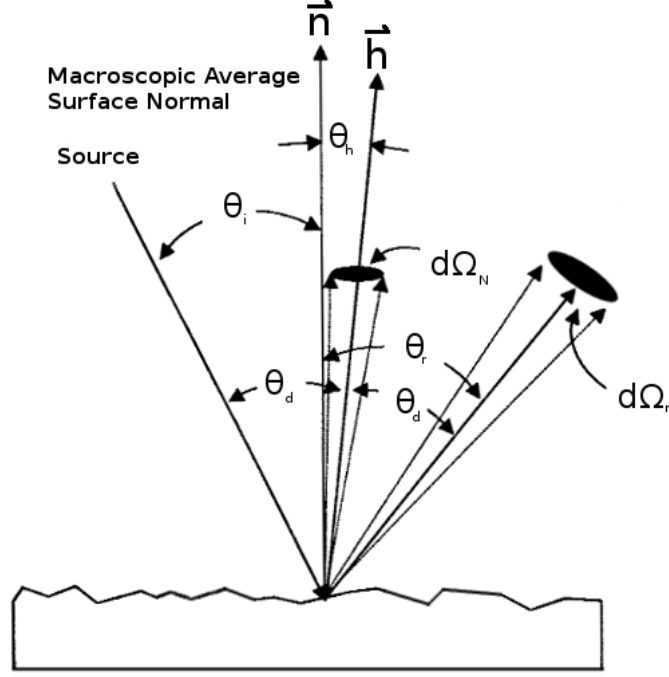


Figure 2. Microfacets and coordinate system geometry in two dimensions

The angle between the macrosurface normal $\vec{n}$ and $\vec{h}$ is θ_h . The angle between the incident light and $\vec{h}$ is θ_d .

Now that coordinates have been established, BRDF can be defined as the ratio of reflected radiance from a surface to incident irradiance upon a surface [21].

$$f_r(\theta_i, \theta_s, \lambda) = \frac{L_s(\theta_i, \theta_s, \lambda)}{E_i(\theta_i, \lambda)} \quad (5)$$

where L_s is reflected radiance and E_i is incident irradiance.

It is possible for both the incident and scattered light to come from any angle above the horizon. This results in a large range of angles which the BRDF must calculate. In many circumstances, this is undesirable because it requires a lot of computation time. Fortunately, most materials are isotropic, meaning the reflectance does not vary with azimuthal angle, the angle which rotates around $\vec{n}$. Still, only considering angles θ dependence leads to a large calculated space. Many times this is

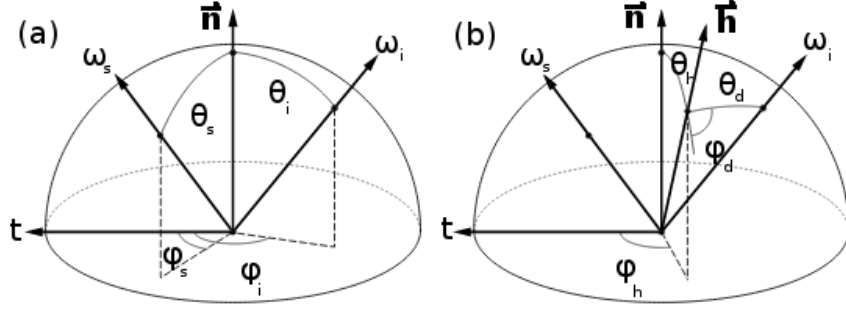


Figure 3. Rusinkiewicz coordinates in three dimensions with ω_i and ω_s shown. (a) Macroscopic surface coordinates, (b) Microfacet coordinates

avoided by using HDR or DHR. Calculation using DHR, for instance, is simple. Total scatter for one incident angle is integrated over the hemisphere, resulting in a unitless normalized reflectance between 0 and 1. This is done for all incident angles, giving a simple relationship between incident angle and total reflectance. In this instance, using DHR can greatly reduce the complexity of BRDF as shown below

$$f_r(\theta_i, \lambda) = \int_{2\pi} f(\theta_i, \theta_r, \phi_r, \lambda) d\Omega \quad (6)$$

2.2.2 The Generalized Microfacet Model.

In microfacet models, surfaces are approximated as a collection of microfacets. Numerous microfacet models have been proposed, and each describes a very narrow set of materials and wavelengths. Instead of giving several examples of microfacet models, it will be more profitable to describe a generalized form of these models proposed by [7]. In order to understand the generalized form of microfacet models, it is necessary to review its individual components.

Remember that microfacets are tiny flat plane sections that are oriented in various directions. The direction in which the microfacet surface normals point is described by a distribution function D . Different BRDF models use different distributions. One example distribution function is a Gaussian distribution [25]:

$$D_g(\theta_h) = \frac{1}{2\pi\sigma_g^2 \cos^4 \theta_h} \exp \left[-\frac{\tan^2 \theta_h}{2\sigma_g^2} \right] \quad (7)$$

where σ_g is the Gaussian standard deviation, and θ_h is the half angle vector. Gaussian distributions are common. They are often approximated by the Cosine Lobe, which is computationally faster. This form was proposed by Phong [24] and Blinn [4] for computer graphics rendering and was extended to anisotropic surfaces by Ashikhmin in [2]

$$D_c(\theta_h) = \frac{k_c + 2}{2\pi} (\cos \theta_h)^{k_c} \quad (8)$$

where k_c is a shape parameter for the cosine lobe. The coefficient before the cosine normalizes the distribution.

As mentioned in Section 2.1, most models contain a Fresnel term. This is because it takes into account both angular and material dependence in a simple form. The expressions for Fresnel reflectance are often similar to Equations (3) and (4).

If one were to imagine a surface composed of microfacets, one could imagine the possibility of some reflections being obscured by nearby microfacets. This is accounted for by the geometrical attenuation term G . One common expression to describe this was created by Blinn, assuming V-shaped grooves within a material [4]

$$G_v = \min \left[1, \frac{2 \cos \theta_h \cos \theta_s}{\cos \theta_d}, \frac{2 \cos \theta_h \cos \theta_i}{\cos \theta_d} \right] \quad (9)$$

The $\min[]$ function in this expression selects the one case that applies. Either there is no attenuation, the reflection hits another microfacet, or light is blocked from hitting the microfacet [7]. This is not an entirely physical description of the microfacets, but it does describe an extreme case. A more thorough treatment of geometrical attenuation is provided by Heitz in [16].

Another term found in microfacet models is the scattering cross section conversion term σ_{cs} . This term arises because typical scattering equations in the scatter community assume scattering spheres, which are calculated as point sources. This gives an expression in terms of intensity rather than radiance. However, with BRDF models the scattering surfaces are typically assumed to be flat areas [26]. This form comes from [10]

$$\sigma_{cs} = \frac{1}{4 \cos \theta_i \cos \theta_s} \quad (10)$$

A few models contain a volumetric scatter term V . This becomes more significant in certain materials that contain bulk scatterers, or impurities, such as pigments in paint. This form of volumetric scatter was proposed by Sandford and Robertson in [29].

$$V = \frac{F_s(\theta_i)F_s(\theta_s)}{\pi} \quad (11)$$

where F_s is a Fresnel approximation used by Sandford. Lastly, Butler [7] includes a prefactor term P in his generalized microfacet equation to take into account any model-specific terms that are not easily generalized.

Typically, microfacet equations are simply a combination of ρ_s , ρ_v , and ρ_d , which are the specular, volumetric, and diffuse contributions respectively. The specular portion is modified by a multiplication of terms which modify its shape. The generalized form of a microfacet BRDF model is [7]

$$f_r = \rho_s PDFG \sigma_{cs} + \rho_v V + \frac{\rho_d}{\pi} \quad (12)$$

It should be noted that some models will not define all of these terms. A simple microfacet model is Blinn-Phong [4]. Its BRDF equation assumes a cosine lobe mi-

microfacet distribution function as well as a constant diffuse offset. It does not define F , G , or V , and has no extra prefactor terms. Therefore, the Blinn-Phong expression for BRDF is [4]

$$f_r = \rho_s \frac{w+2}{2\pi} \cos^w \theta_h + \frac{\rho_d}{\pi} = \rho_s D_c(\theta_h) + \frac{\rho_d}{\pi} \quad (13)$$

where D_c was given in Equation (8) and w is a fit parameter. Another common model is Cook-Torrance, which is based on the Beckmann distribution. The explicit and compressed forms are shown below [11]

$$\begin{aligned} f_r &= \frac{\rho_s F G_v}{\pi \cos \theta_i \cos \theta_s} \frac{1}{\sigma_g^2 \cos^4 \theta_h} \exp \left[- \left(\frac{\tan \theta_h}{\sigma_g} \right)^2 \right] + \frac{\rho_d}{\pi} \\ &= 4 \rho_s \sigma_{cs}(\theta_i, \theta_s) D_b(\theta_h) F(\theta_d) G_v(\omega_i, \omega_s) + \frac{\rho_d}{\pi} \end{aligned} \quad (14)$$

where G_v in the first line is the same G_v as Equation 9. Most microfacet models, like Blinn-Phong and Cook-Torrance, are very simple and computationally fast. Also, they fit the generalized form shown in Equation (12).

2.2.3 Microfacet Summary.

There are a few good things about microfacet models. First, a lot of analysis has been done on them via comparison [20], and generalization [8]. Second, they are computationally simple. Third, they can be very accurate for a small set of materials and wavelengths.

However, there is one big problem with microfacet models. They do not describe the wavelength dependence of BRDF. This is due to the fact that microfacet models are derived from geometric optics, where wavelength dependence is ignored. They are parameterized equations based on empirical measurements. Microfacet models are created to describe rough materials. They work best at rough ranges because of

their assumption that a surface is made of tiny microfacets. Butler, in [8], showed that the generalized microfacet model is the same form as a rough approximation for the Modified Beckmann-Kirchhoff (MBK) model. The MBK will be discussed further later in this thesis. A smooth material does not have this kind of surface design. Because of this assumption and the fact that they are empirically fit, they cannot describe reflectance outside the range of prior measurement and analysis. Many models have been created for the visible range, but few for the LWIR. Because they rely on parameterized measurements, models which are accurate in the visible are not necessarily accurate at other wavelengths. Butler’s summary of popular microfacet models in [7] does not contain any models with wavelength dependence. Even small changes in wavelength or material properties can drastically affect the accuracy of a microfacet model [9]. Therefore, one would have to make lab measurements and compare a variety of models in order to pick one that performed well in this new wavelength or material range. Also, it is possible that no microfacet model would be sufficiently accurate. This shortcoming indicates that it may be beneficial to take a physically derived approach which takes into account both wavelength and material properties. A BRDF which includes these would have a much broader range of applicability to wavelength and material properties than a microfacet model.

2.2.4 Wave Optics Models.

An alternate approach to microfacet models is the use of wave optics models. Wave optics models follow a physical derivation, including wavelength dependence and surface properties. They result from a scalar solution to Maxwell’s equations, assuming a relatively high frequency. Even with this assumption, the results are far more generally applicable than microfacet models. This is because they contain wavelength dependence, unlike microfacet models. However, the resulting expressions

are much more complicated than microfacet models.

Two surface parameters that are commonly used in these models are surface height σ_s and correlation length l_c . If a surface is assumed to be comprised of bumps of varying heights and separations, these parameters statistically describe the layout of these bumps. Surface height describes how tall these bumps are, on average. Correlation length is a measure of how far apart, on average, these bumps are. Both surface parameters have units of length.

A complete summary and historical review of physical diffraction models is provided by Krywonos [18]. Two, out of many, wave optics models are the Rayleigh-Rice (RR) model, and Beckmann-Kirchhoff (BK) model. Rayleigh-Rice, first proposed in [27], developed theory for polished surfaces. This did very well for polished surfaces. Due to its limited applicability, it will not be discussed further here.

The model used in this document follows the BK derivation [3], which includes a small-angle approximation and assumes Gaussian surface statistics. Harvey and Krywonos in [15] modified this BK model by starting with the functional form provided by O'Donnell [22]

$$f_{BK} = \frac{\pi l_c^2 F_g^2 \exp(-g)}{A_s} \sum_{m=1}^{\infty} \frac{g^m}{m!m} \exp\left(-\frac{\nu_{xy}^2 l_c^2}{4m}\right) \quad (15)$$

where g describes surface roughness, A_s is scattering area, ν_{xy} is a combination of angular terms and surface height, and F_g is the geometrical attenuation factor, not to be confused with the Fresnel term F . Harvey and Krywonos modified this equation in two ways. They included Q , the polarization factor, which is accurate at all angles, and they removed the geometric attenuation factor F_g in favor of a normalization term K . These two modifications allow for calculation at all angles. Also, because of the normalization change, they were also able to rewrite the expression in terms of scattered radiance, which is more appropriate for BRDF.

Krywonos modified this derivation to generalize the BRDF to cover all angles, not just small angles, calling it the MBK scatter theory. In this thesis, this form of the MBK is called Krywonos Modified Beckmann-Kirchhoff (KMBK), given in [18] as

$$f_{KMBK} = \frac{KQ\pi l_c^2}{2\lambda^2} \exp(-g) \sum_{m=1}^{\infty} \frac{g^m}{m!m} \exp\left(-\frac{\nu_{xy}^2 l_c^2}{4m}\right) \quad (16)$$

where λ is wavelength. The KMBK expression will be used in this work for three reasons. First, it was thoroughly compared with the Rayleigh-Rice model for smooth surfaces and the BK model for rough surfaces by Krywonos [18]. It was shown to be very accurate to these accepted models within their regions of applicability. The second reason is that out of the many wave optics models that have been developed, it is more compact, apart from the summation. The third is that this expression can be rewritten in microfacet coordinates which facilitates comparison with microfacet models. The process of coordinate conversion is outlined below.

2.2.5 Coordinate Conversion.

As mentioned before, wave optics models and microfacet models not only are derived with different assumptions, but they are written in different coordinate systems. Work was done by Butler in [8] to convert f_{KMBK} from wave optics coordinates to microfacet coordinates. A few of the relationships he used are listed here. Although not all the terms in Equation (16) will be explained in detail, it is important to understand the roughness factor g . In microfacet coordinates, g is represented by

$$g = \left(\frac{2\pi\sigma_s}{\lambda}\right)^2 (\cos(\theta_i) + \cos(\theta_s))^2 \quad (17)$$

From this equation, it is apparent that when average relative surface height σ_s/λ is large (indicating a rough material) then g is large. Also, when the incident or scattered

angle is small, then g is large. This is consistent with an intuitive understanding of reflection, which says that most surfaces reflect more specularly at grazing angles.

Butler also related the Fresnel term F to a different term which is used for reflection in wave optics models, Q . Like F , Q depends on n , k , and incident, and reflected angles [27].

$$Q_{ss} = \left| \frac{(\tilde{n}^2 - 1) \cos(\phi_s - \pi)}{(\cos \theta_i + \sqrt{\tilde{n}^2 - \sin^2 \theta_i})(\cos \theta_s + \sqrt{\tilde{n}^2 - \sin^2 \theta_s})} \right|^2 \quad (18)$$

$$Q_{sp} = \left| \frac{(\tilde{n}^2 - 1) \sqrt{\tilde{n}^2 - \sin^2 \theta_s} \sin(\phi_s - \pi)}{(\cos \theta_i + \sqrt{\tilde{n}^2 - \sin^2 \theta_i})(\tilde{n}^2 \cos \theta_s + \sqrt{\tilde{n}^2 - \sin^2 \theta_s})} \right|^2 \quad (19)$$

$$Q_{ps} = \left| \frac{(\tilde{n}^2 - 1) \sqrt{\tilde{n}^2 - \sin^2 \theta_i} \sin(\phi_s - \pi)}{(\tilde{n}^2 \cos \theta_i + \sqrt{\tilde{n}^2 - \sin^2 \theta_i})(\cos \theta_s + \sqrt{\tilde{n}^2 - \sin^2 \theta_s})} \right|^2 \quad (20)$$

$$Q_{pp} = \left| \frac{(\tilde{n}^2 - 1) \left(\sqrt{\tilde{n}^2 - \sin^2 \theta_i} \sqrt{\tilde{n}^2 - \sin^2 \theta_s} \cos(\phi_s - \pi) - \tilde{n}^2 \sin \theta_i \sin \theta_s \right)}{(\tilde{n}^2 \cos \theta_i + \sqrt{\tilde{n}^2 - \sin^2 \theta_i})(\tilde{n}^2 \cos \theta_s + \sqrt{\tilde{n}^2 - \sin^2 \theta_s})} \right|^2 \quad (21)$$

$$Q = Q_{ss} + Q_{sp} + Q_{ps} + Q_{pp} \quad (22)$$

It should be noted that some texts define Q as $Q/2$, but this paper will follow the precedent set by Krywonos.

Q and F vary significantly but $Q = 2F$ at the specular angle. First are shown two figures, each plotted for incident angle $\theta_i = 40$. Figure 4 shows angular reflectance for Nickel and Figure 5 shows angular reflectance for glass.

As shown, Q and $2F$ are equal at the specular angle but are quite different away from specular. Second, Figures 6 and 7 show the reflectance curves of Nickel and glass, respectively, at an incident angle of 0 degrees from normal.

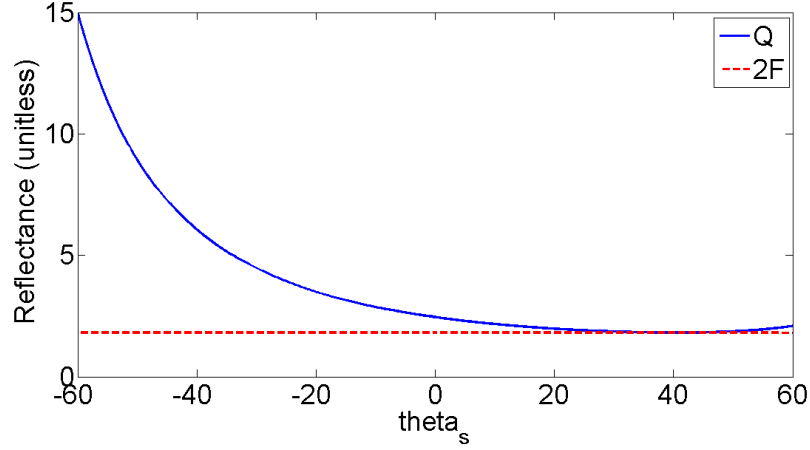


Figure 4. Comparison between $2F$ and Q for Ni ($n = 10$, $k = 12.5$) where $\theta_i = 40$

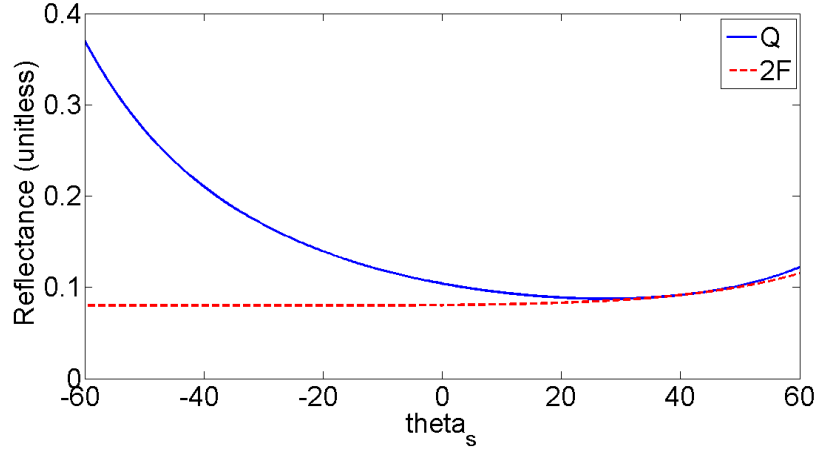


Figure 5. Comparison between $2F$ and Q for glass ($n = 1.5$, $k = 0$) where $\theta_i = 40$

As mentioned previously, microfacet models have various ways of modifying the Fresnel term. Stover found a relationship between F and Q for some limiting cases [31]. Butler found an approximate conversion between these two reflection terms for all angles [8]

$$S = \frac{4 \cos \theta_i \cos \theta_s \cos^4 \theta_h}{(\cos \theta_i + \cos \theta_s)^2} \approx \frac{2F}{Q} \quad (23)$$

The more physically accurate term is Q because F assumes an infinite, flat plane. Q is a rough surface perturbation of F . Therefore, using this direct conversion allows

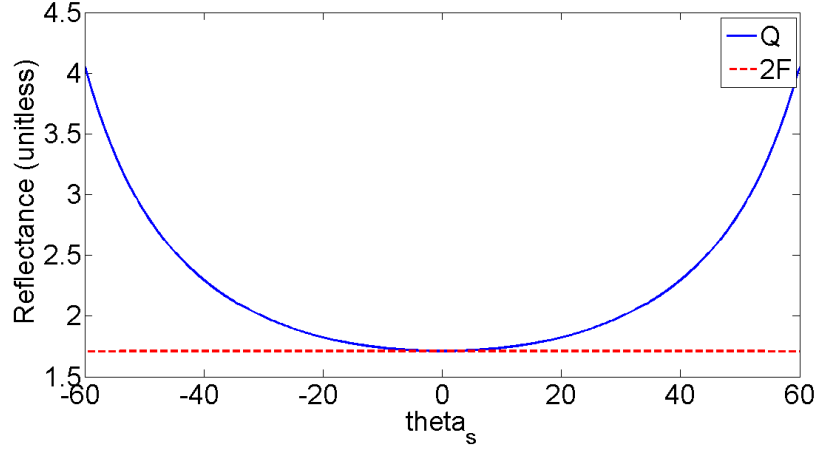


Figure 6. Comparison between $2F$ and Q for Ni ($n = 10$, $k = 12.5$) where $\theta_i = 0$

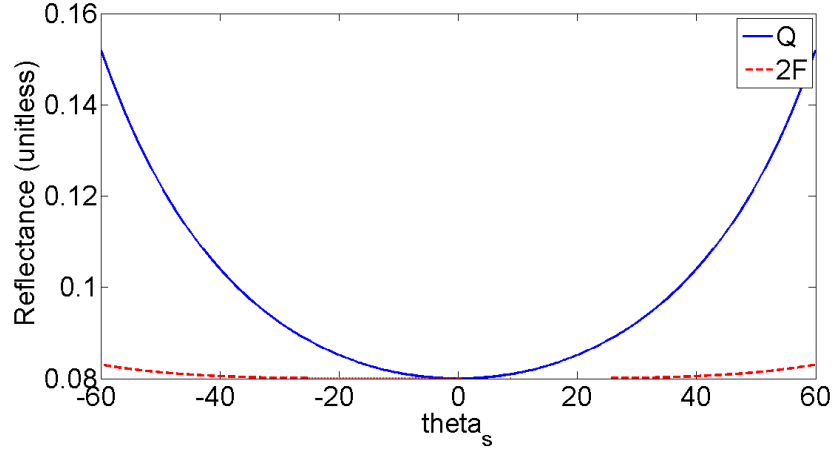


Figure 7. Comparison between $2F$ and Q for glass ($n = 1.5$, $k = 0$) where $\theta_i = 0$

one to convert a BRDF using F into a BRDF in terms of Q . In this case, it was done to convert the KMBK into microfacet coordinates and in similar terms as other microfacet models.

Another pair of terms that needed to be converted are the isotropic angular terms ν_{xy} and η_r , which are given as [8]

$$\nu_{xy}^2 = \left(\frac{2\pi\eta_r}{\lambda} \right)^2 = \left(\frac{2\pi}{\lambda} \right)^2 (\sin^2 \theta_i + \sin^2 \theta_s + 2 \sin \theta_i \sin \theta_s \cos \phi_s) \quad (24)$$

When simplifying the lengthy right hand side of Equation (24) in microfacet co-

ordinates, it is helpful to use the relationship [8]

$$\tan^2 \theta_h = \frac{4 \cos^2 \theta_d - (\cos \theta_i + \cos \theta_s)^2}{(\cos \theta_i + \cos \theta_s)^2} \quad (25)$$

Combining Equations (24) and (25) gives a simplified substitution for ν_{xy} [8]

$$\nu_{xy} = \frac{2\pi}{\lambda} \tan^2 \theta_h (\cos \theta_i + \cos \theta_s)^2 \quad (26)$$

This conversion is used on the exponential of the KMBK, Equation (16), to derive a model in terms of microfacet coordinates only.

The name for the KMBK expression converted to microfacet coordinates will be referred to in this thesis as Butler Modified Beckmann-Kirchhoff (BMBK). It should be noted that the exponent in Butler's original expression [8] was corrected to give this form. This change will be discussed later on.

$$f_{BMBK} = \frac{K\pi l_c^2}{\lambda^2} \frac{FG(\cos \theta_i + \cos \theta_s)^2}{2 \cos \theta_i \cos \theta_s \cos^4 \theta_h} \exp(-g) \times \sum_{m=1}^{\infty} \frac{g^m}{m!m} \exp \left[-\frac{\pi^2 l_c^2}{m\lambda^2} \tan^2 \theta_h (\cos \theta_i + \cos \theta_s)^2 \right] \quad (27)$$

The advantage of this conversion is that the resulting MBK wave optics model can be compared with other microfacet models. Moreover, it allows one to compare it against microfacet models to see where and whether wavelength dependence can be inserted into an existing microfacet model. Notice that, unlike typical microfacet models, wavelength dependence has been preserved in l_c and g . If this behavior could be generalized to fit the general microfacet model form given in Equation (12), perhaps the wavelength scaling could be inserted into existing models.

2.2.6 Approximate Expressions.

Beyond converting the KMBK model into microfacet coordinates, it would be useful to simplify it by simplifying the infinite summation. Butler did this for two extreme cases by developing approximations for the BMBK expression at very smooth and very rough limits [6]. Roughness is a useful property by which to categorize reflectance properties because BRDFs behave very differently in rough and smooth cases. In this thesis, g is used to describe roughness. The region defined as very smooth is where $g \leq 0.025$ [18]. Very rough is defined as $g \geq 800$ [18]. These are approximated so that the error in the summation is less than 1/1000th of the summation. Approximating a very smooth surface is done by simply truncating all terms of the series except the first one by observing that each term has an increasing exponential dependence g^m . Therefore, the terms fall off quickly in importance. Also, $\exp(-g)$ is assumed to be about 1. The approximation for very smooth surfaces is

$$f_{pol} = \frac{(2\pi)^3 FG (\cos \theta_i + \cos \theta_s)^4}{4 \cos \theta_i \cos \theta_s} \left(\frac{\sigma_s^2 l_c^2}{\lambda^4 \cos^4 \theta_h} \right) \exp \left[- \left(\frac{\pi l_c}{\lambda} \right)^2 \tan^2 \theta_h (\cos \theta_i + \cos \theta_s)^2 \right] \quad (28)$$

Figures 8 and 9 show the full BMBK expression compared to the polished approximation at 0 and 40 degrees incident angle, respectively.

The approximation for very rough surfaces is found by taking the limit of the summation [6].

$$f_{vr} = \frac{KFG}{4 \cos \theta_i \cos \theta_s} \left(\frac{l_c^2}{2\pi \sigma_s^2 \cos^4 \theta_h} \right) \exp \left[- \left(\frac{l_c}{2\sigma_s} \right)^2 \tan^2 \theta_h \right] \quad (29)$$

Figures 10 and 11 show the full expression compared to the polished approximation at 0 and 40 degrees incident angle, respectively.

For very smooth or very rough surfaces, these approximate expressions are ade-

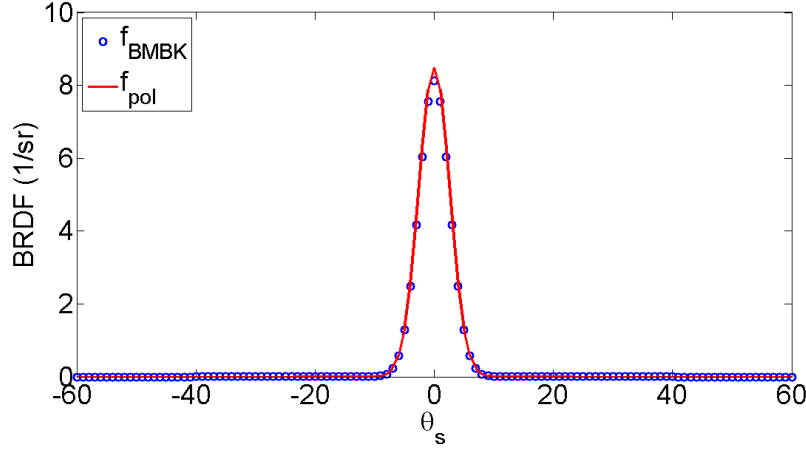


Figure 8. Polished approximation where $\theta_i = 0$, $l_c/\lambda = 5$, and $\sigma_s/\lambda = 0.02$

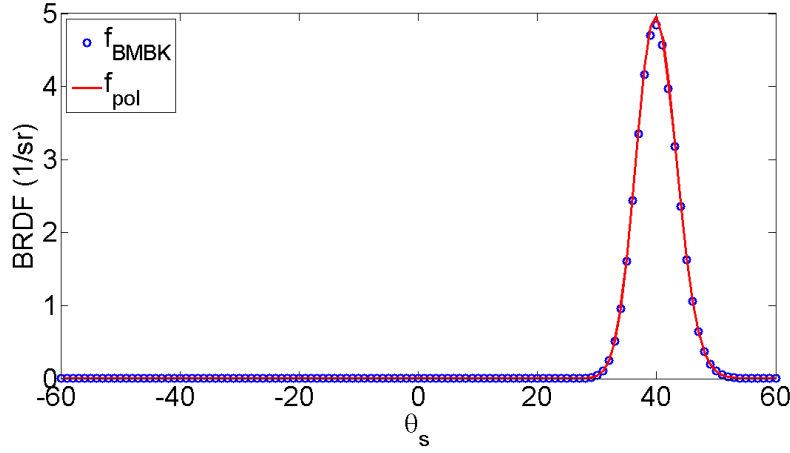


Figure 9. Polished approximation where $\theta_i = 40$, $l_c/\lambda = 5$, and $\sigma_s/\lambda = 0.02$

quate because they are both accurate and computationally simple. However, most materials fall in between these extremes. Therefore, for most materials, the full BMBK expression must be used. A full analysis on how many terms are necessary as a function of g was done by Krywonos in [18]. In order to calculate the number of terms needed, he found the significance of each term as a function of g and m . He observed that for a given value of g , the first terms were very small. Successive terms would increase in magnitude, until eventually they declined. Therefore, there is a cutoff where terms no longer contribute a consequential amount to the sum. This is

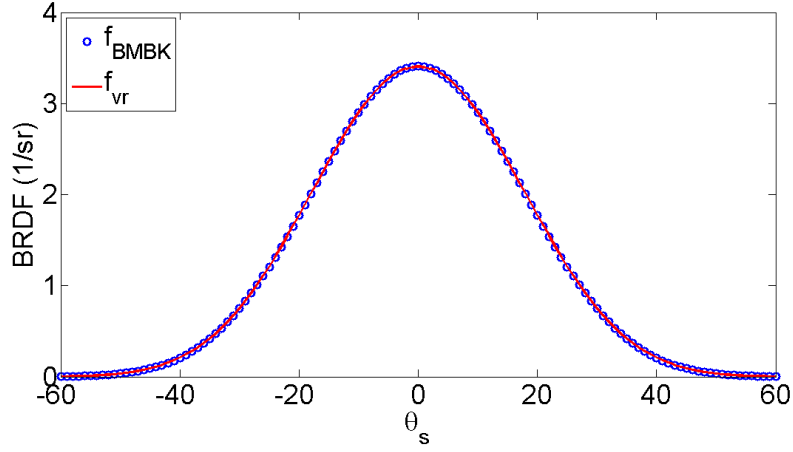


Figure 10. Very rough approximation where $\theta_i = 0$, $l_c/\lambda = 20$, and $\sigma_s/\lambda = 2$

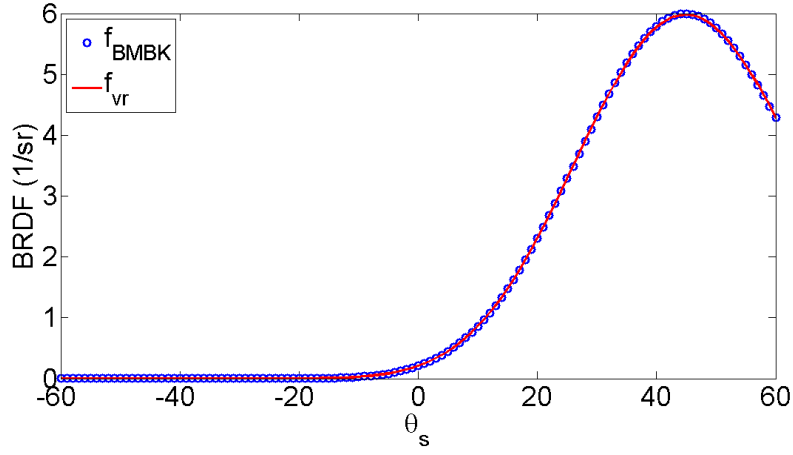


Figure 11. Very rough approximation where $\theta_i = 40$, $l_c/\lambda = 20$, and $\sigma_s/\lambda = 2$

apparent by inspecting the summation within the BMBK, Equation (27). At large m , the factorial in the denominator dominates the expression, bringing successive terms to 0. Krywonos was able to quantify the number of terms needed. He described the number of terms, N , to be [18]

$$N = 1.1g + 40 \quad (30)$$

Therefore, when g is large, the number of terms needed is large. But when g is very small, the infinite summation can be evaluated to 1/1000 precision in 40

terms. However, even when “only” 40 terms are needed, the computation is lengthy, especially for HSI applications which require calculation for numerous wavelengths simultaneously. Therefore, an approximation for this summation is required for the BMBK expression to be useful for materials that are neither very smooth nor very rough.

In this chapter, some physical principles involving electromagnetic radiation were discussed. These were then connected to principles in BRDF- Fresnel reflection, surface parameters, and surface statistic distributions. The relationship between wave optics BRDF terms and microfacet terms were explored and used to rewrite Krywonos’ form of the MBK into Butler’s form of the MBK. Butler’s form is in microfacet coordinates- the same coordinates used in most BRDF models. The BMBK was shown to be computationally difficult because of the summation, see Equation (27). As a best case, this summation requires 40 terms. As a worst case, it requires about 1000 terms. Both of these are unacceptably slow computationally. If this BRDF model is to be used in HSI applications, it is necessary that this summation be approximated into a computationally simple form. Lastly, the BMBK model is approximated in both the very rough and the polished cases, Equations (29, 28). This gives a bound for which values of g the BMBK must be approximated.

III. Approximating the BRDF

Now that the importance of a fast BRDF model which works at different wavelengths has been established and relevant background details have been reviewed, in this chapter steps are taken to simplify a wave optics BRDF model. Some of the steps taken were unfruitful but are helpful in understanding the problem and directing decision-making in the future. First will be shown the shortcomings of using a combination of the polished and very rough approximations. Second, some progress on mathematical simplifications of the summation within the BMBK expression will be shown. Third, an analysis is presented concerning the ranges of variables that need to be solved. Fourth, an approximation via replacing the terms of the summation with a distribution is proposed. However, before these steps are taken, it is necessary to establish the accuracy of prior work in this area.

3.1 Model Comparison

There were two steps taken to verify Butler's work. The first was to computationally compare Butler's form of the MBK to Krywonos' version of MBK. To accomplish this, a testing environment was created and individual components in these equations were compared to each other. In order to investigate visually this difference and the models in general, a MATLAB® GUI was made, shown in Figure 12.

The GUI allowed for easy control of several BRDF situations. It contains drop down menus where the user can choose which BRDF equations to compare. Also, it gives the user control of several BRDF variables: incident and reflected angles, l_c/λ , and σ_s/λ . Moreover, pertinent information and error statistics calculated in the code are listed as output in the text boxes. For instance, the minimum and maximum g values are reported. Also, Relative Root Mean Square (RMS) and Spectral Angle

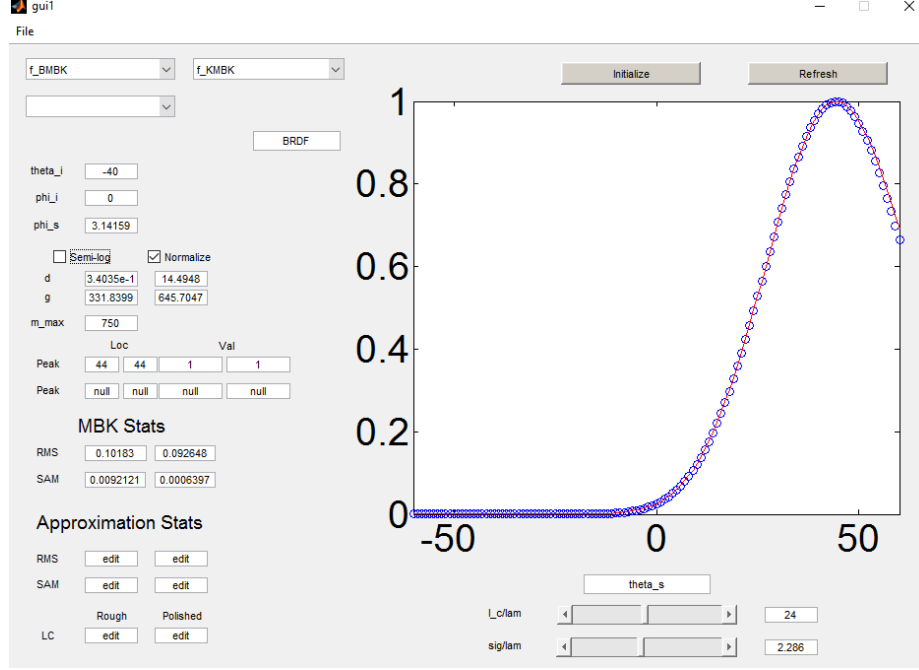


Figure 12. MATLAB® GUI showing plotting options and resulting plots

Mapper (SAM) [13] statistics between the BMBK and KMBK are shown in both regular and log space. Analysis in this thesis largely used RMS calculations to compute error or difference between models. The expression for RMS error is

$$\text{RMS} = \sqrt{\text{mean}\left(\frac{y_t - y_a}{y_t}\right)^2}, \quad (31)$$

where $\text{mean}()$ takes the average of the argument, y_t is the true function, and y_a is the approximation function.

Lastly, the GUI allows for an optional normalized representation (which is useful for comparing shape without magnitude) and an optional semilog representation (which is useful for comparing non-specular regions). The screenshot in Figure 12 shows the normalized BRDF. This is the environment in which the models were compared.

Once the environment was set up and the code for each MBK expression was written, comparing terms was not too difficult. For instance, ν_{xy} was compared

with its longer microfacet form, see Equation (24), at a variety of angles. These comparisons showed that the terms did not lose accuracy when converted to another coordinate system. One exception was that the exponential terms of the BMBK and KMBK, when plotted against each other, were different. After looking again at the mathematics of converting from wave optics coordinates to microfacet coordinates, it was determined that there was a small error in Butler’s exponential term. Presumably, η_r was used instead of ν_{xy} and λ was accidentally inserted under the l_c term. This was corrected, yielding the BMBK Equation (27).

The only term that caused a difference between Krywonos’ form and Butler’s form of the MBK was the use of F , the Fresnel term. The BMBK model uses F instead of Q because it is mathematically simpler. These terms are not exactly equivalent, as mentioned before. The approximation for Q in terms of F and some angles, shown in Equation (23), was substituted for Q . Because the mathematically simpler F is used in the BMBK model, the BRDF curves may be different in magnitude by a small factor. The approximation for Q used is most accurate near specular angle. Therefore, the error will be largest in the diffuse portion of scatter. The diffuse portion tends to be low in magnitude relative to the specular, meaning that error relative to peak is very low. It can be shown visually that the BRDF curves overlap very well in the specular peak and in the diffuse portions. This is accomplished by a direct comparison alongside a semilog comparison. For a specular material with $l_c/\lambda = 5$, $\sigma_s/\lambda = 0.02$, see Figures [13, 14, 15]. For a diffuse material with $l_c/\lambda = 20$, $\sigma_s/\lambda = 2$, see Figures [16, 17, 18]

This analysis verifies the BMBK model. The second step to verify Butler’s work was to analyze the quality of the polished Equation (28) and very rough Equation (29) approximations. It was found by Krywonos [18] that the very rough approximation held for g values of about 800 and greater. The polished approximation held for g

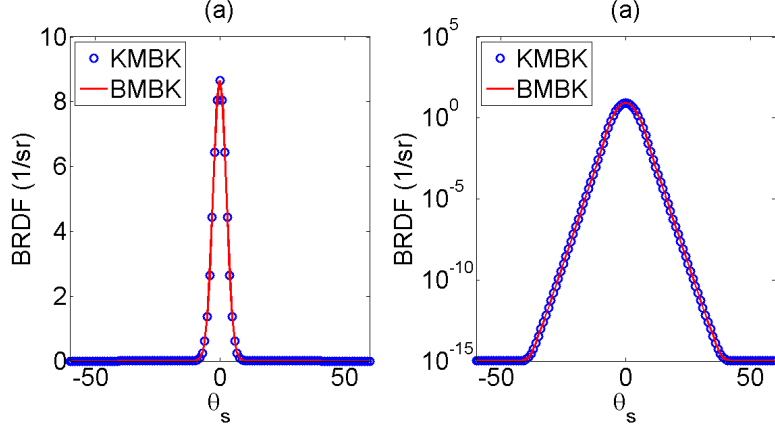


Figure 13. Comparison between KMBK and BMBK where $\theta_i = 0^\circ$, $l_c/\lambda = 5$, $\sigma_s/\lambda = 0.02$. (a) raw form (b) semilog

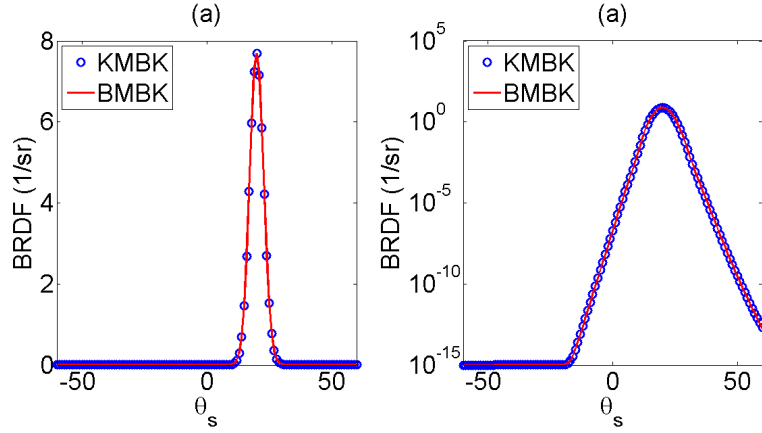


Figure 14. Comparison between KMBK and BMBK where $\theta_i = 20^\circ$, $l_c/\lambda = 5$, $\sigma_s/\lambda = 0.02$. (a) raw form (b) semilog

values of 0.025 and smaller, but only at specular angles. Krywonos only analyzed the approximations in wave optics coordinates, but the approximations in microfacet coordinates had the same behavior.

Comparing the KMBK and BMBK models showed that Butler's equation was accurate. The next step in using his form is to approximate the summation while preserving its dependence on λ . This will make it faster and therefore useful for HSI applications.

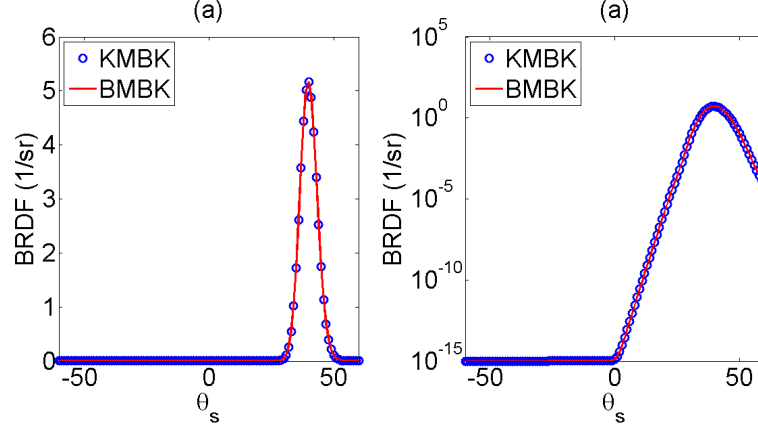


Figure 15. Comparison between KMBK and BMBK where $\theta_i = 40$, $l_c/\lambda = 5$, $\sigma_s/\lambda = 0.02$. (a) raw form (b) semilog

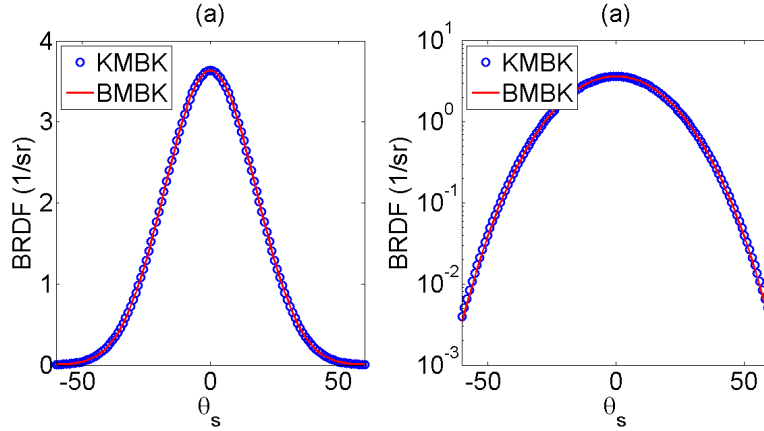


Figure 16. Comparison between KMBK and BMBK where $\theta_i = 0^\circ$, $l_c/\lambda = 20$, $\sigma_s/\lambda = 2$. (a) raw form (b) semilog

3.2 Linear and Exponential Combination Approximations

As mentioned previously, both Krywonos' and Butler's MBK expressions are computationally cumbersome because they contain a very large summation. Two helpful approximations at extreme values of roughness g have been discussed. The polished expression (28) holds for $g \leq 0.025$ and the very rough expression (29) holds for $g \geq 800$. One approach to simplifying the BMBK model is by using some combination of the polished and very rough approximations within the region $0.025 < g < 800$. Two combination approaches were taken.

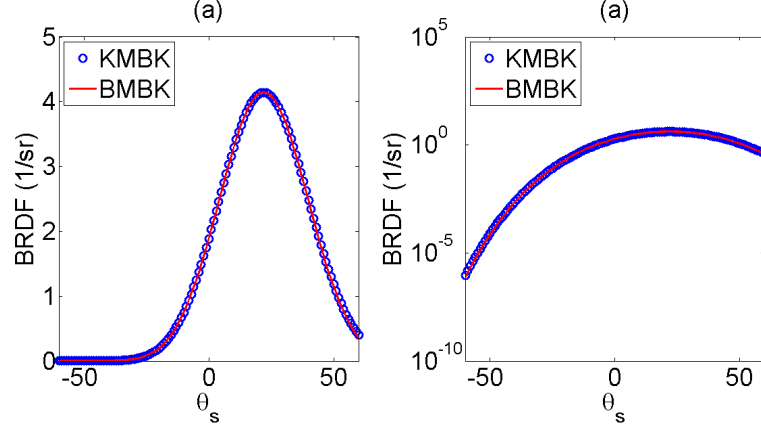


Figure 17. Comparison between KMBK and BMBK where $\theta_i = 20^\circ$, $l_c/\lambda = 20$, $\sigma_s/\lambda = 2$. (a) raw form (b) semilog

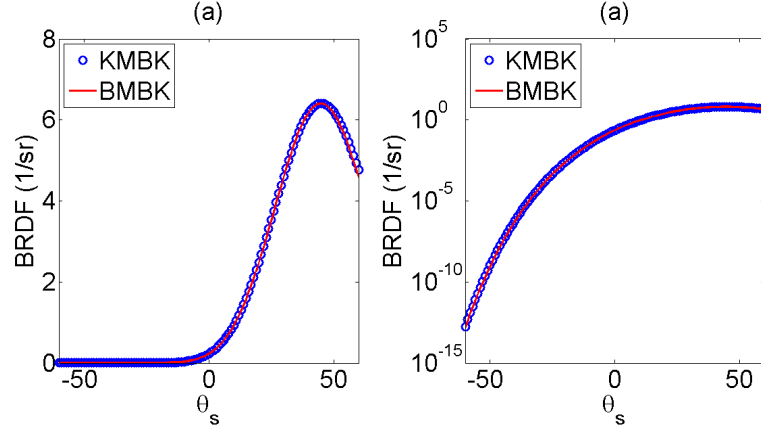


Figure 18. Comparison between KMBK and BMBK where $\theta_i = 40^\circ$, $l_c/\lambda = 20$, $\sigma_s/\lambda = 2$. (a) raw form (b) semilog

3.2.1 Linear Combination.

The first method pursued was defining the BRDF as a linear combination of the polished and smooth approximations. This is done by assuming that within this range of g , the expression can be approximated by a linear combination of the polished and rough expressions according to the form

$$f_{lc} = a_{lc}f_{vr} + (1 - a_{lc})f_{pol} \quad (32)$$

where a ranges from 0 to 1. At $a_{lc} = 0$, the combination is simply the polished expression, and at $a_{lc} = 1$, the combination is simply the very rough expression.

One potential problem with this approach is that the polished approximation is only meant to describe BRDF for extremely polished surfaces. That is to say, σ_s is assumed to be small in this approximation, but for large g values, σ_s is large. Therefore its behavior for intermediate and large g values is unpredictable. For instance, it could yield a shape uncharacteristic of any material. Therefore its behavior was analyzed and found to be similar regardless of g value. This means that though it is inaccurate at large g values, it yields a useful extreme that may be combined with another shape to give an accurate BRDF. A similar analysis was done on the very rough approximation. It was found to give a broad peak regardless of g . Therefore the very rough approximation is also inaccurate at intermediate g values but can be used as an extreme in combination.

When optimized to match the sum at very small g values, it yielded the polished approximation, $a_{lc} = 0$ as expected. However, very soon, at $g \sim 8$ and greater (σ_s/λ), the best approximation was entirely the very rough contribution, $a_{lc} = 1$. This quick jump is shown in Figure 19.

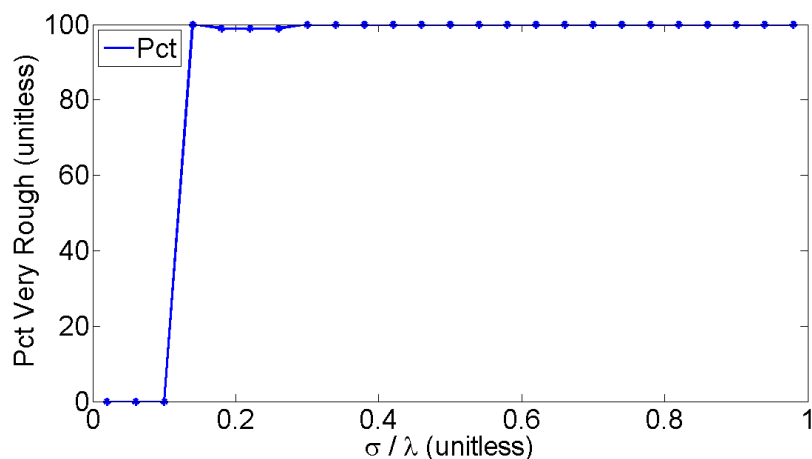


Figure 19. Very rough percentage contribution where $\theta_i = 40^\circ$, $l_c/\lambda = 15$

Besides having over-simplistic behavior, the relative error at these intermediate g values was very large, as shown in Figure 20. The linear combination has a magnitude much higher than the BMBK. This is because both the polished approximation and the very rough approximation have a much larger magnitude than the BMBK at those values of l_c/λ and σ_s/λ , as shown in Figure 21.

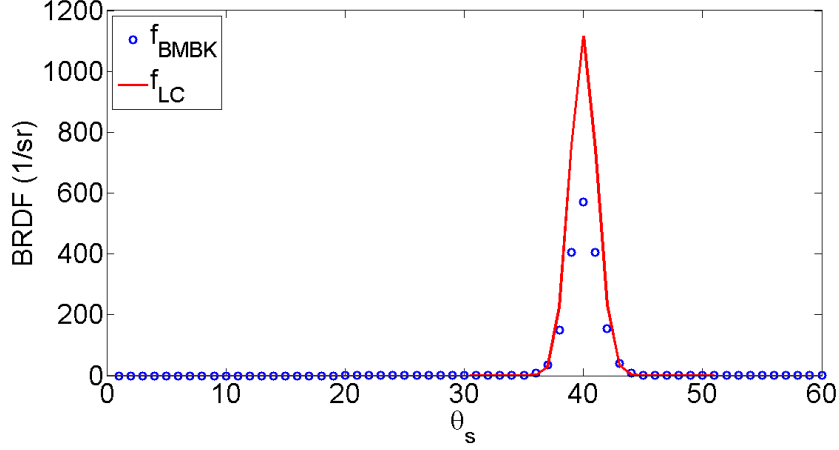


Figure 20. Linear combination approximation where $\theta_i = 40^\circ$, $l_c/\lambda = 15$, $\sigma_s/\lambda = 0.1$

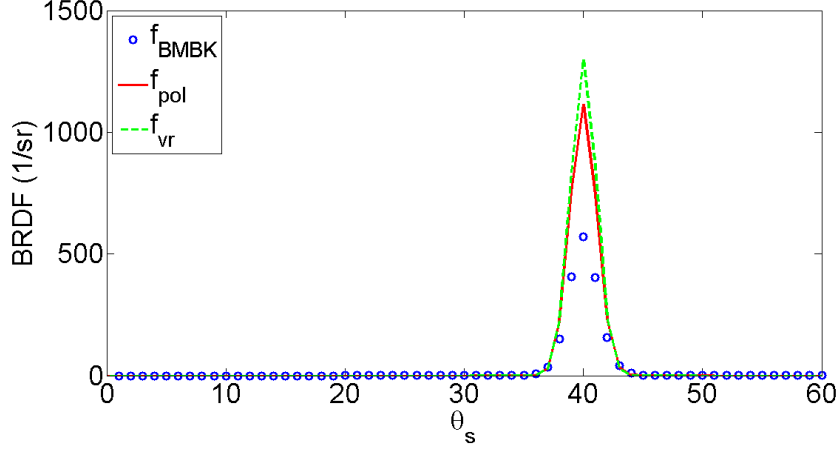


Figure 21. Approximation forms f_{pol} , and f_{vr} both have much higher magnitude than BMBK where $\theta_i = 40^\circ$, $l_c/\lambda = 15$, $\sigma_s/\lambda = 0.1$

Poor fitting and extreme behavior with σ_s/λ makes the linear combination a bad approximation. There may, however, be an alternative combination which could work.

3.2.2 Exponential Combination.

A slightly different approach was to parameterize contributions from the coefficients and exponentials of each equation separately. Coefficient, in this case, means anything before the $\exp()$ term. With some algebra and collecting of terms, the polished and very rough approximations can be rewritten as

$$\begin{aligned} f_{pol} &= Ag \exp(-hg) \\ f_{vr} &= A \frac{1}{g} \exp(-h) \end{aligned} \quad (33)$$

where A is a collection of common terms. Therefore, a combination of polished and rough equations, which is called an exponential combination could be written as

$$f_{ec} = A \frac{1}{g} g^{2a_{ec}} \exp(-hg_{ec}^b) \quad (34)$$

where a_{ec} and b_{ec} range independently from 0 to 1. This allows the coefficients and exponentials to receive different weighting, allowing for greater flexibility than the linear combination. Unfortunately, this approach showed similar behavior and results as the linear combination, see Figure 22.

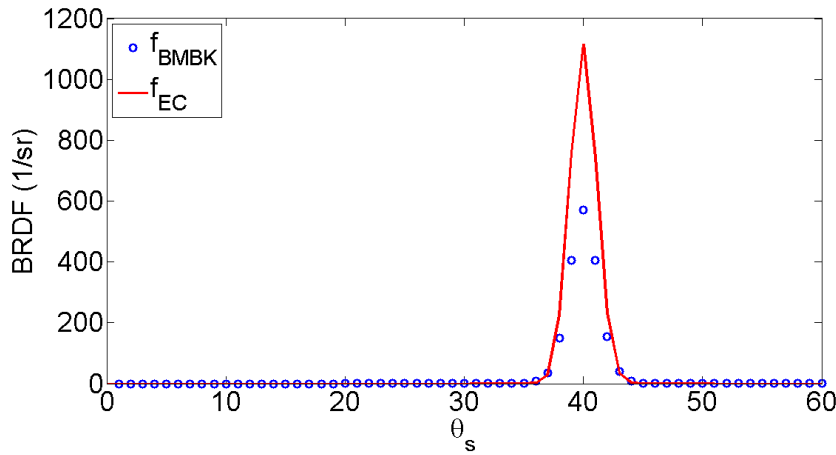


Figure 22. Exponential combination approximation where $\theta_i = 40^\circ$, $l_c/\lambda = 15$, $\sigma_s/\lambda = 0.1$

Notice how similar this is to Figure 20. The exponential combination appears to have the same problem as the linear combination: both the polished and very rough approximations are much higher in magnitude than the BMBK in the region of medium roughness. Because both combination approximations failed, the reason why becomes more clear. σ_s and l_c are quite different for rough and smooth surfaces. However, in these combinations, only one value of σ_s and one value of l_c were used. Neither the polished nor very rough approximation was useful because the values of σ_s and l_c are outside the range of applicability. Because of this problem, combination techniques were abandoned in favor of more promising approaches.

3.3 Mathematical Simplification

The previous approximation techniques were simple initial guesses and did not work. Any other guesses at approximation via combination seemed far-fetched and complicated, so a different approach was taken. Mathematical simplifications and approximation were attempted. The difficulty in the BMBK model is the large summation, so it may be useful to look more closely at it. Here it is shown explicitly on its own.

$$Y = \sum_{m=1}^{\infty} \frac{g^m}{m!m} \exp \left[-\frac{\pi^2 l_c^2}{m\lambda^2} (\cos \theta_i + \cos \theta_s)^2 \tan^2 \theta_h \right] \quad (35)$$

Because this expression already contains g dependence, it can be simplified by writing the exponential in terms of g as well

$$Y = \sum_{m=1}^{\infty} \frac{g^m}{m!m} \exp \left[-\frac{l_c^2}{4m\sigma_s^2} g \tan^2 \theta_h \right] \quad (36)$$

This expression can be shortened further by defining the coefficient of g as h

$$h = \frac{l_c^2}{4\sigma_s^2} \tan^2 \theta_h \quad (37)$$

These simplifications yield the expression below, which is equivalent to Equation (35)

$$Y = \sum_{m=1}^{\infty} \frac{g^m}{m!m} \exp \left[-\frac{hg}{m} \right] \quad (38)$$

It is important to know whether or not this series converges. If not, no exact analytic solution can be found. If it does converge, then an analytic solution exists. A simple way to show convergence is the ratio test [1]. We will take the ratio of the n th and $n + 1$ terms

$$L = \lim_{n \rightarrow \infty} \left| \frac{Y_{n+1}}{Y_n} \right| \quad (39)$$

If $L < 1$, the series converges. To check this, first the ratio of the inside of the summations is taken

$$L = \lim_{m \rightarrow \infty} \left| \frac{\frac{g^{m+1}}{(m+1)!(m+1)} \exp\left(-\frac{hg}{m+1}\right)}{\frac{g^m}{m!m} \exp\left(-\frac{hg}{m}\right)} \right| \quad (40)$$

This can be simplified into the form

$$L = \lim_{m \rightarrow \infty} \left| \frac{mg}{(m+1)^2} \exp \left(\frac{hg}{m(m+1)} \right) \right| = 0 \quad (41)$$

At infinity, the exponential term approaches 1 and the fraction approaches 0. Therefore, $L < 1$ and the series converges, so an analytical solution might be found. A few different approaches of finding the solution to the summation are detailed below.

3.3.1 Integration.

One possible solution is to convert this summation into an integral. In order to convert this expression into an integral, a limit needs to be taken at infinity. That is, our expression needs to fit the form

$$Y = \lim_{n \rightarrow \infty} \sum_{m=1}^{\infty} \frac{g^m}{m!m} \exp \left[-\frac{hg}{m} \right] n \frac{1}{n} \quad (42)$$

The reason $\frac{1}{n}$ is introduced is that a continuous integrating variable is needed which approaches 0 as n approaches infinity. It should be noted that $\frac{1}{m}$ is an unacceptable integrating variable because it is discrete. Therefore a continuous variable must be introduced. The current form of this expression is trivial, however, because the limit approaches our initial expression. One way to approach this problem is to use a substitution. It would have to introduce an integrating variable and leave behind an expression with an analytic limit. One substitution that could be used is $t = m/n$, giving the form

$$Y = \lim_{n \rightarrow \infty} \sum_{m=1}^{\infty} \frac{g^{tn}}{t\Gamma(tn+1)} \exp \left[-\frac{hg}{tn} \right] \frac{1}{n} \quad (43)$$

where Γ is the Gamma function, a continuous version of the factorial. The success with this substitution is that we have an integrating variable $\frac{1}{n}$. The problem with this substitution is that there is n dependence in the numerator and denominator. Also, both remain after a derivative is taken if L'Hôpital's rule were used. Therefore, this expression would only be integrable if the dependence on n could be eliminated. No substitutions were found to satisfy an elimination of the n dependence.

If one were found, the expression to be integrated would probably be difficult to solve analytically. One helpful step that might be taken is to notice that there is a derivative buried inside the expression, taking the form of

$$du = \frac{d}{dg} \exp\left(-\frac{hg}{m}\right) = -\frac{h}{m} \exp\left(-\frac{hg}{m}\right) \quad (44)$$

Therefore, the expression could be rewritten as a multiplication of terms and a derivative. If such a form existed, it might be solved by integration by parts, where dv would necessarily be

$$dv = \frac{g^m}{m!m} \quad (45)$$

Unfortunately, the non-derivative term dv includes an exponent and a factorial, which would have to be integrated to accommodate this approach by defining v . Again, in order to solve it this way, a useful substitution would have to be introduced and then the applicability of integration by parts would have to be re-evaluated because the substitution may change the form of the equation.

3.3.2 Taylor Series Expansion.

Another possible mathematical approach is simplifying this expression by using a Taylor series expansion. A Taylor series expansion was done around $g = 0$ because g determines how smooth or rough a material is, which drastically affects BRDF. Expanding around 0 was done as an initial step. The solution about 0 should be simpler to solve than any other constant, making it a good first step. This expansion yielded this expression, where the first three terms are shown explicitly

$$g + g^2\left(\frac{1}{4} - h\right) + g^3\left(\frac{1}{18} - \frac{h}{8} + \frac{h^2}{2}\right) + \dots \quad (46)$$

This expansion can be rewritten as

$$Y = \sum_{n=1}^{\infty} g^n \sum_{c=0}^{n-1} \frac{-h^c}{c!(n-c)^{c+1}(n-c)!} \quad (47)$$

In doing the Taylor expansion, instead of one summation, now there are two. Moreover, these new expressions are not more easily integrable than our original expression; there are now two factorials. A literature search of discrete distributions yielded nothing which matched Equation (47). If one were found, a convergence identity would probably be found.

3.3.3 Substitution.

The factorial is in the denominator and the m in the exponent is also in the denominator. In order to change this, it may seem helpful to use a change of variables, such as $y = 1/m$, but it should be noted that

$$\frac{1}{(\frac{1}{m})!} \neq m! \quad (48)$$

In fact, $1/m$ is not an integer, so the Gamma function would have to be used. Also, any other substitution involving the argument of the factorial being a non-integer would require use of the Gamma function. Using the Gamma function for m is outlined in the next section.

3.3.4 Mathematical Approximation.

These examples show that the presence of $m!$ makes simplification difficult. Also, it leads to very large values for g^m and $m!$ during numerical analysis. MATLAB® was used extensively for analysis in this research. Its largest double-precision number is 1.7977×10^{308} . The range of m needed to do a thorough analysis is 1 to 920, but double-precision is surpassed by 171!. For these reasons, a few analytical approaches

were taken to change the factorial dependence. It should also be noted that the g^m term also causes numerical difficulty at large g or large m . The maximum value of g analyzed in this research was 800. At this value, g^m increases beyond double precision limits when $m = 107$. Therefore, both of these difficulties must be addressed to facilitate error analysis. One approach was to use the definition of the Gamma function to rewrite the factorial in terms of an integral [14].

$$\Gamma(m+1) = \int_0^\infty x^m \exp(-x) dx \quad (49)$$

This substitution, along with some simplification, yields the equation

$$Y = \sum_{m=1}^{\infty} \frac{1}{m \int_0^\infty (\frac{x}{g})^m \exp(-x + \frac{hg}{m}) dx} \quad (50)$$

The integral in the denominator does not have an analytical solution. Also, this summation would have to be converted to an integral, as mentioned previously. This form does not make it any simpler to convert the summation into an integral.

Another approach to eliminate the factorial is to use an approximation of the factorial function. Two approximations that are explored are the Stirling and the Gosper expressions. These approximations stem from a common derivation which involves an infinite series form of the factorial. The Stirling approximation truncates the series after the first term via a limit [14], whereas the Gosper approximation approximates all subsequent terms [32]. The Stirling approximation is written as

$$m! \approx m^m \exp(-m) \sqrt{2\pi m} \quad (51)$$

The Gosper approximation is written as

$$m! \approx m^m \exp(-m) \sqrt{2\pi m + \frac{\pi}{3}} \quad (52)$$

These approximations decrease in relative error as m increases. Relative error was calculated using the simple equation

$$\text{RelErr} = \frac{|y_t - y_a|}{y_t} \quad (53)$$

where y_t is the true function and y_a is the approximation.

The relative error and behavior of these approximations are shown in Table 1 and Figures [23, 24]

Table 1. Relative Errors in Stirling and Gosper Approximations

x	Stirling	Gosper
1	0.0779	0.0040
2	0.0405	0.0013
3	0.0273	0.0006
4	0.0206	0.0004
5	0.0165	0.0002

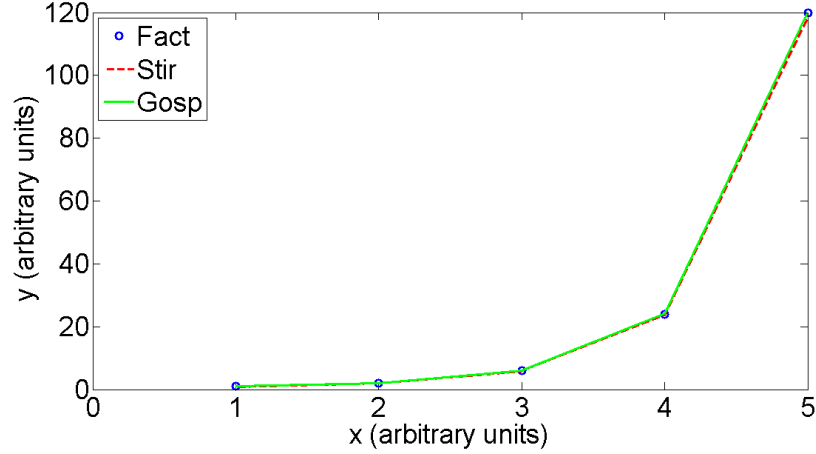


Figure 23. The factorial and the Stirling and Gosper approximations

The Gosper approximation can be substituted into the summation expression, giving

$$Y = \sum_{m=1}^{\infty} \left(\frac{ge}{m}\right)^m \frac{1}{m} \exp\left[-\frac{hg}{m}\right] \frac{1}{\sqrt{2\pi m + \frac{\pi}{3}}} \quad (54)$$

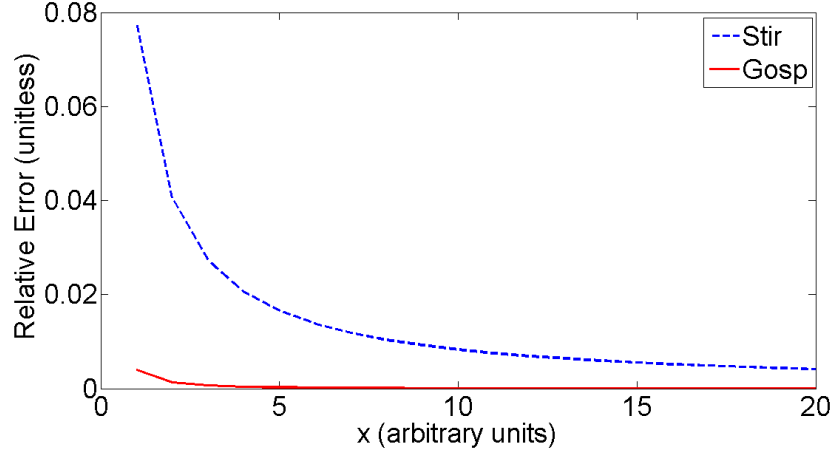


Figure 24. Relative error of the Stirling and Gosper approximations

This accomplishes two useful things. First, our sum is no longer expressed in terms of a factorial. Second, the g^m term is no longer prone to exceed double precision limits because as m increases, the quantity being exponentiated decreases. Changing the expression into this approximation may facilitate analytical simplification, although none has been found at this point.

Purely mathematical approaches to simplify the sum did not yield simpler forms. However, a few approaches have been ruled out and the Stirling or Gosper approximations have been proposed as potential steps in future analytical work.

3.4 Analysis of Variable Ranges

Because mathematical simplification did not yield results, a combination of mathematical and numerical approaches was taken to approximate the sum. For this analysis, the form of the BMBK used was Equation (38). Once in this form, the summation is expressed in terms of only three variables: h , g , and m . It is useful in numerical work to determine what ranges these three variables can have.

Because σ_s , the surface parameter, is in the denominator of h , h can approach infinity. $\tan^2 \theta_h$ can range from 0 to infinity. θ_h is the half-angle between incident and

reflected angles. If non-grazing, non-backscatter is assumed, then $\tan^2 \theta_h$ is less than 2. However, because of σ_s in the denominator, this term still can approach infinity. Because $\tan^2 \theta_h$ is in the numerator, h can also be as small as 0. Therefore, it appears as though h needs to be analyzed from 0 to infinity. A similar analysis can be done on g . Since σ_s/λ can be 0, the smallest possible value of g is 0. The largest value also depends on σ_s/λ , which can be very large. However, for very small and very large g , there are already analytical approximations Equations (28) and (29). Therefore, g must be analyzed in the region not already covered by approximations, so between 0.025 and 800. Lastly, the number of terms of m which must be kept is determined by Equation (30), the largest possible result being under 1000.

What stands out among these ranges is the upper limit of h , which is infinity. Having a variable range from 0 to infinity poses a problem: one would have to find asymptotic behavior of the sum when this variable approaches infinity as well as non-asymptotic behavior when h is small. This is not trivial because the behavior would depend on h , g , and m . Therefore, it would be beneficial if a narrower range were found.

First, the components of h will be analyzed. The farther off from specular the scattered angle is, the larger $\tan^2 \theta_h$ is. Therefore, h will also become larger for angles away from specular. This means that if the range of h was limited, the errors would be most significant far from specular. To quantify this analysis, the summation was looped for incident and scattered angles. If h was greater than 280, then the summation was approximated as 0. The relative error was then taken. When g was very small (≤ 0.025), this limited range of h performed poorly. However, for $g \leq 0.025$, the polished approximation can be used. When l_c/λ and g are fixed, the incident and reflected angles can vary. Because the KMBK derivation assumed non-grazing angles, the angles currently examined are $\theta_i \leq 80$ and $\theta_s \leq 80$. A contour

plot of the greatest relative error for any of these angles is shown in Figure 25

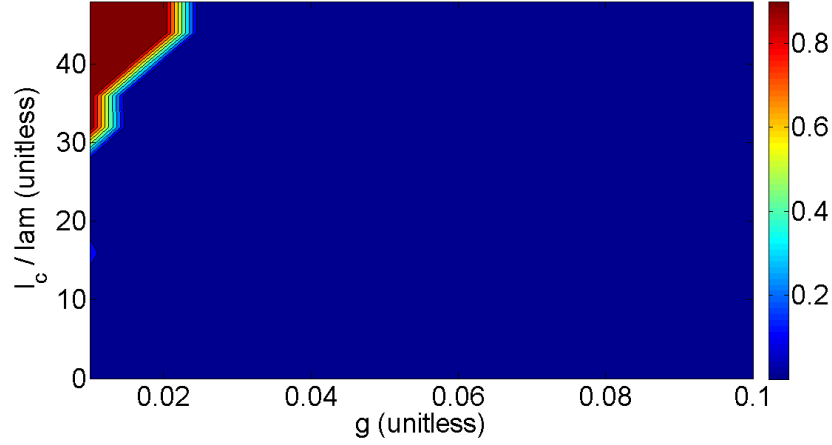


Figure 25. As g increases, limited h produces less error

Larger g ensures a smaller error when h is limited. It can be hard to discern quality of fit from a contour plot like this. The greatest error when $g \geq 0.025$ was found when $g = 0.025$, $l_c/\lambda = 32$, and $\theta_i = 40^\circ$. The BMBK and the h cutoff are shown in Figure 26.

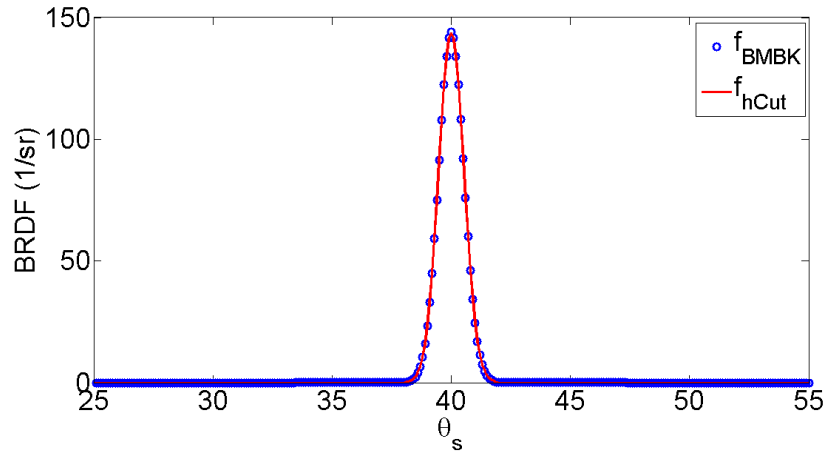


Figure 26. Low error at large g , small h

It may seem counter-intuitive that large g means a lower h value is acceptable. However, this behavior results because a large g indicates a rough surface and rough surfaces have more significant off-specular behavior. However, h has an inverse square

relationship with σ_s . This relationship dominates the magnitude of h when σ_s , and therefore g , is large. So, h will be small at large σ_s . Therefore, limiting the range of h to $h \leq 100$ will have smallest error at large g . There is also a slight angular dependence because $\tan^2 \theta_h$ depends on incident and scattered angle. It is smallest near specular, where $\tan^2 \theta_h = 0$ and is largest off specular, where $\tan^2 \theta_h \approx 2$. So, as mentioned before, limiting the range of h yields greatest error at diffuse angles.

Relative error was calculated comparing the summation using all possible values of h and the summation using $h \leq 100$. This was done for all possible values of $\frac{\sigma_s}{\lambda}$, $\frac{l_c}{\lambda}$, θ_i , and θ_s . The maximum relative errors for all $\frac{l_c}{\lambda}$ values were calculated as a function of $g(\sigma_s, \theta_i, \theta_s)$. Table 2 shows the maximum relative errors as a function of g .

Table 2. Error in limiting the range of h to $h \leq 10$ and $h \leq 100$

g	Max Rel Err $h \leq 10$	Max Rel Err $h \leq 100$
0.01	1	0.4796
0.1	0.487	0.004
1	0.0071	0.003
10	7.4E-4	1.6E-5

This shows that for $g \geq 1$, a range of $h \leq 100$ can be used. This means that h now has an upper bound of 100. Keep in mind that the non-grazing assumption has been made. Table 3 shows the variable ranges.

Table 3. Useful ranges as determined by variable ranges

Variable	Lower Limit	Upper Limit
m	1	1000
h	0	100
g	0.025	800

This is a great improvement over the prior range. However, if the range of g is narrowed, then the range of h is narrowed even further. It will be shown later in this thesis that an even smaller range of analysis is desirable. This was done where h was limited to $h \leq 10$. The result was that if g is limited to a range of $g \geq 1$, the largest

error in the cutoff method is 0.0071. This slightly narrowed region of g yields a much smaller region of h , which facilitates analysis. Table 4 shows the new ranges of the variables. It should be noted that if this range were solved for, there would be a gap in applicability where $0.025 \leq g \leq 1$ because the polished approximation is invalid in this range. However, if a suitable approximation for $1 \leq g$ were found, then perhaps a simple approximation could be found for the range not covered.

Table 4. Useful ranges assuming a large g

Variable	Lower Limit	Upper Limit
m	1	1000
h	0	10
g	1	800

It may be useful to give a visual understanding of how h and g behave as a function of θ_s . First, Figure 27 shows g for a specular material.

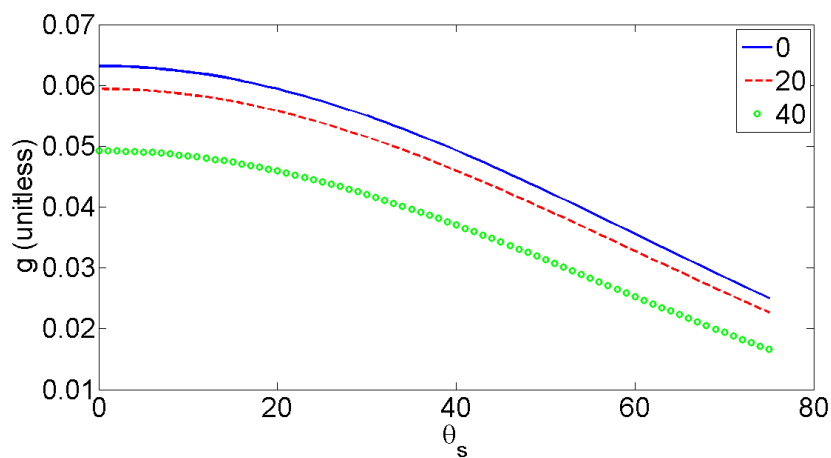


Figure 27. Behavior of g for a specular material where $\sigma_s/\lambda = 0.02$ and $l_c/\lambda = 5$ at three incident angles, 0, 20, and 40

As expected, for a specular material, g is small. For a diffuse material, g will be larger, as shown in Figure 28

A BRDF curve is usually shown as a function of θ_s . The variable g is also a function of θ_s . Therefore, for one BRDF curve, there will be several values of g . Both

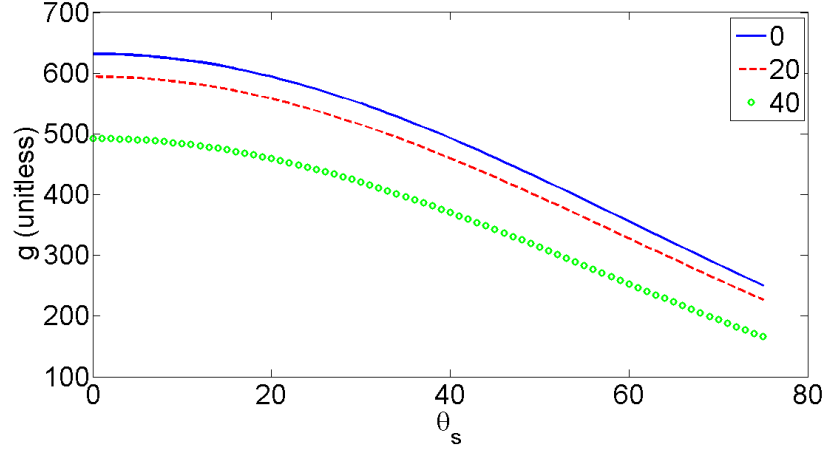


Figure 28. Behavior of g for a diffuse material where $\sigma_s/\lambda = 2$ and $l_c/\lambda = 20$ at three incident angles, 0, 20, and 40

normalized g and normalized BRDF are plotted together to show this relationship in Figure 29.

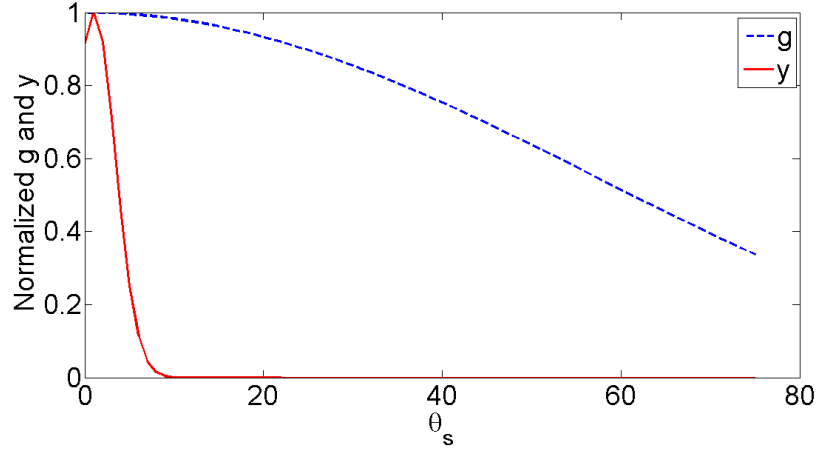


Figure 29. Behavior of normalized g and normalized BRDF a diffuse material where $\sigma_s/\lambda = 2$, $l_c/\lambda = 20$, and $\theta_i = 20^\circ$

It is shown that g is largest at near-normal angles and diminishes at grazing angles.

The variable h can be shown visually as well. For a specular material, h will range from 0 to very large values, as shown in Figure 30

The most significant contribution to the BRDF happens at the specular peak. At the specular peak, $h = 0$. This can be shown with a semilog plot as well. To observe the

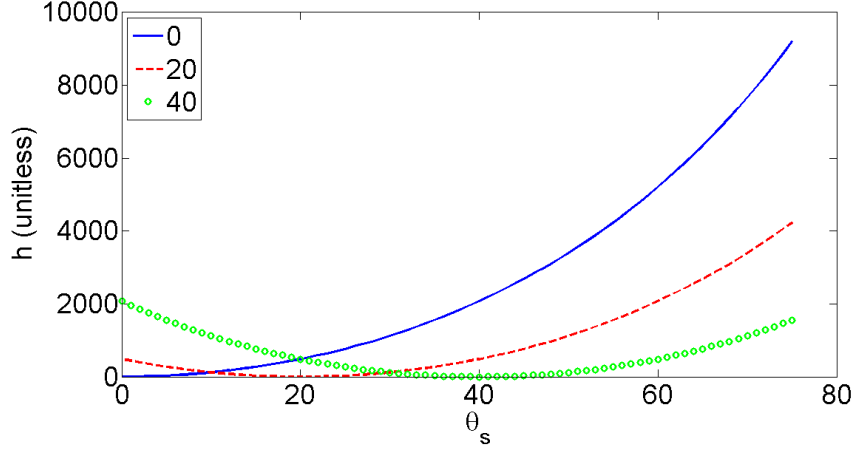


Figure 30. Behavior of h for a specular material where $\sigma_s/\lambda = 0.02$ and $l_c/\lambda = 5$ at three incident angles, 0, 20, and 40

behavior when h is small, see Figure 31

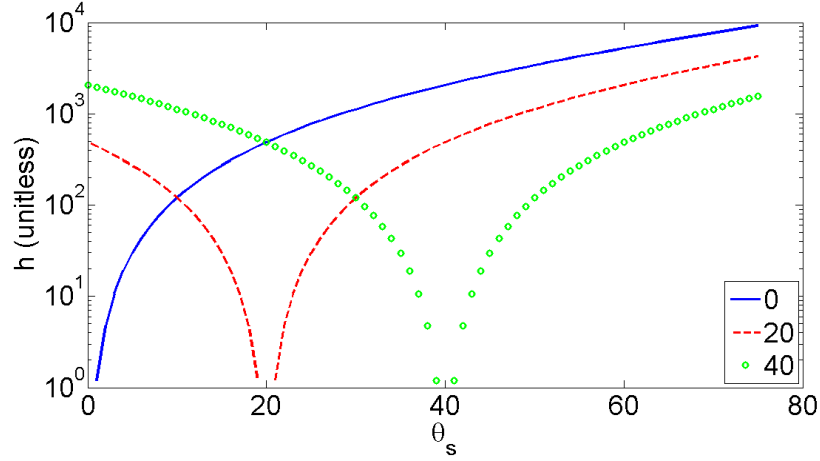


Figure 31. Semilog behavior of h for a specular material where $\sigma_s/\lambda = 0.02$ and $l_c/\lambda = 5$ at three incident angles, 0, 20, and 40

The fact that h becomes very large for a specular material should not be surprising because it was shown that h can only be assumed small for diffuse surfaces. Figure 32 shows how h varies by angle for a diffuse surface.

This shows that h stays small for a diffuse surface. Remember that $h = 0$ at the specular angle. One practical implication of these two facts is that smaller values of h are the most significant, especially for diffuse materials.

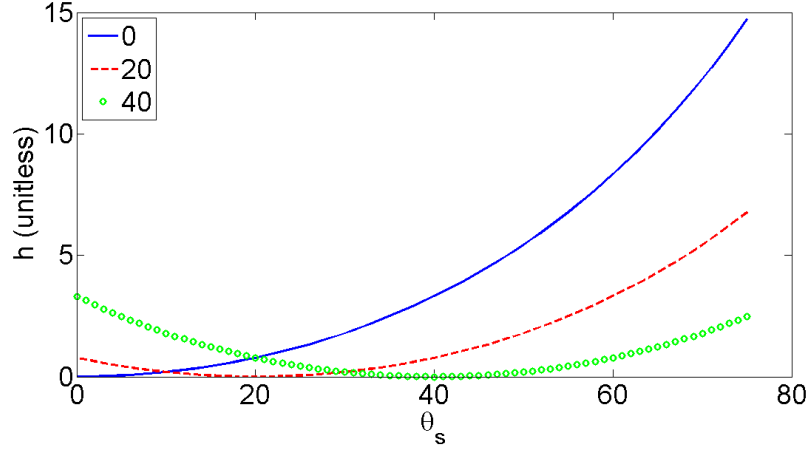


Figure 32. Behavior of h for a diffuse material where $\sigma_s/\lambda = 2$ and $l_c/\lambda = 20$ at three incident angles, 0, 20, and 40

Another implication of this analysis is that it shows what forms of the BMBK can be used for specific g values. It was discussed previously that the BMBK can pose numerical problems in its original form. When calculating the BMBK for small values of g , the original expression was used. The factorial of m and g^m are not problematic for small g . Remember that small g means small m by Equation (30). At the large, problematic ranges of g , the Gosper approximation was used as shown in Equation (54). Remember, the Gosper approximation increased in accuracy with m .

It should be noted that Krywonos' derivation assumed non-grazing angles of θ_i and $\theta_s < 75$. This analysis was done up to θ_i and $\theta_s < 75$. Therefore all relevant angular dependence has been taken into account.

In summary, by looking at the relationship between variables h and g , narrower ranges of analysis can be found and approaches to numerical calculation can be justified. A smaller range makes analysis more practical.

3.5 Distribution Approximation

Now that ranges for θ_i , θ_s , σ_s , and l_c have been found, approximating the summation at non-grazing angles may be simpler. It may be possible to approximate the sum or its terms by using a distribution. This would be advantageous because distributions are well-known and have been analyzed extensively. They often have approximate forms which could be useful. The individual terms of the summation are expressed as y

$$y = \frac{g^m}{m!m} \exp\left(-\frac{hg}{m}\right) \quad (55)$$

3.5.1 Maximums of the Terms of the Summation.

Many distributions contain some form of the peak value or location. Generally speaking, maximums happen when the first derivative is zero and the second derivative is negative, indicating a negative concavity. In the case of this summation, it was found to only contain one maximum and minima only at $m = 0$ or $m = \infty$. This means that the first derivative will find the location of the maximum. The first derivative of y was taken with respect to m using Mathematica®. This was done over m because the terms of the summation are being approximated by a distribution. This means that m is acting as an independent variable. The derivative was taken, giving

$$\frac{dy}{dm} = \frac{1}{m^3(m!)^2} \exp\left(-\frac{hg}{m}\right) g^m \left[m!(hg - m + m^2 \log g) - m^2 \Gamma(m+1) \psi(m+1) \right] \quad (56)$$

This derivative can be simplified in a few ways. First, the gamma function can be simplified to a factorial because m is an integer. Second, the derivative can be set

equal to 0 in order to solve for the maximum values. Then the common numerator and denominator can be factored out.

$$\frac{dy}{dm} = m!(hg - m + m^2 \log g) - m^2 m! \psi(m+1) = 0 \quad (57)$$

Now it can be simplified by dividing by a common $m!$ term. Also, the definition of ψ , the polygamma function, can be used

$$\psi(m+1) = \frac{\Gamma'(m+1)}{\Gamma(m+1)} = \frac{m!(-\gamma + \sum_{j=1}^m \frac{1}{j})}{m!} = -\gamma + \sum_{j=1}^m \frac{1}{j} \quad (58)$$

where γ is the Euler–Mascheroni constant. If this definition is substituted back into the first derivative, it becomes

$$\frac{dy}{dm} = hg - m + m^2 \log g - m^2 \left(-\gamma + \sum_{j=1}^m \frac{1}{j} \right) = 0 \quad (59)$$

The results of the summation in Equation (59) are known as harmonic numbers [30]. They are approximated by

$$H \approx \log m + \gamma + \frac{1}{2m} - \frac{1}{12m^2} \quad (60)$$

This approximation has a maximum relative error of 0.006. If this is substituted back into Equation (59), the first derivative now appears like

$$\frac{dy}{dm} = hg - m + m^2 \log g - m^2 \left(\log m + \frac{1}{2m} - \frac{1}{12m^2} \right) = 0 \quad (61)$$

In this form, the first derivative does not contain a gamma function or a summation. However, it is still not analytically solvable. If a solution for m were found, it might be used within a distribution. This would result in the mean or peak parameter being mathematically derived instead of fitted.

3.5.2 Gaussian Approximation.

There are numerous distribution functions that may approximate the terms of the summation. One, however, stands out as a possibility. The Gaussian distribution is suitable to approximate the terms of the summation because the terms are roughly symmetric and Gaussian surface statistics were assumed. The Gaussian form used in this research was simply

$$y_{\text{gauss}} = a_g \exp \left[- \left(\frac{x - \mu}{\sigma} \right)^2 \right]. \quad (62)$$

where σ , the Gaussian standard deviation, should not be confused with neither σ_s , the surface height, nor σ_{cs} , the cross-section conversion term.

As analysis was done, it was noted that the terms y seemed to have a Gaussian shape for a fixed h and g . For example, when Equation (55), where $h = 0$ and $g = 200$, is plotted against a Gaussian curve, a close fit with an R^2 value of 0.99 is obtained, see Figure 33

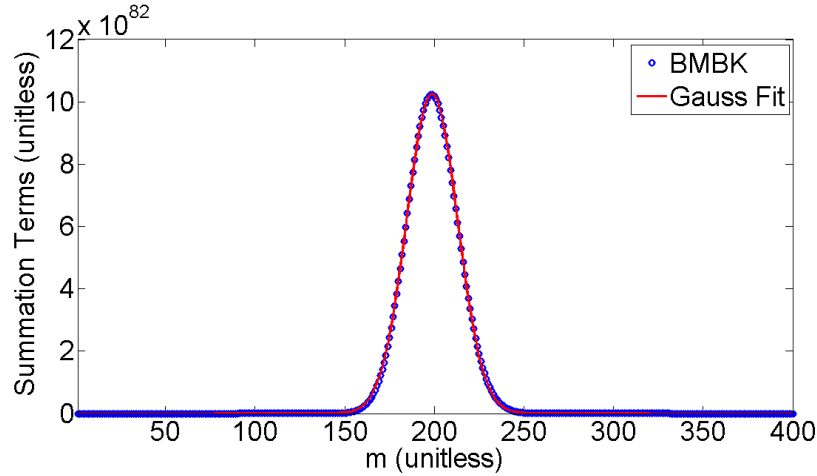


Figure 33. BMBK summation terms with a Gaussian fit. The terms of the sum appear Gaussian when $h = 0$ and $g = 200$

Because this looks Gaussian, it might be possible to approximate the terms of the summation as Gaussians. In order to fit Gaussians to different values of h and

g , the height a , mean μ , and standard deviation σ would vary as functions of h and g . However, further inspection suggested that the Gaussian distribution may not accurately describe the terms of the summation. In some ranges of h and g , the fit is good, but in others, it is very inaccurate. For example, when Equation (55), where $h = 20$ and $g = 2$, is plotted against a Gaussian curve, a bad fit with an R^2 value of 0.96 is obtained, see Figure 34.

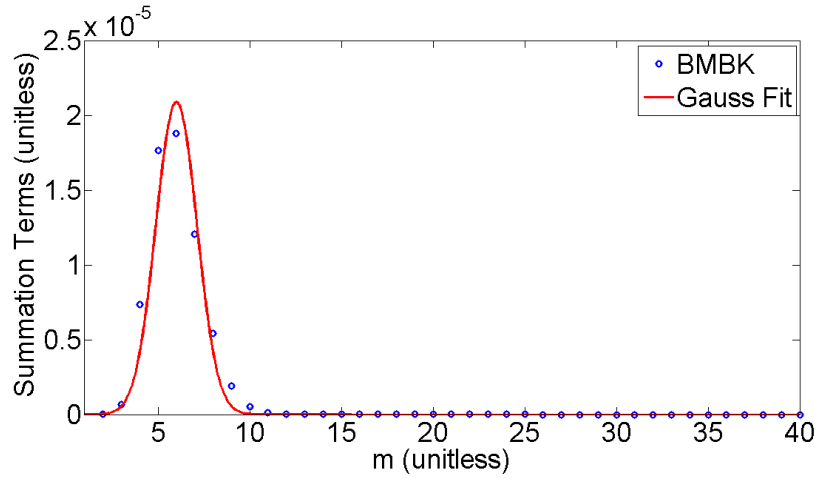


Figure 34. BMBK summation terms with a Gaussian fit. The terms of the sum do not appear Gaussian, where $h = 20$ and $g = 2$

A complete analysis of where the Gaussian approximation could be made was performed by comparing the Gaussian approximation to Equation (55). It was found that the summation could be accurately approximated using a Gaussian distribution for all the variable ranges shown in Table 4, essentially $g \geq 10$. In this range, the smallest that σ_s/λ can be is 0.2516. The largest relative error between the original summation and the Gaussian summation in this range was 0.018. An example of how accurately the Gaussian distribution was able to describe the inside of the summation was shown in Figure 33. A successful fitting would result in a few gains. First, the factorial and g^m would be eliminated. Second, the new form of the summation may be closer to an analytical solution.

In order for this approximation to be useful, a_g , μ , and σ , the parameters of the Gaussian, had to be described as functions of h and g . This means that for specific h and g values, the Gaussian would be only a function of m . Then the Gaussian equation could be substituted in for y . In order to find suitable functional forms of a_g , μ , and σ , a few steps were taken. First, Gaussian fits were done for a normalized BRDF at varying h and g values. One of those fits is shown in Figure 35.

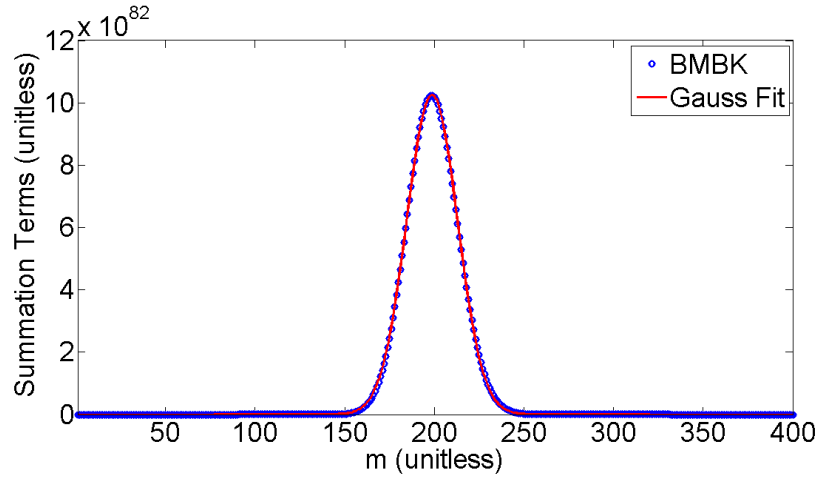


Figure 35. BMBK summation terms with a Gaussian fit. The terms of the sum appear Gaussian when $h = 0$ and $g = 200$

Similar to Figure 33, the terms of the summation appear to match a Gaussian distribution. For these values of h and g , the mean of the Gaussian was 267 and the standard deviation was 22.

The next step was that the parameters μ and σ from these fits were extracted into two dimensional matrices. They are two dimensional because they depend on h and g . Figures [36, 37] show the Gaussian mean as a function of h and g as a contour plot and mesh plot, respectively. Figure 38 shows a mesh plot of the standard deviation.

The last step involves the mean and standard deviation being described as functions of h and g . If this can be done, then the summation can be written as a Gaussian. The mean and standard deviation were fitted as a function of h and g

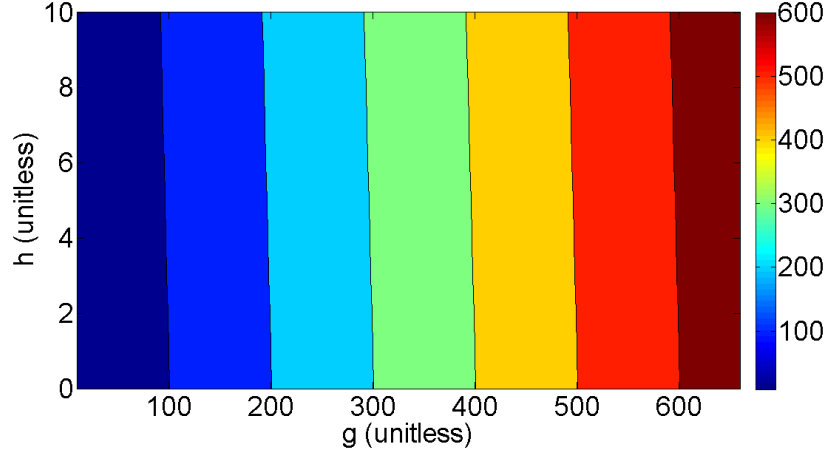


Figure 36. Contour representation of the Gaussian mean

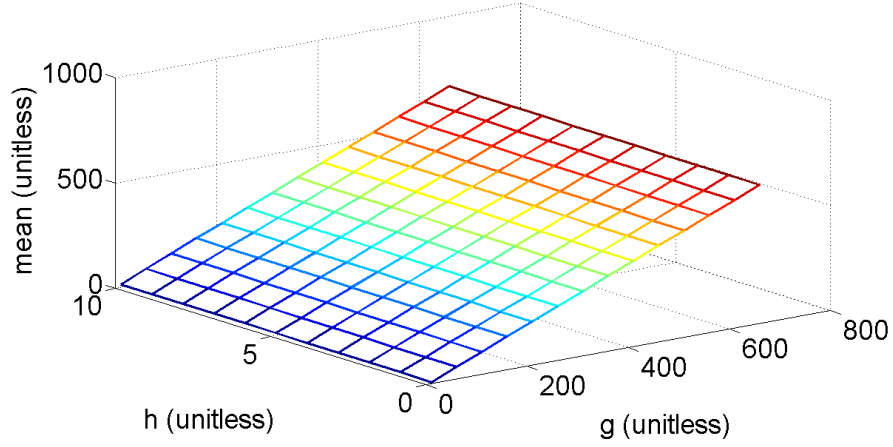


Figure 37. Mesh grid representation of the Gaussian mean

using the MATLAB® CFtool. This is a GUI designed for one and two dimensional fitting. A screenshot is shown in Figure 39.

After analyzing several possible functional forms, the following equation was found for μ , the Gaussian mean.

$$\mu = -1 + p_{\mu 1}h + g \quad (63)$$

where $p_{\mu 1} = 0.91$. Alternatively, using physical variables, the mean can be expressed

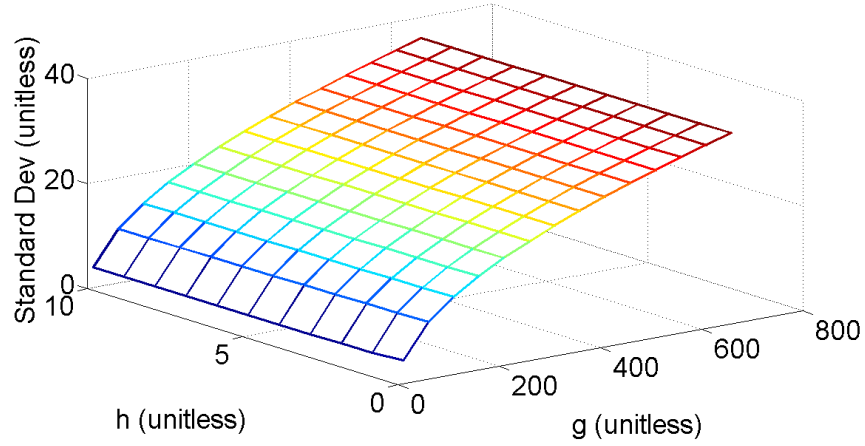


Figure 38. Mesh grid representation of σ , the Gaussian standard deviation parameter

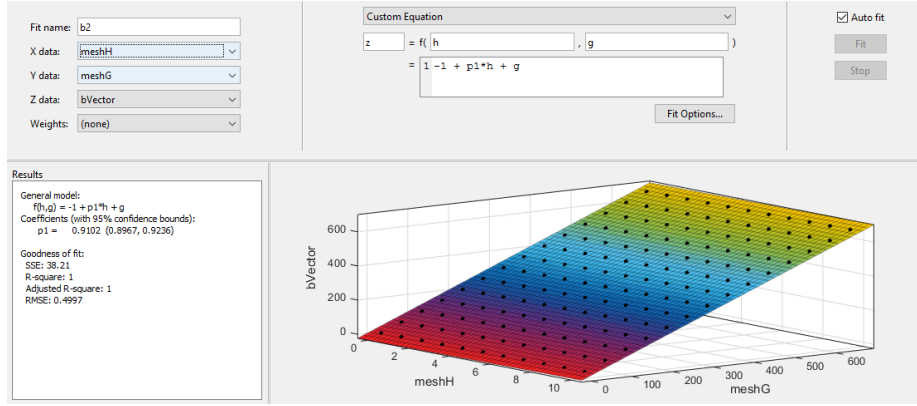


Figure 39. MATLAB® CFTool representation of Gaussian mean. The custom fitting equation used is shown near the top

$$\mu = -1 + p_{\mu 1} \frac{l_c^2}{4\sigma_s^2} \tan^2 \theta_h + \left(\frac{2\pi\sigma_s}{\lambda} \right)^2 (\cos(\theta_i) + \cos(\theta_s))^2 \quad (64)$$

It should be noted, that if an expression for the locations of the maxima were found, as discussed in the previous section, an analytical expression for μ would be obtained. This would be preferable to a fitted form because it would contain no approximation. Figure 40 shows typical behavior of the mean as a function of θ_s .

The mean depends largely on g . This dependence can be seen by comparing Figure 40 with Figure 28. The shape of g drives the shape of μ . This behavior is accentuated as

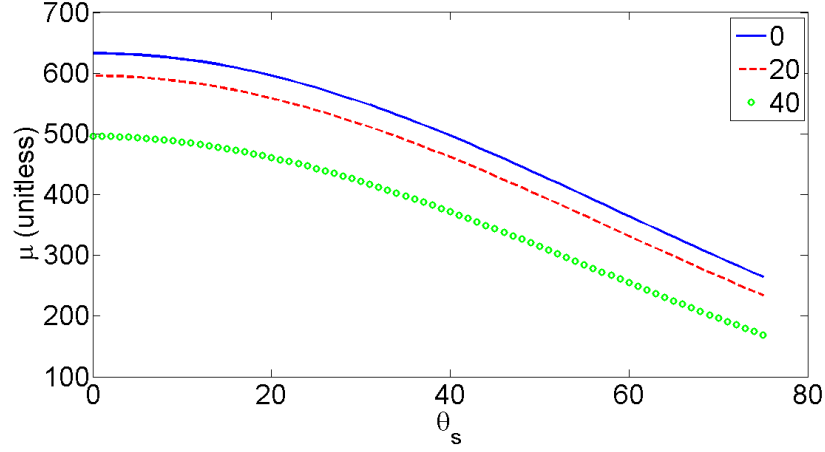


Figure 40. Behavior of μ for a specular material where $\sigma_s/\lambda = 1$ and $l_c/\lambda = 20$ at three incident angles, 0° , 20° , and 40°

a surface becomes more diffuse. Figure 41 shows the mean of a more diffuse surface, where $\sigma_s/\lambda = 2$

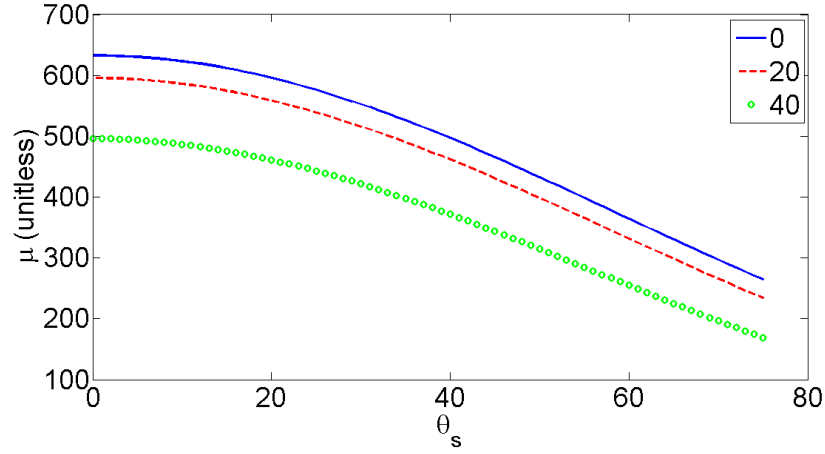


Figure 41. Behavior of μ for a specular material where $\sigma_s/\lambda = 2$ and $l_c/\lambda = 20$ at three incident angles, 0° , 20° , and 40°

The fitted functional form for σ was

$$\sigma = p_{\sigma 1} \sqrt{h} + p_{\sigma 2} \sqrt{g} \quad (65)$$

where $p_{\sigma 1} = -\frac{1}{2\pi} = -0.16$ and $p_{\sigma 2} = \sqrt{2} = 1.4$. This can also be written as

$$\sigma = -\frac{1}{2\pi} \frac{l_c^2}{4\sigma_s^2} \tan^2 \theta_h + \sqrt{2} \left(\frac{2\pi\sigma_s}{\lambda} \right)^2 (\cos(\theta_i) + \cos(\theta_s))^2 \quad (66)$$

The visual behavior of σ can be seen in Figure 42

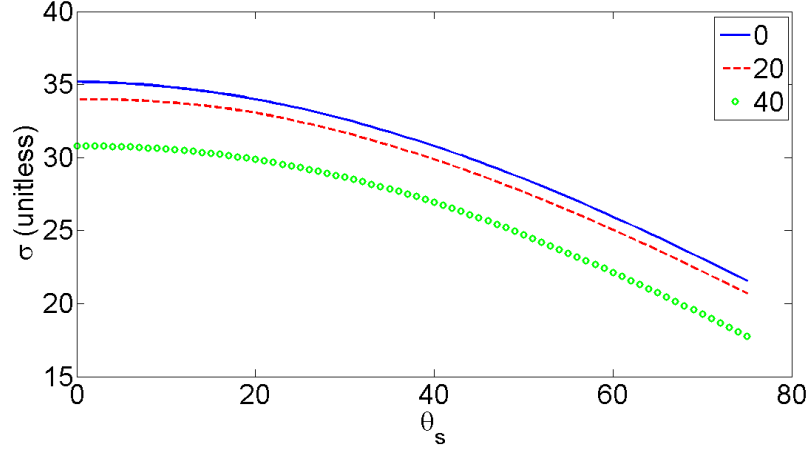


Figure 42. Behavior of Gaussian standard deviation for a specular material where $\sigma_s/\lambda = 1$ and $l_c/\lambda = 20$ at three incident angles, 0° , 20° , and 40°

As expected, the standard deviation depends largely on g . Figure 43 shows the standard deviation for a more diffuse surface.

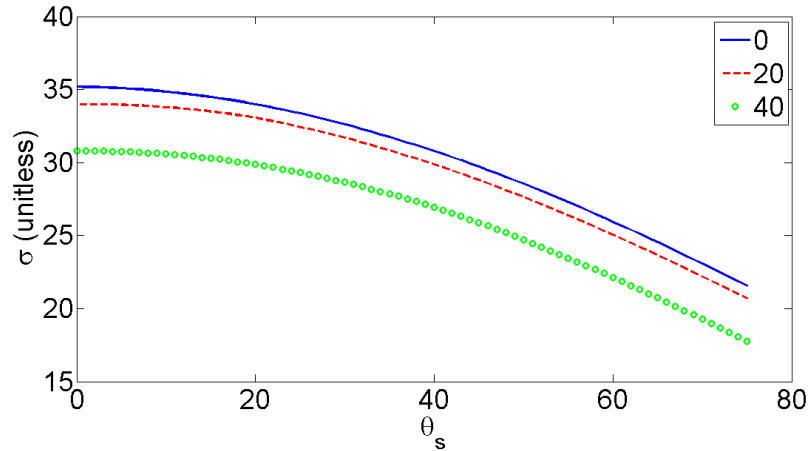


Figure 43. Behavior of Gaussian standard deviation for a specular material where $\sigma_s/\lambda = 2$ and $l_c/\lambda = 20$ at three incident angles, 0° , 20° , and 40°

Instead of fitting for the Gaussian height parameter a_g , it was estimated by sub-

stituting μ into the equation for y . This gives the value of y at the mean of the Gaussian, which is equivalent to finding the height of the Gaussian. The original expression for y and Gosper approximation are shown.

$$a_g = \frac{g^\mu}{\mu! \mu} \exp\left(-\frac{hg}{\mu}\right) \approx \left(\frac{ge}{\mu}\right)^\mu \frac{1}{\mu \sqrt{2\pi m + \frac{\pi}{3}}} \exp\left(\frac{-hg}{\mu}\right) \quad (67)$$

The Gosper approximation can be used on this expression because large g is assumed.

Once functional forms for the mean and standard deviation were found in terms of h and g , the Gaussian approximation could be used within the summation, in place of y .

$$Y_{\text{Gauss}} = a_g \sum_{m=1}^{\infty} \exp\left[-\left(\frac{m-\mu}{\sigma}\right)^2\right] \quad (68)$$

The lower limit of m could be written as 0 or $-\infty$ if convenient because the terms below 1 are insignificant when $g \geq 10$. Also, the upper limit could be changed to anything larger than 1000 because terms above 1000 are insignificant as well. Substituting this form of the summation into the BMBK expression yields Equation (69)

$$f_{\text{Gauss}} = \frac{K\pi l_c^2}{\lambda^2} \frac{FG(\cos\theta_i + \cos\theta_s)^2}{2\cos\theta_i \cos\theta_s \cos^4\theta_h} \exp(-g) \times a_g \sum_{m=1}^{\infty} \exp\left[-\left(\frac{m-\mu}{\sigma}\right)^2\right] \quad (69)$$

The amount of calculation within the summation can be reduced by pulling out terms that do not depend on m .

$$f_{\text{Gauss}} = \frac{aK\pi l_c^2}{\lambda^2} \frac{FG(\cos\theta_i + \cos\theta_s)^2}{2\cos\theta_i \cos\theta_s \cos^4\theta_h} a_g \exp\left[-g - \left(\frac{\mu}{\sigma}\right)^2\right] \sum_{m=1}^{\infty} \exp\left[\frac{-m^2 + 2m\mu}{\sigma^2}\right] \quad (70)$$

This will reduce calculation time. Moreover, $a_g \exp \left[-g - \left(\frac{\mu}{\sigma} \right)^2 \right]$ is *part* of the wavelength dependence due to the summation.

This parameterized Gaussian form succeeds in simplifying the original summation in two ways. First, the summation now does not contain a factorial expression. Second, there is no large exponentiation that approaches infinity like g^m did in the original expression. Its largest relative error when $g \geq 10$, compared with the BMBK, is 0.018.

If μ and σ , the mean and standard deviation, were perfectly described as a function of h and g , then the Gaussian approximation is very accurate. Figure 44 shows the relative error between the original summation and the Gaussian approximation.

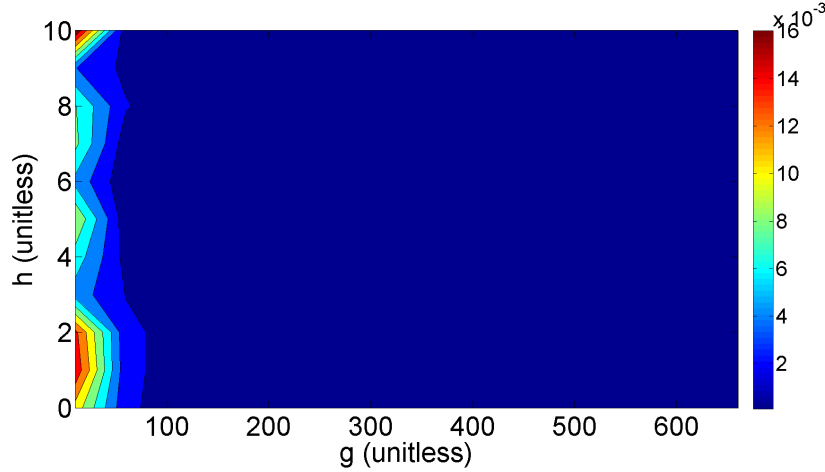


Figure 44. Relative error between the original summation and the Gaussian approximation

The largest relative error between the original and approximated summations is 0.018. Also the error in the approximation diminishes as g increases. Figure 45 shows how the maximum relative error in the Gaussian summation decreases with g .

This means that if a smaller relative error is desired, a smaller range of g could be used. It should be noted that the errors just reported are assuming an ideal parameterization of the mean, standard deviation, and height. The problem with

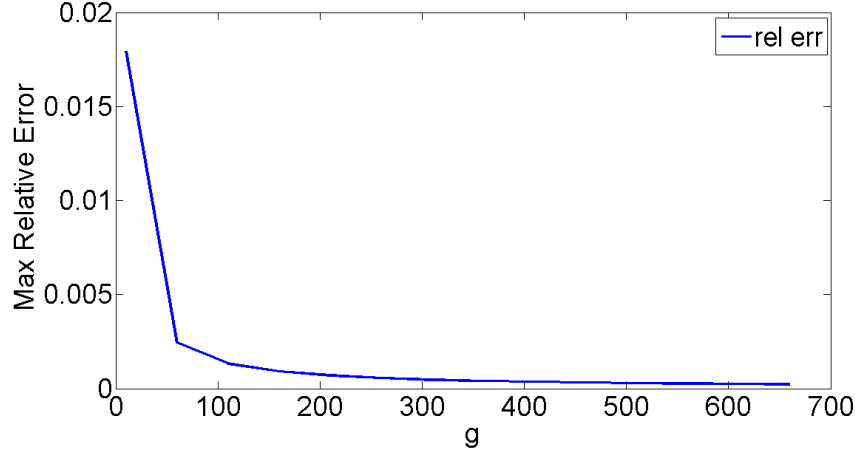


Figure 45. Maximum relative error between the original summation and the Gaussian approximation

the Gaussian approximation was that once a Gaussian form of the summation was written, Equation (68), it could neither be written into an integral nor solved directly.

3.5.3 Limiting the Number of Terms Needed.

The mean and standard deviation have been described mathematically. Because a Gaussian drops off quickly away from the peak location, it may be possible to limit the number of terms needed to describe the summation. In order to explore this, the number of terms needed will be explored with respect to standard deviation. The summation over a few terms around the peak will be compared to the full summation. Figure 46 shows the full summation compared to: the largest term, the sum of terms within $0.25 * \sigma$ of the peak, and the sum of terms within $1 * \sigma$ of the peak.

It should be noted that the peak of the summation does not occur at the specular angle. In fact, as θ_i increases, the location of the peak of the summation *decreases*. This behavior is seen when Figure 46 is seen in conjunction with Figure 47 which has an incident angle of 40.

When more terms are included, the abbreviated sum becomes closer to the full

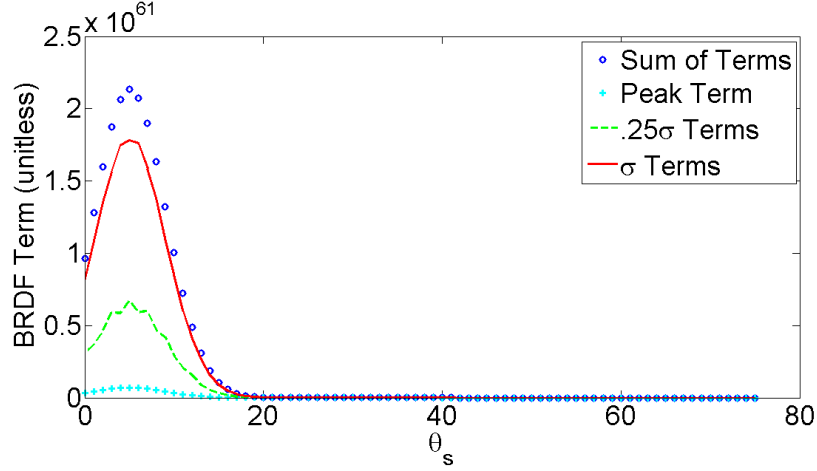


Figure 46. Full summation compared to summing over a limited number of terms where $\sigma_s/\lambda = 1$ and $l_c/\lambda = 20$, and $\theta_i = 20^\circ$

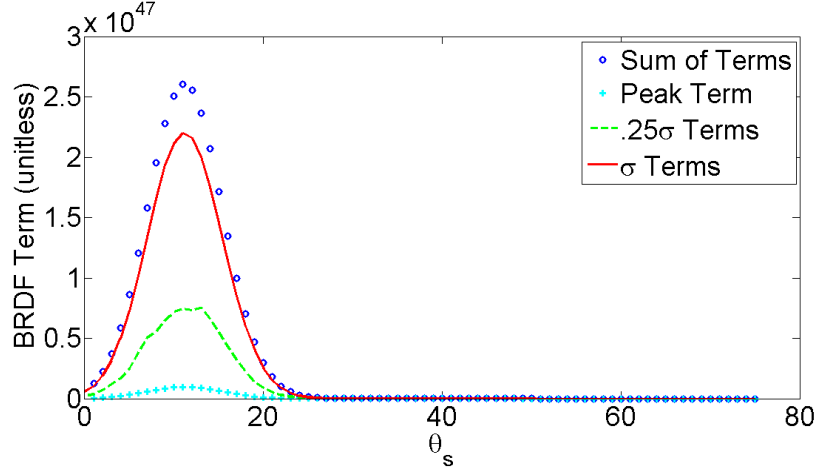


Figure 47. Full summation compared to summing over a limited number of terms where $\sigma_s/\lambda = 1$ and $l_c/\lambda = 20$, and $\theta_i = 40^\circ$. The peak has moved to a lower scattered angle.

sum. Figure 48 shows the full sum and two abbreviated sums.

The relative error of these two abbreviated sums is shown in Figure 49.

As shown in both of the previous figures, the accuracy increases as we include more terms near the mean. The question then becomes how many terms do we want to keep at the expense of computation time? Assuming that keeping 3 standard deviations is acceptable, a new equation for the number of terms needed can be found. This is

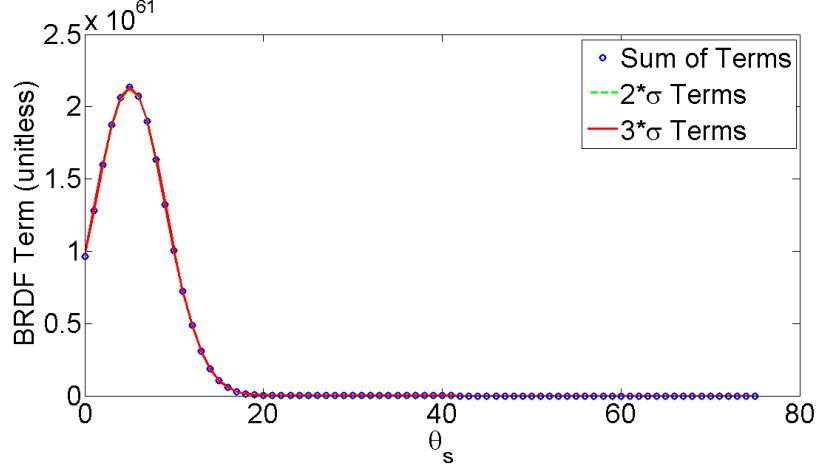


Figure 48. Full summation compared to summing over a limited number of terms where $\sigma_s/\lambda = 1$ and $l_c/\lambda = 20$, and $\theta_i = 40^\circ$. Both $2 * \sigma$ and $3 * \sigma$ appear accurate.

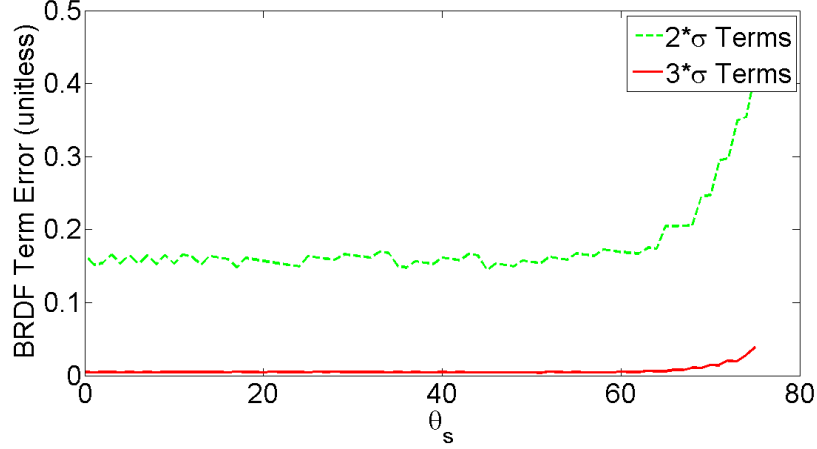


Figure 49. Relative errors of full summation compared to summing over a limited number of terms where $\sigma_s/\lambda = 1$ and $l_c/\lambda = 20$, and $\theta_i = 40^\circ$. $3 * \sigma$ appears to be much more accurate.

done by subtracting $\mu - 3 * \sigma$, the lowest index needed, from $\mu + 3 * \sigma$, the highest term needed.

$$N_{\text{Gauss}} = 8.5\sqrt{g} \quad (71)$$

This expression results in a smaller number of terms needed than Krywonos' original expression for the entire range of the Gaussian approximation: $g \geq 10$.

Figure 50 shows the number of terms needed graphically in terms of g .

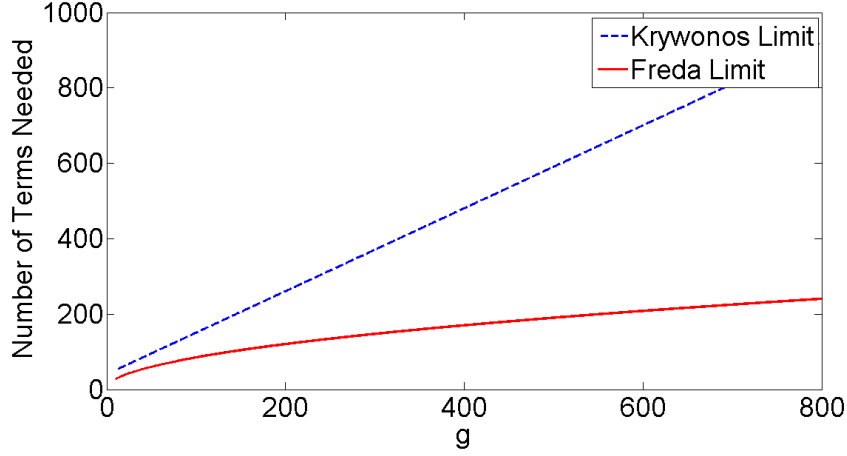


Figure 50. Comparison between Krywonos' and new expression for number of terms needed as a function of g . The new limit is much smaller at large g .

This relationship can also be expressed in terms of θ_s , shown in Figure 51.

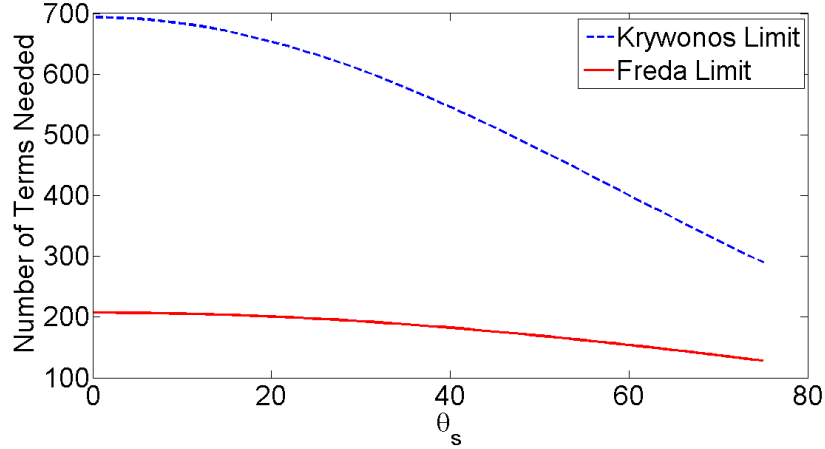


Figure 51. Comparison between Krywonos' and new expression for number of terms needed as a function of θ_s where $\sigma_s/\lambda = 2$ and $l_c/\lambda = 20$, and $\theta_i = 20^\circ$. The new limit is much smaller at small θ_s .

The new expression for the number of terms needed results in a much smaller number, meaning that the summation can be calculated more quickly.

3.5.4 Error Function Approximation.

There is no property for a discrete summation of Gaussians that allows one to reduce this summation into a single term. However, as the number of terms needed increases, a discrete Gaussian looks increasingly like a continuous Gaussian. This can be shown by comparing Figure 34 with Figure 35. The first figure shows the terms of the sum where $g = 2$, which is much smoother than the second figure, where $g = 200$. As roughness increases, more terms are contained within the Gaussian peak. Therefore there may be a threshold whereat one may approximate the discrete Gaussian as a continuous Gaussian.

The cumulative density function of a Gaussian is the Error function [5].

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) \quad (72)$$

Figure 52 shows the shape of the Error function.

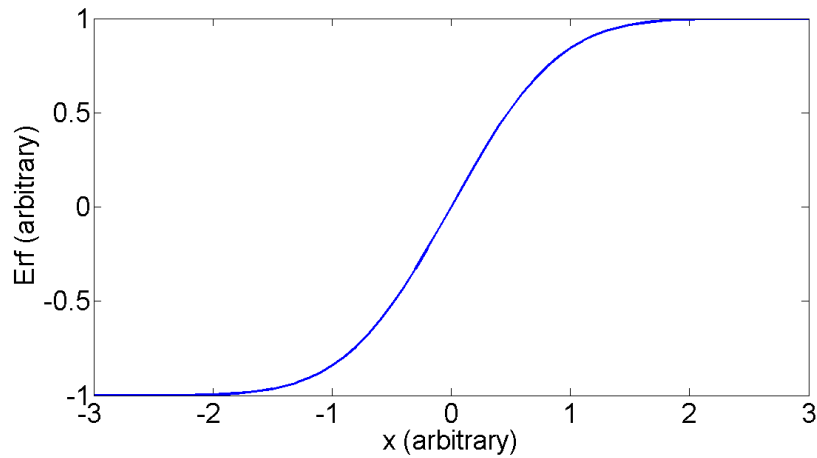


Figure 52. Shape of Error function

When describing the area under a continuous Gaussian function, the Cumulative Density Function (CDF) can be written in terms of the Error function [5].

$$\text{CDF}_{\text{gauss}} = a_e * \left[1 + \text{erf}\left(\frac{x - \mu}{\sigma}\right) \right] \quad (73)$$

where a_e is the height of the Error function.

As is shown in Figure 52, the Error function asymptotically approaches a single value. This value is a_e : the height of the Error function is the area under the entire curve of a continuous Gaussian. Because the Gaussian was defined as Equation (7), one must use a conversion relationship to obtain the Error function height from the Gaussian height. This simple relationship between the height of the Gaussian and the height of the Error function can be very useful.

$$Y_{\text{erf}} = a_e = a_g \sigma \sqrt{\pi} \quad (74)$$

Remember that a_g was defined in Equation (67).

Because a_e is the asymptotic limit of the Gaussian area, it is the only parameter needed to describe the area. It depends only on the height and the Gaussian standard deviation. This makes sense if one considers that the area under a Gaussian is independent of the location of the mean. However, recall that in order to obtain the height, the fitted location of the mean was used. If the analytic expression of the mean were used, then this approach would be made more accurate.

An important step in using the Error function approximation is to find the region where assuming continuity is accurate. To find this region, we explore all values within the range where the Gaussian approximation was accurate: $0.25 \leq \sigma_s/\lambda \leq 2$, $0 \leq l_c/\lambda \leq 50$, $0^\circ \leq \theta_i \leq 75^\circ$, $0^\circ \leq \theta_s \leq 75^\circ$. When doing analysis on these bounds, it was found that $0.5 \leq \sigma_s/\lambda$ was more acceptable. Also, it should be noted that in analyzing the Gaussian approximation, relative error was used. The reason that relative error has been abandoned and RMS error has been adopted is that the Error

function approximation had large relative error off-specular. RMS error is defined in Equation (31). The RMS error statistic puts more weight on larger values- i.e. those in the specular peak. The largest RMS errors within these bounds is shown in Figure 53.

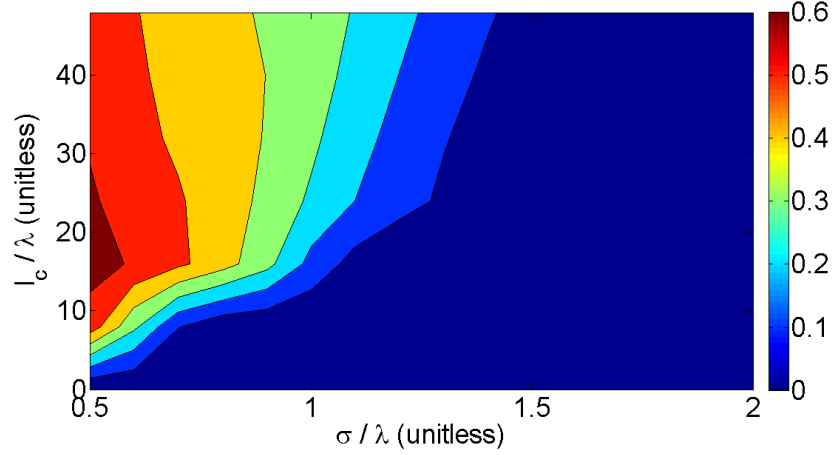


Figure 53. Maximum RMS error using the Error function within a limited range of σ_s/λ , l_c/λ , θ_i , and θ_s

In order to give a visual understanding of the magnitude of this RMS error, the BRDF is plotted where the RMS error is largest (about 0.6), in Figure 54

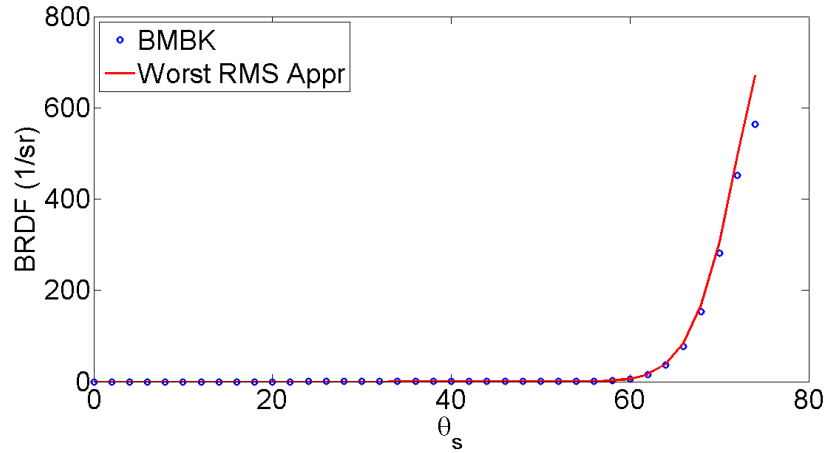


Figure 54. BRDF Comparison at maximum RMS error using the Error function where $\sigma_s/\lambda = 0.5$, $l_c/\lambda = 16$, and $\theta_i = 75^\circ$

This shows that the error is worst at grazing angles. The largest errors at $\theta_i = 0^\circ, 40^\circ$,

and 60° are shown in Figures 55, 56, and 57. For all of these incident angles, the largest error occurred at $\sigma_s/\lambda = 0.5$ and $l_c/\lambda = 16$.

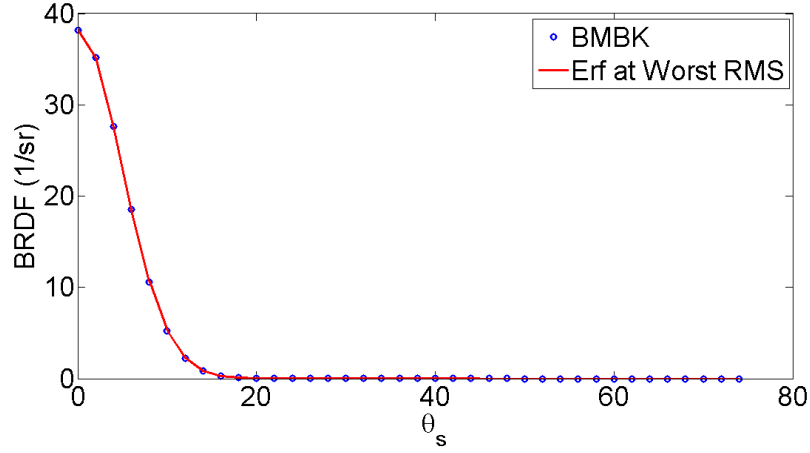


Figure 55. BMBK compared with error function approximation at the location of the largest RMS error for $\theta_i = 0^\circ$, where $\sigma_s/\lambda = 0.5$, and $l_c/\lambda = 16$.

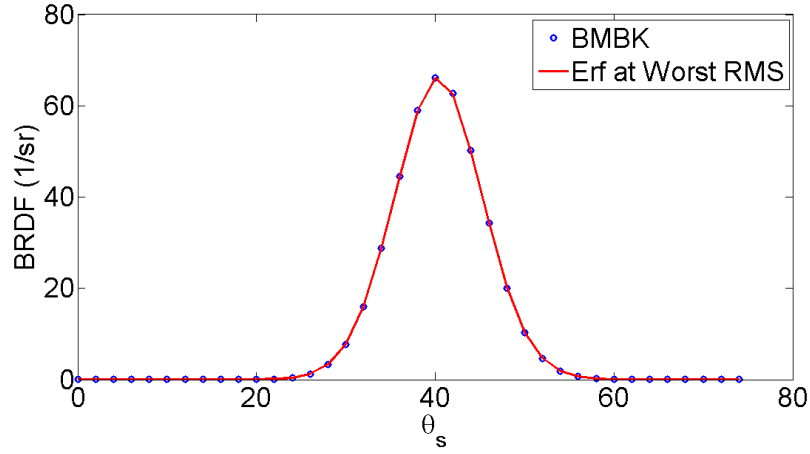


Figure 56. BMBK compared with error function approximation at the location of the largest RMS error for $\theta_i = 40^\circ$, where $\sigma_s/\lambda = 0.5$, and $l_c/\lambda = 16$.

These Figures show that the error increases when angles are more grazing.

If one considers this amount of error to be acceptable, new variable ranges can be established. Now $0.5 \leq \sigma_s/\lambda$, which, assuming that incident and scattered angles can range from 0 to 75 degrees, can be written as $g \geq 40$. This is a more narrow range of

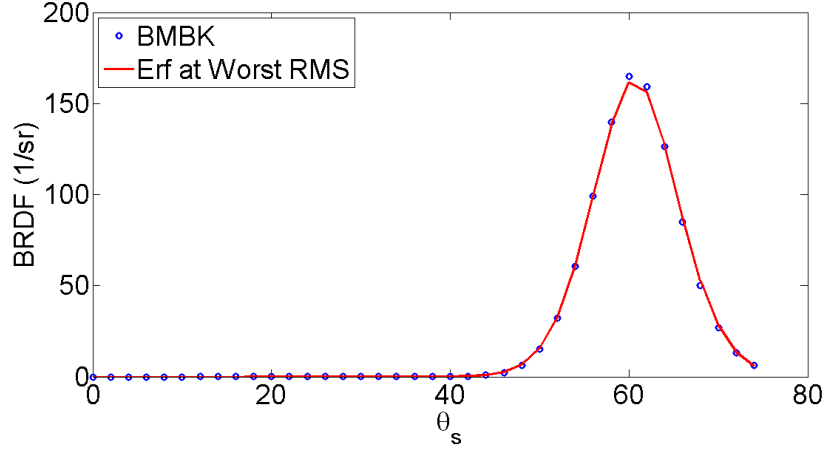


Figure 57. BMBK compared with error function approximation at the location of the largest RMS error for $\theta_i = 60^\circ$, where $\sigma_s/\lambda = 0.5$, and $l_c/\lambda = 16$.

g than the Gaussian approximation but it is a much simpler form. Once one decides on an acceptable amount of error, one can now write the BMBK as:

$$f_{\text{erf}} = \frac{K \pi l_c^2}{\lambda^2} \frac{FG(\cos \theta_i + \cos \theta_s)^2}{2 \cos \theta_i \cos \theta_s \cos^4 \theta_h} \times Z \quad (75)$$

where Z is defined below.

$$Z = \exp(-g) a_g \sigma \sqrt{\pi} = a_g \sqrt{\pi} \exp(-g) (p_{\sigma_2} \sqrt{g} + p_{\sigma_1} \sqrt{h}) \quad (76)$$

Z can be expanded in more fundamental terms, given below.

$$Z = a_g \sqrt{\pi} \exp \left[-\left(2\pi \frac{\sigma_s}{\lambda}\right)^2 (\cos(\theta_i) + \cos(\theta_s))^2 \right] \left[\frac{1}{2} p_{\sigma_1} \left(\frac{l_c \tan \theta_h}{\sigma_s} \right) + 2\pi p_{\sigma_2} \frac{\sigma_s}{\lambda} (\cos(\theta_i) + \cos(\theta_s)) \right] \quad (77)$$

Reducing the summation into one term accomplishes the goal of reducing computation time. With the BRDF written as only one term, this BRDF model could be used within an HSI model.

Remember that the generalized microfacet model was defined in Equation (12),

shown again below.

$$f_r = \rho_s PDFG \sigma_{cs} + \rho_v V + \frac{\rho_d}{\pi} \quad (78)$$

The Error function approximation BRDF f_{erf} can be written in the form of f_r . First, let there be no diffuse or volumetric contribution. Next, we observe that σ_{cs} is present in f_{erf} . The remaining terms comprise the distribution D and the wavelength scaling, now introduced as L .

$$D_{erf} L_{erf} = \frac{2K\pi l_c^2 (\cos \theta_i + \cos \theta_s)^2}{\lambda^2 \cos^4 \theta_h} \times Z \quad (79)$$

This leaves all wavelength dependence to D and L . It may be possible to use $D * Z$ as a starting point for inserting wavelength dependence into other microfacet models.

3.5.5 Gaussian Approximation Summary.

In review, the Gaussian approximation involved a few steps. First, the ranges of analysis were set. It should be understood that the smaller the range to be described, the more accurate the Gaussian approximation will be. Then, each term in the summation provided a data point as a function of index m . These data points were connected to form a curve. Next, a Gaussian was fitted to match the curve. These steps were done for all relevant h and g values. For each curve, there was a Gaussian distribution. For each Gaussian distribution, there was a mean, height, and standard deviation. That means that the mean, height, and standard deviation could be described in terms of h and g . These values for mean, height, and standard deviation were then substituted into the Gaussian. This Gaussian expression was then put inside the summation. This resulted in Equation (68). The difficult part of the

BMBK was the summation. Therefore, if the summation can be simplified, the model is greatly helped. The achievement of this new form in Equation (68) is that there is now neither a g^m term nor a $m!$ term in the summation. This makes computation less likely to result in a number beyond machine precision. The problem with this new form is that it still requires the calculation of hundreds of terms. Therefore, it would be advantageous if the new form of the summation could be solved analytically to yield a closed-form result. That is, if the summation could be solved. Unfortunately, a literature search produced no solution for a discrete summation of Gaussians over the independent variable. If one were to be found, either by a literature review or by analysis, then a closed-form solution to the summation could be found. However, an approximation of continuity allows one to use the Error function to describe the Gaussian summation in the form of the general microfacet model. Putting the summation in terms of a closed-form solution brings the accuracy and flexibility of the BMBK model within the realm of possibility for HSI calculations.

3.6 Summation Results

Several attempts were made to simplify the summation within the BMBK. First, the linear and exponential combination approaches were taken. They failed because they were used outside of their region of applicability. That is, they were analyzed using g values for which they do not approximate the summation. Next, some mathematical simplifications and approximations were explored. First, integration was shown to be unfruitful because the summation could not successfully be rewritten with a continuous limit. Next, the Taylor series expansion of the summation was taken but led to a double summation with no known convergence. The last mathematical approach was to look at the Stirling and Gosper approximations to eliminate the factorial and g^m . They were shown to be accurate increasingly with g . Before

numerical analysis was done, the necessary ranges to cover were analyzed, resulting in a much smaller region of interest. The most general case is shown in Table 3.

The last and most fruitful method of simplifying the summation was to describe it in terms of a distribution. Because the terms looked Gaussian, a Gaussian distribution was used. The resulting summation had no known solution, but allowed useful analysis. The summation terms were now written as Gaussians with defined means and standard deviations. These parameters were used to locate the range of m values required to produce an accurate summation result. This relationship is shown in Equation (71). This newer limit of terms is much smaller than Krywonos' limit but it is only valid where $g \geq 10$. Furthermore, an analytical, single-term expression using an assumption of continuity, was given in Equation (75) for $g \geq 40$. Also, this was compared to the general microfacet model, showing that Z contains all the wavelength dependence.

These attempts are not a complete solution but indicate appropriate directions of future work.

IV. Conclusion

It would be of great advantage to the HSI community if a closed-form approximation could be found for the BMBK expression (27). It would allow for BRDF calculations to be both fast and accurate. A closed-form approximation to the summation was found for rough surfaces only. This was compared to the generalized microfacet model. It may be possible to insert this wavelength-dependence into other microfacet models. No suitable approximation for polished surfaces was found. However, there were several approaches taken, some of which may yield a solution.

Current BRDF models are either flexible and computationally slow, or they are inflexible and computationally fast. The origin of both wave-optics and microfacet models were discussed, along with their limitations of applicability. The need for a fast BRDF model which could be used for a variety of wavelengths and materials was explained. Lastly, former work in the area of approximating a wave-optics model to be more computationally fast was reviewed.

There were several approaches taken to approximating the summation. The first was by combining the polished and very rough approximations in both linear and exponential combinations, section 3.2. It was shown that this method could not work because the summation, at some points, was not bounded by the approximations. Next, mathematical simplifications of the expression within the summation were attempted in section 3.3. Unfortunately, none of these simplifications put the summation into a form which could be integrated or written without a summation. Combinations of the current approximations and mathematical simplification did not yield any results.

Two areas of work gave results which may be continued profitably. The first is the analysis of what variable ranges ought to be used, done in section 3.4. It was found that in any case h could be assumed to be less than 100. Limiting the

space of what variable values need to be analyzed will simplify any future work on the summation within the BMBK, expressed in Equation (38). The second area of potential progress was the distribution approximation, done in section 3.5. It was shown that the maximum relative error in the Gaussian approximation was less than 2 percent. Although this is very low error, it may be possible for it to improve further if an expression for the mean were solved for a certain range of m . Also, an expression was derived which simplified the Gaussian summation into a single term. However, this approximation is only accurate at $g \geq 40$.

4.1 Future Work

The most promising avenue of further research is the Gaussian approximation, detailed in section 3.5.2. In summary, the expression within the summation, called Y , can be approximated as a Gaussian distribution for some ranges of h and g . This approach is considered promising because it was very accurate and because it may be improved further if a mathematical expression for the mean could be derived. The problem with the Gaussian approximation was that the infinite summation of Gaussians could not be simplified into something shorter for the entire range that it accurately described. However, since the Gaussian distribution is widely used, there may be some analysis of a discrete summation which could be applied to this research. Also, it may be possible to develop this kind of relationship. Because the Gaussian approximation is so accurate, pursuing one of these paths may be appropriate. Those conducting this research should remember that the smaller the range of variables h and g analyzed, the more accurate the Gaussian approximation will be.

Because the assumption of continuity was accurate for a relatively large region of g , further analysis ought to be done on it. The BRDF was compared to the generalized microfacet model, accentuating the fact that all wavelength dependence

is held within Z . It may be possible to insert Z into other microfacet models. This option ought to be explored.

Another idea to pursue is that of the combination approach. Current combination approaches are inappropriate because they assume a single value for σ_s and a single value for l_c . It is more physically appropriate to let these have a different value in the polished approximation from the value in the very rough approximation. Either the linear or the exponential combination could be used.

One more idea will be discussed. Because the very rough approximation describes the convergence of the summation, some terms could be subtracted from the very rough approximation in order to make that approximation valid for lower g values, as shown in Equation (80)

$$Y_{\text{subtr}} = f_{\text{vr}} - \frac{K\pi l_c^2}{\lambda^2} \frac{FG(\cos \theta_i + \cos \theta_s)^2}{2 \cos \theta_i \cos \theta_s \cos^4 \theta_h} \exp(-g) \times \sum_{m=1000}^1 \frac{g^m}{m!m} \exp \left[-\frac{hg}{m} \right] \quad (80)$$

This might work well because the very rough approximation is comparable to the generalized microfacet model. Therefore, using this method, one would begin at a known solution and slowly expand into a more accurate, general expression. One could start at $m = 1000$ and gradually subtract terms until $m = 1$ in order to explore how it impacts the very rough approximation.

4.2 Summary

BRDF is important to LWIR HSI in the Air Force because it describes angular and material dependence. BRDF has been explained intuitively and mathematically. Specifically, microfacet models were explained and a general expression was given. This understanding was applied to the conversion of a wave optics BRDF model

to a microfacet BRDF model, and making it faster without losing generalization of wavelength. The difficulty was in making it faster. Within the expression, there is an infinite summation. Several attempts to simplify it were taken, some of which were more fruitful than others. The most progress was made in using a Gaussian approximation for the terms of the summation. Some recommendations of how to continue simplifying this model have been proposed. If this summation of Gaussians could be solved in a closed-form solution for all materials, the BMBK expression would be both fast, accurate, and describe wavelength scaling. The wavelength scaling could possibly be introduced into existing microfacet models. The BMBK expression would be immediately useful for HSI applications.

Bibliography

1. Tom Apostol. *Mathematical Analysis*. Addison-Wesley, second edi edition, 1974.
2. Michael Ashikhmin and Peter Shirley. An Anisotropic Phong BRDF Model. In *Journal of Graphics Tools*, volume 5, pages 25–32, 2000. ISBN 1568812469. doi: 10.1080/10867651.2000.10487522.
3. P. Beckmann and A. Spizzichino. *The Scattering of Electromagnetic Waves from Rough Surfaces*. Artech House, norwood, m edition, 1987.
4. J. F. Blinn. Models of Light Reflection for Computer Synthesized Pictures. In *Proc. 4th annual conference on computer graphics and interactive techniques*, 1977.
5. Mary Boas. *Mathematical Methods in the Physical Sciences*. John Wiley & Sons, 2006. ISBN 978-0-471-19826-0.
6. Samuel Butler. *Experimental and Theoretical Basis for a Closed-Form Spectral BRDF Model*. PhD thesis, Air Force Institute of Technology, 2015.
7. Samuel D Butler and Michael A Marciniak. Robust Categorization of Microfacet BRDF Models to Enable Flexible Application-specific BRDF Adaptation. *SPIE*, 9205(2):1–11, 2014.
8. Samuel D Butler, Stephen E Nauyoks, and Michael A Marciniak. Comparison of Microfacet BRDF Model to Modified Beckmann-Kirchhoff BRDF Model for Rough and Smooth Surfaces. *Optics Express*, 23(22):759–769, 2015.
9. Samuel D. Butler, Stephen E. Nauyoks, and Michael A. Marciniak. Experimental measurement and analysis of wavelength-dependent properties of the BRDF. *SPIE*, 9611:96110G, 2015.
10. Samuel D Butler, Stephen E Nauyoks, and Michael A Marciniak. Experimental Analysis of Bidirectional Reflectance Distribution Function Cross Section Conversion Term in Direction Cosine Space. *Optics Letters*, 40(11):2445–2448, 2015.
11. Robert L. Cook and Kenneth E. Torrance. A reflectance model for computer graphics. *ACM SIGGRAPH Computer Graphics*, 15(3):307–316, 1981. ISSN 00978930. doi: 10.1145/965161.806819.
12. Gregg Crockett. 20031017 157. Technical Report September, 2003.
13. M. T. Eismann. *Hyperspectral Remote Sensing*. SPIE Press, 2012.
14. Orin (Union College) Farrell and Bertram Ross. *Solved Problems*. Macmillan Company, New York, 1963.

15. J. E. Harvey and A. Krywonos. Unified Scatter Model for Rough Surfaces at Large Incident and Scatter Angles. In *SPIE 6672: 66720C*, 2007.
16. Eric Heitz. Understanding the Masking-Shadowing Function in Microfacet-Based BRDFs. *Inria*, 3(February):50, 2014.
17. John David Jackson. *Classical Electrodynamics Third Edition*. 1998. ISBN 047130932X.
18. Andrey Krywonos. *Predicting Surface Scatter Using a Linear Systems Formulation of Non-Paraxial Scalar Diffraction*. PhD thesis, University of Central Florida, 2006.
19. E. Mendez. Fundamentals of Low Observable Technology.
20. Addy Ngan, Frédo Durand, and Wojciech Matusik. Experimental Analysis of BRDF Models. *Proceedings of the Eurographics Symposium on Rendering*, pages 117–126, 2005.
21. F E Nicodemus. Reflectance nomenclature and directional reflectance and emissivity. *Applied optics*, 9(6):1474–1475, 1970. ISSN 0003-6935. doi: 10.1364/AO.9.001474.
22. K. A. O’Donnel and E. R. Mendez. Experimental study of scattering from characterized random surfaces. *Journal of the Optical Society of America*, 4:1194–1205, 1987.
23. Leno Pedrotti and Frank Pedrotti. *Optics and Vision*. Addison-Wesley, Upper Saddle River, 1998.
24. Bui Tuong Phong. Illumination for computer generated pictures. *Communications of the ACM*, 18(6):311–317, 1975.
25. R. G. Priest and S. R. Meier. Polarimetric microfacet scattering theory with applications to absorptive and reflective surfaces. *Opt. Eng.*, 41:988–993, 2002.
26. Richard G Priest and Thomas a Germer. Polarimetric BRDF in the Microfacet Model: Theory and Measurements. *Proc. 2000 Military Sensing Symposia Specialty Group on Passive Sensors*, 1(March):169–181, 2002.
27. S. O. Rice. Reflection of electromagnetic waves from slightly rough surfaces. *Communications in Pure and Applied Mathematics*, 4:351–378, 1951.
28. Szymon M. Rusinkiewicz. A New Change of Variables for Efficient BRDF Representation. *Rendering Techniques, Proceedings of Eurographics Rendering Workshop*, pages 11–22, 1998.

29. B. Sandford and L. Robertson. Infrared Reflectance Properties of Aircraft Paint. In *Proc. IRIS Targets, Backgrounds, and Discrimination*, 1985.
30. James Stewart. *Calculus Early Transcendentals*. Brooks/Cole Publishing, fourth edition, 1999.
31. John C Stover. *Optical Scattering: Measurement and Analysis*. SPIE Press, third edit edition, 2005.
32. Eric Weisstein. Stirling's Approximation. URL <http://mathworld.wolfram.com/StirlingsApproximation.html>.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

The public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number. **PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.**

1. REPORT DATE (DD-MM-YYYY) 05-25-2016			2. REPORT TYPE Master's Thesis		3. DATES COVERED (From — To) Jan 2015 — Jun 2016	
4. TITLE AND SUBTITLE Microfacet Wavelength Scaling of the BRDF					5a. CONTRACT NUMBER	
					5b. GRANT NUMBER	
					5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S) Freda, Samuel E					5d. PROJECT NUMBER	
					5e. TASK NUMBER	
					5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Air Force Institute of Technology Graduate School of Engineering and Management (AFIT/EN) 2950 Hobson Way WPAFB OH 45433-7765					8. PERFORMING ORGANIZATION REPORT NUMBER AFIT-ENP-16-J-015	
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) Department of Engineering Physics 2950 Hobson Way WPAFB OH 45433-7765 Email: Michael.Eismann@us.af.mil ATTN: Dr. Michael Eismann					10. SPONSOR/MONITOR'S ACRONYM(S) AFRL/RV	
					11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION / AVAILABILITY STATEMENT DISTRIBUTION STATEMENT A: APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.						
13. SUPPLEMENTARY NOTES This work is declared a work of the U.S. Government and is not subject to copyright protection in the United States						
14. ABSTRACT The bidirectional reflectance distribution function (BRDF) describes realistic scattering of light off materials by relating incident irradiance to outbound radiance. One popular class of BRDF models assumes a surface is comprised of tiny microfacets. The drawback of microfacet BRDFs is that they often do not contain specific material parameters and neglect wavelength effects. Wave optics BRDF expressions, however, can describe wavelength effects at the expense of being more computationally cumbersome. Previous work of following a Beckmann-Kirchhoff derivation of BRDF, then relating wave optics BRDF coordinates to microfacet coordinates led to a complicated, but versatile, BRDF. In this work, the infinite summation found via this derivation is investigated. This involves algebraic simplification of the expression inside the infinite summation and curve fitting to find a functional approximation to this summation. Some methods which may accomplish this are detailed. The relationships between a wave optics and microfacet BRDF are expected to eventually lead to a simple closed-form BRDF model that more accurately describes wavelength-dependent effects and which will be fast enough to be usable in remote sensing applications.						
15. SUBJECT TERMS BRDF, Microfacet, Wavelength-scaling, Wave-optics						
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON	
a. REPORT	b. ABSTRACT	c. THIS PAGE			Maj Samuel D Butler, AFIT/ENP	
U	U	U	UU	93	19b. TELEPHONE NUMBER (include area code) (937) 255-3636, x4385; sbutler@afit.edu	