ROBUST GOAL PROGRAMMING AND RISK ASSESSMENT USING CARDINALITY-CONSTRAINED AND STRICT ROBUSTNESS VIA ALTERNATIVE UNCERTAINTY SETS

DISSERTATION

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AFIT-ENS-DS-17-S-035

DEPARTMENT OF THE AIR FORCE

AIR UNIVERSITY

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Wright-Patterson Air Force Base, Ohio

AFIT-ENS-DS-17-S-035

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DISSERTATION

Presented to the Faculty

Department of Operational Sciences

Graduate School of Engineering and Management

Air Force Institute of Technology

Air University

Air Education and Training Command

In Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy in Operations Research

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September 2017

AFIT-ENS-DS-17-S-035

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Abstract

In today's dynamic and ever-changing world, data uncertainty is inevitable. The most popular methods for addressing this uncertainty utilize stochastic optimization or robust optimization (RO). Because garnering a true probability distribution can be fleeting, the primary focus of this dissertation is in the RO realm, whose implementation utilizes uncertainty sets to consider uncertainty in data.

We first introduce the notion of robust goal programming (RGP) which utilizes cardinality-constrained robustness via interval-based uncertainty sets. We offer a new extension to the foundational structure of RGP by considering different uncertainty sets to implement. Within this context, we compare interval-based and norm-based uncertainty sets using cardinality-constrained robustness as well as strict robustness using ellipsoidal uncertainty sets. The aforementioned methods are demonstrated for a simple instance from the literature where the results are summarized and conclusions are made regarding the proposed RGP models when likened to a similar RGP model seen in the literature. Further, the suitability of each RGP model is offered when a decision maker's (DM) risk preference or computing availability are taken into consideration. Inferences are made regarding the effectiveness of each uncertainty set in the context of solutions that are relatively unaffected by data uncertainty – that is, robust solutions.

We also investigate several risk elicitation methods in the context of maximizing expected utility in an attempt to approximate a DM's risk preference. To this end, the notion of rigorously eliciting a DM's risk *a priori* for RO parameters is offered. More

specifically, three theorems are provided to show that a DM’s risk preference can be mapped to appropriate RO model parameters, and that a risk-neutral point exists over a bounded interval. Using piecewise exponential decay functions derived from conceptually-motivated differential equation models, we demonstrate how to mathematically map a DM’s risk preference to RO parameters for each of four different risk elicitation methods. To that end, we provide a novel mapping methodology that links the sub-disciplines of risk and RO.

Lastly, we apply an RGP model utilized in concert with the risk mapping methodology to the United States Transportation Command (USTRANSCOM) rate setting problem. USTRANSCOM is responsible for the technical direction and supervision of over $7 billion [1] of annual passenger, cargo, mobility, and personal property movements in support of the Department of Defense (DoD). Transporting people and material with both organic and contracted assets, USTRANSCOM supports DoD organizations and agencies on a reimbursable basis, annually setting and charging rates for air and liner (i.e., sea) transport for their customers and reimbursing the transportation providers accordingly. TCJ8 (Financial Management and Program Analysis), a staff organization within USTRANSCOM, annually sets air and liner shipping rates specific to each combination of origin, destination, commodity type, booking terms, and container size for the upcoming fiscal year (FY). As a government entity, USTRANSCOM seeks to neither make a profit nor operate at a loss in any given FY, although they do maintain a fiscal buffer known as the Transportation Working Capital Fund (TWCF) to prevent the possibility of running a deficit in a given year from incurring a debt. Current rate setting methodologies are relatively straightforward; TCJ8 adjusts average costs computed from invoices to date in a

current year, and subsequently adjusts the average costs for inflation, overhead expenses to manage and administer the process, and aggregation of selected origin-destination pairs by region. This process assumes existing data (i.e., inflation rates, overhead adjustment rates, invoice costs) to be deterministic, resulting in process inaccuracies that contribute to unexpected surpluses or deficits each FY. Due to the increasing number of rates being set and the low density of shipping cost records available for each rate being set, USTRANSCOM also seek to account for the variance inherent in other parameters, with the objectives of managing the TWCF balance and preventing large individual changes to rates from year-to-year. Considering the different goals and inherent parametric variance, we attempt to improve the USTRANSCOM rate setting process by incorporating a DM's risk preference via *a priori* analysis in concert with RGP techniques, identifying a process to set robust shipping rates for DoD customers.

Dedicated to my wife and children

Acknowledgments

I would like to thank both of my advisors, Dr. Jeffery D. Weir and Dr. Brian J. Lunday, for their constant guidance, mentorship, and patience. They both have inspired me to be a better officer, husband, and father. I am forever indebted to them.

I would also like to thank my committee member, Dr. Kenneth M. Hopkinson, for taking the time to support me throughout this endeavor.

Robert W. Hanks

Table of Contents

Table Page

List of Figures

List of Tables

I. Introduction

Robust optimization (RO) is a field in the study of optimization which considers uncertain data using set theory, and specifically, uncertainty sets. Accounting for parametric variability in this manner is helpful as a true probability distribution is not required. RO concepts are first introduced by Soyster [2] when he formulates a linear program which accounts for data uncertainty via an uncertainty set. Soyster [2] considers strict robustness, wherein for any realization of uncertain data, the resulting solution will remain feasible. Considered the most conservative approach, strict robustness is just one of multiple levels of robustness that are considered when using RO. How to represent the uncertain data via uncertainty sets is yet another pillar of RO that must be deliberated. Some uncertainty sets utilize the decision maker's (DM) views on risk, while others simply take into account the maximum deviation from a nominal value. In the end, RO is about compromising between feasibility and optimality. This dissertation research contemplates various RO models when dealing with data of stochastic or uncertain nature in an attempt to not only broaden the scope of RO, but to also increase its practicality to real-world problems, such as the United States Transportation Command (USTRANSCOM) rate setting problem.

This dissertation outlines the research contributions which address the theoretical foundation and practical expansion of various RO models. Chapter II provides a literature review of pertinent subject areas such as multi-objective optimization (MOO), uncertainty modeling methods, decision analysis (DA), and rate setting practices. Chapter III considers a relatively new RO model, first presented by Kuchta [3], called robust goal programming (RGP), wherein we adapt new RGP models with varied levels of robustness and uncertainty sets. Chapter IV discusses a variety of ways to elicit a DM's risk preference *a priori*, wherein a mathematical mapping methodology is offered to account for a DM's views on risk for various RO parameters. Chapter V serves as a proof of principle regarding the theoretical contributions of this research as applied the USTRANSCOM rate setting problem. Finally, Chapter VI discusses overall conclusions generated from this dissertation research along with recommendations for future work. Although Chapters III, IV, and V are individual journal manuscripts, the overarching contribution for this research is evident in each one – the extension of RO and its direct applicability to everyday problems.

More specifically, we examine in Chapter II MOO techniques such as *a priori*, *a posteriori*, and no articulation methods. We provide a comparison of uncertainty modeling methods such as stochastic optimization and RO, wherein we introduce a variety of robustness levels, as well as different uncertainty sets utilized in practice. Furthermore, we discuss different RO models witnessed in the literature, as well as an RGP formulation. We transition from RO to DA wherein we offer expected utility maximization axioms, utility function measures, and different risk elicitation methods. Finally, we offer a brief discussion vis-à-vis rate setting methodologies.

In Chapter III we examine the RGP construct. Instead of strict, or minimax robustness, RGP utilizes a less-conservative approach, first presented by Bertsimas and Sim [4], [5] called cardinality-constrained robustness wherein only a subset of parameters endure variability. Applying this level of robustness, Kuchta [3] utilizes interval-based uncertainty sets with goal programming concepts. In our attempt to extend RGP principles, we offer an RGP model that implements cardinality-constrained robustness, but instead of interval-based uncertainty sets, we utilize norm-based uncertainty sets via both the L_1 and L_2-norms. Moreover, we couple the RGP-mindset with strict robustness and ellipsoidal uncertainty sets in the event a DM is unsure of the amount of parametric variability present. In the end, we compare our three conceptually-motivated, alternative RGP formulations to the existing RGP model via a simple test instance presented by Kuchta [3] when considering a DM's risk *a posteriori*.

Chapter IV's discussion incorporates RO, but also presents the topic of DA and risk assessment. We take into account the fact that *a posteriori* analysis, although useful, is sometimes intractable. To build upon this idea, we consider four popular risk elicitation methods and in particular, the bomb risk elicitation task (BRET), first presented in Crosetto and Filippin [6]. We propose three proofs that serve as the theoretical foundation for a mathematical mapping methodology, which enables a DM's risk preference in the DA realm to be mapped to an RO risk parameter. We demonstrate graphically, using all four risk elicitation methods, the sigmoid shape of the mapping methodology. In order to validate the mapping methodology, we consider the raw data used by Crosetto and Filippin [6] to refine the BRET's risk preference categories and utilize the same test instance used by Kuchta [3]. Applying the RGP model described by Hanks *et al.* [7], we demonstrate the usefulness of the proposed mapping methodology and *a priori* analysis in terms of robust solutions.

Chapter V provides an in-depth summary of the USTRANSCOM rate setting problem, considering liner (sea) rates exclusively. We discuss the current rate setting methodology adopted by TCJ8 (Financial Management and Program Analysis) and the challenges that result from disaggregation of rates and sparse data. A proposed rate setting methodology is then discussed wherein an RGP model, and specifically the cardinality-constrained robustness via L_2-norm uncertainty sets [7], is invoked. Thanks to TCJ8, who provide the liner data, we conduct analysis on FY12-FY16 data wherein we compare and contrast two inflation-adjusted, weighted average rate setting methods, as well as the proposed RGP method. We find that the RGP method outperforms both of the weighted-average models in terms of the TCJ8-specific goals. However, due to the sparsity in the data and a lack of knowledge of all parameters, we present possible directions for future research that will assist TCJ8 optimize their process of setting liner rates for USTRANSCOM.

Finally, Chapter VI provides an overall conclusion to this dissertation research, with ample discussion regarding recommendations for future research endeavors. A final summary regarding our research contributions is also offered.

In summary, the contributions of this dissertation research are many. Upon researching pertinent literature, we introduce three conceptually-motivated, alternative RGP formulations that account for data uncertainty via either cardinality-constrained or strict robustness; demonstrate that the existing RGP model is equivalent to one of our proposed RGP models; and compare the applicability of each RGP model given a DM’s risk preference. Moreover, we provide a means to mathematically map a DM’s risk preference to a variety of RO model risk parameters via a mapping methodology and *a priori* analysis. Finally, we reveal the effectiveness of utilizing RGP techniques in unison with our mapping methodology as applied to the USTRANSCOM rate setting problem.

II. Literature Review

The topics concerning this literature review are extensive. To begin, multi-objective optimization (MOO) methodologies are presented, with an emphasis on goal programming (GP), along with a brief discussion regarding scaling. Stochastic optimization (SO) and robust optimization (RO) concepts are then compared and contrasted, with a much more extensive dialogue pertaining to RO. Decision analysis (DA) areas are covered, including expected utility theory and risk elicitation methods. Finally, discoveries in rate setting are discussed.

2.1 MOO

Multi-objective optimization (MOO), or sometimes referred to as vector optimization, considers the process of systematically or simultaneously solving a collection of objective functions [8]. In general, MOO methodologies can be classified as *a priori, a posteriori,* or no articulation, all of which consider a DM's expertise in a different manner. This sub-section outlines each of these MOO approaches, while providing a more extensive discussion on GP, as it is of interest to this dissertation research.

2.1.1 A priori methods

A priori methods are considered when a DM can explicitly express their preferences or goals in terms of relative importance. Often times, these preferences are quantified via parameters in the form of coefficients, exponents, limits, etc. depending on the *a priori* methodology used [8]. There are many *a priori* methods presented by Marler and Arora [8], wherein an interested reader is directed. Additionally, an interested reader is referred to the following references regarding each specific *a priori* method: the weighted-sum [9], [10], [11], [12], [13], [14], [15], preemptive or lexicographic [16], [17], [18], [19], weighted minimax [10], [20], [21], [22], exponential weighted criterion [23], weighted product [24], bounded objective function or ε-constraint [25], [26], [27], [28], [18], [29], [30], [31], [32], [33], physical programming [34], [35], [36], [37], [38], and finally, GP.

The basic premise of GP dates back to 1955 when Charnes *et al.* [39] did a study on executive compensation, where the concept of minimizing deviation d_j away from a goal t_j for each objective function $F_j(x)$ is first presented. Seminal articles regarding GP [40], [41], [42] can also be viewed in the literature. The objective of GP is to minimize total deviation $\sum_{j=1}^{k} \left| d_j \right|$. However, the absolute value placed on the total deviation can present computational issues. Consequently, the total deviation can be presented as positive and negative parts such as $d_j = d_j^+ - d_j^-$ where $d_j^+ \geq 0, d_j^- \geq 0,$ and $d_j^+ d_j^- = 0$ [8]. In this context, d_j^+ and d_j^- represent overachievement and underachievement for each j^{th} objective, respectively. Accordingly, it follows that the most general GP formulation is outlined in (1)-(6) [8]:

$$Minimize \sum_{i=1}^{k} (d_i^+ + d_i^-) \tag{1}$$

$$subject\ to\ F_j(x) - d_j^+ + d_j^- = t_j, \ \ j = 1, 2, \dots, k \tag{2}$$

$$\boldsymbol{Ax} = \boldsymbol{b} \tag{3}$$

$$d_j^+, d_j^- \geq 0, \ \ j = 1, 2, \dots, k \tag{4}$$

$$d_j^+ d_j^- = 0, \ \ j = 1, 2, \dots, k \tag{5}$$

$$\boldsymbol{x} \in \boldsymbol{\mathcal{X}} \tag{6}$$

Different variations of GP have been presented in the literature that combines GP with other methodologies. GP has been used in conjunction with assigning relative weights to each deviational goal, called Archimedean GP or weighted GP [43]. Additionally, Charnes and Cooper [43] introduce the concept of preemptive GP, wherein the deviational goals are ordered in terms of priority and minimized lexicographically. Other methods that incorporate GP are multigoal programming [44], goal attainment methods via a weighted minimax approach [25], and reference GP [45]. More recently, GP has been implemented using fuzzy logic. Fuzzy logic was developed by Lotfi Zadeh [46] to account for imprecise data or when a problem is not well defined. The use of fuzzy logic is a means to make solving more practical problems, wherein data is either unknown or fluctuates between a continuous or discrete range of values. GP has also been applied in concert with RO [3], [7].

The applications of GP are diverse. Schniederjans [47] discusses nine functional areas to which GP is readily applied: accounting, agriculture, economics, engineering, finance, government, international, management, and marketing. More recent applications of GP have been witnessed in manure management systems pertaining to sustainability development [48] as well as problems where a DM’s judgments are provided as incomplete interval additive reciprocal comparison matrices [49].

2.1.2 A posteriori methods

Unlike *a priori* methods, *a posteriori* methods are beneficial when the DM cannot explicitly state their preferences before analysis begins. *A posteriori* methods are sometimes called generate-first-choose-later approaches because an algorithm is used to generate the Pareto optimal set first, wherein the DM then chooses the option they like best [50]. An interested reader is referred to [8] for an excellent survey of *a posteriori* methods, as well as the following references: physical programming [36], [50], normal boundary intersection [51], [52], [53], and normal constraint [54], [12].

2.1.3 No articulation of preferences

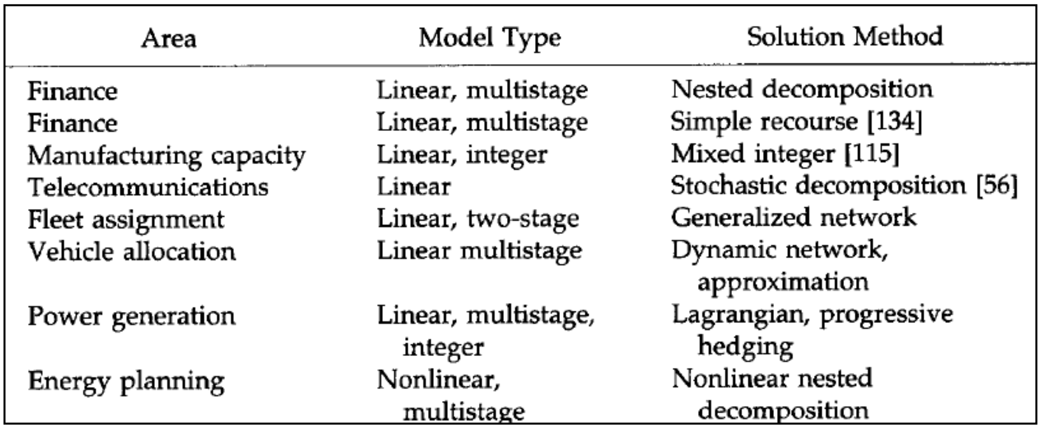
When a DM cannot accurately define or determine their preferences, whether before or after analysis, are classified as MOO methods with no articulation of preferences. The methods having no articulation of preferences presented in this section are generally relaxations or simplifications of *a priori* methods previously discussed, wherein Marler and Arora [8] provide a general overview of each of these methods, along with the following references: exponential sum methods [55], [18], [56], Nash arbitration and objective product [57], [58], and Rao’s method [59], [60], [57].

2.2 Uncertainty modeling

This portion of the literature review briefly introduces SO, along with SO application areas. Then, an in-depth discussion is provided regarding RO models, to include various levels of robustness and uncertainty sets. Furthermore, arenas in which RO methods have been applied is offered.

2.2.1 Stochastic optimization

SO is a widely applied optimization method when dealing with problems contaminated with uncertain data. SO assumes that the uncertainty in data can be accessed via probability distributions. The concept of SO is complex as it incorporates mathematical concepts such as nonlinear calculus, abstract optimization, statistical techniques, and probability theory [61]. The origins of SO can be traced as far back as Dantzig's original paper [62] and in more recent book publications [63], [61], [64], and the survey article [65]. In addition to these well-known sources, Birge [65] presents different methodologies that use SO algorithms to solve a variety of application areas, as seen in Figure 1 below. Moreover, SO has also been researched in the form of Markov decision processes as applied in aircraft path planning, meteorology, traffic reporting, and dynamic programming [66].



Area	Model Type	Solution Method
Finance	Linear, multistage	Nested decomposition
Finance	Linear, multistage	Simple recourse [134]
Manufacturing capacity	Linear, integer	Mixed integer [115]
Telecommunications	Linear	Stochastic decomposition [56]
Fleet assignment	Linear, two-stage	Generalized network
Vehicle allocation	Linear multistage	Dynamic network, approximation
Power generation	Linear, multistage, integer	Lagrangian, progressive hedging
Energy planning	Nonlinear, multistage	Nonlinear nested decomposition

Figure 1: Stochastic Programming Examples [65]

Application areas in which SO and RO are utilized in unison have been witnessed in multi-expert decision making [67], scheduling [68], location problems [69], and the evaluation of queueing networks, auctions, and pricing models [70]. For an extensive list of current application areas regarding RO coupled with SO, an interested reader is direct to the survey article [71].

2.2.2 Why RO?

When a problem is contaminated with uncertain data, it can be difficult to decipher which uncertainty modeling technique to apply. Ben-Tal *et al.* [66] discuss how a large amount of RO approaches consider the notion of minimizing the maximum deviation. When this is in fact the case, SO is often times regarded as the less conservative method if the following are true: 1) The uncertain data are of stochastic nature, 2) The true probability distribution is known and understood, and 3) There is a willingness to accept probabilistic guarantees by chance constraints [66]. However, Ben-Tal *et al.* [66] warn that sometimes even if the data are in fact stochastic in nature, that finding a true probability distribution is difficult. Because of this, SO is often times "forced to operate on oversimplified guesses for the actual probability distributions" [66]. Put another way, SO is often difficult to implement for two reasons: 1) Finding a true probability distribution for uncertain data is extremely challenging and can be misleading, and 2) As the number of potential scenarios increase, the SO model becomes combinatorial and computational intractability [4]. Consequently, we examine RO techniques herein.

2.2.3 Robust optimization

The basic premise regarding RO is the ability to account for data uncertainty, while also ensuring all solutions are "desirable". RO techniques were first introduced by Soyster [2] in 1973 when he presents the concept of minimax robustness via inexact linear programming (LP) and set theory. Yet, for years after Soyster's seminal article, data uncertainty was mainly dealt with reactively via sensitivity analysis or proactively via SO. Reactively considering data uncertainty ignores the uncertainty upfront and addressing it after the fact, which generally does not work well in practice [72]. Similarly, and as previously discussed, finding a true probability distribution is a limiting factor when utilizing SO. To account for these problems, Mulvey *et al.* [73] presents a framework for robust mathematical programming that classify solutions via uncertainty sets as either solution robust (close to optimal for all scenarios) or model robust (almost feasible for all scenarios). Finally, in the late 1990's, the theoretical findings of Soyster's method are expounded with the well-known articles [74], [72], [75], [76], and in the book [66]. It was this literature that also built the framework for less conservative robust approaches.

The tractability of RO in different problem settings have made it very popular in the last 15 years. RO concepts have been discussed in large scale systems and applied to aircraft allocation [73]. Furthermore, robust strategies are presented on how to locate facilities with uncertain data [77]. Additionally, RO concepts have been applied in logistics, inventory management, and finance [71]. For an extensive summary of RO concepts and application areas, see [71], [69], [78].

2.2.3.1 Levels of robustness

There are various levels of robustness in terms of conservatism. These levels of robustness are presented below, with an emphasis on strict and cardinality-constrained robustness. Other widely applied levels of robustness are as follows: adjustable [73], [79], [80], light [81], [82], recoverable [83], and regret or deviation [84], [85], [86]. For a detailed list of all levels of robustness, an interested reader is directed to the survey article [87].

2.2.3.1.1 Strict

Strict robustness is often times regarded as classic RO, minimax optimization, or simply RO [87]. Given an uncertain optimization problem, a solution is considered strictly robust if it is feasible for all scenarios in the uncertainty set [87]. Soyster [2] is the first to consider such an idea in the realm of LP. Strict robustness is applicable in such applications as building a bridge that must be stable at all times, independent of the amount of traffic that occurs at any given time [87]. Although the practicality of strict robustness is limited by it's over conservatism, it still serves as the foundation of RO and the starting point for more conservative RO relaxations.

The strict robustness methodology can best be explained when considering the below LP model [2]:

$$Maximize\ c'x \tag{7}$$

$$\sum_{j=1}^{n} A_j x_j \leq b,\ \ \forall A_j \in K_j, j = 1, \ldots, n \tag{8}$$

$$x_j \geq 0,\ \ \forall j \tag{9}$$

Each uncertainty set K_j is considered a convex set. Soyster also assumes in his formulation that the uncertainty in the A matrix is column-wise, or every column of A_j belongs to the same uncertainty set K_j. Soyster [2] then demonstrates that formulation described in (7)-(9) is in fact equivalent to the below formulation, wherein $\bar{a}_{ij} = sup_{A_j \in K_j}(A_{ij})$:

$$Maximize\ c'x \tag{10}$$

$$\sum_{j=1}^{n} \bar{A}_j x_j \leq b \tag{11}$$

$$x_j \geq 0, \ \ \forall j \tag{12}$$

2.2.3.1.2 Cardinality-constrained

Cardinality constrained robustness achieves a more conservative level of robustness by simply shrinking the uncertainty set as proposed by Bertsimas and Sim [4], [5]. The authors claim that it is unlikely for all coefficients of one constraint to change simultaneously to their worst-case scenario values and instead offer that only a percentage (Γ_i) of them will change at one time. Thus Bertsimas and Sim [4], [5] restrict the amount coefficients that are allowed to change at any one time in this robustness methodology, which is sometimes referred to as $\mathit{\Gamma}$-robustness [87]. For every constraint i, there is a corresponding robustness measure $\Gamma_i, i = 0,1, \dots, m$ that take on values in the interval $[0, |J_i|]$, where $J_i = \{j | \hat{a}_{ij} > 0\}$. The parameter Γ_i is used to adjust robustness in terms of conservatism, whereas J_i are the set of coefficients $a_{ij}, j \in J_i$ that are vulnerable to parameter uncertainty in the interval $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$. Bertsimas and Sim continue by explaining that it is not probable that every $a_{ij}, j \in J_i$ will change with each. In the case where each Γ_i is not restricted to integer values, up to $\lfloor \Gamma_i \rfloor$ of the coefficients will change at their maximum allowable deviation and exactly one coefficient $\hat{a}_{it}$ will change by at most$(\Gamma_i - \lfloor\Gamma_i\rfloor)\hat{a}_{it}$. When the Γ_i values are integer, every coefficient that deviates from their nominal value will do so at a maximum allowable value (e.g.$(\Gamma_i - \lfloor\Gamma_i\rfloor)\hat{a}_{it} = 0$). It is interesting to note that if $\Gamma_i = |J_i|$ then the solution is equivalent to Soyster's strict robustness method [4], [5].

Berstimas and Sim continue by defining a non-linear, robust counterpart that can implement the concepts driving cardinality-constrained robustness. They then use duality theory and concepts of strong duality to define a linear robust counterpart, wherein p_{ij} is the penalty associated with each $i = 1, \dots, m$ and $j = 1, \dots, n$ and b_i, $i = 1, \dots, m$ is the right hand side for each penalty constraint [4], [5].

$$Maximize\ c'x \tag{13}$$

$$subject\ to\ \sum_j a_{ij}x_j + z_i\Gamma_i + \sum_{j \in J_i} p_{ij} \le b_i,\ \ \forall i \in M, \tag{14}$$

$$z_i + p_{ij} \ge \hat{a}_{ij}x_j,\ \ \forall\ i, j \in J_i, \tag{15}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{16}$$

$$x_j, d_i^+, d_i^-, p_{ij}, z_i \ge 0,\ \ \forall\ \ i \in M,\ \ j \in N \tag{17}$$

By proving there is in fact a linear, robust counterpart, Bertsimas and Sim significantly improve the computational tractability of their cardinality-constrained robustness technique. This technique has been extended to uncertainty sets using norms [88] and combinatorial optimization problems [89].

2.2.3.2 Uncertainty sets

Data uncertainty is a reality in almost every real world problem. There are generally two types of uncertainty: microscopic uncertainty such as numerical or measurement errors and macroscopic uncertainty such as forecasting errors, disturbances, or change in environment conditions [87]. For optimization problems that incorporate uncertain data, the nominal scenario is simply the most typical or likely behavior of the uncertain data. Computing these scenarios depend entirely on the type of data uncertainty, but can sometimes be easily determined [87]. Using notation presented by Goerigk and Schöbel [87], optimization problems accounting for uncertainty take the form below, wherein $\xi \in \mathbb{R}^M$ signifies different scenarios that can occur, $F(\cdot, \xi), : \mathbb{R}^n \rightarrow \mathbb{R}^m$ represents the m constraints for each scenario, $f(\cdot, \xi): \mathbb{R}^n \rightarrow \mathbb{R}$ characterizes the objective function for each scenario, and $\mathcal{X} \subseteq \mathbb{R}^n$ defines the variable space:

$$Minimize\ f(x, \xi) \tag{18}$$

$$subject\ to\ F(x, \xi) \leq 0, \tag{19}$$

$$x \in \mathcal{X} \tag{20}$$

The values represented for each scenario are most often not known and thus represented as an uncertainty set $\mathcal{U} \subseteq \mathbb{R}^M$, wherein $\xi \in \mathcal{U}$ [87]. There are many different uncertainty sets that can be used when addressing uncertain data. The implementation of these uncertainty sets are of interest because not all are the same and depending on which one is used, the level of robustness or optimality can be greatly affected. To that end, this dissertation research is particularly interested in interval-based, norm-based, and ellipsoidal uncertainty sets. Other uncertainty sets include: finite [87], polytopic [87], [76], [79], [82], [90], as well as constraint-wise [87]. Furthermore, an interested reader is referred to Bertsimas and Sim [91] on how to construct uncertainty sets and Natarajan *et al.* [92] regarding how to build uncertainty sets while considering risk measures.

2.2.3.2.1 Interval-based uncertainty sets

Interval-based uncertainty sets are considered a special case of polytopic uncertainty sets wherein the polytope is a box $\mathcal{U} = \left[\underline{\xi_1}, \overline{\xi_1}\right] \times \left[\underline{\xi_2}, \overline{\xi_2}\right] \times \ldots \times \left[\underline{\xi_M}, \overline{\xi_M}\right]$ with 2^M extreme points $(\xi_1, \xi_2, \ldots, \xi_M)^t$ wherein $\xi_i \in \left[\underline{\xi_i}, \overline{\xi_i}\right]$[87]. Thus, when implementing interval-based uncertainty sets, the nominal value is bounded above and below by some allowable deviational amount. Interval-based uncertainty sets are used in conjunction with cardinality-constrained robustness [4], [5] and various extensions [3], [7], [93].

2.2.3.2.2 Norm-based uncertainty sets

Bertsimas *et al.* [88] discuss norm-based uncertainty and demonstrate how uncertainty sets using general norms are equivalent to applying the dual norm. Moreover, the authors validate that the L_1-norm and L_∞-norm result in linear optimization problems, whereas the L_2-norm produces a second-order cone problem (SOCP) [88]. The general form for a norm-based uncertainty set is $\mathcal{U} = \left\{\xi \in \mathbb{R}^M : \left\|\xi - \hat{\xi}\right\| \leq \alpha\right\}$ for parameter $\alpha \geq 0$, wherein $\hat{\xi} \in \mathcal{U}$ is the nominal scenario and the robust counterpart is dependent on the problem at hand [87].

2.2.3.2.3 Ellipsoidal uncertainty sets

Ellipsoidal uncertainty sets enable the DM to adjust the size of the ellipsoid using the parameter Ω. This allows the DM to trade off between optimality and robustness in the form of risk assessment. Ellipsoidal uncertainty sets are utilized by Ben-Tal and Nemirovski [75], [76], as well as El Ghaoui *et al.* [74], [94]. Further, Ben-Tal and Nemiroski [72] demonstrate that if an uncertainty set is ellipsoidal then different convex optimization problems and their robust counterparts are computationally tractable in polynomial time. Ben-Tal and Nemiroski [75] further present that the robust counterpart using ellipsoidal uncertainty sets is equivalent to an SOCP. The general form for ellipsoidal uncertainty sets is $\mathcal{U} = \left\{ \xi \in \mathbb{R}^M : \sqrt{\sum_{i=1}^{M} \left(\xi_i^2 / \sigma_i^2 \right)} \leq \Omega \right\}$ for parameter $\Omega \geq 0$ [87].

2.2.4 MOO and RO

Up until this point, the discussion of RO has been in the realm of single objective optimization. MOO coupled with RO is generally a new concept and one whose theoretical understanding is relatively new. Ide and Schobel [95] conduct a survey article of MOO and RO techniques and concepts, as well as an extensive summary of application areas in. Ehrgott *et al.* [96] provide an in depth survey article regarding minmax robustness for MOO problems. Kuchta [97] proposes a new robust methodology for solving MOO problems, which acts as a two-stage process. MOO and RO have been applied via the multi-expert, multi-criteria robust weighted sum approach, which minimizes the worst-case weighted sum of objectives [67]. Moreover, Kuchta [98] discusses a robust, bicriteria approach to project management.

2.2.4.1 RGP

A more recent development which utilizes MOO in concert with RO techniques is a methodology known as RGP [3]. As the name suggests, RGP couples GP techniques with RO ideals wherein a basic GP model is utilized with cardinality-constrained robustness and interval-based uncertainty sets. The GP formulation utilized by Kuchta [3] is depicted in (21)-(25) below:

$$Minimize \quad \sum_{i=1}^{m} w_i d_i^+ \tag{21}$$

$$subject\ to \quad \sum_{j=1}^{n} a_{ij} x_j - d_i^+ + d_i^- = t_i, \quad \forall\, i \in M, \tag{22}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{23}$$

$$\boldsymbol{x} \geq 0, \tag{24}$$

$$d_i^+, d_i^- \geq 0, \quad \forall\, i \in M. \tag{25}$$

Although Kuchta assumes that each weight $w_i = 1$, the general GP formulation compensates for different scales pertaining to constraints, preemption, or a hierarchy of goals. Furthermore, when considering the cardinality-constrained model seen in (13)-(17) as well as the GP formulation in (21)-(25), the below RGP model is generated.

$$Minimize \quad \sum_{i=1}^{m} d_i^+ \tag{26}$$

$$subject\ to \quad \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} p_{ij} + k_i z_i - d_i^+ + d_i^- = t_i, \ \ \forall\, i \in M, \tag{27}$$

$$z_i + p_{ij} \geq \sigma_{ij}x_j, \ \ \forall\, i \in M, \ \ j \in N, \tag{28}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{29}$$

$$x_j, d_i^+, d_i^-, p_{ij}, z_i \geq 0, \ \ \forall\ \ i \in M, \ \ j \in N. \tag{30}$$

Unlike the RO model Bertsimas and Sim [4], [5] present, instead of using Γ_i as a measure of robustness, Kuchta uses $K = (k_i)_{i=1}^{m}$ wherein $k_i, i = 1, \ldots, m$ is chosen by the DM and determines how many of the coefficients in the i^{th} constraint can change. Thus, k_i serves as the DM's source for choosing their level of conservatism. Kuchta [3] labels this the K-robust method. Another subtle difference is that Bertsimas and Sim do not assume integer values for Γ_i, whereas Kuchta does for k_i.

Regarding practical use, RGP has recently been applied to portfolio selection [93], supply chain network design [99], and routing problems [100].

2.3 Decision Analysis

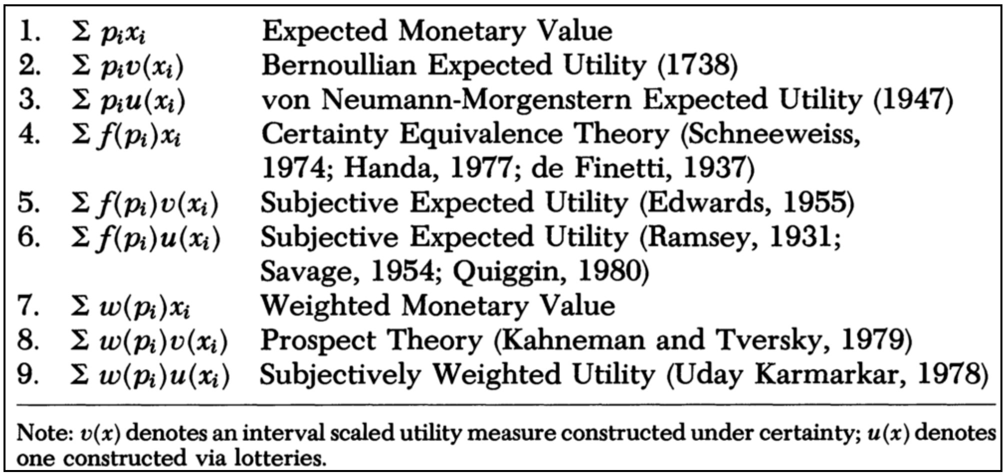
More times than not, individuals believe they are good DM's, or that they make competent decisions. However, individuals make poor decisions routinely, even after careful consideration of the outcomes. DA considers the theory and methodology of making decisions in a formal setting (i.e., DA establishes a discipline for decision making norms). As DA has been studied for several years, there are many facets that comprise the theoretical construct. These theories, along with a variety of utility functions and risk elicitation methods, will be discussed in this section.

2.3.1 Expected utility theory

There are generally two approaches when considering DA: prescriptive and descriptive. A prescriptive approach identifies actions as they *should be*, whereas a descriptive mindset identifies reality and actions as *they actually are*.

Evaluating a decision prescriptively often correlates with the precepts of expected utility theory (EUT), wherein the goal is to maximize expected utility. Bernoulli [101] first presented the idea of computing utilities for strategic purposes using EUT whereas von Neumann and Morgenstern [102] identified axioms, which form the conjectural foundation of EUT. These axioms govern how an individual *should* act when making choices involving some sort of risk or uncertainty. If an individual accepts the EUT axioms to be true, they also believe: (1) it is possible to construct a utility function to evaluate different outcomes or consequences, and (2) decisions should be made corresponding to maximizing expected utility [103]. The EUT axioms, sometimes referred to as the von Neumann and Morgenstern axioms, are briefly discussed in Clemen and Reilly [103], and more extensively in Mongin [104], wherein an interested reader is directed.

There has been much scrutiny regarding the EUT axioms' validity, with inconsistencies such as loss aversion [103], [105] and the Allais Paradox [106], [103], [107]. Because of this, many EUT variants have evolved, wherein the most predominately used methods are subjective EUT [104], [108], [109], [110], [111] and prospect theory [112], [104]. For a comprehensive list of EUT variants, see [101], [113] as well as Figure 2 below.



1.	$\Sigma\, p_i x_i$	Expected Monetary Value
2.	$\Sigma\, p_i v(x_i)$	Bernoullian Expected Utility (1738)
3.	$\Sigma\, p_i u(x_i)$	von Neumann-Morgenstern Expected Utility (1947)
4.	$\Sigma\, f(p_i) x_i$	Certainty Equivalence Theory (Schneeweiss, 1974; Handa, 1977; de Finetti, 1937)
5.	$\Sigma\, f(p_i) v(x_i)$	Subjective Expected Utility (Edwards, 1955)
6.	$\Sigma\, f(p_i) u(x_i)$	Subjective Expected Utility (Ramsey, 1931; Savage, 1954; Quiggin, 1980)
7.	$\Sigma\, w(p_i) x_i$	Weighted Monetary Value
8.	$\Sigma\, w(p_i) v(x_i)$	Prospect Theory (Kahneman and Tversky, 1979)
9.	$\Sigma\, w(p_i) u(x_i)$	Subjectively Weighted Utility (Uday Karmarkar, 1978)

Note: $v(x)$ denotes an interval scaled utility measure constructed under certainty; $u(x)$ denotes one constructed via lotteries.

Figure 2: Popular EUT Variants [113]

Even though there are many generalizations of EUT that either use a subset of the original EUT axioms, or different axioms altogether, the original EUT axioms are still considered at least as good per Hey and Orme [113]. Although EUT comes with "noise", it still outperforms most generalized EUT variants [113]. More specifically, Hey and Orme [113] conclude:

> "…we are tempted to conclude by saying that our study indicates that behavior can be reasonably well modelled as "EU plus noise". Perhaps we should now spend some time on thinking about the noise, rather than about even more alternatives to EU?"

Furthermore, Crosetto and Filippin [114] conclude that "…when it comes to measuring risk attitudes, the theoretical framework usually adopted to map the choices in the tasks is EUT."

2.3.2 Utility functions

In DA, decisions are typically made in the context of some sort of monetary gain or loss. However, a single monetary scale is sometimes misleading or does not provide enough information to make a decision. Because of this, utility functions are often derived and applied, especially when a decision involves uncertainty or risk. This section discusses the exponential and power utility functions, as they are the most widely accepted in practice. For a more comprehensive list of utility functions, please see [115], [116], [117], [118].

2.3.2.1 The exponential utility function

The most commonly applied utility function is the exponential utility function, as it has properties that make it easy to use and understand [119]. One such incentive to using the exponential utility function is that it follows satisfies the delta property. The delta property makes an individual’s initial wealth independent of their risk preference. As a result, if the delta property is applicable over a monetary interval, then an individual’s utility curve can be categorized using just one parameter (i.e., the risk aversion coefficient) over the same interval [120]. Being able to generate a utility function that approximates an individual’s risk preference over some bounded interval saves an enormous amount of time and makes the risk elicitation process that much simpler. Moreover, the delta property has simplifying implications on an individual’s buying and selling prices, as well as on the value of clairvoyance [120].

There are many different forms of the exponential utility function, wherein the most used form incorporates risk odds (i.e., risk attitude parameter, risk-aversion coefficient γ, or the risk tolerance ρ. Table 1 below demonstrates the relationship between risk odds, risk aversion, and risk tolerance and how they correspond to risk preference categorization. Of note, a risk-neutral individual has risk odds $r = 1$, denoted $r(1)$. When using the exponential utility function, it is common to let $r(1) = e^{\gamma}$, where $\gamma = ln(r(1))$ and $\rho = \frac{1}{\gamma} = \frac{1}{ln(r(1))}$ [120].

Table 1: Risk Categories for Different Parameters [120]

	Risk Preferring	Risk Neutral	Risk Averse
Risk Odds, r	$r < 1$	$r = 1$	$r > 1$
Risk Aversion, γ	$\gamma < 0$	$\gamma = 0$	$\gamma > 0$
Risk Tolerance, ρ	$\rho < 0$	$\rho = \infty$	$\rho > 0$

When using the risk-aversion coefficient, the general model is denoted in Equation (31), wherein x denotes payoff (or wealth) and a and b are constants [120]:

$$u(x) = a + be^{-(\gamma x)} \tag{31}$$

Another important feature the exponential utility function is that it incorporates a measure of absolute risk aversion, first presented by Pratt [121] and Arrow [122]. Given a twice differentiable Bernoulli utility function $u(\cdot)$ generally dealing with monetary *total* gains/losses, the Arrow-Pratt measure of absolute risk aversion is defined, wherein $c(x)$ is some function with respect to wealth x [121], [122]:

$$c(x) = \frac{-u''(x)}{u'(x)} \tag{32}$$

Using this absolute risk aversion measure, an individual’s risk preference can be determined given some wealth, or income value x. Yet when this measure of absolute risk aversion inhibits a person whose risk preference is independent of initial wealth (i.e., a deltaperson), the exponential utility function is often implemented as it exhibits constant absolute risk-aversion (CARA) [121], [122], [119].

2.3.2.2 Power utility function

Similar to the exponential utility function, the power utility function is also well received among decision theorists due primarily to the fact it can be utilized with the requirement to assess a single parameter. Furthermore, the power utility, otherwise known as the isoelastic utility function [115], is also popular due to the Arrow-Pratt measure of constant relative risk aversion, or CRRA. The Arrow-Pratt measure for CRRA is defined in Equation (33) below, wherein $c(x)$ is some function with respect to wealth x, and s is some constant with respect to x [121], [122]:

$$g(x) = xc(x) = \frac{-xu''(x)}{u'(x)} = s \tag{33}$$

Common forms of the power utility function are $u(x) = x^r$, $u(x) = x^{1-r}$, or even $u(x) = \frac{x^{1-r}}{(1-r)}$, as they meet the CRRA definition. These specific power utility functions are often used in conjunction with many risk elicitation methods.

2.3.3 Risk elicitation methods

Risk and uncertainty are inevitable in decision making. Moreover, accurately assessing a DM’s risk preference is important to develop individually-tailored DM recommendations.

Risk elicitation methods generally either incorporate a questionnaire or survey, a lottery, or both. The lottery method is the most prominent in the literature when estimating risk preferences wherein many variations are presented. The recent survey articles [123], [124] offer a well-rounded summary of the different elicitation methods as seen in the literature and in practice, as well as describing the various lottery formats. Although Harrison and Rutström [125] segregate risk elicitation methods into five different design types in, the following section aims to consolidate risk elicitation methods into two general categories: simple and complex. Furthermore, while there are many risk elicitation procedures, the procedures mentioned below are prevalent in the literature and the most commonly used in practice, as well as fast and generally easy to implement [124].

2.3.3.1 Simple designs

Simple elicitation designs are those that are "substantially easier for participants to understand" [123]. The simple designs of interest in this dissertation research are outlined below, but the interested reader is referred to [123], [124], [125] as well as the following list pertaining to: questionnaires [126], [127], [128], [129], the Gneezy and Potters method [130], [131], and the balloon analogue risk task [132], [133], [134], [135], [136], [137].

2.3.3.1.1 The bomb risk elicitation task

The bomb risk elicitation task (BRET) is an interactive, choice-based game with a goal of eliciting risk preferences in a simple fashion. The BRET, first examined by Crosetto and Filippin [6], presents its subjects with a 10x10 square, wherein each cell represents a box.

Of the 100 boxes, one box contains an imaginary bomb programmed to explode *after* the participant has selected the number of boxes they wish to collect. The BRET is conducted on a computer (although a static version is also available) and once the participant selects "Start", a box is collected for every second up until the participant hits the "Stop" button. The participant is credited with some constant monetary value after each box is collected. On the screen, the participant is notified of the amount of boxes opened, as well as the amount remaining along with a running total of the amount of money accumulated. The BRET interface can be seen in Figure 3.

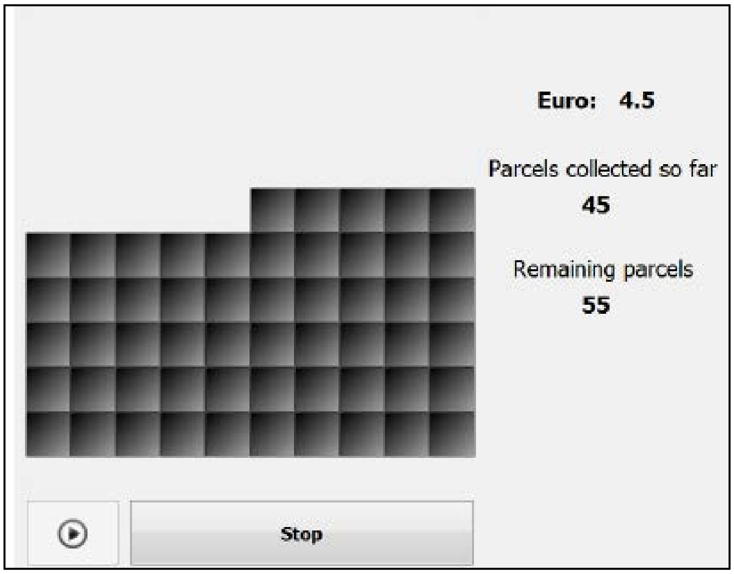


Figure 3: The BRET Interface After 45 Seconds [6]

At any time during the procedure, the participant can hit "Stop" and in doing so, the number of boxes collected, $q \in [0,100]$ is determined. Likewise, the position of the bomb, $b \in [1,100]$ is confirmed after k is known by drawing a number from $1 - 100$ from a hat. Defining q_i^* as the number of boxes collected for participant i, if $q_i^* < b$ then participant i is allowed to keep the accumulated earnings. But, if $q_i^* \geq b$ then participant

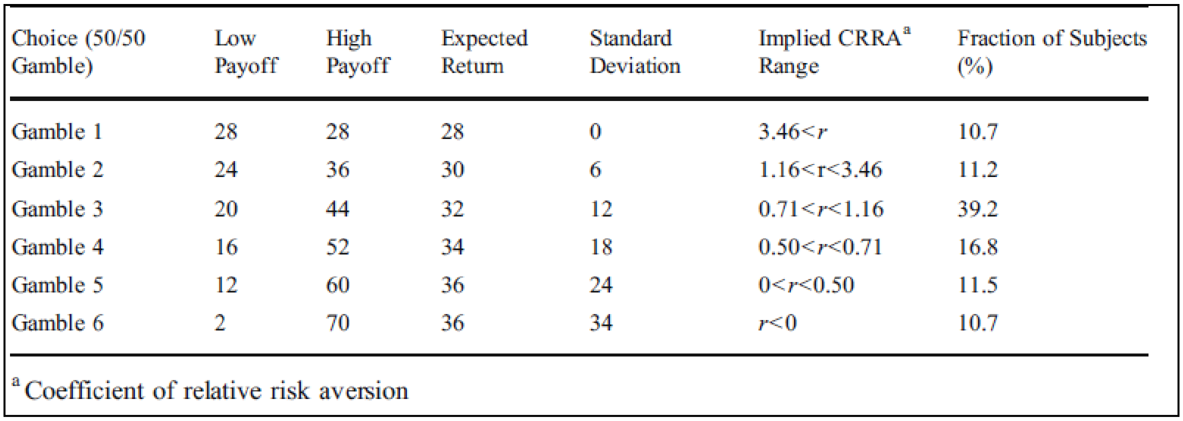
i collected a box containing the bomb and loses all earnings. Knowing the amount of boxes collected for each participant is the primary factor in predicting risk preference

Crosetto and Filippin [6] assume CRRA while using the utility function $u(x) = x^r$, where r is the coefficient of relative risk aversion and x corresponds to wealth. Consequently, the risk-neutral individual should choose to collect $k^* = 50$ boxes. Collecting less than 50 boxes summarizes someone who is risk-averse, whereas collecting more than 50 boxes signifies risk-seeking behavior. The corresponding BRET r-values can be viewed in Appendix A.

For a list of reasons to utilize the BRET method, as well as some of its disadvantages, an interested reader is deferred to [6], [124].

2.3.3.1.2 The Eckel and Grossman method

The Eckel and Grossman method [138] is considered an ordered lottery selection technique whose primary focus is to easily elicit an assortment of risk preferences in an attempt to estimate parameters for a utility function [123]. The ordered lottery forces the individual to make exactly one choice, wherein the probability for each lottery is 50/50. There has been variation in the actual number of lottery presented, but in general there is between 5-7 different gambles for the participants to choose from. However, because the original gamble structure presented in [138] does not differentiate risk-neutral and risk-seeking behavior, Figure 4 demonstrates an alternative gamble structure of the Eckel and Grossman method as presented in Dave *et al.* [139]:



Choice (50/50 Gamble)	Low Payoff	High Payoff	Expected Return	Standard Deviation	Implied CRRA[a] Range	Fraction of Subjects (%)
Gamble 1	28	28	28	0	$3.46<r$	10.7
Gamble 2	24	36	30	6	$1.16<r<3.46$	11.2
Gamble 3	20	44	32	12	$0.71<r<1.16$	39.2
Gamble 4	16	52	34	18	$0.50<r<0.71$	16.8
Gamble 5	12	60	36	24	$0<r<0.50$	11.5
Gamble 6	2	70	36	34	$r<0$	10.7

[a] Coefficient of relative risk aversion

Figure 4: Eckel and Grossman Alternate Lottery Structure [139]

Once the ordered lottery is completed and an individual has chosen one specific lottery, the Eckel and Grossman method estimates a risk parameter using the common CRRA assumption and the utility function $u(x) = x^{1-r}$, wherein r is the coefficient of relative risk aversion and x corresponds to wealth. Individuals with an $r > 0$ are considered risk-averse, whereas those individuals with an $r < 0$ or an $r = 0$ are classified as risk-seeking and risk-neutral, respectively. Once an individual chooses the lottery (or is indifferent between two lotteries), an interval (or specific) r-value is determined. The r-value intervals are outlined in Figure 4 above. As a result, the risk-averse individual should pick lotteries 1-4, whereas the risk-neutral and risk-seeking individuals should select lotteries 5 and 6, respectively.

An interested reader is directed to [140], [141], [139] for examples of when the Eckel and Grossman method is applied to practical problems sets.

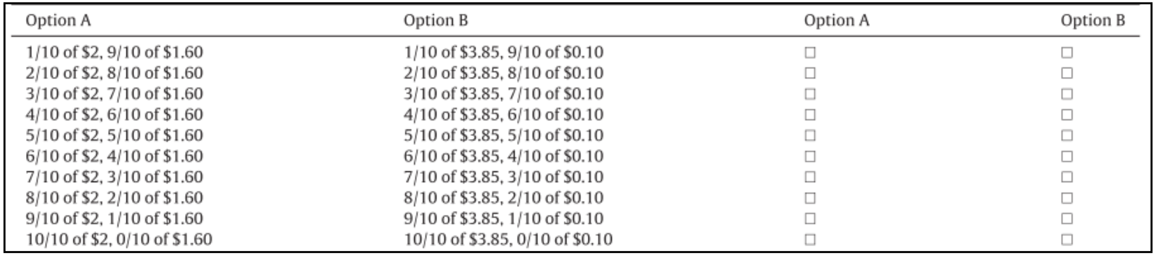
2.3.3.2 Complex designs

Complex designs can be difficult to understand, hard to implement, and might require substantial time to administer. Complex designs often ask individuals to answer a series of questions to help in eliciting an accurate risk assessment. Such complex designs, often synonymous with multiple price list methods (MPL), are described in this section.

2.3.3.2.1 Holt and Laury method

The most influential and widely-used MPL method is known as the Holt and Laury method [142]. The Holt and Laury method ask a participant to choose between two separate lotteries, ten different times, as presented in Table 2 below:

Table 2: Holt and Laury Method [123]



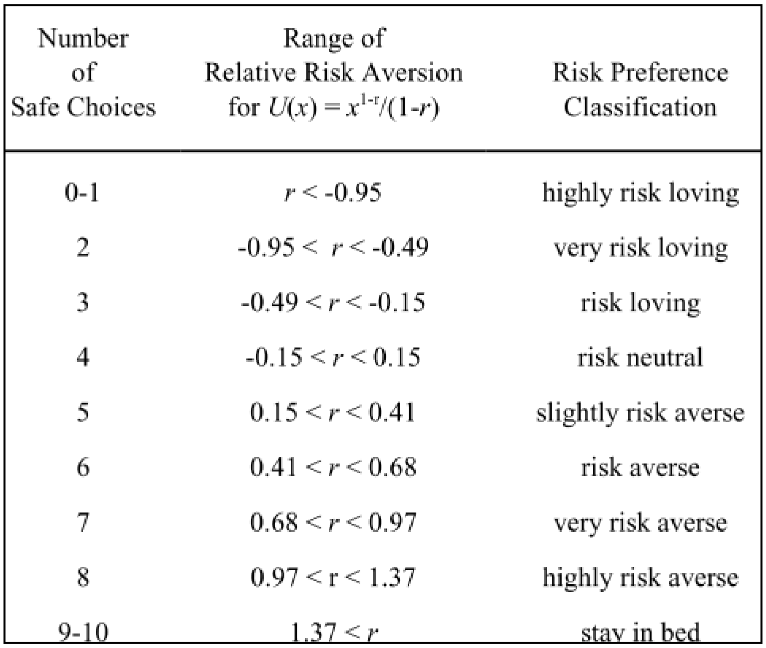
Option A	Option B	Option A	Option B
1/10 of \$2, 9/10 of \$1.60	1/10 of \$3.85, 9/10 of \$0.10	☐	☐
2/10 of \$2, 8/10 of \$1.60	2/10 of \$3.85, 8/10 of \$0.10	☐	☐
3/10 of \$2, 7/10 of \$1.60	3/10 of \$3.85, 7/10 of \$0.10	☐	☐
4/10 of \$2, 6/10 of \$1.60	4/10 of \$3.85, 6/10 of \$0.10	☐	☐
5/10 of \$2, 5/10 of \$1.60	5/10 of \$3.85, 5/10 of \$0.10	☐	☐
6/10 of \$2, 4/10 of \$1.60	6/10 of \$3.85, 4/10 of \$0.10	☐	☐
7/10 of \$2, 3/10 of \$1.60	7/10 of \$3.85, 3/10 of \$0.10	☐	☐
8/10 of \$2, 2/10 of \$1.60	8/10 of \$3.85, 2/10 of \$0.10	☐	☐
9/10 of \$2, 1/10 of \$1.60	9/10 of \$3.85, 1/10 of \$0.10	☐	☐
10/10 of \$2, 0/10 of \$1.60	10/10 of \$3.85, 0/10 of \$0.10	☐	☐

Once a decision has been rendered for all ten lotteries, one lottery is chosen at random and the individual is paid according to the chosen lottery. Since the participants are told beforehand that a random lottery will be chosen at the end of the procedure, their risk preferences are represented truthfully [123]. In fact, Azrieli *et al.* [143] demonstrates that selecting one lottery at random is the only way to theoretically use the Holt and Laury method properly.

The payoffs for lotteries A and B remain constant throughout the risk elicitation process, while the probabilities of the payoffs monotonically increase (decrease) for the higher payoff (lower payoff), when moving down each lotteries' respective row. The payoffs and probabilities for each lottery are constructed in such a way that a reasonable individual *should* have exactly one crossover point – the point at which the participant changes from lottery A to lottery B. This crossover point is then used as the basis for measuring the individuals risk preference.

Once a crossover point has been determined, the Holt and Laury method estimates risk coefficients using the CRRA assumption and the utility function $u(x) = \frac{x^{1-r}}{(1-r)}$, wherein r is the coefficient of relative risk aversion and x represents wealth. For example, if an individual chooses to switch over from lottery A to lottery B on the seventh lottery (ie. they choose lottery A for the first 6 lotteries), then their relative risk coefficient, r will fall in the interval $0.41 < r < 0.68$. The corresponding r values and risk preference classification can be seen in Table 3 where the "Number of Safe Choices" column corresponds to the number of times an individual selects lottery A [142]. Individuals with an $r > 0$ are considered risk-averse, and individuals with an $r < 0$ or an $r = 0$ are classified as risk-seeking and risk-neutral, respectively.

Table 3: Holt and Laury Relative Risk Aversion Interval Values and Risk Preference Classification [142]



Number of Safe Choices	Range of Relative Risk Aversion for $U(x) = x^{1-r}/(1-r)$	Risk Preference Classification
0-1	$r < -0.95$	highly risk loving
2	$-0.95 < r < -0.49$	very risk loving
3	$-0.49 < r < -0.15$	risk loving
4	$-0.15 < r < 0.15$	risk neutral
5	$0.15 < r < 0.41$	slightly risk averse
6	$0.41 < r < 0.68$	risk averse
7	$0.68 < r < 0.97$	very risk averse
8	$0.97 < r < 1.37$	highly risk averse
9-10	$1.37 < r$	stay in bed

For advantages and disadvantages of using the Holt and Laury method, please refer to Charness *et al.* [123] and Crosetto and Filippin [124]. Moreover, for variations of the Holt and Laury method, an interested reader is directed to [144], [145], [123], [124], [125].

2.4 Rate Setting

Rate setting is an extremely complex process that takes much time, discussion, and analysis to conduct. This is also evident in the literature.

Hummels [146] discusses how trade dating all the way back to the Industrial Revolution has been influenced by more than just declining costs. He shows how air shipping grew significantly as a direct result of technological advances from 1955 to 2004. Ocean shipping also experienced a technological revolution, but the corresponding price declines do not coincide with the technological advances. Hummels suggests that ocean shipping costs are somewhat driven by fuel and port costs, as well as port congestion. Hummels concludes by hypothesizing current and future eras of globalization that affect trade and overall shipping prices.

Shneerson [147] provides an extensive discussion on linear (sea) rates. He concludes in his work that liner freight rates can be explained via pricing and demand. Shneerson finds that the most important factors influencing liner rates are the ratio of volume to weight (stowage factor) and the unit value of the commodities being shipped. At the time, it was common practice to label cost as the most influential factor when setting liner rates, but Shneerson shows that the stowage factor instead was the most influential element. However, Brooks and Button [148] demonstrate in a later study that stowage factors can sometimes be a misleading variable when determining liner rates. The authors go on to discuss that the type of customer (such as freight forwarder, consignee, or shipper) can play a significant role in requirement variation on liner rate setting. Moreover, Brooks and Button [148] provide a good discussion on how aggregation of rates can be extremely faulty as the direction of a shipping route directly influences the shipping rate set.

Forkenbrock [149] discusses the notion of external costs for freight trains. He compares the external and private, or direct, costs of freight truckload trucking to that of freight trains. Forkenbrock finds that on a per-ton-mile basis trucking generates up to three times as much external cost than freight trains, where the external cost relative to direct cost is much large for freight trains. He concludes that external costs, such as accidents, emissions, and noise, should be considered in the construction of transportation policy.

Joo *et al.* [150] discover that the direct costs for shipping are dependent upon the shipper's individual contract, whereas the variable costs are remain unchanged, regardless of the contract. The authors discuss that shippers should be aware of unpredictable transportation pricing and prepared to negotiate freight rates early on. To that end, Joo *et al.* [150] conclude that success in the shipping industry is a direct consequence of understanding transportation cost structures, while scaling freight rates at a reasonable price.

Lastly, Skinner *et al.* [151] provide econometric modeling results which demonstrates the amount of variability in rate setting can be explained as follows: 75-80% to distance (e.g., OD pair), 8-10% to geography, and 1-3% to volume.

III. Robust Goal Programming using Different Robustness Echelons via Norm-based and Ellipsoidal Uncertainty Sets

3.1 Introduction

Kutcha [3] presents the conceptual framework for robust goal programming (RGP). In this work, the concept of cardinality-constrained robustness using interval-based uncertainty sets is considered in the realm of goal programming (GP). The formulation presented builds upon prior robust optimization (RO) models in the literature. However, this formulation only considers one type of uncertainty set: interval-based. This paper presents a variant of Kuchta's RGP formulation using norm-based uncertainty in an attempt to make the RGP technique more adaptable to different decision maker (DM) preferences. Moreover, strict robustness using ellipsoidal uncertainty sets under RGP ideals and corresponding results are discussed.

3.1.1 Robust and stochastic optimization

In real world application problems, data uncertainty is a guarantee that, if ignored, can lead to inaccurate results, inefficiencies, or even lives lost. Because of this, garnering robust solutions to problems having inherent uncertainty in parametric data has been notably explored over the last two decades. Sensitivity analysis takes into account uncertain data, but it does so after the fact, or *a posteriori*. Although worthwhile, sensitivity analysis fails in comparison to *a priori* methods that account for uncertain data before developing a solution, i.e., methods such as stochastic optimization (SO) and RO.

SO assumes that the uncertainty in data can be accessed via probability distributions. The concepts of SO can be traced to work by Dantzig [62]. The interested reader is referred to the survey article by Birge [65] for a thorough review of early SO developments. One of the largest applications for which SO is utilized is the field of financial planning and control, wherein financial decision making problems are modeled as stochastic programs [64]. SO has also recently been used regarding capacity expansion where the optimal level of investment and timing are determined to meet future demand [64]. SO has also been applied in the nonlinear programming realm concerning design for manufacturing quality [64]. Even though SO is widely used, it is often difficult to implement for two reasons: (1) finding a true probability distribution for uncertain data is extremely challenging and can be misleading, and (2) as the number of potential scenarios increase, the SO model becomes combinatorically and computationally intractable [4]. Because of this, RO is a preferred technique and has been at the forefront of the research effort.

In contrast to SO, RO does not utilize probability distributions. RO instead explains uncertain data via set theory or uncertainty sets. RO techniques were first introduced by Soyster [2] when he presented the concept of minimax robustness, otherwise known as strict robustness, via inexact linear programming (LP) and set theory. It was not until the late 1990's, nearly 25 years after Soyster's seminal article, that an RO theoretical framework was established with the well-known articles published by Ben-Tal, Nemirovski, El Ghaoui, and various authors. More specifically, El Ghaoui *et al.* [74] consider least-square problems whose constraint matrix and right-hand side vector coefficients are subject to a bounded uncertainty, and they demonstrate how to minimize upper bounds on the optimal worst-case scenario. Similarly, El Ghaoui *et al.* [94] also consider data that is bounded above and below by some deterministic perturbation. They use semidefinite programs, or SDPs, to find robust solutions that minimize the worst-case objective function value while satisfying all constraints, and they justify the necessary and sufficient conditions for a unique, robust solution. Soon thereafter, Ben-Tal and Nemirovski [72] studied convex optimization problems for which the data is subject to an uncertainty set but constraints must still be met – an optimization problem they call 'robust optimization'. Using ellipsoidal uncertainty sets, Ben-Tal and Nemirovski [72] demonstrated that the robust convex optimization problem is either exactly or approximately tractable. Furthermore, Ben-Tal and Nemirovski [75] provided a relationship for interval-based and ellipsoidal uncertainty sets and showed that the robust counterpart to a convex, linear program with ellipsoidal uncertainty yields a conic quadratic program. Ben-Tal and Nemirovski [76] demonstrated various RO methodologies for 90 different LP instances and reveal robust solutions do not always sacrifice optimality for feasibility. Bertsimas *et al.* [88] examined a robust counterpart involving norm-based uncertainty sets. An extensive summary of these various RO techniques can be seen in the book by Ben-Tal *et al.* [66]. The aforementioned RO research expanded the foundation for strict robustness and introduced opportunities for less conservative robustness techniques [87], [152].

RO is widely used in practice. It has been utilized in conjunction with combinatorial optimization to address popular problems such as the minimum spanning tree, the travelling salesman, and the minimum assignment, to name a few [4]. Moreover, Bertsimas and Sim [4] described how RO is used in the shortest path problem and application problems for logistic planning or telecommunications. Gülpınar *et al.* [77] demonstrated how RO can be applied to facility location problems having uncertain demand. Remlil and Rekik [153] showed how RO can be implemented in the winner determination problem to govern winning bids in transportation services. A variation of RO, RGP is seen in Ghahtarani *et al.* [93] for portfolio selection and financial decision making. The RO literature is rich and diverse, and we refer the reader to Mulvey *et al.* [73] for an extensive survey of RO applications.

3.1.2 Echelons of robustness

There are many echelons of robustness that account for the risk preference of a DM. This paper focuses on both strict and cardinality-constrained robustness. However, other robust degrees of conservatism include, light, recoverable, and regret. The interested reader is directed to Goerigk and Schöbel [87] and Bertsimas *et al.* [152] for a comprehensive summary of each echelon of robustness.

3.1.2.1 Strict robustness

The highest level of robustness is strict robustness, wherein a DM believes the worst-case for every scenario will occur – hence the alternative title minimax robustness. The general concept is to identify an optimal solution that is feasible to a convex set, regardless of the deviation that might occur due to a bounded amount of uncertainty in the parameters. Visionary at the time, Soyster's [2] strict robustness methodology is the most conservative, risk-averse robust approach known in the literature, favoring feasibility for any possible uncertainty in data strictly over optimality when less variability may be realized. Although helpful for some problem instances, such an aggressively conservative approach often sacrifices optimality to ensure feasibility. The interested reader is directed to Soyster [2] for specific details regarding the strict robustness formulation.

3.1.2.2 Cardinality-constrained robustness

Bertsimas and Sim [4], [5] use strict robustness ideals to derive a less conservative RO formulation known today as cardinality-constrained robustness, predicated on reducing the size of the interval-based uncertainty set. The authors claim that it is unlikely for all parametric coefficients within one constraint to change simultaneously to their worst-case scenario values, and they instead offer that only a subset of them may change for a realization. Thus, this robustness methodology, sometimes referred to as $\mathit{\Gamma}$-robustness, restricts the number of coefficients that are allowed to change at any one time. For every constraint $i = 0,1,\ldots,m$, there is a corresponding robustness measure Γ_i that takes on values in the interval $[0,|J_i|]$, where $J_i = \{j|\hat{a}_{ij} > 0\}$. The parameter Γ_i is used to adjust robustness in terms of conservatism, whereas J_i are the set of coefficients $a_{ij}, \forall\, j \in J_i$ that are vulnerable to parameter uncertainty in the interval $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$. Bertsimas and Sim [4], [5] justify that it is not probable that every $a_{ij}, \forall\, j \in J_i$ take on uncertainty. In the case wherein each Γ_i is not restricted to integer values, up to $\lfloor\Gamma_i\rfloor$ of the coefficients will change at their maximum allowable deviation, and exactly one coefficient $\hat{a}_{it}$ will change by at most $(\Gamma_i - \lfloor\Gamma_i\rfloor)\hat{a}_{it}$. When the Γ_i values are integer-valued, every coefficient that deviates from their nominal value will do so at a maximum allowable value (i.e., $(\Gamma_i - \lfloor\Gamma_i\rfloor)\hat{a}_{it} = 0$). It is interesting to note that if $\Gamma_i = |J_i|$, the solution is equivalent to Soyster's strict robustness method. Bertsimas and Sim [4], [5] formulate a linear, robust counterpart for cardinality constrained robustness which serves as the foundation for the RGP model used by Kuchta [3], which is presented in Section 3.2.2.

3.1.3 Uncertainty sets

Regardless of the level of robustness chosen, the uncertainty in the data must be represented via an uncertainty set. Interval-based and ellipsoidal uncertainty sets are the main focus for this paper, but this is not an inclusive list. Finite, polytopic, and constraint-wise uncertainty sets are also viable options, as discussed by Goerigk and Schöbel [87].

3.1.3.1 Interval-based uncertainty sets

Interval-based uncertainty sets are widely used when accounting for possible data perturbations. As noted in Bertsimas and Sim [4], [5], the distribution of uncertainty for the objective function coefficient vector and constraint matrix are unknown. Let $N = \{1, \dots, n\}$ be the set of indices for the decision variables and $M = \{1, \dots, m\}$ be the set of indices for the constraints, then each entry $a_{ij} \in A, \forall\, j \in N, i \in M$, is modeled as an independent and bounded random variable $\tilde{a}_{ij}, \forall\, j \in N, i \in M$, that is symmetric in value by $\left[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}\right]$. The model for uncertainty for the objective function coefficient vector $\tilde{c}_j$, $\forall\, j \in N$, takes on values within the bounds $\left[c_j, c_j + d_j\right]$, where d_j is the possible deviation from the nominal value of the objective coefficients. When data is subject to uncertainty in this manner, the uncertainty set is regarded as interval-based.

3.1.3.2 Norm-based uncertainty sets

Norm-based uncertainty sets have been widely used throughout the literature within robust optimization formulations. Although norm-based uncertainty sets and their application can be seen in Goerigk and Schöbel [87] and Bertsimas *et al.* [152], the main source of knowledge is in Bertsimas *et al.* [88]. In particular, the authors show that robust linear programming problems with norm-based uncertainty yield convex programming problems with constraints related to the dual norm. Moreover, the authors show the dual norm of the dual norm to be the original norm, in addition to the robust counterparts of norm-based uncertainty sets.

Of interest to this paper are norms in the form of the L_p-norm, specifically the L_1-norm and the L_2-norm. The L_1-norm is commonly referred to as the "taxicab" norm, Similarly, the L_2-norm can be used as an uncertainty set, but due to its quadratic nature, the resulting problem is a second-order cone problem. It is typically called the Euclidian norm.

When accounting for the robust counterpart using each norm, the amount of allowable deviation ($\hat{a}_{ij}$) from each coefficient's nominal value must be considered [88]. The robust counterparts for the L_1-norm and L_2-norm are as follows, respectively:

$$\sum_{j=1}^{n} \left|\hat{a}_{ij} x_j\right|, \quad \forall\, i \in M \tag{34}$$

$$\sqrt{\sum_{j=1}^{n} \left|\hat{a}_{ij} x_j\right|^2}, \quad \forall\, i \in M \tag{35}$$

These robust counterparts, considered in combination with RGP principles, will be addressed later in Section 3.2.3 to account for data uncertainty.

3.1.3.3 Ellipsoidal uncertainty sets

To reduce for the aggressive conservatism inherent in Soyster's [2] method, the use of ellipsoidal uncertainty sets (i.e., the L_2-norm) is discussed in [74], [94], [76]. The survey articles by Goerigk and Schöbel [87] and Bertsimas *et al.* [152] also provide some insight into ellipsoidal uncertainty sets as well as where-and-when they can be applied. Though ellipsoidal uncertainty sets can be applied to different levels of robustness, for the sake of this paper it will be applied to strict robustness.

Ben-Tal and Nemirovski [75] provide theoretical contributions regarding ellipsoidal uncertainty sets. The basic premise concerning ellipsoidal uncertainty sets allows the DM to control the size of the ellipsoid by changing a parameter θ_i, which is a parametric measure of the DM's risk tolerance. As θ_i increases, so too does the risk aversion of the DM. However, this value of θ_i also allows less-than-maximum or more-than-maximum deviation to occur in relation to interval-based uncertainty sets.

Ben-Tal and Nemirovski [75] show that, given some nominal value p_i^* with an allowable "uncertainty interval" $\Delta_i = [p_i^* - \sigma_i, p_i^* + \sigma_i]$, the uncertainty set is described by the box $B = \{(p_1, \dots, p_n) \mid |p_i - p_i^*| \leq \sigma_i, i = 1, \dots, n\}$, which is precisely the strict robustness approach offered in Soyster [2]. In this case, the box B includes all scenarios which might occur and is equivalent to an interval-based uncertainty set. This is not true for ellipsoidal uncertainty sets. Ben-Tal and Nemirovski [75] demonstrate that the largest volume ellipsoid *contained in* the box B exists when $\theta_i = 1$ and the smallest volume ellipsoid *containing* the box B occurs when $\theta_i = \sqrt{n}$. Although it is possible to have $\theta_i >$

$\sqrt{n}$, we will assume without loss of generality (w.l.o.g.) in this paper that $0 \leq \theta_i \leq \sqrt{n},\ \forall i = 1, \dots, n$.

Figures 5-7 illustrate the feasible regions for different values of θ_i, respecitvely when comparing interval-based and ellipsoidal uncertainty sets. Figure 5 shows the smallest ellipsoid contained in the box B, when $\theta_i = 1$. The center of the ellipsoid is the nominal solution, when no deviation occurs (i.e., $\theta_i = 0$). The further from the center of the ellipse, the worse the solution is with regard to parametric variation. Thus, when $\theta_i \leq 1$, the ellipsoidal uncertainty set will *always* generate solutions that are also in the interval-based uncertainty set described by the box B, but the converse is not true. The shaded region shows solutions that can be generated using the interval-based uncertainty set, but not the ellipsoidal uncertainty set when $\theta_i \leq 1$.

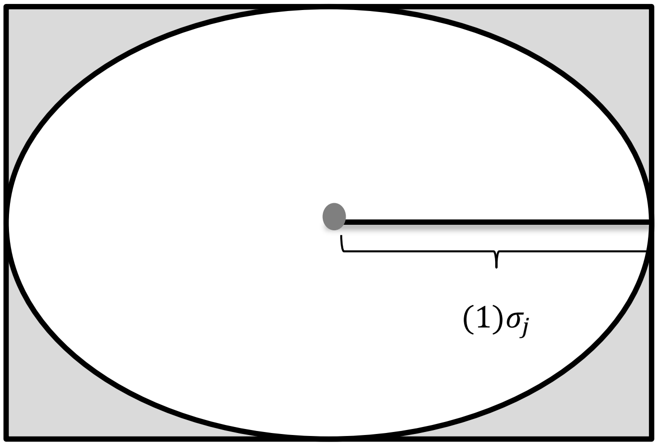


Figure 5: Interval versus Ellipsoidal Uncertainty Sets, $\theta \leq 1$

However, as Figure 6 illustrates, when $1 < \theta_i < \sqrt{n}$, there are solutions that are feasible to the interval-based uncertainty set B that are not feasible to the ellipsoidal uncertainty set, where the converse is also true.

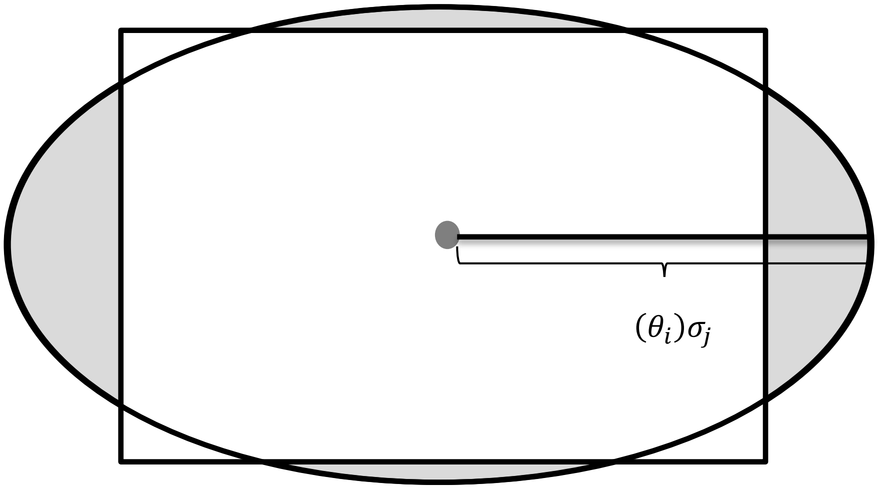


Figure 6: Interval versus Ellipsoidal Uncertainty Sets, $\mathbf{1} < \boldsymbol{\theta} < \sqrt{\boldsymbol{n}}$

Likewise, when $\theta_i = \sqrt{n}$, the box B is bounded by the ellipse where every possible solution in the box B is contained within the ellipse. The shaded region in Figure 7 denotes the feasible region for the ellipsoidal uncertainty set that is not feasible to the interval uncertainty set.

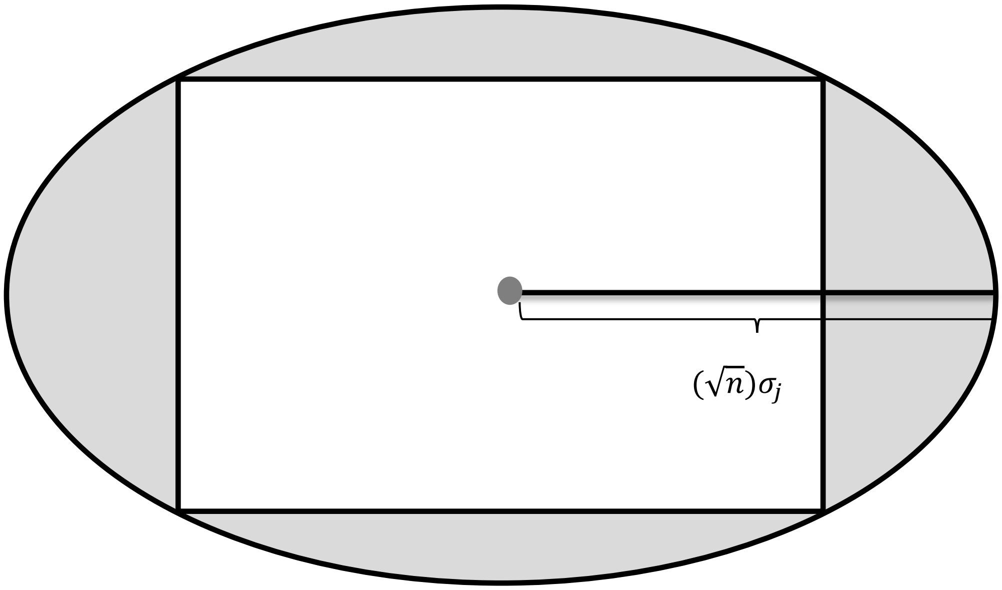


Figure 7: Interval versus Ellipsoidal Uncertainty Sets, $\boldsymbol{\theta} = \sqrt{\boldsymbol{n}}$

In order to properly represent the ellipsoidal uncertainty set in terms of robustness, a robust counterpart is required. Assuming θ_i is the same for each constraint (i.e., θ), Ben-Tal and Nemirovski [75] provide the following ellipsoidal robust counterpart for a Markovitz portfolio optimization formulation::

$$Maximize \quad \sum_{i=1}^{m} p_i^* x_i - \theta\left(\sqrt{\sum_{i=1}^{m}\left|\sigma_i^2 x_i^2\right|}\right) \tag{36}$$

$$subject\ to \ \ \boldsymbol{Ax} = \boldsymbol{b} \tag{37}$$

The objective function (36) maximizes the expected return of the portfolio, $\sum_{i=1}^{m} p_i^* x_i$, as perturbed by uncertainty induced by the portfolio's variance, given the per unit mean and standard deviation on each investment vehicle, p_i^* and σ_i, as well as the robustness parameter θ. Constraint (37) enforces side constraints on investments.

3.1.4 Goal programming

GP is a subset of multi-objective optimization (MOO). MOO is a broad field of optimization that includes *a priori, a posteriori,* and no articulation methods. GP is just one of many different techniques that comprise MOO solution methods, and Marler and Arora [8] provide an excellent overview of MOO practices and their application areas.

The basic premise of GP dates back to 1955 when Charnes *et al.* [39] studied executive compensation, wherein the concept of minimizing deviation d_j away from a goal t_j for each function $F_j(x)$ was first presented. Seminal articles regarding GP can also be seen in [40], [41], [42]. The objective of GP is to minimize the total absolute deviation from a set of goals $\sum_{j=1}^{k}|d_j|$. However, the absolute value function can present computational challenges, and thus the total deviation is often presented as the positive and negative components of deviation from a goal, such as $d_j = d_j^+ - d_j^-$ where $d_j^+ \geq 0, d_j^- \geq 0$, and $d_j^+ d_j^- = 0$ ensures only one component is non-zero valued [8]. In this context, d_j^+ and d_j^- represent overachievement and underachievement, respectively, for each j^{th} objective.

Different variations of GP have been presented in the literature that combines GP with other methodologies. GP has been utilized in conjunction with assigning relative weights to each deviational goal, called Archimedean GP or simply weighted GP as first presented by Charnes and Cooper [43]. Additionally, Charnes and Cooper [43] also set forth the concept of preemptive (or lexicographic) GP, wherein the deviational goals are ordered in terms of priority and minimized lexicographically. Other methods that incorporate GP are multigoal programming [44], goal attainment methods via a weighted minimax approach [25], and reference GP [45].

3.1.5 RO used with SO, MOO

The tractability of RO in different problem settings has made it very popular in the last 15 years, as documented in recent survey articles [71], [154]. The effectiveness of RO is far-reaching, as it has been used in conjunction with SO [67], [155], as well as MOO problems [156], [96], [67], [157], [97].

3.1.5.1 RGP

RGP is first introduced in the literature by Kuchta [3]. The RGP construct uses interval-uncertainty sets with cardinality-constrained robustness in terms of GP. Kuchta applies the RGP methodology to a toy problem, for which the worst possible deviation to the optimal solution for different scenarios is offered. In 2013, the Kuchta model was applied to portfolio selection by Ghahtarani and Najafi [93]. More recent RGP literature can be seen for supply chain management [99], as well as for an intermodal routing problem [100]. Otherwise, the conceptual foundation of RGP has not been researched, to the knowledge of these authors.

3.1.6 Major contributions

This paper's contributions are three-fold: (1) to provide three conceptually-motivated RGP formulations that vary in the way data uncertainty is modeled, (2) relate our work to the initial RGP formulation introduced by Kuchta [3], where we identify one of our methods to be equivalent to Kuchta's model, and (3) compare the parameter-induced sizes of the respective formulations for the suitability of each method, depending on the DM's risk preferences and the computational capability available. Beyond the theoretical contributions, this work also provides a comparison of the varying RGP models as applied to the instance described by Kuchta [3].

3.1.7 Structure of Paper

Section 3.2 discusses model formulations using cardinality-constrained robustness via interval-based (Kuchta's method) and norm-based uncertainty sets in addition to a strict robustness formulation regarding ellipsoidal uncertainty sets. Section 3.3 applies each of these techniques to solve a numerical example set forth by Kuchta [3] and provides comparative analysis of the results. Conclusions, recommendations, and suggestions for potential future work are discussed in Section 3.4.

3.2 Model Formulation

To better follow what is traditionally presented in the literature, the below definitions are offered. Kuchta's model will be presented using the definitions stated, followed by norm-based uncertainty set models pertaining to cardinality-constrained robustness in the terms of RGP. Finally, a strict robustness model using ellipsoidal uncertainty sets is discussed in the context of RGP.

3.2.1 Notation

Prior to setting fourth the various formulations we examine, it is necessary to introduce the related sets, parameters, and decision variables. The notation is defined as follows:

Sets

- M: set of all goals, $i = 1, \dots, m$.
- N: index on all primary decision variables, $j = 1, \dots, n$.
- S_i: set of indices of all coefficients subject to deviation in goal, $i = 1, \dots, m$, where $S_i \subseteq N$.

Note that our use of i and j to respectively index sets N and M necessarily departs from their usage in previous literature [75] to accommodate for modeling of multiple goals and decision variables.

Parameters

- w_i: weight attributed to each goal, $i = 1, \ldots, m$.
- t_i: target values for goal, $i = 1, \ldots, m$.
- a_{ij}: nominal value corresponding to each goal, $i = 1, \ldots, m$ and primary decision variable, $j = 1, \ldots, n$.
- k_i: number of products for each goal, $i = 1, \ldots, m,$ subject to deviation (uncertainty), where $k_i \leq n$.
- f_i: remainder value when k_i is non-integer, such that $f_i = k_i - \lfloor k_i \rfloor$, $i = 1, \ldots, m,$ where $k_i \leq n$.
- σ_{ij}: maximum deviation corresponding to each $a_{ij}, i = 1, \ldots, m,\ j = 1, \ldots, n$.
- θ_i: DM risk assessment parameter for each goal, $i = 1, \ldots, m$.

Decision variables

- x_j: amount of decision variable j produced, $j = 1, \ldots, n$.
- t_i: target value, $i = 1, \ldots, m$.
- d_i^+: amount of deviation above target value, $i = 1, \ldots, m$.
- d_i^- :amount of deviation below target value, $i = 1, \ldots, m$.
- p_{ij}: penalty associated with each goal, $i = 1, \ldots, m$ and decision variable, $j = 1, \ldots, n$ (included only for completion concerning Kuchta’s method formulation).
- z_i: scalar value that takes into account p_{ij} (included only for completion concerning Kuchta’s method formulation).
- p_i: penalty associated with each goal, $i = 1, \ldots, m$.

3.2.2 Cardinality-constrained robustness via interval uncertainty (Kuchta's model)

The GP model examined by Kuchta [3] is presented in (38)-(42). As many GP model formulation variations can be found in the literature, it should be noted that Kuchta does not examine absolute deviation as previously introduced; his formulation minimizes the weighted sum of positive deviations from goals [3].

$$Minimize \quad \sum_{i=1}^{m} w_i d_i^+ \tag{38}$$

$$subject\ to \quad \sum_{j=1}^{n} a_{ij} x_j - d_i^+ + d_i^- = t_i, \quad \forall\, i \in M, \tag{39}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{40}$$

$$\boldsymbol{x} \geq 0, \tag{41}$$

$$d_i^+, d_i^- \geq 0, \quad \forall\, i \in M. \tag{42}$$

The objective function (38) seeks to minimize the total weighted overachievement for goals $i \in M$. Although the objective function (38) accounts for varying goal importance via w_i, it is assumed w.l.o.g. for testing herein that $w_i = 1, \forall\, i \in M$ (i.e., each goal is of equal importance). Constraint (39) calculates, for a given solution $x_j, \forall\, j \in N$, the respective positive and negative deviations from each goal $i \in M$. Constraint (40) enforces constraints not related to the decision maker's goals, and Constraints (41) and (42) enforce non-negativity restrictions on the decision variables.

Kuchta [3] also introduces the concept of a K-robust solution, which is based on the theory of cardinality-constrained robustness. Instead of using Γ_i as a measure of robustness, Kuchta uses $K = (k_i)_{i=1}^{m}$ where $k_i, i = 1, \dots, m$ are integer values that cannot exceed n, assuming each coefficient in each constraint is subject to deviation. The parameter k_i is chosen by the DM and determines how many of the coefficients in the i^{th} constraint can change. Thus, k_i serves as the DM's source for choosing their level of conservatism.

Kuchta [3] assumes that the only coefficients taking on deviation are the cost coefficients in the original objective function. Thus, the interval of uncertainty is one that takes into account deviation that will negatively influence the attainment of a goal, namely $[a_{ij} + \sigma_{ij}]$. Kuchta [3] assumes that the coefficients susceptible to deviation are only goals, not coefficients contained in $\boldsymbol{Ax} = \boldsymbol{b}$ (i.e., the hard constraints will not deviate).

Applying Bertsimas and Sim's [4], [5] model, Kuchta arrives at the robust counterpart of the K-robust methodology, seen below [3]:

$$Minimize \quad \sum_{i=1}^{m} d_i^+ \tag{43}$$

$$subject\ to \quad \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} p_{ij} + k_i z_i - d_i^+ + d_i^- = t_i, \quad \forall\, i \in M, \tag{44}$$

$$z_i + p_{ij} \geq \sigma_{ij}x_j, \quad \forall\, i \in M, \ \ j \in N, \tag{45}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{46}$$

$$x_j, d_i^+, d_i^-, p_{ij}, z_i \geq 0, \;\; \forall \;\; i \in M, \;\; j \in N. \tag{47}$$

The objective function (43) minimizes the sum of positive deviations from goals $i \in M$. Constraint (44) calculates the respective positive and negative deviations from each goal for a given solution $x_j, \forall \;\; j \in N$, and the maximum variability it induces, $\sum_{j=1}^{n} p_{ij} + k_i z_i$, given a robustness parameter k_i, and which is bounded for each combination of decision variable and goal via Constraint (45). Finally, Constraints (46) and (47) enforce non-negativity restrictions on all decision variables.

Kutcha's method is computationally tractable as the number of constraints is $(m + mn)$, regardless of the number of coefficients subject to deviation. However, as stated before, Kuchta's method only allows for interval-based uncertainty, limiting its utility for a DM.

3.2.3 Cardinality-constrained robustness via norm-based uncertainty sets

Consider below the L_1-norm and L_2-norm RGP formulations in (48)-(52) and (53)-(57), respectively:

$$Minimize \;\; \sum_{i=1}^{m} d_i^+ \tag{48}$$

$$subject\ to \;\; \sum_{j=1}^{n} a_{ij} x_j + p_i - d_i^+ + d_i^- = t_i, \;\; \forall \, i \in M, \tag{49}$$

$$p_i \geq \sum_{j \in S_i} \left|\sigma_{ij} x_j\right| + \left|f_i \sigma_{iq} x_q\right|, \;\; \forall \, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i, i \in M, \tag{50}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{51}$$

$$x_j, d_i^+, d_i^- p_i \geq 0, \;\; \forall\, i \in M, \;\; j \in N. \tag{52}$$

Similar to Kuchta's formulation, the objective function (48) minimizes the sum of positive deviations from goals $i \in M$, and Constraint (49) calculates the positive and negative deviations (i.e., d_i^+ and d_i^-) from each goal. In contrast to Kuchta's formulation, the total induced variability is represented by p_i for each goal within Constraint (49). Constraint (50) applies L_1-norm uncertainty sets as discussed earlier in Equation (34) to the possible sets of coefficients subject to parametric uncertainty (i.e., $S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i$), where the least upper bound (i.e., infimum) penalty, $p_i, \forall\, i \in M$ is calculated for every combination of $\sum_{j \in S_i} \left|\sigma_{ij} x_j\right| + \left|f_i \sigma_{iq} x_q\right|$. Constraint (51) represents additional constraints on the decision space that are neither related to the DM's goals nor considered with respect to possible parametric uncertainty. Finally, Constraint (52) ensures all decision variables are non-negative.

$$Minimize \sum_{i=1}^{m} d_i^+ \tag{53}$$

$$subject\ to \;\sum_{j=1}^{n} a_{ij} x_j + p_i - d_i^+ + d_i^- = t_i, \;\; \forall\, i \in M, \tag{54}$$

$$p_i \geq \sqrt{\sum_{j \in S_i} \left(\sigma_{ij} x_j\right)^2 + \left(f_i \sigma_{iq} x_q\right)^2}, \;\; \forall\, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i, i \in M, \tag{55}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{56}$$

$$x_j, d_i^+, d_i^- p_i \geq 0, \;\; \forall\, i \in M, \;\; j \in N. \tag{57}$$

The only difference between the L_2-norm and L_1-norm formulations is the imposition of Constraint (55) in lieu of Constraint (50). Constraint (55) utilizes L_2-norm uncertainty sets as presented in Equation (35), wherein the penalty $p_i, \forall\, i \in M$ is induced via the greatest lower bound over every combination of possible subsets of variables taking on uncertainty, for each goal.

Unlike Kuchta's RGP formulation, the computational tractability of the RGP formulations presented in (48)-(52) and (53)-(57) is not assured because their complexity is directly influenced by the number of coefficients subject to deviation. For instance, the number of constraints needed for Kuchta's method is independent of k_i, whereas the number of constraints required for the L_1-norm and L_2-norm RGP formulations both rely upon the k_i-values. Moreover, the L_1-norm and L_2-norm RGP formulations result in nonlinearities, which also increase complexity. Nevertheless, Formulations (48)-(52) and (53)-(57) broaden the scope of the RGP construct, and the utilization of different uncertainty sets allows more problem types to be explored. In Section 3.3, we present and compare the robust solutions generated via a numerical example.

3.2.4 Strict robustness using ellipsoidal uncertainty sets

Using strict robustness techniques via ellipsoidal uncertainty sets, the following RGP formulation results, in which positive deviations are collectively minimized.

$$Minimize \quad \sum_{i=1}^{m} d_i^+ \tag{58}$$

$$subject\ to \quad \sum_{j=1}^{n} a_{ij}x_j + \theta_i \left(\sqrt{\sum_{j=1}^{n} \sigma_{ij}^2 x_j^2} \right) - d_i^+ + d_i^- = t_i, \ \ \forall\ i \in M, \tag{59}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{60}$$

$$0 \le \theta_i \le \sqrt{n}, \ \ \forall\ i \in M, \tag{61}$$

$$x_j, d_i^+, d_i^- \ge 0, \ \ \forall\ i \in M, \ \ j \in N. \tag{62}$$

The differences between this formulation and the cardinality constraint L_2-norm formulation occur in Constraints (59) and (61). Constraint (59) calculates the positive and negative deviations (i.e., d_i^+ and d_i^-) from each goal with the *strict* assumption that *every* coefficient $a_{ij}, \forall\ i \in M, j \in N$ will deviate from their respective nominal values by σ_{ij}. Moreover, Constraint (59) weights this deviation using a robustness parameter $\theta_i, \forall\ i \in M$, that is bounded via Constraint (61), as per Ben-Tal and Nemirovski [75].

Clearly, the RGP formulation presented in (58)-(62) does not assume that each coefficient subject to deviation take on the maximum value, which is not the case in the previous aforementioned formulations. Formulation (58)-(62) exploits strict robustness, but it differs from Soyster's methodology. Soyster assumes a maximum deviation for each coefficient, whereas the RGP formulation using strict robustness via ellipsoidal uncertainty allows for less-than-maximum and more-than-maximum deviations.

3.3 Testing and Analysis

In this section, the results for the cardinality-constrained robustness, norm-based uncertainty and strict robustness, ellipsoidal uncertainty are discussed in comparison to the cardinality-constrained robustness, interval-based uncertainty used in Kuchta [3].

3.3.1 Numerical Example

To illustrate his RGP Formulation, Kuchta [3] provides the below numerical.

A company manufactures three divisible products. Let $x_j, j = 1, 2, 3$ *denote the amount of the respective products to be manufactured in the coming period. Here is the matrix* $a_{ij}, i = 1, 2, 3, 4$ *and* $j = 1, 2, 3$ *where:*

a) a_{1j} $(j = 1, 2, 3)$ *represent the most possible (normal) amount of material needed to manufacture the* j^{th} *product.*

b) a_{2j} $(j = 1, 2, 3)$ *represent the most possible (normal) amount of human work needed to manufacture the* j^{th} *product.*

c) a_{3j} $(j = 1, 2, 3)$ *represent the most possible (normal) amount of machine time needed to manufacture the* j^{th} *product.*

d) a_{4j} $(j = 1, 2, 3)$ *represent the most possible (normal) selling price needed of the* j^{th} *product (multiplied by -1 due to minimization problem).*

Table 4: Nominal/Target Values for Each Product [3]

	$j = 1$	$j = 2$	$j = 3$	**Target Values**, t_i
$i = 1$	*3*	*7*	*5*	*200*
$i = 2$	*6*	*5*	*7*	*200*
$i = 3$	*3*	*6*	*5*	*200*
$i = 4$	*-28*	*-40*	*-32*	*-1500*

To account for uncertainty in data in an effort to demonstrate his RGP formulation, Kuchta [3] applies interval uncertainty which includes a possible 10% variation to each of the cost coefficients' nominal value. It is important to note that only one-sided deviation from a target value that will negatively impact the solution is considered.

Table 5: Normal Values/Possible Deviations for each Product [3]

	$j = 1$		$j = 2$		$j = 3$		**Target Values**, t_i
	a_{i1}	σ_{i1}	a_{i2}	σ_{i2}	a_{i3}	σ_{i3}	
$i = 1$	*3*	*0.3*	*7*	*0.7*	*5*	*0.5*	*200*
$i = 2$	*6*	*0.6*	*5*	*0.5*	*7*	*0.7*	*200*
$i = 3$	*3*	*0.3*	*6*	*.6*	*5*	*0.5*	*200*
$i = 4$	*-28*	*2.8*	*-40*	*4*	*-32*	*3.2*	*-1500*

Furthermore, to use the proposed RGP formulation, the DM's views on conservatism must be understood. To look at varying levels of conservatism, the k_i-values in Table 6 are used, where each $k_i \leq 3$ because there are three products considered per goal.

Table 6: Different Conservatism Scenarios

Level of Conservatism	**DMs Views on Possible Scenarios**	$(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}, \boldsymbol{k_4})$
Lowest	Coefficients remain at nominal value	$(0, 0, 0, 0)$
↓	Every coefficient is affected in terms of the sales price goal	$(0, 0, 0, 3)$
	At least one coefficient for each goal is affected	$(1, 1, 1, 1)$
	At least one coefficient for each goal is affected, while every coefficient is affected in terms of the sales price goal	$(1, 1, 1, 3)$
	At least two coefficients for each goal is affected	$(2, 2, 2, 2)$
Highest	All coefficients for each goal is affected	$(3, 3, 3, 3)$

It is imperative to understand that although this particular example assumes k_i-values that are integer, the RGP formulations seen in (48)-(52), (53)-(57), and (58)-(62) and allow for continuous values of k_i, making their application that much more substantial.

3.3.2 Results and discussion

Before a thorough discussion regarding the aforementioned RGP formulations as applied to the described numerical example ensues, it should be noted that Kuchta's method presented in (43)-(47) and the RGP formulation using the L_1-norm described in (48)-(52) are clearly equivalent. Kuchta's formulation presented in (43)-(47) utilizes the interval-based uncertainty set within which the decision variables z_i and k_i, for $i = 1, \dots, m$, identify the largest penalty p_{ij} at optimality. By comparison, (48)-(52) also seeks to identify the largest penalty p_{ij} at optimality, but does so by considering every combination of $\sigma_{ij}x_j$, given some k_i. These respective formulations require a different number of constraints for a given instance, which we discuss in sub-section 3.3.3, but we forgo the presentation of optimal solutions using (48)-(52) in the following section because they are identical to those attained via (43)-(47). To that end, a formal proof is offered in Appendix A showing that an optimal solution to Kutcha's RGP formulation (43)-(47) is feasible, has the same objective function value, and is optimal to the L_1-norm RGP formulation (48)-(52).

Using the software LINGO, version 11.0 and its global solver, the detailed results can be seen in Tables 7-8. These results yield insights into how each RGP methodology performs in terms of robust solutions.

Table 7: Scenario-based Optimal Solutions for Cardinality-constrained Robustness Using Both Interval Uncertainty and Norm-based Uncertainty

		(k_1, k_2, k_3, k_4) Scenarios						
		$(0,0,0,3)$	$(1,1,1,1)$	$(1,1,1,3)$	$(2,2,2,2)$	$(3,3,3,3)$	St Dev	Range
Interval	x_1	41.7	28.2	36.9	56.1	56.8	12.4	28.6
	x_2	12.5	19.7	15.8	1.4	1.9	8.3	18.3
	x_3	0.0	0.0	0.0	1.0	0.0	0.5	1.0
	$\sum_{i=1}^{m} d_i^+$	125.0	136.2	172.2	187.3	187.5	29.3	62.5
	d_1^+	12.5	36.6	32.6	1.6	2.1	16.6	35.0
	d_2^+	112.5	84.8	122.8	185.7	185.4	45.4	101.0
	d_3^+	0.0	14.9	16.8	0.0	0.0	8.7	16.8
	d_4^+	0.0	0.0	0.0	0.0	0.0	0.0	0.0
L_2-norm	x_1	35.5	28.2	33.7	34.6	34.6	2.9	7.3
	x_2	15.6	19.7	16.8	16.2	16.2	1.6	4.1
	x_3	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\sum_{i=1}^{m} d_i^+$	106.5	136.2	149.0	158.6	158.6	21.7	52.1
	d_1^+	15.6	36.6	30.6	32.4	32.4	8.1	21.0
	d_2^+	90.9	84.8	106.3	111.0	111.0	12.2	26.2
	d_3^+	0.0	14.8	12.1	15.1	15.1	6.5	15.1
	d_4^+	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 8: Optimal Solutions for Strict Robustness Using Ellipsoidal Uncertainty and Varying θ-values

	Theta, θ						
	0.1	0.5	1	1.5	$\sqrt{3}$	St Dev	Range
x_1	23.4	33.6	34.6	32.3	30.6	4.5	11.2
x_2	21.4	15.4	16.2	17.2	16.9	2.3	6.0
x_3	0.0	0.0	0.0	2.5	5.0	2.2	5.0
$\sum_{i=1}^{m} d_i^+$	70.7	105.1	158.6	215.4	241.3	71.8	170.6
d_1^+	21.6	16.0	32.4	52.8	61.0	19.5	45.1
d_2^+	49.1	89.2	111.0	128.9	138.3	35.6	89.2
d_3^+	0.0	0.0	15.1	33.7	42.0	19.2	42.0
d_4^+	0.0	0.0	0.0	0.0	0.0	0.0	0.0

The $(1, 1, 1, 1)$ scenario for the interval-based and L_2-norm uncertainty sets with regard to cardinality-constrained robustness provide identical results. These results occur because the respective formulations for this particular scenario are equivalent. The

$(1, 1, 1, 1)$ scenario using Kuchta's method is the same as using the L_∞-norm uncertainty set. The robust counterpart for the L_∞-norm, as presented in (63), is identical to allowing the worst possible deviation for each goal, which is what occurs using Kuchta's method for the $(1, 1, 1, 1)$ scenario.

$$max(|\hat{a}_{i1}x_1|, |\hat{a}_{i2}x_2|, \dots, |\hat{a}_{in}x_n|), \quad \forall\, i \in M \tag{63}$$

Using cardinality-constrained robustness via the L_2-norm uncertainty set, the $(2, 2, 2, 2)$ scenario and minimax, $(3, 3, 3, 3)$ scenario yield the same conclusion. This can be attributed to the fact that the third product is not manufactured in this instance (i.e., $x_3 = 0$). The $(3, 3, 3, 3)$ scenario for cardinality-constrained robustness using L_2-norm uncertainty sets generates identical results for the strict robustness using ellipsoidal uncertainty when $\theta = 1$. Once again, when observing the individual formulations, these two instances are equivalent and thus should yield the same results. As the number of coefficients allowed to take on uncertainty increases, the objective function value increases. The same is true for when values of θ increase. This is comports with intuition because, as both the values in the k-vector and θ increase, so too does the DM's conservatism, which should result in a less robust solution.

For cardinality-constrained robustness, the L_2-norm uncertainty set outperforms the interval-based uncertainty set in terms of range for the objective function value, amount of products produced, and individual deviational variables. The same can be said for standard deviation. As expected, with strict robustness using ellipsoidal uncertainty, i.e., when $\theta < 1$, it provides solutions having lesser deviations from target values when compared to any other scenario listed in Table 7. On the contrary, when $\theta > 1$, the solutions are worse than solutions listed in Table 7. This can be attributed to the allowable feasible region corresponding to each θ-value.

Regardless of the type of robustness used, the ellipsoidal and L_2-norm uncertainty sets used with the varying echelons of robustness seem to provide much more robust answers for the amount of products manufactured when compared to Kuchta's method, as illustrated in Tables 7-8 as well as Figures 8-9. Additionally, Figure 10 depicts how the L_2-norm uncertainty set outperformed Kuchta's method in terms of overall deviation.

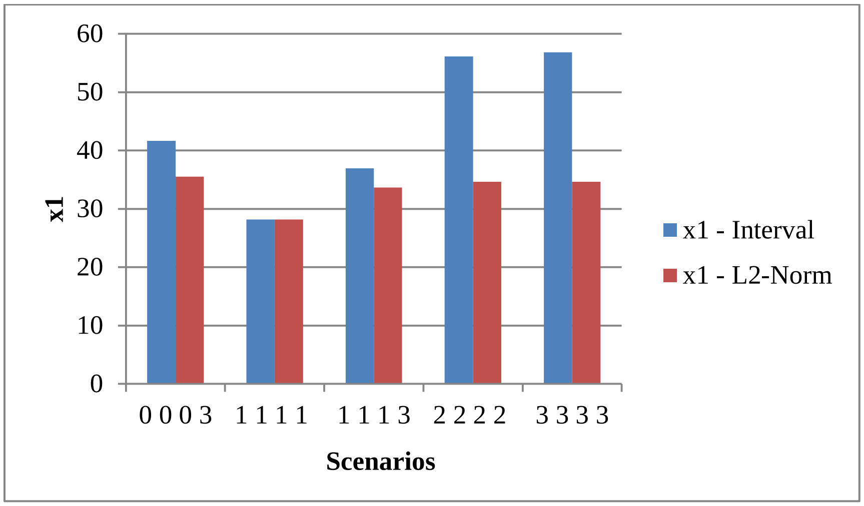


Figure 8: x_1^* for Interval and L_2-Norm Uncertainty Formulations

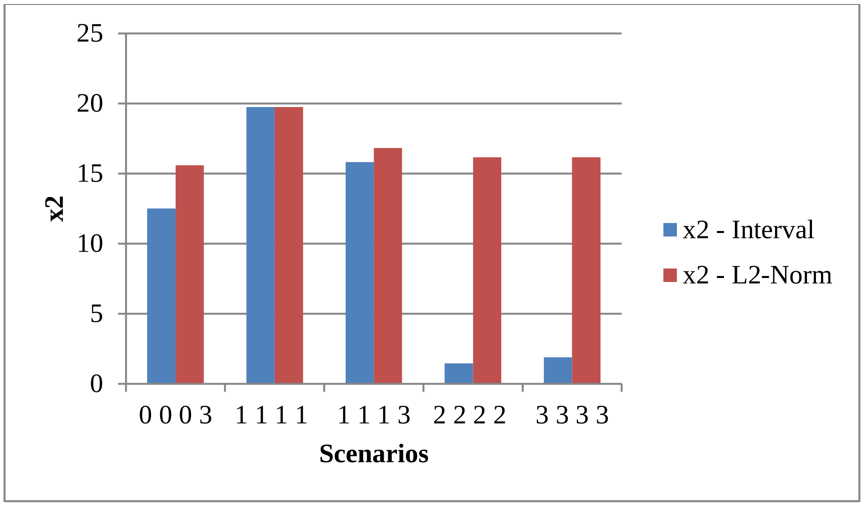


Figure 9: x_2^* for Interval and L_2-Norm Uncertainty Formulations

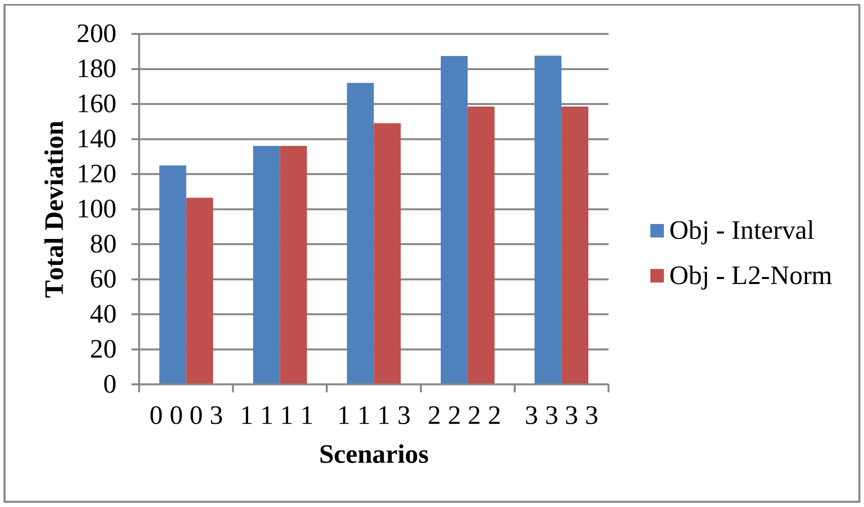


Figure 10: Interval versus L_2-Norm, Objective Function Value

It appears that, as the level of conservatism increases, the cardinality-constrained method using either the interval-based or the L_1-norm uncertainty sets tend to meet more of the goals, while taking on larger deviations of the goals not met. In contrast, as the level of conservatism increases, the strict robustness method via an ellipsoidal uncertainty set tends to spread out the total deviation among more goals in an attempt to remain more robust in terms of products manufactured.

Based on the results, the L_2-norm uncertainty set used with a cardinality-constrained robustness mindset outperformed Kuchta's method for this instance in terms of total undesirable deviation in the objective function value, as well as deviation in the amount of products made, regardless of scenario. Interesting results were found, showing the equivalency of Kuchta's method to the L_∞-norm for the $(1, 1, 1, 1)$ scenario. Moreover, the strict robustness concept using ellipsoidal uncertainty sets performed well in terms of robustness of the products being manufactured, even more so than Kuchta's method.

3.3.3 Discussion

It is important to discuss at this juncture that the utilization of a particular uncertainty set is often dependent on the situation at hand or on the risk assessment of the DM. Table 9 summarizes the RGP methods discussed:

Table 9: RGP Formulation Comparison

Robustness Echelon	**Uncertainty Set**	**# of constraints**	**Non-linear**	**DM View of Risk**		
				Risk Seeking	**Between**	**Risk Averse**
				$k_i \leq 1$	$1 < k_i < n-1$	$k_i \geq n-1$
Cardinality-Constrained	Interval	$m + mn$	No		x	
	L_1-norm	$m + \sum_{i=1}^{m} (n-1)\binom{n}{\lfloor k_i \rfloor}$	Yes	x		x
	L_2-norm	$m + \sum_{i=1}^{m} (n-1)\binom{n}{\lfloor k_i \rfloor}$	Yes	x		x
Strict	Ellipsoidal	m	Yes	x	x	x

When considering the merits and shortcomings of cardinality-constrained robustness, it is interesting to note that Kuchta's model results in the same number of constraints, regardless of the amount of coefficients subject to uncertainty. This is in stark contrast to the proposed formulations using norm-based uncertainty sets, as the number of possible constraints depends on k_i. When $k_i \leq 1$ or $k_i \geq n-1$, the number of constraints for norm-based uncertainty is actually less than Kuchta's model, as illustrated in Figures 11-12. Therefore, when a DM is either extremely risk averse or risk seeking, the computational efficiency using norm-based uncertainty sets will be superior to Kuchta's method, regardless of the size of the problem, at least in terms of the number of constraints included in the formulation.

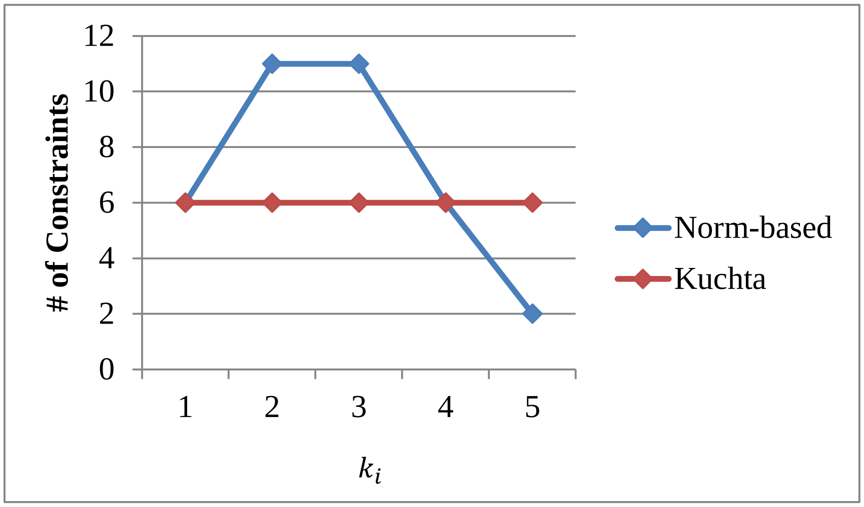


Figure 11: Constraint Comparison Assuming $n = 5, m = 1$

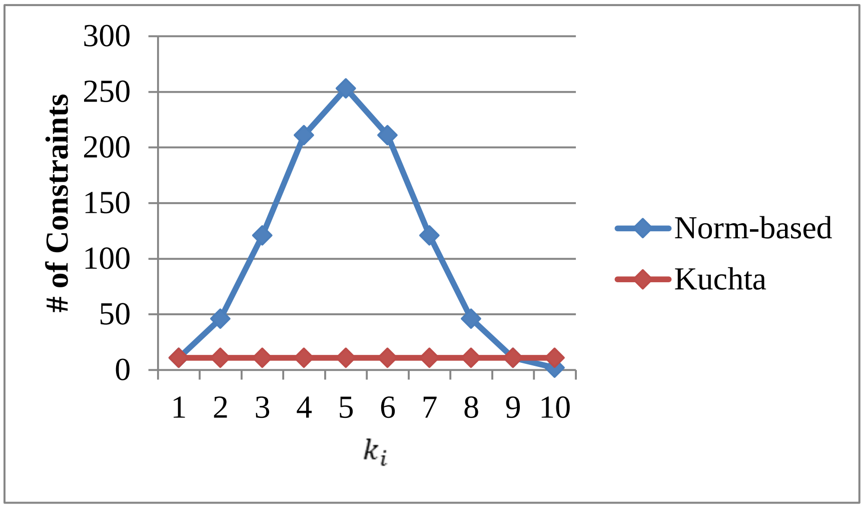


Figure 12: Constraint Comparison Assuming $n = 10, m = 1$

The proposed methodologies using norm-based uncertainty sets are not ideal, as they do not meet the necessary and sufficient conditions for convexity and, consequently, cannot guarantee an optimal solution with more readily available and inexpensive solvers. However, with today's advanced technology and optimization solvers, such nonconvex, nonlinear problems do not pose as much of a problem as they once did. For instance, LINGO has the option to use and implement a global solver in their software. This global solver can easily convert any nonconvex, nonlinear problem into multiple convex, linear subproblems for which it conducts an exhaustive search using well known branch-and-bound techniques to arrive at a global optimal solution [158]. Although our use of the LINDO global solver for the instances solved herein assured the attainment of global optimal solutions, we offer that several commercially available global optimization solvers (e.g., BARON [159], SCIP [160]) are readily available and capable of addressing nonlinear and nonconvex optimization problems. For the budget-constrained analyst, we also offer that the NEOS Optimization Server [161], [162], [163] provides free access to several global optimization solvers that can be invoked using the AMPL [164] and/or GAMS [165] modeling languages.

Yet another option for invoking the proposed methodologies as a way to garner robust solutions includes transforming the ellipsoidal constraints into equality constraints, classifying the convexity of the resulting objective function, and – if appropriate – using CPLEX to solve the formulation having a convex, quadratic objective function and linear constraints. One useful method to classify the convexity is to examine the eigenvalues of the Hessian matrix for the objective function. Given the eigenvalues, $\lambda_i, i = 1, \dots, m,$ the convexity can be determined via Table 10. If each of the eigenvalues is non-negative, then global optimality is assured via commercial solvers using readily available convex optimization algorithms; otherwise, it cannot be guaranteed in the absence of specialized software and/or customized algorithms.

Table 10: Convexity Classification via Determinant Analysis

Hessian Eigenvalues	Hessian Classification	Objective Function Convexity Classification
$\lambda > 0, \;\; i = 1, \ldots, m$	Positive Definite	Strictly Convex
$\lambda \geq 0, \;\; i = 1, \ldots, m$	Positive Semi-Definite	Convex
$\lambda < 0, \;\; i = 1, \ldots, m$	Negative Definite	Strictly Concave
$\lambda \leq 0, \;\; i = 1, \ldots, m$	Negative Semi-Definite	Concave
Does not meet above requirements	Indefinite	Neither Convex nor Concave

To that end, with current technology such as the aforementioned global and commercial solvers, the norm-based uncertainty set methods should be considered for execution for all instances of k_i, including when $1 < k_i < n - 1, \forall\, n > 2$, because it has been shown that the norm-based, specifically the L_2-norm, uncertainty sets produce more robust solutions when compared to Kuchta's methodology.

Finally, for strict robustness using ellipsoidal uncertainty, the computational docility is good and its nonlinearity can be addressed by invoking a global or nonlinear, nonconvex commercial solver. Yet, the biggest advantage to using this methodology is that it gives the DM the luxury of less or more than maximum deviation. Moreover, applying this methodology can be beneficial when the deviational amounts for each coefficient are not known, or not accurate.

3.4 Conclusion

RGP is a relatively new idea to the field of RO. There has been much research regarding RO individually, as well as RO techniques coupled with MOO; however, RGP has not been extensively researched. Cardinality-constrained robustness using interval-based uncertainty is the only technique that has been offered to date for RGP. This paper shows how using norm-based, and specifically the L_2-norm, uncertainty sets can yield more robust solutions within the RGP framework. Moreover, this paper also discusses the computational efficiency of the different modeling approaches in the context of the number of constraints required, as well as when implementation is best suited for each methodology discussed.

As of now, there is no literature that suggests how to set RGP risk parameters…parameters that play a vital role in producing accurate and robust solutions. Future work concerning this very topic is ongoing to properly quantify a DM’s risk attitude via expected utility principles in an attempt to map that value to an RGP parameter such as θ_i or k_i, given the uncertainty set used. Consequently, a general model combining utility theory and RGP concepts will eventually be obtained.

It would also be intriguing to incorporate another level of robustness or a different uncertainty set as it pertains to the RGP framework. This paper considers a simple example; of merit would be to examine the relative benefits of norm-based uncertainty in RGP when applied to a larger problem, such as the portfolio selection problem described by Ghahtarani and Najafi [93].

IV. Mapping a Decision Maker's Risk Preference to Robust Optimization Parameters

4.1 Introduction

Data uncertainty is prevalent in the contemporary business environment. While assumptions can be made to identify deterministic values for decision-making models, ignoring variability induces error; instead, uncertainty in data can be accounted for via *a priori* methods such as stochastic optimization (SO) or robust optimization (RO). Stochastic optimization, which is also commonly referred to as stochastic programming, accounts for data uncertainty via probability distributions and is widely used. However, implementing SO can be combinatorically and computationally challenging, and identifying a true probability distribution is often elusive [4]. Because of these shortcomings, RO is a useful method when considering uncertain data and will be the primary focus of this paper.

4.1.1 RO variants

Unlike SO, which implements probability distributions to account for data uncertainty, RO considers uncertainty in data via uncertainty sets. RO techniques were first introduced by Soyster [2] when the author presented a minimax RO formulation using set theory, later known as strict robustness. Although formative, Soyster's [2] strict robustness methodology is the most conservative, risk-averse RO approach in the literature because it strictly favors feasibility for any possible uncertainty in data over optimality when less variability may be realized.

It was not until nearly 25 years after Soyster's methodology was introduced that an RO theoretical framework was established, allowing for less conservative robustness techniques, as presented in [72], [75], [76], [74], and [94], among other works. For more information regarding RO and its practical applications, an interested reader is referred to Ben-Tal *et al.* [66] as well as Mulvey *et al.* [73].

The aforementioned RO research expanded the foundation for strict robustness and introduced opportunities for less conservative robustness techniques. This paper focuses on a subset of these less conservative RO methods, but the interested reader is directed to M. Goerigk and Schöbel [87] and Bertsimas *et al.* [152] for comprehensive reviews of the many RO variants propounded in the literature.

Cardinality-constrained robustness, a less-conservative level of robustness, is predicated on the concept of reducing the size of an interval-based uncertainty set. Bertsimas and Sim [4], [5] set forth cardinality-constrained robustness as a viable method, as it is unlikely for all parametric coefficients within one constraint to realize their worst-case case deviation simultaneously; rather, only a subset of them will vary at any given time. Consequently, the cardinality-constrained robustness formulation restricts the number of coefficients that can vary simultaneously [4],[5]. More specifically, for every constraint $i = 0,1, \dots , m$ in a linear program, there is a robustness measure k_i that regulates the required level of robustness (i.e., level of conservatism) and takes on values in the interval $[0, |S_i|]$, wherein S_i is the set of coefficients $a_{ij}, \forall\, j \in S_i$ that are vulnerable to parameter uncertainty in the interval $\left[a_{ij} - \sigma_{ij}, a_{ij} + \sigma_{ij}\ \right]$, and wherein $\sigma_{ij} > 0$ indicates the maximum deviation of parameter a_{ij}. (Of note, these definitions utilize the conventions utilized herein and as adopted by [7] vice [4] and [5]). Bertsimas and Sim [4],[5] justify that it is not probable that every $a_{ij}, \forall\, j \in S_i$ take on uncertainty. It also an important distinction that, when k_i is not restricted to integer values, up to $\lfloor k_i \rfloor$ of the coefficients will change by σ_{ij}, and exactly one coefficient a_{iq} will change by at most $(k_i - \lfloor k_i \rfloor)\sigma_{iq}$. Conversely, when the k_i-values are integer-valued, every coefficient that does deviate from their nominal value will do so at a maximum allowable value.

With a view towards application, cardinality-constrained robustness has also been extended to goal programming (GP) via robust goal programming (RGP), as examined by Kuchta [3] and Hanks *et al.* [7], wherein the former utilizes interval-based uncertainty sets and the latter norm-based uncertainty sets. Herein, we introduce an RGP model using L_1-norm cardinality-constrained uncertainty sets to provide context for ensuing conceptual discussions, as well as for a numerical example presented in Section 4.4. The interested reader is referred to Hanks *et al.* [7] for the other RGP models, to include an L_2-norm cardinality-constrained RGP model and an RGP modeling variant imposing strict robustness via ellipsoidal uncertainty sets.

$$Minimize \quad \sum_{i=1}^{m} d_i^+ \tag{64}$$

$$subject\ to \quad \sum_{j=1}^{n} a_{ij}x_j + p_i - d_i^+ + d_i^- = t_i, \quad \forall\, i \in M, \tag{65}$$

$$p_i \geq \sum_{j \in S_i} \left|\sigma_{ij}x_j\right| + \left|f_i\sigma_{iq}x_q\right|, \quad \forall\, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N\backslash S_i, i \in M, \tag{66}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{67}$$

$$x_j, d_i^+, d_i^- p_i \geq 0, \;\; \forall\, i \in M, \;\; j \in N. \tag{68}$$

The objective function (64) minimizes the sum of positive deviations from goals $i \in M$, and Constraint (65) calculates the positive and negative deviations (i.e., d_i^+ and d_i^-) from each goal, where the total induced variability is represented by p_i for each goal. Constraint (66) applies L_1-norm uncertainty sets to the possible sets of coefficients subject to parametric uncertainty (i.e., $S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i$), where the least upper bound (i.e., infimum) penalty, $p_i, \forall\, i \in M$ is calculated for every combination of $\sum_{j \in S_i} |\sigma_{ij} x_j| + |f_i \sigma_{iq} x_q|$, wherein f_i represents the remainder value when k_i is non-integer (i.e., $f_i = k_i - \lfloor k_i \rfloor$). Constraint (67) characterizes additional constraints on the decision space that are neither related to the DM's goals nor considered with respect to possible parametric uncertainty. Lastly, Constraint (68) ensures all decision variables are non-negative.

Yet another modified RO model is described in Hanks *et al.* [7], where strict robustness is used in conjunction with ellipsoidal uncertainty sets. Ellipsoidal uncertainty sets have been utilized to reduce the assumed degree of conservatism compared to that imposed via strict robustness, where the robustness measure θ_i, $i = 1, \dots, m$, adjusts the level of conservatism. Ellipsoidal uncertainty sets have been examined by various authors [74], [75], [76]. The principle underlying ellipsoidal uncertainty sets is the allowance for a DM to adjust θ_i-values to better represent their risk tolerance [75]. Thus, θ_i enables less-than-maximum or more-than-maximum deviation to occur in relation to interval-based uncertainty sets. Given some nominal value p_i^* with an allowable "uncertainty interval" $\Delta_i = [p_i^* - \sigma_i, p_i^* + \sigma_i]$, the uncertainty set is described by the box $B = \{(p_1, \dots, p_n): |p_i - p_i^*| \leq \sigma_i, i = 1, \dots, n\}$. Ben-Tal and Nemirovski [75] proceed to show that the largest volume ellipsoid *contained in* the box B exists when $\theta_i = 1$, and the smallest volume ellipsoid *containing* the box B occurs when $\theta_i = \sqrt{n}$. Although it is possible to have $\theta_i > \sqrt{n}$, we assume w.l.o.g. in this paper that $0 \leq \theta_i \leq \sqrt{n}, i = 1, \dots, n$.

4.1.2 Risk and RO

The formal concepts relating the fields of risk and RO have been presented sparingly in the literature. Natarajan *et al.* [92] discuss how risk measures such as standard deviation, worst-case value-at-risk (VaR), and conditional value-at-risk (CVaR) can be derived from an uncertainty set. Similarly, Bertsimas and Brown [91] demonstrate a methodology for constructing an uncertainty set given an original risk measure. However, both articles consider risk measures as they pertain to investments or portfolio management (as their definitions of risk stem from monetary descriptions), widely limiting application beyond fiscal matters. (There is additional literature regarding risk measures in finance, and the interested reader is direct to the literature reviews [92] and [91], as it is not the focus of this paper.)

In a more general setting, it is intuitive that the level of conservatism enforced via uncertainty sets and varying RO techniques corresponds to the risk aversion of a DM. When coupling risk and RO, there are generally two basic approaches to generate stable solutions – *a posteriori* and *a priori*. The *a posteriori* approach identifies possible parametric values for varying uncertainty sets and RO models in which alternative solutions corresponding to respectively more or less conservative assumptions are generated. Among these alternative solutions, a subset of them may then be presented to the DM for consideration and possible implementation. This *a posteriori* tactic is used with RO ideals in various articles [4], [5], [3], [7]. Conversely, the *a priori* approach classifies a DM's risk preference before analysis begins and uses it to inform RO risk parameters (such as k_i or θ_i) for different uncertainty sets. In doing so, this procedure identifies a single solution for which the level of conservatism regarding assumptions about uncertainty corresponds to the DM's risk preference. Such an approach has yet to be developed in conjunction with RO principles.

4.1.3 Motivation

To date, associating risk within RO formulations has been implemented via either parametric or *a posteriori* analysis. Accounting for risk in this manner is troublesome for many reasons. First, presenting a DM with a list of solutions and corresponding levels of risk introduces partiality, as the DM can select what they perceive to be the best solution, but it might not reflect their true risk preference. Second, the number of potential solutions can be overwhelming, even for smaller problem instances. Finally, the amount of time required for such *a posteriori* analysis can be cumbersome.

In contrast, the aforementioned *a priori* method requires both the elicitation of a DM's risk preference and the use of it to parameterize the uncertainty set to be used in an RO model. As a result of this analysis, a DM is presented with a single solution. To date, we have found no such approach detailed in the literature, and the absence of it is a shortcoming that undermines the applicability of uncertainty sets as discussed in the literature. This paper remedies this deficiency by formally relating the sub-disciplines and demonstrating a related procedure.

This paper demonstrates how to mathematically map risk parameters identified in the context of utility theory to risk parameters utilized in varying RO constructs. Moreover, this paper offers a new risk preference categorized scale for the relative risk aversion coefficient, r, when using the bomb risk elicitation task (BRET) [6].

The remainder of the paper is organized as follows. Section 4.2 offers a brief literature review regarding decision analysis (DA), including expected utility theory (EUT) and different utility functions implemented in practice. Popular risk elicitation methods are discussed, and recommendations are provided regarding which to use for different circumstances. Section 4.3 presents three theorems regarding how a DM's risk tolerance can be mapped to RO risk parameter values, followed by a mathematical mapping methodology. Section 4.4 offers a new risk preference category scale when using the BRET and demonstrates it on a test instance to verify the proposed mapping methodology works appropriately. The ensuing results are provided, along with a related discussion. Section 4.5 concludes the work and identifies possible extensions.

4.2 Literature review

The following topics provide the underlying theoretical framework for the mathematical mapping herein that relates in an applied manner the sub-disciplines of risk and RO.

4.2.1 DA

DA considers the theory and methodology of making decisions with a formal structure (i.e., it establishes a discipline for decision making norms). DA has been studied for several years, and there are many components within its theoretical construct. Related to this study, expected utility theory (EUT) along with some popular EUT variants are discussed in this section.

EUT is generally used as a prescriptive measure (i.e., identifies actions as they *should be*) rather than a descriptive measure (i.e., identifies actions as they *really are*). EUT's primary goal is to maximize expected utility. Bernoulli [101] is the first to present EUT when he tried explaining the Petersburg paradox, but he did so without a way to formally quantify a utility measure. Not until over 200 years later did von Neumann and Morgenstern [102] identify a set of axioms to measure utility and establish the foundation for EUT.

The EUT axioms are rules governing how an individual *should* act when considering a choice involving risk or uncertainty. In general, an individual who follows the EUT axioms believes: (1) it is possible to construct a utility function to evaluate different outcomes or consequences, and (2) decisions should be made to maximize expected utility [103]. The EUT axioms, which are sometimes called the von Neumann and Morgenstern axioms, are further detailed by von Neumann and Morgenstern [102] and Clemen and Reilly [103].

Since their inception, the EUT axioms have been highly debated due to their paradoxical nature. These paradoxes, discussed in the next section, have been the source of dialogues regarding the validity of EUT, which is the basis of multiple EUT variants. Of the EUT axioms under the most scrutiny, invariance and independence are at the forefront.

There is much literature regarding contradictions that stem from the EUT axioms. One of the more readily discussed paradoxes is the called the *framing effect*; an individual's risk attitude or preference can change depending on how a question is framed [103]. For example, by simply framing a question in terms of money saved instead of money spent, an individual might answer differently two questions that relate choices having the same expected utilities. Similarly, another argument made against the EUT axioms is the idea that an individual's wealth or status quo plays a factor in their risk preference. Together, framing effects and wealth status show that some individuals tend to be more risk-averse when dealing with gains and less risk-seeking when encountering losses – a phenomenon known as *loss aversion*. Consequently, the notion arising from loss aversion presents a conflict to the invariance EUT axiom, as it states that only payoffs and associated probabilities are required to determine a DM's risk preference [103].

To remedy paradoxes resulting from EUT axioms, variations of EUT and relaxations of EUT axioms have been presented in the literature. The more prominent EUT variants are Subjective EUT (SEUT) and Prospect Theory (PT). As further dialogue concerning alternatives to EUT is outside the scope of this paper, the interested reader is directed to comprehensive survey articles Schoemaker [105] and Hey and Orme [113] for an extensive list of EUT variants.

Despite the detracting paradoxes and alternative theories, EUT and the von Neumann and Morgenstern axioms are the most used and widely accepted way to model behavior and risk [113]. Although utilizing EUT and the associated axioms induces "noise," "when it comes to measuring risk attitudes [or modeling behavior], the theoretical framework usually adopted to map the choices in the tasks is EUT" [114]. Because of this, EUT and expected utility maximization are exclusively considered in this paper.

4.2.2 Utility functions

In DA, decisions are typically made in the context of some sort of monetary gain or loss. However, a single monetary scale is sometimes misleading or does not provide enough information to make a decision. Because of this, utility functions are often derived and applied, especially when a decision involves uncertainty or risk. Many different utility functions have been utilized in practice [115] that take into consideration a DM's risk preference. Of the utility functions discussed in Gerber and Pafum [115], the exponential and power utility functions are the most widely used.

Utilizing the exponential utility function for decisions involving risk has many incentives. One such incentive is that it satisfies the delta property – a property that makes an individual's initial wealth independent of their risk attitude [120]. Because of this, an individual's initial wealth is considered negligible when dealing with decisions involving uncertainty. This leads to another important characteristic of the delta property: if the delta property is applicable over a monetary interval, then an individual's utility curve can be categorized using just one parameter (i.e., a risk aversion measure or risk odds) over the same interval [120].

There are many variations of the exponential utility function, but Equation (69) presents the general model, wherein x denotes the payoff (or wealth), r denotes the individual's risk odds (i.e., risk attitude parameter), and both a and b are constants [120]. Of interest, equivalent and commonly applied forms of the exponential utility function to evaluate risk are expressed in the book by Howard and Abbas [120] such as $u(x) = a - be^{-\gamma x}$ and $u(x) = a - be^{-x/\rho}$, where γ is the risk aversion coefficient and ρ represents a risk tolerance parameter.

$$u(x) = a - br^{-x} \tag{69}$$

Another significant feature of the exponential utility function is that it incorporates a measure of absolute risk aversion first presented Pratt [121] and Arrow [122]. Given a twice differentiable Bernoulli utility function $u(\cdot)$ generally dealing with monetary total gains and losses, the Arrow-Pratt measure of absolute risk aversion is defined appropriately in Equation (70), wherein $c(x)$ is a function of wealth x [121], [122]:

$$c(x) = \frac{-u''(x)}{u'(x)} \tag{70}$$

Using this absolute risk aversion measure, an individual's risk attitude can be determined for a given amount of wealth, or income value, x. Moreover, when a person exhibits a risk preference independent of initial wealth (i.e., a deltaperson), the absolute risk measure and the exponential utility function are applied in concert, resulting in what is known as constant absolute risk-aversion (CARA)[121], [122], [119]. For instance, $u(x) = 1 - e^{-(\gamma x)}$ denotes a utility function having CARA since $c(x) = \frac{-u''(x)}{u'(x)} = \frac{-(-\gamma^2 e^{-\gamma x})}{\gamma e^{-\gamma x}} = \gamma$, which is constant with respect to x.

Like the exponential utility function, the power utility function (or isoelastic utility function) comes in many forms, as described in Gerber and Pafum [115]. Similar to the exponential utility function, the power utility function can be implemented with a requirement to estimate only one parameter. Moreover, the power utility function is popular because it meets the Arrow-Pratt measure of constant relative risk aversion (CRRA). The Arrow-Pratt measure for CRRA, wherein $c(x)$ is some function with respect to wealth x, and s is some constant with respect to x, is defined as shown in Equation (71) [121], [122]:

$$g(x) = xc(x) = \frac{-xu''(x)}{u'(x)} = s \tag{71}$$

The CRRA assumption is commonly used in risk elicitation methods that adopt a form of the power utility function [138], [142], [6]. As an example, Crosetto and Filippin [6] present the BRET that uses the power utility function $u(x) = x^r$, wherein r represents the risk preference of a DM (i.e., the CRRA coefficient). It follows that $g(x) = xc(x) = \frac{-xr(r-1)x^{r-2}}{rx^{r-1}} = 1 - r$. Thus, the BRET can assume CRRA. Similar results apply to the other risk elicitation methods discussed.

4.2.3 Risk elicitation methods

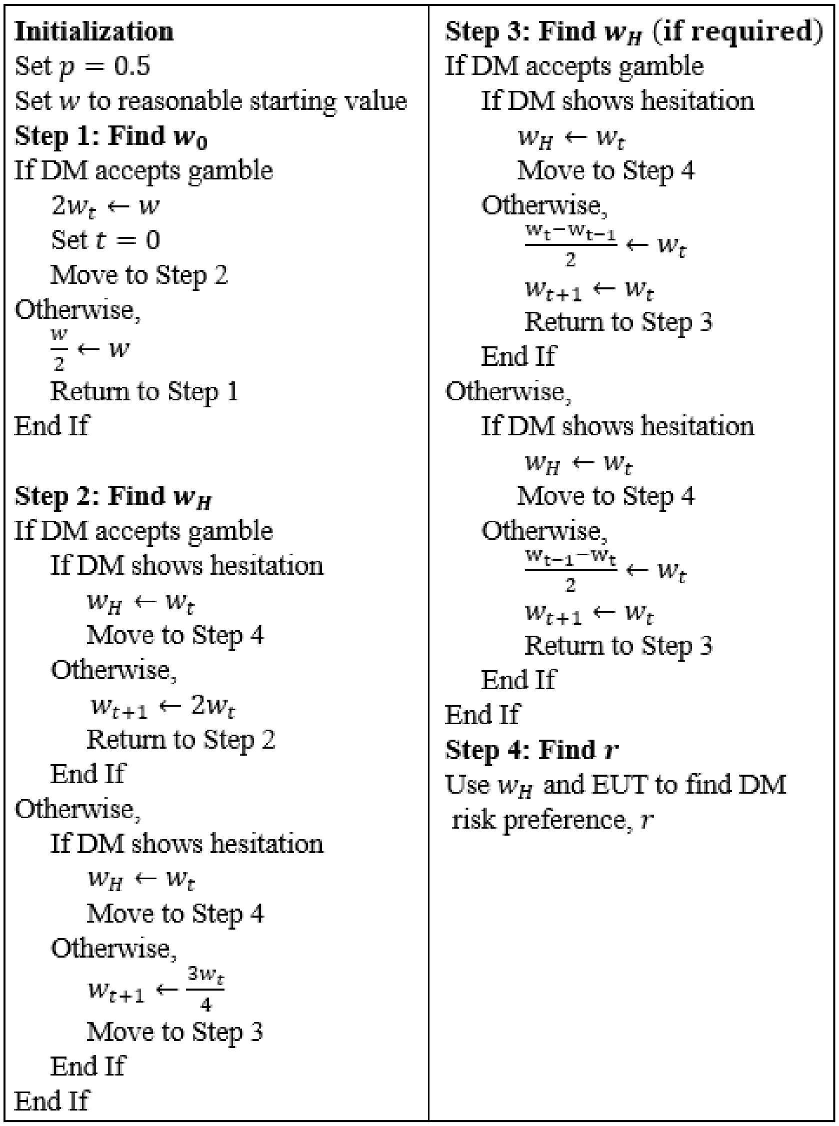
For a DM to make an informed decision, they must have a strong understanding of their risk preference. However, DM's often do not know their *true* risk preference, which is the reason much research is devoted to risk elicitation methods. A number of risk elicitation methods exist in the literature, as summarized by [125], [123], [124]. Considering the risk elicitation methods discussed by the aforementioned authors, a general overview of the more widely used risk elicitation methods is presented in Table 11. With regard to this paper, the methods discussed in greater detail are the bisection method used in conjunction with the exponential utility function, and the Eckel-Grossman (EG), Holt-Laury (HL), and BRET methods, with which a variety of power utility functions are utilized.

Table 11: Common Risk Elicitation Methods

		DM readily available		DM not readily available
Elicitation Method		Interview	Activity/Automated	Questionnaire
Design	Simple	• Gneezy and Potters • EG • Bisection	• Balloon Analogue Risk Task* • BRET*	• German Socio-Economic Panel Study • Domain-Specific Risk-Taking Scale
	Complex	• HL		

* Method can also be used when DM is not readily available.

In mathematics, a bisection method is applied as a root finding algorithm. When used in eliciting risk, the bisection method is a way to iteratively partition winning and losing payoffs in a series of lotteries until the DM hesitates when answering a question regarding preference between two alternative outcomes, each of which differs in its EUT and its best and/or worst-case outcome under uncertainty. Such a hesitation indicates indifference between the two alternatives and signifies that the payoffs associated with the alternatives can be utilized to calculate an individual's risk preference via EUT. The bisection method can be applied via different utility functions with various risk elicitation methods, although it is most commonly applied to ascertain the risk attitude parameter for the exponential utility function. Although multiple versions of the bisection method are available with varying probability values [120], the general process is depicted in Figure 13, where there is an equal probability p of winning w and of losing $w/2$. For reference, w_0 signifies the initial winning value and w_H denotes the value at which hesitation is observed. In this study, we utilize the bisection method in conjunction with the exponential utility function of the form $u(w) = 1 - r^{-w}$, wherein the parameter r signify the risk odds. Upon finding w_H, EUT is implemented to find r, wherein if $r < 1, r = 1,$ or $r > 1$, a DM is labeled a risk seeking, risk neutral, or risk averse individual, respectively [120].



```
Initialization
Set $p = 0.5$
Set $w$ to reasonable starting value
Step 1: Find $w_0$
If DM accepts gamble
    $2w_t \leftarrow w$
    Set $t = 0$
    Move to Step 2
Otherwise,
    $\frac{w}{2} \leftarrow w$
    Return to Step 1
End If

Step 2: Find $w_H$
If DM accepts gamble
    If DM shows hesitation
        $w_H \leftarrow w_t$
        Move to Step 4
    Otherwise,
        $w_{t+1} \leftarrow 2w_t$
        Return to Step 2
    End If
Otherwise,
    If DM shows hesitation
        $w_H \leftarrow w_t$
        Move to Step 4
    Otherwise,
        $w_{t+1} \leftarrow \frac{3w_t}{4}$
        Move to Step 3
    End If
End If
```

```
Step 3: Find $w_H$ (if required)
If DM accepts gamble
    If DM shows hesitation
        $w_H \leftarrow w_t$
        Move to Step 4
    Otherwise,
        $\frac{w_t - w_{t-1}}{2} \leftarrow w_t$
        $w_{t+1} \leftarrow w_t$
        Return to Step 3
    End If
Otherwise,
    If DM shows hesitation
        $w_H \leftarrow w_t$
        Move to Step 4
    Otherwise,
        $\frac{w_{t-1} - w_t}{2} \leftarrow w_t$
        $w_{t+1} \leftarrow w_t$
        Return to Step 3
    End If
End If
Step 4: Find $r$
Use $w_H$ and EUT to find DM
 risk preference, $r$
```

Figure 13: Lottery System Bisection Algorithm [120]

The EG method [138] is an ordered lottery schema that presents individuals with 5-7 lotteries, each of which has two choices with equal probabilities. Although the original lottery structure seen in Eckel and Grossman [138] presents individuals with five lotteries, many EG method variations include 1-2 additional lotteries to better distinguish between risk-neutral and risk-seeking behavior, as presented in Table 12. The EG method forces each individual to make exactly one lottery choice, where each lottery's expected value linearly increases as the low payoff and high payoff values respectively decrease and increase. For a given lottery, the EG method assumes CRRA and classifies the r-value into different intervals while using the utility function $u(x) = x^{1-r}$ to generate a risk curve, wherein x corresponds to wealth. Individuals for which $r > 0$ are considered risk-averse, whereas those individuals for which $r < 0$ or $r = 0$ are classified as risk-seeking or risk-neutral, respectively.

Table 12: Eckel and Grossman Alternate Lottery Structure [139]

Choice (50/50 Gamble)	Low Payoff	High Payoff	Expected Return	Standard Deviation	Implied Coefficient of Relative Risk Aversion, r	Fraction of Subjects (%)
Gamble 1	28	28	28	0	$3.46 < r$	10.7
Gamble 2	24	36	30	6	$1.16 < r < 3.46$	11.2
Gamble 3	20	44	32	12	$0.71 < r < 1.16$	39.2
Gamble 4	16	52	34	18	$0.50 < r < 0.71$	16.8
Gamble 5	12	60	36	24	$0 < r < 0.50$	11.5
Gamble 6	2	70	36	34	$r < 0$	10.7

The BRET is a simple risk elicitation method and is first presented by Crosetto and Filippin [6]. The BRET is administered via an interactive, choice-based game that can be played with or without a computer. Assuming the computer version is used, an individual is presented with a 10x10 unit grid wherein each square within the grid represents a box. Of these 100 boxes, there is an imaginary bomb contained in one of them, the location of which is randomly selected before experimentation begins and is not revealed until the test is completed. The individual selects “Start” and, for every second that passes, a box is collected (e.g., 15 seconds pass, 15 boxes collected) until the “Stop” button is pressed. After each box is collected, the individual is credited with some constant monetary value. On the computer screen, the individual is notified of the number of boxes collected, as well as the accumulated money. Once the individual hits

“Stop”, the location of the bomb is revealed. If the boxes collected by the individual include the bomb, they lose all accumulated earnings; otherwise, they depart with the accrued money [6]. Denote the number of boxes to be collected $v \in \{0,1,\dots,100\}$, the position of the bomb $b \in \{1,2,\dots,100\}$, and the number of boxes collected v^* for an individual. Thus, if $v^* < b$ the individual is allowed to keep the accumulated earnings, but if $v^* \geq b$, then the individual has collected box v containing the bomb b and loses all earnings. Crosetto and Filippin [6] assume CRRA while using the utility function $u(x) = x^r$, wherein r is the coefficient of relative risk aversion and x corresponds to wealth. Thus, a risk-neutral individual will collect exactly 50 boxes, and those individuals who collect fewer than (more than) 50 are deemed to be risk-averse (risk-seeking). Refer to Table 26 in Appendix for a list of v-values and associated r-values.

Unlike the three aforementioned methods, the HL method is considered to be a complex design as it elicits risk using a series of questions known as a multiple price list method. The HL method [142] is the most widely used risk elicitation method [123]. The HL method asks an individual to sequentially choose between two lotteries, ten different times, as depicted in Table 13. When moving down the columns for both Lotteries A and B, the payoffs remain constant, while the probabilities for the higher and lower payoffs monotonically decrease and increase, respectively.

Table 13: The HL Method Lottery System [123]

Lottery A	Lottery B
1/10 of \$2, 9/10 of \$1.60	1/10 of \$3.85, 9/10 of \$0.10
2/10 of \$2, 8/10 of \$1.60	2/10 of \$3.85, 8/10 of \$0.10
3/10 of \$2, 7/10 of \$1.60	3/10 of \$3.85, 7/10 of \$0.10
4/10 of \$2, 6/10 of \$1.60	4/10 of \$3.85, 6/10 of \$0.10
5/10 of \$2, 5/10 of \$1.60	5/10 of \$3.85, 5/10 of \$0.10
6/10 of \$2, 4/10 of \$1.60	6/10 of \$3.85, 4/10 of \$0.10
7/10 of \$2, 3/10 of \$1.60	7/10 of \$3.85, 3/10 of \$0.10
8/10 of \$2, 2/10 of \$1.60	8/10 of \$3.85, 2/10 of \$0.10
9/10 of \$2, 1/10 of \$1.60	9/10 of \$3.85, 1/10 of \$0.10
10/10 of \$2, 0/10 of \$1.60	10/10 of \$3.85, 0/10 of \$0.10

Before the experiment begins, each individual is instructed that a random lottery will be chosen at the end of the experiment with corresponding payoffs rendered. In doing this, the individual's risk preferences are represented truthfully and accurately [123]. Once all lottery decisions have been made, a clear crossover point – the point at which the individual changes from preferring Lottery A to preferring Lottery B – is determined. This crossover point is then used to measure the individual's risk preference. The HL method assumes CRRA while using the utility function $u(x) = \frac{x^{1-r}}{(1-r)}$, wherein r is the coefficient of relative risk aversion, and x represents wealth. Using the crossover point and the utility function, an experimenter can classify an individual's risk preference using Table 14, wherein the "number of safe choices" refers to the number of times lottery A is chosen. For instance, if an individual switched over after the second lottery (i.e., they choose Lottery A for the first two lotteries), then their relative risk coefficient falls in the interval $-0.95 < r < -0.49$ and they are classified as "very risk loving".

Table 14: HL Method Relative Risk Aversion Coefficient and Risk Preference Classification [142]

Number of Safe Choices	Range of Relative Risk Aversion for $u(x) = x^{1-r}/(1-r)$	Risk Preference Classification
0-1	$r < -0.95$	Highly Risk Loving
2	$-0.95 < r < -0.49$	Very Risk Loving
3	$-0.49 < r < -0.15$	Risk Loving
4	$-0.15 < r < 0.15$	Risk Neutral
5	$0.15 < r < 0.41$	Slightly Risk Averse
6	$0.41 < r < 0.68$	Risk Averse
7	$0.68 < r < 0.97$	Very Risk Averse
8	$0.97 < r < 1.37$	Highly Risk Averse
9-10	$1.37 < r$	Stay in Bed

4.2.4 Risk elicitation flowchart

Understanding the difference between CARA and CRRA is important, especially when identifying an appropriate risk elicitation method to implement for a given application. To that end, the flowchart depicted in Figure 14 is offered as algorithm to determine an appropriate course of action concerning the CARA and CRRA assumptions, utility functions, and risk elicitation methods discussed in this paper.

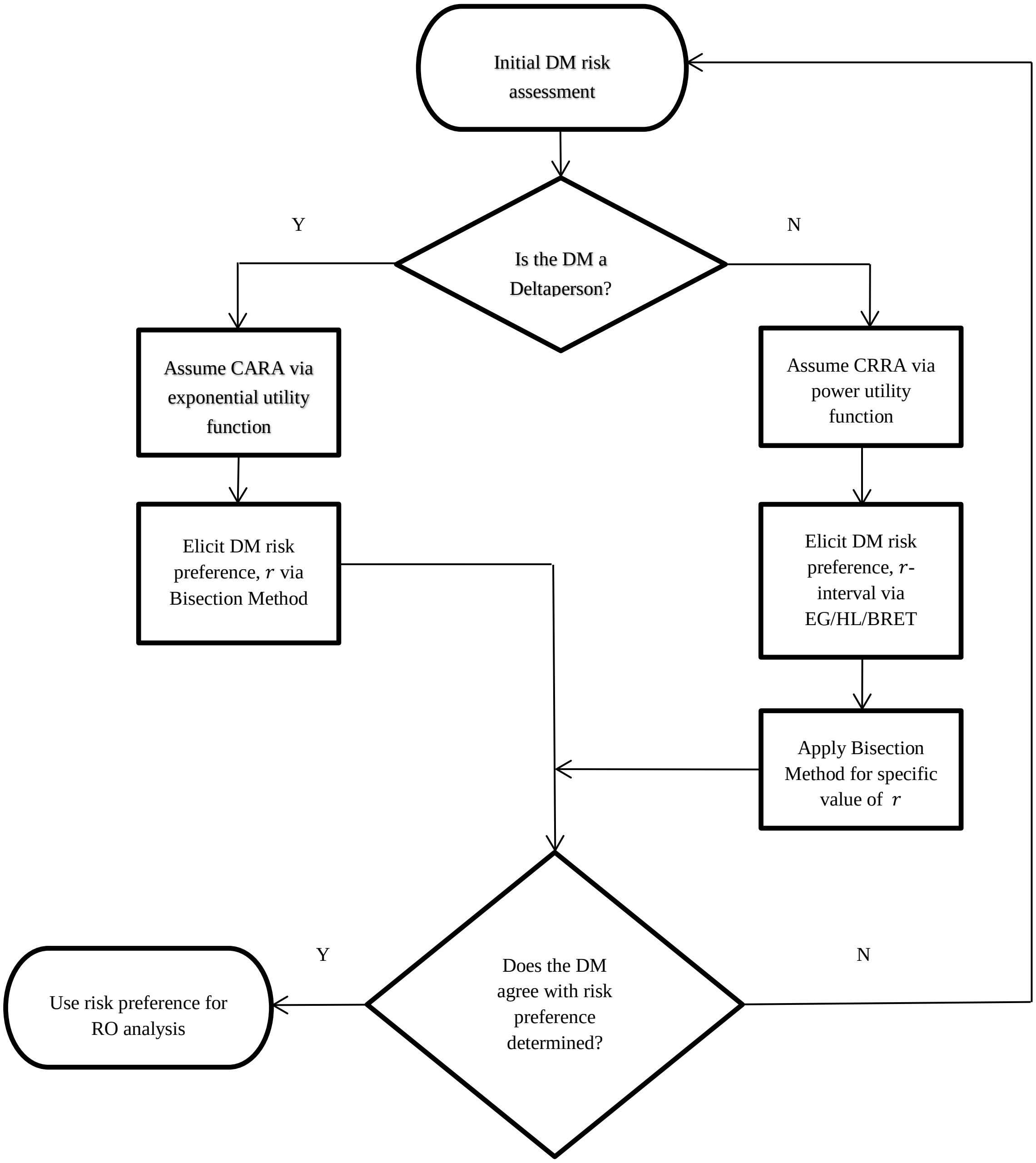


Figure 14: Risk Elicitation and Utility Function Flowchart

4.3 Mapping DMs risk preference to RO parameters

Rigorously eliciting a DM's risk preference is an important task. However, doing so provides little information using techniques found in the literature regarding associative k_i- or θ_i-values for use in RO. This section demonstrates how to map a risk preference r

to an RO risk parameter k_i or θ_i. It should be noted that this particular mapping methodology is valid when applied in concert with risk elicitation methods and utility functions that can simultaneously account for both risk-seeking and risk-averse behavior, as discussed in the aforementioned risk elicitation methods. If the risk elicitation and corresponding utility function *cannot* simultaneously consider these risk behaviors, the mentioned mapping methodology is not directly applicable.

4.3.1 Mapping existence theorem

Clearly, the risk parameter r cannot simply be used as an RO parameter, specifically k_i – the subjective parameter determining the number of coefficients affected by deviation. Instead, a proof is required to demonstrate that r can be mapped to some value of $k_i(r)$. Since the bisection, EG, and HL methods have $r \in (-\infty, \infty)$ and the BRET uses $r \in (0, \infty)$, we discuss the following cases for bounding r, preliminary to proving that either range of risk elicitation values can be mapped to values of a bounded function $k_i(r)$.

Case 1: $\boldsymbol{r} \in (-\infty, \infty)$. Let $S \subset \mathbb{R}$, $k_i(r): S \to \mathbb{R}$, denote S' as the derived set of S (i.e., the collection of all accumulation points of S), and define the open ball of radius ε centered at zero as $B(0, \varepsilon)$. Let $r \in (-\infty, \infty)$ where $r_{min} \in (-\infty, 0)$ and $r_{max} \in (0, \infty)$. Clearly r is not bounded as $\nexists$ a radius $q > 0$ such that $S \subseteq B(0, \varepsilon)$. Consequently, consider w.l.o.g. an extremely large negative number $M^- \in S'$ and an extremely large positive number $M^+ \in S'$ such that $r_{min} \in [M^-, 0)$ and $r_{max} \in (0, M^+]$.

Case 2: $\boldsymbol{r} \in (0, \infty)$. Let $S \subset \mathbb{R}$, $k_i(r): S \to \mathbb{R}$, denote S' as the derived set of S, and define the open ball of radius ε centered at zero as $B(0, \varepsilon)$. Let $r \in (0, \infty)$ where

$0 < r_{min} < r_{max} < \infty$. Clearly, r is bounded below by $r = 0$ but not bounded above as $\nexists$ a radius $q > 0$ such that $S \subseteq B(0, \varepsilon)$. Consider w.l.o.g. an extremely large positive number P such that $M^{-} = (1/P) \in S'$ and an extremely large positive number $M^{+} \in S'$ such that $0 < M^{-} \leq r_{min} < r_{max} \leq M^{+}$.

THEOREM 1. *Given* $r \in S = [M^{-}, M^{+}]$ *and a bounded function* $k_i(r): S \to [0, n]$, *we have* $\lim_{r \to M^{-}} k_i(r) = 0$ *exists when* $\forall\, \varepsilon > 0, \exists\, \delta > 0$ *such that* $|k_i(r) - 0| < \varepsilon, \forall\, r \in S, r \neq M^{-}$ *and* $|r - M^{-}| < \delta$.

PROOF. We have that $r \neq M^{-}$ and $|r - M^{-}| < \delta\ \forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $|k_i(r) - 0| < \varepsilon, \forall\, r \in S$. Via the triangle inequality, we attain $|k_i(r) - 0| \leq |k_i(r) - (r - M^{-})| + |(r - M^{-}) - 0| < \varepsilon$. Thus, let $\delta = \frac{\varepsilon}{|k_i(r) - (r - M^{-})|}$, indicating that $\lim_{r \to M^{-}} k_i(r) = 0$. □

COROLLARY 1. *Given* $r \in S = [M^{-}, M^{+}]$ *and a bounded function* $k_i(r): S \to [0, n]$, *we have* $\lim_{r \to M^{+}} k_i(r) = n$ *exists when* $\forall\, \varepsilon > 0, \exists\, \delta > 0$ *such that* $|k_i(r) - n| < \varepsilon, \forall\, r \in S, r \neq M^{+}$ *and* $|r - M^{+}| < \delta$.

PROOF. Follows from the proof to Theorem 1, letting $\delta = \frac{\varepsilon}{|k_i(r) - (r - M^{+} + n)|}$. □

Therefore, both the risk preference $r \in (-\infty, \infty)$ and the risk preference $r \in (0, \infty)$ can be mapped into the RO framework where $0 \leq k_i(r) \leq n$.

4.3.2 Risk neutrality existence theorem

Given the results of Theorem 1 and Corollary 1, it appropriately follows to prove that risk neutrality exists in $k_i(r)$. Consider the following:

Case 1. Define $r \in (-\infty, \infty)$ as a risk preference parameter value elicited for a continuous and monotonically increasing utility function $u(x)$. Since the risk preference $r \in (-\infty, \infty)$ can be mapped into the RO framework where $0 \leq k_i(r) \leq n$, consider an extremely large negative number $M^- \in S'$ and an extremely large positive number $M^+ \in S'$ and assume w.l.o.g. that $r \in [M^-, M^+]$. Further, denote $r_s = M^-$ as the maximum risk-seeking value and $r_a = M^+$as the maximum risk-averse value.

Case 2. Alternatively, define $r \in (0, \infty)$ as a risk preference parameter value elicited for a continuous and monotonically increasing utility function $u(x)$. Since the risk preference $r \in (0, \infty)$ can be mapped into the RO framework where $0 \leq k_i(r) \leq n$, consider an extremely large positive number P such that $M^- = (1/P) \in S'$ and an extremely large positive number $M^+ \in S'$ and assume w.l.o.g. that $r \in [M^-, M^+]$. Further, denote $r_a = M^-$ as the maximum risk-averse value and $r_s = M^+$ as the maximum risk-seeking value.

LEMMA 1. For Case 1, the risk neutral value is defined by the Arrow-Pratt measure for CARA and CRRA as $c(x; r_n) = 0$ and $g(x; r_n) = 0$, respectively. By definition, $c(x; r_s) < 0$ and $c(x; r_a) > 0$ when assuming CARA, and, likewise, $g(x; r_s) < 0$ and $g(x; r_a) > 0$ when assuming CRRA. Thus, define the risk neutral value r_n such that $r_s < r_n < r_a$.

LEMMA 2. For Case 2, the risk neutral value is defined by the Arrow-Pratt measure for CARA and CRRA as $c(x; r_n) = 0$ and $g(x; r_n) = 0$, respectively. By definition, $c(x; r_s) < 0$ and $c(x; r_a) > 0$ when assuming CARA, and, likewise, $g(x; r_s) < 0$ and

$g(x; r_a) > 0$ when assuming CRRA. Thus, define the risk neutral value r_n such that $r_a < r_n < r_s$.

THEOREM 2. *Let* $k_i(r)$ *be a continuous function and let* r_n *be defined as the risk-neutral value such that* $r_s < r_n < r_a$. *If the mapping* $k_i(r)\colon (r_s,\ r_a) \subset \mathbb{R} \to c$ *where* $c \in [0, n]$ *exists, then* $k_i(r_n) = h$ *and* $k_i(r_s) < h < k_i(r_a)$.

PROOF. Let $A = \{r \in (r_s,\ r_a)\ |k_i(r) < h\}$, let $r_n = sup\ A$, and let $\varepsilon = |k_i(r_n) - h|$. Because $k_i(r)$ is a continuous function, $\exists\ \delta > 0$ such that $|r - r_n| < \delta$ indicating that $|k_i(r) - k_i(r_n)| < \varepsilon$. By contradiction, assume that $k_i(r_n) < h$. Then $\left|k_i\left(r_n + \frac{\delta}{2}\right) - k_i(r_n)\right| < \varepsilon$, so $k_i\left(r_n + \frac{\delta}{2}\right) < k_i(r_n) + \varepsilon = h$. However, $r_n + \frac{\delta}{2} \in A$, indicating that $r_n \neq \sup A$. Further, by contradiction assume that $k_i(r_n) > h$. Then, since $r_n = sup\ A$, $\exists\ \bar{r} \in A$, where $r_n - \delta < \bar{r} < r_n$. But since $|r - r_n| < \delta$ and $|k_i(r) - k_i(r_n)| < \varepsilon$, $k_i(\bar{r}) > k_i(r_n) - \varepsilon = h$, indicating that $\bar{r} \notin A$. Therefore, $\exists\ r_n \in (r_s,\ r_a)$ such that $k_i(r_n) = h$ and $k_i(r_s) < h < k_i(r_a)$. □

THEOREM 3. *Let* $k_i(r)$ *be a continuous function and let* r_n *be defined as the risk-neutral value such that* $r_a < r_n < r_s$. *If the mapping* $k_i(r)\colon (r_a,\ r_s) \subset \mathbb{R} \to c$ *where* $c \in [0, n]$ *exists, then* $k_i(r_n) = h$ *and* $k_i(r_a) < h < k_i(r_s)$.

PROOF. Follows from the proof to Theorem 2, letting $A = \{r \in (r_a,\ r_s)\ |k_i(r) < h\}$. □

4.3.3 Mapping methodology for $\boldsymbol{k_i(r)}$

The mapping methodologies corresponding to different risk elicitation methods and their respective utility functions are described in this section. Specifically, model assumptions are made regarding the function $k_i(r)$, and the mapping methodologies are conditioned on whether a DM is a deltaperson.

4.3.3.1 Assumptions

The risk preference parameter-values elicited vary with different risk elicitation methods because each method adopts different assumptions (i.e., CARA, CRRA, etc.) in conjunction with a utility function. For example, when using the exponential utility function, r signifies risk odds, but when using a power utility function, r represents the CRRA coefficient. Since the value of $k_i(r)$ depends solely on the value of r, the rate of change of r is assumed to be directly proportional to the value $k_i(r)$. In making this assumption, the domain for r-values is considered for each risk elicitation method. Moreover, considering Theorem 1 and Corollary 1, $\lim_{r \to M^-} k_i(r) = 0$ and $\lim_{r \to M^+} k_i(r) = n$. As such, an exponential decay function is appropriate to map the risk-to-RO parameters for either a risk-averse or a risk-seeking individual, particularly when considering extremely strong attitudes. Consequently, $k_i(r)$ should be modeled as a sigmoid function, as the tails of a sigmoid function exhibit the characteristic of exponential decay. Leveraging differential equations [166] to model exponential decay shown in Equation (72), insights concerning the graph of $k_i(r)$ can be inferred, where φ signifies the constant of proportionality and n equates to the maximum value of $k_i(r)$.

$$\frac{dk_i(r)}{dr} = \begin{cases} \varphi \cdot k_i(r), & r \leq r_n \\ \varphi \cdot (n - k_i(r)), & r > r_n \end{cases} \tag{72}$$

To formally utilize Equation (72) to derive a specific $k_i(r)$-mapping, additional assumptions are required, as there is no data available concerning this particular solution.

First, it is shown via Theorem 2 and Corollary 2 that there exists a risk neutral point such that $k_i(r_n) = h$, where $k_i(r_s) < h < k_i(r_a)$, or $k_i(r_a) < h < k_i(r_s)$, respectively. However, Theorem 2 is an existence proof and does not imply an *actual value* for h. This begs the question – what is the correct value for h? As it currently stands, the risk neutral point h is problem specific and depends on the expertise of the DM and/or subject matter expert. Yet, current RO practice denotes the most risk-averse person as $k_i = n$ and suitably, the most risk-seeking individual as $k_i = 0$. Accordingly, it follows that a risk-neutral individual is characterized as $k_i = \frac{n}{2}$. Therefore, it is assumed herein that $k_i(r_n) = h = \frac{n}{2}$ (i.e., the inflection point of the sigmoid function and the midpoint of the vertical axis).

Lastly, to properly parametrize $k_i(r)$, at least three points are required. Since $k_i(r_n) = \frac{n}{2}$ is assumed to be the risk-neutral point, the r- and $k_i(r)$-values corresponding to two additional points near the tails of $k_i(r)$ can reasonably be assumed per Theorem 1. Thus, assume that $k_i(M^-)$ and $k_i(M^+)$ are sufficiently close to 0 and n, respectively.

4.3.3.2 Mapping $(r_s, r_a) \rightarrow [0, n]$ and $(r_a, r_s) \rightarrow [0, n]$

When considering the aforementioned assumptions, the mathematical mapping $(r_s, r_a) \rightarrow [0, n]$ can be derived. It is clear that $k_i(r)$ must be a piecewise function conditioned on the value of $r_n \in (r_s, r_a)$ when applying the bisection, EG, and HL methods. Thus, by exploiting Equation (72), the risk function $k_i(r; h, r_n)$, Equation (73), is generated when risk is elicited using these particular methods.

$$k_i(r; h, r_n) = \begin{cases} he^{\varphi(r-r_n)}, & r \leq r_n \\ n - he^{\varphi(r_n-r)}, & r > r_n \end{cases} \tag{73}$$

Of note, the risk neutral point $k_i(r_n) = h$ is denoted in the general sense, even though the assumption is made that $k_i(r_n) = h = \frac{n}{2}$ for this paper.

Similarly, when eliciting risk using the BRET, the mapping $(r_a, r_s) \rightarrow [0, n]$ is slightly more complex as risk is represented on a different scale and the risk parameter is $0 \leq r < \infty$, which differs from the other risk elicitation methods. Thus, the risk function $k_i(r; h)$ when using the BRET is offered in Equation (74) below, wherein $r_n^{BRET} = 1$.

$$k_i(r; h) = \begin{cases} n - he^{\varphi(1-r)}, & 0 \leq r \leq 1 \\ he^{\varphi(r-1)}, & r > 1 \end{cases} \tag{74}$$

Leveraging Equations (73) and (74) as well as the reference points described in Table 15, the original $k_i(r)$-functions corresponding to each risk elicitation method are depicted in Figures 15-18, where Figure 19 depicts the BRET on a reverse r-scale. Of note, each figure displays results from an example instance with seven decision variables. Table 15 denotes ρ as a predetermined parameter that determines the "sufficiently close" to 0 and n measure. In this paper, we assume $\rho = 0.01$ (i.e., if $n = 7$, then 6.99 and 0.01 are considered "sufficiently close" for $k_i(M^-)$ and $k_i(M^+)$, respectively). Furthermore, the M^-- and M^+-values for the bisection method are problem specific and depend on the values of a and b; however, they should result in symmetric values around r_n as the piecewise exponential utility function is symmetric in nature. For notational clarity, we define $r_n^{(*)}$, where $(*)$ references the risk elicitation method applied.

Table 15: Risk elicitation Reference Points for RO Mapping for $k_i(r_n) = h = \frac{n}{2}$ and $\rho = 0.01$

Utility function	Risk elicitation method	r_n^*	M^-	M^+	$k_i(r_n^*)$	$k_i(M^-)$	$k_i(M^+)$
Exponential	Bisection	$r_n^B = 1$	-0.62	2.62	$n/2$	0.01	$n - 0.01$
Power	EG	$r_n^{EG} = 0$	-1	3.46	$n/2$	0.01	$n - 0.01$
	HL	$r_n^{HL} = 0$	-0.95	1.37	$n/2$	0.01	$n - 0.01$
	BRET	$r_n^{BRET} = 1$	68.275	0	$n/2$	0.01	$n - 0.01$

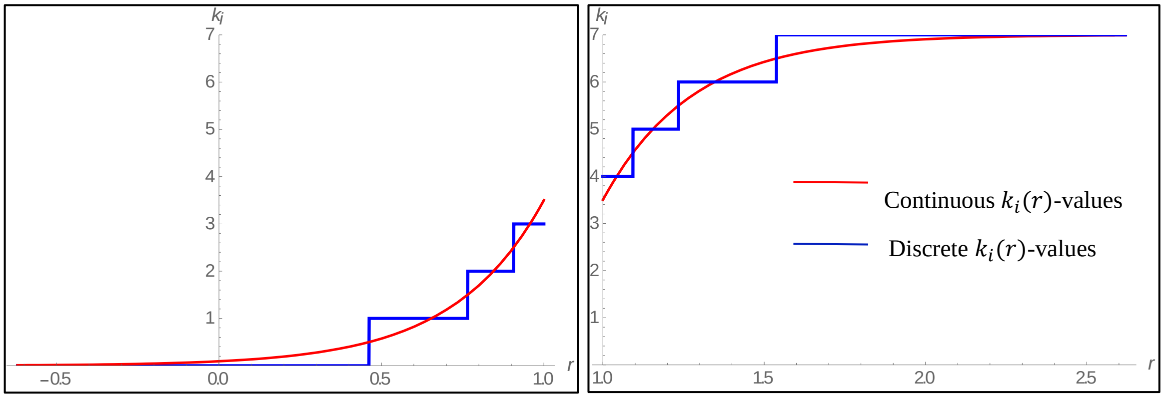


Figure 15: $k_i(r)$-mapping via Bisection Method for $r \in (M^-, r_n)$ (left) and $r \in (r_n, M^+)$ (right)

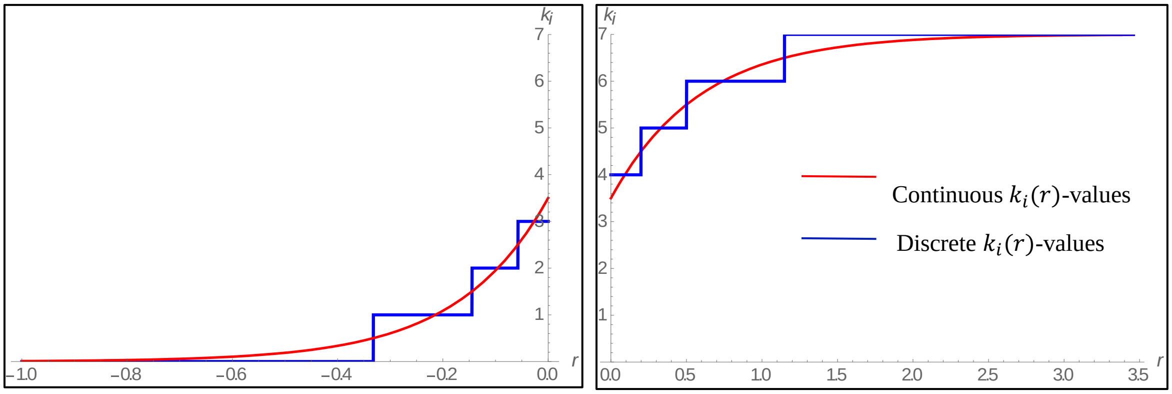


Figure 16: $k_i(r)$-mapping via EG Method for $r \in (M^-, r_n)$ (left) and $r \in (r_n, M^+)$ (right)

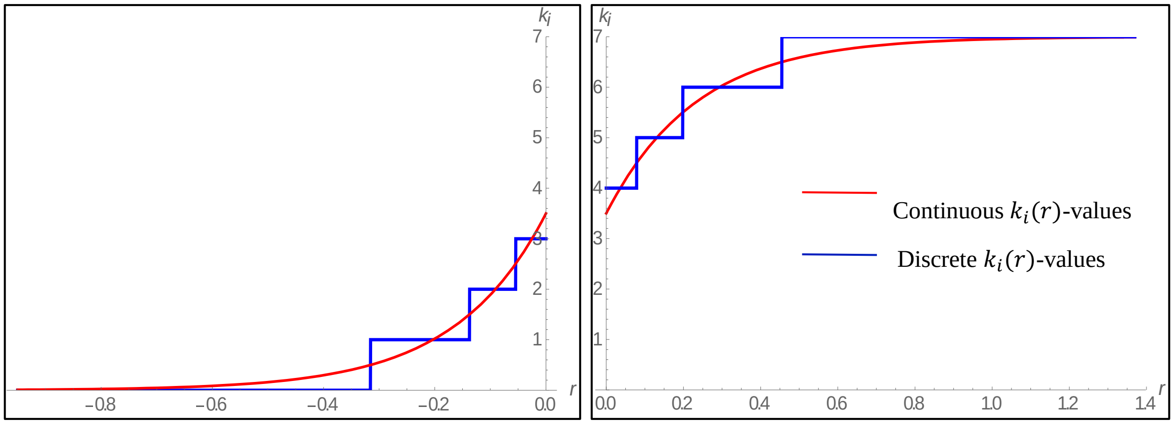


Figure 17: $k_i(r)$-mapping via HL Method for $r \in (M^-, r_n)$ (left) and $r \in (r_n, M^+)$ (right)

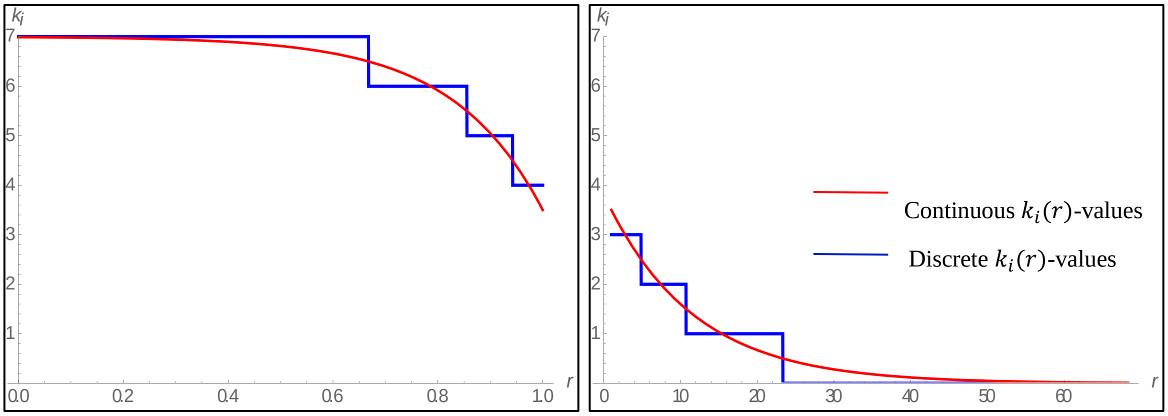


Figure 18: $k_i(r)$-mapping via BRET for $r \in (M^-, r_n)$ (left) and $r \in (r_n, M^+)$ (right)

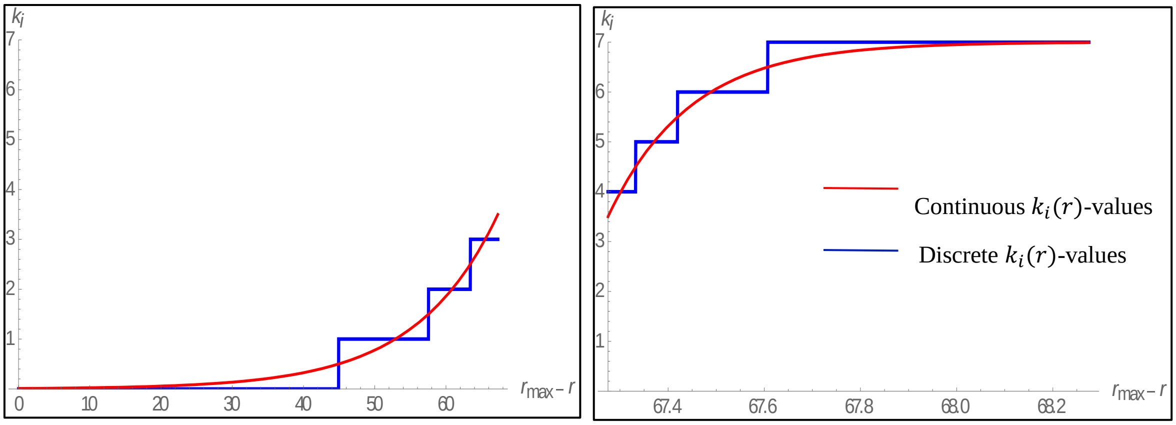


Figure 19: Reverse $k_i(r)$-mapping via BRET for $r \in (M^+, r_n)$ (left) and $r \in (r_n, M^-)$ (right)

4.3.4 Mapping methodology for $\boldsymbol{\theta_i(r)}$

Another RO methodology to account for uncertainty in data is discussed by Hanks *et al.* [7] where the parameter θ_i considers the DM's risk. Current RO methodologies suggest that the most risk averse or risk seeking individuals will adopt the values $\theta_i = 0$ and $\theta_i = \sqrt{n}$, respectively. Consequently, we assume that $\theta_i = \frac{\sqrt{n}}{2}$ constitutes risk neutrality. Because the domain for $\theta_i \in \left[0, \sqrt{n}\right]$, and since $\left[0, \sqrt{n}\right] \subseteq [0, n]$, $\theta_i \in k_i$. Consequently, Theorems 1-3 apply to the RO methods that utilize θ_i as a risk parameter. It follows that $\theta_i(r_n) = d$ and $\theta_i(r_s) < d < \theta_i(r_a)$ when applying the bisection, EG, or HL methods, and $\theta_i(r_a) < d < \theta_i(r_s)$ when employing the BRET. Similarly, the actual value of d is generally determined by the DM, but a sufficient assumption is to let $\theta_i(r_n) = d = \frac{\sqrt{n}}{2}$. The $\theta_i(r; d, r_n)$ mapping functions for the bisection, EG, and HL methods, as well as the $\theta_i(r; d)$ function for the BRET method are respectively presented as Equations (75) and (76).

$$\theta_i(r; d, r_n) = \begin{cases} de^{\varphi(r - r_n)}, & r \leq r_n \\ \sqrt{n} - de^{\varphi(r_n - r)}, & r > r_n \end{cases} \tag{75}$$

$$\theta_i(r; d) = \begin{cases} \sqrt{n} - de^{\varphi(1 - r)}, & 0 \leq r \leq 1 \\ de^{\varphi(r - 1)}, & r > 1 \end{cases} \tag{76}$$

4.4 Testing and analysis

The following results and ensuing discussion consider a demonstration of the mapping methodology for both $k_i(r)$ and $\theta_i(r)$ as applied to the problem instance examined by Kuchta [3] and using RGP methodologies offered by Hanks *et al.* [7]. Due to its ease of implementation, we use the BRET to generate a DM's risk preference while assuming

$k_i(r) = k_j(r), \ \forall i \neq j$ (i.e., all goals are associated with the same DM risk preference). To that end, two instances are discussed in more detail regarding the discrete case for $k_i(r)$, wherein different goals can have varying risk levels, as well as the continuous case for $k_i(r)$. However, prior to such a discussion, categorizing varying levels of risk preference is required.

4.4.1 Categorizing risk

Crosetto and Filippin [6] consider three categories of risk in a BRET setting where, if $v^* < 50$, $v^* = 50$, or $v^* > 50$ an individual is labeled risk averse, risk neutral, or risk seeking, respectively (recall v^* denotes the number of boxes collected by an individual). However, having just three risk preference categories for over 100 choices is coarsely aggregated and can mislead interpretations of the meaning of the test. To account for this, the number of risk categories seen in the HL method are contemplated. The HL method is chosen over the other risk elicitation methods because the HL has a more refined taxonomy of risk categories. Furthermore, Crosetto and Filippin [124] discuss that when estimating risk in an EUT setting, the BRET and HL methods are the preferred choice. Because this paper considers EUT maximization to estimate risk, modifying the BRET risk preference classifications by referring to the HL method is the clear choice.

The HL method takes into consideration nine risk preference categories, where five are classified as risk averse, one risk neutral, and three risk seeking [142]. The HL method accounts for the notion that most people are risk averse, which explains why there are two more risk averse categories than risk seeking ones. Moreover, because a risk neutral r-value is more skewed toward the maximum risk seeking r-value, a wider range of risk averse classifications exists with the HL taxonomy. Herein, we consider the BRET range of 1-99 boxes, each with associated r-values, where risk neutrality is considered to exist at the median. We make the assumption that no person will stop at 0 boxes (as it is equivalent to them not partaking in the experiment) nor will they stop at 100 boxes (as it assures a loss of accumulated earnings). Consequently, and unlike the HL method, the BRET's associated risk preference classifications are symmetric around risk neutral. Because of this, we too consider nine risk preference categories for the BRET, but we do so with an equal number of risk averse and risk seeking categories, in addition to a risk neutral category.

A challenge in classifying the nine risk preference categories for the BRET is identifying the test values corresponding to transitions between risk preference categories. With the help of Crosetto and Filippin [6], who provided the raw data used to create the BRET, the below analysis is conducted, starting with the histogram seen in Figure 20.

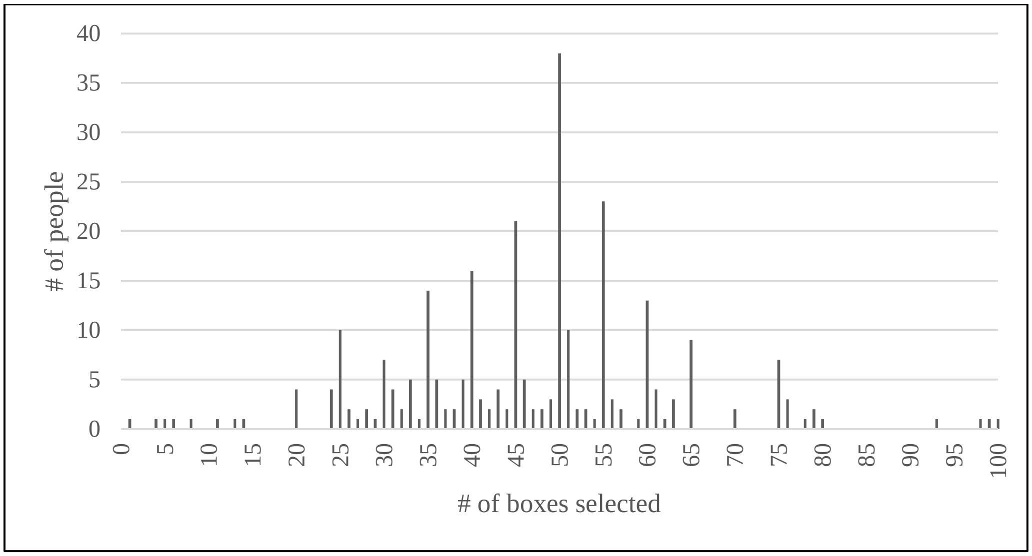


Figure 20: BRET Histogram of the Number of Boxes Selected for 271 Individuals

When analyzing the raw data in Figure 20, one observation is clear: individuals tend to stop at interval increments of five. A total of 167 out of 271 (i.e., 65%) of the BRET participants stopped the time on an interval of five seconds (or boxes). Moreover, when categorizing risk preference for the BRET, we make some inferences regarding the mindset of the individuals. For instance, an artifact of wanting to be classified as slightly risk seeking can be attested to those individuals who stopped at 51 seconds, whereas those individuals who waited slightly longer can be classified as risk seeking. Accounting for these trends observed in the raw data, refined BRET risk preference categories are presented in Table 16. Per Crosetto and Filippin [6], "the bulk of subjects is either risk averse, risk neutral, or slightly risk loving". In accordance with their findings, 50.18% of the individuals under the refined BRET taxonomy are categorized as risk averse, slightly risk averse, risk neutral, or slightly risk neutral.

Table 16: Refined BRET Risk Preference Categories

Number of boxes collected (or stopping times), v^*	Range of Relative Risk Aversion Coefficient, r^{BRET}	Risk Preference Classification
$\{1, 2, \dots, 24\}$	$[0.000, 0.324]$	Highly risk averse
$\{25, 26, \dots 34\}$	$(0.324, 0.526]$	Very risk averse
$\{35, 36, , \dots 44\}$	$(0.526, 0.801]$	Risk averse
$\{45, 46, \dots 49\}$	$(0.801, 0.980]$	Slightly risk averse
$\{50\}$	$(0.980, 1.020]$	Risk neutral
$\{51\}$	$(1.020, 1.061]$	Slightly risk seeking
$\{52, 53, \dots, 59\}$	$(1.061, 1.469]$	Risk seeking
$\{60, 61, \dots 74\}$	$(1.469, 2.921]$	Very risk seeking
$\{75, 76, \dots, 99\}$	$(2.921, 68.275]$	Highly risk seeking

Based on these justifications, we apply the proposed BRET risk preference categories to a test instance to verify that the mapping methodology works as expected.

4.4.2 Results

To confirm that the aforementioned mapping methodology works properly, the mathematical mapping procedure is applied to a simple test instance examined by Kuchta [3]. Instead of using the minimum or maximum value for each corresponding risk preference classification, the midpoint for each r-interval, $r_{midpoint}^{BRET}$ is considered for the ensuing analysis. For visualization, Table 17 reports the $r_{midpoint}^{BRET}$ and the respective $k_i(r_{midpoint}^{BRET})$ for each risk preference classification using the L_1-norm RGP model discussed in Section 4.1.

Table 17: BRET $r \rightarrow k_i(r)$-values for Proof of Concept (L_1-norm)

Level of Conservatism	Highly Risk Averse	Very Risk Averse	Risk Averse	Slightly Risk Averse	Risk Neutral	Slightly Risk Seeking	Risk Seeking	Very Risk Seeking	Highly Risk Seeking
r_{median}^{BRET}	0.162	0.425	0.664	0.891	1	1.041	1.265	2.195	35.598
$k_i(r_{median}^{BRET})$	2.977	2.916	2.722	2.133	1.5	1.495	1.471	1.372	0.114

4.4.2.1 Scaling issues

The objective function (77) and goal constraints (78)-(81) as presented in [3] are displayed as follows, where x_1, x_2, and x_3 correspond to Products 1-3, respectively. Moreover, Constraints (78)-(81) respectively relate to the amount of material needed, the amount of human work necessary, the amount of machine time required, and the selling price for each of the three products. The coefficients on each x_j-variable signify the expected amount of each product, $j = 1, 2, 3$, corresponding to each goal $i = 1, 2,3,4$. As presented earlier, the least upper bound penalty p_i is determined by additional constraints that consider coefficients subject to change with varying uncertainty sets, given some robustness parameter k_i (or θ_i). In this particular instance, the maximum possible deviation is 10% of each coefficients expected, or nominal, value.

$$Minimize \quad d_1^+ + d_2^+ + d_3^+ + d_4^+ \tag{77}$$

$$subject\ to\ 3x_1 + 7x_2 + 5x_3 + p_1 - d_1^+ + d_1^- = 200 \tag{78}$$

$$6x_1 + 5x_2 + 7x_3 + p_2 - d_2^+ + d_2^- = 200 \tag{79}$$

$$3x_1 + 6x_2 + 5x_3 + p_3 - d_3^+ + d_3^- = 200 \tag{80}$$

$$-28x_1 - 40x_2 - 32x_3 + p_4 - d_4^+ + d_4^- = -1500 \tag{81}$$

Kuchta [3] incorporates weights $w_i, \forall\, i \in M$ into his formulation but, for the small problem instance described, assumes all $w_i = 1$, suggesting each goal is equally important. Yet, this is not the case given the respective goal's target values. For instance, when looking at Constraint (81), the target values for Goal 4 are an entire order of magnitude higher than the others. As a result, $d_4^+ = 0$ at optimality regardless of the DM's risk preference (i.e., the fourth goal is *always* met). In this regard, the target values pertaining to each specific goal act as implicit weights wherein, the larger the target value, the more important the goal. In turn, this portends scaling concerns that affect how much of each product should be manufactured for the problem instance seen in Kuchta [3].

Scaling issues are commonly witnessed in GP due to such factors as varying units or conflicting goals, so it is not a surprise to see the same issues arise in the RGP construct. In this particular instance, it appears that not only are the target values for the fourth goal much larger than the previous three goals, satisfying it induces greater deviation from satisfying Goals 1-3. As formulated, Goal 4 is 7.5 times more important than the other goals (i.e. 1500/200). To remedy this, multiplying the first three deviational variables in the objective function by 7.5 should alleviate the scaling issue. Yet, when the problem is solved in this fashion, the only goal that takes on any deviation at optimality is Goal 4, which seems counterintuitive. However, because the satisfaction of Goals 1-3 and the satisfaction of Goal 4 are in conflict, the optimal solution to the math program does not meet one goal (i.e., Goal 4) in favor of meeting three goals (i.e., Goals 1-3). So, the number of goals working in concert can also lead to issues in GP. As a result, we assign weights of $w_1 = w_2 = w_3 = 1$ to the first three goals and $w_4 = \frac{3}{7.5}$ for the fourth goal. In doing so, the problem is properly scaled and takes into consideration the number of conflicting goals as well as scaling issues for the respective goals' magnitudes.

It is important to comment that, in practice, one would not generally consider how many goals are conflicting since a DM will likely want to meet as many goals as possible. However, we find it appropriate to demonstrate the utility of our mapping tool vis-à-vis different DM risk preferences and their respective robustness for possible outcomes.

4.4.3 Analysis

The analysis performed in this section is conducted using three RGP models from Hanks *et al.* [7] in conjunction with the proposed mapping methodology. Moreover, the ensuing analysis considers the scaled problem formulation for which the objective is to minimize total scaled deviation. The following discussions illustrate the relative robustness of different risk preferences via the outcomes resulting for optimal solutions under the range of observed variability in parametric data.

4.4.3.1 Cardinality constrained robustness via the L_1-norm

When using cardinality constrained robustness with the L_1-norm, there were zero instances of risk preference parameters for which $x_3^* > 0$, given the $r_{midpoint}^{BRET}$-values used. Moreover, as a result of having only three decision variables, some risk preference categories and their corresponding solutions are approximately equal and, as such, some risk categories are aggregated for a better graphical representation of the analysis. In the case where risk preference categories are coupled, the $r_{midpoint}^{BRET}$-value is used over the respective r^{BRET} range. Thus, the remaining discussion considers the products x_1 and x_2, as well as total scaled deviation, $\left(\text{i.e.}, d_1^+ + d_2^+ + d_3^+ + \frac{3d_4^+}{7.5}\right)$, as applied to the aggregated risk preference categories, seen in Table 18 and Figures 21-22.

Table 18: Aggregated BRET $r \rightarrow k_i(r)$-values and Corresponding Solution Set (L_1-norm)

	Most RA-SRA	RN-RS	VRS	HRS	Most RS
r_{median}^{BRET}	0.000	1.225	2.195	35.598	62.875
$k_i(r_{median}^{BRET})$	3.000	1.475	1.372	0.114	0.000
x_1^*	8.658	25.765	25.868	23.279	20.833
x_2^*	25.974	18.035	18.107	21.449	22.917

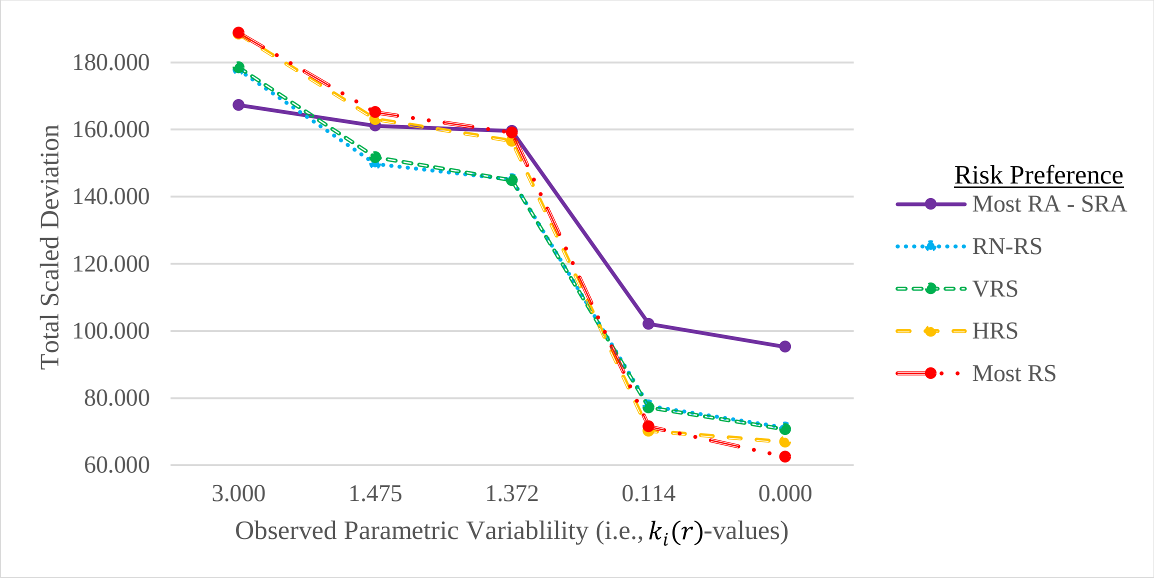


Figure 21: Total Scaled Deviation for Varying Risk Preference Solution Sets (L_1-norm)

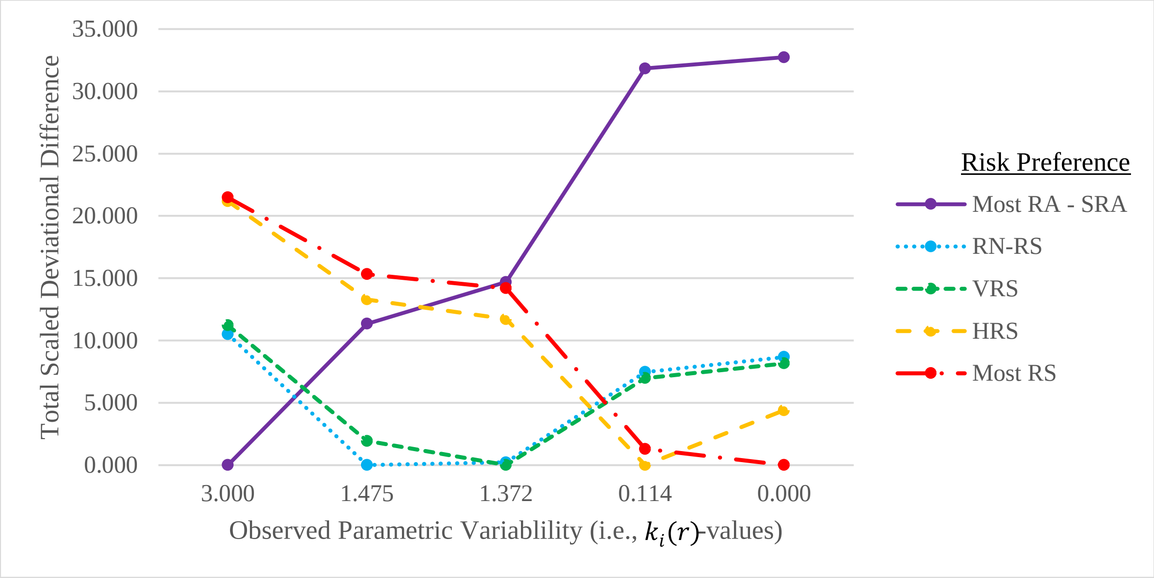


Figure 22: Difference in Best-case and Worst-case Total Scaled Deviation for Varying Risk Preference Solution Sets (L_1-norm)

Figure 21 demonstrates the impact on total scaled deviation given the solution set of an assumed risk preference (i.e., the lines described in the legend) and the solution set of observed parametric uncertainty (i.e., the x-axis). Knowing such information enables the DM to draw inferences regarding the impact that their particular risk preference has on a solution, which may or may not validate what the DM believes to be their true risk preference. Moreover, Figure 21 assists in answering the question, “Does our mapping methodology work as expected?” and, based on the testing results, the observable trends in Figure 21 support the validity of our mapping methodology for many reasons. First, the trends in total scaled deviations are monotonically decreasing as a DM’s risk preference changes from the most risk averse to the most risk seeking. Second, the mathematical mapping generates $k_i(r)$-values appropriate for the risk preference elicited, as the total scaled deviation for the varying risk classifications falls between the most risk averse and most risk seeking values. Furthermore, Figure 21 illustrates the value (or luck) of accurately predicting variability, as well as the negative consequences of not predicting the correct level of parametric variability. For example, if a DM’s risk preference and corresponding solution are accurate with the true, observed data (i.e., clairvoyance), they are rewarded with the lowest total scaled deviation. Likewise, a DM is penalized with higher total scaled deviation if their assumed solution set (corresponding to their elicited risk preference) does not align with the observed value.

Similarly, Figure 22 illustrates the importance of correctly predicting the number of coefficients that will fluctuate, where correctly doing so always yields the best objective function value. More specifically, however, Figure 22 plots the difference in total scaled deviation of the best solution set when compared to the other, un-clairvoyant solution sets. In doing so, the DM has valuable knowledge concerning how their particular risk preference and respective solution set affects robustness regarding total scaled deviation. To that end, Figure 22 illustrates the value of not adapting either an extremely risk averse or extremely risk seeking preference. This is evident in the more varied results (i.e., the larger difference in total scaled deviation when compared to the best possible instance) seen with the most risk averse DM as well as the most risk seeking DM. Conversely, a DM whose risk preference is closer to risk neutral is willing to accept more total scaled deviation for the sake of robustness. If a DM is confident in their risk preference, they should entrust their true risk preference and corresponding solution set in the hopes they are correct; because if they are, they will achieve the lowest total scaled deviation. If a DM is not confident in their risk preference, they should elect a solution mirroring a less extreme risk preference in an attempt to minimize the worst-case, total scaled deviation.

4.4.3.2 Varied RGP models

To further examine the proposed mapping methodology, we apply it in concert with cardinality constrained robustness via L_2-norm uncertainty sets, as well as with strict robustness and ellipsoidal uncertainty sets. As with the L_1-norm analysis, the cardinality constrained by means of the L_2-norm model uses the same $r_{midpoint}^{BRET}$-values and correspondingly the same aggregated risk preference classifications, with results depicted in Figures 23-24.

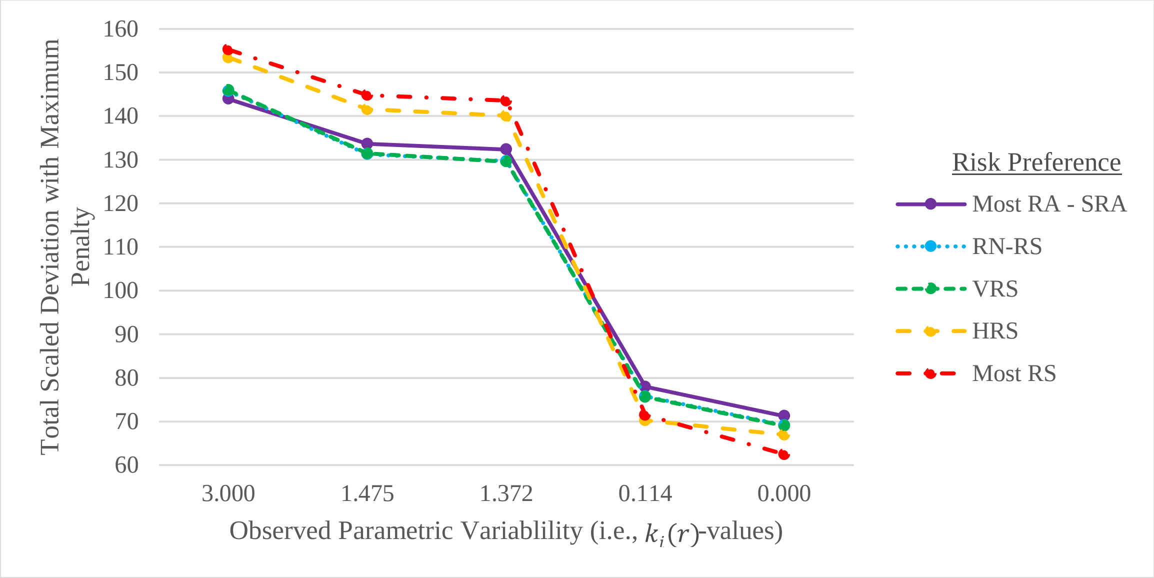


Figure 23: Total Scaled Deviation for Varying Risk Preference Solution Sets (L_2-norm)

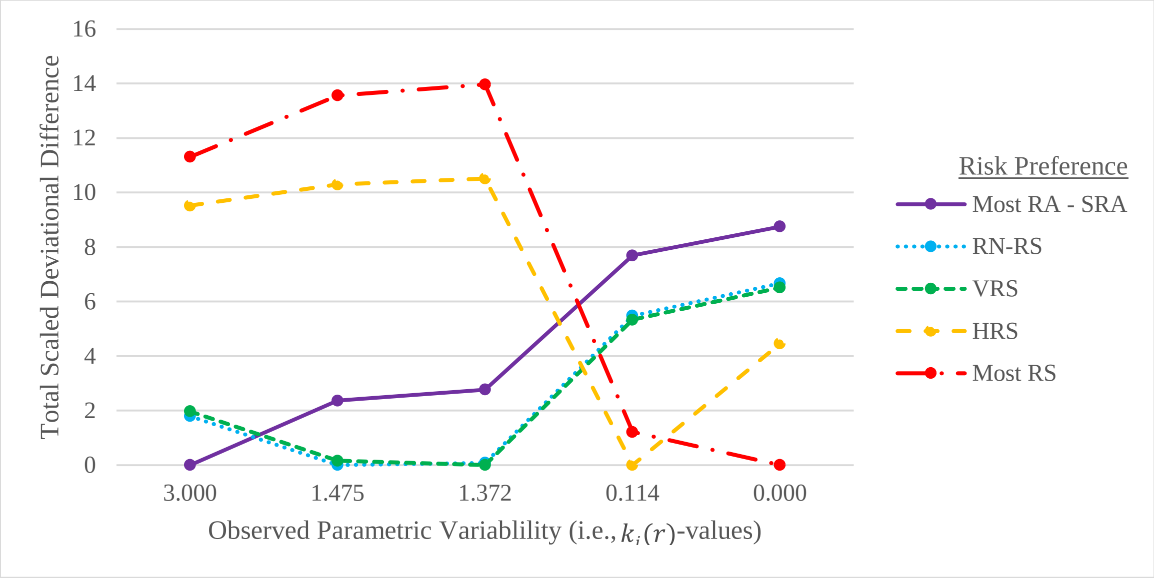


Figure 24: Difference in Best-case and Worst-case Total Scaled Deviation for Varying Risk Preference Solution (L_2-norm)

Alternatively, the strict robustness via ellipsoidal uncertainty sets model results in more varying solutions and, as a result, the risk preference categories are aggregated from nine to six categories instead of five categories, with results illustrated in Figures 25-26.

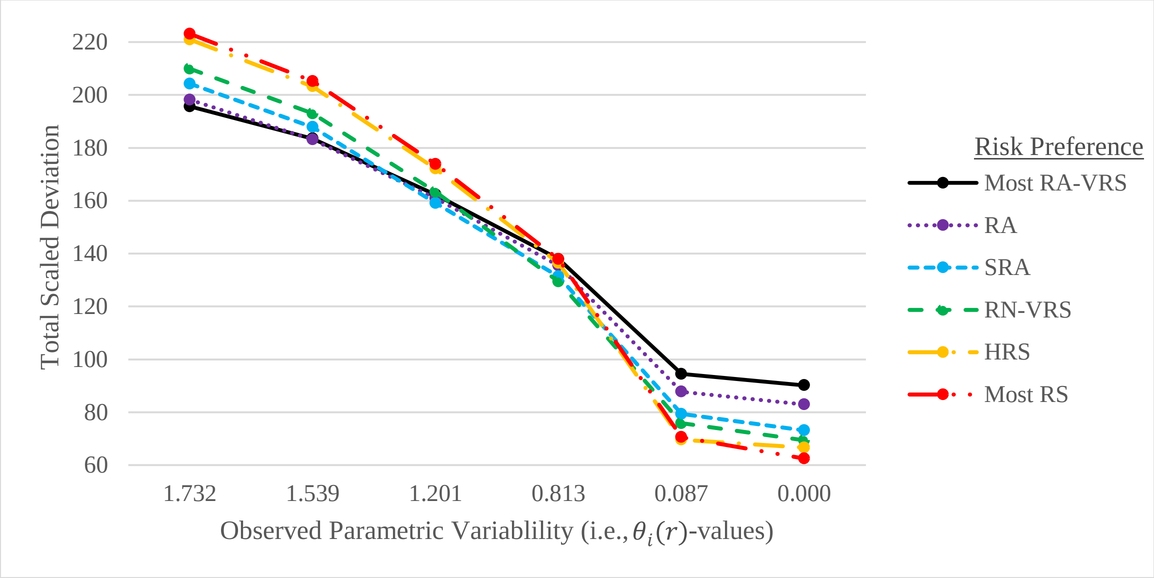


Figure 25: Total Scaled Deviation for Varying Risk Preference Solution Sets (Ellipsoidal)

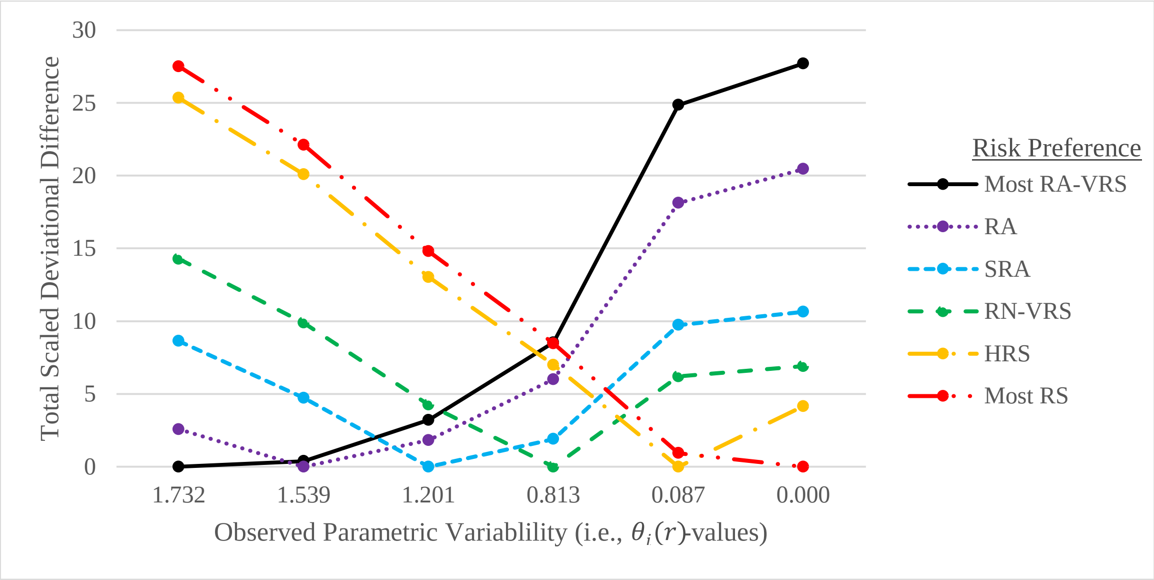


Figure 26: Difference in Best-case and Worst-case Total Scaled Deviation for Varying Risk Preference Solution (Ellipsoidal)

As before, both the L_2-norm and ellipsoidal set results show monotonically decreasing total scaled deviation as a DM's risk preference becomes more risk seeking. Additionally, the value of accurate foresight is again evident and extremely beneficial, as predicting the correct degree of parametric variability will achieve the lowest total scaled deviation for both RGP models. Finally, these results indicate that extreme risk attitudes incur much more varied results than more risk neutral attitudes.

4.4.4 Discussion

As described earlier, *a posteriori* analysis is a burden and can be inefficient. For the simple instance presented, there are three decision variables, all of which have parameters subject to uncertainty, and only four goals. In the simplest discrete case wherein the DM's risk preference is applied equally to all goals, there are $4^4 = 256$ (i.e., m^{n+1}) possible unique solutions. This number increases when the DM has differing risk preferences for various goals, suggesting the *a posteriori* approach is challenging to implement. To that end, when the RO risk parameters are continuous, the *a posteriori* parametric analysis becomes intractable, yielding an infinite number of combinations for parametric variations. As such, the proposed mapping methodology that enables *a priori* analysis when considering a DM's risk preference is the only logical way forward for RGP and RO, in general.

4.5 Conclusion

RO is an established and well known field in the discipline of operations research. Accounting for a DM's risk preference for RO models has been studied in literature, but not in a manner heretofore that begets practical application. This paper offers a mapping methodology that accounts for a DM's true risk preference by way of any of several widely accepted risk elicitation methods. Using the BRET and the proposed risk preference categories, a small instance in the literature is offered to demonstrate the effectiveness of the proposed mapping methodology across a variety of RO models.

As an area for future research, we note that the parameterization of each of our differential equations based mappings required the assumption of three $(r, k(r))$-points that respectively represent extreme views for risk averse and risk seeking preferences, as well as a risk neutral preference. Although these selected points accurately represent mappings derived using risk attitude test limits from the literature, there exists potential for improvement. More specifically, a more accurate mapping may be attainable for specific DM risk preferences at intermediate risk preferences via extensive research and testing using human subjects.

Moreover, because the empirical results of this study suggest that a risk-neutral risk-attitude demonstrates the least variability (i.e., the most robust), it would be of merit to theoretically prove this phenomenon.

V. Applying Risk Assessment and Robust Goal Programming to the United States Transportation Command's Rate Setting Problem

5.1 Introduction

United States Transportation Command (USTRANSCOM) is responsible for the technical direction and supervision for all Department of Defense (DoD) passenger, cargo, mobility, and personal property movements wherein they provide global mobility solutions during both peacetime and wartime environments. USTRANSCOM pays for said movements via a reimbursable operation using shipment rates identified by TCJ8 (e.g. Financial Management and Program Analysis) and charged to DoD customers, where the rates are determined using a combination of expected shipment costs, expected overhead costs, and projected workload. Upon providing their transportation services, government and contracted benefactors are reimbursed by USTRANSCOM. Unlike third party logistics (3PLs), such as FedEx or UPS, for which the primary mission is to make a profit, USTRANSCOM's principal goal is to execute the shipment of DoD goods and services neither at a profit, nor a surplus. If the rates set forth by TCJ8 are not accurate with respect to the actual shipping costs, plus overhead charges reimbursed amount (i.e., the actual costs) paid to the transportation provider, a surplus or deficit results. In the event of a deficit or surplus, the Transportation Working Capital Fund (TWCF) serves as a monetary safeguard to guarantee USTRANSCOM has the funds required for normal operation. Although TCJ8 is responsible for setting both air and liner rates, this study investigates the current rate setting methodology and proposes a new weighted average calculation, as well as a robust goal programming (RGP) model regarding liner rates, exclusively.

5.1.1 Current rate setting methodology

Any DoD liner shipment of goods or services has fixed costs (e.g., maintenance, government employee wages) and variable costs (e.g., tonnage, fuel based on distance traveled). The fixed (or overhead) costs are accounted for via a net, relative adjustment to every inflation-adjusted average cost (i.e., the accumulated operating result (AOR)), whereas the variable costs are calculated by considering 2-1/2 years-worth of data to calculate a weighted average cost. These variable costs depend on the distance between the origin and destination (i.e., OD pair), commodity type, booking terms, and container size.

TCJ8 is accountable for setting transportation rates for their DoD customers, and the rates must be set prior to each fiscal year (FY). The primary goal when setting these rates is to obtain a net operating result (NOR) of zero profit, where the NOR is simply the rates charged to DoD customers by USTRANSCOM minus the costs spent by USTRANSCOM to reimburse government or contracted transportation agencies variable costs as well as fixed costs to administer the process (i.e., overhead costs). In other words, obtaining a $0 NOR will result in a steady TWCF balance.

TCJ8 configures liner rates using a 2-1/2-year, weighted average from the previous FY’s costs. Each weighted average cost is then adjusted by a refresh rate and an AOR factor to estimate the next FY’s rate. The refresh rate considers yearly inflation that is predetermined and written into a contract based on evaluation of the most utilized carrier’s historical data. The AOR factor considers the amount of surplus or deficit (i.e., NOR) for a two-year period and offsets that same amount when setting future rates to recover overhead costs. The AOR factor adjusts rates to both recoup overhead costs and offset the net gains (or losses) made over the previous two years of operations. In essence, the AOR is a recoupment factor that accounts for yearly overhead costs and the last two yeas’ net results.

Previously, TCJ8 applied an aggregated rate setting methodology wherein each rate combination was grouped by OD pair, and the direction of travel was not accounted for. This process assumed that the direction of flow between an OD pair did not affect its cost, and it was previously integrated into TCJ8’s rate setting methodology as a means to reduce the number of unique rates that must be determined annually. Since FY16, TCJ8 set aside this assumption and now sets rates unique to the direction of travel for each OD pair as a measure to ensure rates are equitable for its DoD customers. However, with the disaggregation of OD pairs, less data is available to determine each respective rate in the upcoming fiscal year. As a result, TCJ8 currently considers 2-1/2-years of historical cost data for any OD pair, commodity type, booking term, container size combination to address the issue of data sparsity. Yet, considering data over a longer period of time also increases the amount of uncertainty observed in the data. Currently, TCJ8 does not actively account for such uncertainty. As a way forward, TCJ8 should evaluate data uncertainty via well-known robust optimization (RO) techniques. Furthermore, in an attempt to meet their multiple goals, TCJ8 should utilize RO in concert with goal programming (GP) to not only assess data variability, but also meet their rate setting objectives.

5.1.2 Proposed methodology

The proposed methodology for solving the disaggregated rate setting problem relies solely on the overall task set forth by USTRANSCOM and executed by TCJ8 – to set *robust* shipping rates annually while meeting the *goal* of attaining an NOR of $0. Thus, the implementation of RO techniques are offered in conjunction with GP models via a relatively new method known as RGP. The remainder of this subsection motivates the use of RGP by discussing the need for GP models that account for data uncertainty.

RO models data uncertainty via uncertainty sets in a manner different from stochastic optimization (SO), which requires known probability distributions for uncertainty in parametric data. With USTRANSCOM's disaggregation of shipping rates, very little historic data is available, which makes garnering a true probability distribution extremely difficult [4], [5]. Likewise, accounting for the variability inherent in the disaggregated rates via uncertainty sets alleviates the need for probability distributions – precisely the reason RO is recommended in lieu of SO. Meanwhile, GP is the chosen multi-objective approach because USTRANSCOM has multiple goals and associated target values. However, GP is generally thought to be a deterministic approach, whereas the USTRANSCOM rate setting problem is clearly stochastic in nature. To account for both uncertainty in the data as well as USTRANSCOM's specific goals, RGP is recommended as the modeling technique for TCJ8 and demonstrated herein.

5.1.3 Contributions

This study's contribution is to provide and demonstrate a methodology for TCJ8 to set liner shipping rates that will account for specific USTRANSCOM goals (i.e., maintaining the TWCF balance and reducing rate volatility for customers by minimizing the variation of each shipping rates from year-to-year).

This paper demonstrates the applicability of the RGP methods in the literature, specifically the cardinality constrained via L_2-norm uncertainty sets model as set forth by Hanks *et al.* [7]. Moreover, this paper offers a new methodology that leverages the aforementioned RGP model to solve future transportation rate-related problems, setting robust shipping rates that address uncertainty in the cost-related data while seeking to meet two USTRANSCOM goals.

This paper is organized as follows. Section 5.2 provides a review of the related literature from the respective areas of rate setting and RGP. Section 5.3 presents a USTRANSCOM-specific RGP formulation, given historic liner cost-data and selected sources of uncertainty. Considering a range of decision maker (DM) risk attitudes, Section 5.4 applies the model for setting FY16 rates, comparing the outcomes to currently utilized rate setting methods, and Section 5.5 concludes the work and identifies possible extensions for future research.

5.2 Literature review

Due to its proprietary nature, the published literature related to rate setting practices is sparse, whereas the literature related to RGP models is nascent and developing. The research herein builds upon recent RGP modeling developments as well as the technical paper by Hanks *et al.* [167] that informs the parameterization of such RGP models for a given DM's risk preference (i.e., the degree of risk aversion).

5.2.1 Rate setting

Rate setting is an extremely complex process that takes much time, discussion, and analysis to complete. However, most of the literature related to rate setting is proprietary because 3PL’s want to have a competitive advantage in the market and, consequently, useful published information is limited. However, a few works pertaining to liner rate setting methodologies, policies, and theoretical inferences do influence this study of the USTRANSCOM rate setting problem.

Shneerson [147] extensively discussed liner rates and made conclusions that liner freight rates can be explained via pricing and demand. The author identified the most important factors influencing liner rates to be the ratio of volume to weight (i.e., stowage factor) and the unit value of the commodities being shipped. At the time, it was common practice to label cost as the most influential factor when setting liner rates, but Shneerson showed that the stowage factor is the most influential element. However, in a later study done by Brooks and Button [148], it is demonstrated that stowage factors can sometimes be a misleading variable when determining liner rates. The authors identified that the type of customer (e.g., freight forwarder, consignee, or shipper) can vary the requirement related to liner rate setting. Moreover, the authors discussed extensively how the aggregation of rates can yield inaccurate rates, as the direction of a shipping route directly influences the cost of transport.

Also closely related to this study is work by Skinner *et al.* [151], wherein the authors provided econometric modeling results showing the amount of variability in rate setting can be attributed as follows: 75-80% to distance (or OD pair), 8-10% to geography, and 1-3% to volume.

Beyond these works, there are several recent studies on shipping costs and rate setting (e.g., see [149], [146], and [150]), but none are sufficiently related to inform the specific problem considered herein.

5.2.2 RGP

RGP is a relatively new area in the field of RO that was first introduced by Kuchta [3] to combine cardinality-constrained robustness and interval-based uncertainty with popular GP techniques. When compared to Soyster's [2] strict robustness, cardinality-constrained robustness [4], [5] is a less conservative RO model predicated on the idea that not all coefficients subject to parametric uncertainty will realize their worst-case deviational values. More explicitly, consider a robustness measure k_i, for every constraint $i = 1,2,\dots,m$, of a linear program, that regulates the required level of conservatism and takes on values in the interval $[0,|S_i|]$, wherein S_i is the set of coefficients $a_{ij}, \forall\, j \in S_i$ that are vulnerable to parametric uncertainty in the interval $\left[a_{ij} - \sigma_{ij}, a_{ij} + \sigma_{ij}\ \right]$ (i.e., interval-based uncertainty set), and $\sigma_{ij} > 0$ designates the maximum allowable deviation of parameter a_{ij}. Bertsimas and Sim [4], [5] justified that it is not likely that every $a_{ij}, \forall\, j \in S_i$ take on uncertainty. In the event k_i is not integer-valued, up to $\lfloor k_i \rfloor$ of the coefficients may deviate by σ_{ij}, and exactly one coefficient a_{iq} will deviate by at most $(k_i - \lfloor k_i \rfloor)\sigma_{iq}$. Conversely, when the k_i-values are discrete (i.e. integer-valued), every coefficient that does deviate from their nominal value may do so at a maximum allowable value.

Applying parametric analysis for varying values of k_i in combination with an *a posteriori* assessment of a DM's risk preference, Hanks *et al.* [7] extended Kuchta's [3] RGP model via cardinality-constrained robustness and norm-based uncertainty sets. The model that applies the L_2-norm to account for data uncertainty is presented below [7], and an interested reader is referred that work for alternative RGP models that utilize either strict robustness or ellipsoidal uncertainty sets.

$$Minimize \sum_{i=1}^{m} d_i^+ \tag{82}$$

$$subject\ to \sum_{j=1}^{n} a_{ij}x_j + p_i - d_i^+ + d_i^- = t_i,\ \ \forall\, i \in M, \tag{83}$$

$$p_i \geq \sqrt{\sum_{j \in S_i} \left(\sigma_{ij}x_j\right)^2 + \left(f_i\sigma_{iq}x_q\right)^2},\ \ \forall\, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N\backslash S_i, i \in M, \tag{84}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{85}$$

$$x_j, d_i^+, d_i^- p_i \geq 0,\ \ \forall\, i \in M,\ \ j \in N. \tag{86}$$

The objective function (82) minimizes the sum of positive deviations from goals $i \in M$, and Constraint (83) calculates the positive and negative deviations (i.e., d_i^+ and d_i^-) from each goal, where the total induced variability is represented by p_i for each goal. Constraint (84) applies L_1-norm uncertainty sets to the possible sets of coefficients subject to parametric uncertainty (i.e., $S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i$), where the least upper bound (i.e., infimum) penalty, $p_i, \forall\ i \in M$ is calculated for every combination of $\sqrt{\sum_{j \in S_i} \left(\sigma_{ij} x_j\right)^2 + \left(f_i \sigma_{iq} x_q\right)^2}$, and wherein f_i represents the remainder value when k_i is not integer-valued (i.e., $f_i = k_i - \lfloor k_i \rfloor$). Constraint (85) characterizes additional constraints on the decision space that are neither related to the DM's goals nor considered with respect to possible parametric uncertainty. Finally, Constraint (86) ensures all decision variables are non-negative.

To date, applications of RGP in the literature include portfolio selection [93], supply chain network design [99], and intermodal routing problems [100].

5.2.3 RGP and risk assessment

There have been a few persistent shortcomings related to the application of RGP models in the literature to date. First, the RGP models apply parametric analysis to consider possible combinations of k_i-values, which can be computationally expensive when the number of decision variables or the number of goal-related constraints are high. Second, the risk preference of the DM is not taken into consideration until after-the-fact (i.e., *a posteriori*). Consequently, the DM must decide which solution set is best among many alternative optimal solutions, each of which represents a different risk attitude; selecting in this manner is a daunting-if-not-impossible task for a DM, and the presentation of the myriad of solution options can potentially bias the identification of their true risk preference. Lastly, when $k_i \notin \mathbb{Z}^+$ the current RGP methodologies are intractable because they will yield an infinite number of combinations of parametric variations.

To further develop the practical nature of RGP, Hanks *et al.* [167] account for a DM's risk preference via *a priori* analysis using a variety of rigorous risk elicitation methods that can then be mapped to an RGP risk parameter (i.e., k_i). More specifically, the authors present three theorems that enable a mathematical mapping of risk attitude to RGP model parameters by utilizing piecewise differential equation models. Hanks *et al.* [167] develop mappings for several risk elicitation techniques: the bisection, Holt-Laury (HL), and Eckel-Grossman (EG) methods, as well as the Bomb Risk Elicitation Task (BRET). The mapping used when eliciting risk via the bisection, HL, and EG methods is presented in Equation (87), whereas the mapping applied in conjunction with the BRET is offered in Equation (88).

$$k_i(r|h, r_n) = \begin{cases} he^{\varphi(r-r_n)}, & r \leq r_n \\ n - he^{\varphi(r_n - r)}, & r > r_n \end{cases} \tag{87}$$

$$k_i(r|h) = \begin{cases} n - he^{\varphi(1-r)}, & 0 \leq r \leq 1 \\ he^{\varphi(r-1)}, & r > 1 \end{cases} \tag{88}$$

By eliciting the DM's risk *a priori,* Equations (87) or (88), as appropriate, one can generate a corresponding risk parameter k_i as a function of the quantitative risk attitude test measurement r, wherein h denotes the assumed risk neutral point in the RGP realm and r_n is the r-value corresponding to a risk neutral attitude. In doing so, a robust solution is identified using the existing RGP methods that will not only provide the DM with a single solution, the solution set will also correspond to the DM's measured risk preference.

5.3 Methodology

The USTRANSCOM rate setting problem is complex and requires preliminary explanation. Section 5.3.1 offers the notation utilized in the USTRANSCOM rate setting problem, including various sets, parameters, and decision variables. The current methodology for setting yearly liner rates (as implemented by TCJ8) is explained in Section 5.3.2, as well as a proposed methodology in Section 5.3.3 that uses an alternative calculation of weighted average costs to determine liner rates. In Section 5.3.4, important model assumptions are presented, especially with regard to parametric variability within the model. The section concludes by setting forth a variant of the RGP model described in (82)-(86) as it pertains to the USTRANSCOM rate setting problem.

5.3.1 Notation

Prior to setting forth the various formulations we examine, it is necessary to introduce the associated sets, parameters, decision variables, and functions. The notation is defined as follows:

Sets

- $O = \{1, 2, \dots, 59\}$: set of all origins for liner shipments, indexed as $o \in O$.
- $D = \{1, 2, \dots, 59\}$: set of all destinations for liner shipments, indexed as $d \in D$.
- $C = \{01, 02, 03, 04, 06, 07, 08, 09, 11, 12, 13\}$: set of all commodity types carried within liner shipments, indexed as $c \in C$.
- $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$: booking term classification for liner shipments, indexed as $b \in B$.

- $S = \{0, 20, 40\}$: container size classification for shipments, indexed as $s \in S$, where $s = 0$ denotes a breakbulk shipment, $s = 20$ signifies a shipment container size less than 40 feet, and $s = 40$ is a shipment container greater than or equal to 40 feet.
- $G = \{1, 2\}$: set of all USTRANSCOM goals, indexed as $g \in G$.
- $T = \{12, 13, 14, 15, 16\}$: set of all FY's, indexed as $t \in T$.

Parameters

- n_{odcbs}^{t}: number of observations/shipments from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$.
- $(m_i)_{odcbs}^{t}$: the amount of measurement tons for each observation/shipment $i = 1, 2, \dots, n_{odcbs}^{t}$ from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$.
- $(q_i)_{odcbs}^{t}$: cost per measurement ton for each observation/shipment $i = 1, 2, \dots, n_{odcbs}^{t}$ (does not consider refresh rates) corresponding to shipments from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$.
- $(wac)_{odcbs}^{t}$: weighted average cost for each measurement ton (does not consider refresh rates) corresponding to shipments from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size

classification $s \in S$ during FY $t \in T$. Of note, this calculation does NOT consider refresh rate and is used exclusively to calculate the weighted calculation.

- $(cwac)_{odcbs}^{t}$: current calculation for the weighted average cost for each measurement ton (while considering previous year's refresh rates) corresponding to shipments from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$.
- p_{odcbs}^{t}: number of observations/shipments considered *for the* $(cwac)_{odcbs}^{t}$ *calculation* from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$. Over a 2-1/2 year period, $\sum_{t=t-3}^{t-1} p_{odcbs}^{t} = \{0, 1, 2, 3\}$.
- $(pwac)_{odcbs}^{t}$: proposed calculation for the weighted average cost for each measurement ton (while considering previous refresh rates) corresponding to shipments from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$.
- α_{RR}^{t}: refresh rate adjustment for any given rate set in FY $t \in T$.
- α_{AOR}^{t}: AOR/overhead adjustment for any given rate set in FY $t \in T$.
- $\left(w_g\right)_{odcbs}$: weight attributed to each goal, $g = \{1, 2\}$ for all shipments from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$.
- $\left(k_g\right)_{odcbs}$: number of products for each goal, $g = \{1, 2\}$ for all shipments from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking

term $b \in B$ and container size classification $s \in S$ subject to deviation (uncertainty), where $(k_g)_{odcbs} \in [0,1]$ (due to the separable nature of each specific rate $o \in O, d \in D, c \in C, b \in B, s \in S$).

- σ_{RR}^{t}: the allowable amount of parametric uncertainty observed in the refresh rate during FY $t \in T$.
- σ_{AOR}^{t}: the allowable amount of parametric uncertainty observed in the AOR factor during FY $t \in T$.
- $\left(\sigma^{t}_{\frac{1}{pwac}}\right)_{odcbs}$: the allowable amount of parametric uncertainty witnessed in $\frac{1}{pwac}$ corresponding to a shipment from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$.
- $\sigma^{t}_{\frac{1}{1+\alpha_{RR}}}$: the amount of parametric uncertainty observed in $\frac{1}{1+\alpha_{RR}}$ during FY $t \in T$.
- $\sigma^{t}_{\frac{1}{1+\alpha_{AOR}}}$: the amount of parametric uncertainty observed in $\frac{1}{1+\alpha_{AOR}}$ during FY $t \in T$.
- $(\sigma_g^t)_{odcbs}$: the combined amount of parametric uncertainty observed in the *coefficient* pertaining to goal $g = \{1, 2\}$ corresponding to a shipment from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$.

Decision Variables

- x_{odcbs}^{t}: rate (\$/metric ton) for DoD customers from origin $o \in O$ to destination $d \in D$ transporting commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$ (i.e., using *current* USTRANSCOM rate setting methodology).
- $\hat{x}_{odcbs}^{t}$: rate (\$/metric ton) for DoD customers from origin $o \in O$ to destination $d \in D$ transporting commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$ (i.e., using *proposed* rate setting methodology).
- $\left(p_g\right)_{odcbs}$: penalty associated to a shipment from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$ pertaining to each goal $g \in G$.
- $\left(d_{gA}^{+}\right)_{odcbs}$: amount of deviation *above* a target value for $\left(p_g\right)_{odcbs}$ corresponding to a shipment from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$ pertaining to each goal $g \in G$.
- $\left(d_{gA}^{-}\right)_{odcbs}$: amount of deviation *below* a target value for $\left(p_g\right)_{odcbs}$ corresponding to a shipment from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$ pertaining to each goal $g \in G$.

- $\left(d_{gB}^{+}\right)_{odcbs}$: amount of deviation *above* a target value for $-\left(p_g\right)_{odcbs}$ corresponding to a shipment from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$ pertaining to each goal $g \in G$.
- $\left(d_{gB}^{-}\right)_{odcbs}$: amount of deviation *below* a target value for $-\left(p_g\right)_{odcbs}$ corresponding to a shipment from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY $t \in T$ pertaining to each goal $g \in G$.

Functions

- $(TIAC)_{odcbs}^{t-1} = \left(\sum_{i=1}^{n_{odcbs}^{t-3}}(1+\alpha_{RR}^{t-3})(1+\alpha_{RR}^{t-2})(m_i)_{odcbs}^{t-3}(q_i)_{odcbs}^{t-3}\right) + \left(\sum_{i=1}^{n_{odcbs}^{t-2}}(1+\alpha_{RR}^{t-2})(m_i)_{odcbs}^{t-2}(q_i)_{odcbs}^{t-2}\right) + \left(\sum_{i=1}^{n_{odcbs}^{t-1}}(m_i)_{odcbs}^{t-1}(q_i)_{odcbs}^{t-1}\right)$, wherein $(TIAC)_{odcbs}^{t-1}$ represents the total inflation-adjusted cost for every shipment from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY's $t-3, t-2,$ and $t-1$.
- $(TM)_{odcbs}^{t-1} = \left(\sum_{i=1}^{n_{odcbs}^{t-3}}(m_i)_{odcbs}^{t-3}\right) + \left(\sum_{i=1}^{n_{odcbs}^{t-2}}(m_i)_{odcbs}^{t-2}\right) + \left(\sum_{i=1}^{n_{odcbs}^{t-1}}(m_i)_{odcbs}^{t-1}\right)$, wherein $(TM)_{odcbs}^{t-1}$ calculates the number of mtons shipped from origin $o \in O$ to destination $d \in D$ of commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY's $t-3, t-2,$ and $t-1$.

- $\left(\frac{1}{TIAC}\right)_{odcbs}^{t-1} = \left(\sum_{i=1}^{n_{odcbs}^{t-3}}(1+\alpha_{RR}^{t-3})(1+\alpha_{RR}^{t-2})(m_i)_{odcbs}^{t-3}\frac{1}{(q_i)_{odcbs}^{t-3}}\right) + \left(\sum_{i=1}^{n_{odcbs}^{t-2}}(1+\alpha_{RR}^{t-2})(m_i)_{odcbs}^{t-2}\frac{1}{(q_i)_{odcbs}^{t-2}}\right) + \left(\sum_{i=1}^{n_{odcbs}^{t-1}}(m_i)_{odcbs}^{t-1}\frac{1}{(q_i)_{odcbs}^{t-1}}\right)$, wherein $\left(\frac{1}{TIAC}\right)_{odcbs}^{t-1}$ represents the total inflation-adjusted *inverse* cost for every shipment from origin $o \in O$ to destination $d \in D$ carrying commodity type $c \in C$ with booking term $b \in B$ and container size classification $s \in S$ during FY's $t-3, t-2,$ and $t-1$.

5.3.2 Current rate setting methodology

As a means of comparison, it is important to understand how TCJ8 currently calculates yearly liner rates for USTRANSCOM. Due to the sparsity of liner-specific data, TCJ8 calculates the weighted average cost (WAC) of each shipment over the past 2-1/2 years, which is then inflation-adjusted (via the refresh rate) for each respective year (CWAC). In order to implement this methodology, TCJ8 first calculates the WAC for each shipment, applies the refresh rate to each respective year, and takes the average over those inflation-adjusted costs to calculate the CWAC via Equations (89) and (90), sequentially:

$$(wac)_{odcbs}^{t} = \frac{\sum_{i=1}^{n_{odcbs}^{t}}(m_i)_{odcbs}^{t}(q_i)_{odcbs}^{t}}{\left(\sum_{i=1}^{n_{odcbs}^{t}}(m_i)_{odcbs}^{t}\right)}, \forall\ o \in O, d \in D, c \in C, b \in B, s \in S, t \in T \tag{89}$$

$$(cwac)_{odcbs}^{t-1} = \frac{(1+\alpha_{RR}^{t-3})(1+\alpha_{RR}^{t-2})(wac)_{odcbs}^{t-3} + (1+\alpha_{RR}^{t-2})(wac)_{odcbs}^{t-2} + (wac)_{odcbs}^{t-1}}{\sum_{t=t-3}^{t-1} p_{odcbs}^{t}}, \quad \forall\ o \in O, d \in D, c. \in C, b \in B, s \in S, t \in T \tag{90}$$

TCJ8 then adjusts each CWAC using the current FY's refresh rate and AOR factor to set the next FY's rate, as shown in Equation (91).

$$x_{odcbs}^{t} = (1+\alpha_{AOR}^{t-1})(1+\alpha_{RR}^{t-1})(cwac)_{odcbs}^{t-1},\ \forall\, o \in O, d \in D, c \in C, b \in B, s \in S, t \in T \tag{91}$$

The current rate setting methodology utilized by TCJ8 is relatively new and was initially altered to improve the overall effectiveness of the rate setting process. However, even with this new methodology, there exists the potential for improvement to meet the two goals of minimizing deviation away from $0 NOR and setting shipping rates that are robust from year-to-year, given uncertainty in the cost data.

5.3.3 Alternative rate setting methodology

Instead of calculating the average of the inflation-adjusted WAC (i.e. the CWAC), the proposed rate setting calculations take into consideration every movement over the previous two years as well as six months of the current year, and inflation-adjusts *each* movement into current year dollars. In doing so, a more representative WAC is generated – which is denoted as the PWAC (i.e., proposed weighted average cost).

$$(pwac)_{odcbs}^{t-1} = \frac{(TIAC)_{odcbs}^{t-1}}{(TM)_{odcbs}^{t-1}}, \forall\, o \in O, d \in D, c \in C, b \in B, s \in S, t \in T \tag{92}$$

Similarly to TCJ8's current methodology, the PWAC is then adjusted by both the refresh rate and the AOR factor to set the upcoming FY's liner rate, as per Equation (93).

$$\hat{x}_{odcbs}^{t} = (1+\alpha_{AOR}^{t-1})(1+\alpha_{RR}^{t-1})(pwac)_{odcbs}^{t-1}, \forall\, o \in O, d \in D, c \in C, b \in B, s \in S, t \in T \tag{93}$$

We recommend TCJ8 adopt the alternative rate setting methodology because it computes a weighted average cost over 2-1/2 years of inflation-adjusted shipment costs, whereas the current methodology computes an evenly weighted average of 2-1/2 FY-specific weighted average costs.

However, even if TCJ8 were to adopt the PWAC calculation and alternative rate setting methodology, they would still not likely succeed in meeting their goals because the many sources of variability are not being taken into consideration. As such, the remainder of this study proposes and demonstrates an RGP formulation to set liner rates that does account for parametric variability in the PWAC calculation, as well as uncertainty in the computation of the refresh rate and the AOR factor.

5.3.4 RGP formulation

The RGP formulation presented in this section is a modified version of the RGP model described above in (82)-(86) and as set forth by Hanks *et al.* [7]. However, it is appropriate to first review the relevant RGP modeling assumptions, followed by discussing the parametric variability observed in the USTRANSCOM rate setting problem and, finally, presenting an RGP model to assist TCJ8 in their liner rate setting endeavors.

5.3.4.1 RGP model assumptions

The following assumptions are made regarding the implementation of the RGP model concerning the USTRANSCOM rate setting problem:

- All $odcbs$-combinations are subject to parametric uncertainty.

- The rate setting process is separable by $odcbs$-combination. This assumption is valid because, if each rate is robust and contributes to neither a deficit nor a surplus to the TWCF, then the aggregation of rates will share the same attributes. Thus $\left(k_g\right)_{odcbs} \in [0,1]$ for our modeling implementation. In turn, the rate setting process being separable alleviates the need for the remainder value $\left(f_g\right)_{odcbs}$ as described in Section 5.2.2.
- Due to a lack of understanding (regarding USTRANSCOM DM risk), it is assumed that $\left(k_g\right)_{odcbs} = 1$ portrays a risk-averse DM, whereas $\left(k_g\right)_{odcbs} = 0.5$ and $\left(k_g\right)_{odcbs} = 0$ respectively denote a risk-neutral and risk-seeking DM. As a result, direct implementation of a risk-elicitation method as described in Section 0 is not required for this demonstration.
- The DM's risk parameter is the same for every $odcbs$-combination during FY $t \in T$.
- Each goal is equally important (i.e., $(w_1)_{odcbs} = (w_2)_{odcbs}$) to the DM, and the units of deviation for the respective goals are well scaled.
- The refresh rate is normally distributed as $\alpha_{RR} \sim N(\mu_{RR}, \sigma_{RR})$.
- The AOR is normally distributed as $\alpha_{AOR} \sim N(\mu_{AOR}, \sigma_{AOR})$.
- The current NOR is zero, as a more detailed result is not currently known for the purpose of this study.

5.3.4.2 Goal constraints

The first goal for the USTRANSCOM rate setting problem is to minimize deviation from \$0 NOR. In other words, TCJ8 wants to set liner rates such that next FY's rate is as close to the previous FY's cost-adjusted PWAC-based rate as possible, per Equation (94).

$$\hat{x}^{t}_{odcbs} = (1 + \alpha^{t-1}_{AOR})(1 + \alpha^{t-1}_{RR})(pwac)^{t-1}_{odcbs}, \forall\, o \in O,\, d \in D,\, c \in C,\, b \in B,\, s \in S,\, t \in T \qquad (94)$$

However, the RGP model described in (82)-(86) demands that parametric uncertainty be considered only as a cost coefficient. Thus, because the uncertainty is considered in the parameters α^{t-1}_{AOR}, α^{t-1}_{RR}, and $(pwac)^{t-1}_{odcbs}$, the NOR goal constraint seen in Equation (94) is transformed as follows:

$$\left(\frac{1}{1 + \alpha^{t-1}_{AOR}}\right)\left(\frac{1}{1 + \alpha^{t-1}_{RR}}\right)\left(\frac{1}{(pwac)^{t-1}_{odcbs}}\right)\hat{x}^{t}_{odcbs} = 1, \forall\, o \in O, d \in D, c \in C, b \in B, s \in S, t \in T \qquad (95)$$

Conducting this transformation does require more work when calculating certain parameters, specifically $\left(\frac{1}{(pwac)^{t-1}_{odcbs}}\right)$. Calculating the $(pwac)^{t-1}_{odcbs}$ and then taking the reciprocal is merely an approximation, whereas the calculation set forth in Equation (96) is more appropriate, where each $\frac{1}{(q_i)^{t}_{odcbs}}$ is considered for a 2-1/2-year period.

$$\frac{1}{(pwac)^{t-1}_{odcbs}} = \frac{\left(\frac{1}{TIAC}\right)^{t-1}_{odcbs}}{(TM)^{t-1}_{odcbs}}, \forall\, o \in O, d \in D, c \in C, b \in B, s \in S, t \in T \qquad (96)$$

Similarly, the second goal of interest to USTRANSCOM and TCJ8 is to set liner rates that are robust from year-to-year, setting the rate as close to the cost adjusted rate from last year.

$$\hat{x}_{odcbs}^{t} = (1 + \alpha_{RR}^{t-1})\hat{x}_{odcbs}^{t-1}, \forall\ o \in O, d \in D, c \in C, b \in B, s \in S, t \in T \qquad (97)$$

To ensure the parameter subject to uncertainty is represented in the coefficient, Equation (97) is rewritten as:

$$\left(\frac{1}{1 + \alpha_{RR}^{t-1}}\right)\hat{x}_{odcbs}^{t} = \hat{x}_{odcbs}^{t-1}, \forall\ o \in O, d \in D, c \in C, b \in B, s \in S, t \in T \qquad (98)$$

In addition to these goal constraints, there are penalty constraints that must be considered to account for possible parametric variability.

5.3.4.3 Parametric variability

It is assumed that there are three sources of variability present in the USTRANSCOM rate setting problem. The first parameter subject to uncertainty is the refresh rate such that $\alpha_{RR}^{t} \sim N(\mu_{RR}^{t}, \sigma_{RR}^{t})$, where μ_{RR}^{t} is the average refresh rate over the past three years and σ_{RR}^{t} is the corresponding standard deviation over the same three years. Given these values, a Monte Carlo simulation is applied for 10,000 replications to compute $\frac{1}{1+\alpha_{RR}^{t}}$, from which the average and standard deviation are computed.

The second parameter that is considered to model uncertainty is the AOR factor. The parametric variability witnessed in the AOR factor is calculated in the exact same fashion as the refresh rate, wherein $\alpha_{AOR}^{t} \sim N(\mu_{AOR}^{t}, \sigma_{AOR}^{t})$,$\mu_{AOR}^{t}$ is the average AOR during the previous three years, and σ^t_{AOR} is the analogous standard deviation over the same three years. A Monte Carlo simulation of 10,000 replications is conducted to identify the mean and standard deviation for $\frac{1}{1+\alpha^t_{AOR}}$.

Finally, the third factor contributing to variability in the RGP model is considered in the $\frac{1}{(pwac)^{t-1}_{odcbs}}$ calculation. Unlike the refresh rate and AOR factor, $\frac{1}{(pwac)^{t-1}_{odcbs}}$ is dependent on each $o \in O, d \in D, c \in C, b \in BT, cs \in C$. Because the $\frac{1}{(pwac)^{t-1}_{odcbs}}$ calculation considers a weighted average, the weighted standard deviation is computed to account for the variability witnessed in $\frac{1}{(pwac)^{t-1}_{odcbs}}$, per Equation (99).

$$\left(\sigma^t_{\frac{1}{pwac}}\right)_{odcbs} = \sqrt{\frac{(n^t_{odcbs})\sum_{i=1}^{n^t_{odcbs}}[(m_i)^t_{odcbs}]\left(\frac{1}{(q_i)^t_{odcbs}} - \frac{1}{pwac^t_{odcbs}}\right)^2}{(n^t_{odcbs}-1)\sum_{i=1}^{n^t_{odcbs}}(m_i)^t_{odcbs}}}, \forall\, o \in O, d \in D, c \in C, b \in B, s \in S, t \in T \tag{99}$$

Although each source of uncertainty is important, the combined uncertainty comprised in the coefficient of the NOR goal (i.e., $\left(\frac{1}{1+\alpha^{t-1}_{AOR}}\right)\left(\frac{1}{1+\alpha^{t-1}_{RR}}\right)\left(\frac{1}{(pwac)^{t-1}_{odcbs}}\right)$) merits additional discussion. To identify the combined effect of the various uncertainties, the variance of multiple random variables is considered using the formula described in Equation (100).

$$Var(XY) = E[X^2Y^2] - (E[XY])^2 = Var(X)Var(Y) + Var(X)[E(Y)]^2 + Var(Y)[E(X)]^2 \tag{100}$$

However, Equation (100) only considers the variance of two random variables, but we are interested in the combined variance of three random variables. As a result, first calculate V$ar\left(\left(\frac{1}{1+\alpha_{AOR}^{t-1}}\right)\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\right)$ using Equation (100) and define the following:

$$X = \left(\left(\frac{1}{1+\alpha_{AOR}^{t-1}}\right)\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\right) \tag{101}$$

$$Y = \left(\frac{1}{(pwac)_{odcbs}^{t-1}}\right) \tag{102}$$

Because the $Var(X) = Var\left(\left(\frac{1}{1+\alpha_{AOR}^{t-1}}\right)\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\right)$, three variables are condensed into two wherein we can apply Equation (100) to get the parametric variability coinciding with the coefficient of the NOR goal, per Equation (103).

$$(\sigma_1^t)_{odcbs} = \sqrt{Var\left(\left(\frac{1}{1+\alpha_{AOR}^{t-1}}\right)\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\left(\frac{1}{(pwac)_{odcbs}^{t-1}}\right)\right)} \tag{103}$$

Addressing the parametric variability witnessed in the second goals' coefficient is a much simpler process. Because the second goal depends entirely on the allowable variability of the refresh rate, the parametric variability pertaining to the coefficient of the yearly, robust rates' goal is outlined in Equation (104) below.

$$(\sigma_2^t)_{odcbs} = \sqrt{Var\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)} \tag{104}$$

Of note, $(\sigma_2^t)_{odcbs}$ is equivalent to $\sigma^t_{\frac{1}{1+\alpha_{RR}}}$.

5.3.4.4 USTRANSCOM RGP model formulation

The RGP method that enforces cardinality-constrained robustness via L_2-norm uncertainty sets described in (82)-(86) considers one-sided variability. For the USTRANSCOM rate setting problem, two-way deviation must be taken into consideration. Furthermore, because of the separable nature of the USTRANSCOM rate setting problem and the fact that $\left(k_g\right)_{odcbs} \in [0,1]$, the RGP model discussed in (82)-(86) is actually equivalent to the cardinality-constrained RGP model that uses L_1-norm uncertainty sets [7]. This equivalency is demonstrated below, along with the final RGP model that is applied to solve the USTRANSCOM rate setting problem.

$$Minimize \;\; (w_1)_{odcbs}((d_1^-)_{odcbs} + (d_1^+)_{odcbs}) + \; (w_2)_{odcbs}((d_2^-)_{odcbs} + (d_2^+)_{odcbs}) \tag{105}$$

$$subject\ to \;\; \left(\frac{1}{1+\alpha_{AOR}^{t-1}}\right)\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\left(\frac{1}{(pwac)_{odcbs}^{t-1}}\right)\hat{x}_{odcbs}^{t} + (p_1)_{odcbs} - (d_1^+)_{odcbs} + (d_1^-)_{odcbs} = 1 \tag{106}$$

$$\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\hat{x}_{odcbs}^{t} + (p_2)_{odcbs} - (d_2^+)_{odcbs} + (d_2^-)_{odcbs} = \hat{x}_{odcbs}^{t-1}, \tag{107}$$

$$(p_1)_{odcbs} \geq \sqrt{|(k_1)_{odcbs}(\sigma_1^t)_{odcbs}\hat{x}_{odcbs}^{t}|^2}, \tag{108}$$

$$(p_2)_{odcbs} \geq \sqrt{|(k_2)_{odcbs}(\sigma_2^t)_{odcbs}\hat{x}_{odcbs}^{t}|^2}, \tag{109}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{110}$$

$$\left(d_g^-\right)_{odcbs}, \left(d_g^+\right)_{odcbs}, \left(p_g\right)_{odcbs}, \hat{x}_{odcbs}^{t} \geq 0. \tag{111}$$

When referencing Constraints (108) and (109), it is evident that the square root of a non-negative squared term $\left(\sqrt{(\cdot)^2}\right)$ is the term itself $(\cdot)$ and is therefore equivalent to the cardinality-constrained model that uses L_1-norm uncertainty sets, as portrayed in (112)-(118):

$$Minimize\ \ (w_1)_{odcbs}((d_1^-)_{odcbs} + (d_1^+)_{odcbs}) + \ (w_2)_{odcbs}((d_2^-)_{odcbs} + (d_2^+)_{odcbs}) \tag{112}$$

$$subject\ to\ \ \left(\frac{1}{1+\alpha_{AOR}^{t-1}}\right)\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\left(\frac{1}{(pwac)_{odcbs}^{t-1}}\right)\hat{x}_{odcbs}^{t} + (p_1)_{odcbs} - (d_1^+)_{odcbs} + (d_1^-)_{odcbs} = 1, \tag{113}$$

$$\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\hat{x}_{odcbs}^{t} + (p_2)_{odcbs} - (d_2^+)_{odcbs} + (d_2^-)_{odcbs} = \hat{x}_{odcbs}^{t-1}, \tag{114}$$

$$(p_1)_{odcbs} \geq |(k_1)_{odcbs}(\sigma_1^t)_{odcbs}\hat{x}_{odcbs}^{t}|, \tag{115}$$

$$(p_2)_{odcbs} \geq |(k_2)_{odcbs}(\sigma_2^t)_{odcbs}\hat{x}_{odcbs}^{t}|, \tag{116}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{117}$$

$$\left(d_g^-\right)_{odcbs}, \left(d_g^+\right)_{odcbs}, \left(p_g\right)_{odcbs}, \hat{x}_{odcbs}^{t} \geq 0. \tag{118}$$

Because Constraint (118) ensures all decision variables are non-negative, there is no requirement for Constraints (114) and (115) to utilize the absolute value notation. However, the USTRANSCOM rate setting problem considers both positive and negative deviation from goals, and because we assume $(w_1)_{odcbs} = (w_2)_{odcbs} = 1$, the below RGP formulation is offered, which accounts for two-way deviation.

$$\begin{aligned} Minimize\ (d_{1A}^-)_{odcbs} + (d_{1A}^+)_{odcbs} + (d_{1B}^-)_{odcbs} + (d_{1B}^+)_{odcbs} + (d_{2A}^-)_{odcbs} + (d_{2A}^+)_{odcbs} + (d_{2B}^-)_{odcbs} \\ + (d_{2B}^+)_{odcbs} \end{aligned} \tag{119}$$

$$subject\ to\ \ \left(\frac{1}{1+\alpha_{AOR}^{t-1}}\right)\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\left(\frac{1}{(pwac)_{odcbs}^{t-1}}\right)\hat{x}_{odcbs}^{t} + (p_1)_{odcbs} - (d_{1A}^+)_{odcbs} + (d_{1A}^-)_{odcbs} = 1, \tag{120}$$

$$\left(\frac{1}{1+\alpha_{AOR}^{t-1}}\right)\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\left(\frac{1}{(pwac)_{odcbs}^{t-1}}\right)\hat{x}_{odcbs}^{t} - (p_1)_{odcbs} - (d_{1B}^+)_{odcbs} + (d_{1B}^-)_{odcbs} = 1, \tag{121}$$

$$\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\hat{x}_{odcbs}^{t} + (p_2)_{odcbs} - (d_{2A}^{+})_{odcbs} + (d_{2A}^{-})_{odcbs} = \hat{x}_{odcbs}^{t-1}, \tag{122}$$

$$\left(\frac{1}{1+\alpha_{RR}^{t-1}}\right)\hat{x}_{odcbs}^{t} - (p_2)_{odcbs} - (d_{2B}^{+})_{odcbs} + (d_{2B}^{-})_{odcbs} = \hat{x}_{odcbs}^{t-1}, \tag{123}$$

$$(p_1)_{odcbs} \geq (k_1)_{odcbs}(\sigma_1^t)_{odcbs}\hat{x}_{odcbs}^{t}, \tag{124}$$

$$(p_2)_{odcbs} \geq (k_2)_{odcbs}(\sigma_2^t)_{odcbs}\hat{x}_{odcbs}^{\,t}, \tag{125}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{126}$$

$$\left(d_g^{-}\right)_{odcbs}, \left(d_g^{+}\right)_{odcbs}, \left(p_g\right)_{odcbs}, \hat{x}_{odcbs}^{t} \geq 0. \tag{127}$$

Furthermore, it should be noted that Constraint (126) is included for additional side-constraints in the general-sense; however, the USTRANSCOM rate setting problem does not currently require side constraints.

5.4 Testing and analysis

The RGP model in (119)-(127) is applied to solve the USTRANSCOM liner rate setting problem for FY16 using FY12-FY15 cost data and evaluated using FY16 cost data. For comparison, both the CWAC and PWAC methods are also applied and evaluated for FY16 using the same data. Before any mention of the results, however, caveats are offered regarding the shortcomings of the FY12-FY16 cost data.

There are some notable caveats to the quantitative results of any of the methods considered in this study, despite a notably illustrative demonstration of the RGP model's application vis-à-vis the PWAC and CWAC methods. While conducting this study, there were substantial issues regarding the data used by TCJ8, as well as the impact of the AOR factor on the overall rate setting process.

There is substantial consideration related to data [168] and in particular the “five V’s”: volume (amount of data), velocity (speed at which data is generated), variety (various types of data), value (possible insights created from data), and finally, veracity (accuracy of data) [169]. In this regard, problems discovered in the USTRANSCOM data were two-fold: 1) the veracity of the data, and 2) the sparsity of the data. It was noticed that there were certain shipments of data that had negative-valued measurement tons (mtons), or even zero-valued mtons, but still had a cost associated with it. Table 19 reports the number of shipments reported within each FY that had negative-valued or zero-valued mtons, but an associated cost.

Table 19: Veracity Issues in TCJ8 Data

	# of shipments with $mtons = 0$	# of shipments with $mtons < 0$
FY12	0	2
FY13	0	0
FY14	0	8
FY15	729	18
FY16	588	8

Even though the percentage of these particular shipments is low relative to the total number of shipments within a given FY, including this type of data in the calculation of the next FY’s rate for any $odcbs$-combination can detrimentally affect the outcomes (i.e., the NOR, robust rates) of the rate setting process. Because of this, the overall precision of the USTRANSCOM data is problematic.

Of equal if not more importance is the issue of sparse data. To address this concern, TCJ8 has recently adapted the CWAC method, which considers 2-1/2 years-worth of data with the goal of leveraging more cost-data for any given $odcbs$-combination. Although it leverages 2-1/2 years of liner costs to address the sparsity of data (as opposed to using six months of data), this study has demonstrated that, even when utilizing the RGP, PWAC, or CWAC methods, the amount of cost data is still insufficient to produce accurate estimates for *every* liner rate in the next FY. For example, in calculating the FY16 rates using the RGP, PWAC and CWAC methods, there were 2880 distinct $odcbs$-combinations. When looking at each observed $odcbs$-combination receipt in FY16, there were 1,641 individual combinations. Yet, of these 1,641 rates, only 1,133 were in common with the 2,880 estimated rates set forth by the RGP, PWAC and CWAC methods (see Table 20). Consequently, 30.96% of the $odcbs$-combinations and 13.68% of the total mtons shipped in FY16 are not being taken into consideration for the NOR calculation because there is no basis of comparison (i.e., there were 508 observed $odcbs$-combinations in FY16 that are not present in FY15). Ultimately, the lack of collective rates year-to-year is a direct result of sparsity seen in the data, wherein the only way to address this issue is added processes (e.g., consult a subject matter expert) to the TCJ8 rate setting methodology.

Table 20: FY16 RGP Rate versus FY16 Actual Rate Data Comparison

Actual FY16 Data		Captured in FY16 RGP Data		% Captured in FY16 RGP Data	
$odcbs$-combinations	mtons	$odcbs$-combinations	mtons	$odcbs$-combinations	mtons
1,641	2,363,507.28	1,133	2,040,077.20	69.04%	86.32%

In this regard, according to TCJ8, the FY16 direct costs recovered by liner rates was $599.8 million, whereas the non-direct (overhead) cost for liner rates was $62.1 million. However, because the RGP method accounts for approximately 86.32% of the total mtons shipped via liner in FY16, we assume 86.32% of the total FY16 liner costs and adjust accordingly, per Table 21. Applying this adjusted price for this study is essential in reporting a true NOR comparison of the three rate setting methodologies.

Table 21: FY16 Liner Cost Data

	Actual FY16 Cost Data		Adjusted FY16 Cost Data	
	Direct	Non-Direct	Direct	Non-Direct
	$599,800,000	$62,100,000	$517,747,360	$53,604,720
Total	$661,900,000		$571,352,080	

One final topic regarding this study concerns the AOR factor. As it is currently understood, the AOR factor is dependent upon the total amount of shipments over the course of a FY, wherein the AOR factor acts a cost recoupment factor in the hopes to garner a $0 NOR by distributing overhead costs proportionally to all $odcbs$-combinations within a FY. At this time, the actual AOR calculation is not fully understood, which is concerning due to its high variability over recent FY's (as seen in Table 22) and large impact on the rate setting process.

Table 22: Recent AOR Factors

	FY08	FY09	FY10	FY11	FY12	FY13	FY14	FY15
AOR Factor	20%	-7%	12.2%	5.5%	8%	8%	22.3%	38.3%

In an effort to limit the AOR factor's impact, it is highly recommended that it only be applied to the yearly $odcbs$-combinations that are historically reliable, consistently priced, and high in cost. For all other $odcbs$-combinations (i.e., lack consistency, low cost, etc.), the rates should be inflation-adjusted using the refresh rate alone. With this subtle change, the overhead costs will continue to be recovered and the AOR factor will have a far less influence on the variability in rates, and ultimately the NOR.

The NOR calculation considers the total revenue generated for each rate setting methodology and subtracts the adjusted total cost described in Table 21. The results are presented in Tables 23-24 regarding the NOR for varying levels of risk preference, k_g. More specifically, Table 23 compares the NOR of all rate setting methods for parametric variability up to one standard deviation. In an attempt to capture the impact on the NOR when more variability is induced, Table 24 demonstrates the effect on the NOR for parametric variability up to three standard deviations.

Table 23: NOR Comparison for Varying Risk Preferences (one-standard deviation)

		Rate Setting Methodology		
	Risk Preference	RGP	PWAC*	CWAC*
Total NOR	$k = 1$	$14,414,801	$21,334,878	$20,376,676
Total NOR	$k = 0.5$	$17,873,244	$21,334,878	$20,376,676
Total NOR	$k = 0$	$21,149,331	$21,334,878	$20,376,676

*The PWAC and CWAC methods are independent of a DM's risk-preference.

Table 24: RGP Method NOR Comparison for Varying Risk Preference and Allowable Variability

		Allowable Parametric Variability		
	Risk Preference	σ	2σ	3σ
Total NOR	$k = 1$	$14,414,801	$14,407,706	$14,406,755
Total NOR	$k = 0.5$	$17,873,244	$17,771,427	$17,759,605
Total NOR	$k = 0$	$21,149,331	$21,149,331	$21,149,331

Table 23 illustrates that each of the three rate setting methods easily recover the adjusted FY16 total costs, wherein the RGP method is the closest to achieving the goal of $0 NOR. In fact, if a DM is risk-averse, the RGP model provides USTRANSCOM with roughly $6 million less profit than the PWAC or CWAC methods, respectively. The RGP method also outperforms the other rate setting models when a DM is risk-neutral, but the difference in NOR is decreased dramatically. Finally, if a DM assumes there is no parametric variability (i.e., the DM is risk-seeking), the RGP method generates roughly $800K more in revenue when compared to the CWAC method. Therefore, the NOR gets worse as a DM's risk preference tends towards more risk-seeking behavior because of the significant variability observed in the actual FY15 and FY16 rates. Similar results are documented by Hanks *et al.* [167] wherein a DM who assumed a large amount of parametric variability (i.e., a risk-averse DM) is rewarded with a lower NOR because the observed parametric variability is in fact significant. Similarly, the risk-seeking DM predicts that there will be no variability in the data, but this assumption is clearly not accurate and, as a result, the NOR is pointedly worse than the NOR realized by the risk-averse DM. Similar results ensue when the allowed parametric variability increases by one or two standard deviations, as depicted in Table 24. As a result, because of the complexity observed in the USTRANSCOM rate setting problem, parametric variability is a certainty – and when RGP is implemented, a risk-averse attitude should be put into effect to reduce the NOR.

While considering the second goal of setting robust rates annually, the difference between the rates set by each methodology and the inflation-adjusted rate (i.e., $(1+\alpha_{RR}^{15})\hat{x}^{15}$) is evaluated. In particular, Figures 27-28 illustrate the average and standard deviation of the rate difference, respectively, whereas Table 25 offers a summary of the minimum, maximum, and range of the rate difference for each rate setting methodology.

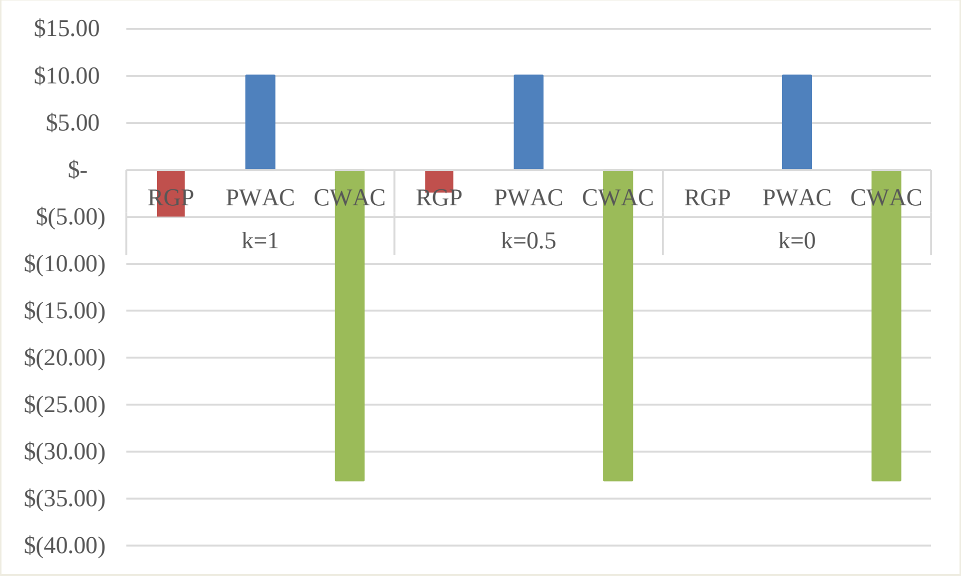


Figure 27: Average of the Rate Difference for Varying Methods and Risk Preferences

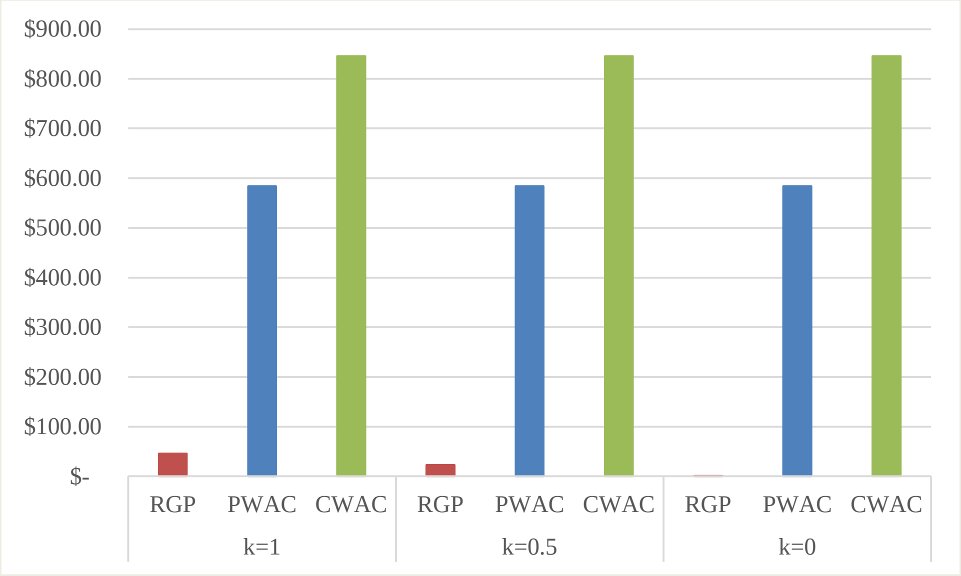


Figure 28: Standard Deviation of the Rate Difference for Varying Methods and Risk Preferences

Table 25: Minimum, Maximum, and Range of the Rate Difference for Varying Methods and Risk Preferences

		Rate Setting Methodology		
Risk Preference		RGP	PWAC*	CWAC*
$k = 1$	Minimum	-$1,543	-$3,140	-$26,170
	Maximum	$1	$17,735	$6,486
	Range	$1,544	$20,875	$32,656
$k = 0.5$	Minimum	-$776	-$3,140	-$26,170
	Maximum	$1	$17,735	$6,486
	Range	$777	$20,875	$32,656
$k = 0$	Minimum	$0	-$3,140	-$26,170
	Maximum	$0	$17,735	$6,486
	Range	$0	$20,875	$32,656

*The PWAC and CWAC methods are independent of a DM's risk-preference.

Figures 27-28 confirm that the RGP method yields a lower average of the rate difference as well as a considerably lower standard deviation of the rate difference as compared to results corresponding to either the PWAC or the CWAC method. Moreover, Table 25 reports that the RGP method has a lower minimum, maximum, and range of the rate difference when compared to the other rate setting methodologies. As the DM's preference towards risk transitions from risk-aversion to risk-seeking, all metrics (i.e., average, standard deviation, minimum, etc.) decrease. This confirms intuition because, as a DM becomes less risk-averse, they assume less parametric variability and robust rates are more easily obtained.

5.5 Conclusion and recommendations

USTRANSCOM's mission is to "provide full-spectrum, global mobility solutions and enabling capabilities to their customers in peace and war" [170]. TCJ8 plays a significant role in fulfilling this mission in how they set yearly liner rates, which can impact contractual negotiations, fiscal planning, and overall readiness. Currently, the rate setting methodology being utilized by TCJ8 to set their yearly liner rates is not as effective as it could be due to underlying assumptions about the deterministic nature of historical average cost data. They currently use a 2-1/2-year average of the WAC and cost-adjust each $odcbs$-combination using a refresh (inflation) rate, as well as a highly variable AOR factor. However, using this methodology is resulting in a large NOR with increasing variability of year-to-year liner rates. This study demonstrates an RGP methodology that considers parametric variability in an attempt to minimize deviation from $0 NOR and last year's liner rates. Although the RGP model demonstrates that it can reduce the NOR and produce robust shipping rates, its effectiveness is limited by the veracity and sparsity of the data, as well as the overall importance of the AOR factor.

We propose several recommendations to improve the liner rate setting methodology utilized by TCJ8. First, TCJ8 should take active measures to ensure their cost record data is complete and accurate. Second, a modified process should leverage more than 2-1/2 years of historical, inflation-adjusted data to address the inability of the current process to set each specific rate. Third, TCJ8 should conduct additional research regarding the AOR factor and its impact on the rate setting process. Fourth, TCJ8 should investigate how unequal weighting of USTRANSCOM goals effects the RGP model's solution. Finally, TCJ8 should try and automate the overall rate setting process [171].

As a follow-on study, we suggest TCJ8 recoup their overhead costs differently in an attempt to reduce the impact of the AOR factor on the TWCF balance. More specifically, we recommend recouping overhead costs by marginally increasing the inflation-adjusted costs (via an AOR factor) on the highest volume of shipping combinations that have low uncertainty in the workload forecasts, while using only inflation-adjusted costs to calculate rates for all other shipping combinations. In doing so, the AOR factor’s influence on any one rate will not be substantial, and the variability in the NOR incurred resulting from uncertainty in parts of the liner forecast will be reduced.

Moreover, as a future study we suggest that TCJ8 consider recouping their overhead costs by evaluating a weighting schema for the various TCJ8 goals in regards to the RGP methodology as applied to the USTRANSCOM rate setting problem.

VI. Conclusions and Recommendations

This dissertation research encompasses primarily the field of RO, but also comprises other disciplines such utility theory, risk elicitation methods, and rate setting. The theoretical contributions pertaining to this dissertation research are significant. First, we present three unique RGP models which utilize various levels of robustness and uncertainty sets. We also offer an explanation of when to apply each of the proposed and existing RGP models in lieu of a DM's risk preference. Then, we present an original mapping methodology which converts a DM's risk preference to various RO risk parameters. Finally, we demonstrate the usefulness of applying RGP in concert with the mapping methodology for the USTRANSCOM rate setting problem.

6.1 Conclusions

There are many RGP models discussed in this dissertation research, starting with the seminal article by Kuchta [3] wherein he leverages RO concepts to account for the deterministic nature of GP. More specifically, Kuchta [3] accounts for data uncertainty via cardinality-constrained robustness and interval-based uncertainty sets. We make RGP more robust by offering three original RGP formulations: 1) cardinality-constrained robustness via L_1-norm uncertainty sets, 2) cardinality-constrained robustness applied with L_2-norm uncertainty sets, and 3) strict robustness by ellipsoidal uncertainty sets. In the comparative analysis regarding these RGP models, the cardinality-constrained robustness and L_2-norm uncertainty set model performed the best regarding variation observed in the products manufactured, as well as overall deviation given various scenarios. Furthermore, the RGP model using strict robustness via ellipsoidal uncertainty sets demonstrated less variability than Kuchta's RGP model. To that end, with today's advanced computers and accessibility to commercial global solvers, the RGP models proposed in this research, although nonlinear, should be considered for implementation, regardless of the DM's risk preference, as our RGP models generate more robust solutions than Kuchta's RGP model.

As a means to improve upon the *a posteriori* assessments applied in the RGP literature, we provide a mathematical mapping methodology to account for a DM's risk preference *a priori*. Unambiguously, we provide three theorems that serve as the theoretical framework of the mapping methodology. Utilizing the limit definition of a function, we prove that a DM's risk preference can be mapped to an RO parameter. Given this, and the fact that $r_s < r_n < r_a$, we then apply the Central Limit Theorem to various domains of r proving there exists some risk-neutral point such that $k_i(r_n) = h$ and $k_i(r_s) < h < k_i(r_a)$. Given these theorems and the fact that the function $k_i(r)$ is dependent exclusively upon r, we invoke differential equations that model exponential decay for the varying risk elicitation methods. In an effort to parameterize $k_i(r)$, we assume a risk-neutral point, as well as two points near in the tails of $k_i(r)$. The BRET was utilized to validate the mapping methodology. Beforehand, however, we examined the raw data utilized by Crosetto and Filippin [6]. In doing so, we were able to discretize the original BRET risk preference categories from three to nine, wherein we applied our mapping methodology using all our original RGP models to a simple instance witnessed in the literature. The results from our study were encouraging as they did in fact validate the workings of our mapping methodology. First, as a DM's risk preference transferred from risk-aversion to risk-seeking, the total scaled deviations monotonically decreased. Furthermore, the $k_i(r)$-values generated from the mathematical mapping produced were appropriate for the given DM risk preference. Likewise, the results demonstrated the value of a DM’s assumed risk preference and the effects of correctly or incorrectly guessing the level of observed parametric variability. If the DM guessed correctly, they were rewarded with a lower total scaled deviation, whereas, if the DM guessed incorrectly, their total scaled deviation was much higher. These trends in the results were apparent in all three RGP models, further validating the mapping methodology’s viability.

Finally, as a means to show the effectiveness of implementing RGP with the mapping methodology, we apply each in concert to the USTRANSCOM rate setting problem in an attempt to set FY16 liner rates such that we minimize deviation away from their goals of \$0 NOR and robust shipping rates year-to-year. We consider three different methods for comparison – the CWAC (TCJ8’s current method), PWAC, and RGP methods. Allowing for one standard deviation of parametric variability while assuming the DM is risk-averse, the RGP model generates millions of dollars less in NOR profit and results in far less variability when compared to the rates generated by the PWAC and CWAC methods. However these results are not consistent when a DM becomes more risk-seeking. In fact, when a DM is risk-seeking, the RGP method does worse than the CWAC method! Yet, this is not alarming, and in fact confirms intuition. For instance, we know, based on the actual FY15 and FY16 liner receipts, that the cost data is extremely variable. Thus, the DM who assumes the USTRANSCOM data is *not* highly variable (i.e., risk-seeking DM) will, as a consequence, not be rewarded with a good solution, and vice versa. Therefore, these results follow exactly what we should expect. Moreover, due to the high variability in this particular problem, we investigate the effects of allowing two or three standard deviations in the data. The results illustrate that allowing for three standard deviations in the parameters will generate better NOR results – again, results that confirm intuition. In regards to robust rates year-to-year, irrespective of the level of DM risk preference, the RGP model outperforms the CWAC and PWAC methods in average, standard deviation, minimum, maximum, and range regarding the rate difference in the FY16 liner rate set for each methodology versus the FY16 inflation-adjusted rate. In light of these findings, the RGP methodology applied in concert with the mapping methodology was a success.

6.2 Recommendations

The proposed RGP models consider norm-based and ellipsoidal uncertainty sets used with cardinality-constrained and strict robustness as a theoretical extension to Kutcha's RGP model. It would be interesting to see further extensions applying a variety of robustness and uncertainty set combinations. Furthermore, we apply the RGP model which utilizes cardinality-constrained robustness and L_2-norm uncertainty sets to the USTRANSCOM rate setting problem, however, due to our problem formulation and the fact that $k_i \in [0,1]$, it was equivalent to using L_1-norm uncertainty sets. It would be of merit to apply an RGP model which utilizes cardinality-constrained robustness and L_2-norm uncertainty sets to one of the largest RGP application areas, such as portfolio selection. It would also be proper to test our strict robustness and ellipsoidal uncertainty set model on a problem wherein the DM does not have a full understanding of or confidence in the allowed parametric variability.

Considering the original nature of our mapping methodology, we had to assume three $(r, k(r))$-points to parameterize the function, $k_i(r)$. The data points that are assumed to be "sufficiently close" in the tails are not as much a concern as the assumed risk-neutral point, or $k_i(r_n) = \frac{n}{2}$ and $\theta_i(r_n) = \frac{\sqrt{n}}{2}$. As a future endeavor to increase the validity of the mapping methodology, testing of human subjects with a variety of risk preferences is required. In doing so, a true risk neutral point can be determined. Furthermore, our mapping methodology is applied to the simple test instance as provided in Kuchta [3] and also as a proof of principle to the USTRANSCOM rate setting problem, wherein $k_i(r) \in [0,1]$. It would be of value to apply the mapping methodology, in unison with some RO methodology, wherein a DM's true risk preference is elicited using one of the four risk elicitation methods discussed in this research, to a practical problem having multiple decision variables.

Finally, TCJ8 should take active measures to ensure secure data veracity. Due to the sparse data inherent in the USTRANSCOM rate setting problem, TCJ8 should also consider introducing an extra process in an effort to set all shipping combinations. Furthermore, due to its variability over recent years, it appears that the AOR factor is overemphasized and causing rate setting inaccuracies. To address this, we recommend TCJ8 marginally apply the AOR factor to the inflation-adjusted rates on shipping combinations that are high in volume and demonstrate low variability in cost year-to-year. In regards to the RGP methodology, it would be interesting to see the affect various weights have on the outcome of the rate setting problem. Lastly, TCJ8 should consider automating the rate setting process [171], which will not only save money, but time.

6.3 Summary

In summary, this dissertation research provides three novel RGP methods that combine cardinality-constrained robustness and norm-based uncertainty sets, as well as strict robustness and ellipsoidal uncertainty sets. Attempting to make these RGP models more practical, we also offer a unique mapping methodology that considers a DM's risk preference *a priori* and relates this risk preference in the DA realm to an RO risk parameter. Finally, we demonstrate the applicability of the RGP model that applies cardinality-constrained robustness via L_2-norm uncertainty sets in unison with the aforementioned mapping methodology to the USTRANSCOM rate setting problem.

In completing this dissertation research, we have expounded the RO theoretical framework, served as a conduit between two sub-disciplines concerning risk assessment, and demonstrated the practical use of such findings.

Appendix A

Table 26: The BRET CRRA coefficient values [6]

v	r	v	r	v	r	v	r
1	$0.000 \leq r \leq 0.014$	26	$0.343 \leq r \leq 0.360$	51	$1.021 \leq r \leq 1.061$	76	$3.082 \leq r \leq 3.255$
2	$0.015 \leq r \leq 0.025$	27	$0.361 \leq r \leq 0.379$	52	$1.062 \leq r \leq 1.105$	77	$3.256 \leq r \leq 3.444$
3	$0.026 \leq r \leq 0.036$	28	$0.380 \leq r \leq 0.398$	53	$1.106 \leq r \leq 1.150$	78	$3.445 \leq r \leq 3.651$
4	$0.037 \leq r \leq 0.046$	29	$0.399 \leq r \leq 0.418$	54	$1.151 \leq r \leq 1.197$	79	$3.652 \leq r \leq 3.878$
5	$0.047 \leq r \leq 0.058$	30	$0.419 \leq r \leq 0.438$	55	$1.198 \leq r \leq 1.247$	80	$3.879 \leq r \leq 4.129$
6	$0.059 \leq r \leq 0.069$	31	$0.439 \leq r \leq 0.459$	56	$1.248 \leq r \leq 1.298$	81	$4.130 \leq r \leq 4.406$
7	$0.070 \leq r \leq 0.080$	32	$0.460 \leq r \leq 0.481$	57	$1.299 \leq r \leq 1.352$	82	$4.407 \leq r \leq 4.715$
8	$0.081 \leq r \leq 0.092$	33	$0.482 \leq r \leq 0.503$	58	$1.353 \leq r \leq 1.409$	83	$4.716 \leq r \leq 5.062$
9	$0.093 \leq r \leq 0.104$	34	$0.504 \leq r \leq 0.526$	59	$1.410 \leq r \leq 1.469$	84	$5.063 \leq r \leq 5.453$
10	$0.105 \leq r \leq 0.117$	35	$0.527 \leq r \leq 0.550$	60	$1.470 \leq r \leq 1.531$	85	$5.454 \leq r \leq 5.898$
11	$0.118 \leq r \leq 0.129$	36	$0.551 \leq r \leq 0.574$	61	$1.532 \leq r \leq 1.597$	86	$5.899 \leq r \leq 6.410$
12	$0.130 \leq r \leq 0.142$	37	$0.575 \leq r \leq 0.599$	62	$1.598 \leq r \leq 1.666$	87	$6.411 \leq r \leq 7.003$
13	$0.143 \leq r \leq 0.155$	38	$0.600 \leq r \leq 0.625$	63	$1.667 \leq r \leq 1.739$	88	$7.004 \leq r \leq 7.700$
14	$0.156 \leq r \leq 0.169$	39	$0.626 \leq r \leq 0.652$	64	$1.740 \leq r \leq 1.816$	89	$7.701 \leq r \leq 8.530$
15	$0.170 \leq r \leq 0.183$	40	$0.653 \leq r \leq 0.680$	65	$1.817 \leq r \leq 1.898$	90	$8.531 \leq r \leq 9.534$
16	$0.184 \leq r \leq 0.197$	41	$0.681 \leq r \leq 0.709$	66	$1.899 \leq r \leq 1.985$	91	$9.535 \leq r \leq 10.776$
17	$0.198 \leq r \leq 0.212$	42	$0.701 \leq r \leq 0.739$	67	$1.986 \leq r \leq 2.077$	92	$10.777 \leq r \leq 12.351$
18	$0.213 \leq r \leq 0.226$	43	$0.740 \leq r \leq 0.769$	68	$2.078 \leq r \leq 2.174$	93	$12.532 \leq r \leq 14.412$
19	$0.227 \leq r \leq 0.242$	44	$0.770 \leq r \leq 0.801$	69	$2.175 \leq r \leq 2.278$	94	$14.413 \leq r \leq 17.229$
20	$0.243 \leq r \leq 0.257$	45	$0.802 \leq r \leq 0.834$	70	$2.279 \leq r \leq 2.389$	95	$17.230 \leq r \leq 21.309$
21	$0.258 \leq r \leq 0.273$	46	$0.835 \leq r \leq 0.869$	71	$2.390 \leq r \leq 2.508$	96	$21.31 \leq r \leq 27.76$
22	$0.274 \leq r \leq 0.290$	47	$0.870 \leq r \leq 0.904$	72	$2.509 \leq r \leq 2.636$	97	$27.761 \leq r \leq 39.532$
23	$0.291 \leq r \leq 0.307$	48	$0.905 \leq r \leq 0.941$	73	$2.637 \leq r \leq 2.773$	98	$39.533 \leq r \leq 68.274$
24	$0.308 \leq r \leq 0.324$	49	$0.942 \leq r \leq 0.980$	74	$2.774 \leq r \leq 2.921$	99	$r \geq 68.275$
25	$0.325 \leq r \leq 0.342$	50	$0.981 \leq r \leq 0.1.02$	75	$2.922 \leq r \leq 3.081$		

Appendix B

Rewrite (43)-(47) as Problem **P1** depicted below:

$$\mathbf{P1}\text{: } \underset{\boldsymbol{x},\boldsymbol{d}^{+},\boldsymbol{d}^{-},\boldsymbol{p},\boldsymbol{z}}{Minimize} \sum_{i=1}^{m} d_i^{+} \tag{B.1}$$

$$subject\ to\quad \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} p_{ij} + k_i z_i - d_i^{+} + d_i^{-} = t_i,\ \ \forall\, i \in M, \tag{B.2}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{B.3}$$

$$x_j, d_i^{+}, d_i^{-}, p_{ij}, z_i \geq 0,\ \ \forall\ i \in M,\ \ j \in N. \tag{B.4}$$

Now, consider the following formulation, formally shown as (48)-(52), labeled P2, which eliminates the z-variables and is established in Theorem 1 below to yield an equivalent mode of solving Problem P2:

$$\mathbf{P2}\text{: } \underset{\boldsymbol{x},\boldsymbol{d}^{+},\boldsymbol{d}^{-},\boldsymbol{p}}{Minimize} \sum_{i=1}^{m} d_i^{+} \tag{B.5}$$

$$subject\ to\quad \sum_{j=1}^{n} a_{ij}x_j + p_i - d_i^{+} + d_i^{-} = t_i,\ \ \forall\, i \in M, \tag{B.6}$$

$$p_i \geq \sum_{j \in S_i} |\sigma_{ij}x_j| + |f_i \sigma_{iq} x_q|,\ \ \forall\, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i, i \in M, \tag{B.7}$$

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{B.8}$$

$$x_j, d_i^+, d_i^-, p_i \geq 0, \ \ \forall \ i \in M, \ \ j \in N. \tag{B.9}$$

Theorem. An optimal solution to Problem P1 yields an optimal solution to Problem P2 under the following transformation:

$$(d_i^+, d_i^-)_{P2} = (d_i^+, d_i^-)_{P1}, \ \ \forall \ i \in M,$$

$$\left(x_j\right)_{P2} = \left(x_j\right)_{P1}, \ \ \forall \, i \in M,$$

$$(p_i)_{P2} = \left(\sum_{j \in N} p_{ij} + k_i z_i\right), \ \ \forall \, i \in M.$$

Proof. There are three parts of this proof that show an optimal solution to Problem P1 under the aforementioned transformation: (a) is feasible to Problem P2, (b) has the same objective function value as Problem P2, and (c) is optimal to Problem P2, respectively.

(a) When taking into consideration feasibility, the non-negativity of decision variables for Problems P2 and P1 is self-evident, given constraints $x_j, d_i^+, d_i^-, p_i \geq 0, \ \forall \ i \in M, \ j \in N$ and $x_j, d_i^+, d_i^-, p_{ij}, z_i \geq 0, \ \forall \ i \in M, \ j \in N$, respectively. Clearly constraints $\boldsymbol{Ax} = \boldsymbol{b}$ for Problems P2 and P1 are equivalent. Likewise, constraint $\sum_{j=1}^{n} a_{ij} x_j + p_i - d_i^+ + d_i^- = d_i, \ \forall \, i \in M$ corresponds to constraint $\sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n} p_{ij} + k_i z_i - d_i^+ + d_i^- = d_i, \forall \, i \in M$. Further, regarding constraint $p_i \geq \sum_{j \in S_i} \left|\sigma_{ij} x_j\right| + \left|f_i \sigma_{iq} x_q\right|, \ \forall \, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i, i \in M$, consider constraint $z_i + p_{ij} \geq \sigma_{ij} x_j, \ \forall \, i \in M$ aggregated over $j \in N$, as follows:

$$\sum_{j\in N}(z_i + p_{ij}) \geq \sum_{j\in N}\sigma_{ij}x_j, \quad \forall\, i \in M$$

which is equivalent to

$$\sum_{j\in N} p_{ij} + |N|z_i \geq \sum_{j\in N}\sigma_{ij}x_j\,, \quad \forall\, i \in M$$

and which is further equivalent to

$$\sum_{j\in N} p_{ij} + k_i z_i - k_i z_i + |N|z_i \geq \sum_{j\in N}\sigma_{ij}x_j\,, \quad \forall\, i \in M.$$

Thus, it follows

$$z_i(|N| - k_i) + k_i z_i + \sum_{j\in N} p_{ij} \geq \sum_{j\in N}\sigma_{ij}x_j\,, \quad \forall\, i \in M$$

At optimality for Problem P1, constraint $z_i + p_{ij} \geq \sigma_{ij}x_j,\ \ \forall\, i \in M,\ \ j \in N$ is tight [3], where

$$z_i = \min_{j\in N}\{\sigma_{ij}x_j^*\}$$

which results in

$$k_i z_i^* + \sum_{j\in N} p_{ij}^* = -\min_{j\in N}\{\sigma_{ij}x_j^*\}(|N| - k_i) + \sum_{j\in N}\sigma_{ij}x_j^*, \quad \forall\, i \in M.$$

Since we know $p_i \geq \sum_{j\in S_i}|\sigma_{ij}x_j| + |f_i\sigma_{iq}x_q|,\ \ \forall\, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N\backslash S_i, i \in M$ and that the transformation states $p_i = \sum_{j\in N} p_{ij} + k_i z_i,\ \ \forall\, i \in M$, where substituting into Problem P2, at optimality:

$$\sum_{j\in N} p_{ij}^* + k_i z_i^* \geq \sum_{j\in S_i} \left|\sigma_{ij} x_j^*\right| + \left|f_i \sigma_{iq} x_q^*\right|, \;\; \forall\, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i, i \in M$$

It is easily rendered that

$$-\min_{j\in N}\{\sigma_{ij} x_j^*\}(|N| - k_i) + \sum_{j\in N} \sigma_{ij} x_j^* \geq \sum_{j\in S_i} \left|\sigma_{ij} x_j^*\right| + \left|f_i \sigma_{iq} x_q^*\right|, \;\; \forall\, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i, i \in M.$$

Now, assuming $\sigma_{ij} \geq 0$ and defining the infimum, or greatest lower bound, for $(p_i)_{\mathrm{P2}}$ in the constraint $p_i \geq \sum_{j\in S_i} \left|\sigma_{ij} x_j\right| + \left|f_i \sigma_{iq} x_q\right|, \;\; \forall\, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i, i \in M$ via the below argument:

$$\left(\widehat{S}_\iota, \hat{t}\right) = argmax \left\{ \sum_{j\in S_i} \left|\sigma_{ij} x_j^*\right| + \left|f_i \sigma_{iq} x_q^*\right| \right\}, \;\; \forall\, S_i \in N, |S_i| = \lfloor k_i \rfloor, q \in N \backslash S_i, i \in M.$$

Upon substitution, this results in

$$-\min_{j\in N}\{\sigma_{ij} x_j^*\}(|N| - k_i) + \sum_{j\in N} \sigma_{ij} x_j^* \geq \sum_{j\in \widehat{S}_\iota} \sigma_{ij} x_j^* + \left(f_i \sigma_{iq} x_{\hat{q}}^*\right), \;\; \forall\, i \in M$$

which simplifies to

$$\left(f_i \sigma_{iq} x_{\hat{q}}^*\right) + \sum_{j\in N\backslash\{\widehat{S}_\iota \cup \hat{q}\}} \sigma_{ij} x_j^* \geq \min_{j\in N}\{\sigma_{ij} x_j^*\}(|N| - k_i), \;\; \forall\, i \in M,$$

and which holds by inspection.

(b) Problem P2 has the same objective function value as Problem P1 since $(d_\iota^+)_{\mathrm{P2}} = (d_\iota^+)_{P1}, \; \forall \; i \in M.$

(c) Given an optimal solution to Problem P1, assume for contradiction that it is not optimal to Problem P2 under the stated transformation. Thus, $\exists\; \hat{\iota} \in M$ such that

$(d_{\hat{\imath}}^{+})_{P1} < (d_{\hat{\imath}}^{+})_{\mathrm{P2}}$. By constraint $\sum_{j=1}^{n} a_{\hat{\imath}j}x_j + p_{\hat{\imath}} - d_{\hat{\imath}}^{+} + d_{\hat{\imath}}^{-} = d_{\hat{\imath}},\ \ \forall\, \hat{\imath} \in M$ we have a resulting $(\overline{p_{\hat{\imath}}})_{\mathrm{P2}} > (p_{\hat{\imath}})_{\mathrm{P2}}^{*}$. Applying the transformation, $\exists \left(k_{\hat{\imath}}\bar{z}_{\hat{\imath}} + \sum_{j\in N} \bar{p}_{\hat{\imath}j}\right)_{P1} > \left(k_{i}z_{i} + \sum_{j\in N} p_{ij}\right)_{P1}^{*}$ and $\sum_{j=1}^{n} a_{\hat{\imath}j}x_j + \sum_{j=1}^{n} p_{\hat{\imath}j} + k_{\hat{\imath}}z_{\hat{\imath}} - d_{\hat{\imath}}^{+} + d_{\hat{\imath}}^{-} = d_{\hat{\imath}},\ \ \forall\, \hat{\imath} \in M$, corresponding to $\left(\bar{d}_{\hat{\imath}}^{+}\right)_{P1} < \left(\bar{d}_{\hat{\imath}}^{+}\right)_{\mathrm{P2}}$, a contradiction. □

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REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704–0188*

1. REPORT DATE (DD–MM–YYYY) 14-09-2014	2. REPORT TYPE Doctoral Dissertation	3. DATES COVERED (*From* — *To*) Oct 2014-Sep 2017

4. TITLE AND SUBTITLE Robust Goal Programming and Risk Assessment using Cardinality-Constrained and Strict Robustness via Alternative Uncertainty Sets	5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) Hanks, Robert W., Captain, USAF	5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Air Force Institute of Technology Graduate School of Engineering and Management (AFIT/EN) 2950 Hobson Way WPAFB OH 45433-7765	8. PERFORMING ORGANIZATION REPORT NUMBER AFIT-ENS-DS-17-S-035
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) United States Transportation Command Joint Distribution Process Analysis Center/TCJ8 ATTN: Tim Landvogt and Jasolyn Evans 508 Scott Dr Scott AFB, IL 62225	10. SPONSOR/MONITOR’S ACRONYM(S) USTRANSCOM, JDPAC, TCJ8 11. SPONSOR/MONITOR’S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT

Distribution Statement A. Approved for Public Release; Distribution Unlimited.

13. SUPPLEMENTARY NOTES

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14. ABSTRACT

Within many disciplinary applications, data uncertainty is problematic to informing parameters for optimization modeling. Although there exist alternative methods to account for such uncertainty, this research considers robust optimization (RO), wherein variability can be estimated but the probability distribution for different outcomes cannot be reasonably approximated. Within this context, this research sets forth three robust goal programming (RGP) models that alternatively combine cardinality-constrained robustness and norm-based uncertainty sets, as well as strict robustness and ellipsoidal uncertainty sets. With a view towards parametrizing these models for any given decision maker (DM), we also propose a mapping methodology that considers a DM’s risk preference *a priori* and relates this risk preference from the decision analysis subdiscipline to an RO risk parameter in the optimization subdiscipline. Finally, we demonstrate the applicability of the RGP model that applies cardinality-constrained robustness via L_2-norm uncertainty sets in unison with the aforementioned mapping methodology to a transportation rate setting problem addressed annually by the United States Transportation Command.

15. SUBJECT TERMS

Robust Optimization, Goal Programming, Risk Elicitation, Mathematical Programming, Rate Setting

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON Dr. Jeffery D. Weir, AFIT/ENS
a. REPORT U	b. ABSTRACT U	c. THIS PAGE U	UU	198	19b. TELEPHONE NUMBER (Include Area Code) (937) 255-3636 x4523 jeffery.weir@afit.edu