



**AN INVESTIGATION OF THE HOMICIDAL
CHAUFFEUR DIFFERENTIAL GAME**

THESIS

Sean Coates, Second Lieutenant, USAF
AFIT-ENG-MS-17-M-015

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GAME

THESIS

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Degree of Masters of Science in Electrical Engineering

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Second Lieutenant, USAF

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Abstract

The Homicidal Chauffeur differential game is often referenced in the field of differential games, but the game's complete solution is not readily available. In this work, the complete solution process of the Homicidal Chauffeur differential game is illustrated for the parameter range initially investigated by Isaacs. The solution is obtained by first deriving and analyzing the solution process of two novel optimal control problems as well as a rudimentary differential game proposed by Isaacs. In the process of deriving the solution to the Homicidal Chauffeur differential game, some salient features of the solution of this foundational differential game are highlighted; these salient features include the differential game solution's translation to realistic plane motion, the differential game solution's connection to optimal control problems, and the implementation of strategies by the Pursuer that prevent the Evader from exploiting the Equivocal Surface. Additionally, in this thesis some popular misconceptions concerning the barrier and the Equivocal Surface are dispelled. This thesis fills a gap in the literature on pursuit-evasion differential games which has direct military applications.

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Table of Contents

	Page
Abstract	iv
Acknowledgements	v
List of Figures	ix
List of Abbreviations	xii
I. Introduction	1
1.1 Problem and Approach	2
1.2 Thesis Organization	4
II. Background	5
2.1 Differential Games	5
2.2 Dubins Car	6
2.3 Homicidal Chauffeur Differential Game	7
2.3.1 Reference frame	8
2.3.2 Cost	10
2.4 Optimal Control	10
2.5 Pontryagin Maximum Principle	12
2.5.1 Dynamics	13
2.5.2 Cost	13
2.5.3 Hamiltonian	14
2.5.4 Hamiltonian Optimization	14
2.5.5 Terminal Condition	16
2.5.6 Transversality Condition	16
2.5.7 PMP - DP Connection	16
2.6 Related Research	20
2.6.1 Rufus Isaacs	21
2.6.2 Anthony Merz	22
2.6.3 William Othling	23
2.6.4 S. Miles and D. Williamson-Noble	24
2.6.5 Urban H.D. Lynch	24
2.6.6 Current Interest	25
2.7 Advancing the Research	26
III. Methodology	29
3.1 Dubins Car Optimal Control Problems	30
3.1.1 The Optimal Control Problem	31
3.2 Dubins Car with a Circular Capture Set	36

	Page
3.2.1 Case 1: $\ell < 1$	36
3.2.2 Primary Optimal Trajectories	37
3.2.3 The Barrier	42
3.2.4 The Universal Line and the Dispersal Line	44
3.2.5 Tributary Optimal Trajectories	44
3.2.6 Switch Line	48
3.2.7 Secondary Optimal Trajectories	52
3.2.8 Case 2: $\ell \geq 1$	55
3.3 Dubins Car with a Line Segment Capture Set	58
3.3.1 Primary Optimal Trajectories	58
3.3.2 The Barrier	64
3.3.3 Tributary Optimal Trajectories	65
3.3.4 Switch Line	67
3.3.5 Secondary Optimal Trajectories	69
3.3.6 Solutions for Additional Capture Set Lengths	76
3.4 Discussion	87
3.4.1 Safe Contact Trajectories	88
3.4.2 Fundamental Solution Structure	88
3.5 Isaacs' Academic Exercise 10.6.1	89
3.5.1 Introduction	89
3.5.2 The Differential Game	90
3.5.3 Optimal Control	93
3.5.4 Exercise 10.6.1	103
3.5.5 Parameterized Exercise 10.6.1	111
3.5.6 Discussion	117
IV. Results	124
4.1 Differential Game and Dynamics	124
4.2 Optimal Control	125
4.2.1 Primary Optimal Trajectories	126
4.2.2 Barrier	130
4.2.3 The Universal Line and the Dispersal Line	136
4.2.4 Tributary Optimal Trajectories	136
4.2.5 Equivocal Surface	142
4.2.6 Secondary Optimal Trajectories	149
4.3 Discussion	173
4.3.1 Equivocal Surface Misconception	174
4.3.2 Equivocal Surface Avoidance	176
4.3.3 Dubins Car Optimal Control Problem Analogue	182
4.3.4 Application	185
V. Conclusion	186

	Page
Appendix A.	190
A An Examination of the SL and ES	190
Appendix B.	192
B Optimal Control Along the ES	192
Appendix C.	194
C Alternative Methods for Constructing the ES	194
C.1 Alternative ES Construction (1)	194
C.2 Alternative ES Construction (2)	196
Appendix D.	200
D The Evader’s Tributary Option at B	200
Appendix E.	202
E Consideration for the Novel Homicidal Chauffeur Differential Game	202

List of Figures

Figure	Page
1 Reduced State Space	9
2 Isccas' Outer Barrier Solution	21
3 Isaacs' Inner Barrier Solution	21
4 Merz' Parameter Space	23
5 Circular Capture Set Reference Frame	31
6 Line Segment Capture Set Reference Frame	32
7 Circular Capture Set: Optimal Trajectories (1)	43
8 Circular Capture Set: Optimal Trajectories (2)	48
9 Circular Capture Set: Optimal Trajectories (3)	51
10 Circular Capture Set: Full Optimal Flow Field (1)	55
11 Circular Capture Set: Full Optimal Flow Field (2)	57
12 Circular Capture Set: Full Optimal Flow Field (3)	57
13 Line Segment Capture Set: Optimal Trajectories (1)	65
14 Line Segment Capture Set: Optimal Trajectories (2)	66
15 Line Segment Capture Set: Optimal Trajectories (3)	69
16 Line Segment Capture Set: Optimal Trajectories (4)	72
17 Line Segment Capture Set: Optimal Trajectories (5)	73
18 Line Segment Capture Set: Full Optimal Flow Field (1)	76
19 Line Segment Capture Set: Full Optimal Flow Field (2)	77
20 Line Segment Capture Set: Optimal Trajectories (6)	78
21 Circular Arc Capture Set Reference Frame	79
22 Modified Circular Arc Capture Set Reference Frame	80

Figure		Page
23	Line Segment Capture Set: Optimal Trajectories (7)	84
24	Line Segment Capture Set: Full Optimal Flow Field (3)	87
25	Exercise 10.6.1: Isaacs' Optimal Flow Field	91
26	Exercise 10.6.1: Reference Frame	92
27	Exercise 10.6.1: Optimal Flow Field	94
28	Exercise 10.6.1: Equivocal Surface	106
29	Exercise 10.6.1: Capture Set Synthesis	119
30	Homicidal Chauffeur Differential Game: Optimal Trajectories (1)	135
31	Homicidal Chauffeur Differential Game: Optimal Trajectories (2)	140
32	Homicidal Chauffeur Differential Game: Optimal Trajectories (3)	147
33	Homicidal Chauffeur Differential Game: Optimal Trajectories (4)	153
34	Homicidal Chauffeur Differential Game: Optimal Trajectories (5)	156
35	Homicidal Chauffeur Differential Game: Emphasized Uncertainty Region	157
36	Isaacs' Inner Barrier Solution	157
37	Homicidal Chauffeur Differential Game: Critical Secondary Trajectory	158
38	Homicidal Chauffeur Differential Game: Polar Coordinate Reference frame	163
39	Homicidal Chauffeur Differential Game: Optimal Trajectories (6)	170
40	Homicidal Chauffeur Differential Game: Full Optimal Flow Field	171

Figure		Page
41	Homicidal Chauffeur Differential Game: Realistic Motion	172
42	Homicidal Chauffeur Differential Game: Capture Set Synthesis	178
43	Homicidal Chauffeur Differential Game: Non-Classical Optimal Flow Field	184
44	Dubins Car Optimal Flow Field: Circular Capture Set	184
45	SL ES Examination	191
46	Circular Sector Capture Set, $\ell > \mu$	203
47	Circular Sector Capture Set, $\ell < \mu$	204

List of Abbreviations

Abbreviation	Page
UP	Usable Part 9
BUP	Boundary of the Usable Part 9
NUP	Non-Usable Part 9
PMP	Pontryagin Maximum Principle 12
DP	Dynamic Programming 12
PDE	Partial Differential Equation 18
ODE	Ordinary Differential Equation 18
HJI	Hamilton-Jacobi-Isaacs 19
ES	Equivocal Surface 22
SL	Switch Line 36
UL	Universal Line 44
LOS	Line-of-Sight 44
DL	Dispersal Line 44

AN INVESTIGATION OF THE HOMICIDAL CHAUFFEUR DIFFERENTIAL GAME

I. Introduction

Differential games can be described as a mathematical representation of conflict between two or more players. The purpose of solving a differential game is to mathematically express the winning strategies of the players. The military's interest in differential games grew drastically in the 1950's [1] with the popularization of Rufus Isaacs' work. Although work in the field of differential games was done prior to Isaacs, his work was the first application driven contribution to the field. Isaacs demonstrated that it was possible to equate differential games to real-world problems including, but not limited to, optimal missile interception path problems, collision avoidance problems, and optimal pursuit/evasion problems.

Isaacs' most notable work, the Homicidal Chauffeur differential game, falls into the pursuit/evasion differential game category. In the Homicidal Chauffeur differential game, a fast Pursuer, bound by a minimum turning radius, aims to capture a slower Evader that can instantaneously change its heading. The Pursuer, who controls its turning direction relative to the Evader, strives to capture the Evader as quickly as possible while the Evader, who controls its heading relative to the Pursuer, aims to delay or avoid capture altogether. To obtain a strategy to achieve their goals, the Pursuer and Evader must each solve an optimal control problem. The solution to the optimal control problem yields the controls that will minimize/maximize the time-to-capture for the Pursuer and Evader respectively.

Despite being the founder of the Homicidal Chauffeur differential game, Isaacs

was unable to obtain the full optimal control solution of the game. Despite not having a fully formed solution, Isaacs' work on the Homicidal Chauffeur differential game sparked a monumental interest in differential games. The most significant extension of Isaacs' work was the Ph.D. dissertation of Anthony Merz. With the help of his advisor John Breakwell, Merz was able to fully solve and even expand upon the Homicidal Chauffeur differential game developed by Isaacs. When Isaacs first investigated the Homicidal Chauffeur differential game, he did so for a limited range of values for two parameters: the speed ratio between the players, and the radius of the Pursuer's capture set. Merz expanded upon this initial parameter space and was able to obtain optimal control solutions for various regions of the parameter space. To date, Merz' dissertation is the most complete solution to the Homicidal Chauffeur differential game currently available. Despite being the most complete solution to the Homicidal Chauffeur differential game, Merz' dissertation is not readily available in the academic community; a summary of his dissertation can be found in [2].

Isaacs' and Merz' works are widely referenced in the field of differential games. By demonstrating the potential real-world applicability of differential game solutions, Isaacs' and Merz' works influenced the research of more complex differential games. With Isaacs' and Merz' work serving as inspiration; research has been conducted for differential games including, but not limited to, games featuring 3-D models and dynamics, games that incorporate uncertainty and time delays in the players' decision making process, and games that correspond to real-world air-to-air combat scenarios.

1.1 Problem and Approach

Although much work has been accomplished in the field of differential games since the appearance of Isaacs' and Merz' research, due to Isaacs' incomplete solution and the widespread unavailability of Merz' dissertation, there is a gap in the literature

in the field of differential games concerning the solution to the Homicidal Chauffeur differential game. Since the Homicidal Chauffeur differential game has directly influenced research on applying complex differential game solutions to real-world problems, it is important to fully understand the solution process. By providing a work that fully develops and explains the solution of the Homicidal Chauffeur differential game, the foundational work in the field of differential games is strengthened. If a greater understanding of the Homicidal Chauffeur differential game can be achieved, follow-on research can be conducted that improves upon the existing work of more complex differential games: specifically those that have real-world military applications such as air-to-air combat.

This work aims to fill the gap in the differential game literature by solving the Homicidal Chauffeur differential game for the parameter space initially investigated by Isaacs. This work takes the novel approach of obtaining the solution to the Homicidal Chauffeur differential game by first solving the optimal control analogue to the differential game. In this work, the Homicidal Chauffeur differential game is first reduced to an optimal control problem by replacing the mobile Evader of the differential game with a stationary target point. This optimal control problem is effectively the Homicidal Chauffeur differential game for the case of the speed ratio between the players set to zero. Obtaining the solution to this novel optimal control problem yields insights into the Homicidal Chauffeur differential game for the case of a speed ratio greater than zero. Additionally, an investigation of a fundamental differential game posed in [1] is performed. Isaacs posed and obtained the solution to this differential game, Exercise 10.6.1, and used methodology from the solution process in his subsequent development of the Homicidal Chauffeur differential game solution. Exercise 10.6.1 introduces and discusses salient features of differential game solutions that directly impact the formation of the Homicidal Chauffeur differential game solu-

tion. Therefore, understanding the solution process to Exercise 10.6.1 can lead to an enhanced understanding of the Homicidal Chauffeur differential game solution. For this reason, the investigation of Exercise 10.6.1 is included within this work.

In addition to deriving the Homicidal Chauffeur differential game solution in a novel way, this work offers tutorial value by providing the entire process used to obtain the full solution: something that is lacking in both Isaacs' and Merz' works. In this work, the solution to the Homicidal Chauffeur differential game is explicitly derived and presented as an optimal flow field. This optimal flow field depicts trajectories that reveal how the Evader's position, relative to the Pursuer, changes over time as the players apply their optimal control strategies. The solution to the Homicidal Chauffeur differential game is obtained analytically while the optimal flow field is constructed in MATLAB.

1.2 Thesis Organization

Following the introduction, Chapter II presents an overview of topics utilized in this work. Additionally, this chapter discusses related research in an effort to contextualize the contribution of this work in the field of differential games. In Chapter III, the foundational work used to obtain the solution to the Homicidal Chauffeur differential game is presented. These foundation works include two novel optimal control problems as well as the investigation of the academic exercise, Exercise 10.6.1, proposed by Isaacs. In Chapter IV, the full solution to the Homicidal Chauffeur game is presented, followed by an analysis of the results. In Chapter V, the results of the research are summarized and recommendations for continued research are given.

II. Background

Chapter II begins with a background on topics that were utilized in the development of this research. This includes an overview on the topic of differential games, an examination of the Homicidal Chauffeur differential game and the Dubins car game, an overview of Optimal Control, and the development of optimal control laws through Pontryagin’s Maximum Principle. The second section of this chapter discusses related works in order to highlight developments within the field of differential games as well as contextualize the importance and contribution of this work. The majority of the differential games information stems from [1] and [2], while the majority of the Optimal Control and PMP information stems from [3] - [7].

2.1 Differential Games

Differential games are defined as “games in which the position of the player develops continuously in time” [8]. The military’s interest in differential games grew drastically in the early 1950’s [1] with the popularization of Rufus Isaacs’ application driven work. The players in these differential games, whose roles and objectives vary with the game type, have motion based on kinematics models that are founded on real-world dynamics. In order to make the problem more tractable, however, these game kinematics are often simplified versions of real-world dynamics.

To simplify the game to a reasonable level, differential games often operate under various assumptions: for instance, differential games are commonly established as a perfect knowledge game in which the players have instantaneous up-to-date information concerning the prior control inputs of their opponents. Although the players have up-to-date information on their opponents, they do not know how their opponent will react in the future. Based on current knowledge of their opponents, the players will

create a playing strategy based on the game's state variables. The state variables are any variables that are relevant to decision making at each instant in time: for instance, distance and heading between players in a pursuit/evasion game can be set as state variables to influence a player's decision making.

In each differential game there is an objective each player strives to achieve. In a zero sum differential game, such as the Homicidal Chauffeur differential game, players have opposing objectives. As an example, the objective of a surface-to-air missile is to hit its target, if possible, in as little time as possible, while the opposing objective of the aircraft would be to escape, if possible, or delay the time to impact. The objective of this example is to achieve/avoid impact while the payoff is to minimize/maximize the time to impact. With respect to differential games, the payoff (or cost) is the quantity the players apply their controls to affect. In differential games, the players strive to choose the best control possible in order to minimize/maximize the cost and achieve their objective. The Value of the differential game is the cost when all of the players are performing optimally. The termination of the game occurs when certain conditions are met. Referring back to the missile example, the game would terminate if the missile successfully hit the aircraft or the aircraft was able to successfully escape the missile.

2.2 Dubins Car

In [9] L.E. Dubins developed the solution to the problem of finding the shortest path between two points when the curvature of the path is constrained and when designated initial and terminal path tangents exist. Dubins showed that the solution to the problem is a path constructed from maximum curvature and/or straight lines. The solution to the shortest path problem is referred to as the Dubins path. Based on this work, the Dubins car model was devised and implemented in differential

games. The Dubins car is a vehicle model constrained by a minimum turning radius. The Dubins car makes an appearance as the Pursuer in the Homicidal Chauffeur differential game which is described in Section 2.3.

2.3 Homicidal Chauffeur Differential Game

The Homicidal Chauffeur differential game is a zero sum, perfect information, differential game founded by Rufus Isaacs in his original work [1]. Because this differential game was examined by others after Isaacs: Isaacs' version of the differential game is referred to as the *classical* Homicidal Chauffeur differential game. The *classical* game entails a Pursuer, endowed with a circular capture set and subject to a minimal turning radius, pursuing an Evader that can instantaneously change its heading relative to the Pursuer. While the Evader is highly maneuverable compared to the Pursuer, it is slower than the Pursuer.

This differential game is a pursuit/evasion game which is defined as a game that ends when capture occurs. In the Homicidal Chauffeur differential game, capture occurs when the distance between the players is less than the radius (ℓ) of the circular capture set centered on the Pursuer. The capture range ℓ represents the feasible effective range of the Pursuer's weapon. The Pursuer is endowed with this capture radius because in a real-world application, a Purser would only need to get within a certain range of his opponent in order to deploy a weapon and eliminate his opponent. Because the Homicidal Chauffeur differential game is a zero sum perfect information game, each player's speed and turning capabilities, as well as the size of the capture radius, are known to both players. Additionally, due to the Pursuer's speed advantage, in the *classical* parameter space, capture of the Evader is guaranteed.

2.3.1 Reference frame

The *classical* Homicidal Chauffeur differential game takes place in the Euclidean plane, also referred to as the realistic space, and is depicted in Figure 1. The state space is comprised of the Pursuer's (x_P, y_P) position and its heading, and the Evader's (x_E, y_E) position for a total of five states. To simplify the analysis of the game, a reduced 2-D state space is used. In the reduced 2-D state space, based on the game dynamics, equations of motion can be derived that relate the Evader's position to the Pursuer using a rotating reference frame. These equations of motion can then be used to develop trajectories that show how the Evader's position relative to the Pursuer evolves over time. This work develops retrograde trajectories which 'emanate' from the Evader's termination point and evolve backwards in time to the Evader's initial location. The origin of the reference frame is collocated with the Pursuer's (P) instantaneous position and its y-axis is aligned with the Pursuer's velocity vector. The Evader's (E) position in the rotating reference frame is (x, y) . The collocation of the Pursuer with the origin of the reference frame reduces the state space to three states: the Evader's (x, y) position and its heading relative to the Pursuer.

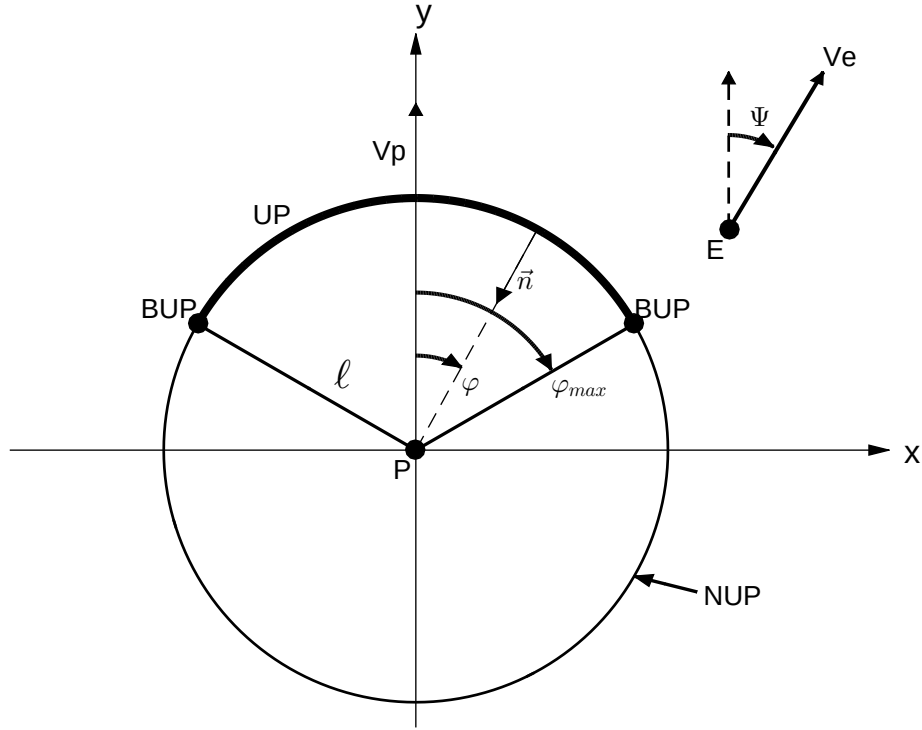


Figure 1. The Reduced State Space (x, y) : The Reference Frame of the Homicidal Chauffeur Differential Game

Referring to Figure 1, in the Homicidal Chauffeur differential game, the Pursuer's velocity is V_P and its control u is a scalar that sets the minimal turning radius R of its turning circle. The Evader's velocity is V_E and its control, the scalar point Ψ , is its instantaneous heading relative to the Pursuer's heading. The Usable Part (UP) of the capture set is the part of the capture set that, when penetrated by the Evader, causes the termination of the game. At termination, the Evader is captured on the UP of the circumference of the capture circle $x^2 + y^2 = \ell^2$. The UP is characterized by $|\varphi| < \varphi_{max}$ where the scalar φ is the bearing of the Evader at the time it is captured. The maximum bearing angle φ_{max} of the Evader at termination specifies the Boundary of the Usable Part (BUP). The Non-Usable Part (NUP) of the capture set is the region of the capture set where the Pursuer is unable to enforce capture

of the Evader. The inward pointing normal to the UP of the capture set is denoted as $\vec{n}$.

2.3.2 Cost

Because of the Pursuer’s speed advantage, the capture of the Evader is guaranteed. Since capture of the Evader is imminent, the focus of the problem is understanding the best course of action each player should take to achieve their goals. In this differential game, the cost the players wish to influence is the time-to-capture. The Pursuer aims to minimize the time-to-capture while the Evader seeks to maximize it. To influence the cost of the game, the Evader controls its heading relative to the Pursuer while the Pursuer controls its turning direction based on the relative position of the Evader.

2.4 Optimal Control

“A control problem is the translation of a control-system design problem into mathematical terms” [4]. In an optimal control problem, a mathematical model is used to determine the best controls to apply in order to achieve an objective. In the case of the Homicidal Chauffeur differential game, the Pursuer’s control u and the Evader’s control Ψ are incorporated in mathematical expressions that yield the Evader’s position relative to the Pursuer. The Pursuer and Evader must each solve an optimal control problem to determine what controls to apply, at each instance of time, in order to achieve their objectives of minimizing and maximizing the time-to-capture respectively. There are several aspects needed to fully define a control problem. These aspects include a full mathematical model of the system, a definition of the desired system output, a definition of the admissible controls, and a performance cost functional.

To properly define a control problem, the system and its constraints must be

expressed as a mathematical model with respect to the control inputs. Typical constraints include event constraints, boundary conditions, path constraints, and terminal constraints. In the case of the Homicidal Chauffeur differential game, the dynamics are considered a path constraint. While it is desired to minimize/maximize the time-to-capture, the players must do so while obeying the game's dynamics and dynamic constraints. The game's dynamics are functions of the Evader's (x, y) coordinate, as well as time, and are subject to the Pursuer's control u and the Evader's control Ψ . The game is also subject to boundary conditions. The initial retrograde trajectory position is limited by the boundary condition that the initial position must be located on the UP of the capture set characterized by φ . The retrograde trajectories are also bound by various terminal conditions. If the trajectories come into contact with the capture set or the y-axis, they are terminated. Additionally the dynamics, which are a function of time, are bound by a maximum time. In addition to the boundary and path constraints there are constraints on the state variables associated with the game's dynamics. The players' velocities are constrained such that the speed ratio μ is subject to $\mu = 0 \leq \frac{V_E}{V_P} < 1$.

The next aspect of a properly defined control problem is a statement on the desired output of the control system. In the Homicidal Chauffeur differential game, the desired outputs are expressions that produce the Evader's position relative to the Pursuer for any retrograde time τ .

The next aspect of a properly defined control problem is a definition of the admissible controls. The Pursuer controls its turning direction based on the Evader's location relative to the Pursuer. The value of the Pursuer's control u is constrained to $-1 < u < 1$ (to be explicitly derived in Chapter III). As stated previously, the Evader controls its instantaneous heading Ψ relative to Pursuer. The Evader's control is a function of time and φ and thus its value is subject to the constraint on the

retrograde time $0 \leq \tau \leq \tau_{max}$ as well as the constraint on φ , $|\varphi| \leq \varphi_{max}$.

The final aspect necessary to fully define a control problem is an establishment of a cost function which acts as a performance index for the players' control choices. A large number of inputs can be chosen that will result in an accomplishment of a goal. The cost function ensures the players are choosing the best possible controls to achieve their objectives. The set of controls that best accomplishes the players' goals is known as the optimal control. In the Homicidal Chauffeur differential game the cost is time; the Pursuer seeks to minimize the time the time-to-capture while the Evader seeks to maximize it.

With a fully defined control problem that includes a mathematical model describing the dynamics and constraints of the problem, a desired output, a definition of the controls and their constraints, and a cost functional, one can obtain an optimal control solution. When applying optimal control to differential games, one can obtain formulas used for making strategic decisions based on mathematical models that are provided game information.

2.5 Pontryagin Maximum Principle

Formulated by Lev Pontryagin, the Pontryagin Maximum Principle (PMP) is often used to solve optimal control problems. The PMP provides a way to solve dynamics problems where the dynamics and/or controls are bound by constraints. With respect to the Homicidal Chauffeur differential game, the PMP can be used to obtain the Pursuer's and Evader's optimal control laws. Although the PMP only provides necessary conditions for an optimal control solution, a connection exists between the PMP and Dynamic Programming (DP), which provides sufficient conditions for optimality. Thus, the use of the PMP in this work is a valid approach to obtain a truly optimal solution. The PMP is presented, in general terms, in this section and its

connection to DP is verified.

2.5.1 Dynamics

The process, with a control function u , is described by the dynamics

$$\dot{x} = f(x, u), \quad x(0) = x_0, \quad 0 \leq t \leq t_f$$

where the terminal time t_f is free and

$$x, x_0 \in \mathbb{R}^n, u(t) \in \Omega \subset \mathbb{R}^m, \quad f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$

$$u(t) \in \Omega \subset \mathbb{R}^m$$

2.5.2 Cost

There are multiple control options available that can transition a state variable from an initial state to a final state, however, each choice of controls has an associated cost. The cost function is defined as

$$J(u(\cdot); x_0) = \min_{u(t) \in \Omega} [h(x(t_f), t_f) + \int_0^{t_f} L(x(t), u(t)) dt]$$

where the integral represents a running cost and the function h defines a cost on the final state. The control $u^*(t)$ that follows an optimal path $x^*(t)$ in a way that minimizes the cost functional J is the problem's optimal control (where $(\cdot)^*$ denotes optimal control). The integrand in the cost functional is defined by

$$L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^1$$

2.5.3 Hamiltonian

The Hamiltonian, developed by Pontryagin, is a mathematical formula used in solving an optimal control problem. The general form of the Hamiltonian is

$$H(x, u, \lambda) = -L(x, u) + \lambda^T f(x, u)$$

where $\lambda \in \mathbb{R}^n$ is the costate, or Lagrange multiplier, and its dynamics are

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

2.5.3.1 Lagrange Multiplier

Lagrange multipliers are used in finding local maximum or minimum of a function that is subject to constraints. From the equations above, it is desired to minimize $L(x, u)$ subject to the constraints $f(x, u)$. At a local maximum or minimum, $\nabla L = \frac{\partial L}{\partial x} = \frac{\partial L}{\partial u} = 0$. The solution to the constrained problem is the intersection of $f(x, u)$ with ∇L . At this intersection point, $L(x, u)$ and $f(x, u)$ will have common tangents and common normals. At this optimum point, it is necessary that the partial derivatives of $L(x, u)$ and $f(x, u)$ be proportional to one another. The Lagrange multiplier λ ensures that these values are proportional. So a necessary condition for optimal performance is

$$\begin{aligned}\frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} &= 0 \\ \frac{\partial L}{\partial u} + \lambda \frac{\partial f}{\partial u} &= 0\end{aligned}$$

2.5.4 Hamiltonian Optimization

To find the control that minimizes the cost functional while obeying the constraints of the problem, the extremum of the Hamiltonian, with respect to u , must be a

maximum.

$$\max_{u \in \Omega} H(x, u, \lambda) \equiv 0$$

If one wished to find the control that maximizes a cost functional, they would simply need to find the control that results in the minimization of the Hamiltonian. For an extensive proof of this concept, one can refer to [5]. When examining the Hamiltonian for an optimal control, there are several necessary conditions the Hamiltonian and control must satisfy to be a candidate for an optimal solution. One of the necessary conditions for optimality, defined by Pontryagin, is

$$H(x^*, u^*, \lambda^*, t) \leq H(x^*, u, \lambda^*, t)$$

for $0 \leq t \leq t_f$ and all admissible controls. Additionally, for a free final time problem such as the Homicidal Chauffeur differential game, a necessary condition for optimality is

$$H(x^*, u^*, \lambda^*, t) = 0, \quad \forall \quad 0 \leq t \leq t_f$$

These are necessary, yet not sufficient, conditions for optimality. In regards to the optimal control: the optimal control must satisfy the PMP, however, there can be non-optimal controls that still satisfy the PMP, so additional optimality conditions are needed. The additional necessary conditions for an optimal control are

$$\dot{x}^*(t) = \frac{\partial H}{\partial \lambda}(x^*, u^*, \lambda^*, t)$$

$$\dot{\lambda}^*(t) = -\frac{\partial H}{\partial x}(x^*, u^*, \lambda^*, t)$$

$$0 = \frac{\partial H}{\partial u}(x^*, u^*, \lambda^*, t)$$

Once again, these conditions are necessary but not sufficient for optimality [3].

2.5.5 Terminal Condition

The termination of the game occurs when the terminal condition is met. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^1$, whereupon the terminal condition is specified by

$$g(x) \Big|_{t_f} = 0$$

The function $g(x)$ specifies the terminal manifold upon which termination occurs. The terminal time is determined from the condition

$$H(x, u, \lambda) \Big|_{t_f} = 0$$

In the Homicidal Chauffeur differential game, the terminal condition of the game is when the Evader penetrates the capture set ℓ , where

$$x^2 + y^2 = \ell^2 \quad \Rightarrow \quad x^2 + y^2 - \ell^2 = 0$$

2.5.6 Transversality Condition

The transversality condition is the state of the costates at the termination of the game. The transversality condition is defined as

$$\lambda(t_f) = a \cdot \vec{n}(x(t_f)), \quad a > 0$$

where $\vec{n}$ is the inward pointing normal to the terminal manifold $g(x) = 0$.

2.5.7 PMP - DP Connection

The PMP as stated is a necessary condition for optimality because it provides optimal trajectory candidates. The PMP was initially conceived for obtaining a

single optimal trajectory that emanates from a specified initial state x_0 and meets a terminal manifold, and it also provides the attendant open-loop optimal control time history. Because the PMP is presented as a necessary condition for optimality, a solution satisfying the PMP does not guarantee that the solution is truly optimal. In the Homicidal Chauffeur differential game, however, it can be shown that the PMP has a direct connection to DP which provides a sufficient condition for optimality. The PMP's connection to DP validates its use in obtaining the full optimal control solution to the Homicidal Chauffeur differential game. The method of DP, as well as its connection to the PMP, is evidenced below.

Isaacs' method of obtaining solutions to optimal control problems and differential games (two-sided optimal control problems), involved utilizing the connection between the PMP and DP. Although Isaacs was unaware of the PMP, he unknowingly derived it himself and showed that its connection to DP guaranteed the optimality of the solution.

Here the method of DP using the Hamiltonian formulation is considered. Now the Hamiltonian

$$H(x, u, V_x) \equiv L(x, u) + V_x \cdot f(x, u)$$

where the Value function $V(x)$ is the cost at the initial state $x \in R^n$. Using the Hamiltonian formulation, the DP equation for autonomous control systems and a free terminal time t_f is

$$\min_u H(x, u, V_x) \equiv 0, \quad 0 \leq t \leq t_f,$$

that is

$$H(x, u^*(x, V_x), V_x) \equiv 0, \quad 0 \leq t \leq t_f \tag{1}$$

where

$$u^*(x, V_x) = \underset{u}{argmin} H(x, u, V_x)$$

Equation (1) is a hyperbolic Partial Differential Equation (PDE) and its solution by the method of characteristics requires the solution of the adjoint Ordinary Differential Equations (ODE)

$$\frac{d}{dt} V_x = -\frac{\partial H}{\partial x}, \quad t_f \geq t \geq 0 \quad (2)$$

The terminal condition for the adjoint vector $V_x \in \mathbb{R}^n$ is established as follows: At time $t = t_f$, where $x = x(t_f)$,

$$V(x) \equiv 0 \quad \forall \quad x \quad s.t. \quad g(x) = 0$$

Let the terminal manifold $g(x)$ be parameterized as follows.

$$x = \zeta(s), \quad s \in \mathbb{R}^{n-1}$$

and since on the UP of the terminal manifold the Value function is 0 and constant, at t_f

$$V_x \cdot \zeta_s \equiv 0$$

Since the vectors ζ_s define the tangent hyperplane to the terminal manifold at $x(t_f)$, the vector V_x is normal to the terminal manifold at $x(t_f)$. With this, the DP transversality condition at t_f is

$$V_x(t_f) = a \cdot \vec{n},$$

where $a \in \mathbb{R}_+^1$ and $\vec{n}$ is an outward pointing vector which is normal to the terminal manifold.

Although the PMP is a necessary condition for optimality, by virtue of its con-

nection to the method of DP, it becomes a sufficient condition for optimality if the following holds; it is required that the PMP provides a unique optimal trajectory candidate for each initial state in the state space. If the trajectories of the constructed optimal flow field(s) do not intersect, then the global optimal flow field has been synthesized and the DP's Value function is C^1 : if this is not the case, and optimal flow fields intersect, singular surfaces must be introduced.

When the global optimal flow field has been successfully synthesized, the optimal control problem has been solved. This is so because the optimal flow field, which covers the whole state space, renders the solution of the Hamilton-Jacobi-Isaacs (HJI) hyperbolic PDE of DP - the Value function. Having the Value function fully defined for every point in the optimal flow field guarantees that the solution is truly optimal. With the Value function fully defined, the global solution of the optimal control problem and the optimal state feedback control law have been obtained by the method of DP.

The Euler Lagrange equations of the PMP/Isaacs' method, used in the construction of the optimal flow field, are the equations of the characteristics of the HJI PDE. If a solution to the HJI hyperbolic PDE is obtained using the method of characteristics, and it is valid in the whole state space, the solution is a sufficient condition for optimality. Because the PMP can be used to construct an optimal flow field, and a full optimal flow field yields the Value function of the DP, guaranteeing optimality, constructing an optimal flow field using the PMP guarantees optimality. Thus, using the method of characteristics to solve the PDE of DP yields optimality.

At this point one should note that the PMP's costate λ and the partial derivatives V_x of the Value function are related as follows: $\lambda = -V_x$ and maximization in the PMP is replaced by minimization in the DP. For further validation of this method, the PMP's connection to DP was already realized in [10] and is further evidenced in

[11] and [12].

Concerning two-sided optimal control problems, where the minimizing player's control is u and the maximizing player's control is v : in a differential game, the Hamiltonian is

$$H(x, u, v, V_x) \equiv L(x, u, v) + V_x^T \cdot f(x, u, v)$$

and, provided the Hamiltonian has a pointwise saddle point in the controls u and v , Equation (1) is replaced by

$$H(x, u^*(x, V_x), v^*(x, V_x), V_x) \equiv 0, \quad 0 \leq t \leq t_f \quad (3)$$

where,

$$u^*(x, V_x) = \underset{u}{\operatorname{argmin}} H(x, u, v, V_x)$$

and

$$v^*(x, V_x) = \underset{v}{\operatorname{argmax}} H(x, u, v, V_x)$$

Equation (3) is Isaacs' Main Equation 2 and the costate vector V_x satisfies the differential system in Equation (2); if the construction of the optimal flow field is successful and the complete state space is covered, the differential game has been solved.

2.6 Related Research

What follows is a review of important research related to the field of differential games. This section discusses the foundational research of the Homicidal Chauffeur differential game, provides examples of attempts at applying solutions to differential games to real-world scenarios, highlights the current interest within the field of differential games, and contextualizes the importance and contribution of this work.

2.6.1 Rufus Isaacs: Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization

In [1] Isaacs introduced the concept of the Homicidal Chauffeur differential game to the world of differential games. The dynamics of the Homicidal Chauffeur differential game are described by second-order differential equations expressed in terms of the speed ratio between the players and the radius of the capture set. For a certain range of these parameters, Isaacs was able to obtain the solution seen below in Figures 2 and 3. Although he was the innovator of this differential game, Isaacs was unable to obtain a full solution to the Homicidal Chauffeur differential game. Underneath the barrier $\mathcal{B}$, which separates the regions of the Pursuer's strategy, there exists a gap in Isaacs' solution as seen in Figure 3. Despite not having a full solution to the Homicidal Chauffeur differential game, this work, which was the first application driven contribution to the field of differential games, sparked a huge interest in the field.

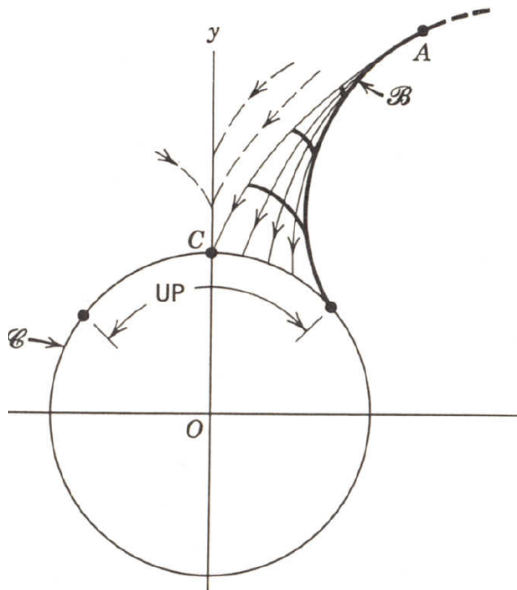


Figure 2. Isaacs' Homicidal Chauffeur Outer Barrier Solution - Reproduced from Fig. 10.3.1 in [1]

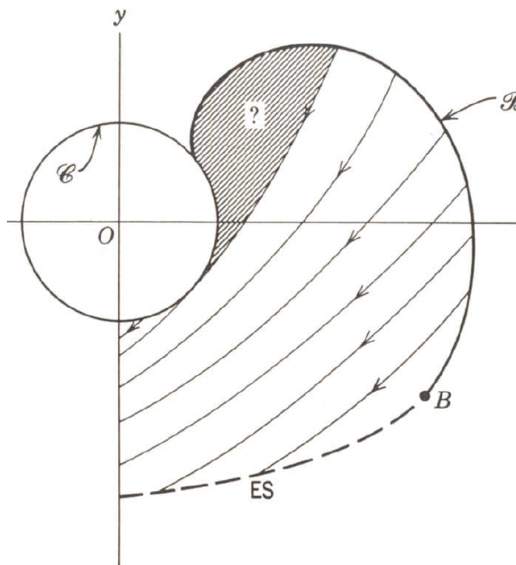


Figure 2. Isaacs' Homicidal Chauffeur Outer Barrier Solution - Reproduced from Fig. 10.3.1 in [1]

2.6.1.1 Exercise 10.6.1

Before deriving the solution to the Homicidal Chauffeur differential game, Isaacs conceived and analyzed the differential game presented as Exercise 10.6.1 in [1]. This differential game was posed as an academic exercise which featured a barrier and introduced the Equivocal Surface (ES) to the field of differential games. Isaacs relied on methodology used in this example problem in his subsequent development of the Homicidal Chauffeur differential game solution.

2.6.2 Anthony Merz: The Homicidal Chauffeur

In an attempt to obtain a full solution to the Homicidal Chauffeur differential game, Anthony Merz and John Breakwell examined the differential game posed by Isaacs in [1]. According to Merz, Isaacs' solution is what results from evaluating the *classical* differential game for the parameter space within regions I and II of Figure 4, where γ and β are the speed ratio and capture radius respectively. Merz expanded upon Isaacs' investigation by examining a larger parameter space which is broken into 20 distinct region as seen in Figure 4. The 20 regions of Merz' full parameter space separate regions where the solution to the differential game is qualitatively different from the solutions generated from the other parameter spaces. In his dissertation, summarized in [2], Merz was able to obtain the most complete solution to the Homicidal Chauffeur differential game that is currently available.

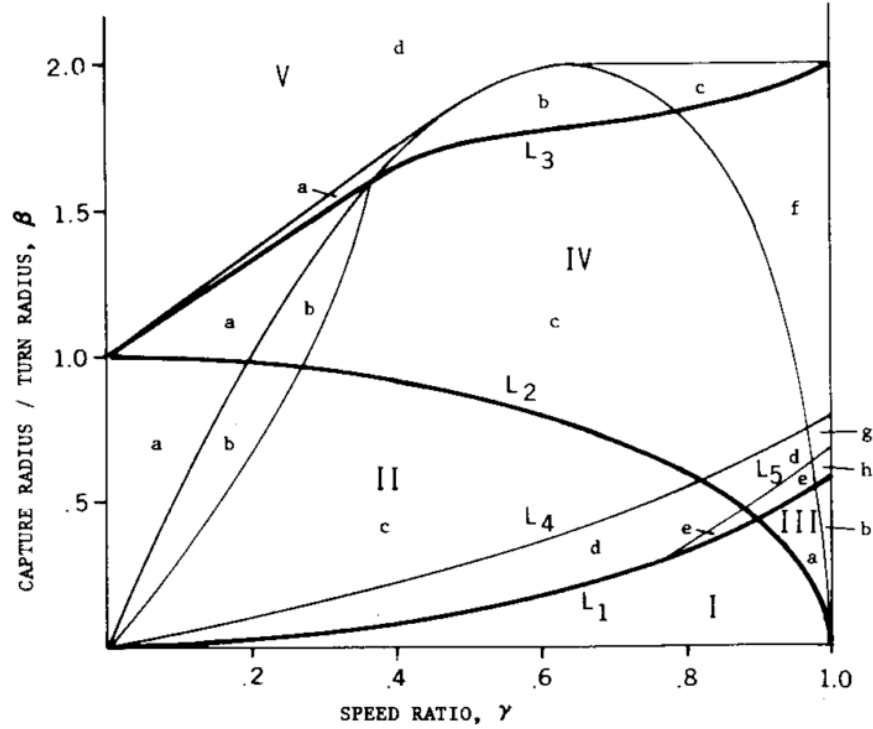


Figure 4. Merz' Parameter Space for the Homicidal Chauffeur Differential Game - Reproduced from Fig. 1 in [2]

2.6.3 William Othling: Application of Differential Game Theory to Pursuit-Evasion Problems of Two Aircraft

In this work a pursuit-evasion differential game is analyzed for a realistic aircraft model bound to 2-D planar motion. First, the author obtained optimal control solutions for simplified models, then by analyzing and applying those solutions to his standard aircraft model, the author was able to get a near-optimal control solution for the most complex of his aircraft models. This research showed that reasonable solutions to complex differential games can be obtained from the analysis of the solutions of simplified models.

2.6.4 S. Miles and D. Williamson-Noble: Toward a Differential Game Solution to a Practical Two-Aircraft Pursuit-Evasion Problem in Three-Dimensional Space

As a follow-on to Othling's dissertation [13], in this work optimal control laws are derived for a pursuit-evasion differential game involving complex aircraft models. Similar to Othling's work, multiple models with varying degrees of complexity are examined. The aircraft models in this work, however, were not limited to 2-D planar motion. A closed-loop optimal control solution was obtained for the simplest of the three aircraft models. Similar to Othling's work, the author used knowledge gained from the optimal control solution of the simplest model in order to obtain a feasible solution for a more complex model. Using this method, a pseudo-optimal solution for the realistic aircraft model was derived. Despite obtaining a pseudo-optimal solution for the most complex aircraft model, the author noted that a 3-D differential game is not a trivial extension to its 2-D analogue. Adding the extra complexity of a third dimension to the differential game made the solution much more difficult to obtain.

2.6.5 Urban H.D. Lynch: Differential Game Barriers and Their Application in Air-To-Air Combat

In this work, the author addresses the lack of applications of differential game solutions. The author states that the reason solutions aren't being applied to real-world scenarios is that the complexity of the solutions makes them difficult to feasibly apply. According to the author, at the time of this paper, people were doing work on practical techniques to solve differential games but the techniques were never reaching the point of actual application.

The purpose of Lynch's paper is to understand the effect of problem parameters in an air-to-air combat differential game solution in order to demonstrate how differ-

ential game solutions can be applied to real-world scenarios. The author varied the problem parameters of the differential game and analyzed their effect on the solution. The author was then able to use the differential game solution to draw general conclusions about air-to-air combat scenarios; from adjusting an aircraft model’s turning capability, weapon systems, and the thrust-to-weight ratio of the aircraft, the author concluded that increasing a fighter aircraft’s thrust-to-weight ratio would yield the greatest increase in the fighter’s combat capability.

2.6.6 Current Interest

A persistent interest in the Homicidal Chauffeur differential game is evidenced by the recent works [14] and [15]. Additionally, an extensive but non-exhaustive list of works involving pursuit-evasion differential games can be found in [16] and [17]. To illustrate the variety of topics surrounding this subject, differential games featuring 3-D models and dynamics, elements of stochastics such as uncertainty, and extra players or players with varying speeds can be found in these works. These works show that there is still a persistent interest and value in understanding differential game solutions.

The work of Miles and Williamson-Nobel showed that increasing the realism of a differential game adds a large degree of difficulty in obtaining the solution; this serves to illustrate the importance of providing a full solution to the *classical* Homicidal Chauffeur differential game. In order to obtain solutions to more complex and realistic differential games, it is important to provide a full solution to the foundational Homicidal Chauffeur differential game.

2.7 Advancing the Research

The *classical* Homicidal Chauffeur differential game’s complete solution, although often referenced in corresponding literature, is not readily available. This is due to Isaacs’ incomplete solution in [1] as well as the fact that Merz’ dissertation is not readily available to the entire community of differential games. Because Isaacs and Merz didn’t include explicit derivations of their solutions in their work, there is no literature available that shows how the full solution to the *classical* Homicidal Chauffeur differential game solution is obtained. Therefore, in the work herein, the complete solution process of the *classical* Homicidal Chauffeur differential game is illustrated for the parameter space initially investigated by Isaacs. In the process of investigating this foundational differential game, some salient features of the solution are highlighted and popular misconceptions of the differential game are dispelled. This work fills a gap in the literature on pursuit-evasion differential games.

The works [13], [18], and [19] were specifically mentioned above because concepts from these works are utilized to find the optimal control solution to the *classical* Homicidal Chauffeur differential game. To mitigate the difficulty of obtaining an optimal solution to a complex game, Othling’s, as well as Miles’/Williamson-Nobel’s, research showed that solutions to differential games are easier to obtain when simpler versions of them are analyzed. Using this strategy to solve the *classical* Homicidal Chauffeur differential game, the differential game can be reduced to an optimal control problem by substituting the mobile Evader for a stationary target point. With the Pursuer modeled as a Dubins car, the differential game is reduced to a Dubins car optimal control problem. The need to solve the Dubins car optimal control problem as posed herein is in fact suggested in Isaacs’ work [1] and the connection of the Dubins problem to the Homicidal Chauffeur differential game is evidenced in [20]. Thus, solving this optimal control problem helps gain insight into the *classical* Homicidal

Chaufeur differential game. Additionally, Othling's and Miles'/Williamson-Nobel's technique of understanding a solution of simplified differential games in order to obtain the solution to a more complex game can be applied to Isaacs' academic exercise 10.6.1. Because this academic exercise is a simple differential game containing the barrier and ES, its solution can be used in order to understand the formation of these features in the more complex Homicidal Chauffeur differential game. In fact, Isaacs relied on solution methodology used in Exercise 10.6.1 in his subsequent development of the *classical* Homicidal Chauffeur differential game solution. Therefore, to gain greater insight into the *classical* Homicidal Chauffeur differential game solution, this foundational differential game example is closely examined in this thesis. Lastly, Lynch's work showed that optimal control solutions, when obtained, are very difficult to practically apply. However, it is possible to make practical recommendations for real-world scenarios based on an analysis of those results. Thus, an analysis on practical applicability is given for the optimal control solution of the *classical* Homicidal Chauffeur differential game.

The original scope of this thesis contained the novel investigation of a modification of the *classical* Homicidal Chauffeur differential game in which the circular capture set of the Pursuer is replaced with a line segment capture set. Due to time constraints, however, an optimal control solution of the novel Homicidal Chauffeur differential game was not obtained. Due to the original scope of this research, to aid in the understanding of this novel differential game, the Dubins car optimal control problem solution was obtained for the case of a Pursuer endowed with a line segment capture set. Although the novel Homicidal Chauffeur differential game solution was ultimately not included in this work, its optimal control analogue where the Pursuer, endowed with a line segment capture set, trying to reach a stationary capture point was still valuable in obtaining the full solution to the *classical* Homicidal Chauffeur differential

game. Therefore, the solution process to this optimal control problem is included in this work.

As introduced in Chapter I, the contributions of this work to the field of differential games, include the derivation of two novel Dubins car optimal control problems, an investigation and novel solution derivation of Isaacs' Exercise 10.6.1, and the derivation/presentation of the explicit solution process of the *classical* Homicidal Chauffeur differential game.

III. Methodology

The goal of this research is to obtain the full optimal control solution for the *classical* Homicidal Chauffeur differential game. To accomplish this, the differential game is first reduced to an optimal control problem by replacing the mobile Evader with a stationary target point. With this substitution, the differential game is effectively reduced to a Dubins car optimal control problem. The Dubins car optimal control problem is then solved for the cases of a Pursuer endowed with a circular capture set and a line segment capture set. The case of a Pursuer endowed with a line segment capture set was initially included in this work to aid in the solving of a novel Homicidal Chauffeur differential game. Although the novel Homicidal Chauffeur differential game solution is not included in this work, its optimal control analogue proved beneficial in obtaining the solution to the *classical* Homicidal Chauffeur differential game: therefore, it is included in this work. Due to their inherent similarities, the Dubins car optimal control problems are introduced together. Following their introduction, the Dubins car optimal control problems are solved independently of one another. The Dubins car optimal control solutions were originally derived in [21] and [22]: their solution processes are outlined herein in order to emphasize their connection to the *classical* Homicidal Chauffeur differential game.

Following the solution development of the optimal control problems, the solution to Exercise 10.6.1 from [1] is obtained. This academic exercise provides an opportunity to closely analyze the construction of a differential game barrier and an ES. Because a barrier and an ES feature in the Homicidal Chauffeur differential game, the methods of obtaining a solution to this academic exercise are directly applicable to obtaining a solution to the *classical* Homicidal Chauffeur differential game. The optimal control solutions to the Dubins car optimal control problems and the solution to Exercise 10.6.1 are then analyzed and used to help obtain the solution of the

classical Homicidal Chauffeur differential game. The methodology and analysis of the optimal control solutions will then be used to derive the optimal control solution of the *classical* Homicidal Chauffeur differential game in Chapter IV.

3.1 Dubins Car Optimal Control Problems

In this section two new versions of the Dubins car optimal control problem are considered: the first is the optimal control of a circular capture set endowed Dubins car and the second is the optimal control of a line segment capture set endowed Dubins car. The capture set endowed Dubins cars are tasked to reach a stationary target point in minimum time. Because the Pursuer of the Homicidal Chauffeur differential game is modeled as a Dubins car, by replacing the mobile Evader of the differential game with a stationary target point, the differential game is effectively reduced to the Dubins car optimal control problem. Thus, the Homicidal Chauffeur differential game dynamics are examined for the case where the speed ratio $\mu = \frac{V_E}{V_P} = 0$. Due to the similarities of the Dubins car optimal control problem and the Homicidal Chauffeur differential game, it stands to reason that the solutions to the optimal control problems are similar to the differential game solution. Thus, the solutions to the optimal control problems are used to gain insight into the *classical* Homicidal Chauffeur differential games where the speed ratio $\mu > 0$.

To facilitate the comparisons between the solutions of the Homicidal Chauffeur differential game and Dubins car optimal control problems, the car and target of the Dubins problems will be referred to as the Pursuer and Evader respectively, despite the stationary nature of the Evader in the optimal control problems.

3.1.1 The Optimal Control Problem

A capture set endowed Pursuer, whose speed is V_P and has a minimum turning radius R , is to be controlled such that in minimum time it reaches a stationary point in the Euclidean plane. In the optimal control problems' differential game analogue, the Pursuer is to reach a mobile Evader whose speed is V_E .

The reference frames of for the Homicidal Chauffeur differential game analogues are shown below in Figures 5 and 6. From the differential game reference frames, the players' dynamics for the differential game are obtained then reduced to the dynamics of the optimal control problem. The reference frames in Figures 5 and 6 abide by the description of the differential game reference frame given in Section 2.3.1.

For the case of a circular capture set, at termination ($t = t_f$), the Evader is 'captured' on the reduced state space capture set $T = \sqrt{x^2(t_f) + y^2(t_f)} = \ell$. The reference frame for the case of a circular capture set is shown in Figure 5; this reference frame mirrors the reference frame of the Homicidal Chauffeur differential game in Figure 1 since the dynamics of the optimal control problem are first derived as differential game dynamics then reduced to the dynamics of the optimal control problem.

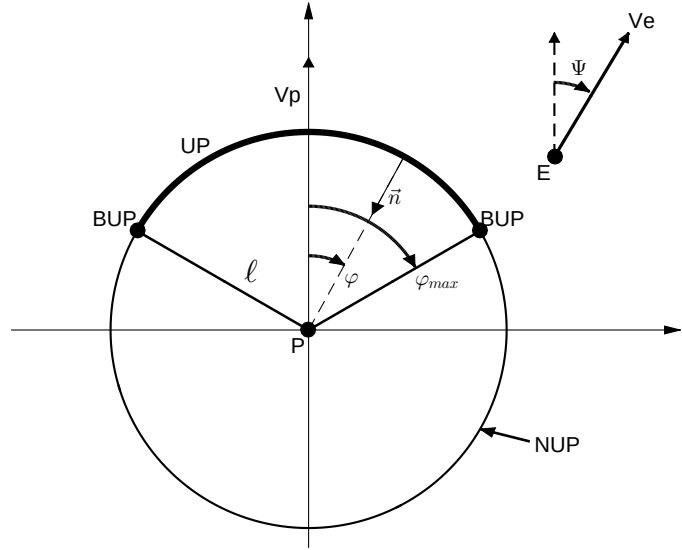


Figure 5. The Reduced State Space (x,y) for a Circular Capture Set

For the case of a line segment capture set, at termination ($t = t_f$), when the Evader is captured, the Evader's relative position will have an x-coordinate of 0, so the capture set in the reduced state space is $T = \{(x, y) | x = 0, \mu \leq y \leq \ell\}$. The reference frame for the case of a line segment capture set is shown in Figure 6.

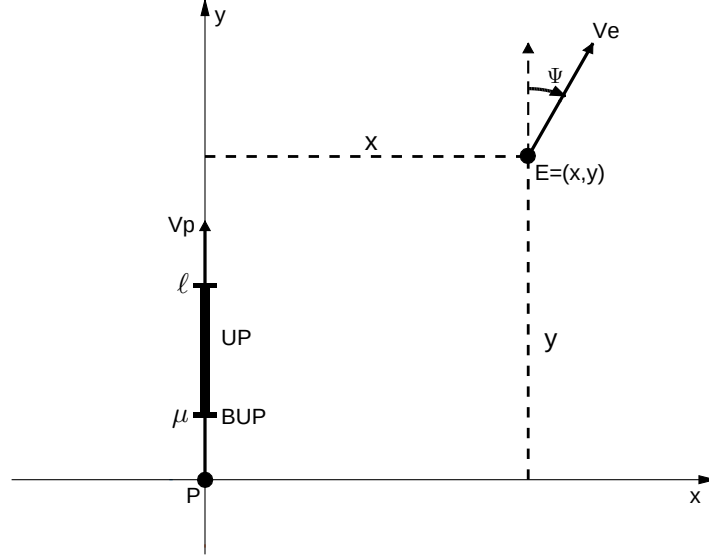


Figure 6. The Reduced State Space (x,y) for a Line Segment Capture Set

To obtain the dynamics of the Dubins car optimal control problem, the dynamics of the Homicidal Chauffeur differential game are derived then reduced to the Dubins car optimal control problem dynamics by setting the speed ratio parameter $\mu = 0$. The Homicidal Chauffeur differential game dynamics are

$$\begin{aligned}\dot{x} &= -\frac{V_P}{R}yu + V_E \sin \Psi \\ \dot{y} &= \frac{V_P}{R}xu - V_p + V_E \cos \Psi\end{aligned}$$

independent of the Pursuer's capture set type. Applying the non-dimensional vari-

ables

$$x \rightarrow \frac{x}{R}, \quad y \rightarrow \frac{y}{R}, \quad t \rightarrow \frac{V_p}{R}t, \quad \ell \rightarrow \frac{\ell}{R}, \quad \mu = \frac{V_E}{V_P}$$

to the dynamics results in the canonical kinematic equations relating the Evader's position relative to the Pursuer

$$\dot{x} = -uy + \mu \sin \Psi, \quad x(0) = x_0 \quad (4)$$

$$\dot{y} = ux - 1 + \mu \cos \Psi, \quad y(0) = y_0, \quad 0 \leq t \leq t_f \quad (5)$$

The dynamics above can be used to solve for expressions of the Evader's (x, y) position for any time (t) . From those expressions, trajectories showing the Evader's relative position change from an initial time to a final time can be obtained.

The Homicidal Chauffeur differential game dynamics are reduced to the Dubins car optimal control problem dynamics by setting the speed ratio $\mu = 0$. It was shown in [23] that it is viable to find a Dubins path using the PMP. Thus, the attendant optimal control problems are solved using the PMP. The Dubins car optimal control problems' dynamics in the reduced state space (x, y) are

$$\dot{x} = -uy, \quad x(0) = x_0 \quad (6)$$

$$\dot{y} = ux - 1, \quad y(0) = y_0, \quad 0 \leq t \leq t_f. \quad (7)$$

Using the dynamics in Equations (4) and (5), the UP and the BUP can be characterized. Anywhere on the UP of the capture set

$$\max_u \min_{\Psi} [\vec{n} \cdot \vec{f}(x, y, u, \Psi)] > 0$$

and, on the BUP

$$\max_u \min_{\Psi} [\vec{n} \cdot \vec{f}(x, y, u, \Psi)] = 0 \quad (8)$$

where $\vec{n}$ is the inward pointing normal to the UP of the capture set. The dynamics $\vec{f}(x, y, u, \Psi)$ are specified by

$$\vec{f} = \begin{pmatrix} -yu + \mu \sin \Psi \\ xu - 1 + \mu \cos \Psi \end{pmatrix}$$

With this, an expression for the UP of the capture set can be found for a circular and line segment capture set.

In the Homicidal Chauffeur differential game with a circular capture set, because the Pursuer enforces penetration of the capture set, the normal to the capture set, when the Evader is initially to the right of the Pursuer, is $\vec{n} = -(\sin \varphi, \cos \varphi)$. From Equation (8), for the case of a circular capture set, on the BUP

$$\max_u \min_{\Psi} [(yu - \mu \sin \Psi) \sin \varphi + (-xu + 1 - \mu \cos \Psi) \cos \varphi] = 0$$

$\Rightarrow$

$$(-x + 1) \cos \varphi + y \sin \varphi - \mu \cdot \max_{\Psi} (\sin \varphi \sin \Psi + \cos \varphi \cos \Psi) = 0$$

$\Rightarrow$

$$(-x + 1) \cos \varphi + y \sin \varphi - \mu = 0 \quad (9)$$

The circular capture set is parameterized according to

$$x = \ell \sin \varphi, \quad y = \ell \cos \varphi$$

Applying this parameterization to Equation (9) yields.

$$\cos\varphi - \max_{\Psi}(\mu\cos(\varphi - \Psi)) = 0$$

Using the above equation, it is found that

$$\varphi_{max} = A\cos\mu \quad (10)$$

Hence, for the special case of a stationary Evader, namely in the Dubins car optimal control problem with a circular capture set, the UP= $\{(x, y)|x^2 + y^2 = \ell^2, \frac{-\pi}{2} < \varphi < \frac{\pi}{2}\}$ and the BUP of the capture set corresponds to $\varphi_{max} = \pm\frac{\pi}{2}$.

In the Homicidal Chauffeur differential game with a line segment capture set, because the Pursuer enforces the Evader's penetration of the capture set, the normal to the capture set, when the Evader is initially to the right of the Pursuer, is $\vec{n} = (-1, 0)$. From Equation (8), for the case of a line segment capture set, on the BUP

$$\max_u \min_{\Psi}[-1(-yu + \mu\sin\Psi)] = 0 \quad (11)$$

$\Rightarrow$

$$\max_u[yu - \mu] = 0 \quad (12)$$

Using the above equation, it is found that on the BUP

$$y = \mu \quad (13)$$

Hence, for the special case of a stationary Evader, namely in the Dubins car optimal control problem with a line segment capture set, the UP= $\{(x, y)|x = 0, 0 < y \leq \ell\}$ and the BUP of the capture set corresponds to $y = 0$.

3.2 Dubins Car with a Circular Capture Set

Based on the initial position of the Evader relative to the Pursuer, three classes of trajectories can result in both the Dubins car optimal control problem and Homicidal Chauffeur differential game: primary, secondary, and tributary trajectories. The trajectories shown encapsulate the motion of both the Pursuer and Evader based on this relative position to one another. For the case of a Dubins car with a circular capture set: expressions for these trajectories, as well as for other components of the optimal control solution (the barrier and Switch Line (SL)), are derived in this section. The optimal flow field is then systematically built as each component of the solution is obtained. To obtain the optimal control solution, the trajectories are derived in the Homicidal Chauffeur differential game and are then reduced to the Dubins car optimal control problem by setting the speed ratio $\mu = 0$.

In the sections below, the solution to the Dubins car optimal control problem is derived for the case of a circular capture set endowed Pursuer. The solution is derived for the case of the Evader initially positioned in the Right-Hand-Plane (RHP) of the reference frame. While the solution is only derived for the case of the Evader initially positioned in the RHP, it is important to note that the optimal flow field in the Left-Half-Plane (LHP) is a mirror reflection of the solution in the RHP.

3.2.1 Case 1: $\ell < 1$

The optimal control problem is parameterized by the non-dimensional capture range ℓ . In this section, the Dubins car optimal control solution is obtained for a capture radius of size $\ell < 1$. The nature of the solution changes as ℓ varies so the optimal control solution is initially obtained for $\ell < 1$ because each component of the optimal flow field is present for this condition. Following this derivation, the optimal control solution is shown for $\ell \geq 1$ in Section 3.2.8.

3.2.2 Primary Optimal Trajectories

Primary optimal trajectories are retrograde trajectories that ‘emanate’ from the UP of the circular capture set $UP = \{(x, y) | x^2 + y^2 = \ell^2, \frac{-\pi}{2} < \varphi < \frac{\pi}{2}\}$. To obtain these optimal trajectories, the Hamiltonian of the Homicidal Chauffeur differential game is examined. From the PMP, the general form of the Hamiltonian is

$$H(x, u, \lambda) = -L(x, u) + \lambda^T f(x, u)$$

and the cost is

$$J(u(\cdot); x_0) = \min_{u(t) \in \Omega} [h(x(t_f), t_f) + \int_0^{t_f} L(x(t), u(t)) dt]$$

In the Homicidal Chauffeur differential game, there is no additional cost on the final state so $h(x(t_f), t_f) = 0$. Thus, the cost functional of the Homicidal Chauffeur differential game is

$$J(u(\cdot); x_0) = \min_{u(t) \in \Omega} \int_0^{t_f} L(x(t), u(t)) dt$$

The cost in the Homicidal Chauffeur differential game is the time-to-capture (t_f). Thus, the integrand of the cost function incorporated into the Hamiltonian is $L = 1$. In the Hamiltonian above, the dynamics $f(x, u)$ are the dynamics in Equations (4) and (5) for the Homicidal Chauffeur differential game. Thus, in the Homicidal Chauffeur differential game the Hamiltonian is

$$H(x, \lambda, u, \Psi) = -1 + \lambda_x(-uy + \mu \sin \Psi) + \lambda_y(ux - 1 + \mu \cos \Psi)$$

$\Rightarrow$

$$H = -1 + u(x\lambda_y - y\lambda_x) - \lambda_y + \mu(\lambda_x \sin \Psi + \lambda_y \cos \Psi) \quad (14)$$

From Equation (14), and according to the PMP, to minimize the time-to-capture, the Pursuer must maximize the Hamiltonian by selecting its control such that

$$u^* = \text{sign}(x\lambda_y - y\lambda_x)$$

The equation above, as well as [9], show that, as long as $x(\tau)\lambda_y(\tau) - y(\tau)\lambda_x(\tau) \neq 0$, it is always optimal for the Pursuer to turn as hard as possible.

From (14), and according to the PMP, to maximize the time-to-capture, the Evader must minimize the Hamiltonian by setting its heading angle Ψ such that

$$\sin\Psi^* = -\frac{\lambda_x}{\sqrt{\lambda_x^2 + \lambda_y^2}}, \quad \cos\Psi^* = -\frac{\lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}} \quad (15)$$

According to the PMP, the costates satisfy the differential equations

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -\lambda_y u \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = \lambda_x u \quad (16)$$

To obtain expressions for the primary optimal trajectories, the dynamics in Equations (4) and (5) will be rewritten in retrograde time $\tau = t_f - t$. At $\tau = 0$ the retrograde trajectories begin at their termination point on the UP of the capture circle and develop backwards to the Evader's initial location. So the termination point of a trajectory in real time is the initial point of a trajectory in retrograde time. The dynamics in retrograde time τ are

$$\dot{x} = uy - \mu \sin\Psi, \quad x(\tau = 0) = x_0 \quad (17)$$

$$\dot{y} = -ux + 1 - \mu \cos\Psi, \quad y(\tau = 0) = y_0, \quad 0 \leq \tau \leq \tau_{max} \quad (18)$$

where $(\dot{\cdot})$ denotes differentiation in retrograde time and $(x_0, y_0) \in UP$. The retro-

grade time τ_{max} denotes the maximum retrograde time a trajectory is valid for; it is determined following the derivation of the trajectory expressions.

The Evader is considered captured at $\tau = 0$ when located on the UP of the capture set. In the RHP, in the region of the primary optimal trajectories, the Pursuer's optimal control $u^* = 1$. Thus, the transversality condition of the PMP yields

$$\lambda_x(\tau = 0) = -a \cdot \sin\varphi, \quad \lambda_y(\tau = 0) = -a \cdot \cos\varphi, \quad a > 0, \quad 0 \leq \varphi < A\cos\mu$$

when the game evolves in the RHP. Using the relationships in Equation (16) yields the harmonic equations

$$\ddot{\lambda}_x + \lambda_x = 0, \quad \ddot{\lambda}_y + \lambda_y = 0$$

With the transversality conditions, the solutions to the harmonic equations yield the costates

$$\lambda_x(\tau) = -a \cdot \sin(\varphi + \tau), \quad \lambda_y(\tau) = -a \cdot \cos(\varphi + \tau) \quad (19)$$

The optimal control law of the Evader can now be expressed in terms of time. Applying the results above to Equation (15) yields

$$\sin(\Psi(\tau)^*) = \sin(\varphi + \tau), \quad \cos(\Psi(\tau)^*) = \cos(\varphi + \tau)$$

From the equation above, it can be determined that, in the region of primary trajectories, the optimal control of the Evader is

$$\Psi^*(\tau) = \varphi + \tau, \quad \tau \geq 0$$

Using this expression for the Evader's optimal control, from the dynamics in Equa-

tions (17) and (18), the primary optimal trajectory dynamics are expressed as

$$\dot{x} = y - \mu \sin(\varphi + \tau), \quad x(\tau = 0) = \ell \sin \varphi \quad (20)$$

$$\dot{y} = -x + 1 - \mu \cos(\varphi + \tau), \quad y(\tau = 0) = \ell \cos \varphi, \quad 0 \leq \varphi < A \cos \mu \quad (21)$$

when the game develops in the RHP. Differentiating Equation (20) yields

$$\ddot{x} = \dot{y}(\tau) - \mu \cos(\varphi + \tau)$$

which yields the non-homogeneous harmonic equation

$$\ddot{x} + x = 1 - 2\mu \cos(\varphi + \tau), \quad x(\tau = 0) = \ell \sin \varphi, \quad \dot{x}(\tau = 0) = \ell \cos \varphi - \mu \sin \varphi$$

Solving the above non-homogeneous harmonic equation, and applying a similar derivation to Equation (21) yields expressions for the Evader's position relative to the Pursuer in the region of primary trajectories

$$x(\tau) = 1 + \ell \sin(\varphi + \tau) - \mu \tau \sin(\varphi + \tau) - \cos(\tau)$$

$$y(\tau) = \ell \cos(\varphi + \tau) - \mu \tau \cos(\varphi + \tau) + \sin(\tau), \quad 0 \leq \tau \leq \tau_{max}, \quad 0 \leq \varphi < A \cos \mu$$

To obtain expressions for the primary trajectories for the Dubins car optimal control problem, $\mu = 0$ is set in the equations above. The result is a family of primary optimal trajectories parameterized by φ and defined as

$$x(\tau) = 1 + \ell \sin(\varphi + \tau) - \cos \tau \quad (22)$$

$$y(\tau) = \ell \cos(\varphi + \tau) + \sin \tau, \quad 0 \leq \tau \leq \tau_{max}, \quad 0 \leq \varphi < \frac{\pi}{2} \quad (23)$$

These trajectories correspond to the scenario where the Evader is initially in the RHP.

A primary trajectory persists until the Pursuer's control switches from 1 to -1 or vice versa in the LHP, as indicated by the switching function $S(\cdot)$ which features in the Hamiltonian and determines the domain of validity of the primary trajectories. From the Hamiltonian in Equation (14), the switching function is

$$S(\tau) = x(\tau)\lambda_y(\tau) - y(\tau)\lambda_x(\tau) > 0, \quad 0 \leq \tau \leq \tau_{max}$$

Substituting Equations (19), (22), and (23) in the switching function above yields

$$S(\tau) = \cos(\varphi + \tau) - \cos\varphi$$

The maximum retrograde time τ_{max} possible, before the Pursuer's control switches, occurs when $S(\tau) = 0$. Applying $S(\tau) = 0$, the switching condition becomes

$$\cos(\varphi + \tau_{max}) = \cos(\varphi)$$

which yields

$$\tau_{max} = 2(\pi - \varphi) \tag{24}$$

So, the family of primary trajectories, in the Dubins car optimal control problem, parameterized by $0 \leq \varphi < \varphi_{max}$, is

$$x(\tau) = 1 + \ell \sin(\varphi + \tau) - \cos\tau \tag{25}$$

$$y(\tau) = \ell \cos(\varphi + \tau) + \sin\tau, \quad 0 \leq \tau \leq 2(\pi - \varphi), \quad 0 \leq \varphi < \frac{\pi}{2} \tag{26}$$

For a retrograde time $\tau > 0$, Equations (25) and (26) can be expressed as

$$\begin{aligned}
& \begin{pmatrix} x-1 \\ y \end{pmatrix} = \begin{bmatrix} \ell \cos \varphi & -(1 - \ell \sin \varphi) \\ 1 - \ell \sin \varphi & \ell \cos \varphi \end{bmatrix} \begin{pmatrix} \sin \tau \\ \cos \tau \end{pmatrix} \\
\Rightarrow & \begin{pmatrix} \sin \tau \\ \cos \tau \end{pmatrix} = \frac{1}{\ell^2 - 2\ell \sin \varphi + 1} \begin{bmatrix} \ell \cos \varphi & 1 - \ell \sin \varphi \\ -(1 - \ell \sin \varphi) & \ell \cos \varphi \end{bmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix} \\
\Rightarrow & \begin{aligned} \sin \tau &= \frac{1}{\ell^2 - 2\ell \sin \varphi + 1} [(x-1)\ell \cos \varphi + (1 - \ell \sin \varphi)y] \\ \cos \tau &= \frac{1}{\ell^2 - 2\ell \sin \varphi + 1} [\ell y \cos \varphi - (1 - \ell \sin \varphi)(x-1)] \end{aligned}
\end{aligned}$$

Using the trigonometric identity $\sin^2 \tau + \cos^2 \tau = 1$ yields

$$\begin{aligned}
& 1 = \frac{1}{(\ell^2 - 2\ell \sin \varphi + 1)^2} [(x-1)^2(\ell^2 - 2\ell \sin \varphi + 1) + y^2(\ell^2 - 2\ell \sin \varphi + 1)] \\
\Rightarrow & (x-1)^2 + y^2 = \ell^2 - 2\ell \sin \varphi + 1, \quad 0 \leq \varphi < \frac{\pi}{2}
\end{aligned}$$

Thus, for the Dubins car endowed with a circular capture set, the primary optimal trajectories in the reduced state space are a family of concentric circular arcs centered at $(1, 0)$ with a radius of $\sqrt{\ell^2 - 2\ell \sin \varphi + 1}$ where $0 \leq \varphi < \frac{\pi}{2}$, as seen in Figure 7.

3.2.3 The Barrier

An abnormal trajectory, the barrier, emanates from the BUP at $\varphi = \varphi_{max}$. Although it behaves as a primary trajectory, the barrier is in fact a locus of initial conditions where the Pursuer's strategy is to turn away from the Evader, as shown by Merz [2]. This means that the barrier is not an optimal trajectory traversable by

the Evader. Thus, if the Evader is initially on or underneath the barrier, the Pursuer's optimal strategy is to initially turn away from the Evader before doubling back and catching the Evader. Thus, the barrier delineates the regions of the Pursuer's optimal strategies.

Following the process used to obtain primary trajectories, although the barrier is not an optimal trajectory, deriving its dynamics reveals that it is equivalent to a primary trajectory emanating from the BUP. Therefore, in the Dubins car optimal control problem where $\mu = 0$, the barrier trajectory emanates from the BUP at $\varphi_{max} = \frac{\pi}{2}$. In the Dubins car optimal control problem, the barrier is a semicircle centered at (1,0) with a radius of

$$\sqrt{\ell^2 - 2\ell \sin \frac{\pi}{2} + 1} = 1 - \ell$$

as seen in Figure 7, where B demarcates the end point of the barrier.

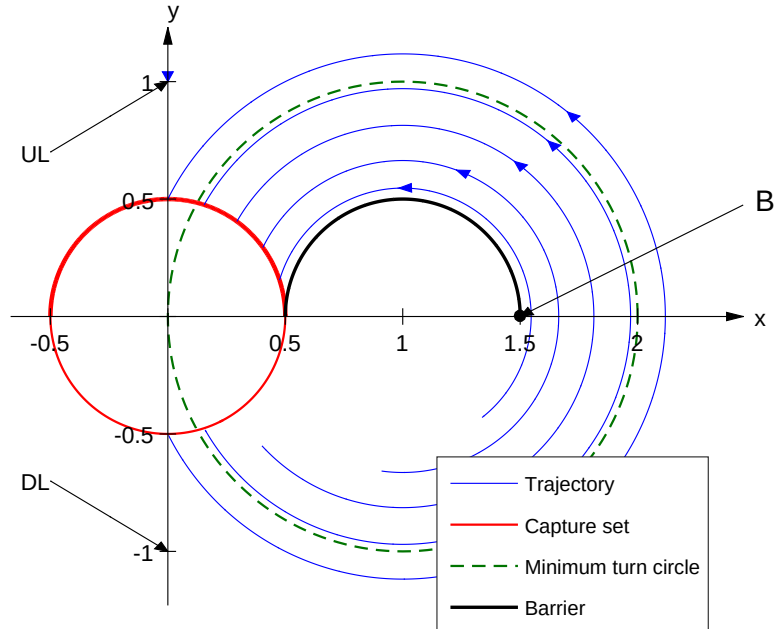


Figure 7. Dubins Car Optimal Control Problem with a Circular Capture Set: Primary Optimal Trajectories in the Optimal Flow Field, $\ell = 0.5$

3.2.4 The Universal Line and the Dispersal Line

In Figure 7 the Universal Line (UL) is $UL = \{(x, y) | x=0, y \geq \ell\}$ and is an advantageous trajectory. In the Dubins car optimal control problem it is apparent that once the Evader is positioned on the UL, it is optimal for the Pursuer to employ Line-of-Sight (LOS) pure pursuit. In Figure 7 the Dispersal Line (DL) is $DL = \{(x, y) | x=0, y \leq -\ell\}$ along the negative y-axis and is a locus along which Pursuer's strategy is not unique. Paths emanating in real time from either side of the DL are optimal. In the Dubins car optimal control problem, the Pursuer may choose his turning direction without consequence since the Evader is stationary and has no control to choose. This would not be the case, however, in the Homicidal Chauffeur differential game.

3.2.5 Tributary Optimal Trajectories

Tributaries are retrograde trajectories that emanate from the UL at $(0, s)$, where $s > \ell$. To obtain the tributary optimal trajectories the costates at $\tau = 0$ must be determined. To this end, the Value function $V(x, y)$ at the point where the trajectories emanate from is established. Now, the Value function on the positive y-axis, where LOS pursuit and evasion takes place, is

$$V(0, y) = \frac{y - \ell}{1 - \mu}$$

By symmetry $V_x \Big|_{x=0} = 0$ and $V_y \Big|_{x=0} = \frac{1}{1 - \mu}$ is calculated.

Based on the connection between the PMP and DP, as well as the work [24], it is known that for this optimal control problem, the two costates λ_x and λ_y are such that $\lambda_x = -V_x$ and $\lambda_y = -V_y$. Thus, when the game evolves in the RHP, the costates

of the retrograde tributary trajectories satisfy

$$\dot{\lambda}_x = \lambda_y, \quad \lambda_x(\tau_1 = 0) = 0$$

$$\dot{\lambda}_y = -\lambda_x, \quad \lambda_y(\tau_1 = 0) = -\frac{1}{1-\mu}, \quad 0 \leq \tau_1$$

Hence

$$\dot{\lambda}_x(\tau_1 = 0) = -\frac{1}{1-\mu}$$

$\Rightarrow$

$$\lambda_x(\tau_1) = -\frac{1}{1-\mu} \sin \tau_1, \quad \lambda_y(\tau_1) = -\frac{1}{1-\mu} \cos \tau_1 \quad (27)$$

where τ_1 denotes the retrograde time measured from the time the tributary trajectory emanates from $(0, s)$, $s > \ell$. The Evader's optimal control is characterized by Equation (15) which, in view of the costates in Equation (27) yields

$$\sin \Psi^* = \sin \tau_1, \quad \cos \Psi^* = \cos \tau_1$$

which is the Evader's optimal heading angle when located on a tributary trajectory. Using the expression for the Evader's optimal control from the retrograde dynamics in Equations (17) and (18), the tributary optimal trajectories are expressed as

$$\dot{x} = y - \mu \sin \tau_1, \quad x(\tau_1 = 0) = 0 \quad (28)$$

$$\dot{y} = -x + 1 - \mu \cos \tau_1, \quad y(\tau_1 = 0) = s, \quad s > \ell \quad (29)$$

when the game develops in the RHP. Differentiating Equation (28) yields

$$\ddot{x} = \dot{y} - \mu \cos \tau_1$$

which yields the non-homogeneous harmonic equation

$$\ddot{x} + x = 1 - 2\mu \cos\tau_1, \quad x(\tau_1 = 0) = 0, \quad \dot{x}(\tau_1 = 0) = s$$

Solving the above non-homogeneous harmonic equation and applying a similar derivation to Equation (29) yields expressions for the Evader's position relative to the Pursuer in the region of tributary trajectories

$$x(\tau_1) = 1 + (s - \mu\tau_1)\sin\tau_1 - \cos\tau_1 \quad (30)$$

$$y(\tau_1) = (s - \mu\tau_1)\cos\tau_1 + \sin\tau_1, \quad 0 \leq \tau_1 \leq \tau_{1max}, \quad s > \ell \quad (31)$$

Reducing Equations (30) and (31) to the Dubins car optimal control problem, where $\mu = 0$, yields

$$x(\tau_1) = 1 + s \cdot \sin\tau_1 - \cos\tau_1 \quad (32)$$

$$y(\tau_1) = s \cdot \cos\tau_1 + \sin\tau_1, \quad 0 \leq \tau_1 \leq \tau_{1max}, \quad s > \ell \quad (33)$$

These trajectories correspond to the scenario where the Evader is initially located in the RHP. The domain of validity of the tributary trajectories is determined by the switching function.

$$S(\tau) = x(\tau_1)\lambda_y(\tau_1) - y(\tau_1)\lambda_x(\tau_1) > 0, \quad 0 \leq \tau_1 \leq \tau_{1max}$$

Substituting in Equations (27), (32), and (33), the switching function for the tributary trajectories becomes

$$S(\tau_1) = \frac{1}{1 - \mu}(1 - \cos\tau_1)$$

The maximum retrograde time τ_{1max} possible, before the Pursuer's control switches,

occurs when $S(\tau) = 0$. Applying $S(\tau) = 0$ to the switching function above yields

$$\cos\tau_{1max} = 1$$

which yields

$$\tau_{1max} = 2\pi$$

So, the family of tributary trajectories, in the Dubins car optimal control problem, parameterized by s , is

$$x(\tau_1) = 1 + s \cdot \sin\tau_1 - \cos\tau_1 \quad (34)$$

$$y(\tau_1) = s \cdot \cos\tau_1 + \sin\tau_1, \quad 0 \leq \tau_1 \leq 2\pi, \quad s > \ell \quad (35)$$

for a retrograde time $\tau_1 \geq 0$. Equations (34) and (35) can be expressed as

$$\begin{pmatrix} x-1 \\ y \end{pmatrix} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \begin{pmatrix} \sin\tau_1 \\ \cos\tau_1 \end{pmatrix}$$

$\Rightarrow$

$$\begin{pmatrix} \sin\tau_1 \\ \cos\tau_1 \end{pmatrix} = \frac{1}{1+s^2} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix}$$

$\Rightarrow$

$$\sin\tau_1 = \frac{1}{1+s^2} [s(x-1) + y]$$

$$\cos\tau_1 = \frac{1}{1+s^2} [sy - (x-1)]$$

Using the trigonometric identity $\sin^2\tau + \cos^2\tau = 1$ yields

$$1 = \frac{1}{(1+s^2)^2} [(x-1)^2(1+s)^2 + y^2(1+s^2)]$$

$\Rightarrow$

$$(x - 1)^2 + y^2 = 1 + s^2$$

Thus, for a Dubins car endowed with a circular capture set, the tributary trajectories in the reduced state space are a family of concentric circular arcs parameterized by $s > \ell$, centered at $(1, 0)$ with a radius of $\sqrt{1 + s^2}$, that ‘terminate’ on the y-axis at $y = -s < -\ell$ as seen in Figure 8.

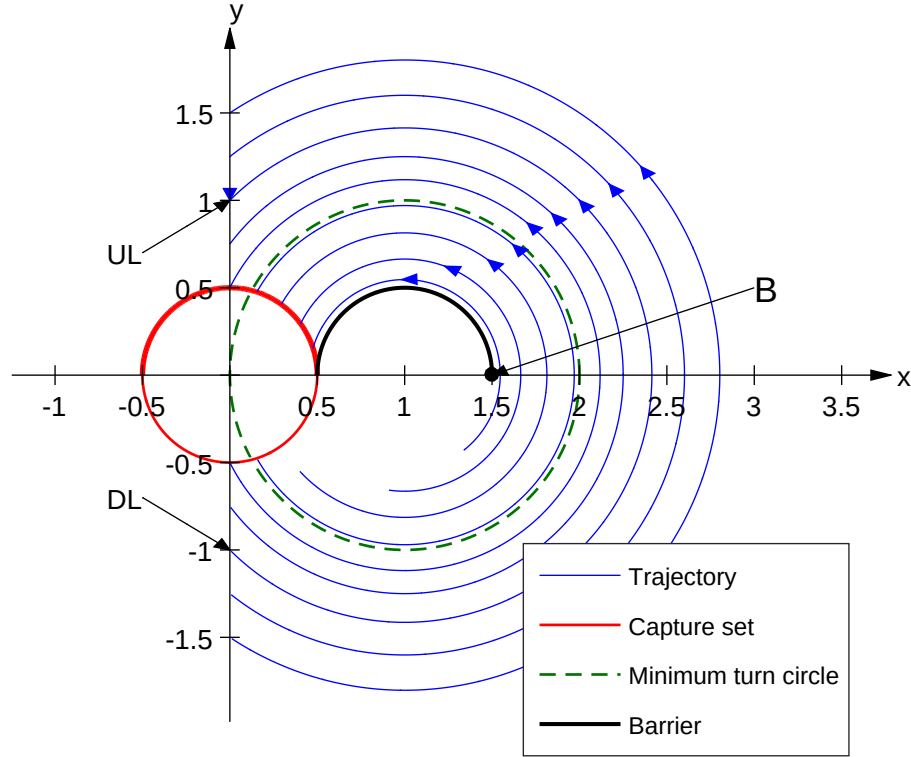


Figure 8. Dubins Car Optimal Control Problem with a Circular Capture Set: Tributary Optimal Trajectories in the Optimal Flow Field, $\ell = 0.5$

3.2.6 Switch Line

In this section, an expression for the SL is developed. The method used to develop the SL is only applicable to the Dubins car optimal control problem where $\mu = 0$. The differential game counterpart to the SL, the ES [1], is fundamentally different from

the SL. Although it is constructed for $\mu = 0$, the SL does *not* coincide with Isaacs' description of the ES evaluated for $\mu = 0$. This is further evidenced in Appendix A.

A primary optimal trajectory persists until the Pursuer's control u switches from 1 to -1, or vice versa in the LHP. Recalling the switching function for primary trajectories

$$S(\tau) = \cos(\varphi + \tau) - \cos\varphi$$

From Equation (24), in the RHP the retrograde primary trajectory persists for $u(\tau)^* = 1$ until the maximum retrograde time $\tau_{max} = 2(\pi - \varphi)$ at which point the Pursuer's control switches to $u(\tau)^* = -1$. Due to the effect of τ_{max} on the primary trajectories, it is possible to construct an SL from the dynamics of the primary trajectories. The SL effectively shows where the Pursuer's control will switch in the optimal flow field. The parametric expression for the SL is derived as follows.

Equations (25) and (26) are valid until they meet the SL at point $B'(\varphi)$, $\frac{\pi}{2} \geq \varphi \geq \bar{\varphi}$, where B' denotes the end point of a primary trajectory evaluated for τ_{max} and $\bar{\varphi}$ demarcates the primary trajectories that contact the SL from those that contact the NUP of the capture set. Since the trajectories encounter the SL at the maximum retrograde time $\tau_{max} = 2(\pi - \varphi)$, the parametric equations for the SL can be derived by evaluating the primary trajectory equations for τ_{max} . Evaluating Equation (25) at τ_{max} yields

$$x_{B'} = 1 + \ell \sin(\varphi + 2(\pi - \varphi)) - \cos(2(\pi - \varphi))$$

$\Rightarrow$

$$x_{B'} = 1 - \ell \sin\varphi - \cos(2\varphi)$$

$\Rightarrow$

$$x_{B'} = (2\sin\varphi - \ell)\sin\varphi$$

Evaluating Equation (26) at τ_{max} yields

$$y_{B'} = \ell \cos(\varphi + 2(\pi - \varphi)) + \sin(2(\pi - \varphi))$$

$\Rightarrow$

$$y_{B'} = \ell \cos \varphi - \sin(2\varphi)$$

$\Rightarrow$

$$y_{B'} = -(2\sin \varphi - \ell) \cos \varphi$$

So the parametric expression for the SL is

$$x_{B'} = (2\sin \varphi - \ell) \sin \varphi \tag{36}$$

$$y_{B'} = -(2\sin \varphi - \ell) \cos \varphi, \quad \bar{\varphi} \leq \varphi \leq \frac{\pi}{2} \tag{37}$$

Any retrograde trajectories that emanate from $0 \leq \varphi < \bar{\varphi}$ do not come into contact with the SL, they terminate on the NUP of the capture set. To fully characterize the SL it is necessary to derive an expression for $\bar{\varphi}$. For $x_{B'} > 0$ and $0 \leq \varphi \leq \frac{\pi}{2}$, the parameter φ must satisfy

$$2\sin \varphi - \ell > 0$$

Because $\bar{\varphi}$ delineates the trajectories that contact the capture set from the trajectories that contact the SL, $\bar{\varphi}$ satisfies

$$2\sin \bar{\varphi} - \ell = \ell$$

$\Rightarrow$

$$\bar{\varphi} = A \sin \ell$$

With this, the primary trajectories that contact the SL are valid for $Asin\ell \leq \varphi \leq \frac{\pi}{2}$. The separation between primary trajectories that contact the SL and primary trajectories that terminate on the capture set can be seen in Figure 9.

Based on the dynamics in Equations (36) and (37), the SL will contact the y-axis for a capture set radius $\ell = 0$. Since the Dubins car is a Pursuer with a circular capture set of radius $\ell > 0$, the SL will *never* terminate on the y-axis. Additionally, the SL is tangent to the barrier at B.

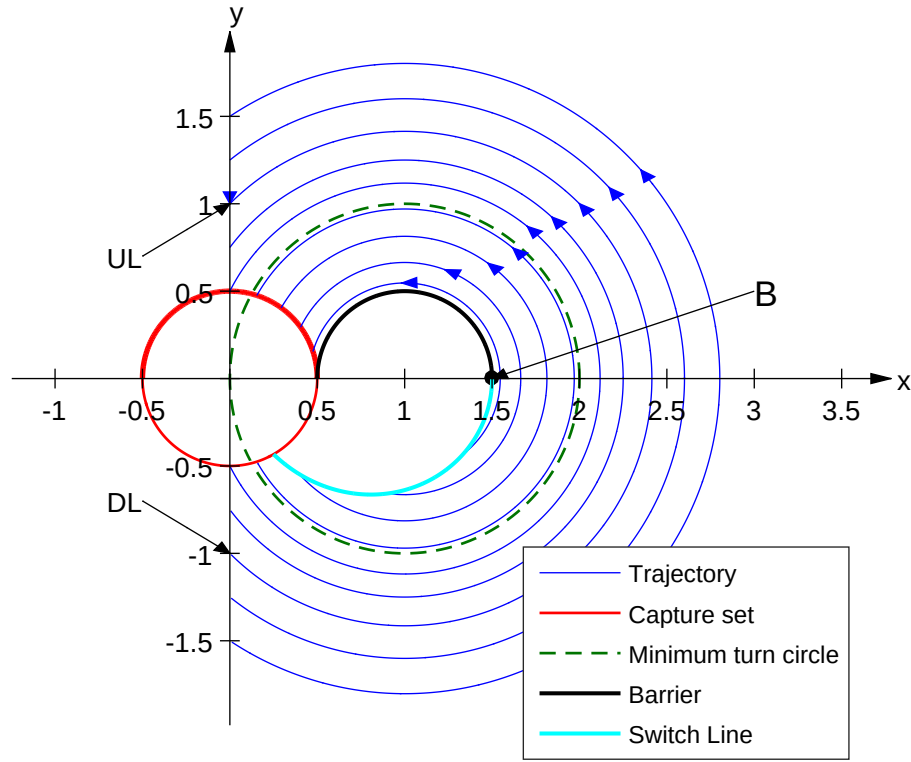


Figure 9. Dubins Car Optimal Control Problem with a Circular Capture Set: Switch Line in the Optimal Flow Field, $\ell = 0.5$

The reader should note that, in the Dubins car optimal control problem, because the tributary trajectory can be developed up to $\tau_{1max} = 2\pi$ and in Figure 9 the tributary trajectories do not contact the SL, the switch of the Pursuer's control is not present on the tributary trajectories.

3.2.7 Secondary Optimal Trajectories

The trajectories included between the barrier and the SL are the secondary optimal trajectories. The secondary trajectories emanate from the SL, in a retrograde manner, and in the RHP correspond to the Pursuer's optimal control being $u^* = -1$. Because the Pursuer's control switches on the SL, the trajectories that come into contact with the SL take on new dynamics that cause them to terminate on the barrier or the NUP of the capture set. Up until now, expressions for the optimal trajectories have been obtained by deriving the expressions using the *classical* Homicidal Chauffeur differential game dynamics and then reducing them to the Dubins car optimal control dynamics by setting $\mu = 0$; due to the complex nature of the secondary trajectories, in this section, the secondary optimal trajectories are derived based on the Dubins car optimal control dynamics in Equations (6) and (7). Applying the Pursuer control $u^* = -1$, Equations (6) and (7) become

$$\dot{x} = -y, \quad x(\tau_2 = 0) = 1 - \cos(2\theta) - \ell \sin\theta \quad (38)$$

$$\dot{y} = x + 1, \quad y(\tau_2 = 0) = \sin(2\theta) - \ell \cos\theta, \quad 0 \leq \tau_2, \quad \frac{\pi}{2} \leq \theta \leq \pi - A \sin(\ell) \quad (39)$$

with the initial position $(x(\tau_2 = 0), y(\tau_2 = 0))$ set as the point on the SL that contacts the barrier end point B. The expressions of the initial conditions are obtained by expressing Equations (36) and (37) in polar coordinates where $\varphi = \pi - \theta$. Additionally, τ_2 denotes the retrograde time measured from the time the secondary trajectory emanates from the SL.

Differentiating Equation (38) yields

$$\ddot{x} + x = -1, \quad x(\tau_2 = 0) = 1 - \cos(2\theta) - \ell \sin\theta, \quad \dot{x}(\tau_2 = 0) = \ell \cos\theta - \sin(2\theta)$$

Solving the above non-homogeneous harmonic equation and applying a similar derivation to Equation (39) yields expressions for the Evader's position relative to the Pursuer in the region of secondary trajectories

$$x(\tau_2) = (\ell \cos \theta - \sin(2\theta)) \sin \tau_2 + (2 - \cos(2\theta) - \ell \sin \theta) \cos \tau_2 - 1$$

$$y(\tau_2) = -(\ell \cos \theta - \sin(2\theta)) \cos \tau_2 + (2 - \cos(2\theta) - \ell \sin \theta) \sin \tau_2$$

These trajectories correspond to the scenario where the Evader is initially located in the RHP. For a retrograde time $\tau_2 \geq 0$, the above equations can be expressed as

$$\begin{pmatrix} x+1 \\ y \end{pmatrix} = \begin{bmatrix} \ell \cos \theta - \sin(2\theta) & 2 - \cos(2\theta) - \ell \sin \theta \\ 2 - \cos(2\theta) - \ell \sin \theta & -(\ell \cos \theta - \sin(2\theta)) \end{bmatrix} \begin{pmatrix} \sin \tau \\ \cos \tau \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \sin \tau \\ \cos \tau \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} \ell \cos \theta - \sin(2\theta) & 2 - \cos(2\theta) - \ell \sin \theta \\ 2 - \cos(2\theta) - \ell \sin \theta & -(\ell \cos \theta - \sin(2\theta)) \end{bmatrix} \begin{pmatrix} x+1 \\ y \end{pmatrix}$$

where

$$\Delta = [(\ell - 3\sin \theta)^2 + \cos^2 \theta]$$

$\Rightarrow$

$$\sin \tau = \frac{1}{\Delta} [(x+1)(\ell \cos \theta - \sin(2\theta)) + y(2 - \cos(2\theta) - \ell \sin \theta)]$$

$$\cos \tau = \frac{1}{\Delta} [(x+1)(2 - \cos(2\theta) - \ell \sin \theta) - y(\ell \cos \theta - \sin(2\theta))]$$

Using the trigonometric identity $\sin^2 \tau + \cos^2 \tau = 1$

$$1 = \frac{1}{((\ell - 3\sin \theta)^2 + \cos^2 \theta)^2} [(x+1)^2((\ell - 3\sin \theta)^2 + \cos^2 \theta) + y^2((\ell - 3\sin \theta)^2 + \cos^2 \theta)]$$

$\Rightarrow$

$$(x+1)^2 + y^2 = (\ell - 3\sin\theta)^2 + \cos^2\theta, \quad \frac{\pi}{2} \leq \theta < \pi - A\sin\ell$$

Thus, for a Dubins car endowed with a circular capture set, the secondary optimal trajectories in the reduced state space are a family of concentric circular arcs centered at $(-1, 0)$ with a radius of $\sqrt{(\ell - 3\sin\theta)^2 + \cos^2\theta}$, $\frac{\pi}{2} \leq \theta < \pi - A\sin\ell$.

The family of retrograde secondary trajectories parameterized by $\frac{\pi}{2} \leq \theta < \bar{\bar{\theta}}$ terminate on the barrier and the retrograde trajectories parameterized by $\bar{\bar{\theta}} \leq \theta < \bar{\theta}$ terminate on the capture circle, where $\bar{\theta} = \pi - A\sin\ell$. The secondary trajectory emanating from $\bar{\bar{\theta}}$ delineates the trajectories that terminate on the capture set from those that terminate on the barrier. This secondary trajectory contacts the capture set on the x-axis at $1+\ell$ which is measured from the center of the secondary trajectories $(-1, 0)$. So $\bar{\bar{\theta}}$ is determined by the condition

$$\rho(\bar{\bar{\theta}}) = 1 + \ell$$

$\Rightarrow$

$$(\ell - 3\sin\bar{\bar{\theta}})^2 + \cos^2\bar{\bar{\theta}} = (1 + \ell)^2$$

$\Rightarrow$

$$4\sin^2\bar{\bar{\theta}} - 3\ell\sin\bar{\bar{\theta}} - \ell = 0$$

$\Rightarrow$

$$\bar{\bar{\theta}} = \pi - A\sin\left(\frac{3\ell + \sqrt{9\ell^2 + 16\ell}}{8}\right)$$

The secondary trajectories which correspond to $\bar{\bar{\theta}} \leq \theta < \pi - A\sin\ell$ will terminate on the capture circle while the secondary trajectories which correspond to $\frac{\pi}{2} \leq \theta < \bar{\bar{\theta}}$ will terminate on the barrier as seen in Figure 16.

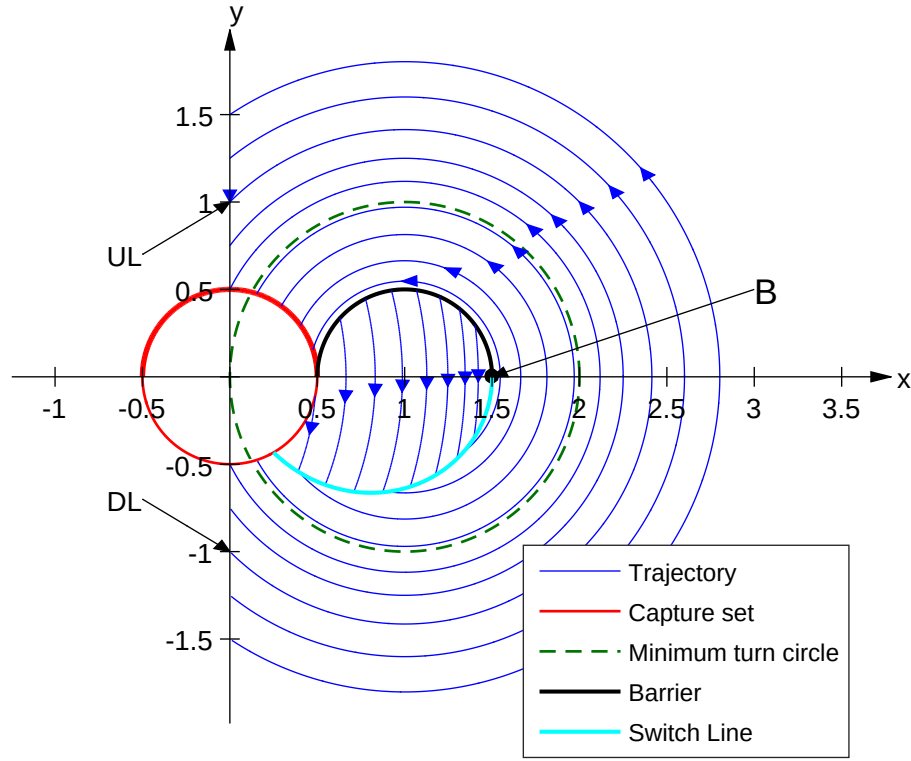


Figure 10. Dubins Car Optimal Control Problem with a Circular Capture Set: Secondary Optimal Trajectories in the Optimal Flow Field, $\ell = 0.5$

With this, the full optimal flow field has been constructed for the Dubins car optimal control problem for the case of a circular capture set endowed Pursuer where $\ell < 1$.

3.2.8 Case 2: $\ell \geq 1$

The case where the parameter $\ell \geq 1$ is now examined. In the Homicidal Chauffeur differential game, the barrier only forms if $\mu^2 + \ell^2 < 1$. Because of the definition of the barrier's radius from Section 3.2.3, the barrier conditional $\mu^2 + \ell^2 < 1$ still applies to the Dubins car optimal control problem. With $\ell \geq 1$ the barrier end point B is encompassed by the capture set, so there is no barrier, secondary trajectories, or SL in the Dubins car optimal control problem when $\mu^2 + \ell^2 < 1$.

As before, the primary trajectories emanating from the UP of the circular capture set are

$$x(\tau) = 1 + \ell \sin(\varphi + \tau) - \cos\tau$$

$$y(\tau) = \ell \cos(\varphi + \tau) + \sin\tau, \quad 0 \leq \tau \leq \tau_{max}, \quad 0 \leq \varphi < \frac{\pi}{2}$$

Similar to the analysis of the case where $\ell < 1$, the primary trajectories are a family of circular arcs centered at $(1, 0)$ with a radius of $\sqrt{\ell^2 - 2\ell \sin\varphi + 1}$ as seen in Figures 11 and 12.

As before, the tributary trajectories emanating from the y-axis (the UL) are

$$x(\tau_1) = 1 + s \cdot \sin\tau_1 - \cos\tau_1$$

$$y(\tau_1) = s \cdot \cos\tau_1 + \sin\tau_1, \quad 0 \leq \tau_1 \leq \tau_{1max}, \quad s > \ell$$

Similar to the case where $\ell < 1$, the trajectories are concentric circles centered at $(1, 0)$ with a radius of $\sqrt{1 + s^2}$ that terminate on the y-axis at $y = -s < -\ell$ as seen in Figures 11 and 12.

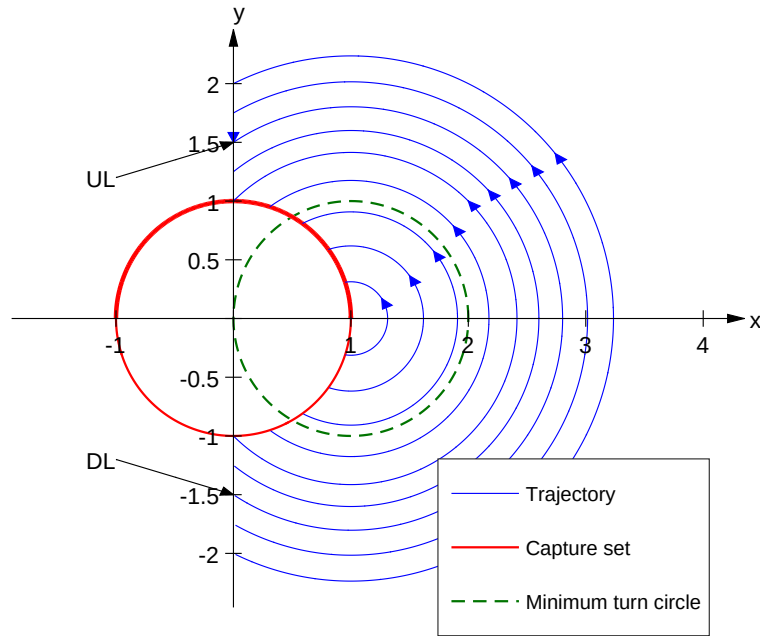


Figure 11. Dubins Car Optimal Control Problem with a Circular Capture Set: Full Optimal Flow Field, $\ell = 1$

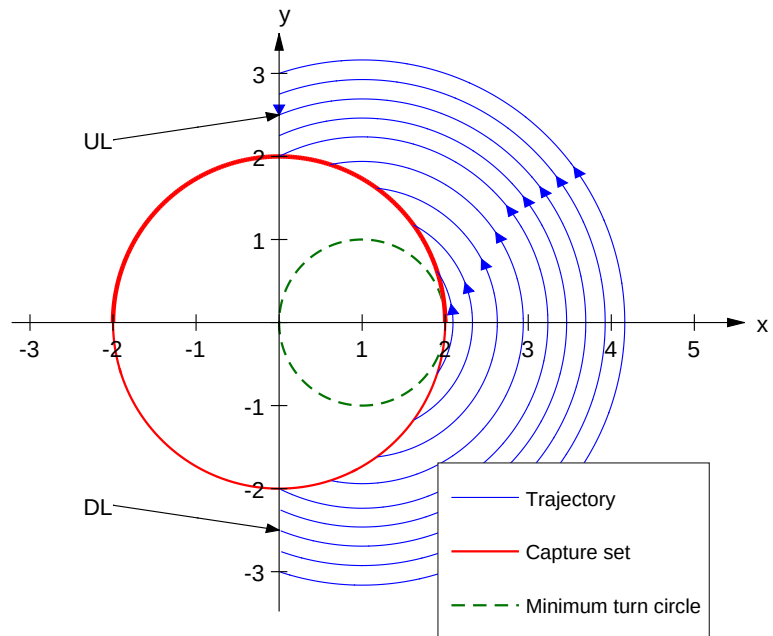


Figure 12. Dubins Car Optimal Control Problem with a Circular Capture Set: Full Optimal Flow Field, $\ell = 2$

With the optimal flow field constructed for $\ell \geq 1$, the full solution to the Dubins car optimal control problem for the case of a Pursuer endowed with a circular capture set has been characterized.

3.3 Dubins Car with a Line Segment Capture Set

As with the circular capture set: the primary, secondary, and tributary optimal trajectories, as well as the barrier and SL are present in the optimal control problem of a Dubins car endowed with a line segment capture set. Similar to Section 3.2, in this section the components of the optimal flow field are systematically built as each component of the solution is derived. As before, the trajectories are derived in the Homicidal Chauffeur differential game and will then be reduced to the Dubins car optimal control problem by setting the speed ratio $\mu = 0$.

In the sections below, the solution to the Dubins car optimal control problem is derived for the case of a line segment capture set endowed Pursuer. The solution is derived for the case of the Evader initially positioned in the RHP of the reference frame. While the solution is only derived for the case of the Evader initially positioned in the RHP, it is important to note that the optimal flow field in the LHP is a mirror reflection of the solution in the RHP.

The solution to this optimal control problem is initially depicted for $\ell = 1$. Following the construction of the optimal flow field, the optimal flow field for additional values of ℓ are shown.

3.3.1 Primary Optimal Trajectories

Primary optimal trajectories are retrograde trajectories that emanate from the UP of the line segment capture set $UP = \{(x, y) | x = 0, 0 < y \leq \ell\}$. To obtain these optimal trajectories, the Hamiltonian of the Homicidal Chauffeur differential game

is examined. Although the nature of the capture set has changed from the optimal control problem in Section 3.2, the problem's dynamics and cost remain the same. Therefore, the Hamiltonian is

$$H(x, \lambda, u, \Psi) = -1 + \lambda_x(-uy + \mu \sin \Psi) + \lambda_y(ux - 1 + \mu \cos \Psi)$$

$\Rightarrow$

$$H = -1 + u(x\lambda_y - y\lambda_x) - \lambda_y + \mu(\lambda_x \sin \Psi + \lambda_y \cos \Psi) \quad (40)$$

From Equation (40), and according to the PMP, to minimize the time-to-capture, the Pursuer must maximize the Hamiltonian by selecting its control such that

$$u^* = \text{sign}(x\lambda_y - y\lambda_x) \quad (41)$$

Despite the new capture set, it is still always optimal for the Pursuer to turn as hard as possible as long as $x(\tau)\lambda_y(\tau) - y(\tau)\lambda_x(\tau) \neq 0$.

From Equation (40), and according to the PMP, to maximize the time-to-capture, the Evader must minimize the Hamiltonian by setting its heading angle Ψ such that

$$\sin \Psi^* = -\frac{\lambda_x}{\sqrt{\lambda_x^2 + \lambda_y^2}}, \quad \cos \Psi^* = -\frac{\lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}} \quad (42)$$

According to the PMP, the costates satisfy the differential equations

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -\lambda_y u \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = \lambda_x u \quad (43)$$

To obtain expressions for the primary optimal trajectories, the dynamics in Equations (4) and (5) will be rewritten in retrograde time $\tau = t_f - t$. The dynamics in

retrograde time τ are

$$\dot{x} = uy - \mu \sin \Psi, \quad x(\tau = 0) = 0 \quad (44)$$

$$\dot{y} = -ux + 1 - \mu \cos \Psi, \quad y(\tau = 0) = y_0, \quad \mu \leq y, \quad 0 \leq \tau \leq \tau_{max} \quad (45)$$

where $(0, y_0) \in UP$. The retrograde time τ_{max} denotes the maximum retrograde time a trajectory is valid for; it is determined following the derivation of the trajectory expressions.

The Evader is considered captured at $\tau = 0$ when located on the UP of the capture set. In the RHP, in the region of the primary optimal trajectories, the Pursuer's optimal control $u^* = 1$. Thus, the transversality condition of the PMP yields

$$\lambda_x(\tau = 0) = -a, \quad \lambda_y(\tau = 0) = 0, \quad a > 0 \quad (46)$$

when the game evolves in the RHP. Using the relationships in Equation (43) yields the harmonic equations

$$\ddot{\lambda}_x + \lambda_x = 0, \quad \ddot{\lambda}_y + \lambda_y = 0$$

With the transversality conditions, the solutions to the harmonic equations yield the costates

$$\lambda_x(\tau) = -a \cdot \cos \tau, \quad \lambda_y(\tau) = a \cdot \sin \tau \quad (47)$$

The optimal control of the Evader can now be expressed in terms of time. Applying the results above to Equation (42) yields

$$\sin(\Psi^*(\tau)) = \cos \tau, \quad \cos(\Psi^*(\tau)) = -\sin \tau$$

From the equation above, it can be determined that, in the region of primary trajec-

tories, the optimal control of the Evader is

$$\Psi^*(\tau) = \frac{\pi}{2} + \tau, \quad \tau \geq 0$$

Using the expression for the Evader's optimal control, from the dynamics in Equations (44) and (45), the primary optimal trajectories are expressed as

$$\dot{x} = y - \mu \cos \tau, \quad x(\tau = 0) = 0 \quad (48)$$

$$\dot{y} = -x + 1 + \mu \sin \tau, \quad y(\tau = 0) = y_f, \quad \mu < y_f \leq \ell \quad (49)$$

when the game evolves in the RHP. Here, y_f is the y-coordinate of the Evader at termination. Differentiating Equation (48) yields

$$\ddot{x} = \dot{y} + \mu \sin \tau$$

which yields the non-homogeneous harmonic equation

$$\ddot{x} + x = 1 + 2\mu \sin \tau, \quad x(\tau = 0) = 0, \quad \dot{x}(\tau = 0) = y_f - \mu \quad (50)$$

Solving the above non-homogeneous harmonic equation, and applying a similar derivation to Equation (49), yields expressions for the Evader's position relative to the Pursuer in the region of primary trajectories

$$x(\tau) = 1 + y_f \sin \tau - (1 + \mu \tau) \cos \tau \quad (51)$$

$$y(\tau) = y_f \cos \tau + (1 + \mu \tau) \sin \tau, \quad 0 \leq \tau \leq \tau_{max}, \quad \mu < y_f \leq \ell \quad (52)$$

To obtain expressions for the primary trajectories for the Dubins car optimal control

problem, set $\mu = 0$ in the equations above. The result is a family of primary optimal trajectories parameterized by y_f and defined as

$$x(\tau) = 1 + y_f \sin \tau - \cos \tau \quad (53)$$

$$y(\tau) = y_f \cos \tau + \sin \tau, \quad 0 \leq \tau \leq \tau_{max}, \quad 0 < y_f \leq \ell \quad (54)$$

These trajectories correspond to the scenario where the Evader is initially in the RHP.

A primary trajectory persists until the Pursuer's control switches from 1 to -1 or vice versa, as indicated by the switching function $S(\cdot)$ which features in the Hamiltonian and determines the domain of validity of the primary trajectories. From the Hamiltonian in Equation (40), the switching function is

$$S(\tau) = x(\tau)\lambda_y(\tau) - y(\tau)\lambda_x(\tau) > 0, \quad 0 \leq \tau \leq \tau_{max}$$

Inserting Equations (47), (53), and (54) into the switching function above yields

$$S(\tau) = a \cdot y_f + a \cdot \sin \tau$$

The maximum retrograde time τ_{max} possible, before the Pursuer's control switches, occurs when $S(\tau) = 0$. Applying $S(\tau) = 0$, the switching condition becomes

$$\sin \tau_{max} = -y_f$$

which yields

$$\tau_{max} = \pi + A \sin(y_f), \quad 0 \leq y_f \leq \ell \quad (55)$$

Obviously Equation (55) is only valid for $0 \leq y_f \leq 1$. The equation above will always be true for $\ell \leq 1$. If the line segment capture set $\ell > 1$, then trajectories which

emanate from $(0, y_f)$, $1 < y_f < \ell$ will reach the negative y-axis. Note that a primary trajectory which emanates from $(0, y_f)$, $y_f \geq 1$ terminates at $(0, -y_f)$. Because the trajectories in the RHP are mirrored by those in the LHP, once a trajectory reaches the y-axis is terminates.

So, the family of primary trajectories parameterized by $\mu < y_f \leq \ell \leq 1$ is

$$x(\tau) = 1 + y_f \sin \tau - \cos \tau \quad (56)$$

$$y(\tau) = y_f \cos \tau + \sin \tau, \quad 0 < y_f \leq 1, \quad 0 \leq \tau \leq \pi + A \sin(y_f) \quad (57)$$

If $\ell > 1$, then for $1 < y_f \leq \ell$, the trajectories (56) and (57) terminate on the negative y-axis. For a retrograde time $\tau \geq 0$, Equations (56) and (57) can be expressed as

$$\begin{pmatrix} x - 1 \\ y \end{pmatrix} = \begin{bmatrix} y_f & -1 \\ 1 & y_f \end{bmatrix} \begin{pmatrix} \sin \tau \\ \cos \tau \end{pmatrix}$$

$\Rightarrow$

$$\begin{pmatrix} \sin \tau \\ \cos \tau \end{pmatrix} = \frac{1}{1 + y_f^2} \begin{bmatrix} y_f & 1 \\ -1 & y_f \end{bmatrix} \begin{pmatrix} x - 1 \\ y \end{pmatrix}$$

$\Rightarrow$

$$\sin \tau = \frac{1}{1 + y_f^2} [(x - 1)y_f + y]$$

$$\cos \tau = \frac{1}{1 + y_f^2} [-(x - 1) + y \cdot y_f]$$

Using the trigonometric identity $\sin^2 \tau + \cos^2 \tau = 1$ yields

$$1 = \frac{1}{(1 + y_f^2)^2} [(x - 1)^2(1 + y_f^2) + y^2(1 + y_f^2)]$$

$\Rightarrow$

$$(x - 1)^2 + y^2 = 1 + y_f^2, \quad 0 < y_f \leq \ell$$

Thus, for a Dubins car endowed with a line segment capture set, the primary optimal trajectories in the reduced state space are a family of concentric circular arcs centered at $(1, 0)$ with a radius of $\sqrt{1 + y_f^2}$ where $0 < y_f \leq \ell$, as seen in Figure 13.

3.3.2 The Barrier

An abnormal trajectory, the barrier, emanates from the BUP at $y_f = \mu$. As discussed in Section 3.2.3, the barrier is a locus of initial conditions where the Pursuer's strategy is to turn away from the Evader. This means that the barrier is not an optimal trajectory traversable by the Evader. Despite the change in the capture set type from Section 3.2.3, the barrier in this optimal control problem delineates the regions of the Pursuer's optimal strategies.

Following the process used to obtain the primary trajectories, although the barrier is not an optimal trajectory, deriving its dynamics reveals that it is equivalent to a primary trajectory emanating from the BUP. Therefore, in the Dubins car optimal control problem, for a Pursuer endowed with a line segment capture set, where $\mu = 0$, the barrier trajectory emanates from the BUP at $y_f = 0$. In the Dubins car optimal control problem the barrier is a semicircle centered at $(1, 0)$ with a radius of 1.

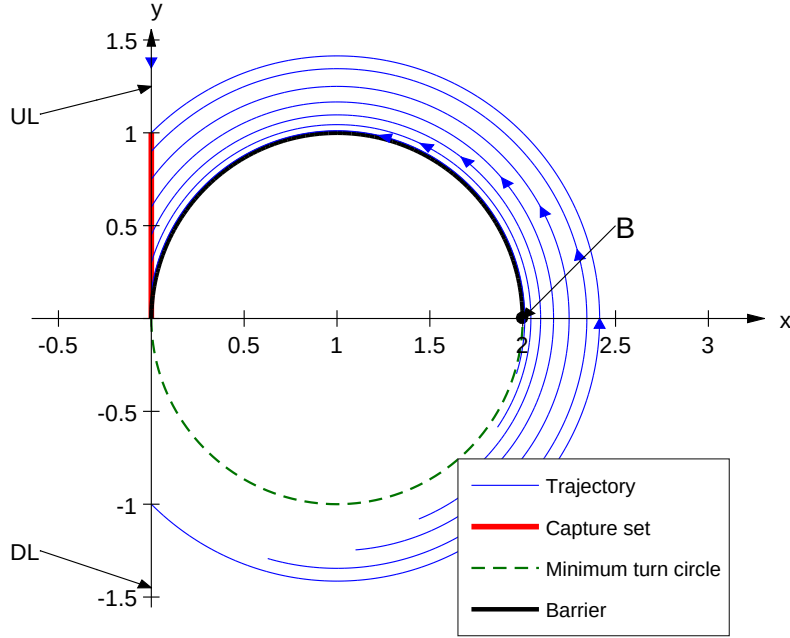


Figure 13. Dubins Car Optimal Control Problem with a Line Segment Capture Set: Primary Optimal Trajectories in the Optimal Flow Field, $\ell = 1$

In the optimal control problem of a Dubins car endowed with a line segment capture set, the nature of the Pursuer's strategies on the UL and DL, shown in Section 3.2.4 remains the same. Once on the UL it is optimal for the Pursuer to employ LOS pure pursuit. In the Dubins car optimal control problem, if the stationary Evader is positioned on the DL, the Pursuer may turn in either direction without consequence.

3.3.3 Tributary Optimal Trajectories

Tributary trajectories are retrograde trajectories that emanate from the UL on the positive y-axis. Because they emanate from the UL, changes to the capture set do not affect the dynamics of the tributary trajectories. Thus, for this optimal control problem, the derivation of expressions for the tributary trajectories is unnecessary since the results of Section 3.2.5 are unchanged by the new capture set. The parametric

representation of the tributary trajectories is still

$$x(\tau_1) = 1 + s \cdot \sin\tau_1 - \cos\tau_1 \quad (58)$$

$$y(\tau_1) = s \cdot \cos\tau_1 + \sin\tau_1, \quad 0 \leq \tau_1 \leq 2\pi, \quad s \geq \ell \quad (59)$$

where

$$(x - 1)^2 + y^2 = 1 + s^2$$

Thus, for a Dubins car endowed with a line segment capture set, the tributary trajectories in the reduced state space are a family of concentric circular arcs parameterized by $s > \ell$, centered at $(1, 0)$ with a radius of $\sqrt{1 + s^2}$, as seen in Figure 14.

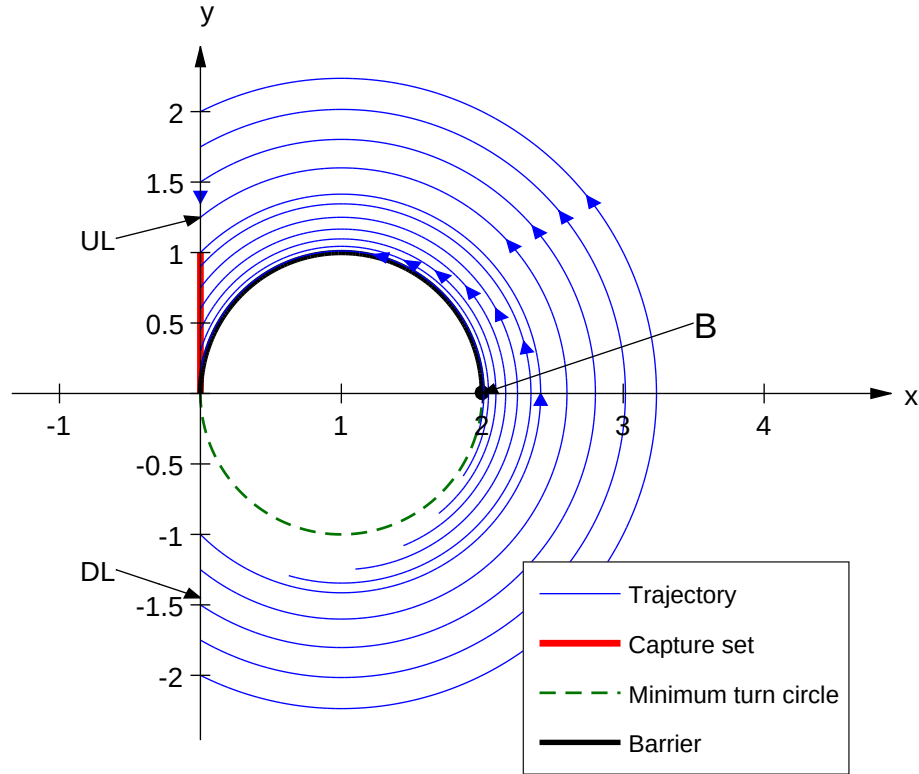


Figure 14. Dubins Car Optimal Control Problem with a Line Segment Capture Set: Tributary Optimal Trajectories in the Optimal Flow Field, $\ell = 1$

3.3.4 Switch Line

In this section, an expression for the SL is developed. The method used to develop the SL is only applicable to the Dubins car optimal control problem where $\mu = 0$. The differential game analogue to the SL, the ES, is fundamentally different from the SL. Although it is constructed for $\mu = 0$, the SL does *not* coincide with an ES constructed using Isaacs' method evaluated for $\mu = 0$.

For the optimal control problem of a Dubins car endowed with a line segment capture set, the SL is derived based on the primary trajectories. Due to the effect of τ_{max} on trajectories, the primary trajectories are used to form an outline of the SL. Since the tributary trajectories can be evaluated up to $\tau_{1max} = 2\pi$, they do not form this outline. For this reason, the construction of the SL is based on the primary trajectories.

A primary trajectory persists until time τ_{max} when the Pursuer's control u switches from 1 to -1 in the RHP, or vice versa in the LHP. Recalling the switching function for primary trajectories

$$S(\tau) = a \cdot y_f + a \cdot \sin\tau$$

In the RHP, the retrograde primary trajectories persist for $u(\tau)^* = 1$ until the maximum retrograde time $\tau_{max} = \pi + A\sin(y_f)$ at which point the Pursuer's control switches to $u(\tau)^* = -1$. The parametric expression for the SL is derived as follows.

Equations (56) and (57) are valid until they meet the SL at point $B'(y_f)$, $0 \leq y_f \leq 1$ where B' denotes the end point of the primary trajectory evaluated for τ_{max} . Since the trajectories encounter the SL at the maximum retrograde time τ_{max} , the parametric equations for the SL can be derived by evaluating the primary trajectory equations for τ_{max} . Evaluating Equations (56) and (57) at $\tau_{max} = \pi + A\sin(y_f)$ yields

$$x_{B'}(y_f) = 1 + y_f \sin(\pi + A\sin(y_f)) - \cos(\pi + A\sin(y_f))$$

$$y_{B'}(y_f) = y_f \cos(\pi + A \sin(y_f)) + \sin(\pi + A \sin(y_f))$$

Simplifying the above equations results in the parametric expression for the SL

$$x_{B'}(y_f) = 1 - y_f^2 + \sqrt{1 - y_f^2} \quad (60)$$

$$y_{B'}(y_f) = -y_f \sqrt{1 - y_f^2} - y_f, \quad 0 \leq y_f \leq \ell \quad (61)$$

With this, the primary trajectories that contact the SL are valid for $0 \leq y_f \leq \ell \leq 1$. When $\ell \geq 1$: evaluating the above equations for $y_f = 1$ reveals that the SL contacts the y-axis at $(0, -1)$ similar to the primary trajectory emanating from $\ell = 1$. Since the end point of the SL will not change for any $\ell \geq 1$, it is known that the primary trajectory emanating from $\ell = 1$ delineates the primary trajectories that contact the SL from those that terminate on the y-axis. Furthermore, it is readily verifiable that the SL is tangent to the barrier at the barrier end point B, as seen in Figure 15.

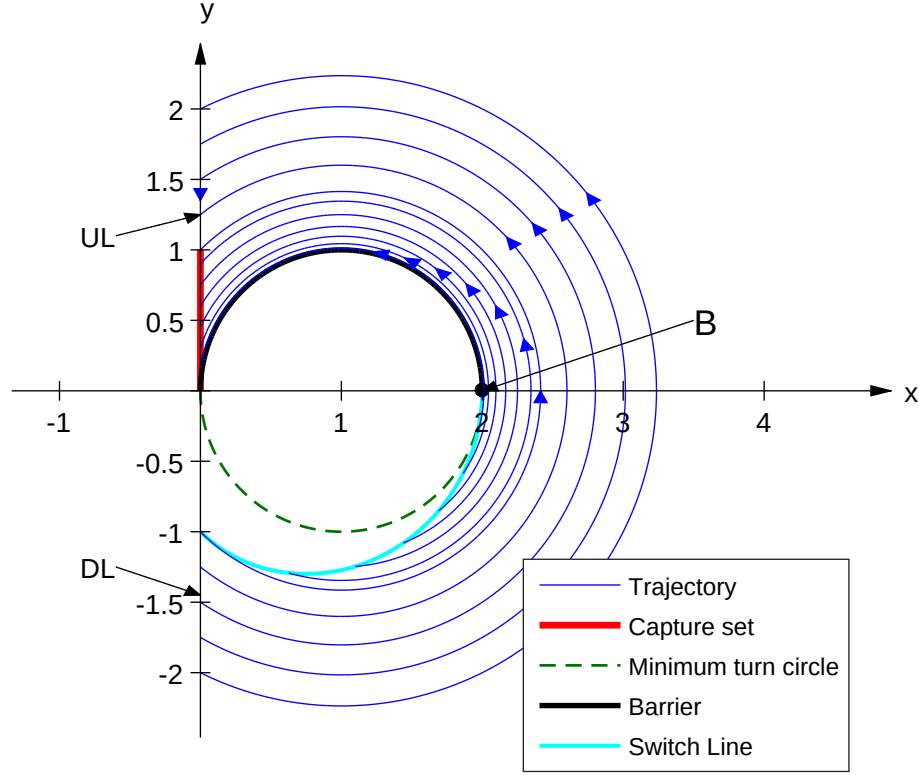


Figure 15. Dubins Car Optimal Control Problem with a Line Segment Capture Set: Switch Line in the Optimal Flow Field, $\ell = 1$

3.3.5 Secondary Optimal Trajectories

The trajectories included between the barrier and the SL are the secondary optimal trajectories. The secondary trajectories emanate in a retrograde manner from the SL and in the RHP correspond to the Pursuer's optimal control being $u^*(\tau) = -1$. Unlike the optimal control problem in Section 3.2, there is no NUP of the capture set in this optimal control problem. Thus, the secondary trajectories emanating from the SL only terminate on the barrier. Up until now, expressions for the optimal trajectories have been obtained by deriving the expressions using the *classical* Homicidal Chauffeur differential game dynamics and then reducing them to the Dubins car optimal control dynamics by setting $\mu = 0$; due to the complex nature of the secondary

trajectories, in this section, the secondary optimal trajectories are derived based on the Dubins car optimal control dynamics in Equations (6) and (7). Applying the Pursuer's control $u^*(\tau) = -1$, Equations (6) and (7) become

$$\dot{x} = -y, \quad x(\tau_2 = 0) = 1 - y_f^2 + \sqrt{1 - y_f^2} \quad (62)$$

$$\dot{y} = x + 1, \quad y(\tau_2 = 0) = -y_f \sqrt{1 - y_f^2} - y_f, \quad 0 \leq y_f \leq \ell \quad (63)$$

when $y_f = 0$, the 'initial' position $(x(\tau_2 = 0), y(\tau_2 = 0))$ is set as the point on the SL that contacts the barrier end point B. Additionally, τ_2 denotes the retrograde time measured from the time the secondary trajectory emanates from the SL.

Differentiating Equation (62) yields

$$\ddot{x} + x = -1, \quad x(\tau_2 = 0) = 1 - y_f^2 + \sqrt{1 - y_f^2}, \quad \dot{x}(\tau_2 = 0) = y_f \sqrt{1 - y_f^2} + y_f$$

Solving the above non-homogeneous harmonic equation and applying a similar derivation to Equation (63) yields expressions for the Evader's position relative to the Pursuer in the region of secondary trajectories

$$x(\tau_2) = [y_f \sqrt{1 - y_f^2} + y_f] \sin \tau + [2 - y_f^2 + \sqrt{1 - y_f^2}] \cos \tau - 1$$

$$y(\tau_2) = [2 - y_f^2 + \sqrt{1 - y_f^2}] \sin \tau - [y_f \sqrt{1 - y_f^2} + y_f] \cos \tau$$

These trajectories correspond to the scenario where the Evader is initially in the RHP.

For a retrograde time $\tau_2 \geq 0$, the above equations can be expressed as

$$\begin{pmatrix} x + 1 \\ y \end{pmatrix} = \begin{bmatrix} y_f \sqrt{1 - y_f^2} + y_f & 2 - y_f^2 + \sqrt{1 - y_f^2} \\ 2 - y_f^2 + \sqrt{1 - y_f^2} & -y_f \sqrt{1 - y_f^2} - y_f \end{bmatrix} \begin{pmatrix} \sin \tau \\ \cos \tau \end{pmatrix}$$

$\Rightarrow$

$$\begin{pmatrix} \sin\tau \\ \cos\tau \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} y_f \sqrt{1-y_f^2} + y_f & 2 - y_f^2 + \sqrt{1-y_f^2} \\ 2 - y_f^2 + \sqrt{1-y_f^2} & -y_f \sqrt{1-y_f^2} - y_f \end{bmatrix} \begin{pmatrix} x+1 \\ y \end{pmatrix}$$

where the determinant

$$\Delta = 4\sqrt{1-y_f^2} - 3y_f^2 + 5$$

$\Rightarrow$

$$\sin\tau = \frac{1}{\Delta} \left[(x+1) (y_f \sqrt{1-y_f^2} + y_f) + y (2 - y_f^2 + \sqrt{1-y_f^2}) \right]$$

$$\cos\tau = \frac{1}{\Delta} \left[(x+1) (2 - y_f^2 + \sqrt{1-y_f^2}) - y (y_f \sqrt{1-y_f^2} + y_f) \right]$$

Using the trigonometric identity $\sin^2\tau + \cos^2\tau = 1$

$$1 = \frac{1}{\left(4\sqrt{1-y_f^2} - 3y_f^2 + 5\right)^2} \left[(x+1)^2 \left(4\sqrt{1-y_f^2} - 3y_f^2 + 5\right) + y^2 \left(4\sqrt{1-y_f^2} - 3y_f^2 + 5\right) \right]$$

$\Rightarrow$

$$(x+1)^2 + y^2 = 4\sqrt{1-y_f^2} - 3y_f^2 + 5, \quad 0 \leq y_f \leq \ell$$

Thus, for a Dubins car endowed with a line segment capture set, the secondary optimal trajectories in the reduced state space are a family of concentric circular arcs centered at $(-1, 0)$ with a radius of $\sqrt{4\sqrt{1-y_f^2} - 3y_f^2 + 5}$ where $0 \leq y_f \leq \ell$ as seen in Figure 16.

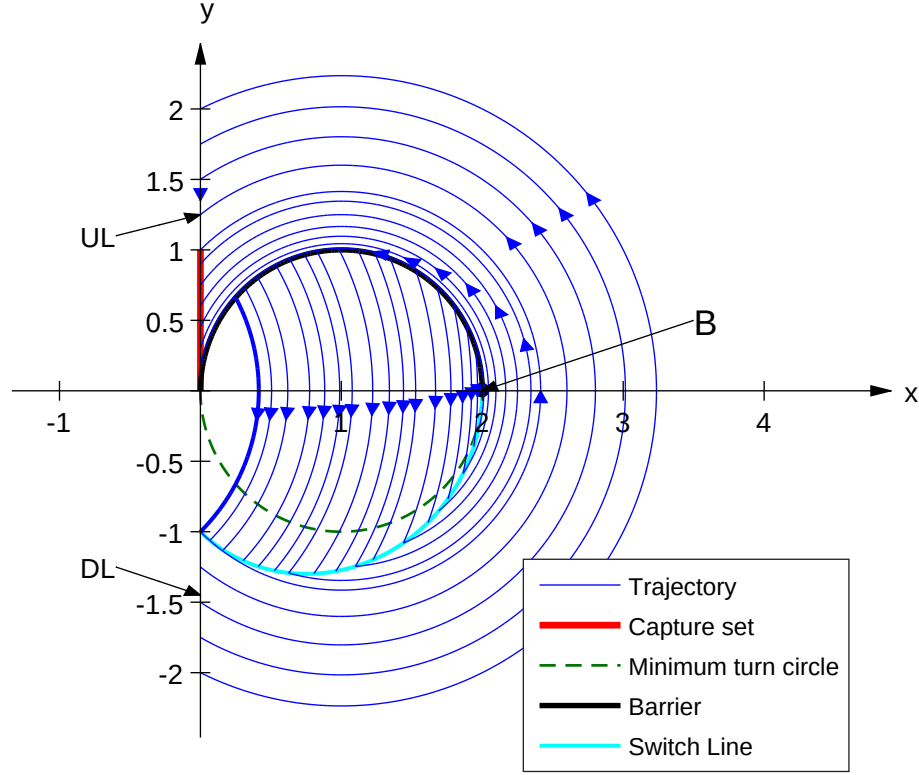


Figure 16. Dubins Car Optimal Control Problem with a Line Segment Capture Set: Secondary Optimal Trajectories in the Optimal Flow Field, $\ell = 1$

Because the radius is $\sqrt{4\sqrt{1 - y_f^2} - 3y_f^2 + 5}$, a final y-coordinate greater than 1 is not feasible. For this reason, the trajectories located in the shaded region in Figure 17 still need to be characterized. Similar to Isaacs' Homicidal Chauffeur differential game [1] there is a hole in the state space that needs to be filled with optimal trajectories.

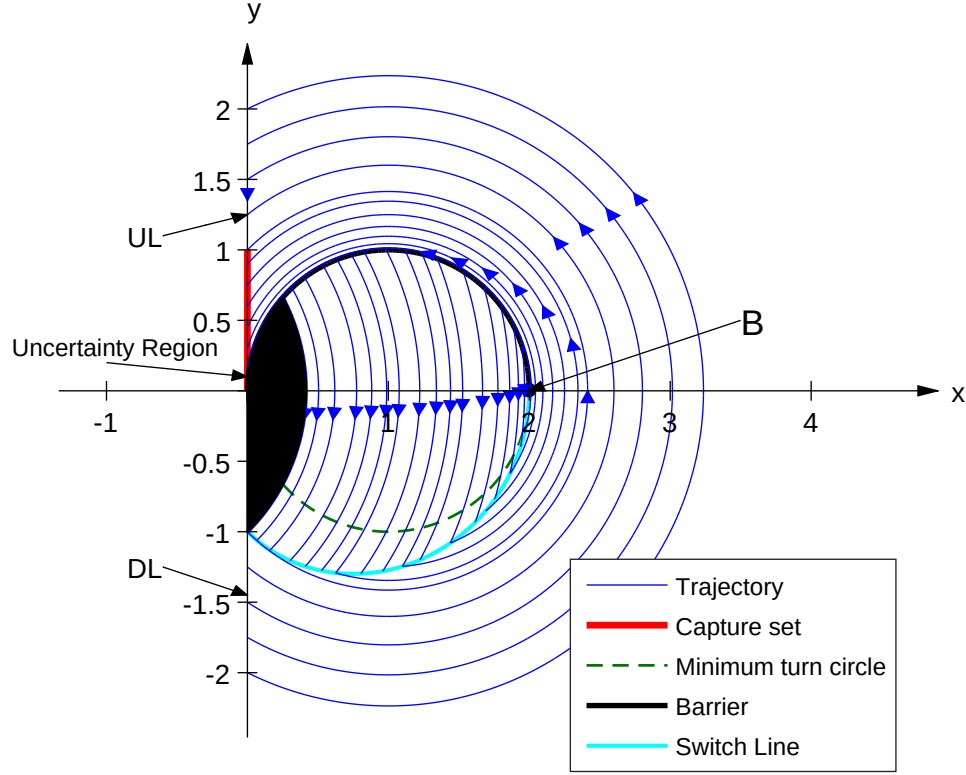


Figure 17. Dubins Car Optimal Control Problem with a Line Segment Capture Set: Uncertainty Region in the Optimal Flow Field $\ell = 1$

To understand the optimal control solution for an Evader initially located in the shaded state space region of Figure 17, consider the following: the negative y-axis segment $0 \geq y \geq -\ell$ is a UL similar to the positive y-axis beyond the line segment capture set. On the negative y-axis UL, a reversal of roles is seen. If the Evader is located on this UL it is optimal for the Pursuer to run from the Evader until the Evader reaches the DL. Once the Evader is on the DL, the Pursuer will resume the strategy of turning hard towards the Evader with $u^* = 1$ in the RHP. The remaining secondary trajectories located in the shaded region of Figure 17 emanate from this UL on the negative y-axis in a retrograde manner and terminate on the barrier.

Similar to the UL on the positive y-axis, on the negative y-axis UL, $V_x \Big|_{x=0} = 0$

and in the Homicidal Chauffeur differential game $V_y|_{x=0} = \frac{1}{1-\mu} \quad \forall \quad 0 > y > -1$ is calculated. Hence, in the Dubins car optimal control problem when the speed ratio parameter $\mu = 0$ the costates of these additional secondary trajectories satisfy

$$\dot{\lambda}_x = \lambda_y u, \quad \lambda_x(\tau_3 = 0) = -V_x|_{x=0} = 0$$

$$\dot{\lambda}_y = \lambda_x u, \quad \lambda_y(\tau_3 = 0) = -V_y|_{x=0} = -\frac{1}{1-\mu} = -1$$

In the region of the secondary trajectories $u^* = -1$. Thus, the costates of these additional secondary trajectories are

$$\lambda_x(\tau_1) = -\frac{1}{1-\mu} \sin \tau_3, \quad \lambda_y(\tau_1) = \frac{1}{1-\mu} \cos \tau_3 \quad (64)$$

Here, τ_3 denotes the retrograde time measured from the time the secondary trajectory emanates from the negative y-axis.

Since, in this region of the RHP, $u^* = -1$, the corresponding retrograde dynamics of the Dubins car optimal control problem are

$$\dot{x} = -y, \quad x(\tau_3 = 0) = 0 \quad (65)$$

$$\dot{y} = x + 1, \quad y(\tau_3 = 0) = s, \quad 0 > s > -1 \quad (66)$$

Differentiating Equation (65) yields the non-homogeneous harmonic equation

$$\ddot{x} + x = 1, \quad x(\tau_3 = 0) = 0, \quad \dot{x}(\tau_3 = 0) = -s$$

Solving the above non-homogeneous equation and applying a similar derivation to Equation (66) yields expressions for the Evader's position relative to the Pursuer in

the additional region of secondary trajectories

$$x(\tau_2) = -s \cdot \sin\tau_3 + \cos\tau_3 - 1$$

$$y(\tau_2) = s \cdot \cos\tau_3 + \sin\tau_3, \quad \tau_3 \geq 0$$

These trajectories correspond to the scenario where the Evader is initially located in the RHP. For a retrograde time $\tau_3 \geq 0$ the above equations can be written as

$$\begin{pmatrix} x+1 \\ y \end{pmatrix} = \begin{bmatrix} -s & 1 \\ 1 & s \end{bmatrix} \begin{pmatrix} \sin\tau_3 \\ \cos\tau_3 \end{pmatrix}$$

$\Rightarrow$

$$\sin\tau_3 = \frac{1}{1+s^2}[-s(x+1) + y]$$

$$\cos\tau_3 = \frac{1}{1+s^2}[sy + (x+1)]$$

Using the trigonometric identity $\sin^2\tau + \cos^2\tau = 1$ yields

$$1 = \frac{1}{(1+s^2)^2}[(x+1)^2(1+s)^2 + y^2(1+s^2)]$$

$\Rightarrow$

$$(x+1)^2 + y^2 = 1 + s^2$$

For a Dubins car endowed with a line segment capture set, the remaining secondary trajectories, which fill the hole in the reduced state space, are a family of concentric circular arcs parameterized by $0 > s > -1$, centered at $(-1,0)$ with a radius of $\sqrt{1+s^2}$ as seen in Figure 18.

Thus, if the Evader is initially in this region, the Pursuer first turns away from the Evader until the Evader ends up directly behind the Pursuer, whereupon the Pursuer

plays LOS evasion until the Evader reaches the DL. At this point, the Pursuer turns toward the Evader until it reaches the capture zone at $(0, y_f)$. The complete optimal flow field for the case of $\ell = 1$ is shown in Figure 18.

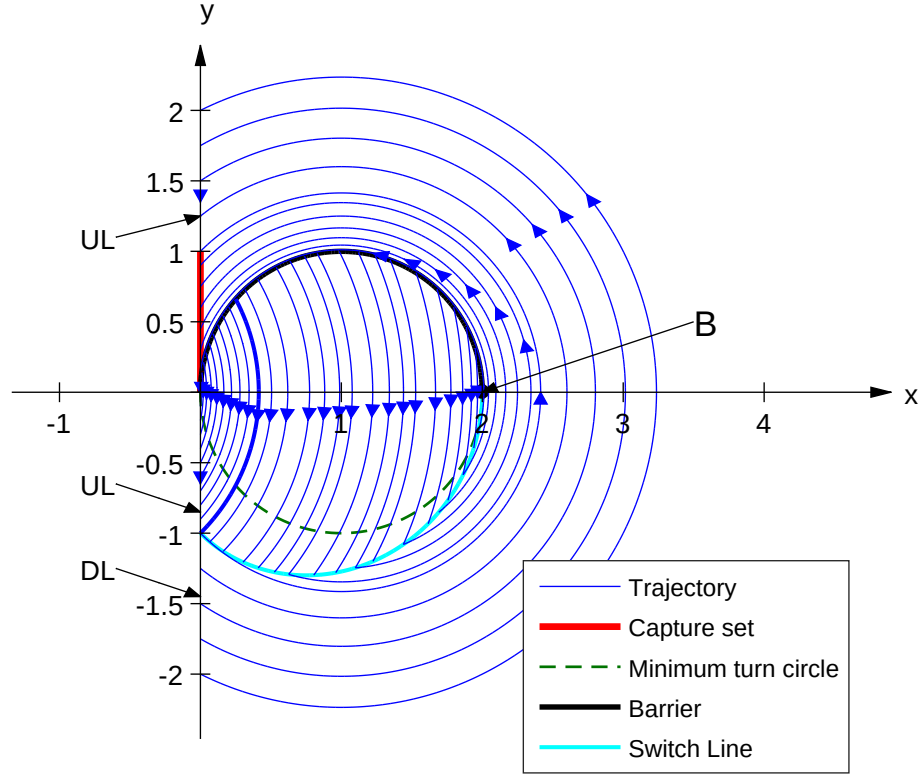


Figure 18. Dubins Car Optimal Control Problem with a Line Segment Capture Set: Secondary Optimal Trajectories in the Optimal Flow Field, $\ell = 1$

3.3.6 Solutions for Additional Capture Set Lengths

When deriving the optimal control solution, the optimal flow field was depicted for the case where the problem parameter $\ell = 1$. If $\ell > 1$, any primary trajectories emanating from $1 \leq y_f \leq \ell$ will terminate on the y-axis at $(0, -y_f)$ instead of contacting the SL as seen in Figure 19.

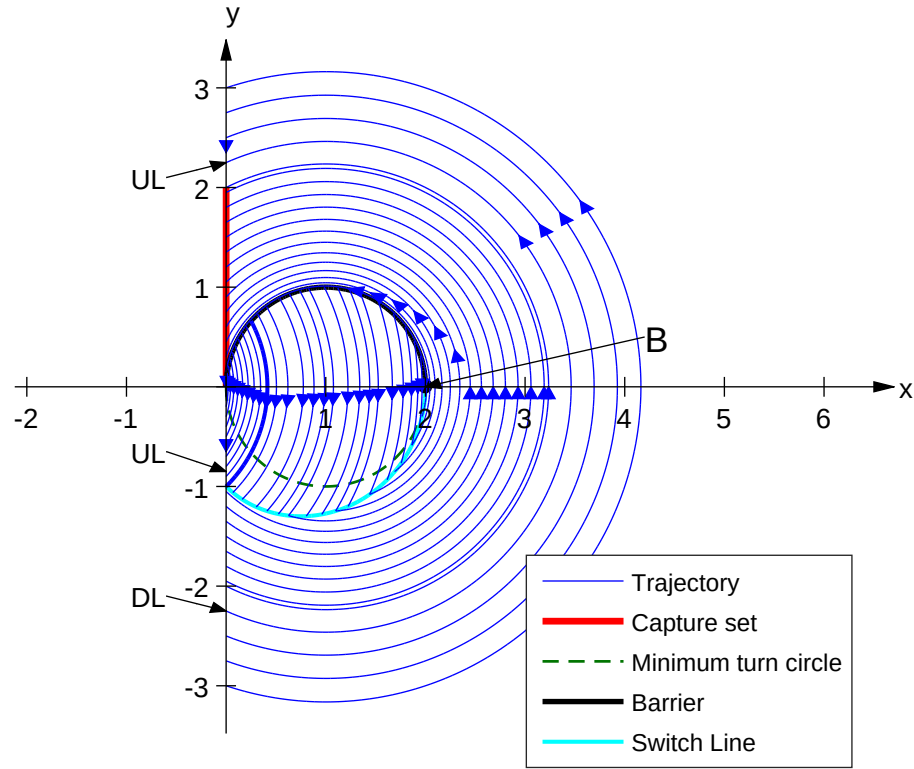


Figure 19. Dubins Car Optimal Control Problem with a Line Segment Capture Set: Full Optimal Flow Field, $\ell = 2$

If $\ell < 1$, due to the SL's dependence on ℓ , a gap is created in the optimal flow field as seen in Figure 20. When $\ell < 1$, the secondary trajectories obtained in Section 3.3.5 no longer completely fill the state space.

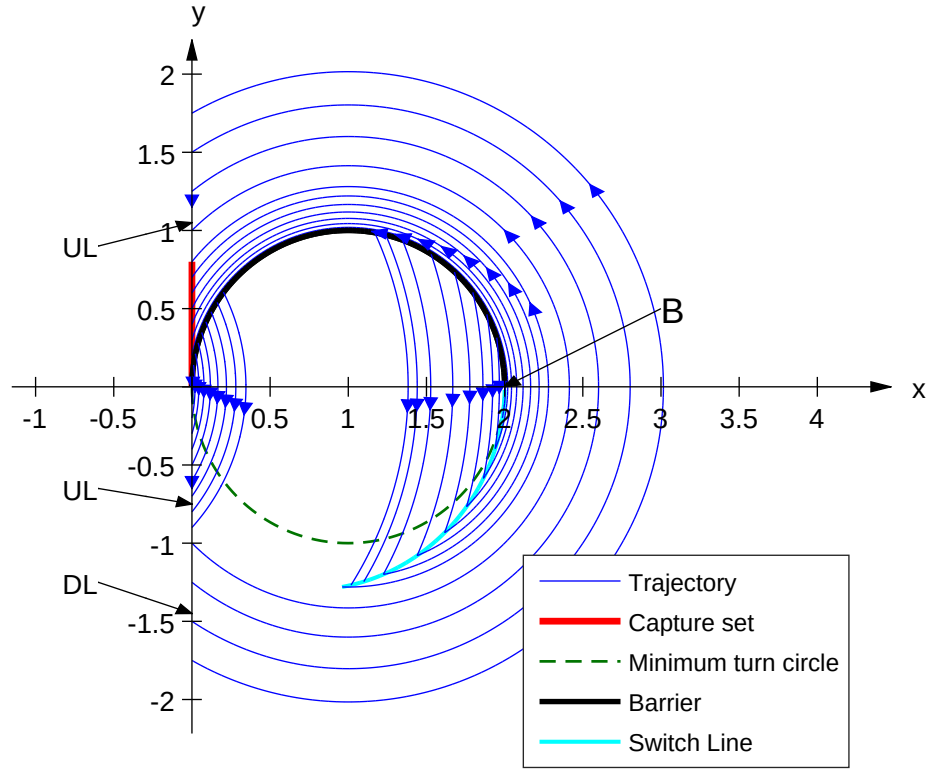


Figure 20. Dubins Car Optimal Control Problem with a Line Segment Capture Set: Optimal Trajectories in the Optimal Flow Field, $\ell = 0.8$

To address this parameter range $\ell < 1$, consider the following: for practical purposes, the capture set in Figure 6 is a 1-D line segment. However, in differential games and optimal control problems the capture set must not have an empty interior. With this in mind, the line segment capture set in Figure 6 must be treated as a circular arc capture set with a circular arc angle $0 < \varepsilon \ll 1$, as seen in Figure 21.

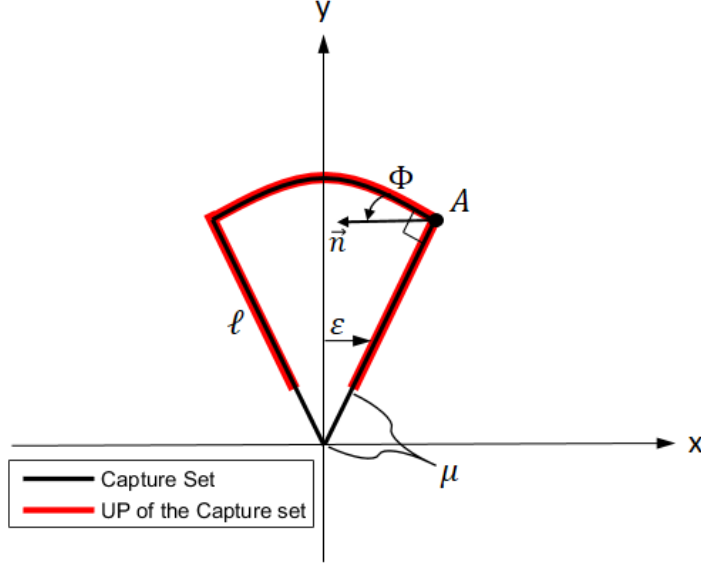


Figure 21. Circular Arc Capture Set Reference Frame

The normals ($\vec{n}$) to the capture set at corner A form a right angled cone. The cone of normals parameterized by the angle $0 \leq \phi < \frac{\pi}{2}$ is characterized as

$$\vec{n} = \begin{pmatrix} -\cos(\varepsilon - \phi) \\ \sin(\varepsilon - \phi) \end{pmatrix}$$

$\Rightarrow$

$$\vec{n} = - \begin{pmatrix} \cos(\phi - \varepsilon) \\ \sin(\phi - \varepsilon) \end{pmatrix}$$

$\Rightarrow$

$$\vec{n} \approx - \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad 0 \leq \phi < \frac{\pi}{2} \quad (67)$$

The circular arc capture set in Figure 21 corresponds to a differential game where $\mu \neq 0$; in the Dubins car optimal control problem $\mu = 0$, thus the entire circular arc capture set for this Dubins car optimal control problem is the UP of the capture set.

Because corners can potentially cause discontinuities in partial differential equa-

tions, it is advantageous to round the corners of the circular arc capture set using circles with a small radius $r = \varepsilon^2 \ell$ as seen in Figure 22.

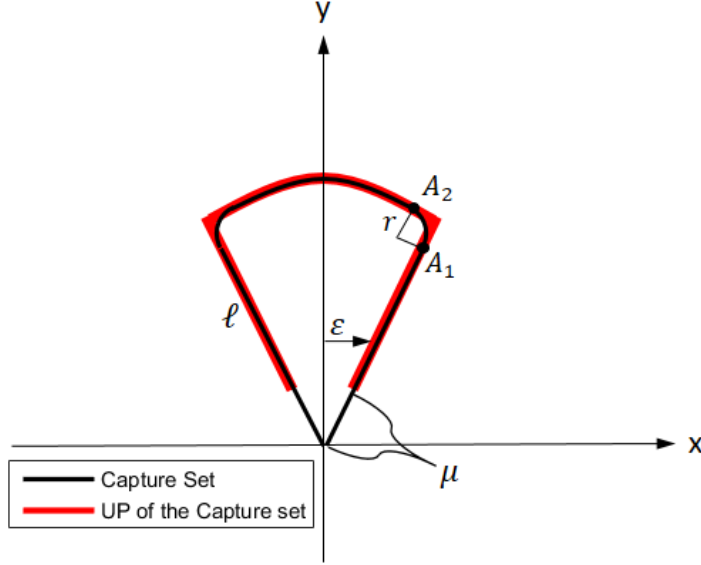


Figure 22. Circular Arc Capture Set Reference Frame with Rounded Corners

The radius of the rounded corner between point A_1 and A_2 . Since the rounded corner between points A_1 and A_2 is included in the UP of the capture set, it is necessary to characterize the trajectories that emanate from this corner. Since these trajectories emanate from the UP of the circular arc capture set, they are primary trajectories. To characterize the primary trajectories emanating from the rounded corner, their attendant costates are first obtained. Utilizing Equation (67), along the rounded corner the transversality condition of the PMP yields

$$\lambda_x(\tau_4 = 0) = -a \cdot \cos\phi, \quad \lambda_y(\tau_4 = 0) = -a \cdot \sin\phi, \quad a > 0$$

where τ_4 denotes the retrograde time measured from the time the primary trajectory emanates from the rounded corner. From the relationships in Equation (43), for a

game evolving in the RHP, the costates satisfy the harmonic equations

$$\ddot{\lambda}_x + \lambda_x = 0, \quad \ddot{\lambda}_y + \lambda_y = 0$$

With the transversality conditions, the solutions to the harmonic equations yield the family of costates

$$\lambda_x(\tau_4) = -a \cdot \cos(\tau_4 - \phi), \quad \lambda_y(\tau_4) = a \cdot \sin(\tau_4 - \phi) \quad (68)$$

From Equations (53) and (54), where $y_f = \ell$, the primary trajectories emanating from the rounded corner are

$$x(\tau_4) = \ell \sin \tau_4 - \cos \tau_4 + 1 \quad (69)$$

$$y(\tau_4) = \ell \cos \tau_4 + \sin \tau_4 \quad (70)$$

The multitude of primary trajectories emanating from the rounded corner would practically identical to the trajectories in Equations (69) and (70). From the Hamiltonian in Equation (40), the switching function is

$$S(\tau_4) = x(\tau_4)\lambda_y(\tau_4) - y(\tau_4)\lambda_x(\tau_4) > 0, \quad 0 \leq \tau_4 \leq \tau_{max}$$

Inserting Equations (68), (69) and (70) into the switching function above yields

$$S(\tau_4) = a \cdot (\tau_4 - \phi)(\ell \sin \tau_4 + 1) + a \cdot (\tau_4 - \phi)(\ell \cos \tau_4 + \sin \tau_4) \quad (71)$$

Evaluating the switching function above for $\tau_4 = 0$ reveals that

$$S(\tau_4 = 0) > 0 \quad \forall \quad 0 \leq \phi < \frac{\pi}{2}$$

which shows that $u^*(\tau_4 = 0) = 1$. From Equation (71)

$$\frac{1}{a}S(\tau_4) = (\ell \sin \tau_4 - \cos \tau_4 + 1)(\sin \tau_4 \cos \phi - \cos \tau_4 \sin \phi) + (\ell \cos \tau_4 + \sin \tau_4)(\cos \tau_4 \cos \phi + \sin \tau_4 \sin \phi)$$

$\Rightarrow$

$$\frac{1}{a}S(\tau_4) = (1 - \cos \tau_4) \sin \phi + (\ell + \sin \tau_4) \cos \phi \quad (72)$$

When $\phi = 0$, the switching function becomes

$$\frac{1}{a}S(\tau_4) = \ell + \sin \tau_4$$

Setting the switching function above to 0 yields

$$\tau_{max} = \pi + A \sin(\ell)$$

which shows that $\phi = 0$ corresponds to the primary trajectory emanating from the point A_1 . In general, from Equation (72), the switching function vanishes when

$$\frac{-(\ell + \sin \tau_4)}{1 - \cos \tau_4} = \tan \phi, \quad 0 \leq \phi < \frac{\pi}{2} \quad (73)$$

If the switching function's effect on the primary trajectory emanating from A is momentarily ignored, it is possible to find the retrograde time τ_5 at which a primary trajectory emanating from the rounded corner ($A_1 \rightarrow A_2$) contacts the negative y-axis. From Equation (69), the primary trajectory that contacts the negative y-axis is subject to

$$\ell \sin \tau_5 - \cos \tau_5 + 1 = 0$$

$\Rightarrow$

$$(\ell \sin \tau_5 + 1)^2 = 1 - \sin^2 \tau_5$$

$\Rightarrow$

$$\sin\tau_5 = -\frac{2\ell}{1+\ell^2}$$

Therefore, contact with the negative y-axis occurs at

$$\tau_5 = \pi + A \sin\left(\frac{2\ell}{1+\ell^2}\right)$$

It is important to note that $\frac{2\ell}{1+\ell^2} < 1$ and if $\ell < 1$, $\frac{2\ell}{1+\ell^2} > \ell$ which means $\tau_5 > \tau_{max}$. For the primary trajectories emanating from the rounded corner ($A_1 \rightarrow A_2$) to continue from the SL and contact the negative y-axis, the following condition must hold:

$$\frac{-(\ell + \sin\tau_4)}{1 - \cos\tau_4} < 0 \quad \forall \quad \tau_{max} < \tau_4 < \tau_5$$

so that a feasible $0 \leq \phi < \frac{\pi}{2}$ exists and Equation (73) is satisfied; when this condition is met, the left hand side of Equation (73) becomes positive meaning there exists some angle $0 \leq \phi < \frac{\pi}{2}$ that causes the switching function to vanish at the retrograde time $\tau_{max} < \tau_4 < \tau_5$. Thus, these primary trajectories that emanate from the rounded corner experience the effects of the switching function at a retrograde time greater than τ_{max} meaning the family of primary trajectories emanating from the rounded corner ($A_1 \rightarrow A_2$) can continue the SL obtained in Section 3.3.4 all the way to the negative y-axis. Since this extension was constructed based on the end points of coalescing primary trajectories where the switching function vanishes ($S(\tau_4) = 0$), this extension is itself a traversable trajectory as shown in Figure 23.

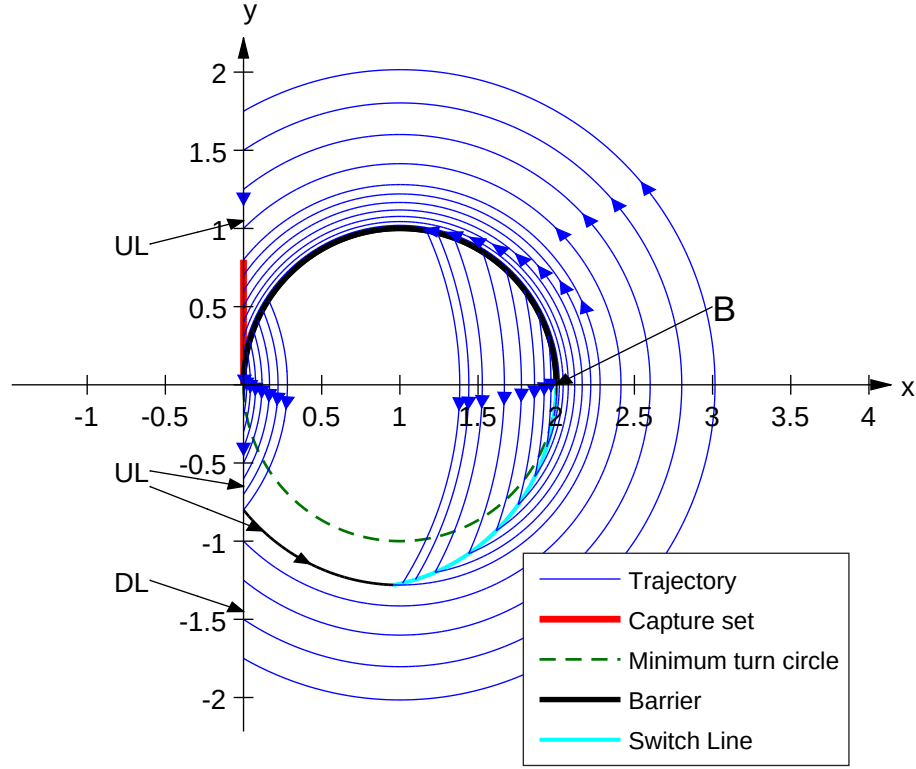


Figure 23. Dubins Car Optimal Control Problem with a Line Segment Capture Set: New Universal Line in the Optimal Flow Field, $\ell = 0.8$

With $S(\tau_4) = 0$, $\tau_{max} < \tau_4 < \tau_5$ along this extension, this surface is effectively a new UL. For an Evader initially positioned on this UL, the Pursuer merely applies his primary trajectory strategy $u^* = 1$ until the Evader contacts the UP of the capture set at $(0, \ell)$ where $0 < \ell < 1$.

With the construction of this new UL, the secondary trajectories that fill the gap in the state space can be obtained. This UL provides a locus of initial conditions for the remaining secondary trajectories. Recalling the dynamics of the retrograde trajectories: the dynamics of the secondary trajectories which will fill the state space gap are

$$\dot{x} = -y, \quad x(\tau_6 = 0) = 1 - \sqrt{1 + \ell^2} \cos \theta \quad (74)$$

$$\dot{y} = x + 1, \quad y(\tau_6 = 0) = -\sqrt{1 + \ell^2} \sin \theta \quad (75)$$

where the initial conditions $(x(\tau_6 = 0), y(\tau_6 = 0))$ are coordinates along the new UL in Figure 23. These initial conditions can be obtained using a polar coordinate frame where θ is the angle measured from the negative y-axis and $1 + \ell^2$ is the radius of the primary trajectory. Here τ_6 denotes the retrograde time measured from the time the secondary trajectory emanates from the new UL. Differentiating Equation (74) yields the non-homogeneous harmonic equation

$$\ddot{x} + x = -1, \quad x(\tau_4 = 0) = 1 - \sqrt{1 + \ell^2} \cos \theta, \quad \dot{x}(\tau_6 = 0) = \sqrt{1 + \ell^2} \sin \theta$$

Solving the non-homogeneous harmonic equation and applying a similar derivation to Equation (75) yields expressions for the Evader's position relative to the Pursuer in the region of the secondary trajectories emanating from the new UL:

$$x(\tau_6) = 2 \cos \tau_6 - \sqrt{1 + \ell^2} \cos(\theta + \tau_6) - 1$$

$$y(\tau_6) = 2 \sin \tau_6 - \sqrt{1 + \ell^2}(\theta + \tau_6), \quad \ell < 1, \quad \bar{\theta} < \theta < \bar{\bar{\theta}}$$

where $\bar{\theta}$ delimits the secondary trajectories emanating from the negative y-axis from the secondary trajectories emanating from the new UL and $\bar{\bar{\theta}}$ delimits the secondary trajectories emanating from the new UL from the secondary trajectories emanating from the SL. This again is a family of circular arcs centered at $(-1, 0)$. To fully characterize the remaining secondary trajectories, it is necessary to characterize $\bar{\theta}$ and $\bar{\bar{\theta}}$.

Since the new UL dynamics are equivalent to the primary trajectories' dynamics, it is known that the new UL will contact the negative y-axis at $(0, -\ell)$. To characterize the angle $\bar{\theta}$, the secondary trajectory initial conditions from Equations (74) and (75)

can be related to the coordinates $(0, \ell)$ yielding

$$0 = 1 - \sqrt{1 + \ell^2} \cos \bar{\theta}$$

$$-\ell = -\sqrt{1 + \ell^2} \sin \bar{\theta}$$

Solving the above equations for $\bar{\theta}$ yields

$$\bar{\theta} = A \cos \left(\frac{1}{\sqrt{1 + \ell^2}} \right) = A \sin \left(\frac{\ell}{\sqrt{1 + \ell^2}} \right)$$

From the parametric expression of the SL in Equations (60) and (61), one can characterize the angle $\bar{\bar{\theta}}$ by evaluating those expressions at the end point of the SL, $y_f = \ell$. These expressions can then be related to the secondary initial conditions yielding

$$1 - \ell^2 + \sqrt{1 - \ell^2} = 1 - \sqrt{1 + \ell^2} \cos \bar{\bar{\theta}}$$

$$-\ell^2 + \sqrt{1 - \ell^2} = -\sqrt{1 + \ell^2} \sin \bar{\bar{\theta}}$$

Solving the above equations for $\bar{\bar{\theta}}$ yields

$$\bar{\bar{\theta}} = A \cos \left(\frac{\ell^2 - \sqrt{1 - \ell^2}}{\sqrt{1 + \ell^2}} \right) = A \sin \left(\frac{\ell \sqrt{1 - \ell^2} + \ell}{\sqrt{1 + \ell^2}} \right)$$

So the family of secondary trajectories parameterized by θ is

$$\dot{x} = -y, \quad x(\tau_6 = 0) = 1 - \sqrt{1 + \ell^2} \cos \theta$$

$$\dot{y} = x + 1, \quad y(\tau_6 = 0) = -\sqrt{1 + \ell^2} \sin \theta$$

for

$$A \sin \left(\frac{\ell}{\sqrt{1 + \ell^2}} \right) < \theta < A \sin \left(\frac{\ell \sqrt{1 - \ell^2} + \ell}{\sqrt{1 + \ell^2}} \right)$$

This being a family of circular arcs centered at $(-1, 0)$. With the characterization of these additional secondary trajectories, the full optimal flow field is constructed as seen in Figure 24.

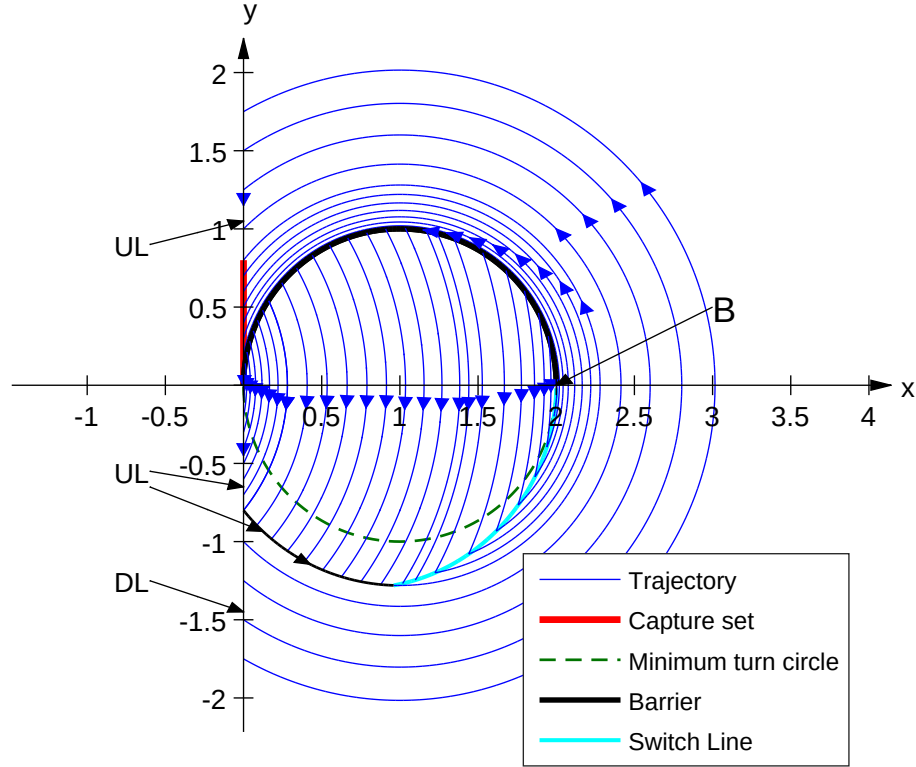


Figure 24. Dubins Car Optimal Control Problem with a Line Segment Capture Set: Full Optimal Flow Field, $\ell = 0.8$

With this, the full solution is obtained for the Dubins car optimal control problem for the case of a line segment capture set endowed Pursuer.

3.4 Discussion

In this section a discussion of the results in Sections 3.2 and 3.3 are presented. This discussion is used to highlight certain features of the Dubins car optimal control solutions that offer insights into the nature of the solution of the *classical* Homicidal

Chauffeur differential game. This discussion, as well as the general solution process of the Dubins car optimal control problems, are used to provide a methodology framework used to develop the solution to the *classical* Homicidal Chauffeur differential game in Chapter IV.

3.4.1 Safe Contact Trajectories

In the Dubins car optimal control problem of Section 3.2, the secondary trajectories of Figure 10 that terminate on the NUP achieve safe contact. Safe contact occurs when the Evader, initially positioned under the barrier comes into contact with, but does not penetrate, the NUP of the circular capture set. Secondary trajectories can also achieve safe contact in the Homicidal Chauffeur differential game and, in [1], results in the gap in Isaacs' solution as seen in Figure 3. For the case of $\mu = 0$ in the Dubins car optimal control problem, a gap in the optimal flow field is not present, as illustrated in Figure 10.

3.4.2 Fundamental Solution Structure

The solutions to the Dubins car optimal control problems, presented herein, provide an expectation for the characteristics of the players' optimal strategies in the *classical* Homicidal Chauffeur differential game solution. Since the problem of steering a capture set endowed Dubins car to reach a target point in minimum time is also the solution of the Homicidal Chauffeur differential game for the case of a speed ratio $\mu = 0$, the optimal strategies of the Homicidal Chauffeur differential game solution can be interpreted by examining the optimal flow fields of Figures 10 and 18.

A Pursuer modeled as a Dubins car will initially turn into the Evader if the Evader is not under the barrier. Once the Pursuer aligns with the Evader, it will implement LOS pursuit until the Evader penetrates the capture set. If the Evader's

initial position is in the state space region under the barrier, the Pursuer will initially swerve away from the Evader until the Evader reaches the SL. The Pursuer will then resume turning towards the Evader and continue pursuit until the Evader is captured.

3.5 Isaacs' Academic Exercise 10.6.1

In this section the differential game Exercise 10.6.1 [1] is analyzed. The herein considered differential game, conceived and analyzed by Isaacs, introduces the concept of the ES to the field of differential games. Additionally, this example problem features the barrier. Isaacs relies on the methodology used in this example in his subsequent development of the *classical* Homicidal Chauffeur differential game in which the formation of the barrier and ES greatly impact the solution. To gain a better understanding of the *classical* Homicidal Chauffeur differential game, the foundational differential game Example 10.6.1 is closely examined in this section.

3.5.1 Introduction

In this differential game, two players have control over a state vector (x, y) . Each player has their own objective concerning the state vector in $\mathbb{R}^2$ that they're jointly controlling. One player, designated herein as the Pursuer, wishes to move the state vector to a terminal set $\subset \mathbb{R}^2$ in minimum time. The opponent, designated as the Evader, wishes to maximize the time it takes for the state vector to reach the final set. To meet their conflicting objectives, each player must solve an optimal control problem. The solutions to these optimal control problems yield the control laws that will minimize/maximize the time-to-capture for the Pursuer and Evader respectively. Similar to the *classical* Homicidal Chauffeur differential game, this differential game contains 2 states. Unlike the *classical* Homicidal Chauffeur differential game, however, this differential game is purely an academic example of a 2 state differential

game and is completely detached of any realistic physics of a pursuit-evasion differential game. This differential game serves as a way to introduce the concept of the ES which is utilized by Isaacs for the subsequent solution of the *classical* Homicidal Chauffeur differential game. The most notable contributions of this differential game solution are the construction of the barrier, the introduction of the ES, and the analysis of their connection. Understanding the theory behind the construction of the barrier and the ES is crucial in understanding their impact on the *classical* Homicidal Chauffeur differential game.

Following this introduction, the differential game is stated. Following the specification of the differential game, the optimal strategies of the Pursuer and Evader are derived in order to characterize the primary trajectories, secondary trajectories, barrier, and ES present in this differential game. A solution is obtained for general and parameterized case; the solutions to the general and parameterized problems were provided by Isaacs and are used to validate the results. Following the derivation of the solution, a discussion is presented on the results.

3.5.2 The Differential Game

In this differential game, a Pursuer strives to pull the state vector $(x, y) \in \mathbb{R}^2$ to a terminal set in minimum time while an Evader tries to maximize the time it takes for the state vector to reach the terminal set. The time the players wish to maximize/minimize is referred to as the time-to-capture.

The action of this game takes place in the Euclidean plane seen in Figure 26. This state space differs slightly from the one used by Isaacs. In Isaacs' optimal solution, depicted in Figure 25, he depicted the capture set as the entire x-axis. However, in differential games the BUP, located to at point O in Figure 25, must be normal to the capture set. If the capture set was indeed the entire x-axis, then the BUP

could not possibly be normal to the capture set, therefore, there would be no barrier and consequently no ES. Hence, a modified capture set is supplanted herein. This modified capture set forms a corner at the origin and continues down the negative y-axis. To be shown in greater detail later: the corner of the capture set allows the BUP to be normal to the capture set. Thus, the differential game is played in the modified state space seen in Figure 26.

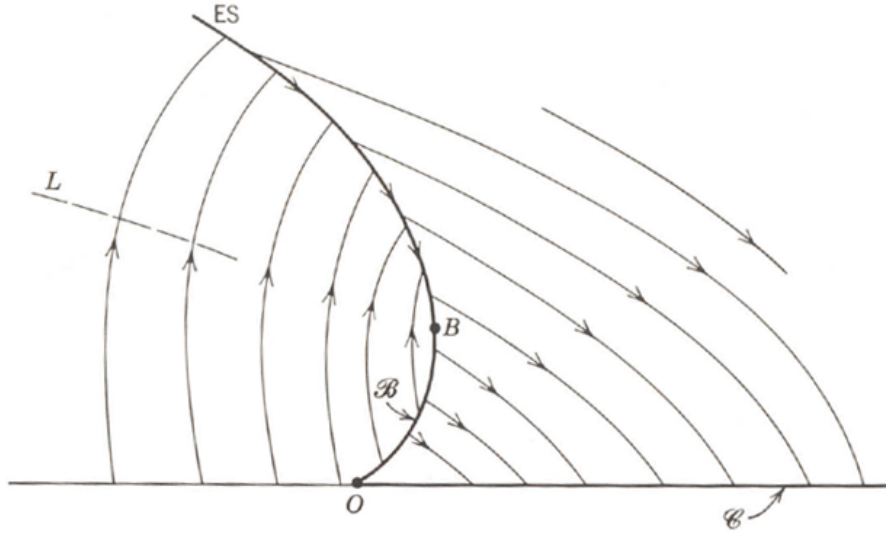


Figure 25. Exercise 10.6.1: Isaacs' Optimal Flow Field - reproduced from Fig. 10.7.3 in [1]

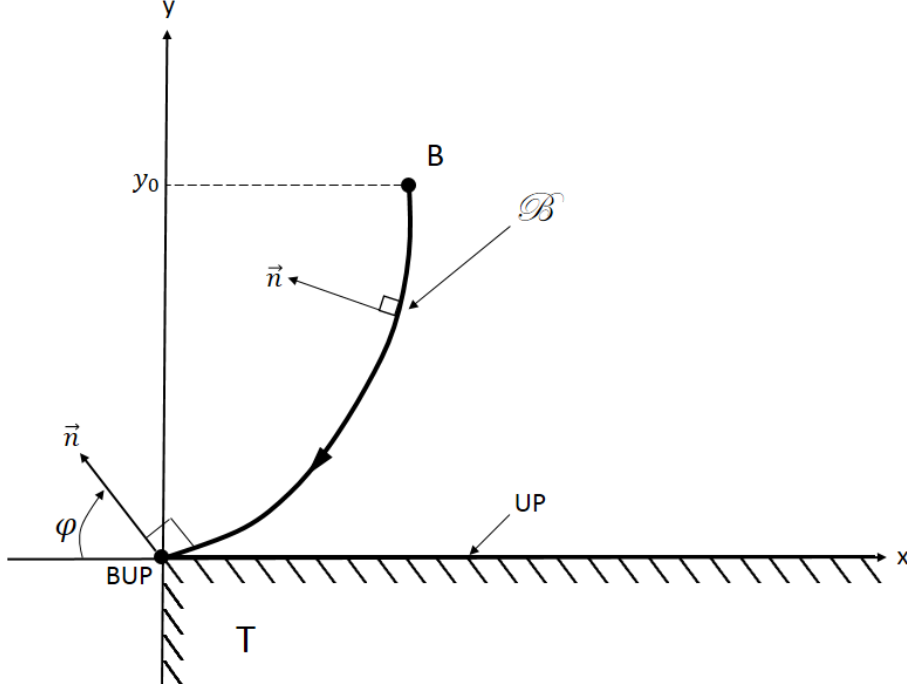


Figure 26. Exercise 10.6.1: Reference Frame

The Pursuer applies his control ϕ to minimize the time-to-capture while the Evader applies his control Ψ to maximize the time-to-capture. At the termination of the game, the state vector will have reached the UP of the terminal set $T = \{(x, y) \mid x \geq 0, y \leq 0\}$. Termination is achieved when the state vector reaches the UP of the capture set, $UP = \{(x, y) \mid y = 0, x > 0\}$. The BUP corresponds to the origin of the reference frame $(0, 0)$. Additionally, $0 \leq \varphi \leq \frac{\pi}{2}$ is the angle that a normal to the capture set at the BUP makes with the negative x-axis. The inward pointing normal $\vec{n}$, corresponds to the barrier $\mathcal{B}$. The state vector's y-coordinate y_0 corresponds to the y-coordinate of the barrier end point B.

The dynamics of this differential game (chosen by Isaacs) are

$$\dot{x} = u(y) + w \sin \Psi \quad (76)$$

$$\dot{y} = -b\phi + w \cos \Psi, \quad -1 \leq \phi \leq 1, \quad t \geq 0 \quad (77)$$

$$u(y) > 0, \quad u'(y) > 0$$

where w and b are simply coefficients. The performance functional of the differential game is

$$\int_0^{t_f} 1 \cdot dt$$

and the terminal manifold is $T = \{(x(t_f), y(t_f)) | x(t_f) > 0, y(t_f) = 0\}$. In retrograde time $\tau = t_f - t$, the dynamics are

$$\dot{x} = -u(y) - w \sin \Psi, \quad x(\tau = 0) = x(> 0) \quad (78)$$

$$\dot{y} = b\phi - w \cos \Psi, \quad y(\tau = 0) = 0, \quad 0 \leq \tau \leq t_f \quad (79)$$

These equations encapsulate both the Evader's and Pursuer's influence on the state vector in the 2-D state space. At $\tau = 0$ the retrograde optimal trajectories begin at their termination point and develop backwards to the states' initial location.

3.5.3 Optimal Control

In this differential game, there are primary and secondary optimal trajectories. Primary optimal trajectories emanate', in a retrograde manner, from the UP of the capture set according to Equations (78) and (79). The secondary optimal trajectories emanate, in a retrograde manner, from the ES. These primary and secondary trajectories are sketched in Figure 27 and are referenced in the forthcoming derivations. The optimal flow field illustrated in Figure 27 is a modified rendition of Isaacs' solution seen in Figure 25. The optimal flow field in Figure 27 corresponds to the modification made to the capture set seen in Figure 26.

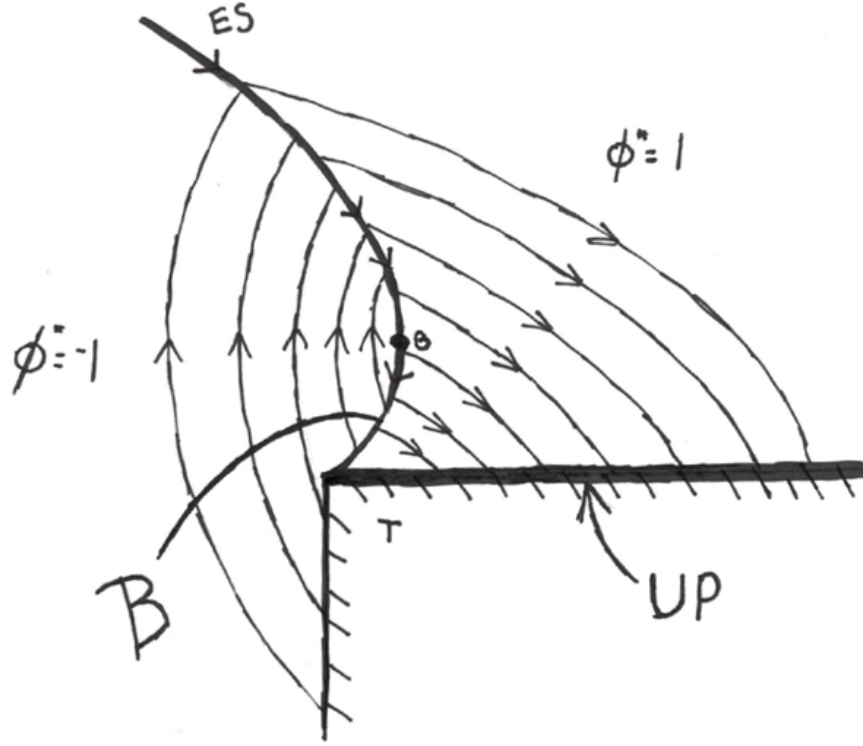


Figure 27. Exercise 10.6.1: Modified Optimal Flow Field Corresponding To The Updated Target Set Definition

3.5.3.1 The Barrier

The barrier is a special trajectory emanating from the BUP whose end point is demarcated by B. The barrier separates the regions of the Pursuer's optimal strategies. Before obtaining the primary and secondary trajectories, it is necessary to fully define the barrier. To derive an expression for the barrier, the Hamiltonian of the differential game is examined. From [1], when examining the barrier there is no integral cost in the Hamiltonian. Thus, using the PMP, the Hamiltonian for the barrier is

$$H = \lambda_x[u(y) + w \sin \Psi] + \lambda_y[-b\phi + w \cos \Psi]$$

$\Rightarrow$

$$H = \lambda_x u(y) - \lambda_y b\phi + w(\lambda_x \sin\Psi + \lambda_y \cos\Psi) \quad (80)$$

Initially the costate vector $(\lambda_x(\tau = 0), \lambda_y(\tau = 0))$ is a normal to the capture set at the BUP. The costate vector $(\lambda_x(\tau), \lambda_y(\tau))$ points ‘upward’ and to the ‘left’. The Evader, which is interested in pushing the state to the left of the barrier, maximizes the component of the Hamiltonian that his control Ψ impacts:

$$\max_{\Psi} (\lambda_x \sin\Psi + \lambda_y \cos\Psi) \quad (81)$$

At the same time, the Pursuer, which is interested in pulling the state to the ‘right’ of the barrier where the UP is, minimizes the component of the Hamiltonian that his control ϕ impacts:

$$\min_{\phi} (-\lambda_y b\phi) \quad (82)$$

Hence, from Equation (81), the Evader’s control Ψ is such that

$$\sin\Psi^* = \frac{\lambda_x}{\sqrt{\lambda_x^2 + \lambda_y^2}}, \quad \cos\Psi^* = \frac{\lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}} \quad (83)$$

and from Equation (82) the Pursuer’s control ϕ is such that

$$\phi^* = \text{sign}(\lambda_y) \quad (84)$$

Thus, if the state is initially located in the region of the primary trajectories, the Pursuer applies the control $\phi^* = 1$ which pushes the state vector towards the UP. If the state vector is initially located in the region of the secondary trajectories, the Pursuer applies the control $\phi^* = -1$ in order to pull the state vector over the barrier, then the Pursuer proceeds to push the state vector towards the UP with $\phi^* = 1$, as

seen in Figure 27.

Mirroring Isaacs' approach, now that the optimal control of each player has been characterized, it is desired to express the players' optimal controls in terms of the parameter φ . According to the PMP, the costates satisfy the differential equations

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0, \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\lambda_x \cdot u'(y)$$

The transversality condition of the PMP yields

$$\lambda_x(\tau = 0) = -\cos\varphi, \quad \lambda_y(\tau = 0) = \sin\varphi \quad (85)$$

so the costates are

$$\lambda_x(\tau) = -\cos\varphi, \quad \dot{\lambda}_y(\tau) = -u'(y)\cos\varphi$$

and consequently, the Evader's optimal controls are

$$\sin(\Psi^*(\tau)) = -\frac{\cos\varphi}{\sqrt{\cos^2\varphi + \lambda_y^2(\tau)}}, \quad \cos(\Psi^*(\tau)) = \frac{\lambda_y(\tau)}{\sqrt{\cos^2\varphi + \lambda_y^2(\tau)}}$$

Applying the above results to Equations (78)-(80) yields

$$\dot{x} = -u(y) + w \frac{\cos\varphi}{\sqrt{\lambda_y^2 + \cos^2\varphi}}, \quad x(\tau = 0) = 0 \quad (86)$$

$$\dot{y} = b \cdot \text{sign}(\lambda_y) - w \frac{\lambda_y}{\sqrt{\lambda_y^2 + \cos^2\varphi}}, \quad y(\tau = 0) = 0 \quad (87)$$

$$H = u(y) \cdot \lambda_x - b\lambda_y \text{sign}\lambda_y + w\sqrt{\lambda_x^2 + \lambda_y^2} = 0 \quad \forall \quad \tau \geq 0 \quad (88)$$

where,

$$\lambda_x(\tau) = -\cos\varphi \quad (89)$$

$$\dot{\lambda}_y(\tau) = -u'(y)\cos\varphi, \quad \lambda_y(\tau = 0) = \sin\varphi, \quad \tau \geq 0 \quad (90)$$

$\Rightarrow$

$$\dot{\lambda}_y(\tau) < 0 \quad \forall \quad \tau \geq 0$$

Next the validity range of φ is examined. From Figure 26, the capture set has a set of normals at the BUP that form a cone. It is known that since the normal to the barrier at the BUP is an element of that cone, it must lie within the range $0 \leq \varphi \leq \frac{\pi}{2}$. However, the true range of validity for φ is $0 < \varphi < \frac{\pi}{2}$. The region of validity for φ is established below via proof by contradiction.

Claim 1: $\varphi \neq 0$

If $\varphi = 0$, Equations (89) and (90) yield

$$\lambda_x = -1, \quad \lambda_y(\tau = 0) = 0$$

Additionally, from Equation (84)

$$\dot{\lambda}_y < 0 \rightarrow \lambda_y(\tau) < 0 \quad \forall \quad \tau > 0 \rightarrow \phi^* = -1$$

Applying this to Equation (87) yields

$$\dot{y} = -b - w \frac{\lambda_y}{\sqrt{1 + \lambda_y^2}}$$

$\Rightarrow$

$$\dot{y}(0) = -b (< 0)$$

at the same time, from Equation (86)

$$\dot{x} = -u(y) + w \frac{1}{\sqrt{1 + \lambda_y^2}}$$

$\Rightarrow$

$$\dot{x}(\tau = 0) = -u(y(\tau = 0)) + w > 0$$

thus $\dot{x}(\tau = 0) > 0$ and $\dot{y}(\tau = 0) < 0$. If this were true, the barrier would cut into the capture set, which is impossible. Hence, $0 < \varphi$

Claim 2: $\varphi \neq \frac{\pi}{2}$.

If $\varphi = \frac{\pi}{2}$, from Equations (89) and (90), $\lambda_x = 0$ and $\lambda_y = 1 \quad \forall \quad \tau \geq 0$. With the costates defined, using Equation (88), the following is calculated

$$H = -b + w = 0 \quad \Rightarrow \quad w = b$$

but from Isaacs' problem statement $w < b$. Thus, the above result is a contradiction and is invalid. Hence, $0 < \varphi < \frac{\pi}{2}$. With the validity of φ established, an expression for the barrier can be obtained.

Termination of the Barrier

The first step in fully characterizing the barrier is verifying its termination at the barrier end point B, as seen in Figure 26. It is known that $\lambda_y(\tau)$ is a decreasing function because,

$$\dot{\lambda}_y < 0 \quad \forall \quad \tau \geq 0$$

However, λ_y is initially positive:

$$\lambda_y(\tau = 0) = \sin\varphi > 0$$

Hence, there exists a retrograde time $\bar{\tau} > 0$ such that $\lambda_y(\tau) > 0 \quad \forall \quad 0 \leq \tau < \bar{\tau}$. For $\tau < \bar{\tau}$, to obtain the barrier, integrate the set of two ODEs from Equations (86), (87), and (90).

To construct the barrier, it is necessary to determine the state of the barrier at $\tau = \bar{\tau}$. Evaluating the Hamiltonian in Equation (88) at $\tau = \bar{\tau}$, where $\lambda_y(\bar{\tau}) = 0$ and $\lambda(\bar{\tau}) = -\cos\varphi$, yields

$$H(\bar{\tau}) = -u(y)\cos\varphi + w\cos\varphi = 0$$

Since $\varphi \neq \frac{\pi}{2}$, based on Isaacs' problem statement

$$u(y(\bar{\tau})) = w$$

where

$$y(\bar{\tau}) = y_0$$

From Equation (86), the following is calculated

$$\dot{x}(\bar{\tau}) = w - u(y(\bar{\tau}))$$

$\Rightarrow$

$$\dot{x}(\bar{\tau}) = w - w = 0$$

and from Equation (87)

$$\dot{y}(\bar{\tau}) = b$$

Therefore, at retrograde time $\bar{\tau}$, the tangent to the barrier curve is normal to the x-axis as shown in Figure 26.

Because $y(\bar{\tau}) = y_0$, the parameter $0 < \varphi < \frac{\pi}{2}$ is determined by the requirement that the solution of the system of ODEs in Equations (87) and (90) satisfies the two conditions $\lambda_y(\bar{\tau}) = 0$, $y(\bar{\tau}) = y_0$. In this system of equations, there are two equations with the two unknowns $0 < \varphi < \frac{\pi}{2}$ and $\bar{\tau} > 0$. This is a two-point-boundary-value problem.

Next the location of the barrier end point B point shown in Figure 26 is verified. It was shown that $\dot{\lambda}_y$ is a decreasing function and that $\lambda_y(\tau) < 0 \quad \forall \quad \tau > \bar{\tau}$. From Equation (84) it is known that if $\lambda_y(\tau) < 0$

$$\phi^* = -1$$

Applying this to Equation (87) yields

$$\dot{y} = -b - w \frac{\lambda_y}{\sqrt{\lambda_y^2 + \cos^2 \varphi}} < 0$$

So $\forall \quad \tau > \bar{\tau}$, the y-coordinate of the state vector decreases and a cusp is formed. This would result in the barrier folding back on itself, which is absurd. So, the barrier does not extend beyond $\tau > \bar{\tau}$ and in fact terminates at

$$B = (x(\bar{\tau}), y(\bar{\tau}))$$

where

$$y(\bar{\tau}) = y_0$$

With the barrier end point B characterized, the formation of the barrier can be further examined.

Geometric Construction of the Barrier

As an alternative to Isaacs' approach, a differential geometric construction of the barrier is implemented in this section.

In Figure 26, the normal to the barrier is $\vec{n} = (-y', 1)$ where $y' = \frac{dy}{dx}$ and the vector $\vec{n}$ points to the left. With the Pursuer applying controls to pull the state to the right and the Evader applying controls to pull the state to the left, the following

holds

$$0 = \max_{\Psi} \min_{\phi} [(-y', 1) \cdot (u(y) + w \sin \Psi, -b\phi + w \cos \Psi)]$$

$\Rightarrow$

$$0 = \max_{\Psi} \min_{\phi} [-y' u(y) - b\phi + w(\cos \Psi - y' \sin \Psi)] \quad (91)$$

Hence, the players select their controls such that

$$\cos \Psi^* = \frac{1}{\sqrt{1 + y'^2}}, \quad \sin \Psi^* = -\frac{y'}{\sqrt{1 + y'^2}}, \quad \phi^* = 1$$

Applying these optimal controls to Equation (91) yields

$$0 = -u(y)y' - b + w\sqrt{1 + y'^2}$$

$\Rightarrow$

$$b^2 + u^2 y'^2 + 2bu y' = w^2 + w^2 y'^2$$

$\Rightarrow$

$$(w^2 - u^2)y'^2 - 2bu y' + w^2 - b^2 = 0$$

$\Rightarrow$

$$y' = \frac{bu(y) + w\sqrt{u^2 + b^2 - w^2}}{w^2 - u^2}$$

Thus, the differential equation for the barrier is.

$$\frac{dx}{dy} = \frac{w^2 - u^2}{bu(y) + w\sqrt{u^2 + b^2 - w^2}}, \quad x(0) = 0$$

$\Rightarrow$

$$x(y) = \int_0^y \frac{w^2 - u^2(y)}{bu(y) + w\sqrt{u^2 + b^2 - w^2}} dy, \quad 0 \leq y \leq y_0 \quad (92)$$

However, Isaacs' solution for the barrier is

$$x(y) = \int_0^y \frac{w\sqrt{u^2(y) + b^2 - w^2} - bu(y)}{b^2 - w^2} dy$$

By manipulating Isaacs' solution, it can be shown that his solution matches Equation (92):

$$x(y) = \int_0^y \frac{1}{b^2 - w^2} \frac{w\sqrt{u^2(y) + b^2 - w^2} - bu(y)}{bu(y) + w\sqrt{u^2(y) + b^2 - w^2}} \cdot bu(y) + w\sqrt{u^2(y) + b^2 - w^2} dy$$

$\Rightarrow$

$$x(y) = \int_0^y \frac{1}{b^2 - w^2} \frac{w^2(u^2(y) + b^2 - w^2) - b^2u^2(y)}{bu(y) + w\sqrt{u^2(y) + b^2 - w^2}} dy$$

$\Rightarrow$

$$x(y) = \int_0^y \frac{1}{b^2 - w^2} \frac{w^2u^2(y) + w^2b^2 - w^4 - b^2u^2(y)}{bu(y) + w\sqrt{u^2(y) + b^2 - w^2}} dy$$

$\Rightarrow$

$$x(y) = \int_0^y \frac{1}{b^2 - w^2} \frac{b^2(w^2 - u^2(y)) - w^2(w^2 - u^2(y))}{bu(y) + w\sqrt{u^2(y) + b^2 - w^2}} dy$$

$\Rightarrow$

$$x(y) = \int_0^y \frac{1}{b^2 - w^2} \frac{(b^2 - w^2)(w^2 - u^2(y))}{bu(y) + w\sqrt{u^2(y) + b^2 - w^2}} dy$$

$\Rightarrow$

$$x(y) = \int_0^y \frac{w^2 - u^2(y)}{bu(y) + w\sqrt{u^2 + b^2 - w^2}} dy, \quad 0 \leq y \leq y_0$$

which matches the solution in Equation (92). With the characterization of the barrier end point B, as well as the expression for the barrier in Equation (92), the barrier of this differential game is fully defined.

3.5.4 Exercise 10.6.1

In Isaacs' book [1], Exercise 10.6.1 is the problem that introduces the concept of the ES to differential games. In Exercise 10.6.1, with the barrier already defined, the ES is constructed and the primary and secondary trajectories are obtained, thereby completing the optimal flow field. In this section, the solution to the differential game is obtained for a general case.

The dynamics are specified in Equations (76) and (77) with

$$b > w > u(0), \quad \exists \quad y_0 > 0 \quad s.t. \quad w = u(y_0)$$

To determine the primary and secondary optimal trajectories, the Hamiltonian is examined. Recalling the differential game's Hamiltonian

$$H = -1 + \lambda_x[u(y) + w \sin \Psi] + \lambda_y[-b\phi + w \cos \Psi] \quad (93)$$

The Evader wishes to find the optimal control to minimize the Hamiltonian which will in turn maximize the time-to-capture

$$\min_{\Psi} H \Rightarrow \sin \Psi^* = -\frac{\lambda_x}{\sqrt{\lambda_x^2 + \lambda_y^2}}, \quad \cos \Psi^* = -\frac{\lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}} \quad (94)$$

Conversely, the Pursuer wishes to find the optimal control to maximize the Hamiltonian which will minimize the time-to-capture.

$$\max_{\phi} H \Rightarrow \phi^* = -\text{sign}(\lambda_y) \quad (95)$$

From the PMP

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 \quad (96)$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y} = -u'(y)\lambda_x \quad (97)$$

3.5.4.1 Primary Optimal Trajectories

Based on the above results, expressions for the primary optimal trajectories can be derived. The primary optimal trajectories emanate from the UP of the capture set at $\tau = 0$, as seen in Figure 1. From Figure 27, the following is concluded

$$\lambda_x(\tau = 0) = 0, \quad \lambda_y(\tau = 0) = -a, \quad a > 0$$

Therefore, along the primary optimal trajectories, the costates are

$$\lambda_x(\tau) = 0, \quad \lambda_y(\tau) = -a \quad \forall \quad \tau \geq 0$$

Thus, from Equation (95), in the field of primary trajectories

$$\phi^* = 1$$

Applying the above costates to Equation (94) yields

$$\sin \Psi^* = -\frac{0}{\sqrt{0 + (-a)^2}} = 0, \quad \cos \Psi^* = \frac{a}{\sqrt{0 + (-a)^2}} = 1$$

Hence, in the region of the primary trajectories, $\Psi^* = 0$.

Applying the above results to the dynamics in Equations (76) and (77) yields the dynamics of the primary trajectories

$$\dot{x} = u(y) \quad (98)$$

$$\dot{y} = w - b \quad (99)$$

Additionally, in the primary trajectory domain the Value function is

$$V(x, y) = \frac{y}{b - w}$$

$\Rightarrow$

$$V_x = 0, \quad V_y = \frac{1}{b - w} \quad (100)$$

3.5.4.2 Equivocal Surface

The ES is a trajectory emanating from the barrier end point B and is subject to different dynamics than the primary and secondary trajectories. The ES and barrier partition the state space into Zone 1 and Zone 2, as shown in Figure 28. On the ES, where the Pursuer's optimal control $\phi \neq 1$, the Evader has two distinct optimal strategy options: the Evader can traverse the ES or break away into Zone 1. The ES, as it applies to this academic differential game, can be seen in Figure 28.

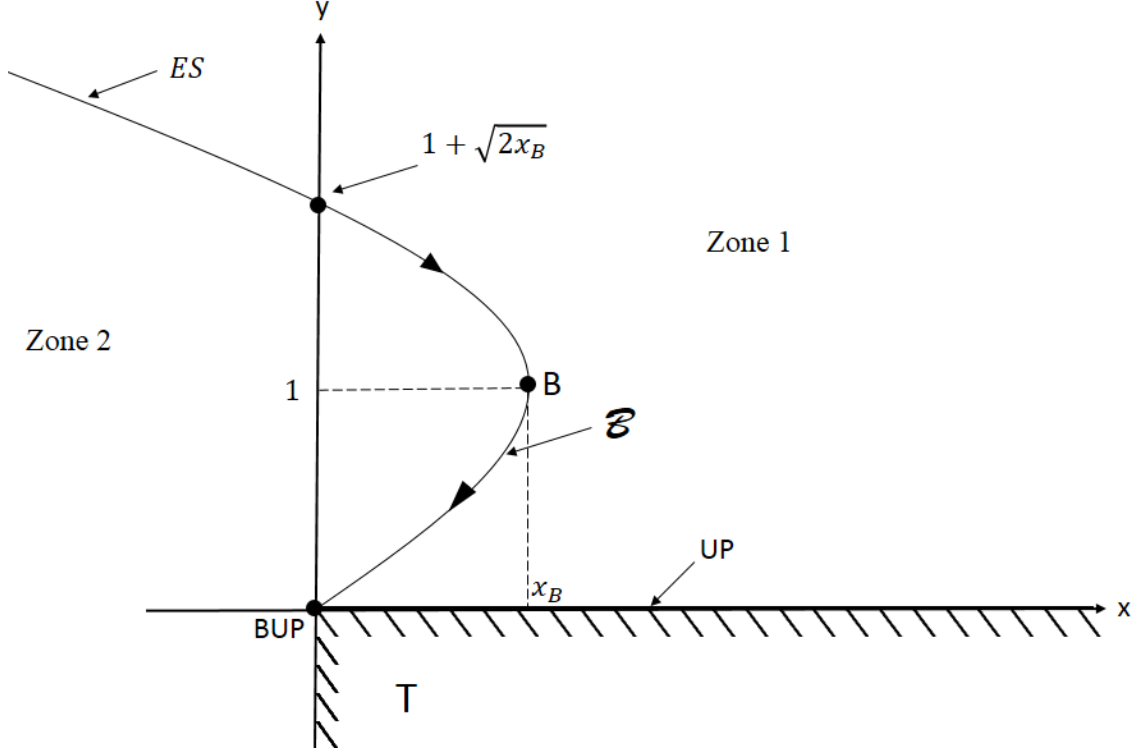


Figure 28. Exercise 10.6.1: Optimal Flow Field Depicting the Barrier, ES, and the Separation of Zone 1 and Zone 2

To construct the ES, it is first necessary to determine the Evader's optimal traversing control on the ES. Since the Pursuer chooses the slope of the ES, it is optimal for the Evader to traverse the ES as slowly as possible. The Evader minimizing his rate of descent along the ES is tantamount to $\min_{\Psi} \dot{x}$. Therefore, from Equation (76)

$$\check{\Psi} = \underset{\Psi}{\operatorname{argmin}}[u(y) + w \sin \Psi]$$

This yields the Evader's optimal control $\check{\Psi} = -\frac{\pi}{2}$ ¹.

To determine the Pursuer's control on the ES, the ES condition established by Isaacs is used. This condition stipulates that the payoff accrued to the Evader is the same whether the Evader chooses to traverse the ES or, alternatively, chooses to enter

¹Utilizing Isaacs' method. A more robust analytical derivation of the Evader's optimal control on the ES can be found in Appendix B.

Zone 1. The ES condition is

$$0 = -1 - V_x \dot{x} - V_y \dot{y}$$

where V_x and V_y are partial derivatives of the Value function from Equation (100).

Applying the dynamics from Equations (76) and (77) to the above equation yields

$$0 = -1 - V_x[u(y) + w \sin \check{\Psi}] - V_y[-b\check{\phi} + w \cos \check{\Psi}]$$

and applying the Evader's optimal control on the ES, $\check{\Psi} = -\frac{\pi}{2}$, yields

$$0 = -1 - V_x[u(y) - w] + V_y b\check{\phi}$$

Solving for the Pursuer's optimal control yields

$$\check{\phi} = \frac{V_x[u(y) - w] + 1}{bV_y}$$

Applying the Value function derivatives from Equation (100) yields the Pursuer's optimal control

$$\check{\phi} = 1 - \frac{w}{b} \quad (< 1) \tag{101}$$

Hence, the dynamics along the ES are

$$\dot{x} = u(y) + w \sin \check{\Psi}$$

$$\dot{y} = -b\check{\phi} + w \cos \check{\Psi}$$

$\Rightarrow$

$$\dot{x} = w - u(y), \quad x(\tau = 0) = x_B$$

$$\dot{y} = b - w, \quad y(\tau = 0) = y_B = y_0$$

where (x_B, y_B) , is the location of the barrier end point B. Note that because $u(y_0) = w$

$$\left. \frac{dx}{dy} \right|_B = 0$$

which shows that the ES joins the barrier smoothly at its end point B, as seen in Figure 28.

To obtain an expression for the ES, integrate the differential equation

$$\frac{dx}{dy} = \frac{w - u(y)}{b - w}, \quad x(y_0) = x_B, \quad y \geq y_0$$

$\Rightarrow$

$$x(y) = x_B + \frac{w}{b - w}(y - y_0) - \frac{1}{b - w} \int_{y_0}^y u(y) dy$$

An expression for the ES is obtained by integration:

$$x(y) = x_B - \frac{w}{b - w}y_0 + \frac{1}{b - w}[wy - \int_{y_0}^y u(y) dy] \quad (102)$$

3.5.4.3 Secondary Optimal Trajectories

The secondary trajectories emanate from the ES at $\tau = 0$, as sketched in Figure 27. Similar to the primary trajectories, the Hamiltonian and PMP are used to establish the secondary trajectories. Equations (93)-(97) still apply for the secondary trajectories, thus

$$\lambda_x \equiv constant = \lambda_{x_0}$$

$$\dot{\lambda}_y = u'(y)\lambda_{x_0}, \quad \lambda_y(\tau = 0) = \lambda_{y_0}, \quad \tau \geq 0$$

However, λ_{x_0} and λ_{y_0} have to be determined.

In the following derivation, an alternative approach to Isaacs' is used to generate the secondary trajectories. Lemma 10.5.2 concerning the continuity of the Evader's optimal control, as the state vector reaches the ES, is used herein. Lemma 10.5.2 states that the optimal traversing strategy of the Evader on the ES is continuous with the Evader's optimal strategy on the secondary trajectories. Since the Evader's optimal control $\check{\Psi}$ along the ES is known, the Evader's optimal control on the secondary trajectories upon reaching the ES at $\tau = 0$ is known:

On the ES

$$\check{\Psi} = -\frac{\pi}{2} \Rightarrow \sin\check{\Psi} = -1, \cos\check{\Psi} = 0 \quad (103)$$

Since on the ES, at $\tau = 0$, the control of the Evader is continuous with the secondary trajectories, from Equation (103) it is known that the optimal control on the secondary trajectories at $\tau = 0$ satisfies

$$\sin\Psi^* = \sin\check{\Psi} = -1, \cos\Psi^* = \cos\check{\Psi} = 0 \quad (104)$$

Applying this to Equation (94) yields

$$-\frac{\lambda_{x0}}{\sqrt{\lambda_{x0}^2 + \lambda_{y0}^2}} = -1, \quad -\frac{\lambda_{y0}}{\sqrt{\lambda_{x0}^2 + \lambda_{y0}^2}} = 0 \quad (105)$$

So, close to and slightly below the ES

$$\lambda_{x0} = a > 0, \quad \lambda_{y0} = 0$$

Hence, along the secondary optimal trajectories the costates are

$$\lambda_x(\tau) = a$$

$$\dot{\lambda}_y = u'(y)a, \quad \lambda_y(\tau = 0) = 0, \quad \tau \geq 0$$

Since

$$u'(y) > 0, \quad a > 0, \quad \lambda_y(\tau) > 0 \quad \forall \quad \tau > 0$$

from Equation (95), in Zone 2 the Pursuer's optimal control is $\phi^* = -1$ and from Equations (104) and (105) it is found that

$$\sin(\Psi^*(\tau)) = -\frac{a}{\sqrt{a^2 + \lambda_y^2(\tau)}}, \quad \cos(\Psi^*(\tau)) = -\frac{\lambda_y}{\sqrt{a^2 + \lambda_y^2(\tau)}} \quad (106)$$

The retrograde form of Equations (76) and (77) are

$$\dot{x} = -u(y) - w \sin \Psi$$

$$\dot{y} = b\phi - w \cos \Psi$$

Substituting the Evader's controls from Equation (106) into the retrograde dynamics yields

$$\dot{x} = -u(y) + w \frac{a}{\sqrt{a^2 + \lambda_y^2(\tau)}}, \quad x(\tau = 0) \in ES \quad (107)$$

$$\dot{y} = -b + w \frac{\lambda_y}{\sqrt{a^2 + \lambda_y^2(\tau)}}, \quad y(\tau = 0) \in ES \quad (108)$$

where

$$\dot{\lambda}_y = u'(y)a, \quad \lambda_y(\tau = 0) = 0, \quad \tau \geq 0 \quad (109)$$

Because $\lambda_y(\tau = 0) = 0$, $\lambda_y(\tau)$ is homogeneous in a : this means a can be set to 1 since it has no impact on the problem. Thus, consider

$$\lambda_y(\tau) = \zeta(\tau) \cdot a$$

With this

$$\dot{y} = -b + w \frac{\zeta a}{\sqrt{a^2 + \zeta a^2}}$$

$\Rightarrow$

$$\dot{y} = -b + w \frac{\zeta}{\sqrt{1 + \zeta^2}}$$

Additionally,

$$\dot{\lambda}_y = \dot{\zeta} a = u'(y) a$$

$\Rightarrow$

$$\dot{\zeta} = u'(y)$$

So in Equations (107)-(109), a can be set to 1 which is tantamount to setting $\lambda_y = \frac{1}{a} \lambda_y$. Applying $\lambda_y = \frac{1}{a} \lambda_y$ to Equations (107)-(109) yields

$$\dot{x} = -u(y) + w \frac{1}{\sqrt{1 + \lambda_y^2(\tau)}}, \quad x(\tau = 0) = x_{ES} \quad (110)$$

$$\dot{y} = -b + w \frac{\lambda_y}{\sqrt{1 + \lambda_y^2(\tau)}}, \quad y(\tau = 0) = y_{ES} \quad (111)$$

$$\dot{\lambda}_y = u'(y), \quad \lambda_y(0) = 0, \quad \tau \geq 0 \quad (112)$$

where the initial state $(x_{ES}, y_{ES}) \in ES$. Thus, the family of secondary trajectories is parameterized by $y_{ES} \geq y_B = y_0$.

3.5.5 Parameterized Exercise 10.6.1

To provide an example of the barrier and the newly introduced ES, Isaacs provided the parameterized Exercise 10.6.1 [1]. The results of this exercise are expressions for the primary and secondary optimal trajectories as a function of the retrograde time

τ as well an expression for the ES. Isaacs set the parameters to

$$b = 3, \quad w = 2, \quad u(y) = 1 + y \quad (113)$$

It is known that $u(y_0) = w$ and from there $y_0 = 1$. Additionally it can be found that $u'(y) = 1$.

3.5.5.1 Primary Optimal Trajectories

First, the primary trajectories in Equations (98) and (99) are evaluated for the parameters in Equation (113)

$$\dot{x} = 1 + y$$

$$\dot{y} = -1$$

In retrograde time τ , the parameterized primary optimal trajectories are

$$\dot{x} = -1 - y, \quad x(\tau = 0) = \eta$$

$$\dot{y} = 1, \quad y(\tau = 0) = 0$$

$\Rightarrow$

$$y(\tau) = \tau \quad \Rightarrow \quad \dot{x}(\tau) = -\tau - 1$$

$\Rightarrow$

$$x(\tau) = -\tau - \frac{1}{2}\tau^2 + \eta$$

$\Rightarrow$

$$x(y; \eta) = -\frac{1}{2}y^2 - y + \eta, \quad \eta > 0$$

Numerically evaluating Equation (92) for the parameters above yields

$$x_B = -\frac{3}{2} + \eta_B = 0.181316$$

Therefore, the primary trajectory emanating from the barrier end point B contacts the x-axis at

$$\eta_B = \frac{3}{2} + x_B = 1.681316$$

3.5.5.2 Equivocal Surface

To obtain the ES, the values in Equation (113) are applied to Equations (111) and (112) yielding

$$\dot{y} = -3 + 2\frac{\lambda_y}{\sqrt{1 + \lambda_y^2}}, \quad y(\tau = 0) = y_{ES} \quad (114)$$

$$\dot{\lambda}_y = 1, \quad \lambda_y(0) = 0, \quad \tau \geq 0$$

For $\lambda_y(\tau) = \tau$, Equation (114) becomes

$$\frac{dy}{d\tau} = -3 + 2\frac{\tau}{\sqrt{1 + \tau^2}}, \quad y(\tau = 0) = y_{ES}, \quad \tau \geq 0$$

To obtain an expression for the state's y-coordinate as a function of the retrograde time τ , the equation above is integrated to yield

$$y(\tau) = y_{ES} - 2 - 3\tau + 2\sqrt{1 + \tau^2} \quad (115)$$

Next an expression for the state's x-coordinate for any retrograde time τ is derived.

Applying the parameters set by Isaacs to Equation (110) yields

$$\dot{x} = -1 - y + 2\frac{1}{\sqrt{1 + \tau^2}}, \quad x(0) = x_{ES}, \quad \tau \geq 0$$

Integrating the expression above yields

$$x(\tau) = x_{ES} + (1 - y_{ES})\tau + \frac{3}{2}\tau^2 - 2 \int_0^\tau \frac{\tau}{\sqrt{1 + \tau^2}} d\tau \quad (116)$$

where integration by parts is applied to find

$$\int_0^\tau \frac{\tau}{\sqrt{1 + \tau^2}} dt = \frac{1}{2}[\tau\sqrt{1 + \tau^2} - \sinh^{-1}(\tau)]$$

Applying the results to Equation (116) yields

$$x(\tau) = x_{ES} + (1 - y_{ES})\tau + \frac{3}{2}\tau^2 - \tau\sqrt{1 + \tau^2} + \sinh^{-1}(\tau), \quad \tau \geq 0 \quad (117)$$

Equations (115) and (117) are expressions for the ES that will be used in the following section to obtain expressions for the secondary optimal trajectories. The parametric evaluation of the ES is accomplished by applying the problem parameters in Equation (113) to Equation (102) which yields

$$x_{ES}(y_{ES}) = x_B - 2 + 2y_{ES} - \int_1^{y_{ES}} (1 + y) dy$$

$\Rightarrow$

$$x_{ES}(y_{ES}) = x_B - \frac{1}{2} + y_{ES} - \frac{1}{2}y_{ES}^2, \quad y_{ES} \geq 1$$

$\Rightarrow$

$$x_{ES}(y_{ES}) = x_B - \frac{1}{2}(1 - y_{ES})^2 \quad (118)$$

Obtaining an expression for the ES can add an extra degree of complexity to solving a differential game. For this reason, two additional methods for constructing the ES of Exercise 10.6.1 can be found in Appendix C; these additional methods can potentially be applied to more complex differential games in order to obtain a full solution.

For completeness, the ability of the Evader to move the state vector into Zone 1 from any point along the ES, is shown.

From Equations (101) and (113), if the state is on the ES for the parameterized problem, the Pursuer's optimal control is

$$\check{\phi} = \frac{1}{3} \quad (119)$$

On the ES, when $\check{\phi} = \frac{1}{3}$ and the Evader employs its Zone 1 optimal control $\Psi^* = 0$, the dynamics of the differential game

$$\vec{f} = \begin{pmatrix} 1 + y + 2\sin\Psi \\ -3\phi + 2\cos\Psi \end{pmatrix} \quad (120)$$

become

$$\vec{f} = \begin{pmatrix} 1 + y \\ 1 \end{pmatrix}$$

and the normal to the ES, pointing into Zone 1, is $\vec{n} = (1, y - 1)$ for $y > 1$.

$$\vec{n} \cdot \vec{f} = 2y > 0 \quad (121)$$

Thus, when the state is on the ES, if the Evader switches to its Zone 1 penetration optimal strategy, the state will immediately enter Zone 1.

3.5.5.3 Secondary Optimal Trajectories

The secondary optimal trajectories can be obtained using the parameterized ES. Substituting Equation (118) into Equation (117) yields

$$x(\tau, y_{ES}) = x_B - \frac{1}{2}(1 - y_{ES})^2 + (1 - y_{ES})\tau + \frac{3}{2}\tau^2 - \tau\sqrt{1 + \tau^2} + \sinh^{-1}(\tau) \quad (122)$$

where

$$\sinh^{-1}(\tau) = \ln(\tau + \sqrt{1 + \tau^2})$$

The time-to-go is $s = \frac{y_{ES}}{b - w}$, so for Exercise 10.6.1 $s = y_{ES}$. Hence, Equations (115) and (122) become

$$x(\tau, s) = x_B - \frac{1}{2}(1 - s)^2 + (1 - s)\tau + \frac{3}{2}\tau^2 - \tau\sqrt{1 + \tau^2} + \sinh^{-1}(\tau) \quad (123)$$

$$y(\tau, s) = s - 2 - 3\tau + 2\sqrt{1 + \tau^2} \quad (124)$$

These expressions match Isaacs' expression for the secondary trajectories.

With the primary optimal trajectories, secondary optimal trajectories, and ES fully characterized, the complete optimal flow field is constructed as illustrated in Figure 27.

From the results above, it is possible to determine the optimal trajectory that emanates from the BUP=(0,0). Using Equations (123) and (124), a system of two equations in the unknowns $\tau > 0$ and $s > 1$ can be created

$$0 = x_B - \frac{1}{2}(1 - s)^2 + (1 - s)\tau + \frac{3}{2}\tau^2 - \tau\sqrt{1 + \tau^2} + \ln(\tau + \sqrt{1 + \tau^2}) \quad (125)$$

$$0 = s - 2 - 3\tau + 2\sqrt{1 + \tau^2} \quad (126)$$

Solving Equation (126) for s yields

$$s = 2 + 3\tau - 2\sqrt{1 + \tau^2}$$

Applying the equation above to Equation (125) yields

$$4\tau + 8\tau^2 - 2\sqrt{1 + \tau^2} - 7\tau\sqrt{1 + \tau^2} - \ln(+\sqrt{1 + \tau^2}) = x_B - \frac{5}{2} \approx -2.3187 \quad (127)$$

With this, when τ is known, s can be obtained which reveals the y-coordinate at which the trajectory emanating from the BUP contacts the ES.

3.5.5.4 The parameter “a”

The parameter representing a is inconsequential, but, as a matter of interest it can be obtained as follows. Considering the Hamiltonian in Equation (93) at time $\tau = 0$ where $\lambda_x = a(> 0)$, $\lambda_y = 0$ yields

$$H = -1 + a[u(y_{ES}) - w] = 0$$

Solving for a yields

$$a = \frac{1}{u(y_{ES}) - w}$$

Based on the values in Equation (113)

$$a = \frac{1}{y_{ES} - 1}, \quad y_{ES} > 1 \tag{128}$$

3.5.6 Discussion

In this section, a discussion on the ES and barrier are given. Because the ES adds an extra degree of difficulty to solving differential games, it is desirable to develop strategies for the Pursuer that preclude the Evader from exploiting the ES in an attempt to achieve safe contact. In this section, three methods that prevent the Evader from exploiting the ES are presented, one of which is suggested by Isaacs [1]. The three methods are as follows:

1. Capture Set Synthesis: The Pursuer can guarantee the state vector's contact of the UP by synthesizing its optimal control strategy as if the capture set were slightly smaller than it is in reality. By allowing the Evader to push the

state vector along the synthesized barrier, the Pursuer ensures the state vector contacts the UP of the true capture set.

2. Isaacs' Approach: If the state vector is initially located in Zone 2; the Pursuer plays $\phi^* = -1$ until the state vector reaches the ES. If the Pursuer continues to play $\phi^* = -1$ for a short time afterwards, the state vector is forced into Zone 1, despite the optimal control of the Evader.
3. ES Traversal: If the Pursuer allows the state vector to traverse the ES, once the state contacts the barrier end point B, by playing $\phi^* = -1$ for a short interval, the Pursuer effectively forces the state vector into Zone 1, despite the optimal control of the Evader.

Applying any of these methods requires the Pursuer to incur a small cost in optimality, the benefit, however, is guaranteed termination of the game. Additionally, these methods can be applied to more complex differential games in order to reduce the difficulty in obtaining an optimal control solution.

3.5.6.1 Method 1: Capture Set Synthesis

If the Evader enforces the ES traversal strategy and reaches B, based on the Pursuer's choice of control, the Evader can force the state down the barrier and achieve a safe contact at the BUP. If the Evader can enforce a safe contact, the game does not terminate and is therefore prolonged. The Pursuer can prevent this outcome by synthesizing its optimal strategy as if the capture set were slightly smaller than it actually is. When the Evader pushes the state vector along the barrier of the synthesized capture set, in an attempt to contact the BUP, the state vector instead contacts the UP of the true capture set, as seen in Figure 29. Although this strategy causes the Pursuer to incur a small additional cost, it guarantees capture in the face

of optimal play by the Evader and eliminates a neutral outcome.

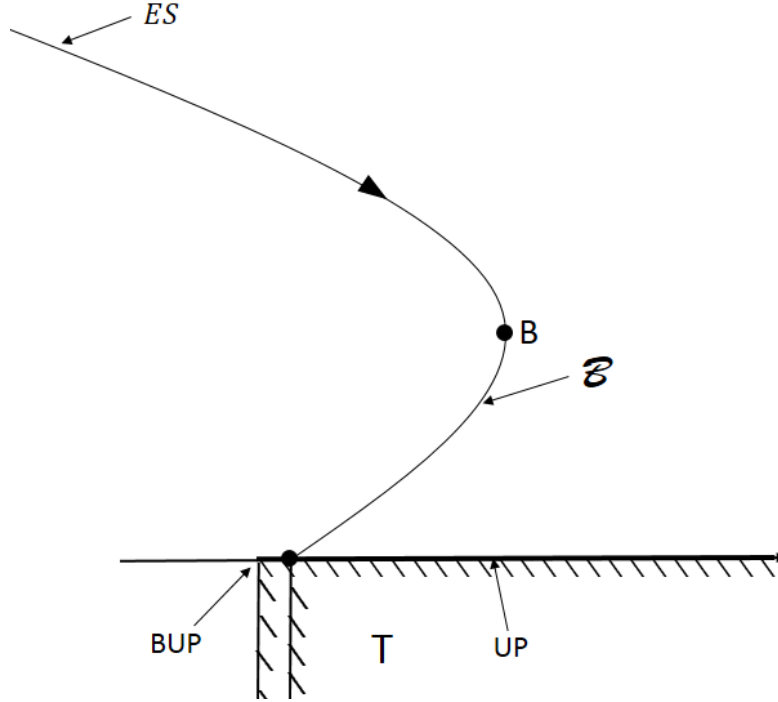


Figure 29. Exercise 10.6.1: The Result of the Pursuer Synthesizing an Optimal Control Solution for a Smaller Capture Set

3.5.6.2 Method 2: Isaacs' Approach

An alternative approach to guaranteeing capture, recommended by Isaacs, is one where upon the ES, the Pursuer causes the state vector to cross the ES by continuing to play the Zone 2 strategy $\phi^* = -1$ for a short time. Once the state crosses the ES, the Pursuer then switches its control to the Zone 1 strategy $\phi^* = 1$. Playing $\phi^* = -1$ for an additional small interval after the state contacts the ES causes the Pursuer to sacrifice some optimality and incur an additional cost in order to ensure that the state will end up on the UP of the capture set. For this strategy to work, the permeability condition must be met, that is

$$\vec{n} \cdot \vec{f}'(-1, \Psi) > 0$$

where $\vec{n}$ is the normal to the ES pointing into Zone 1.

$$\vec{n} = \left(\frac{1}{\sqrt{2(x_B - x)}}, 1 \right), \quad x_B > x$$

To push the state vector into Zone 1, the Pursuer's optimal control is $\phi^* = -1$. From Section 3.5.4.2, when the state vector is on the ES, the Evader's optimal control is $\check{\Psi} = -\frac{\pi}{2}$. Thus, the permeability condition is

$$\vec{n} \cdot \vec{f}(\phi^* = -1, \check{\Psi} = -\frac{\pi}{2}) > 0$$

Applying $\phi^* = -1$, $\check{\Psi} = -\frac{\pi}{2}$ to the dynamics in Equation (120) yields

$$\vec{f} = \begin{pmatrix} 1 + y - w \\ b \end{pmatrix} = \begin{pmatrix} y - 1 \\ 3 \end{pmatrix}$$

Hence

$$\vec{n} \cdot \vec{f}(\phi^* = -1, \check{\Psi} = -\frac{\pi}{2}) = 3 + \frac{y - 1}{\sqrt{2(x_B - x)}}$$

From Equation (118)

$$y - 1 = \sqrt{2(x_B - x)}$$

$\Rightarrow$

$$\vec{n} \cdot \vec{f}(\phi^* = -1, \check{\Psi} = -\frac{\pi}{2}) = 4 > 0$$

Therefore, when the Evader plays his ES traversal strategy, if the Pursuer plays the Zone 2 strategy of $\phi^* = -1$, the Pursuer can force the state vector off of the ES into Zone 1. Additionally, if the Evader plays the Zone 1 optimal strategy $\Psi^* = 0$, the

dynamics are

$$\vec{f} = \begin{pmatrix} 1 + y \\ b + w \end{pmatrix} = \begin{pmatrix} 1 + y \\ 5 \end{pmatrix}$$

$\Rightarrow$

$$\vec{n} \cdot \vec{f}(\phi^* = -1, \Psi^* = 0) = 5 + \frac{1 + y}{\sqrt{2(x_B - x)}}, \quad x < x_B$$

Thus, the permeability condition is satisfied if the Pursuer plays $\phi^* = -1$ when the state is on the ES. However, it is known that the ES is not a permeable surface because there is a corner. If the Pursuer chooses the optimal control $\phi^* = 1$ when the state is on the ES, the dynamics become

$$\vec{f} = \begin{pmatrix} 1 + y - w \\ -3 \end{pmatrix} = \begin{pmatrix} y - 1 \\ -3 \end{pmatrix}$$

Examining the permeability condition when the Pursuer plays $\phi^* = 1$ and the Evader plays the ES traversal strategy yields

$$\vec{n} \cdot \vec{f}(\phi^* = 1, \check{\Psi} = -\frac{\pi}{2}) = \frac{y - 1}{\sqrt{2(x_B - x)}} - 3$$

$\Rightarrow$

$$= 1 - 3 = -2 < 0$$

Thus, the Pursuer is unable to pull the state over the ES by utilizing the Zone 1 optimal control $\phi^* = 1$.

3.5.6.3 Method 3: ES Traversal

There is a strategy available to the Pursuer that allows the Pursuer to ensure the state reaches the UP of the capture set even if the state travels down the ES to the barrier end point.

The normal pointing into Zone 1 at B is $\vec{n} = (1, 0)$. When the Evader forces the state to traverse the ES, if the Pursuer allows the state to traverse the barrier until it reaches B, to ensure capture, the Pursuer chooses the optimal control $\phi^* = -1$, whereupon

$$\vec{f} = \begin{pmatrix} 1 + y + 2\sin\Psi \\ 3 + 2\cos\Psi \end{pmatrix} \quad (129)$$

$$\vec{n} \cdot \vec{f} = 2(1 + \sin\Psi) \geq 0 \quad \forall \quad \Psi \quad (130)$$

Additionally the y-component of $\vec{f}$ is $3 + 2\cos\Psi > 0 \quad \forall \quad \Psi$. Thus, when the state reaches B, if the Pursuer plays $\phi^* = -1$ at B and for a short interval (t_+) afterwards, the state is pushed into Zone 1. At time t_+ the state is s.t. $y > 1$, so the state is either in the interior of Zone 1 or on the ES. If the state is located on the ES, the additional small time interval where $\phi^* = -1$ will push the state into Zone 1 as shown by Method 2. If the Pursuer utilizes this strategy, the state will always reach the UP of the capture set. Similar to the previous implementation strategy, the Pursuer sacrifices some optimality by playing $\phi^* = -1$ at B and for some short time afterwards. This strategy is just another method of ensuring the state reaches the UP of the capture set that Isaacs did not investigate.

If the state has reached B and the Pursuer were to play the Zone 1 optimal strategy $\phi^* = 1$, the Evader would be able to force the state down the barrier all the way to the BUP. At the BUP, the Pursuer's optimal strategy is the Zone 2 control $\phi^* = -1$, so if the state reaches the BUP, the Pursuer's control returns the state to Zone 2 and then to the ES. Thus, the Evader can preclude the state from reaching the UP of the capture set if the Pursuer plays $\phi^* = 1$ at the barrier end point B. However, the Evader is unable to utilize this strategy as long as the Pursuer plays $\phi^* = -1$ at B and for a short time afterwards.

This chapter presents the foundational work used to obtain the solution of the

classical Homicidal Chauffeur differential game. These preliminary works include two novel Dubins car optimal control problems, as well as Exercise 10.6.1 posed by Isaacs in his work [1]. The Dubins car optimal control problems act as simplified analogues to the *classical* Homicidal Chauffeur differential game by replacing the mobile Evader of the differential game with a stationary target point. Obtaining the solution to the optimal control problems validates a methodology that can be used to obtain the solution to the *classical* Homicidal Chauffeur differential game. The optimal control solutions also provide an expectation for the characteristics of the players' optimal strategies in the *classical* Homicidal Chauffeur differential game. By examining Exercise 10.6.1, a greater understanding concerning the formation and effect of a barrier and ES on the solution of a differential game is achieved. Additionally, the investigation of Exercise 10.6.1 produced three methods of preventing the Evader from exploiting the ES that can be used to simplify the solution of the *classical* Homicidal Chauffeur differential game. The examination of the optimal control problems and Exercise 10.6.1 provide a starting point, as well an approach, to solve the *classical* Homicidal Chauffeur differential game.

IV. Results

In this Chapter the solution to the *classical* Homicidal Chauffeur differential game is derived. Utilizing the Dubins car optimal control solutions, as well as the solution to Exercise 10.6.1, the solution process of the *classical* Homicidal Chauffeur differential game is derived and outlined herein. The approach taken to obtain a solution mirrors the approach of Sections 3.2 and 3.3, however, the stationary target point of the previous sections is now replaced with a mobile Evader. Following a statement of the differential game, the optimal flow field is systematically constructed as each component of the solution is derived. Following the derivation of the full solution, a discussion of the results is presented.

4.1 Differential Game and Dynamics

In the herein considered differential game, a Pursuer modeled as Dubins car, whose speed is V_P and with a minimum turning radius R , is to be steered such that in minimum time it comes within a range ℓ from an Evader with simple motion whose speed is V_E . The Pursuer wants to minimize the time-to-capture while the Evader strives to maximize it. In this differential game the speed ratio $0 \leq \mu = \frac{V_E}{V_P} < 1$. In the *classical* region of the (μ, ℓ) parameter space, capture of the Evader is guaranteed in the whole state space. To minimize/maximize the time-to-capture the Pursuer and Evader must each solve an optimal control problem based on their position relative to one another.

The *classical* Homicidal Chauffeur differential game takes place in the Euclidean plane referred to as the realistic space. The reference frame of the *classical* Homicidal Chauffeur differential game, and all components contained therein, were presented in Section 2.3.1 and shown in Figure 1.

Since the Dubins car optimal control problems were solved by deriving the *classical* Homicidal Chauffeur differential game dynamics, then reducing them to optimal control dynamics by setting $\mu = 0$, the dynamics derived in Section 3.1.1 are valid for this differential game. Therefore, the kinematic equations of the *classical* Homicidal Chauffeur differential game are

$$\dot{x} = -uy + \mu \sin \Psi, \quad x(0) = x_0 \quad (131)$$

$$\dot{y} = ux - 1 + \mu \cos \Psi, \quad y(0) = y_0, \quad 0 \leq t \leq t_f. \quad (132)$$

Recalling the process used to characterize the BUP and UP of the Dubins car optimal control problem for the case of a circular capture set: the BUP in the differential game occurs at $\varphi_{max} = A \cos \mu$ and the UP = $\{(x, y) | x^2 + y^2 = \ell^2, -\varphi_{max} \leq \varphi \leq \varphi_{max}\}$. The Evader is captured on the reduced state space capture set $T = \sqrt{x^2(t_f) + y^2(t_f)} = \ell$.

4.2 Optimal Control

Based on the initial position of the Evader relative to the Pursuer, three classes of trajectories can result in the Homicidal Chauffeur differential game: primary, secondary, and tributary trajectories. These trajectories encapsulate the motion of both the Pursuer and Evader and depict how the Evader's position changes relative to the Pursuer. Expressions for these trajectories, as well as for other components of the optimal flow field (the barrier and ES), are derived in this section. The optimal flow field is then systematically built as each component of the solution is obtained.

Similar to the Dubins car optimal control problems derived previously: the solution to the Homicidal Chauffeur differential game can change as the parameters are varied. Changes to the parameters can produce solutions that are qualitatively different from one another. In the *classical* parameter range, however, the solution

is qualitatively the same for any set of parameters, its appearance just varies slightly as parameters are changed. Therefore, in the sections below, the optimal flow field is depicted for the parameters $\ell = 0.5$, $\mu = 0.5$, which are well within the *classical* parameter range examined by Isaacs.

The solution is presented for the case of the Evader initially positioned in the RHP. While the solution is only derived for the case of the Evader initially positioned in the RHP, the optimal flow field in the LHP is a mirror reflection of the solution in the RHP.

4.2.1 Primary Optimal Trajectories

Primary optimal trajectories are retrograde trajectories that emanate from the UP= $\{(x, y) | x^2 + y^2 = \ell^2, -\varphi_{max} < \varphi < \varphi_{max}\}$ of the circular capture set. To obtain these optimal trajectories, the Hamiltonian of the Homicidal Chauffeur differential game is examined. Using the PMP, the Hamiltonian is

$$H(x, \lambda, u, \Psi) = -1 + \lambda_x(-uy + \mu \sin \Psi) + \lambda_y(ux - 1 + \mu \cos \Psi)$$

$\Rightarrow$

$$H = -1 + u(x\lambda_y - y\lambda_x) - \lambda_y + \mu(\lambda_x \sin \Psi + \lambda_y \cos \Psi) \quad (133)$$

From Equation (133), and according to the PMP, to minimize the time-to-capture, the Pursuer must maximize the Hamiltonian by selecting its control such that

$$u^* = \text{sign}(x\lambda_y - y\lambda_x) \quad (134)$$

Thus, as long as the switch function $S(\tau) = x(\tau)\lambda_y(\tau) - y(\tau)\lambda_x(\tau) \neq 0$, it is always optimal for the Pursuer to turn as hard as possible.

From Equation (133), and according to the PMP, to maximize the time-to-capture,

the Evader must minimize the Hamiltonian by setting its heading angle Ψ such that

$$\sin\Psi^* = -\frac{\lambda_x}{\sqrt{\lambda_x^2 + \lambda_y^2}}, \quad \cos\Psi^* = -\frac{\lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}} \quad (135)$$

Thus, the players' optimal controls are characterized by Equations (134) and (135).

According to the PMP, the costates satisfy the differential equations

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -\lambda_y u, \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y} = \lambda_x u \quad (136)$$

To obtain expressions for the primary optimal trajectories, the dynamics in Equations (131) and (132) will be rewritten in retrograde time $\tau = t_f - t$. The dynamics in retrograde time τ are

$$\dot{x} = uy - \mu \sin\Psi, \quad x(\tau = 0) = x_0 \quad (137)$$

$$\dot{y} = -ux + 1 - \mu \cos\Psi, \quad y(\tau = 0) = y_0, \quad 0 \leq \tau \leq \tau_{max} \quad (138)$$

The retrograde time τ_{max} denotes the maximum retrograde time a trajectory is valid for; it is determined following the derivation of the trajectory expressions.

The Evader is considered captured at $\tau = 0$ when located on the UP of the capture set. In the RHP, in the region of the primary optimal trajectories, the Pursuer's optimal control $u^* = 1$. Thus, the transversality condition of the PMP yields

$$\lambda_x(\tau = 0) = -a \cdot \sin\varphi, \quad \lambda_y(\tau = 0) = -a \cdot \cos\varphi, \quad a > 0, \quad 0 \leq \varphi < A\cos\mu$$

when the game evolves in the RHP. The relationships in Equation (136) yields the harmonic equations

$$\overset{\infty}{\lambda}_x + \lambda_x = 0, \quad \overset{\infty}{\lambda}_y + \lambda_y = 0$$

With the transversality conditions, the solutions to the harmonic equations yield the costates

$$\lambda_x(\tau) = -a \cdot \sin(\varphi + \tau), \quad \lambda_y(\tau) = -a \cdot \cos(\varphi + \tau) \quad (139)$$

The optimal control law of the Evader can now be expressed in terms of time. Applying the results above to Equation (135) yields

$$\sin(\Psi^*(\tau)) = \sin(\varphi + \tau), \quad \cos(\Psi^*(\tau)) = \cos(\varphi + \tau)$$

From the equation above, it can be determined that the optimal control of the Evader, in the region of primary trajectories, is

$$\Psi^*(\tau) = \varphi + \tau, \quad \tau \geq 0$$

Using the expression for the Evader's optimal control, from the retrograde dynamics in (137) and (138), the primary optimal trajectory dynamics are expressed as

$$\dot{x} = y - \mu \sin(\varphi + \tau), \quad x(\tau = 0) = \ell \sin \varphi \quad (140)$$

$$\dot{y} = -x + 1 - \mu \cos(\varphi + \tau), \quad y(\tau = 0) = \ell \cos \varphi, \quad 0 \leq \varphi < A \cos \mu \quad (141)$$

when the game develops in the RHP. Differentiating Equation (140) yields

$$\ddot{x} = \dot{y}(\tau) - \mu \cos(\varphi + \tau)$$

which yields the non-homogeneous harmonic equation

$$\ddot{x} + x = 1 - 2\mu \cos(\varphi + \tau), \quad x(\tau = 0) = \ell \sin \varphi, \quad \dot{x}(\tau = 0) = \ell \cos \varphi - \mu \sin \varphi$$

Solving the above differential equation, and applying a similar derivation to Equation (141) yields expressions for the Evader's position relative to the Pursuer in the region of primary trajectories in the RHP.

$$x(\tau) = 1 + (\ell - \mu\tau)\sin(\varphi + \tau) - \cos(\tau) \quad (142)$$

$$y(\tau) = (\ell - \mu\tau)\cos(\varphi + \tau) + \sin(\tau), \quad 0 \leq \tau \leq \tau_{max}, \quad 0 \leq \varphi < A\cos\mu \quad (143)$$

A primary trajectory persists until the Pursuer's control switches from 1 to -1 or vice versa in the LHP, as indicated by the switching function $S(\cdot)$ which features in the Hamiltonian and in part determines the domain of validity of the primary trajectories. From the Hamiltonian in Equation (133), the switching function is

$$S(\tau) = x(\tau)\lambda_y(\tau) - y(\tau)\lambda_x(\tau) > 0, \quad 0 \leq \tau \leq \tau_{max}$$

Substituting Equations (139), (142), and (143) in the switching function above yields

$$S(\tau) = \cos(\varphi + \tau) - \cos\varphi$$

The maximum retrograde time τ_{max} possible, before the Pursuer's control switches, occurs when $S(\tau) = 0$. Applying $S(\tau) = 0$, the switching condition becomes

$$\cos(\varphi + \tau_{max}) = \cos(\varphi)$$

which yields

$$\tau_{max} = 2(\pi - \varphi) \quad (144)$$

So, the family of primary trajectories parameterized by $0 \leq \varphi < \varphi_{max}$ is

$$x(\tau) = 1 + (\ell - \mu\tau)\sin(\varphi + \tau) - \cos(\tau) \quad (145)$$

$$y(\tau) = (\ell - \mu\tau)\cos(\varphi + \tau) + \sin(\tau), \quad 0 \leq \tau \leq 2(\pi - \varphi), \quad 0 \leq \varphi < \text{Acos}\mu \quad (146)$$

With τ_{max} known it is possible to obtain an expression for the end point of a primary trajectory. Using $B(\varphi)$ to denote the φ -trajectory's end point and evaluating Equations (145) and (146) for $\tau_{max} = 2(\pi - \varphi)$ yields

$$x(B(\varphi)) = 1 - [\ell - 2\mu(\pi - \varphi)]\sin\varphi - \cos(2\varphi)$$

$$y(B(\varphi)) = [\ell - 2\mu(\pi - \varphi)]\cos\varphi - \sin(2\varphi)$$

$\Rightarrow$

$$x(B(\varphi)) = [2\sin\varphi + 2\mu(\pi - \varphi) - \ell]\sin\varphi$$

$$y(B(\varphi)) = -[2\sin\varphi + 2\mu(\pi - \varphi) - \ell]\cos\varphi$$

With this, it can be shown that the trajectory end point lies on a straight line through the origin whose slope is $-\tan\varphi$

$$\frac{x(B(\varphi))}{y(B(\varphi))} = -\frac{\sin\varphi}{\cos\varphi} = -\tan\varphi$$

4.2.2 Barrier

In the *classical* Homicidal Chauffeur differential game, where the problem parameters are subject to $\mu^2 + \ell^2 < 1$, an abnormal trajectory, the barrier, emanates from the BUP at $\varphi = \varphi_{max}$. If $\mu^2 + \ell^2 > 1$, the barrier trajectory is immediately invalidated as its development path causes it to initially penetrate the capture set

before evolving outward.

To construct the barrier, the Hamiltonian in Equation (133) is examined, excluding the running cost term. Thus, the Hamiltonian used to construct the barrier is

$$H = -u(x\lambda_y - y\lambda_x) - \lambda_y + \mu(\lambda_x \sin \Psi + \lambda_y \cos \Psi) \quad (147)$$

Following the process used to obtain expressions for the primary optimal trajectories, one also obtains the expressions of the barrier in parametric form

$$x(\tau) = 1 + (\ell - \mu\tau)\sin(\varphi_{max} + \tau) - \cos\tau \quad (148)$$

$$y(\tau) = (\ell - \mu\tau)\cos(\varphi_{max} + \tau) + \sin\tau, \quad 0 \leq \tau \leq \tau_{max} \quad (149)$$

Although the expressions for the barrier are equivalent to the primary optimal trajectory expressions in Equations (145) and (146), evaluated for $\varphi = \varphi_{max}$, the barrier is not an optimal trajectory. Although its construction mirrors a primary trajectory's construction, the barrier is in fact a locus of initial conditions where the Pursuer's optimal strategy is to turn away from the Evader, as shown by Merz [2]. In the *classical* Homicidal Chauffeur differential game, if the Evader is initially positioned on or beneath the barrier, the Pursuer's optimal strategy is to initially turn away from the Evader before doubling back and catching the Evader.

Differentiating Equations (148) and (149) with respect to τ and evaluating at $\tau = 0$ yields

$$\dot{x}(\tau = 0) = \ell \cos \varphi_{max} - \mu \sin \varphi_{max}$$

$$\dot{y}(\tau = 0) = 1 - \ell \sin \varphi_{max} - \mu \cos \varphi_{max}$$

$\Rightarrow$

$$\dot{x}(\tau = 0) = \ell \cos(A \cos \mu) - \mu \sin(A \cos \mu)$$

$$\dot{y}(\tau = 0) = 1 - \ell \sin(A \cos \mu) - \mu \cos(A \cos \mu)$$

Using the above equations, the slope of the barrier at the BUP is

$$\left. \frac{dy}{dx} \right|_{\tau=0} = \frac{-\ell \sqrt{1 - \mu^2} + 1 - \mu^2}{\ell \mu - \mu \sqrt{1 - \mu^2}}$$

$\Rightarrow$

$$\left. \frac{dy}{dx} \right|_{\tau=0} = -\frac{\sqrt{1 - \mu^2}}{\mu} = -\tan \varphi_{max} \quad (150)$$

which reveals that, at the BUP, the barrier is tangent to the capture circle.

Because the barrier is developed as a primary trajectory which corresponds to φ_{max} , it is governed by the switching function as seen on the primary trajectories. If the barrier formation condition $u^2 + \ell^2 < 1$ is met, the barrier's trajectory stops at time $\tau_{max} = 2(\pi - \varphi_{max})$ when the switch function changes sign. The position the barrier stops at is evaluated by expressing Equations (148) and (149) in terms of τ_{max} . Those equations are shown below, where the subscript “B” denotes the point at which the barrier stops. The coordinates of the end point of the barrier are

$$x_B = 1 - [\ell - 2\mu(\pi - \varphi_{max})] \sin \varphi_{max} - \cos(2\varphi_{max})$$

$$y_B = [\ell - 2\mu(\pi - \varphi_{max})] \cos \varphi_{max} - \sin(2\varphi_{max})$$

$\Rightarrow$

$$x_B = 1 - [\ell - 2\mu(\pi - A \cos \mu)] \sqrt{1 - \mu^2} + 1 - 2\mu^2$$

$$y_B = [\ell - 2\mu(\pi - A \cos \mu)] \mu - 2\mu \sqrt{1 - \mu^2}$$

$\Rightarrow$

$$x_B = \sqrt{1 - \mu^2} [-\ell + 2\mu(\pi - A \cos \mu) + 2\sqrt{1 - \mu^2}] \quad (151)$$

$$y_B = -\mu[-\ell + 2\mu(\pi - A\cos\mu) + 2\sqrt{1-\mu^2}] \quad (152)$$

$\Rightarrow$

$$\frac{x_B}{y_B} = -\frac{\sqrt{1-\mu^2}}{\mu} = -\tan\varphi_{max}$$

Differentiating Equations (148) and (149) with respect to τ and evaluating at $\tau = 2(\pi - \varphi_{max})$ yields

$$\dot{x}|_B = -\sin(2\varphi_{max}) + [\ell - 2\mu(\pi - \varphi_{max})]\cos\varphi_{max} + \mu\sin\varphi_{max}$$

$$\dot{y}|_B = \cos(2\varphi_{max}) + [\ell - 2\mu(\pi - \varphi_{max})]\sin\varphi_{max} - \mu\cos\varphi_{max}$$

$\Rightarrow$

$$\dot{x}|_B = -2\mu\sqrt{1-\mu^2} + [\ell - 2\mu(\pi - A\cos\mu)]\mu + \mu\sqrt{1-\mu^2}$$

$$\dot{y}|_B = 2\mu^2 - 1 + [\ell - 2\mu(\pi - A\cos\mu)]\sqrt{1-\mu^2} - \mu^2$$

Using the above equations the slope of the barrier at B is

$$\left.\frac{dx}{dy}\right|_B = \frac{\mu[\ell - 2\mu(\pi - A\cos\mu) - \sqrt{1-\mu^2}]}{\sqrt{1-\mu^2}[\ell - 2\mu(\pi - A\cos\mu) - \sqrt{1-\mu^2}]}$$

Provided that $\ell \neq 2\mu(\pi - A\cos\mu) + \sqrt{1-\mu^2}$, $0 \leq \varphi \leq A\cos\mu$

$$\left.\frac{dy}{dx}\right|_B = \frac{\sqrt{1-\mu^2}}{\mu} = \tan\varphi_{max}$$

However, for a barrier to be present in the first place

$$\mu^2 + \ell^2 < 1$$

$\Rightarrow$

$$\ell < \sqrt{1 - \mu^2}$$

As long as those conditions are met, from the equation above and Equation (150)

$$\left. \frac{dy}{dx} \right|_{\tau=0} / \left. \frac{dy}{dx} \right|_B = -1$$

This reveals that the tangent to the capture circle at the BUP and the tangent to the barrier at point B are orthogonal.

There is a point A at which all the primary trajectories intersect: all primary trajectories intersect at the retrograde time $\tau = \frac{\ell}{\mu}$, because at this single point the trajectories are not a function of φ . Evaluating the primary trajectories in Equations (145) and (146) for $\tau = \frac{\ell}{\mu}$, yields

$$x\left(\frac{\ell}{\mu}\right) = 1 - \cos\left(\frac{\ell}{\mu}\right)$$

$$y\left(\frac{\ell}{\mu}\right) = \sin\left(\frac{\ell}{\mu}\right)$$

Thus all primary trajectories intersect at the point

$$A = \left(1 - \cos\left(\frac{\ell}{\mu}\right), \sin\left(\frac{\ell}{\mu}\right)\right)$$

Since the barrier is included in the set of primary trajectories, in the *classical* Homicidal Chauffeur differential game, the primary trajectories terminate at point A on the barrier. The point A is also on the Pursuer's minimum turning circle. Hence, in the *classical* Homicidal Chauffeur differential game, A is also at the intersection of the barrier and the minimal turning circle. This is not so if the speed ratio parameter is outside of the *classical* range, for example, $0 < \mu \ll 1$.

The primary trajectories, barrier, and their intersection point A are shown in Figure 30.

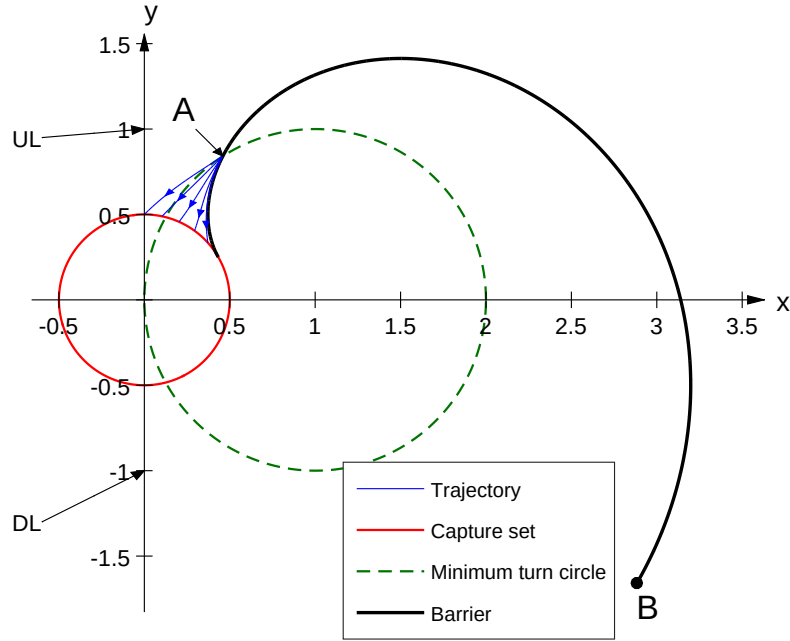


Figure 30. Homicidal Chauffeur Differential Game: Primary Optimal Trajectories in the Optimal Flow Field; $\ell = 0.5$, $\mu = 0.5$

Since point $A \in \mathcal{B}$, the Pursuer's optimal strategy at point A is $u^* = -1$. Therefore, point A is *not* a state where the Evader has multiple options. The Evader would be able to traverse any primary trajectory if the Pursuer played $u^* = 1$ at A : however, this is not the Pursuer's optimal strategy when the Evader is located at point A because point A is located on the barrier which is a locus of turn away points for the Pursuer. It is true, however, that the vicinity of point A , slightly above the barrier, is a region where the Pursuer's strategy and ensuing optimal trajectory are very sensitive to the Evader's position relative to the Pursuer.

4.2.3 The Universal Line and the Dispersal Line

In Figure 30 the UL and DL are similar in nature to the UL and DL of the Dubins car optimal control problems of Chapter III.

Once the Evader is on the UL it is optimal for both the Pursuer and Evader to play pure line-of-sight (LOS) pursuit and evasion, where $UL = \{(x, y) | x = 0, y \geq \ell\}$. The UL is a singular optimal trajectory where the switch function $S(\tau) \equiv 0$. The UL is the only singular trajectory in the *classical* Homicidal Chauffeur differential game. The DL along the negative y-axis is a locus along which the players' strategies are not unique, where $DL = \{(x, y) | x = 0, y \leq -\ell\}$. Paths emanating in real time from either side of the DL are optimal. In the Homicidal Chauffeur differential game, on the DL, the Evader's choice of control is dependent on the Pursuer's choice of control. Since the players have opposing goals, and essentially encounter a control selection stalemate on the DL, Isaacs proposed the use of an Instantaneous Mixed Strategy (IMS) in which both players randomly choose to turn left or right for a small window of time, then, once away from the DL, apply their optimal strategies.

4.2.4 Tributary Optimal Trajectories

Tributaries are retrograde trajectories that emanate in a retrograde sense from the UL at $(0, s)$, where $s > \ell$. To obtain the tributary optimal trajectories the costates at $\tau = 0$ must be determined. To this end, the Value function $V(x, y)$ at the point where the trajectories emanate from is established. Now, the Value function on the positive y-axis, where LOS pursuit and evasion takes place, is

$$V(0, y) = \frac{y - \ell}{1 - \mu}$$

By symmetry $V_x|_{x=0} = 0$ and we calculate $V_y|_{x=0} = \frac{1}{1-\mu}$. Based on the connection between the PMP and DP, the two costates λ_x and λ_y are such that, at $\tau = 0$, $\lambda_x = -V_x$ and $\lambda_y = -V_y$. Thus, when the game evolves in the RHP, the costates of the retrograde tributary trajectories satisfy

$$\dot{\lambda}_x = \lambda_y, \quad \lambda_x(\tau_1 = 0) = 0$$

$$\dot{\lambda}_y = -\lambda_x, \quad \lambda_y(\tau_1 = 0) = -\frac{1}{1-\mu}, \quad 0 \leq \tau_1$$

Hence

$$\dot{\lambda}_x(\tau_1 = 0) = -\frac{1}{1-\mu}$$

$\Rightarrow$

$$\lambda_x(\tau_1) = -\frac{1}{1-\mu} \sin \tau_1, \quad \lambda_y(\tau_1) = -\frac{1}{1-\mu} \cos \tau_1 \quad (153)$$

where τ_1 denotes the retrograde time measured from the time the trajectory emanates from the y-axis $(0, s)$, $s > \ell$. The Evader's optimal control is characterized by Equation (135) which, in view of the costates in Equation (153), yields

$$\sin(\Psi^*(\tau_1)) = \sin \tau_1, \quad \cos(\Psi^*(\tau_1)) = \cos \tau_1 \quad (154)$$

which is the Evader's optimal heading angle along a tributary trajectory. Using the expression for the Evader's optimal control, from the retrograde dynamics in Equations (137) and (138), the tributary optimal trajectories are expressed as

$$\dot{x} = y - \mu \sin \tau_1, \quad x(\tau_1 = 0) = 0 \quad (155)$$

$$\dot{y} = -x + 1 - \mu \cos \tau_1, \quad y(\tau_1 = 0) = s, \quad s > \ell \quad (156)$$

Differentiating Equation (155) yields

$$\ddot{x} = \dot{y} - \mu \cos \tau_1$$

which yields the non-homogeneous harmonic equation

$$\ddot{x} + x = 1 - 2\mu \cos \tau_1, \quad x(\tau_1 = 0) = 0, \quad \dot{x}(\tau = 0) = s$$

Solving the above non-homogeneous harmonic equation and applying a similar derivation to Equation (156) yields expressions for the Evader's position relative to the Pursuer in the region of tributary trajectories.

$$x(\tau_1) = 1 + (s - \mu \tau_1) \sin \tau_1 - \cos \tau_1 \quad (157)$$

$$y(\tau_1) = (s - \mu \tau_1) \cos \tau_1 + \sin \tau_1, \quad 0 \leq \tau_1 \leq \tau_{1max}, \quad s > \ell \quad (158)$$

These trajectories correspond to the scenario where the Evader is initially in the RHP. The domain of validity of the tributary trajectories is determined by the switching function

$$S(\tau) = x(\tau_1) \lambda_y(\tau_1) - y(\tau_1) \lambda_x(\tau_1) > 0, \quad 0 \leq \tau_1 \leq \tau_{1max}$$

Substituting in Equations (153), (157), and (158), the switching function for the tributary trajectories becomes

$$S(\tau_1) = \frac{1}{1 - \mu} (1 - \cos \tau_1) \quad (159)$$

The maximum retrograde time τ_{max} possible, before the Pursuer's optimal control switches, occurs when $S(\tau) = 0$. Applying $S(\tau) = 0$ to the switching function above

yields

$$\cos\tau_{1max} = 1$$

which yields

$$\tau_{1max} = 2\pi$$

So, the family of tributary trajectories, parameterized by $s > \ell$, is

$$x(\tau_1) = 1 + (s - \mu\tau_1)\sin\tau_1 - \cos\tau_1 \quad (160)$$

$$y(\tau_1) = (s - \mu\tau_1)\cos\tau_1 + \sin\tau_1, \quad 0 \leq \tau_1 \leq 2\pi, \quad s > \ell \quad (161)$$

Yet, unlike in optimal control, the tail end of these trajectories may no longer be part of an optimal flow field. This is due to the effect of the ES, derived in Section 4.2.5. While the SL occurs at the natural termination point of the trajectories, the ES can interrupt a trajectory before the trajectory naturally terminates. The tributary trajectories are shown in Figure 31.

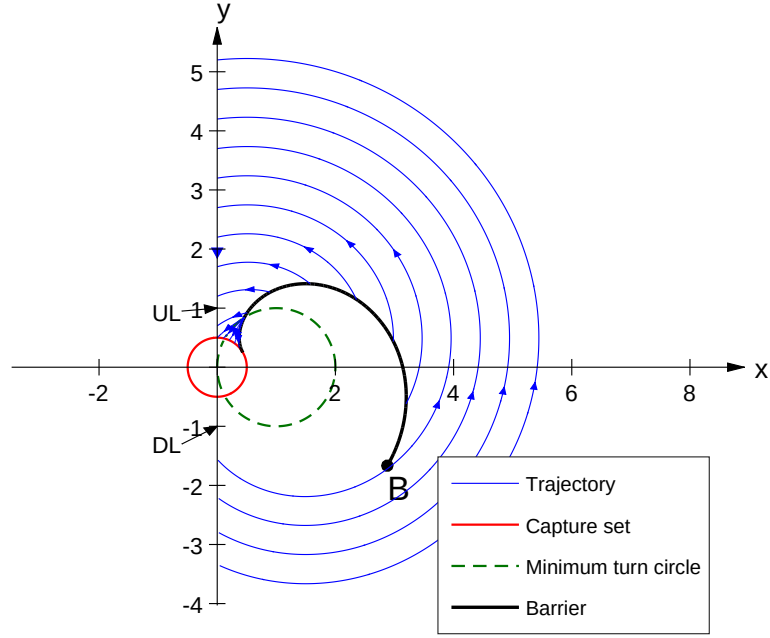


Figure 31. Homicidal Chauffeur Differential Game: Tributary Optimal Trajectories in the Optimal Flow Field, $\ell = 0.5$, $\mu = 0.5$

With the tributary trajectories and barrier constructed, it is possible to determine the tributary that meets the barrier at its end point B. Similar to finding the trajectory that contacted the BUP in Exercise 10.6.1 in Section 3.5.5.3: to solve for the tributary starting position $(0, s)$, $s > \ell$, that will cause the tributary to meet the barrier at point B, a system of two equations in τ_1 and s is built using Equations (151) and (152) in conjunction with Equations (160) and (161) yielding.

$$\sqrt{1 - \mu^2}[-\ell + 2\mu(\pi - A\cos\mu) + 2\sqrt{1 - \mu^2}] = 1 + (s - \mu\tau_1)\sin\tau_1 - \cos\tau_1 \quad (162)$$

$$-\mu[-\ell + 2\mu(\pi - A\cos\mu) + 2\sqrt{1 - \mu^2}] = (s - \mu\tau_1)\cos\tau_1 + \sin\tau_1 \quad (163)$$

To solve for s , multiply the first equation by $\cos\tau_1$ and the second equation by $\sin\tau_1$

then subtracting yields

$$a \cdot \cos\tau_1 + b \cdot \sin\tau_1 = \cos\tau_1 - 1$$

where

$$a(\mu, \ell) = \sqrt{1 - \mu^2}[-\ell + 2\mu(\pi - A\cos\mu) + 2\sqrt{1 - \mu^2}]$$

$$b(\mu, \ell) = \mu[-\ell + 2\mu(\pi - A\cos\mu) + 2\sqrt{1 - \mu^2}] = (s - \mu\tau_1)\cos\tau_1 + \sin\tau_1$$

Thus

$$(1 - a)\cos\tau_1 - b \cdot \sin\tau_1 = 1$$

Note

$$\frac{a}{b} = \frac{\sqrt{1 - \mu^2}}{\mu}$$

$\Rightarrow$

$$(1 - a)\cos\tau_1 - \frac{\mu}{\sqrt{1 - \mu^2}}a \cdot \sin\tau_1 = 1$$

Introducing the angle $\eta(\mu, \ell)$ subject to

$$\sin\eta = \frac{\frac{\mu}{\sqrt{1 - \mu^2}}a}{\sqrt{(1 - a)^2 + \frac{\mu^2}{1 - \mu^2}a^2}}, \quad \cos\eta = \frac{1 - a}{\sqrt{(1 - a)^2 + \frac{\mu^2}{1 - \mu^2}a^2}}$$

$\Rightarrow$

$$\cos\eta \cdot \cos\tau_1 - \sin\eta \cdot \sin\tau_1 = 1$$

$\Rightarrow$

$$\cos(\tau_1 + \eta) = \frac{1}{\sqrt{(1 - a)^2 + \frac{\mu^2}{1 - \mu^2}a^2}}$$

$\Rightarrow$

$$\tau_1 = 2\pi - A \cos \left(\frac{1}{\sqrt{(1-a)^2 + \frac{\mu^2}{1-\mu^2}a^2}} \right) - \eta \quad (164)$$

From Equation (163)

$$-b = (s - \mu\tau_1)\cos\tau_1 + \sin\tau_1$$

$\Rightarrow$

$$s = \frac{-(b + \sin\tau_1)}{\cos\tau_1} + \mu\tau_1 \quad (165)$$

The critical tributary trajectory that meets the barrier end point B is parameterized by s given by Equation (165), where

$$\tau_1 = 2\pi - A \cos \left(\frac{1}{\sqrt{(1-a)^2 + \frac{\mu^2}{1-\mu^2}a^2}} \right) - \eta$$

4.2.5 Equivocal Surface

The ES is an exclusive surface to differential games and has no analogue in optimal control. Its complexity adds an additional degree of difficulty to obtaining an optimal control solution in a differential game. The ES acts as a threshold that prevents trajectories from intersecting each other while also maintaining the optimality of traveling along a specific trajectory. For example, evaluating the tributary trajectories range of validity using the switching function reveals that the tributary trajectory are valid up to $\tau_{1max} = 2\pi$. If the tributary trajectories were to develop up to $\tau_{1max} = 2\pi$, a portion of the trajectory would then intersect with the secondary trajectories. Thus, the ES is constructed in order to separate the families of trajec-

tories where an intersection would occur. This separation causes the ES to interrupt a trajectory, as discussed in Section 4.2.4.

The ES emanates from the barrier end point B and is subject to different dynamics than the previously established trajectories. An Evader located on the ES has two distinct optimal strategies at each point on the ES: the Evader can choose to traverse the ES or depart from the ES along a tributary trajectory to the UL and then for the point $(0, \ell)$ on the UP of the capture set. In order to construct the ES, the Pursuer's and Evader's optimal controls are examined for the Evader's ES traversal strategy.

If traveling along the ES, the Evader strives to minimize its velocity along the ES. This, according to Lemmas 10.5.1 and 10.2.1 from [1], demands that the Evader points towards the Pursuer such that

$$\sin \Psi_{ES}^* = -\frac{x}{r}, \quad \cos \Psi_{ES}^* = -\frac{y}{r} \quad (166)$$

where r is the distance between the Evader and Pursuer

$$r = \sqrt{x^2 + y^2}$$

With this optimal control, the Evader tries to come closer to the Pursuer by chasing the Pursuer. To find the Pursuer's control u when the Evader is traveling on the ES, the Hamiltonian is examined. Applying the Evader's optimal control in Equation (166) to the Hamiltonian in Equation (133) yields

$$\lambda_x \sin \Psi_{ES}^* + \lambda_y \cos \Psi_{ES}^* = -\frac{1}{r}(x\lambda_x + y\lambda_y)$$

Based on Equation (153) for the costates along the ES, the equation above simplifies

to

$$\frac{1}{1-\mu} \frac{1}{r} (x \sin \tau_1 + y \cos \tau_1) \quad (167)$$

Substituting Equations (153), (159), and (167) into the Hamiltonian yields

$$-1 + \frac{1}{1-\mu} \cos \tau_1 + \frac{\mu}{1-\mu} \frac{1}{r} (x \sin \tau_1 + y \cos \tau_1) = \frac{1}{1-\mu} (\cos \tau_1 - 1) u_{ES}$$

$\Rightarrow$

$$\cos \tau_1 - 1 + \mu \frac{1}{r} (r + x \sin \tau_1 + y \cos \tau_1) = (\cos \tau_1 - 1) u_{ES}$$

Solving for the Pursuer's control u_{ES} yields

$$u_{ES} = 1 - \mu \frac{r + x \sin \tau_1 + y \cos \tau_1}{r(1 - \cos \tau_1)} \quad (168)$$

From Equations (157) and (158) for the right tributary trajectories, one backs out τ_1 in terms of x and y [1]:

$$\cos \tau_1 = \frac{1 - x + hy}{h^2 + 1}, \quad \sin \tau_1 = \frac{y - h(1 - x)}{h^2 + 1} \quad (169)$$

where

$$h^2 = x^2 + y^2 - 2x$$

Applying the above to Equation (168) yields

$$1 - \cos \tau_1 = \frac{h^2 + x - hy}{1 + h^2}, \quad r + x \sin \tau_1 + y \cos \tau_1 = \frac{r(1 + h^2) + hr^2 + y - hx}{1 + h^2}$$

$\Rightarrow$

$$u_{ES} = 1 - \mu \frac{r(1 + h^2) + hr^2 + y - hx}{r^2 - x - hy} \quad (170)$$

Note that Isaacs' derivation of u_{ES} in [1] is slightly different

$$u_{ES} = 1 - \mu \frac{1}{r} \frac{r^2 - x + rh}{r - y} \quad (171)$$

but the two expressions can be reconciled as follows:

$$\frac{r(1 + h^2) + hr^2 + y - hx}{r^2 - x - hy} = \frac{r^2 - x + rh}{r - y}$$

$\Rightarrow$

$$(r(1 + h^2) + hr^2 + y - hx)(r - y) = (r^2 - x - hy)(r^2 - x + rh)$$

$\Rightarrow$

$$r^2 h^2 = r^4 - 2r^2 x$$

with $h^2 = r^2 - 2x$

$$r^4 - 2r^2 x = r^4 - 2r^2 x$$

Therefore, the Pursuer's optimal control along the ES in Equations (170) and (171) are equivalent. Based on the Pursuer's optimal control in Equation (171), when the Evader is located on the ES, it is no longer optimal for the Pursuer to turn as hard as possible.

When the Evader traverses the ES, the Pursuer must solve a max/min control problem to obtain an optimal control. This results in the Pursuer utilizing a stroboscopic/discriminatory strategy of the form $u_{ES}(\Psi)$. Because the Pursuer's control relies upon knowledge of the Evader's instantaneous control, this is not a satisfactory state of affairs.

Now that the control of the Pursuer, during the Evader's traversal of the ES, is obtained, the ES traversal trajectory can be calculated using the Pursuer's control in Equation (170) and the Evader's optimal control in Equation (166). Utilizing

Equations (137) and (138), the retrograde dynamics along the ES are

$$\dot{x}_{ES} = y_{ES}u_{ES} + \mu \frac{x_{ES}}{r_{ES}}, \quad x_{ES}(\tau_2) = x_B \quad (172)$$

$$\dot{y}_{ES} = -x_{ES}u_{ES} + 1 + \mu \frac{y_{ES}}{r_{ES}}, \quad y_{ES}(\tau_2) = y_B, \quad 0 \leq \tau_2 \quad (173)$$

where x_{ES} and y_{ES} are the (x, y) coordinates of the Evader located on the ES.

$$\frac{dy_{ES}}{dx_{ES}} = \frac{1 - x_{ES}u_{ES} + \mu \frac{y_{ES}}{r_{ES}}}{y_{ES}u_{ES} + \mu \frac{x_{ES}}{r_{ES}}}, \quad y_{ES}(x_B) = y_B, \quad x_{ES} \leq x_B$$

where, τ_2 is the retrograde time starting from the end point B of the barrier at which the barrier stops.

One can construct the ES trajectory by numerical integration of the above differential equation by utilizing the Euler method of integrating ordinary differential equations. Using the Euler method, for some function

$$\dot{x}(\tau) = f(\tau, x(\tau)), \quad x(\tau = 0) = x_0$$

the value of $x(\tau)$ can be determined for incremental changes of τ starting from the initial value $x(\tau = 0)$. Hence, for some incremental step Δ where $\tau_n = \tau_0 + n \cdot \Delta$

$$x_{n+1} = x_n + \Delta \cdot f(\tau_n, x_n)$$

When constructing the ES, it is known that the ES will develop towards and eventually terminate on the y-axis. Thus, from the initial condition $y_{ES}(\tau_2) = y_B$, the y-coordinate of the ES can be determined as the x-coordinate is incrementally changed from $x_{ES}(\tau_2) = x_B$ to $x_{ES}(\tau_2) = 0$. The ES constructed in this manner is shown in Figure 32.

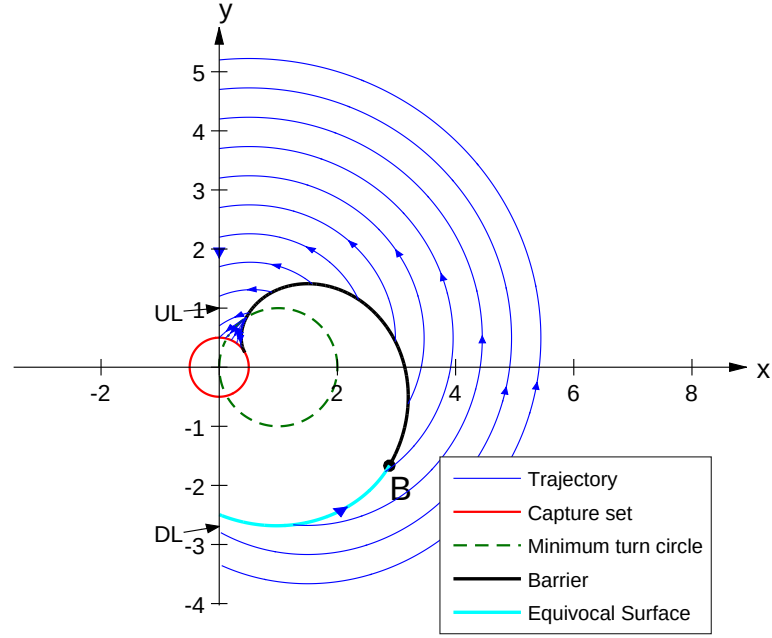


Figure 32. Homicidal Chauffeur Differential Game: Equivocal Surface in the Optimal Flow Field, $\ell = 0.5$, $\mu = 0.5$

For completeness, the Evader's ability to exit the ES into the field of tributary trajectories is examined. Here it is shown that the ES is truly equivocal.

Using Equations (172) and (173), the tangent anywhere on the ES can be defined as

$$\vec{t}(\tau_2) = (\dot{x}, \dot{y}), \quad 0 \leq \tau_2 \leq \tau_{2max}$$

where τ_{2max} is subject to $x(\tau_{2max}) = 0$. Hence, the normal to the tangent, pointing into the region of the tributary trajectories, is

$$\vec{n}(\tau_2) = (-\dot{y}, \dot{x})$$

$\Rightarrow$

$$\vec{n}(\tau_2) = \begin{pmatrix} x_{ES}u_{ES} - 1 - \mu \frac{y_{ES}}{r_{ES}} \\ y_{ES}u_{ES} + \mu \frac{x_{ES}}{r_{ES}} \end{pmatrix}$$

To show that the Evader can move from the ES to the region of the tributary trajectories, the dynamics in the realistic plane are examined. Recalling the forward time dynamics from Equations (131) and (132)

$$\vec{f} = \begin{pmatrix} -yu + \mu \sin \Psi \\ xu - 1 + \mu \cos \Psi \end{pmatrix}$$

To exit the ES, into the region of the tributary trajectories, the Evader applies its optimal control from the region of the tributary trajectories. Applying the controls in Equation (154) to the dynamics above yields

$$\vec{f} = \begin{pmatrix} -yu + \mu \sin \tau_1 \\ xu - 1 + \mu \cos \tau_1 \end{pmatrix}$$

Utilizing Equation (169), the Evader's tributary trajectory optimal control strategy can be expressed in terms of its position on the ES. Applying Equation (169), along with the Pursuer's optimal control strategy $u^* = u_{ES}$ from Equation (171), to the dynamics above yields

$$\vec{f} = \begin{pmatrix} -y_{ES}u_{ES} + \mu \frac{y - h(1-x)}{h^2 + 1} \Big|_{ES} \\ x_{ES}u_{ES} - 1 + \mu \frac{1-x+hy}{h^2 + 1} \Big|_{ES} \end{pmatrix}$$

For the Evader to exit the ES into the region of the tributary trajectories as it wishes, the following must hold true

$$\vec{n} \cdot \vec{f} > 0 \quad \forall \quad 0 \leq \tau_2 \leq \tau_{2max}$$

A numerical analysis reveals that, for any position on the ES (x_{ES}, y_{ES}) , $\vec{n} \cdot \vec{f} > 0$, thereby verifying that the Evader has two distinct optimal control options along

the ES that are available at every point along the surface. Because the Evader has two optimal paths to choose when located on the ES, the ES is not a singular optimal trajectory. This equivocal phenomenon, and the resulting surface, is unique to differential games and has no counterpart in the theory of optimal control.

4.2.6 Secondary Optimal Trajectories

The secondary optimal trajectories are the trajectories included between the barrier and the ES. The secondary trajectories in the RHP correspond to the Pursuer's optimal control being $u^* = -1$. Because the Pursuer's control switches after crossing the ES, the secondary retrograde trajectories that emanate from the ES take on new dynamics that cause them to terminate on the barrier or the NUP of the capture set. In order to obtain the secondary trajectories, it is necessary to obtain the Evader's optimal control in this region. To obtain the Evader's optimal control, the costates in the domain of the secondary trajectories are determined.

Similar to Section 3.5.4.3, Lemma 10.5.2 can be used to obtain the costates of the secondary trajectories. Recall Lemma 10.5.2 states that the optimal traversing control of the Evader on the ES is continuous with its optimal control on the secondary trajectories. Since the Evader's optimal control along the ES is known, the Evader's optimal control on the secondary trajectories upon reaching the ES at time $\tau_3 = 0$ is known; here τ_3 denotes the retrograde time measured from the time the secondary trajectory emanates from the ES, with $\tau_3 = 0$ corresponding to the trajectory's initial location on the ES.

Utilizing Lemma 10.5.2, the Evader's ES traversing strategy in Equation (166) and optimal strategy in Equation (135), at time $\tau_3 = 0$ are equal. Thus

$$\frac{\lambda_{x_0}}{\sqrt{\lambda_{x_0}^2 + \lambda_{y_0}^2}} = \frac{x_{ES}}{r}, \quad \frac{\lambda_{y_0}}{\sqrt{\lambda_{x_0}^2 + \lambda_{y_0}^2}} = \frac{y_{ES}}{r}$$

and using the above relationships, the costates of the secondary trajectories at time $\tau_3 = 0$ are

$$\lambda_y(\tau_3 = 0) = \frac{y_{ES}}{x_{ES}}\lambda_{x_0}, \quad \lambda_x(\tau_3 = 0) = \lambda_{x_0} \quad (174)$$

In the region of the secondary trajectories, in the RHP, the Pursuer's optimal control $u^* = -1$. Applying this to Equation (136) yields

$$\dot{\lambda}_x = \lambda_y \rightarrow \dot{\lambda}_x = -\lambda_y \quad (175)$$

$$\dot{\lambda}_y = -\lambda_x, \rightarrow \dot{\lambda}_y = \lambda_x \quad (176)$$

Using the relationships above yields the harmonic equations for the costates

$$\ddot{\lambda}_x + \lambda_x = 0, \quad \lambda_x(\tau_3 = 0) = \lambda_{x_0}, \quad \dot{\lambda}_x(\tau_3 = 0) = -\frac{y_{ES}}{x_{ES}}\lambda_{x_0}$$

$$\ddot{\lambda}_y + \lambda_y = 0, \quad \lambda_y(\tau_3 = 0) = \frac{y_{ES}}{x_{ES}}\lambda_{x_0}, \quad \dot{\lambda}_y(\tau_3 = 0) = \lambda_{x_0}$$

With the transversality conditions in Equation (174), the solutions to the harmonic equations from above yield the costates

$$\lambda_x(\tau_3) = \lambda_{x_0}(\cos\tau_3 - \frac{y_{ES}}{x_{ES}}\sin\tau_3), \quad \lambda_y(\tau_3) = \lambda_{x_0}(\sin\tau_3 + \frac{y_{ES}}{x_{ES}}\cos\tau_3) \quad (177)$$

To obtain an expression for the Evader's optimal control along the secondary trajectories, the costates in Equation (177) are applied to Equation (135).

$$\sin(\Psi^*(\tau_3)) = \frac{-\lambda_{x_0} \left(\cos\tau_3 - \frac{y_{ES}}{x_{ES}}\sin\tau_3 \right)}{\sqrt{\lambda_{x_0}^2 \left[\left(\cos\tau_3 - \frac{y_{ES}}{x_{ES}}\sin\tau_3 \right)^2 + \left(\sin\tau_3 + \frac{y_{ES}}{x_{ES}}\cos\tau_3 \right)^2 \right]}}$$

$$\cos(\Psi^*(\tau_3)) = \frac{-\lambda_{x_0} \left(\sin\tau_3 + \frac{y_{ES}}{x_{ES}} \cos\tau_3 \right)}{\sqrt{\lambda_{x_0}^2 \left[\left(\cos\tau_3 - \frac{y_{ES}}{x_{ES}} \sin\tau_3 \right)^2 + \left(\sin\tau_3 + \frac{y_{ES}}{x_{ES}} \cos\tau_3 \right)^2 \right]}}$$

Simplifying the expressions above yields the Evader's optimal control

$$\sin(\Psi^*(\tau_3)) = \frac{\frac{y_{ES}}{x_{ES}} \sin\tau_3 - \cos\tau_3}{\sqrt{1 + \frac{y_{ES}^2}{x_{ES}^2}}}, \quad \cos(\Psi^*(\tau_3)) = \frac{-\frac{y_{ES}}{x_{ES}} \cos\tau_3 - \sin\tau_3}{\sqrt{1 + \frac{y_{ES}^2}{x_{ES}^2}}} \quad (178)$$

To obtain expressions for the Evader's position relative to the Pursuer, the Pursuer's optimal control along the secondary trajectories $u^* = -1$, is applied to the dynamics in Equations (137) and (138) yielding the secondary dynamics

$$\dot{x} = -y - \mu \sin\Psi, \quad x(0) = x_0 \quad (179)$$

$$\dot{y} = x + 1 - \mu \cos\Psi, \quad y(0) = y_0, \quad 0 \leq \tau \leq \tau_{max} \quad (180)$$

Using the expressions for the Evader's optimal control in Equation (178), the dynamics for the secondary optimal trajectories emanating from the ES are

$$\dot{x} = -y - \mu \left(\frac{\frac{y_{ES}}{x_{ES}} \sin\tau_3 - \cos\tau_3}{\sqrt{1 + \frac{y_{ES}^2}{x_{ES}^2}}} \right), \quad x(\tau_3 = 0) = x_{ES} \quad (181)$$

$$\dot{y} = x + 1 + \mu \left(\frac{\frac{y_{ES}}{x_{ES}} \cos\tau_3 + \sin\tau_3}{\sqrt{1 + \frac{y_{ES}^2}{x_{ES}^2}}} \right), \quad y(\tau_3 = 0) = y_{ES}, \quad (x_{ES}, y_{ES}) \in ES \quad (182)$$

Differentiating Equation (181) yields

$$\ddot{x} = -\dot{y} - \mu \left(\frac{\frac{y_{ES}}{x_{ES}} \cos \tau_3 + \sin \tau_3}{\sqrt{1 + \frac{y_{ES}^2}{x_{ES}^2}}} \right)$$

which yields the non-homogeneous harmonic equation

$$\ddot{x} + x = -1 - 2\mu \left(\frac{\frac{y_{ES}}{x_{ES}} \cos \tau_3 + \sin \tau_3}{\sqrt{1 + \frac{y_{ES}^2}{x_{ES}^2}}} \right), \quad x(\tau_3 = 0) = x_{ES}, \quad \dot{x}(\tau_3 = 0) = -y_{ES} + \frac{\mu}{\sqrt{1 + \frac{y_{ES}^2}{x_{ES}^2}}}$$

Solving the above non-homogeneous harmonic equation, and applying a similar derivation to Equation (182) yields expressions for the Evader's position relative to the Pursuer in the region of secondary trajectories emanating from the ES.

$$x(\tau_3) = -y_{ES} \sin \tau_3 + (x_{ES} + 1) \cos \tau_3 - \mu \tau_3 \left(\frac{\frac{y_{ES}}{x_{ES}} \sin \tau_3 - \cos \tau_3}{\sqrt{1 + \frac{y_{ES}^2}{x_{ES}^2}}} \right) - 1$$

$$y(\tau_3) = y_{ES} \cos \tau_3 + (x_{ES} + 1) \sin \tau_3 + \mu \tau_3 \left(\frac{\frac{y_{ES}}{x_{ES}} \cos \tau_3 + \sin \tau_3}{\sqrt{1 + \frac{y_{ES}^2}{x_{ES}^2}}} \right)$$

The secondary trajectories emanating from the ES are shown in Figure 33.

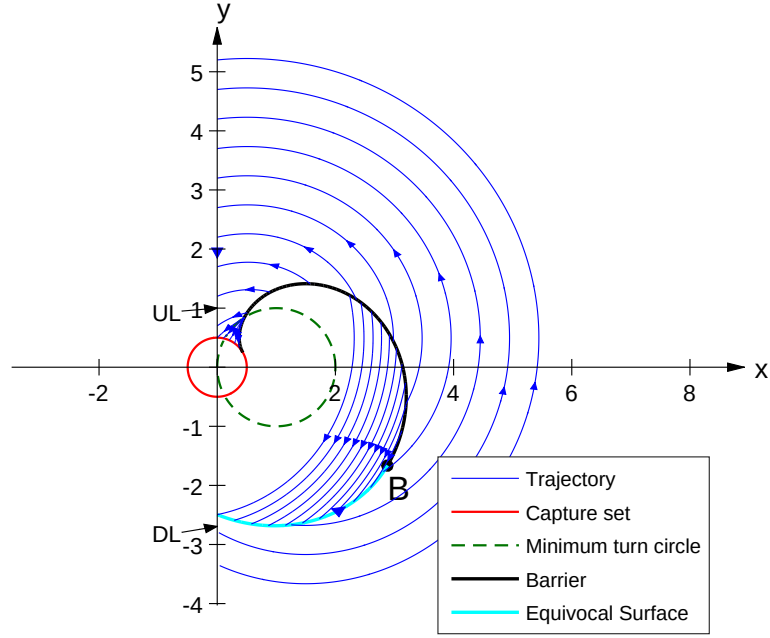


Figure 33. Homicidal Chauffeur Differential Game: Secondary Optimal Trajectories in the Optimal Flow Field, $\ell = 0.5$, $\mu = 0.5$

Similar to the secondary trajectories in Section 3.3.5, the secondary trajectories emanating from the ES do not fill the entire state space between the barrier and the ES. To further fill the gap in the (reduced) state space, it is necessary to obtain the secondary trajectories emanating from the negative y-axis segment $-\ell > y > y_{ES}(x = 0)$. Here $y_{ES}(x = 0)$ denotes the y-coordinate of the ES where it meets the y-axis at $x_{ES} = 0$.

The negative y-axis segment $-\ell > y > y_{ES}(x = 0)$ is a UL similar to the positive y-axis beyond the circular capture set. On the negative y-axis UL, a reversal of roles is seen. If the Evader is located on this UL it is optimal for the Evader to follow the Pursuer in the realistic plane, that is, employ LOS pursuit while the Pursuer runs away from the Evader. In this scenario, it is optimal for the Pursuer to run from the Evader to increase the Evader-Pursuer separation until the Evader reaches the ES at $(0, y_{ES}(x = 0))$; once the Evader is on the ES at $(0, y_{ES}(x = 0))$, the Pursuer will

ideally employ the strategy $u_{ES}(< 1)$ of turning towards the Evader.

To obtain expressions for the secondary trajectories emanating from the negative y-axis UL, the costates of the trajectories are obtained. As shown in Section 3.3.5, the expressions for the costates on the negative y-axis UL are similar to the costates along the positive y-axis UL. Therefore, by utilizing the tributary trajectory costates from Equation (153), it is known that on the negative y-axis UL, the costates satisfy

$$\lambda_x(\tau_1 = 0) = 0, \quad \lambda_y(\tau_1 = 0) = -\frac{1}{1 - \mu} \quad (183)$$

Using these transversality conditions, the relationships in Equations (175) and (176) yields the harmonic equations

$$\ddot{\lambda}_x + \lambda_x = 0, \quad \lambda_x(\tau_4 = 0) = \lambda_{x_0}, \quad \dot{\lambda}_x(\tau_4 = 0) = \frac{1}{1 - \mu}$$

$$\ddot{\lambda}_y + \lambda_y = 0, \quad \lambda_y(\tau_4 = 0) = -\frac{1}{1 - \mu}, \quad \dot{\lambda}_y(\tau_4 = 0) = 0$$

Where τ_4 denotes the retrograde time measure from the time the secondary trajectory emanates from the negative y-axis UL. With the transversality conditions in Equation (183), the solutions to the harmonic equations yield the costates

$$\lambda_x(\tau_4) = \frac{1}{1 - \mu} \sin \tau_4, \quad \lambda_y(\tau_4) = -\frac{1}{1 - \mu} \cos \tau_4 \quad (184)$$

To obtain an expression for the Evader's optimal control along the secondary trajectories, the costates in Equation (184) are applied to Equation (135).

$$\sin(\Psi^*(\tau_4)) = \frac{-\frac{1}{1 - \mu} \sin \tau_4}{\sqrt{\left(\frac{1}{1 - \mu} \sin \tau_4\right)^2 + \left(-\frac{1}{1 - \mu} \cos \tau_4\right)^2}}$$

$$\cos(\Psi^*(\tau_4)) = \frac{\frac{1}{1-\mu}\cos\tau_4}{\sqrt{\left(\frac{1}{1-\mu}\sin\tau_4\right)^2 + \left(-\frac{1}{1-\mu}\cos\tau_4\right)^2}}$$

Simplifying the equations above yields expressions for the Evader's optimal control along the secondary trajectories emanating from the negative-y-axis

$$\sin(\Psi^*(\tau_4)) = -\sin\tau_4, \quad \cos(\Psi^*(\tau_4)) = \cos\tau_4$$

Using the expressions for the Evader's optimal control, when the Evader is initially in the RHP, the dynamics in Equations (179) and (180) for the secondary optimal trajectories emanating from the negative y-axis UL are

$$\dot{x} = -y + \mu\sin\tau_4, \quad x(\tau_4 = 0) = 0 \quad (185)$$

$$\dot{y} = x + 1 - \mu\cos\tau_4, \quad y(\tau_4 = 0) = s, \quad -\ell > s > y_{ES}(x = 0) \quad (186)$$

Differentiating Equation (185) yields

$$\ddot{x} = -\dot{y} - \mu\cos\tau_4$$

which yields the non-homogeneous harmonic equation

$$\ddot{x} + x = -1 + 2\mu\cos\tau_4, \quad x(\tau_4 = 0) = 0, \quad \dot{x}(\tau_4 = 0) = -s, \quad -\ell > s > y_{ES}(x = 0)$$

Solving the above non-homogeneous harmonic equation, and applying a similar derivation to Equation (186) yields expressions for the Evader's position relative to the Pursuer in the region of secondary trajectories emanating from the negative y-axis

UL

$$x(\tau_4) = -(s - \mu\tau_4)\sin\tau_4 + \cos\tau_4 - 1 \quad (187)$$

$$y(\tau_4) = (s - \mu\tau_4)\cos\tau_4 + \sin\tau_4, \quad -\ell > s > y_{ES}(x = 0) \quad (188)$$

The secondary trajectories emanating from the negative y-axis UL are shown in Figure 34.

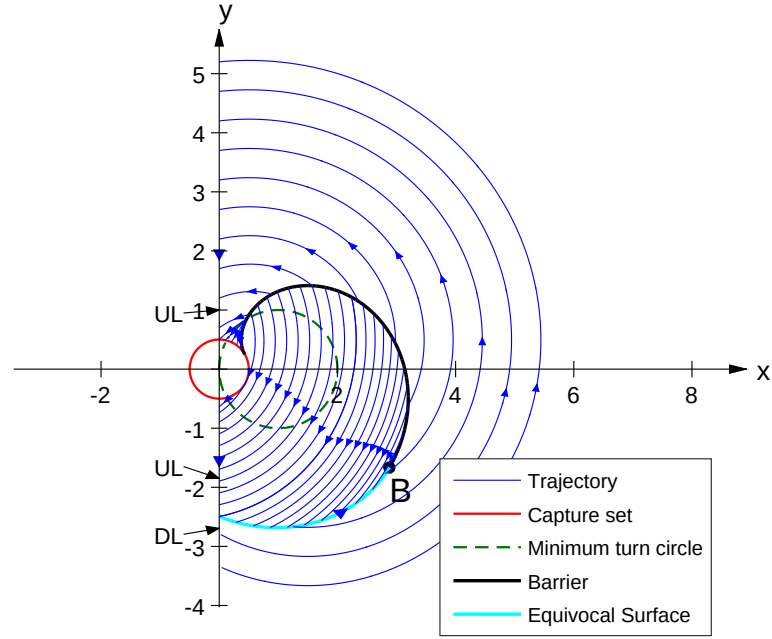


Figure 34. Homicidal Chauffeur Differential Game: Secondary Optimal Trajectories in the Optimal Flow Field, $\ell = 0.5$, $\mu = 0.5$

Removing the tributary trajectories from Figure 34 for clarity, one can clearly see in Figures 35 and 36 that the optimal flow field constructed thus far is the limit to what Isaacs documented in [1]. To extend Isaacs' results, the secondary trajectories in the shaded region of Figure 35 must be constructed.

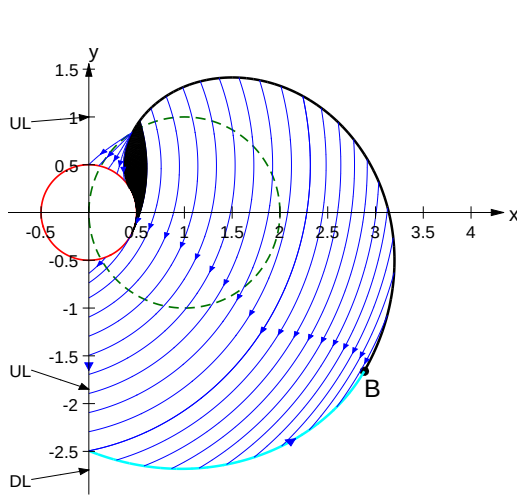


Figure 35. Homicidal Chauffeur Differential Game: Uncertainty Region in the Optimal Flow Field, $\ell = 0.5$, $\mu = 0.5$

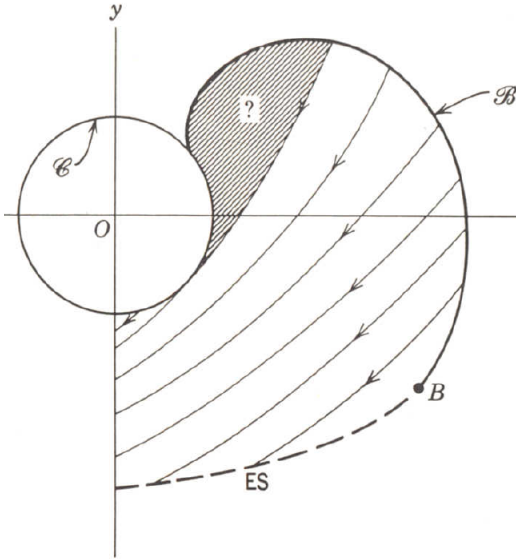


Figure 36. Isaacs' Homicidal Chauffeur Differential Game Solution - Reproduced from Fig. 10.9.4 in [1]

Before constructing the trajectories in this shaded region, it is important to first obtain the critical secondary trajectory that achieves safe contact. Here, safe contact refers to a trajectory coming into contact with the NUP of the capture set. On the NUP of the capture set, the Pursuer is not able to force the Evader's penetration of capture set so the Evader's contact of the NUP of the capture set is considered safe. There is a critical secondary trajectory that emanates from the negative y-axis UL that momentarily achieves safe contact with, but does not terminate on, the NUP of the capture set, as seen in Figures 36 and 37. This critical secondary trajectory effectively delineates the secondary trajectories previously obtained from those in the shaded region.

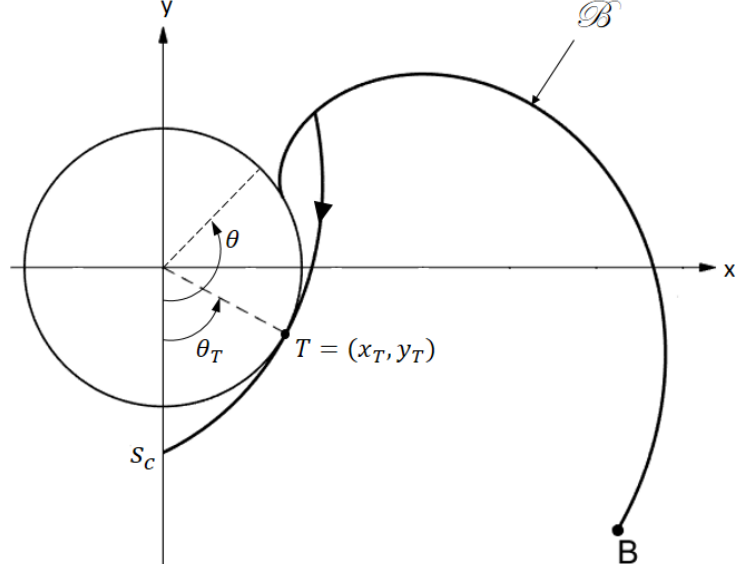


Figure 37. Homicidal Chauffeur Differential Game: Critical Secondary Trajectory Reference

To obtain this critical secondary trajectory, the secondary trajectory emanating from the negative y-axis UL is examined at the location $T = (x_T, y_T)$ where the trajectory contacts the capture circle. Using Equations (187) and (188)

$$x_T = -s_c \cdot \sin\tau_c + \mu\tau_c \sin\tau_c + \cos\tau_c - 1 \quad (189)$$

$$y_T = s_c \cdot \cos\tau_c - \mu\tau_c \cos\tau_c + \sin\tau_c \quad (190)$$

where τ_c is the retrograde time at which the critical secondary trajectory is tangent to the capture circle and s_c is the coordinate on the negative y-axis from which the critical secondary trajectory emanates. With the capture set parameterized by

$$x = \ell \sin\theta, \quad y = -\ell \cos\theta$$

the angle θ_T corresponds to the point (x_T, y_T) on the capture circle that is contacted by the critical secondary trajectory.

To characterize the critical secondary trajectory, it is necessary to determine its emanating point on the y-axis, that is, s_c . This is accomplished by examining the point of the trajectory that contacts the capture circle. Since Equations (189) and (190) describe the critical secondary trajectory at the point it contacts the capture circle, the following holds

$$\ell \sin(\theta_T) = -s_c \cdot \sin\tau_c + \mu\tau_c \sin\tau_c + \cos\tau_c - 1 \quad (191)$$

$$-\ell \cos(\theta_T) = s_c \cdot \cos\tau_c - \mu\tau_c \cos\tau_c + \sin\tau_c, \quad 0 < \theta < \frac{\pi}{2} \quad (192)$$

$\Rightarrow$

$$\tan(\theta_T) = \frac{-\ell \sin(\theta_T)}{-\ell \cos(\theta_T)} = \frac{1 + s_c \cdot \sin\tau_c - \cos\tau_c - \mu\tau_c \sin\tau_c}{s_c \cdot \cos\tau_c - \mu\tau_c \cos\tau_c + \sin\tau_c} \quad (193)$$

Taking the derivative of Equations (187) and (188) with respect to τ_c yields

$$\left. \frac{dx}{d\tau} \right|_{\tau_c} = -s_c \cdot \cos\tau_c + \mu \sin\tau_c + \mu\tau_c \cos\tau_c - \sin\tau_c$$

$$\left. \frac{dy}{d\tau} \right|_{\tau_c} = -s_c \cdot \sin\tau_c + \cos\tau_c - \mu \cos\tau_c + \mu\tau_c \sin\tau_c$$

$\Rightarrow$

$$\left. \frac{dy}{dx} \right|_{\tau_c} = \frac{(1 - \mu)\cos\tau_c + \mu\tau_c \sin\tau_c - s_c \cdot \sin\tau_c}{\mu\tau_c \cos\tau_c - (1 - \mu)\sin\tau_c - s_c \cdot \cos\tau_c} \quad (194)$$

Therefore,

$$\frac{dy}{dx} = \frac{-x_T}{y_T} = \tan(\theta_T)$$

Applying Equations (193) and (194) to the relationship above yields

$$\frac{(1 - \mu)\cos\tau_c + \mu\tau_c \sin\tau_c - s_c \cdot \sin\tau_c}{\mu\tau_c \cos\tau_c - (1 - \mu)\sin\tau_c - s_c \cdot \cos\tau_c} = \frac{1 + s_c \cdot \sin\tau_c - \cos\tau_c - \mu\tau_c \sin\tau_c}{s_c \cdot \cos\tau_c - \mu\tau_c \cos\tau_c + \sin\tau_c} \quad (195)$$

Additionally, on the capture circle $x_T^2 + y_T^2 = \ell^2$. Thus

$$(1 + s_c \cdot \sin\tau_c - \cos\tau_c - \mu\tau_c \sin\tau_c)^2 + (s_c \cdot \cos\tau_c - \mu\tau_c \cos\tau_c + \sin\tau_c)^2 = \ell^2 \quad (196)$$

With Equations (195) and (196), a system of two equations in the two unknowns s_c and τ_c exists. Simplifying Equation (195) yields

$$(1 - \mu)\sin\tau_c - \mu\tau_c \cos\tau_c + \mu^2\tau_c = (\mu - \cos\tau_c)s_c$$

$\Rightarrow$

$$(s_c - \mu\tau_c)(\mu - \cos\tau_c) - (1 - \mu)\sin\tau_c = 0$$

$\Rightarrow$

$$s_c - \mu\tau_c = \frac{(1 - \mu)\sin\tau_c}{\mu - \cos\tau_c} \quad (197)$$

Simplifying Equation (196) yields

$$(s_c - \mu\tau_c)^2 + 2(1 - \cos\tau_c) + 2(s_c - \mu\tau_c)\sin\tau_c = \ell^2 \quad (198)$$

Applying Equation (197) to Equation (198) yields

$$\frac{(1 - \mu)^2 \sin^2\tau_c}{(\mu - \cos\tau_c)^2} + 2(1 - \cos\tau_c) + 2\left(\frac{(1 - \mu)\sin\tau_c}{\mu - \cos\tau_c}\right) = \ell^2$$

$\Rightarrow$

$$\frac{(1 - \mu)^2 \sin^2\tau_c}{(\mu - \cos\tau_c)^2} + 2\left(\frac{(1 - \mu)\sin\tau_c}{\mu - \cos\tau_c}\right) = \ell^2 + 2\cos\tau_c - 2$$

$\Rightarrow$

$$\left[\frac{(1 - \mu)\sin\tau_c}{\mu - \cos\tau_c} + \sin\tau_c\right]^2 = \ell^2 + 2\cos\tau_c - 2 + \sin^2\tau_c$$

$\Rightarrow$

$$\left[\frac{(1-\mu)}{\mu - \cos\tau_c} + 1 \right]^2 \sin\tau_c^2 = \ell^2 - (1 - \cos\tau_c)^2$$

This solution must satisfy $\cos\tau_c > 1 - \ell(> 0)$ and $0 < \tau_c < \frac{\pi}{2}$. Obtaining the equation in terms of $\cos\tau$

$$\left[\frac{(1-\mu)}{\mu - \cos\tau_c} + \frac{\mu - \cos\tau_c}{\mu - \cos\tau_c} \right]^2 = \frac{\ell^2 - (1 - \cos\tau_c)^2}{\sin\tau_c^2}$$

$\Rightarrow$

$$\frac{(1 - \cos\tau_c)^2}{(\mu - \cos\tau_c)^2} = \frac{\ell^2 - (1 - \cos\tau_c)^2}{\sin\tau_c^2} \quad (199)$$

Simplifying Equation (199) yields the cubic equation in $\cos\tau_c$

$$2\mu\cos\tau_c^3 + (\ell^2 - \mu^2 - 4\mu - 1)\cos\tau_c^2 + 2(\mu^2 - \ell^2\mu + \mu + 1)\cos\tau_c + \ell^2\mu^2 - \mu^2 - 1 = 0 \quad (200)$$

After obtaining the solution to the cubic equation above, it is possible to obtain the retrograde time τ_c at which the critical secondary trajectory contacts the capture set. Having calculated τ_c , the critical angle θ_T can be obtained. With Equation (193) expressed as

$$\tan(\theta_T) = \frac{1 + (s_c - \mu\tau_c)\sin\tau_c - \cos\tau_c}{(s_c - \mu\tau_c)\cos\tau_c + \sin\tau_c}$$

applying Equation (197) and simplifying yields

$$\tan(\theta_T) = \frac{1 - \mu\cos\tau_c}{\mu\sin\tau_c} \quad (201)$$

From the above equation, the critical angle θ_T can be obtained. With θ_T and τ_c known, Equation (191) or (192) can be evaluated for s_c which yields the emanating

point on the negative y-axis of the critical secondary trajectory:

$$s_c = \frac{\cos\tau_c - 1 - \ell\sin(\theta_T)}{\sin\tau_c} + \mu\tau_c = \frac{-\ell\cos(\theta_T) - \sin\tau_c}{\cos\tau_c} + \mu\tau_c$$

From Equation (150), the angle from the origin to the end point of the NUP of the capture set in the RHP can be expressed as

$$\tan\theta = \frac{\sqrt{1-\mu^2}}{\mu}$$

If the end point of the NUP of the capture set, in the RHP where $\theta = A\tan\frac{\sqrt{1-\mu^2}}{\mu} = \varphi_{max}$, can ever correspond to the point (x_T, y_T) where the critical secondary trajectory contacts the capture set, from Equation (201), the following would hold

$$\frac{\sqrt{1-\mu^2}}{\mu} = \frac{1 - \mu\cos\tau}{\mu\sin\tau}$$

The above condition is true for $\cos\tau = \mu$. The value $\cos\tau = \mu$ satisfies Equation (200) if

$$\mu^4 - 2\mu^3 + 2\mu^2 - 1 = 0$$

This only holds true, however, for $\mu = 1$. Therefore, in the *classical* parameter range of the Homicidal Chauffeur differential game, the end point of the NUP of the target set, where $\theta = \varphi_{max}$, will never correspond to the point where the critical secondary trajectory contacts the capture circle. In the *classical* parameter range $\theta_T > \varphi_{max}$.

With the critical secondary trajectory obtained, it is now possible to synthesize the optimal trajectories in the shaded region of Figure 35. The trajectories in this region emanate from a point on the NUP of the capture set and terminate on the barrier. In the realistic plane, this equates to the Evader moving on a straight line toward the NUP of the circular capture set, then actively enforcing safe contact for

as long as possible. Once the Evader can no longer maintain safe contact, the state proceeds to the negative y-axis along the critical secondary trajectory. To obtain the trajectories in this region, it is first necessary to characterize the safe contact region.

To simplify the process of obtaining these trajectories, it is convenient to use polar coordinates as seen in Figure 38.

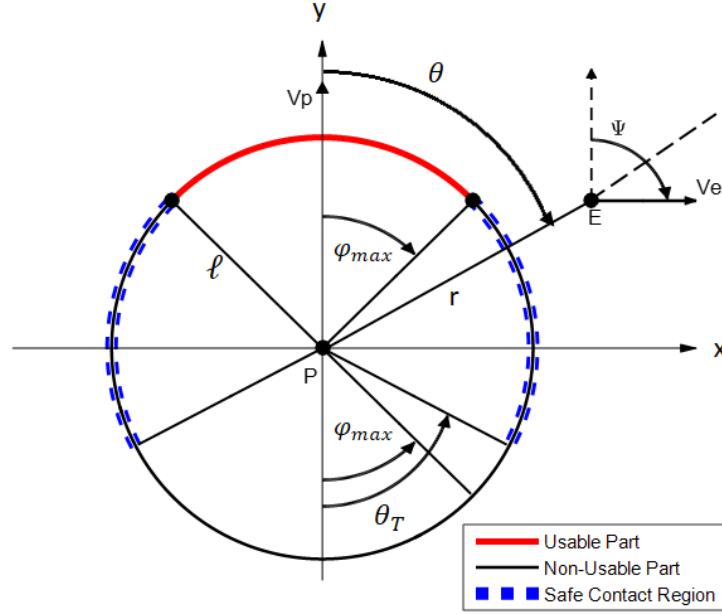


Figure 38. Homicidal Chauffeur Differential Game: Polar Coordinate Reference Frame

In this reference frame, r denotes the distance from the Pursuer to the Evader and θ denotes the bearing of the Evader during safe contact $r(\tau) = \ell$. The outer limit of the safe contact region, the BUP, corresponds to the angle $\varphi_{max} = A \cos \mu$. The NUP, in the RHP, corresponds to the bearings $\varphi_{max} < \theta < \pi$.

It was shown above that $\theta_T > \varphi_{max}$, therefore, the safe contact region corresponds to $\varphi_{max} < \theta < \pi - \theta_T$. Once the Evader establishes safe contact, it is optimal for the Evader to maintain safe contact for as long as possible before being forced by the Pursuer into the region of the secondary trajectories emanating from the negative y-axis UL. When the safe contact trajectories reach the bearing $\theta = \pi - \theta_T$, the

Evader can no longer maintain contact with the NUP of the capture set and continues along the critical secondary trajectory to the negative y-axis. The critical secondary trajectory acts as a "turnpike" for the secondary trajectories exiting the safe contact region at (x_T, y_T) which continue along that trajectory.

Note that the angle θ used in the following derivation is measured from the positive y-axis where, conversely, the critical angle θ_T of the critical secondary trajectory was obtained from a reference frame where the angle was measured from the negative y-axis as seen in Figure 37.

To convert the dynamics of the differential game to polar coordinates (r, θ) , the Cartesian coordinates (x, y) are set to

$$x = r \sin \theta, \quad y = r \cos \theta \quad (202)$$

Taking the derivative of the above equations yields

$$\dot{x} = \dot{r} \sin \theta + r \cos \theta \cdot \dot{\theta}$$

$$\dot{y} = \dot{r} \cos \theta - r \sin \theta \cdot \dot{\theta}$$

$\Rightarrow$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} \sin \theta & r \cos \theta \\ \cos \theta & -r \sin \theta \end{bmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix}$$

$\Rightarrow$

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = -\frac{1}{r} \begin{bmatrix} -r \sin \theta & -r \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \quad (203)$$

Recalling the differential game dynamics

$$\dot{x} = -yu + \mu \sin \Psi, \quad x(0) = x_0$$

$$\dot{y} = xu - 1 + \mu \cos \Psi, \quad y(0) = y_0$$

and applying the polar coordinate definitions in Equation (202) yields

$$\dot{x} = \mu \sin \Psi - u r \cos \theta$$

$$\dot{y} = \mu \cos \Psi + u r \sin \theta - 1$$

Applying the above equations to Equation (203) yields

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = -\frac{1}{r} \begin{bmatrix} -r \sin \theta & -r \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix} \begin{pmatrix} \mu \sin \Psi - u r \cos \theta \\ \mu \cos \Psi + u r \sin \theta - 1 \end{pmatrix}$$

$\Rightarrow$

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \mu \cos(\Psi - \theta) - \cos \theta \\ -u + \frac{1}{r} [\sin \theta + \mu \sin(\Psi - \theta)] \end{pmatrix}$$

Thus, the dynamics of the Homicidal Chauffeur differential game expressed in polar coordinates are

$$\dot{r} = \mu \cos(\Psi - \theta) - \cos \theta, \quad r(0) = r_0 \tag{204}$$

$$\dot{\theta} = \frac{1}{r} [\mu \sin(\Psi - \theta) + \sin \theta] - u, \quad \theta(0) = \theta_0 \tag{205}$$

$$\bar{\theta} = A \cos \mu$$

Once the Evader achieves safe contact, in order to enforce it ($\dot{r} \equiv 0$), while the Pursuer is turning away, the Evader employs the state feedback strategy.

$$\Psi(\ell, \theta) = \theta + A \sin \left(\frac{1}{\mu} \cos \theta \right) - \frac{\pi}{2} \tag{206}$$

As a proof that this is indeed the Evader's control that maintains safe contact, the

following is evaluated

$$\cos(\Psi(\ell, \theta) - \theta) = \frac{1}{\mu} \cos \theta \quad (207)$$

Applying this to Equation (204) yields

$$\dot{r} = 0$$

Therefore, Equation (206) is the Evader's optimal control to which enforces safe contact with the Pursuer's capture set. Next, the following is calculated

$$\sin(\Psi(\ell, \theta) - \theta) = -\cos \left(A \sin \left(\frac{1}{\mu} \cos \theta \right) \right)$$

$\Rightarrow$

$$\sin(\Psi(\ell, \theta) - \theta) = -\sqrt{1 - \frac{1}{\mu^2} \cos^2 \theta} \quad (208)$$

Thus, Equation (205) becomes

$$\dot{\theta} = \frac{1}{r} \left(\sin \theta - \sqrt{\mu^2 - \cos^2 \theta} \right) - u$$

During safe contact the distance between the Pursuer and Evader is ℓ and, when the game evolves in the RHP, the Pursuer's optimal control $u^* = -1$. Thus

$$\dot{\theta} = \frac{1}{\ell} \left(\sin \theta - \sqrt{\mu^2 - \cos^2 \theta} \right) + 1, \quad \varphi_{max} < \theta < \pi - \theta_T$$

Note that $\dot{\theta} > 0$ because $\sin \theta > \sqrt{\mu^2 - \cos^2 \theta} \quad \forall \quad \theta$. With this, it is obvious that during safe contact, the Evader moves towards the rear of the Pursuer until the bearing $\pi - \theta_T$ is reached, whereupon the Evader's safe region contact is broken off and the state proceeds along the critical secondary trajectory toward the negative y-axis.

To obtain expressions for the trajectories emanating from the safe contact region, for any retrograde time τ_5 , it is necessary to characterize the Evader's optimal control along the trajectories. Here τ_5 denotes the retrograde time measured from the time the secondary trajectory emanates from the safe contact region. Utilizing Equations (207) and (208), the Evader's optimal control at time $\tau_5 = 0$ is calculated:

$$\sin(\Psi(\ell, \theta)) = \sin(\Psi(\ell, \theta) - \theta + \theta)$$

$\Rightarrow$

$$\sin(\Psi(\ell, \theta)) = \sin(\Psi(\ell, \theta) - \theta)\cos\theta + \cos(\Psi(\ell, \theta) - \theta)\sin\theta$$

$\Rightarrow$

$$\sin(\Psi(\ell, \theta)) = \frac{1}{\mu}\cos\theta \left(\sin\theta - \sqrt{\mu^2 - \cos^2\theta} \right) \quad (209)$$

Likewise,

$$\cos(\Psi(\ell, \theta)) = \cos(\Psi(\ell, \theta) - \theta + \theta)$$

$\Rightarrow$

$$\cos(\Psi(\ell, \theta)) = \cos(\Psi(\ell, \theta) - \theta)\cos\theta - \sin(\Psi(\ell, \theta) - \theta)\sin\theta$$

$\Rightarrow$

$$\cos(\Psi(\ell, \theta)) = \frac{1}{\mu} - \frac{1}{\mu}\sin\theta \left(\sin\theta - \sqrt{\mu^2 - \cos^2\theta} \right) \quad (210)$$

Right before the secondary trajectories contact the safe contact region, Equation (135) holds, that is

$$\sin\Psi^* = -\frac{\lambda_x}{\sqrt{\lambda_x^2 + \lambda_y^2}}, \quad \cos\Psi^* = -\frac{\lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}}$$

Hence, from Equations (209) and (210), since the trajectory that ends in contact is

tangential to the capture circle,

$$\lambda_x(\tau_5 = 0; \theta) = a \cdot \cos\theta \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \quad (211)$$

$$\lambda_y(\tau_5 = 0; \theta) = -a \left[1 + \sin\theta \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \right], \quad a > 0 \quad (212)$$

Using these transversality conditions, the relationships in Equations (175) and (176) the following harmonic equations are solved

$$\begin{aligned} \ddot{\lambda}_x + \lambda_x &= 0, & \lambda_y(\tau_5 = 0; \theta) &= -a \left[1 + \sin\theta \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \right] \\ & & \dot{\lambda}_y(\tau_5 = 0; \theta) &= a \cdot \cos\theta \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \end{aligned}$$

$$\begin{aligned} \ddot{\lambda}_y + \lambda_y &= 0, & \lambda_x(\tau_5 = 0; \theta) &= -a \left[1 + \sin\theta \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \right] \\ & & \dot{\lambda}_x(\tau_5 = 0; \theta) &= a \cdot \cos\theta \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \end{aligned}$$

With the transversality conditions in Equations (211) and (212), the solutions to the harmonic equations yield the costates

$$\lambda_x(\tau_5; \theta) = \sin\tau + \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \cos(\tau_5 - \theta) \quad (213)$$

$$\lambda_y(\tau_5; \theta) = -\cos\tau + \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \sin(\tau_5 - \theta) \quad (214)$$

To obtain an expression for the Evader's optimal control along the secondary trajectories emanating from the safe contact region, the costates in Equations (213) and (214) are applied to Equation (135). It can be shown that

$$\lambda_x^2(\tau_5, \theta) + \lambda_y^2(\tau_5, \theta) = \mu^2$$

Therefore, the expressions for the Evader's optimal control along the secondary tra-

jectories emanating from the safe contact region are

$$\sin(\Psi^*(\tau_5, \theta)) = -\frac{1}{\mu} \left[\sin\tau_5 + \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \cos(\tau_5 - \theta) \right]$$

$$\cos(\Psi^*(\tau_5, \theta)) = -\frac{1}{\mu} \left[-\cos\tau_5 + \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \sin(\tau_5 - \theta) \right]$$

Using the expressions for the Evader's optimal control, when the Evader is initially in the RHP, the dynamics in Equations (179) and (180) for the secondary optimal trajectories emanating from the safe contact region are

$$\dot{x} = -y + \sin\tau_5 + \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \cos(\tau_5 - \theta), \quad x(\tau_5 = 0) = \ell \sin\theta \quad (215)$$

$$\dot{y} = x + 1 - \cos\tau_5 + \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \sin(\tau_5 - \theta), \quad y(\tau_5 = 0) = \ell \cos\theta \quad (216)$$

differentiating Equation (215) yields

$$\ddot{x} = -\dot{y} + \cos\tau_5 - \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \sin(\tau_5 - \theta)$$

which yields the on-homogeneous harmonic equation

$$\ddot{x} + x = -1 + 2\cos\tau_5 - 2 \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta \right) \sin(\tau_5 - \theta)$$

which is subject to the initial conditions

$$x(\tau_5 = 0) = \ell \sin\theta, \quad \dot{x}(\tau_5 = 0) = \left(\sqrt{\mu^2 - \cos^2\theta} - \sin\theta - \ell \right) \cos\theta$$

Solving the above non-homogeneous harmonic equation and applying a similar derivation to Equation (216) yields expressions for the Evader's position relative to the Pursuer in the region of the secondary trajectories emanating from the safe contact

region

$$x(\tau_5, \theta) = -\ell \sin(\tau_5 - \theta) + \tau_5 \sin \tau_5 + \cos \tau_5 + \left(\sqrt{\mu^2 - \cos^2 \theta} - \sin \theta \right) \cdot \tau_5 \cdot \cos(\tau_5 - \theta) - 1$$

$$y(\tau_5, \theta) = \ell \cos(\tau_5 - \theta) - \tau_5 \cos \tau_5 + \sin \tau_5 + \left(\sqrt{\mu^2 - \cos^2 \theta} - \sin \theta \right) \cdot \tau_5 \cdot \sin(\tau_5 - \theta)$$

The secondary trajectories emanating from the safe contact region are shown in Figure 39.

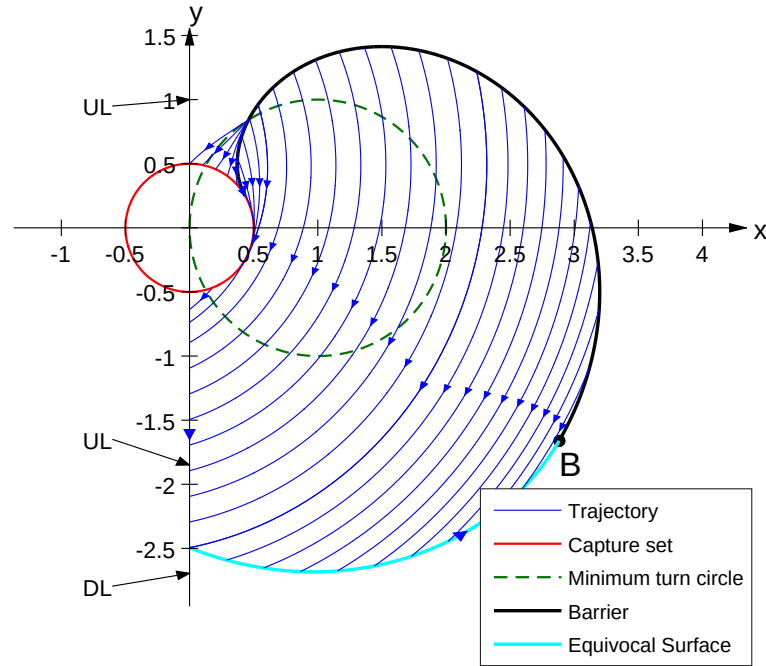


Figure 39. Homicidal Chauffeur Differential Game: Secondary Optimal Trajectories in the Optimal Flow Field, $\ell = 0.5$, $\mu = 0.5$

During safe contact, the bearing of the Evader is subject to

$$\dot{\theta} = \frac{1}{\ell} \left(\sin \theta - \sqrt{\mu^2 - \cos^2 \theta} \right) + 1, \quad \theta(0) = \theta_0 (> A \cos \mu)$$

$\Rightarrow$

$$\frac{dt}{d\theta} = \frac{\ell}{\ell + \sin \theta - \sqrt{\mu^2 - \cos^2 \theta}}$$

$\Rightarrow$

$$t(\theta) = \int_{\theta_0}^{\theta} \frac{\ell}{\ell + \sin\theta - \sqrt{\mu^2 - \cos^2\theta}} d\theta$$

With this, the complete optimal control solution to the *classical* Homicidal Chauffeur differential game has been derived. Each component of the optimal flow field has been constructed and can be seen in Figure 40, where the complete solution of the *classical* Homicidal Chauffeur differential game for the parameters $\ell = 0.5$, $\mu = 0.5$ is shown.

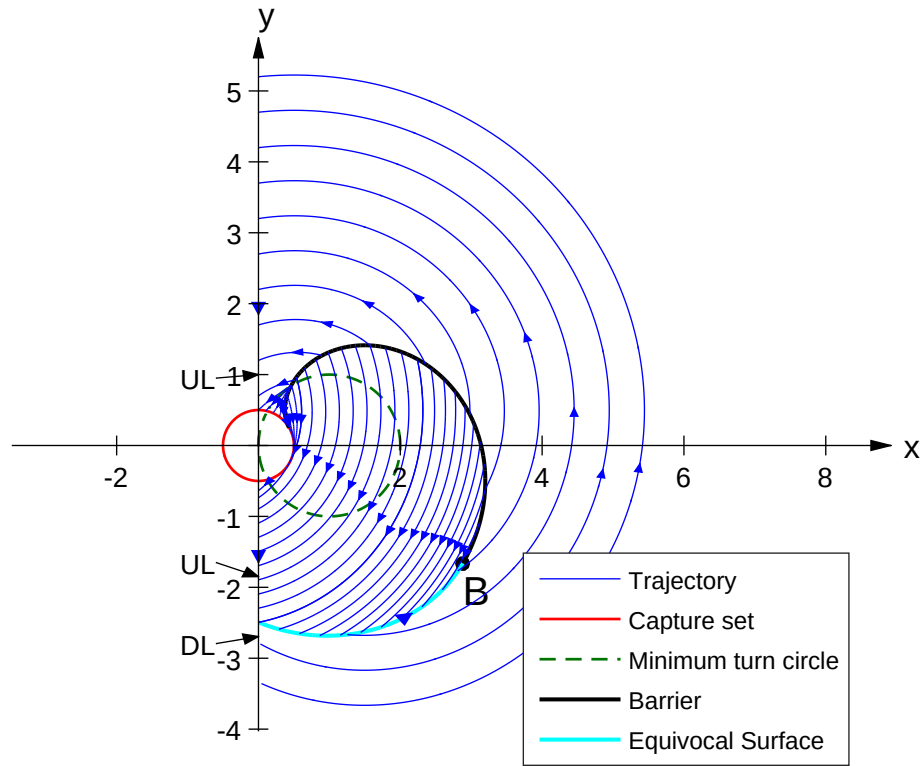


Figure 40. Complete Optimal Flow Field for the *Classical* Homicidal Chauffeur Differential Game, $\ell = 0.5$, $\mu = 0.5$

In Figure 41, in order to contextualize this solution, the motion of the players in the realistic plane are related to the full optimal flow field depicted in Figure 40. The motions of the players in the realistic plane can be determined by examining their

optimal control strategies that were obtained for each component of the optimal flow field.

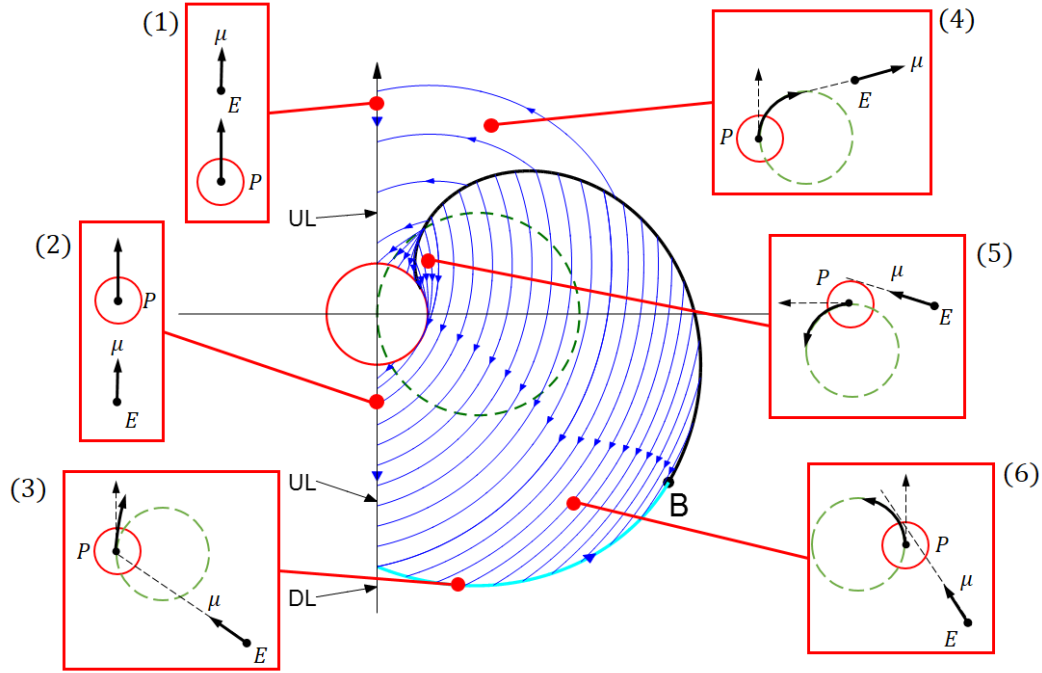


Figure 41. Homicidal Chauffeur Differential Game: Pursuer-Evader Motion in the Realistic Plane

1. On the positive y-axis UL: the Pursuer and Evader play pure LOS pursuit-evasion until the Evader penetrates the capture circle at $(0, \ell)$.
2. On the negative y-axis UL: the Pursuer and Evader play pure LOS pursuit-evasion, with the Pursuer running from the Evader. The Pursuer continues running from the Evader until the Evader contacts the ES at $(0, y_{ES})$. Upon the Evader's contact of the ES, the Pursuer plays the optimal strategy $u^* = u_{ES}$.
3. On the ES: The Evader has two optimal strategies on this surface. If the Evader chooses to leave the ES: the Evader applies the tributary trajectory region strategy of running along the right tangent to the Pursuer's turn circle. If the Evader chooses to traverse the ES: the Evader points towards the Pursuer. For

the Evader’s traversal strategy, the Pursuer turns toward the Evader. Contrary to the Pursuer’s strategy everywhere else on the optimal flow field: if the Evader is on the ES, the Pursuer does not turn as hard as possible.

4. In the region of the primary and tributary trajectories: The Pursuer turns toward the Evader until the Evader contacts either the UP of the capture set or the positive y-axis UL. The Evader, changes its heading angle based on its relative position to the Pursuer. To delay capture, the Evader runs away from the Pursuer along the tangent to the Pursuer’s turn circle.
5. In the region of the secondary trajectories that contact the safe contact region: The Pursuer turns away from the Evader until the Evader traverses the critical secondary trajectory and contacts the negative y-axis UL. The Evader runs along the right tangent to the Pursuer’s capture circle towards the Pursuer.
6. In the region of the secondary trajectories: For the remaining secondary trajectory variants derived in Section 4.2.6, the Pursuer’s and Evader’s strategies do not change. The Pursuer turns away from the Evader until the Evader contacts the negative y-axis UL or the ES. The Evader runs along the tangent to the Pursuer’s turn circle. By pointing toward the Pursuer, the Evader increases the time it takes to enter a region where the Pursuer’s optimal control changes. These strategies apply to the barrier as well since it is a locus of turn away points for the Pursuer.

4.3 Discussion

In this section, discussions inspired by the *classical* Homicidal Chauffeur differential game are presented. The topics discussed in this section include: the correction of a popular misconception concerning the ES, methods from Section 3.5.6 for elim-

inating the Evader's use of the ES are applied to the *classical* Homicidal Chauffeur differential game, the connection of the *classical* Homicidal Chauffeur differential game's solution to its Dubins car optimal control analogue solution in Section 3.2 is evidenced, and finally the *classical* Homicidal Chauffeur differential game solution is connected to a modern application.

4.3.1 Equivocal Surface Misconception

In Merz' work, it is stated that along the ES, the switch function vanishes, that is $S(\tau) \equiv 0$. However, this statement is incorrect. One could have erroneously arrived at this conclusion using the following flawed logic: when obtaining the optimal trajectories of the Homicidal Chauffeur differential game, the Evader's optimal control which is characterized by Equation (135), and which also applies to the secondary optimal trajectories, is subject to

$$\tan \Psi^* = \frac{\lambda_x}{\lambda_y}$$

On the ES, the Evader's optimal control law is given by Equation (166), hence

$$\tan \Psi_{ES}^* = \frac{x}{y}$$

Because Lemma 10.5.2 guarantees that $\Psi^* \rightarrow \Psi_{ES}^*$, when the Evader approaches the ES from the secondary optimal trajectories where the costates are λ_x and λ_y , one is tempted to conclude that $\frac{\lambda_x}{\lambda_y} = \frac{x}{y}$. Applying this to the switching function

$$S(\tau) = x(\tau)\lambda_y(\tau) - y(\tau)\lambda_x(\tau)$$

yields

$$S(\tau) = \frac{\lambda_x(\tau)}{\lambda_y(\tau)} y(\tau) \lambda_y(\tau) - y(\tau) \lambda_x(\tau) \equiv 0$$

However, on the ES proper, the costates of the ES are the costates of the tributary trajectories. Hence, Merz' argument is incorrect and can be disproved by examining the switch function along the tributary trajectories

$$S(\tau_1) = \frac{1}{1 - \mu} (1 - \cos \tau_1)$$

According to Merz, $S(\tau) \equiv 0$ should hold along the ES. Examining the switch function above, however, reveals that the switch function cannot meet that condition along the ES. The retrograde time that causes the switch function to vanish is $\tau_1 = 2\pi$. At $\tau_1 = 2\pi$ the tributary trajectories will have crossed the ES, as is evident from a comparison of Figures 31 and 32. Therefore, since the tributary trajectories naturally terminate on the negative y-axis, the introduction of the ES invalidates the portion of the optimal tributary trajectories beyond the ES. Unlike in optimal control, the costates are not continuous across the ES. The ES is a line of discontinuity of the costates. The costates $\lambda_{secondary} - \lambda_{tributary} = \alpha \vec{n}$ where $\vec{n}$ is the normal to the ES pointing into the domain of the secondary optimal trajectories and $\alpha < 0$. The costates are continuous, however, while approaching the ES from the tributary trajectories. The tributary costates are the costates along the ES and then upon crossing the ES into the domain of the secondary trajectories there is a jump in the costate values. Therefore, the Value function in the Homicidal Chauffeur differential game is not C^1 on the ES; the Value function has a corner on the ES.

4.3.2 Equivocal Surface Avoidance

There is a belief that, even though the barrier is open, there is a strategy available to the Evader that would allow him to avoid capture. Isaacs stated that, under optimal play by both players, if the Evader were to traverse the ES to the barrier end point B, the optimal control of the Pursuer $u^* = 1$ would then allow the Evader, whose heading Ψ is normal to the barrier, to traverse the barrier until it achieved safe contact at the BUP. Achieving safe contact would then position the Evader in the region of the secondary trajectories where it would once again end up on the ES to repeat the barrier strategy and achieve safe contact at the BUP once more, effectively forcing a draw. To avoid an on-going game, Isaacs suggested a strategy for the Pursuer that would prevent the Evader from utilizing the ES to travel to the point B, thus guaranteeing the capture of the Evader.

Merz correctly points out that the Evader is unable to travel along the barrier, because when the Evader is on the barrier, the Pursuer plays $u^* = -1$ which pulls the state under the barrier into the domain of the secondary optimal trajectories. Nevertheless, as discussed in Section 3.2.3, the ES adds a degree of difficulty to achieving an optimal control solution. Therefore, Isaacs' suggestion of synthesizing a strategy for the Pursuer that prevents the Evader from exploiting the ES is still desirable. In Section 3.5.6, the following three methods of precluding the Evader's ES traversal strategy were analyzed.

1. Capture Set Synthesis: The Pursuer can guarantee capture of the Evader by synthesizing its optimal strategy as if the capture set were slightly smaller than it is in reality. The Pursuer can then allow the Evader to traverse the smaller synthesized barrier by playing $u^* = 1$. This would result in the Evader contacting the real UP of the capture set.

2. Isaacs' Approach: For an Evader initially located in the region of the secondary trajectories, the Pursuer plays its optimal control $u^* = -1$ until the Evader reaches the ES. The Pursuer then continues playing $u^* = -1$ for a short interval afterwards, in order to force the Evader into the region of the tributary trajectories. If the Evader is positioned on the negative y-axis UL, the Pursuer simply plays LOS evasion, running away from the Evader until the Evader crosses the ES at $(0, y_{ES})$ and contacts the DL.
3. ES traversal: For an Evader allowed to traverse the ES until it reaches the barrier end point B: once the Evader contacts B, the Pursuer plays $u^* = -1$ for a short interval in an attempt to force the Evader into the region of the tributary trajectories.

All three of these strategies entail the Pursuer incurring a small cost in optimality in order to capture of the Evader as well as reduce the complexity of obtaining an optimal flow field by negating the Evader's ES traversal strategy.

4.3.2.1 Method 1: Capture Set Synthesis

In this section, the Pursuer's ability to capture the Evader, by synthesizing its optimal control strategy as if the capture set were slightly smaller than it is in reality, is examined. By synthesizing a smaller capture set of radius $\ell - \epsilon$, $0 < \epsilon \ll 1$, an optimal flow field that corresponds to the reduced capture set radius is created. The fact that the Evader believes that the radius of the capture circle is ℓ does not affect its optimal control. If the Pursuer were to allow the Evader to travel along the synthesized barrier, the Evader would penetrate the UP of the true capture set by trying to contact the BUP of the synthesized barrier. The viability of this method is evidenced by examining an optimal flow field that utilizes a synthesized capture set.

As seen in Figure 42, if the Evader were to travel along the synthesized barrier, the Evader would contact the UP of the true capture set.

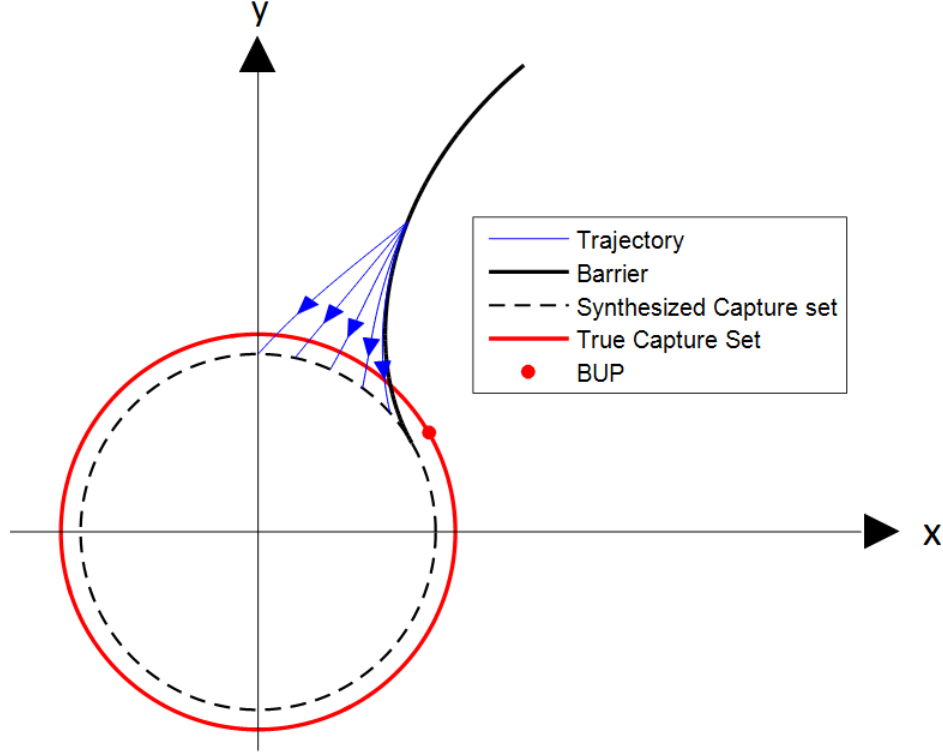


Figure 42. Application of the Capture Set Synthesis Method to the Homicidal Chauffeur Differential Game

Figure 42 depicts a synthesized capture set of radius $\ell = 0.45$ and a true capture set of radius $\ell = 0.5$. These values were chosen in order to clearly show the validity of this method of capture. This method is valid for any synthesized capture radius (ℓ_s) within the range $0 < \ell_s < \ell$. However, to mitigate the sacrifice of optimality, the Pursuer would logically choose a synthesized capture radius very close to the true capture radius.

4.3.2.2 Method 2: Isaacs' approach

In this section, the Pursuer's ability to push the Evader off of the ES is examined. Recall from Section 4.2.5, the normal along the ES pointing into the region of the

tributary trajectories can be characterized as

$$\vec{n}(\tau_2) = \begin{pmatrix} x_{ES}u_{ES} - 1 - \mu\frac{y_{ES}}{r_{ES}} \\ y_{ES}u_{ES} + \mu\frac{x_{ES}}{r_{ES}} \end{pmatrix}$$

Since the Pursuer is pushing the Evader from the ES, into the region of the tributary trajectories, the forward time dynamics are examined. Recalling the forward time dynamics from Equations (131) and (132)

$$\vec{f} = \begin{pmatrix} -yu + \mu\sin\Psi \\ xu - 1 + \mu\cos\Psi \end{pmatrix}$$

To push the Evader off of the ES, the Pursuer turns away from the Evader by applying the optimal control $u^* = -1$. When the Evader applies the ES traversal strategy, from Equation (166), the dynamics above become

$$\vec{f} = \begin{pmatrix} y_{ES} - \mu\frac{x_{ES}}{r_{ES}} \\ -x_{ES} - 1 - \mu\frac{y_{ES}}{r_{ES}} \end{pmatrix}$$

For the Pursuer to push the Evader off of the ES, into the region of the tributary trajectories, the following must hold true

$$\vec{n} \cdot \vec{f} > 0 \quad \forall \quad 0 \leq \tau_2 \leq \tau_{2max}$$

A numerical analysis of the equations above reveals that, for any position (x_{ES}, y_{ES}) on the ES, $\vec{n} \cdot \vec{f} > 0$, thereby verifying that when the Evader applies the traversal strategy along the ES, the Pursuer can push the Evader from the ES at any time by turning away from the Evader with $u^* = -1$.

4.3.2.3 Method 3: ES traversal

In this section, the Pursuer's ability to push the Evader off of the barrier end point B is examined for two cases: the Evader employing the traversal strategy and the barrier strategy. The first case examined is the Evader applying the traversal strategy at B. The normal at B, pointing into the region of the tributary trajectories, is

$$\vec{n} = (\sqrt{1 - \mu^2}, -\mu)$$

Since the Pursuer is pushing the Evader from B, into the region of the tributary trajectories, the forward time dynamics are examined

$$\vec{f} = \begin{pmatrix} -yu + \mu \sin \Psi \\ xu - 1 + \mu \cos \Psi \end{pmatrix}$$

To push the Evader off of B, the Pursuer turns away from the Evader by applying the optimal control $u^* = -1$. When the Evader applies the traversal strategy, from Equation (166), the dynamics above become

$$\vec{f} = \begin{pmatrix} y_B - \mu \frac{x_B}{\sqrt{x_B^2 + y_B^2}} \\ -x_B - 1 - \mu \frac{y_B}{\sqrt{x_B^2 + y_B^2}} \end{pmatrix}$$

where (x_B, y_B) is the Evader's position on the barrier end point from Equations (151) and (152). For the Pursuer to push the Evader off of B, into the region of the tributary trajectories, the following must hold true

$$\vec{n} \cdot \vec{f} > 0$$

A numerical analysis of the equations above reveals that, at the barrier end point (x_B, y_B) , $\vec{n} \cdot \vec{f} = 0$. Therefore, the Pursuer is unable to push the Evader off of B if the Evader applies the traversal strategy.

Next, the ability of the Pursuer to force the Evader off of B is examined for the case of the Evader applying the barrier strategy. As before, the normal at B, pointing into the region of the tributary trajectories, is

$$\vec{n} = (\sqrt{1 - \mu^2}, -\mu)$$

Since the Pursuer is pushing the Evader off of B, into the region of the tributary trajectories, the forward time dynamics are examined

$$\vec{f} = \begin{pmatrix} -yu + \mu \sin \Psi \\ xu - 1 + \mu \cos \Psi \end{pmatrix}$$

To push the Evader off of B, the Pursuer turns away from the Evader by applying the optimal control $u^* = -1$. When the Evader applies the barrier strategy, the Evader heads in a direction normal to B and into the domain of the secondary trajectories. At B, when the Evader applies the barrier strategy,

$$\sin \Psi_B = -\sqrt{1 - \mu^2}, \quad \cos \Psi_B = \mu$$

Applying the Pursuer's and Evader's controls to the dynamics above yields,

$$\vec{f} \Big|_B = \begin{pmatrix} y_B - \mu \sqrt{1 - \mu^2} \\ -x_B - 1 + \mu^2 \end{pmatrix}$$

For the Pursuer to push the Evader off of B, into the region of the tributary trajec-

tories, the following must hold true

$$\vec{n} \cdot \vec{f} \Big|_B > 0$$

A numerical analysis of the equations above reveals that, at the barrier end point (x_B, y_B) , $\vec{n} \cdot \vec{f} = 0$. Therefore, the Pursuer is unable to push the Evader off of the barrier end point if the Evader applies the barrier strategy.

Surprisingly, this does not mirror results seen in the analysis of Isaacs' Exercise 10.6.1 in Section 3.5.6, in which the Pursuer was able to force the Evader off of B. In the *classical* Homicidal Chauffeur differential game, if the Pursuer allows the Evader to traverse the ES and contact B, the Pursuer cannot achieve capture by forcing the Evader off of B. Therefore, the Pursuer must not allow the Evader to reach B. The Pursuer can prevent the Evader from reaching B by forcing the Evader off of the ES using Method 2 discussed above.

In the event that the Evader is allowed to reach B, the Evader can choose to enter the region of the tributary trajectories. This is presented in greater detail in Appendix D.

4.3.3 Dubins Car Optimal Control Problem Analogue

In Section 3.2, the optimal control analogue to the *classical* Homicidal Chauffeur differential game was examined. In this optimal control problem, the mobile Evader of the Homicidal Chauffeur differential game was replaced with a stationary target point. The solution of this optimal control problem is, in effect, the solution to the *classical* Homicidal Chauffeur differential game for the case of the speed ratio $\mu = 0$.

In the Dubins car optimal control problem, the ES present in the Homicidal Chauffeur differential game was replaced with an SL. The SL present in the optimal control problem is *not* equivalent to the expression of the ES obtained by setting the speed

ratio $\mu = 0$. Although these two surfaces are not equivalent, Merz showed that for certain small values of μ , the ES can be partially or entirely replaced by a SL. This validates the results obtained in Section 3.2 and also reveals another conclusion of the *classical* Homicidal Chauffeur differential game: for values $0 < \mu \ll 1$, the solution to the *classical* Homicidal Chauffeur differential game is qualitatively the same as the solution to the Dubins car optimal control analogue of the differential game examined in Section 3.2: There is no ES and its place is taken by an SL, as in the Dubins car optimal control problem. It can be shown that, for certain small values of μ , the shaded region of Figure 35 in Isaacs' original solution vanishes and the SL (which replaces the ES) contacts the NUP of the capture set removing the trajectories that emanate from the negative y-axis UL from the optimal flow field. The effect of a small μ in the Homicidal Chauffeur differential game makes the solution qualitatively equal to the solution of the Dubins car optimal control problem as seen in Figures 43 and 44.

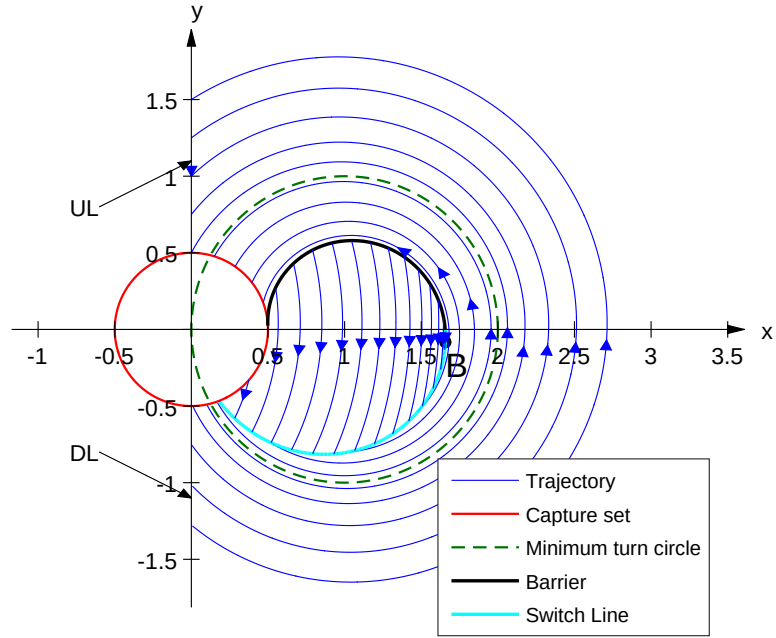


Figure 43. *Non-Classical Homicidal Chauffeur Differential Game: Optimal Flow Field,* $\ell = 0.5, \mu = 0.05$

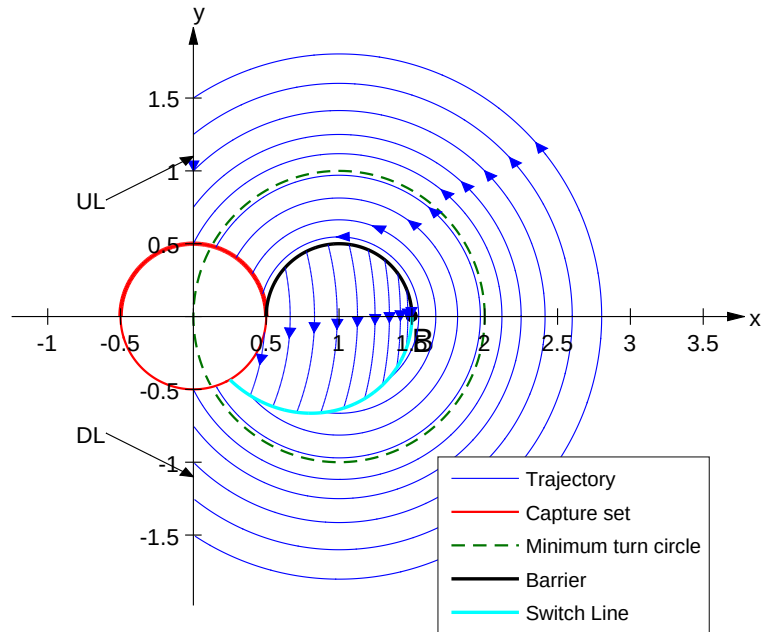


Figure 44. Dubins Car Optimal Control Problem: Optimal Flow Field, $\ell = 0.5, \mu = 0$

4.3.4 Application

This section addresses the applicability of the solution to the *classical* Homicidal Chauffeur differential game to a real-world problem. To demonstrate the utility of the *classical* Homicidal Chauffeur differential game solution, a modern air-to-air combat scenario is examined. In modern air-to-air combat, one of the key components that affects the outcome of an engagement is the stealth capabilities of each aircraft. If a pursuing aircraft (Pursuer) wishes to eliminate an enemy aircraft (Evader), the Pursuer must maximize the usefulness of its stealth capabilities. In [25], it was shown that an aircraft is most susceptible to radar detection when the side of the aircraft was exposed to radar; having the nose or tail of the aircraft facing the radar reduced the radar cross section of the aircraft. Therefore, in order to maximize its stealth advantage, it is advantageous for the Pursuer to minimize the exposure of the aircraft's side to the Evader's radar. The Pursuer can decrease its chance of exposure by approaching the Evader head on. This strategy not only aids in concealing the Pursuer, it also reflects the optimal control strategy of the Pursuer in the *classical* Homicidal Chauffeur differential game. In the *classical* Homicidal Chauffeur differential game, the Pursuer turns as hard as possible towards the Evader in order to achieve pure LOS pursuit as quickly as possible. In the air-to-air combat scenario, by turning hard towards the Evader, the Pursuer minimizes the exposure of the aircraft's side to the Evader while simultaneously utilizing the strategy that minimizes the time it takes to capture the Evader. If a pursuing aircraft were to apply this strategy in a modern air-to-air combat scenario, it would minimize the time of the engagement while maximizing the aircraft's stealth capabilities: both of which are desirable to a pilot.

V. Conclusion

Isaacs' development of the Homicidal Chauffeur differential game inspired a great deal of research that sought to apply the solutions of differential games to real-world problems. Attempting to apply differential game solutions to real-world problems yields complex differential games, and as the first application driven research in the field, Isaacs' work served as a foundation to solve those complex differential games. Despite the impact of the *classical* Homicidal Chauffeur differential game on the field of differential games, the full optimal control solution is not readily available. This is due to Isaacs' incomplete solution and the widespread unavailability of the only work that provides the full solution to the Homicidal Chauffeur differential game, Merz' dissertation. Thus, there is a gap in the field of differential games concerning the solution process of the *classical* Homicidal Chauffeur differential game. While others have modified the Homicidal Chauffeur differential game by adding time delays, multiple players, stochastic elements, etc, there has never been an investigation of the original solution process of the *classical* Homicidal Chauffeur differential game outside of Isaacs' and Merz' work.

This thesis fills the gap in the literature of the field of differential games by providing the full solution process to the *classical* Homicidal Chauffeur differential game. This solution was obtained by examining two optimal control problems as well as an academic exercise postulated by Isaacs. To gain a greater understanding of the Homicidal Chauffeur differential game, two novel Dubins car optimal control problems, that served as simplified differential game analogues, were developed and analyzed. The full solution process was outlined for the Dubins car optimal control problem for the cases of a Pursuer endowed with a circular capture set and a line segment capture set. Additionally, the solution to Exercise 10.6.1, suggested by Isaacs in [1], was obtained in order to gain a greater understanding of two salient features of the

classical Homicidal Chauffeur differential game, the barrier and ES. An analysis of the solution processes of the Dubins car optimal control problems and Exercise 10.6.1 were then used to develop the full solution to the *classical* Homicidal Chauffeur differential game. Once the solution to the Homicidal Chauffer differential game was obtained, the motion of the players in the realistic plane was described in relation to the differential game's optimal flow field. Additionally, a connection between the Homicidal Chauffeur differential game and its Dubins car optimal control analogue was evidenced. This connection helps to validate the methodology of using simplified problems in order to characterize and obtain the solution to more complex problems. Lastly, the Pursuer's strategies that prevent the Evader from exploiting the ES, constructed after obtaining the solution to Exercise 10.6.1, were applied to the Homicidal Chauffeur differential game. These strategies can potentially be used to obtain a nearly-optimal control solution for complex differential games where the ES inhibits the derivation of a tractable solution.

This thesis offers tutorial value by providing the entire process used to obtain the full solution; something that is lacking in both Isaacs' and Merz' work. This thesis also addresses and resolves common misconceptions concerning the barrier and ES present in the *classical* Homicidal Chauffeur differential game solution. This thesis effectively fills a literature gap in the field of differential game, has tutorial value concerning the solution process of two novel Dubins car optimal control problems, Isaacs' Exercise 10.6.1, and the Homicidal Chauffeur differential game, and opens possibilities for continued research.

Suggested thrusts for continued research include the following:

- In an effort to make the investigation of complex differential games easier, methods of reducing the game's degree of difficulty, by eliminating the Evader's ability to exploit the ES, were discussed in Section 3.5.6. Applying any of these

methods in a differential game requires the Pursuer to momentarily deviate from its optimal strategy. By deviating from its optimal strategy, the Pursuer incurs an additional cost. An avenue for continued research could be the investigation of the additional cost the Pursuer incurs by preventing the Evader from exploiting the ES. An analysis on the additional cost incurred versus the benefits of simplifying the problem can be completed for the *classical* Homicidal Chauffeur differential game. The results of the cost analysis would potentially provide validation for using these methods to simplify more complex differential games.

- The methodology of Othling [13] and Miles/Williamson-Nobel [18] was used in the development of this work. Their research showed that solutions to differential games are easier to obtain when simpler versions of them are analyzed. This work provides the solution process of two novel Dubins car optimal control problems, Exercise 10.6.1, and the *classical* Homicidal Chauffeur differential game which can now serve as foundation for the development of more complex differential games. Following Othling's and Miles/Williamson-Nobel's principle, with the solutions derived in this work acting as the simplified versions of problems, more complex versions of these problems can now be examined. Potential modifications to the problems include the introduction of 3-D motion, adding delays in the players' control choices, and expanding upon the *classical* parameter space.
- The original scope of this research included the investigation of the *classical* Homicidal Chauffeur differential game parameters for the case of a Pursuer endowed with a line segment capture set. With the completion of its optimal control problem counterpart in Section 3.3, examining this differential game serves as a logical continuation of this work. An introductory discussion on the

novel Homicidal Chauffeur differential game is presented in Appendix E: the effect of the speed ratio parameter μ on the UP of the capture set depicted in Figure 6 is examined.

Appendix A.

A An Examination of the SL and ES

In this section, it is shown that the SL of the Dubins car optimal control problem in Section 3.2.6 is not equivalent to the ES of the *classical* Homicidal Chauffeur differential game constructed for $\mu = 0$, seen in Section 4.2.5.

In the *classical* Homicidal Chauffeur differential game, for an Evader located on the ES, the Pursuer's optimal control u_{ES} is characterized as

$$u_{ES} = 1 - \mu \frac{1}{r} \frac{r^2 - x + rh}{r - y}$$

If the ES were to be constructed for the case of a stationary Evader ($\mu = 0$), as in the Dubins car optimal control problem, the Pursuer's optimal control for an Evader positioned on the ES would be

$$u_{ES} = 1$$

Applying this optimal control to the dynamics along the ES

$$\dot{x}_{ES} = y_{ES}u_{ES} + \mu \frac{x_{ES}}{r_{ES}}, \quad x_{ES}(\tau_2) = x_B$$

$$\dot{y}_{ES} = -x_{ES}u_{ES} + 1 + \mu \frac{y_{ES}}{r_{ES}}, \quad y_{ES}(\tau_2) = y_B, \quad 0 \leq \tau_2$$

yields

$$\dot{x}_{ES} = y_{ES}, \quad x_{ES}(\tau_2) = x_B$$

$$\dot{y}_{ES} = -x_{ES} + 1, \quad y_{ES}(\tau_2) = y_B, \quad 0 \leq \tau_2$$

$\Rightarrow$

$$\frac{dy_{ES}}{dx_{ES}} = \frac{1 - x_{ES}}{y_{ES}}, \quad y_{ES}(x_B) = y_B, \quad x_{ES} \leq x_B$$

Constructing the ES, utilizing the method seen in Section 4.2.5, produces the ES seen in Figure 45 which depicts the optimal flow field of the Dubins car optimal control problem analogue of the *classical* Homicidal Chauffeur differential game.

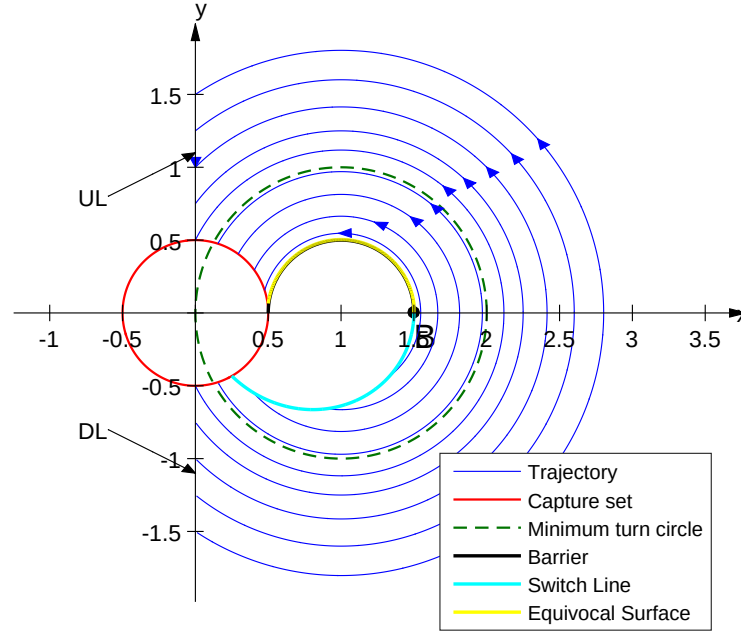


Figure 45. Dubins Car Optimal Flow Field: A Comparison of the Switch Line and the Equivocal Surface Constructed for $\mu = 0$

From Figure 45, it is apparent that the SL of the Dubins car optimal control problem is not equivalent to the ES of the *classical* Homicidal Chauffeur differential game constructed for $\mu = 0$.

Appendix B.

B Optimal Control Along the ES

In this section, an analytical method for the derivation of the Evader's optimal control along an ES, as opposed to Isaacs' geometric approach, is provided for Exercise 10.6.1 in Section 3.5.4.2.

Recall the dynamics of the differential game:

$$\dot{x} = u(y) + w \sin \Psi$$

$$\dot{y} = -b\phi + w \cos \Psi, \quad -1 \leq \phi \leq 1$$

The slope of the ES (c) is chosen by the Pursuer where

$$c = \frac{dy}{dx} = \frac{-b\phi + w \cos \Psi}{u(y) + w \sin \Psi} \quad (217)$$

Solving for the Pursuer's control ϕ yields

$$\phi(\Psi; c) = \frac{w}{b} \cos \Psi - \frac{c}{b} [u(y) + w \sin \Psi] \quad (218)$$

The Evader, interested in moving as slowly as possible on the ES, solves the optimization problem

$$\min_{\Psi} \sqrt{\dot{x}^2 + \dot{y}^2} \quad (219)$$

From Equation (217), inserting $\dot{y} = c\dot{x}$ into the above equation yields

$$\min_{\Psi} (\sqrt{1 + c^2} \cdot \dot{x})$$

Since the Pursuer controls the slope of the ES, the Pursuer can keep it at a constant

value despite the Evader's control choices. So the optimization problem for the Evader becomes

$$\min_{\Psi} \dot{x}$$

and since $\dot{x} = u(y) + w \sin \Psi$

$$\check{\Psi} = -\frac{\pi}{2}$$

Alternatively, from Equation (217), inserting $\dot{x} = \frac{\dot{y}}{c}$ into Equation (219) yields

$$\min_{\Psi} \sqrt{1 + \frac{1}{c^2}} \cdot \dot{y}$$

Since the Pursuer controls the slope of the ES, the above equation becomes

$$\min_{\Psi} \dot{y} = \min_{\Psi} (-b\phi + w \cos \Psi)$$

Inserting Equation (218) into the equation above yields

$$\min_{\Psi} cu(y) + cw \sin \Psi$$

$\Rightarrow$

$$\min_{\Psi} (\sin \Psi)$$

$\Rightarrow$

$$\check{\Psi} = -\frac{\pi}{2}$$

The importance of this alternative approach to the Evader's traversal strategy on the ES is that it can be extended to apply to general differential games where the dynamics are $\dot{x} = f(x, u, v)$

Appendix C.

C Alternative Methods for Constructing the ES

C.1 Alternative ES Construction (1)

In this section an alternative construction of the ES in Exercise 10.6.1 Section 3.5.5.2 is presented. On the ES there is a corner: The costate is not continuous, the Value function $\equiv V(x) \notin C^1$. For the primary trajectory that contacts the ES at the corner, the Value function is

$$V_x = 0, \quad V_y = 1$$

For the secondary trajectory that contacts the ES at the corner, the Value function is

$$V_x = -a, \quad V_y = 0$$

Knowing this, the expression of the ES previously derived is verifiable.

It is known that across the ES

$$V_x^- - V_x^+ = \alpha \cdot \vec{n}, \quad \alpha \in \mathbb{R}^1 \tag{220}$$

where $\vec{n}$ is the vector normal to the ES pointing into Zone 1. From the Value functions described above and Equation (220)

$$\begin{aligned} & \begin{pmatrix} -a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha \cdot \vec{n} \\ \Rightarrow & \begin{pmatrix} a \\ 1 \end{pmatrix} = \alpha \cdot (-\vec{n}) \end{aligned} \tag{221}$$

From Equation (118) the ES is the parabola

$$x = x_B - \frac{1}{2}(y - 1)^2$$

and the normal to this parabola which points into Zone 1 is

$$\vec{n} = \left(\frac{1}{\sqrt{2(x_B - x)}}, 1 \right), \quad x < x_B$$

Applying Equation (221) yields

$$\begin{pmatrix} a \\ 1 \end{pmatrix} = \alpha(x) \begin{pmatrix} -\frac{1}{\sqrt{2(x_B - x)}} \\ -1 \end{pmatrix}$$

$\Rightarrow$

$$\alpha(x) < -1 \quad \forall \quad x < x_B$$

$\Rightarrow$

$$a(x) = \frac{1}{\sqrt{2(x_B - x)}} \quad \forall \quad x < x_B$$

In the secondary flow field along secondary trajectories which terminate on the ES (x_{ES}, y_{ES}) .

$$V_x(x, y) = -\frac{1}{\sqrt{2(x_B - x_{ES})}}$$

$$a = \frac{1}{\sqrt{2(x_B - x)}}$$

From Equation (128)

$$\frac{1}{y - 1} = \frac{1}{\sqrt{2(x_B - x)}}$$

$\Rightarrow$

$$x_B + x = \frac{1}{2}(y - 1)^2 \tag{222}$$

which yields the equation of the ES parabola, as expected.

C.2 Alternative ES Construction (2)

In this section an additional alternative construction of the ES in Exercise 10.6.1 Section 3.5.5.2 is presented. An alternative construction of the ES can be accomplished by utilizing a method established by Bernhard [26]. The method utilizes information on the costates on both sides of a singular surface in order to obtain an expression for the surface. The method is used herein to obtain an expression for the ES in the parametric Exercise 10.6.1.

Based on the work of Bernhard [26] it is known that

$$\begin{pmatrix} \lambda_x^{(2)} \\ \lambda_y^{(2)} \end{pmatrix}_{ES^-} = \begin{pmatrix} \lambda_x^{(1)} \\ \lambda_y^{(1)} \end{pmatrix}_{ES^+} + \alpha \vec{n}, \quad \alpha \in \mathbb{R}^1 \quad (223)$$

where the notation ES^- and ES^+ denote the costates close to the ES in Zone 2 $(\lambda_x^{(2)}, \lambda_y^{(2)})$ and Zone 1 $(\lambda_x^{(1)}, \lambda_y^{(1)})$ respectively. Additionally, $\vec{n} = (-y'_{ES}, 1)$ is the normal to the ES. It was shown in Section 3.5.4.1 that in Zone 1, where the primary optimal trajectories are,

$$V_x = 0, \quad V_y = \frac{1}{w - b}$$

where the parameters $b=3$ and $w=2$. Applying this to Equation (223) yields

$$\begin{pmatrix} \lambda_x^{(2)} \\ \lambda_y^{(2)} \end{pmatrix}_{ES^-} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} -y'_{ES} \\ 1 \end{pmatrix}$$

Thus, the costates close to the ES in Zone 2 satisfy

$$\lambda_x^{(2)} = -\alpha y'_{ES} \quad (224)$$

$$\lambda_y^{(2)} = -1 + \alpha \quad (225)$$

From Equation (93), the Hamiltonian is

$$H = -1 + \lambda_x(1 + y + 2\sin\Psi) + \lambda_y(3 + 2\cos\Psi) \quad (226)$$

To minimize the Hamiltonian, the Evader's control is such that

$$\sin\Psi^* = -\frac{\lambda_x}{\sqrt{\lambda_x^2 + \lambda_y^2}}, \quad \cos\Psi^* = -\frac{\lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}} \quad (227)$$

Applying the optimal control above to Equation (226) yields

$$0 \equiv -1 + (1 + y)\lambda_x + 3\lambda_y - 2\sqrt{\lambda_x^2 + \lambda_y^2} \quad (228)$$

Consider the time $\tau = y_{ES}$ when the state is on the ES (x_{ES}, y_{ES}) : On the ES at time $\tau = y_{ES}$, $H|_{\tau=y_{ES}} = 0$. Thus, inserting Equations (224) and (225) into the Hamiltonian in Equation (228), yields

$$0 = -1 + (1 + y_{ES}(-\alpha y'_{ES}) + 3(-1 + \alpha) - 2\sqrt{\alpha^2(y'_{ES})^2 + (-1 + \alpha)^2}$$

From Section 3.5.4.2, on the ES the Evader's traversing strategy is $\check{\Psi} = -\frac{\pi}{2}$: this optimal strategy is the same as the optimal trajectory approaches the ES from Zone 2. Hence, from Equations (224), (225), and (227)

$$\sin\Psi^* = -\frac{-\alpha y_{ES}}{\sqrt{\alpha^2(y'_{ES})^2 + (-1 + \alpha)^2}}, \quad \cos\Psi^* = -\frac{-1 + \alpha}{\sqrt{\alpha^2(y'_{ES})^2 + (-1 + \alpha)^2}}$$

and from Section 3.5.4.3

$$\sin\Psi^* = \sin\check{\Psi} = -1, \quad \cos\Psi^* = \cos\check{\Psi} = 0$$

Thus, the scalar $\alpha = 1$. Additionally, it is known that $y'_{ES} < 0$. With the value of α known, from Equations (224) and (225) the costates at time $\tau = y_{ES}$ are

$$\lambda_x^{(2)}(\tau = y_{ES}) = -y'_{ES}, \quad \lambda_y^{(2)}(\tau = y_{ES}) = 0$$

From the PMP and the Hamiltonian in Equation (226), the costates satisfy

$$\dot{\lambda}_y^{(2)} = \lambda_x^{(2)} = -y'_{ES}$$

Thus,

$$\lambda_y(\tau) = y'_{ES}(y_{ES} - \tau), \quad \tau > y_{ES}$$

So $\forall \tau \geq y_{ES}$ the Evader's optimal control in Zone 2 is

$$\sin(\Psi^*(\tau)) = -\frac{1}{\sqrt{1 + (\tau - y_{ES})^2}}, \quad \cos(\Psi^*(\tau)) = -\frac{y_{ES} - \tau}{\sqrt{1 + (\tau - y_{ES})^2}} \quad (< 0)$$

Hence, the Hamiltonian at $\tau = y_{ES}$, coming from Zone 2, is

$$H = [-1 + \lambda_x(1 + y + 2\sin\Psi^*) + \lambda_y(3 + 2\cos\Psi^*)]_{ES} = 0$$

$\Rightarrow$

$$0 = -1 + (-y'_{ES})(1 + y_{ES}) + 2y'_{ES}$$

$\Rightarrow$

$$0 = -1 - y'_{ES} - y_{ES}y'_{ES} + 2y'_{ES}$$

Solving for y'_{ES} yields the differential equation

$$y'_{ES} = \frac{1}{1 - y_{ES}}$$

That is

$$\frac{dy_{ES}}{dx_{ES}} = \frac{1}{1 - y_{ES}}$$

$\Rightarrow$

$$\frac{dx_{ES}}{dy_{ES}} = 1 - y_{ES}, \quad x(1) = x_B$$

Integrating the above equation yields

$$x_{ES}(y_{ES}) = -\frac{1}{2}y_{ES}^2 + y_{ES} + x_B - \frac{1}{2}$$

So the equation of the ES is

$$x_{ES}(y_{ES}) = x_B - \frac{1}{2}(y_{ES} - 1)^2$$

This expression matches the previously obtained ES expressions found in Equations (118) and (222).

Appendix D.

D The Evader's Tributary Option at B

In this section, Evader's ability, in the *classical* Homicidal Chauffeur differential game, to exit the barrier end point B and travel along a tributary trajectory is presented. This is shown by repeating the methodology used in Section 4.3.2. The normal at B, pointing into the region of the tributary trajectories is

$$\vec{n} = (\sqrt{1 - \mu^2}, -\mu)$$

Since the Evader is entering the region of the tributary trajectories from B, the forward time dynamics are examined. Recalling the forward time dynamics from Equations (131) and (132)

$$\vec{f} = \begin{pmatrix} -yu + \mu \sin \Psi \\ xu - 1 + \mu \cos \Psi \end{pmatrix}$$

When the Evader is located on B, the Pursuer applies the optimal control $u^* = -1$. To enter the region of the tributary trajectories from B, the Evader applies the tributary trajectory optimal control strategy from Equation (154). At B, when these optimal strategies are applied

$$\vec{f} = \begin{pmatrix} y_B + \mu \sin \tau_1 \\ -x_B - 1 + \mu \cos \tau_1 \end{pmatrix}$$

where (x_B, y_B) is the Evader's position on B from Equations (151) and (152). For the Pursuer to push the Evader off of B, into the region of the tributary trajectories, the following must hold true

$$\vec{n} \cdot \vec{f} \Big|_B > 0$$

Evaluating the above equation yields

$$\vec{n} \cdot \vec{f} = \sqrt{1 - \mu^2} y_B + \mu \sqrt{1 - \mu^2} \sin \tau_1 + \mu x + \mu - \mu^2 \cos \tau_1$$

$\Rightarrow$

$$\vec{n} \cdot \vec{f} = \mu \sqrt{1 - \mu^2} \sin \tau_1 - \mu^2 \cos \tau_1 + \mu$$

From Section 4.2.4, it is known that there is a critical tributary trajectory that contacts B at the retrograde time τ_1 characterized in Equation (164).

A numerical analysis of the equation above reveals that, at the critical retrograde time in Equation (164), $\vec{n} \cdot \vec{f} > 0$, verifying that when the Evader applies its tributary region optimal control at B, the Evader can move from B into the region of the tributary trajectories.

Appendix E.

E Consideration for the Novel Homicidal Chauffeur Differential Game

The original scope of the research considered examining the novel Homicidal Chauffeur differential game where a Pursuer is endowed with a line segment capture set. Due to time constraints, a solution to this differential game was not obtained: however, the optimal control analogue to the novel differential game was successfully solved in Section 3.3. To motivate further examination of this novel Homicidal Chauffeur differential game, a discussion is presented herein concerning how the introduction of a non-zero speed ratio affects the UP of the line segment capture set of Figure 6.

When considering the novel Homicidal Chauffeur differential game where the capture set is a line segment, the relationship between the problem parameters μ and ℓ becomes an important factor which impacts the nature of the solution. To gain an understanding of the impact of this relationship, it is beneficial to approach the Homicidal Chauffeur differential game with a line segment capture set by considering a more general circular sector capture set with $0 \leq \varepsilon \ll 1$ as seen in Figure 46.

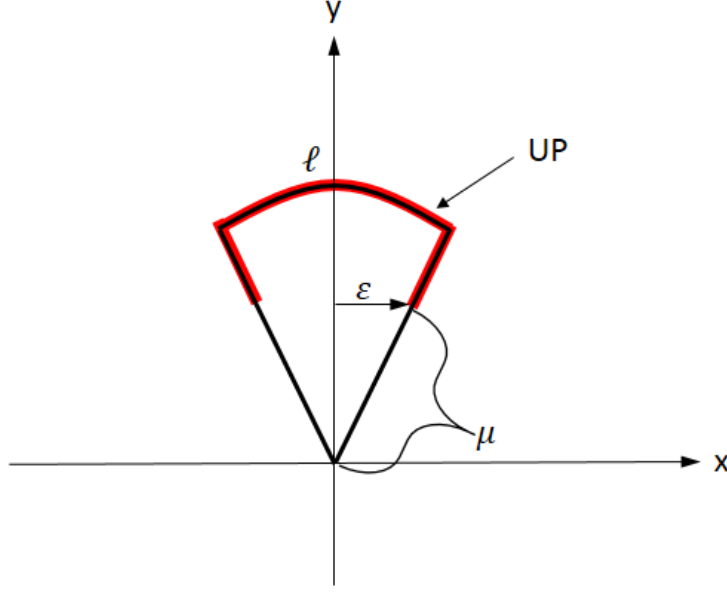


Figure 46. Circular Sector Capture Set, $\ell > \mu$

The figure above shows a circular sector capture set where $\ell > \mu$. For this case the UP is a closed circular arc plus extensions on the “sidelines” up to the points $(\approx \mu(\sin\varepsilon, \cos\varepsilon), \mu(-\sin\varepsilon, \cos\varepsilon))$. However if $\ell < \mu$ the UP of the capture set becomes an open circular arc, as shown in Figure 47. So for the case when $\varepsilon \rightarrow 0$, namely, a line segment capture set with $\ell < \mu < 1$, the UP of the capture set is an empty set. Hence, an optimally playing Evader can avoid capture indefinitely. This holds even if the Evader is initially positioned at ℓ ; if $\ell < \mu$ the Evader is able to turn away from the capture set before the Pursuer can force the Evader to penetrate the capture set.

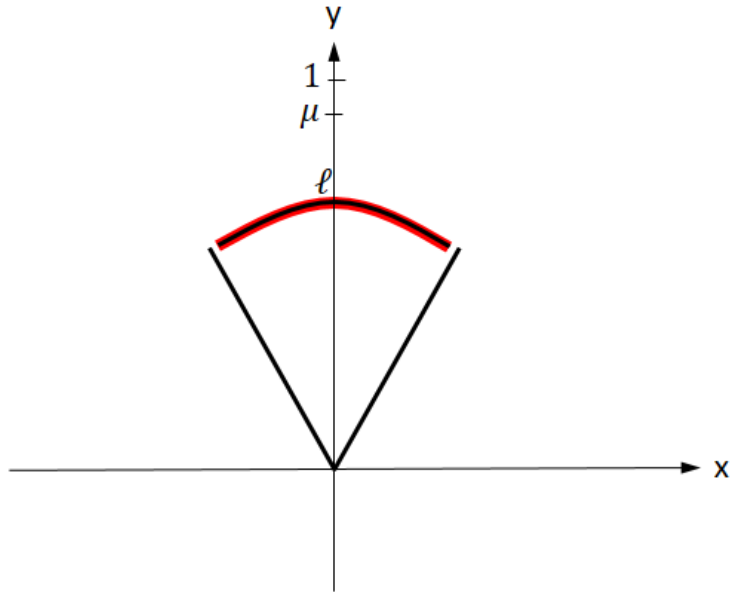


Figure 47. Circular Sector Capture Set, $\ell < \mu$

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14. ABSTRACT The Homicidal Chauffeur differential game is often referenced in the field of differential games, but the game's complete solution is not readily available. In this work, the complete solution process of the Homicidal Chauffeur differential game is illustrated for the parameter range initially investigated by Isaacs. The solution is obtained by first deriving and analyzing the solution process of two novel optimal control problems as well as a rudimentary differential game proposed by Isaacs. In the process of deriving the solution to the Homicidal Chauffeur differential game, some salient features of the solution of this foundational differential game are highlighted and some popular misconceptions are dispelled. This thesis fills a gap in the literature on pursuit-evasion differential games which has direct military applications.						
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