

QUANTIFYING PROPAGATION UNCERTAINTIES IN COMPUTER CODE CHAINS

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14. ABSTRACT We present a novel approach to quantifying uncertainties in the values of the output variables of computer codes used for numerical simulations of scientific and other processes of interest, where these uncertainties are a result of (1) uncertainties in the "true" (real-world) values of the code input variables, (2) uncertainties in the forms and details of the code models, and (3) numerical "uncertainties" arising from the need to perform arithmetic computations via discretization of the often continuum model equations and from the use of finite precision. We apply our uncertainty quantification (UQ) methodology to both a single, stand-alone code as well as to an N-member code chain in which the jth code in the chain produces a value of an output variable having uncertainty quantified in our sense and which output variable value subsequently serves as an input variable value for the (j+1)th successor code in the chain, which (j+1)th code then itself produces a value of an output variable also having uncertainty quantified in our sense. The uncertainties in the values of all input and output variables are represented in our formulation by probability density functions (PDF's). Our entire formulation is analytic and requires (1) two "uncertainty constants" for each code in the chain, (2) knowledge of (or estimates of) upper and lower bounds for the values of all code variables of interest, and (3) the functional relationships between all input and output variables of interest (which may be available either analytically or computationally) in the absence of uncertainty considerations.					
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Quantifying Propagating Uncertainties in Computer Code Chains

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0. Abstract

We present a novel approach to quantifying uncertainties in the values of the output variables of computer codes used for numerical simulations of scientific and other processes of interest, where these uncertainties are a result of (1) uncertainties in the “true” (real-world) values of the code input variables, (2) uncertainties in the forms and details of the code models, and (3) numerical “uncertainties” arising from the need to perform arithmetic computations via discretization of the often continuum model equations and from the use of finite precision. We apply our uncertainty quantification (UQ) methodology to both a single, stand-alone code as well as to an N-member code chain in which the j^{th} code in the chain produces a value of an output variable having uncertainty quantified in our sense and which output variable value subsequently serves as an input variable value for the $(j+1)^{\text{th}}$ successor code in the chain, which $(j+1)^{\text{th}}$ code then itself produces a value of an output variable also having uncertainty quantified in our sense. The uncertainties in the values of all input and output variables are represented in our formulation by probability density functions (PDF’s). Our entire formulation is analytic and requires (1) two “uncertainty constants” for each code in the chain, (2) knowledge of (or estimates of) upper and lower bounds for the values of all code variables of interest, and (3) the functional relationships between all input and output variables of interest (which may be available either analytically or computationally) in the absence of uncertainty considerations.

I. Introduction

The interest in quantification of uncertainties in the values of variables under consideration in numerical simulations of scientific and other processes—those simulations being implemented via digital computers using applicable models of those processes (i.e., via computer “codes”)—is expanding. In [1], a comprehensive framework for such *uncertainty quantification* (UQ) was presented based upon a threefold division of the sources of that uncertainty, namely, (1) uncertainties in the “true” (real-world) values of some (all) of the code input variables and input parameters (the latter regarded as distinct from input variables), (2) uncertainties in the forms and details of the code models,

and (3) numerical “uncertainties” arising from the need to perform arithmetic computations *via* discretization of the often continuum model equations and from the use of finite precision. In this paper we present an approach to UQ that, in addition to being novel in itself, is based upon, rather, a twofold division of uncertainties in which we retain source-(1) as separate but coalesce source-(2) plus source-(3) (denoting the latter as source-(2+3)). Further, the uncertainties in the values of each of our quantities of central interest—computer code input and output variables—are taken by us to be representable by probability distributions (in fact, probability density functions [PDF’s]), whether the associated real-world (physical) variables represented in the code by corresponding code variables be aleatory or epistemic [1] (so that epistemic quantities represented by interval-valued variables are beyond our present scope). We consider both a single stand-alone code and, as well, the more general situation in which several computer codes are linked together in “end-to-end” fashion to form a *chain* of several codes so that some of the outputs of any given code in the chain serve dually as inputs to successor codes in the chain. In this manner, uncertainties in the values of output variables computed at one stage in the chain will in general lead to uncertainties in values of output variables computed in subsequent stages in the chain when the latter variables depend upon (some of) the former ones; this process is called “propagation of uncertainty” through the chain.

Our overall approach, presented in the current and four paragraphs thereafter, is as follows. Foundationally, we take the point of view that the notion of uncertainties in the values taken by code input and output variables serves to provide a quantification of *the degree to which the values assigned to the input variables and the values produced therefrom for the output variables faithfully represent the real-world (non-code) values of the real-world physical quantities that these code variables represent in the code modeling*. To quantitatively develop this real-world-variable/code-variable interrelationship, we first consider a single stand-alone code and, in addition, for it we initially consider any code that admits only one input variable and only one output variable (and no uncertain parameters); further, we consider for that code—again only initially—any allowed single value of the input variable, which single input value results in a single code-computed value for the output variable.

To proceed, we associate with this single computed value of the code output variable a specified (see later) non-sharp PDF (i.e., not a δ -function) for the possible value of the real-world variable represented in the code by that code output variable, when the real-world experiment mimicking that modeled in the code is performed, and with any discrepancy between code and real-world results attributable to code source-(2+3) uncertainty; thus this non-sharp PDF is intended to capture solely the source-(2+3) uncertainty—and not the source-(1) uncertainty—in the code output variable value. This associated PDF is logically extraneous to the code itself and may be specified separately from the latter. (Indeed, when we later formalize this process via probability theory, the random variables therein will correspond to the real-world variables and not to the code variables.) We refer to this process of assigning a non-sharp PDF to the code output variable value as “smearing” of the code output variable value. This procedure provides a quantitative measure of the extent to which the computed code result for the output variable value accurately represents the real-world value of the real-world variable corresponding to that code output variable, even when the specified value of the input variable has no uncertainty associated with it at all; i.e., the procedure attempts to account solely for uncertainty source-(2+3). We then define the uncertainty in the code output variable value (with respect to the true value of the corresponding real-world variable)—when a single value of the code input variable is specified—to be that given by the associated smearing PDF for the corresponding real world variable.

Next, we relax the condition above that imposes no uncertainty at all in the single value of the code input variable and allow for a distribution of possible values “associated with” (see later) that input variable single value, thus enabling capture of code input variable value uncertainty—uncertainty source-(1)—as expressed by some PDF for this uncertainty. That is, we associate with a single value of the code input variable a specified PDF (which may be a δ -function when that input value has no uncertainty at all) for the possible real-world values of the real-world variable represented in the code by that input variable, when the real-world experiment mimicking that modeled in the code is performed, and with any discrepancy between code and real-world results attributable to source-(1) uncertainty—and not to source-(2+3) uncertainty—in the code input variable value; i.e., we allow the value of the input variable to be smeared as well. Thus, in analogy

with the uncertainty of the code output variable value being defined *via* the PDF of a real-world variable, we may also identify the uncertainty in the code input variable value with an uncertainty PDF for the real-world variable represented in the code by the former. (Again, when we later formalize this process *via* probability theory, the random variable therein will correspond to the real-world variable and not to the code variable.) We then coalesce this input variable value uncertainty PDF with the PDF's resulting from the output variable value smearing process for each of the possible input variable smearing values (namely, those in the support of the input-variable- value uncertainty PDF) to arrive at the overall (i.e., comprising sources (1) + (2) + (3)) uncertainty PDF for the output variable (as a weighted superposition of output-smearing PDF's). Once again, *we define the uncertainty in the code output variable value—this time when the code input variable value uncertainty PDF is specified—to be that given by the PDF that is the weighted (via the input variable value uncertainty PDF) superposition of output-smearing PDF's.* This is then the explicit interrelationship between real-world-variable uncertainty and code-variable uncertainty in the general case (of an input variable value uncertainty PDF).

Having established the single-code/single-input-single-output-variable results, we then provide the straight- forward extension of these results to codes having more than one output variable but still a single input variable (and no uncertain input parameters). The further extension to codes having more than one input variable (and possibly also uncertain input parameters) and a single output variable, and the consequent straightforward extension thereof to codes having more than one input variable and more than one output variable, will appear in a follow-on paper.

Finally, for code chains, we consider initially just a pair of codes, i.e., any two-code chain, and apply the above single-code results to it (where each code in the chain has one input variable and one output variable and no uncertain parameters; the extension to chains in which any of the codes has more than one input variable will appear in the follow-on paper alluded to above). We then extend these results to any arbitrary finite-length code chain. This concludes the description of our overall approach.

It is useful at this point to expand the discussion of our fundamental notion, alluded to above, of the smearing of a single computed value of an output variable, this output value

resulting from a single value of an input variable. To reiterate: *We assign to each possible computed value, say ξ , of a code output variable—each such value computed by the code solely from a single value, say ω , of the code input variable—a non-sharp smearing PDF, say f^ξ , that represents the probabilities for the occurrence of real-world values, say x , of the real-world variable corresponding to the code output variable when the value of the real-world input variable value is $w = \omega$ (and the code is mimicking the experiment). Further, as suggested notationally above, this output-smearing PDF depends upon the value (ξ) of the output variable computed by the code for the given value (ω) of the input variable. In general, differing choices for the value taken on by the input variable, which in general result in differing values for the output variable, lead to differing output-smearing PDF assignments for the corresponding real-world quantity. (In case two differing values of the input variable lead to one and the same value of the output variable, we take the output-smearing PDF assignment to be identical for both.) In more detail, our approach associates with a single value, say ψ , of a code-computed output variable, say Ψ —with ψ as computed using a given value of each input variable (where we include here the more general case of several input variables)—a PDF for Ψ that encompasses the possible “real-world” values z of real-world variable Z^* that is represented in the code by Ψ , with the parameters of the PDF dependent upon the computed value ψ of Ψ . If we consider Z^* , hence Ψ , to be positive-valued (which is no loss of generality since *we will consider only bounded input and output variables*) then we stipulate:*

If the value of output variable $\Psi > 0$, as computed by a code using specified values of all input variables, is given by $\psi > 0$, and if Ψ represents in the code the real-world variable $Z^* > 0$ that takes on real-world value $z > 0$ when the values taken by the real-world correspondents of the code input variables are the above-referred-to specified code input variable values, then for some real numbers a and b satisfying $0 < a \leq 1$ and $0 < b < 1$ we have

$$\text{Probability[actual real-world value, } z, \text{ of } Z^* \text{ satisfies } (1 - b)\psi \leq z \leq (1 + b)\psi] = a. \quad (0)$$

Equation (0) quantifies the uncertainty in code output variable value ψ insofar as ψ represents the real-world value z of Z^* under the condition that the set of values of the code input variables and those of the corresponding real-world (“input”) variables are

identical. We call the numbers a and b “*uncertainty constants*” for Ψ with respect to the code; and we say that output variable Ψ is “*smeared*” at ψ via a and b . (In the present paper, we avoid the complication that a and b may depend upon Ψ and/or ψ .) The description “*smeared*” applied to Ψ at ψ is intended to convey the idea that the single value ψ of Ψ may in fact be made to imply information—based upon Eq. (0)—about the probability that any given subset of possible values z of $\mathcal{Z}^*$ may occur, where that subset is any taken from the interval $[z_{\min}, z_{\max}]$ containing all possible values of $\mathcal{Z}^*$ (and—as we assume—any member of $[z_{\min}, z_{\max}]$ is a possible value of $\mathcal{Z}^*$). In other words, “*smeared*” as applied to Ψ refers to a particular type of uncertainty specification PDF for output variable Ψ induced by the value ψ of Ψ , namely any PDF for the possible values z of $\mathcal{Z}^*$ that is consistent with the condition of Eq. (0). (Later, we will specify explicitly some specific such PDF’s.)

We note that in our discussions above, we have actually alluded to two types of smearing of the *output* variable. The first type involves smearing in the case of a single value for the input variable and is the one being discussed in Eq. (0) *ff.* while the second type involves an input-variable-uncertainty-PDF weighted superposition of smeared distributions of the first type. To distinguish between the two we term the first type “*basic smearing*” and the second type—and indeed any such more general uncertainty description that may arise in our developments (e.g., for code-chains)—“*compound smearing*”.

As a final matter in this Introduction, we observe that while the statement in Eq. (0) appears *intuitively* clear, its precise meaning must rest upon a mathematical foundation provided by the presence of some probability space $(\Omega, \mathcal{A}, \mathcal{P})$ in which the bracketed expression in Eq. (0) corresponds to an event $E \in \mathcal{A}$ and “ $\text{Prob}[E]$ ” is given by a probability measure $\mathcal{P}$ on $(\Omega, \mathcal{A})$ as $\mathcal{P}(E)$. We now repair this deficiency.

To begin, let $\mathcal{C}$ be any computer code that provides a quantitative treatment of the behavior of some phenomena (the *scope* of $\mathcal{C}$) at some given level of fidelity (the *modeling assumptions* of $\mathcal{C}$) and suppose that $\mathcal{C}$ comprises a totality of n_1 code input variables U_1 ,

..., U_{n_I} —with U_j taking on values $u_j \in \mathcal{D}_j \subseteq \mathbb{R}$, $j = 1, \dots, n_I$ —and produces therefrom a totality of n_O code output variables $V_1, \dots, V_{n_O}$, with V_k taking on values $v_k \in \mathbb{R}$, $k = 1, \dots, n_O$. In the context of Eq. (0), we are considering $n_O = 1$ with $V_{n_O} = V_1 \equiv \Psi$ and $v_{n_O} = v_1 \equiv \psi$. Next, we introduce some further terminology and notation. An *instantiation* of $\mathcal{C}$ is any phenomena-self-consistent input n_I -tuple $\mathbf{u} \equiv \langle u_1, u_2, \dots, u_{n_I} \rangle \in \prod_{j=1}^{n_I} \mathcal{D}_j$; we denote the set of all instantiations of $\mathcal{C}$ by $\mathcal{I}[\mathcal{C}]$ where, in general, $\mathcal{I}[\mathcal{C}]$ may be only a proper subset of $\prod_{j=1}^{n_I} \mathcal{D}_j$. Further, if $\mathbf{u} \in \mathcal{I}[\mathcal{C}]$ then we denote by $\mathcal{C}[\mathbf{u}]$ the output n_O -tuple $\mathbf{v}_u \equiv \langle v_1(\mathbf{u}), \dots, v_{n_O}(\mathbf{u}) \rangle$ produced by $\mathcal{C}$ when using input *instantiation* $\mathbf{u}$. Analogously, when a physical experiment is performed to examine the phenomena within the scope of $\mathcal{C}$ and the values of the components of instantiation $\mathbf{u}$ are selected as those for the experimental input variables $\hat{U}_1, \dots, \hat{U}_{n_I}$ that are represented in the code by code input variables $U_1, \dots, U_{n_I}$ respectively (with $\mathbf{u}$ thus dually representing values of experimental inputs as well as of code inputs), then we denote by $\mathcal{E}[\mathbf{u}]$ the experimental values taken on by the experimental output variables $\hat{V}_1, \dots, \hat{V}_{n_O}$ that are represented in the code by code output variables $V_1, \dots, V_{n_O}$ respectively, with $\mathcal{E}[\mathbf{u}]$ also being represented by an n_O -tuple denoted by $\hat{\mathbf{v}}_u \equiv \langle \hat{v}_1(\mathbf{u}), \dots, \hat{v}_{n_O}(\mathbf{u}) \rangle$. Of course, repeated experimental trials using one and the same $\mathbf{u}$ will in general not give precisely the same $\hat{\mathbf{v}}_u$'s for each such trial because of experimental error; nevertheless, for each $\mathbf{u} \in \mathcal{I}[\mathcal{C}]$ we define a *unique* $\mathcal{E}[\mathbf{u}]$ to be the average $\hat{\mathbf{v}}$ obtained over “sufficiently many” repeated trials of the experiment using input $\mathbf{u}$ (where the components of $\hat{\mathbf{v}}$ are the average $\hat{v}_k(\mathbf{u})$'s). *In principle*, for every $\mathbf{u} \in \mathcal{I}[\mathcal{C}]$, $\mathcal{C}[\mathbf{u}]$ and $\mathcal{E}[\mathbf{u}]$ may be determined (the circumstance that, in general, the cardinality of $\mathcal{I}[\mathcal{C}]$ is not finite notwithstanding).

We are now prepared to specify a probability space *via* which Eq. (0) is well-defined. As indicated earlier, for the Eq. (0) context we have $n_O = 1$ so that $\mathcal{C}[\mathbf{u}] = \langle v_1(\mathbf{u}) \rangle$ and

$\mathcal{E}[\mathbf{u}] = \langle \hat{V}_1(\mathbf{u}) \rangle$ with $V_1 = \Psi$, $v_1 = \psi$, $\hat{V}_1 = \mathcal{Z}^*$ and $\hat{v}_1 = z$; further we have taken $V_1, \hat{V}_1 > 0$. To proceed, we fix—as per Eq. (0)—some $b \in (0, 1)$ and we imagine a random experiment in which we randomly select at each trial an instantiation, $\mathbf{u}$, from the “box” $\mathcal{J}[\mathcal{Q}]$, without replacement, and observe if $|\mathcal{Q}[\mathbf{u}] - \mathcal{E}[\mathbf{u}]| \leq b |\mathcal{Q}[\mathbf{u}]|$ or if $|\mathcal{Q}[\mathbf{u}] - \mathcal{E}[\mathbf{u}]| > b |\mathcal{Q}[\mathbf{u}]|$, i.e., if $|v_1(\mathbf{u}) - \hat{v}_1(\mathbf{u})| \leq b v_1(\mathbf{u})$ or $|v_1(\mathbf{u}) - \hat{v}_1(\mathbf{u})| > b v_1(\mathbf{u})$; note that $|v_1(\mathbf{u}) - \hat{v}_1(\mathbf{u})| \leq b v_1(\mathbf{u})$ iff $(1 - b)v_1(\mathbf{u}) \leq \hat{v}_1(\mathbf{u}) \leq (1 + b)v_1(\mathbf{u})$. The outcome set for this random experiment can then be taken to be $\Omega = \{“\leq”, “>”\}$. We denote by $E_{\leq}$ the event $E_{\leq} = \{“\leq”\}$ = “the selected $\mathbf{u}$ is such that $|\mathcal{Q}[\mathbf{u}] - \mathcal{E}[\mathbf{u}]| \leq b |\mathcal{Q}[\mathbf{u}]|$ ” and by $E_{>}$ the event $E_{>} = \{“>”\}$ = “the selected $\mathbf{u}$ is such that $|\mathcal{Q}[\mathbf{u}] - \mathcal{E}[\mathbf{u}]| > b |\mathcal{Q}[\mathbf{u}]|$ ”; further, $\mathcal{A} = \{\emptyset, E_{\leq}, E_{>}, \Omega\}$. We now assume that this random experiment exhibits probabilistic behavior: If $S \equiv \langle \mathbf{u}_i^S \rangle_{i=1}^{\infty}$ is any sequence of instantiations in $\mathcal{J}[\mathcal{Q}]$ —that sequence arising from successive draws from box $\mathcal{J}[\mathcal{Q}]$ —and if $\#(S, n; \leq)$ denotes the number of outcomes “ $\leq$ ” observed in finite subsequence $\langle \mathbf{u}_i^S \rangle_{i=1}^n$ of S , then $\lim_{n \rightarrow \infty} [\#(S, n; \leq)/n]$ exists and has value independent of S . We denote this limit by $L_{\leq}$ and take $\mathcal{P}(E_{\leq}) = L_{\leq}$ and $\mathcal{P}(E_{>}) = 1 - L_{\leq}$; i.e., $\mathcal{P}(\mathbf{u} \text{ is such that } |\mathcal{Q}[\mathbf{u}] - \mathcal{E}[\mathbf{u}]| \leq b |\mathcal{Q}[\mathbf{u}]|) = \mathcal{P}(E_{\leq}) = L_{\leq}$. In our case of Eq. (0), for which we further write $\langle v_1(\mathbf{u}) \rangle = \psi_u$ and $\langle \hat{v}_1(\mathbf{u}) \rangle = z_u$, we have $L_{\leq} = a$; so the above becomes $\mathcal{P}(\mathbf{u} \text{ is such that } |\psi_u - z_u| \leq b \psi_u) = a$, i.e., $\mathcal{P}(\mathbf{u} \text{ is such that } (1 - b)\psi_u \leq z_u \leq (1 + b)\psi_u) = a$. We then define

$$\begin{aligned}
 & \text{Prob}[\text{actual real-world value, } z, \text{ of } \mathcal{Z}^* \text{ satisfies } (1 - b)\psi \leq z \leq (1 + b)\psi] \\
 & \equiv \mathcal{P}(\mathbf{u} \text{ is such that } (1 - b)\psi_u \leq z_u \leq (1 + b)\psi_u) \\
 & = a,
 \end{aligned}$$

where by “real-world value” we mean experimental value. And we are thus done.

The remainder of this paper proceeds as follows. In Section II we construct in detail our methodology for implementing our approach to UQ for a single stand-alone code that admits only one input variable and one output variable (and no uncertain parameters), with the input variable taking on any *one (fixed)* of its allowed values. We then extend our results to allow for a (smearing) distribution of possible values of the input variable, with

the values included in that distribution being those in some arbitrary neighborhood of the fixed input variable value indicated above. In Section III we treat code chains, beginning by considering any two-code chain and applying our single-code results of the previous section to it; we then extend these results to any arbitrary finite-length code chain. In Section IV we develop a formalism for a two-code chain to treat the presence in Code-2 itself of a particular type of stochasticity, this stochasticity differing from that arising from UQ considerations and being important in a particular application of interest; we also present there an example of the use of this formalism in that application. In Section V we conclude.

II. A Single Stand-Alone Code

In this section we present in detail our methodology for implementing our approach to UQ for *any* single, stand-alone code that admits only one input variable and one output variable (and no uncertain parameters), with the input variable in the initial stage of this development taking on any *one fixed* (but arbitrary within its allowed range) value, say $\omega_{\S}$, and the code producing therefrom a single value of the output variable, say $\xi_{\omega_{\S}}$. (In the more general context of an input variable value distribution [PDF], the input variable value PDF in this single-input-value case may be regarded as “sharp”, being given by a δ -function peaked at $\omega_{\S}$.) We then extend our results to accommodate a non-sharp smearing PDF for possible input variable values in any neighborhood of $\omega_{\S}$.

Since we are ultimately interested in a code chain (sequence) comprising $N < \infty$ such single-input-single-output codes—which codes we label by index $n = 1, \dots, N$ in the order of their occurrence in the chain—we then label the single, stand-alone code under consideration in this section as “Code- n ”, for the results we derive here hold for every one of the N single codes of the chain.

II.1 Code- n Description

To begin, for the single code under consideration, namely, Code- n , we designate its input variable as $\mathcal{W}_n$, with $\mathcal{W}_n$ having values designated by ω_n and with the set of all values ω_n allowed to $\mathcal{W}_n$ designated by $\mathcal{V}_{\mathcal{W}_n}$ (so that $\omega_n \in \mathcal{V}_{\mathcal{W}_n}$). $\mathcal{W}_n$ represents in Code- n the real-

world variable W_n^* that has values designated by w_n , with the set of all values allowed to W_n^* designated by $\mathcal{V}_{W_n^*}$. We designate the Code- n output variable as $\mathcal{X}_n$, with $\mathcal{X}_n$ having values designated by ξ_n and with the set of all values allowed to $\mathcal{X}_n$ designated by $\mathcal{V}_{\mathcal{X}_n}$. $\mathcal{X}_n$ represents in Code- n the real-world variable X_n^* that has values designated by x_n , with the set of all values allowed to X_n^* designated by $\mathcal{V}_{X_n^*}$. When code output value ξ_n results from code input value ω_n , we then write $\xi_n = \xi_{n;\omega_n}$; similarly, when real-world-value x_n results from measurement of X_n^* when the value of W_n^* is w_n then we write $x_n = x_{n;w_n}$. Note that output variable $\mathcal{X}_n$ is basically smeared at every achievable value of ξ_n via a_n and b_n but input variable $\mathcal{W}_n$ is not smeared at all (at this stage).

Concerning variables $\mathcal{W}_n$ and W_n^* , we suppose that

$$\mathcal{V}_{\mathcal{W}_n} = [\omega_{n, \min}, \omega_{n, \max}] \quad \text{and} \quad \mathcal{V}_{W_n^*} = [w_{n, \min}, w_{n, \max}] \quad (1)$$

where

$$\omega_{n, \min} > 0, \quad \omega_{n, \max} < \infty, \quad \omega_{n, \min} \leq \omega_{n, \max} \quad (2a)$$

and

$$w_{n, \min} > 0, \quad w_{n, \max} < \infty, \quad w_{n, \min} \leq w_{n, \max}; \quad (2b)$$

further, we suppose that $\omega_{n, \min} = w_{n, \min}$ and $\omega_{n, \max} = w_{n, \max}$ so that

$$\mathcal{V}_{\mathcal{W}_n} = \mathcal{V}_{W_n^*} \quad (3)$$

(i.e., any possible real-world value of W_n^* is an admissible input value of $\mathcal{W}_n$).

Concerning the use of distinct symbols ω and w , especially in view of Eq. (3), we offer the following. The symbol ω is used to represent any *fixed input* value of code input variable $\mathcal{W}$, e.g. $\omega_{\S}$ (see the first paragraph of this section for notation), and does not vary in the discussion once selected. On the other hand, the symbol w is used to represent values of virtual input value variations around ω , which variations arise from consideration of the smearing of ω into subsets of $\mathcal{V}_{W_n^*}$, with w *varying* in such subsets. Of course, one of the possible values of w is the value of ω , for which we write $w = \omega$,

but w is allowed to take on values other than that of ω , but the value of ω is fixed once selected. We have in this paper chosen to make $\mathcal{V}_w = \mathcal{V}_{w^*}$ (Eq.(3)) but one could, in more general treatments, choose rather $\mathcal{V}_w \subseteq \mathcal{V}_{w^*}$.

With regard to variables X_n and X_n^* , we suppose that

$$\mathcal{V}_{X_n} = [\xi_{n, \min}, \xi_{n, \max}] \quad \text{and} \quad \mathcal{V}_{X_n^*} = [x_{n, \min}, x_{n, \max}] \quad (4)$$

where

$$\xi_{n, \min} > 0, \quad \xi_{n, \max} < \infty, \quad \xi_{n, \min} \leq \xi_{n, \max} \quad (5a)$$

and

$$x_{n, \min} > 0, \quad x_{n, \max} < \infty, \quad x_{n, \min} \leq x_{n, \max}; \quad (5b)$$

further, we suppose that $x_{n, \min} \leq \xi_{n, \min}$ and $\xi_{n, \max} \leq x_{n, \max}$ so that

$$\mathcal{V}_{X_n} \subseteq \mathcal{V}_{X_n^*} \quad (6)$$

(with proper inclusion holding when, for example, Code- n modeling is deficient in that it cannot produce the full range of possible values for X_n^*). Note that in particular, the third condition in Eq. (5b) disallows the case $x_{n, \min} = x_{n, \max} \equiv x_{n,0}$. On mathematical grounds the equality of $x_{n, \min}$ with $x_{n, \max}$ causes technical difficulties in our formulation; but on physical grounds this case is eliminable because there is no uncertainty in the value of X_n produced by a proper code—that value must always be $\xi_n = x_{n,0}$. (But, consistent with the third condition in Eq. (2b), we *do allow* the case $w_{n, \min} = w_{n, \max}$ [hence also $\omega_{n, \min} = \omega_{n, \max}$].) Finally, we write $\xi_{n; \omega_n} = g_n(\omega_n)$ for function $g_n: \mathcal{V}_{\omega_n} \rightarrow \mathcal{V}_{X_n}$ where we assume here that g_n is strictly positive and continuous on $\mathcal{V}_{\omega_n}$. Then g_n is onto $[\xi_{n, \min}^{g_n}, \xi_{n, \max}^{g_n}]$ where

$$\xi_{n, \min}^{g_n} \equiv \min_{\omega_n \in [\omega_{n, \min}, \omega_{n, \max}]} g_n(\omega_n) = \xi_{n, \min} \quad (7a)$$

and

$$\xi_{n, \max}^{g_n} \equiv \max_{\omega_n \in [\omega_{n, \min}, \omega_{n, \max}]} g_n(\omega_n) = \xi_{n, \max} \quad (7b)$$

and, further, g_n is bounded on $\mathcal{V}_{\omega_n}$: below by $\xi_{n, \min}^{g_n} = \xi_{n, \min} \geq x_{n, \min} > 0$ and above by

$\xi_{n,max}^{g_n} = \xi_{n,max} \leq x_{n,max} < \infty$; note also that $\text{rng } g_n = [\xi_{n,min}^{g_n}, \xi_{n,max}^{g_n}] = [\xi_{n,min}, \xi_{n,max}] = \mathcal{V}_{\mathcal{X}_n}$. We will place an additional technical requirement on g_n in a later section (II.4.1) but we defer discussion of that requirement until then. We point out that distinction here between ξ and x is analogous to that between ω and w discussed earlier.

In terms of the above definitions, the condition of Eq. (0) as applied to Code- n becomes

$$\text{Probability}[(1 - b_n)\xi_{n;\omega_n} \leq x_{n;\omega_n} \mid \omega_n = \omega_n \leq (1 + b_n)\xi_{n;\omega_n}] = a_n, \quad (8)$$

where a_n and b_n are given uncertainty constants for $\mathcal{X}_n$ with respect to Code- n with $0 < a_n \leq 1$ and $0 < b_n < 1$.

We now begin to develop our framework to compute $\text{Prob}(X_n^* \in A \text{ when } \mathcal{X}_n = \xi_n)$ for any $\xi_n \in \mathcal{V}_{\mathcal{X}_n}$ and $A \subseteq \mathcal{V}_{\mathcal{X}_n^*}$. To that end, we introduce a probability space having outcome set $\Omega_n^{\rightarrow} = \mathcal{V}_{\mathcal{X}_n^*}$ representing any possible outcome of an actual real-world measurement of the value of the real-world quantity X_n^* represented in Code- n by $\mathcal{X}_n$. We now need to represent X_n^* in our probability framework and we do so *via* a random variable, X_n , given by $X_n(x_n) = x_n$ for $x_n \in \Omega_n^{\rightarrow}$. (The necessity for X_n in addition to X_n^* and $\mathcal{X}_n$ is that the latter two possess no associated probability space structure in which legitimate manipulations [to be required below] may be performed; but the former, X_n , does. Also, for purists we point out that the σ -algebra, $\mathcal{A}_n^{\rightarrow}$, of events in $\Omega_n^{\rightarrow}$ required for the proper definition of X_n may be taken to be the Lebesgue-measurable subsets of $\mathcal{V}_{\mathcal{X}_n^*}$.) The specification of the probability assignment (measure) in $\Omega_n^{\rightarrow}$ (or, more precisely, in $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow})$), say $\mathcal{P}_n^{\rightarrow}$, may be accomplished by specifying a PDF, f_{X_n} , for X_n . In doing so, we must capture in X_n the notion of uncertainty for $\mathcal{X}_n$ given by Eq. (0) *via* Eq. (8). Then, as described earlier, the PDF for X_n will be ascribed to $\mathcal{X}_n$ in order to define the uncertainty of $\mathcal{X}_1$. Since this procedure demands a given (but arbitrary within $\mathcal{V}_{\mathcal{X}_n}$) Code- n -computed value for output variable $\mathcal{X}_n$ —given in Eq. (8) by $\xi_{n;\omega_n}$ but given in general by simply ξ_n —then the PDF for

X_n must depend upon ξ_n ; we acknowledge this ξ_n -dependence by writing $X_n^{\xi_n}$ in lieu of merely X_n and therefore writing as well $f_{X_n^{\xi_n}}$ in lieu of f_{X_n} and $\mathcal{P}_n^{\rightarrow \xi_n}$ in lieu of $\mathcal{P}_n^{\rightarrow}$. We thus wish to capture for $X_n^{\xi_n}$ the requirement for it dictated by Eq. (8) that, given uncertainty constants a_1 and b_1 with respect to Code- n for X_n then

$$\mathcal{P}_n^{\rightarrow \xi_n}((1 - b_n)\xi_n \leq X_n^{\xi_n} \leq (1 + b_n)\xi_n) = a_n, \quad (9)$$

i.e., that $X_n^{\xi_n}$ is basically smeared at ξ_n via a_n and b_n . (We have here extended the terminology “smeared”—from beyond merely code variables to random variables as well—in order to serve also as a descriptor of the latters’ PDF’s.) Note that for *each*

$n = 1, \dots, N$, we are in fact considering a *family* of probability spaces indexed by $\xi_n \in \mathcal{V}_{X_n}$, namely, $\{(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow}, \mathcal{P}_n^{\rightarrow \xi_n})\}_{\xi_n \in \mathcal{V}_{X_n}}$, where $X_n^{\xi_n}$ is a random variable in $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow}, \mathcal{P}_n^{\rightarrow \xi_n})$ with $X_n^{\xi_n}(x_n) = x_n$ for $x_n \in \Omega_n^{\rightarrow}$; further, each $X_n^{\xi_n}$ is a continuous random variable since each possesses a PDF $f_{X_n^{\xi_n}}$.

Our choice for PDF $f_{X_n^{\xi_n}}$ entails several assumptions and, further, will in fact take on one of four possible forms (“types”) *depending solely upon the values of* $x_{n, \min}$, $x_{n, \max}$, b_n , and ξ_n . (While the $f_{X_n^{\xi_n}}$ type will be independent of a_n , the specification of the functions for the differing $f_{X_n^{\xi_n}}$ ’s will depend explicitly upon a_n , except for the fourth type.) We now discuss these PDF’s.

II.2 PDF’s $f_{X_n^{\xi_n}}(x_n)$ for Fixed $\xi_n \in \mathcal{V}_{X_n}$ as a Function of Variable $x_n \in \mathcal{V}_{X_n^*}$

The four possible forms for $f_{X_n^{\xi_n}}$, termed as *Types A, B, C, D*, and written as $f_{X_n^{\xi_n}}^{\alpha}$, $\alpha = A, B, C, D$, are shown in Fig.1 (with ξ there interpreted to mean ξ_n). These PDF’s must, of course, be defined for all $x_n \in \mathbb{R}$ ($\mathbb{R}$ being the set of all real numbers): we have

$$f_{X_n^{\xi_n}}^{\alpha}(x_n) = 0 \text{ for } x_n \in \mathbb{R} \setminus \mathcal{V}_{X_n^*}.$$

Throughout this section (II.2) we take $\xi_n \in \mathcal{V}_{\mathcal{X}_n}$ to be fixed (but arbitrary within $\mathcal{V}_{\mathcal{X}_n}$). Further, for this ξ_n we have $g_n^{-1}[\{\xi_n\}] \neq \emptyset$ and if we select any one ω_n therein, say $\omega_{n,\$}$, then we may regard $\xi_n = \xi_{n,\omega_{n,\$}}$; i.e., we may regard $\omega_{n,\$}$ as the fixed value of the Code- n input variable. Also, for simplicity of notation within the remainder of this section, we will suppress all subscripts and superscripts “ n ” and write simply $\mathcal{W}$, ω , $\mathcal{X}$, ξ , $\mathcal{W}^*$, w , $\mathcal{X}^*$, x , X , ξ_ω , x_w , $\mathcal{V}_{\mathcal{W}}$, $\mathcal{V}_{\mathcal{W}^*}$, $\mathcal{V}_{\mathcal{X}^*}$, $\mathcal{V}_{\mathcal{X}}$, a , b , g , $\Omega^\rightarrow$, $\mathcal{A}^\rightarrow$, and $\mathcal{P}^\rightarrow^\xi$.

The PDF in Fig. 1-A—applicable only when $x_{\min} < (1 - b)\xi$ and $(1 + b)\xi < x_{\max}$ —has three non-zero regions. A central region captures the requirement of Eq. (9), with respect to probability a , via a uniform distribution for X^ξ in the interval $[(1 - b)\xi, (1 + b)\xi]$. This is, once again, a choice of convenience—other choices, e.g., a central portion of a normal distribution having mean ξ , may be accommodated. Secondly, the left and right “wing” regions divide the remaining probability, $1 - a$, between them, with the left wing encompassing $\phi_X(\xi)(1 - a)$ of that probability, where $0 < \phi_X(\xi) < 1$ is arbitrary, and with the right wing then encompassing the remaining $[1 - \phi_X(\xi)](1 - a)$ of that probability. In general, ϕ_X may also depend upon a , b , $x_{\min}$, $x_{\max}$, and ξ , but we exhibit only the ξ -dependence explicitly. Further, f_{X^ξ} in these two intervals, $[x_{\min}, (1 - b)\xi]$ and $[(1 + b)\xi, x_{\max}]$ respectively, is taken to be linear; yet again, these two choices are merely ones of convenience—other choices can be accommodated. The PDF in Fig.1-B—applicable only when $x_{\min} < (1 - b)\xi$ and $(1 + b)\xi \geq x_{\max}$ —has two non-zero regions. The right region captures the requirement of Eq. (9) via a uniform distribution for X^ξ in the interval $[(1 - b)\xi, x_{\max}]$, assigning probability a to that interval since $[(1 - b)\xi, (1 + b)\xi]$ is the disjoint union $[(1 - b)\xi, x_{\max}] \cup (x_{\max}, (1 + b)\xi]$ and the latter interval is such that $\mathcal{P}^\rightarrow^\xi(X^\xi \in (x_{\max}, (1 + b)\xi]) = 0$; the remaining probability $1 - a$ is assigned to the left region, again via a linear function. The PDF in Fig. 1-C—applicable only when $(1 - b)\xi \leq x_{\min}$ and $(1 + b)\xi < x_{\max}$ —also has two non-zero regions and the specification of f_{X^ξ} in this case is analogous to that in Fig. 1-B. Finally, the PDF in Fig.1-D—applicable only when $(1 - b)\xi$

$\leq x_{min}$ and $(1 + b)^\xi \geq x_{max}$ —has one non-zero region. Since $[(1 - b)^\xi, (1 + b)^\xi]$ is the disjoint union $[(1 - b)^\xi, x_{min}) \cup [x_{min}, x_{max}] \cup (x_{max}, (1 + b)^\xi]$ and $\mathcal{P}^{\rightarrow\xi}(X^\xi \in [(1 - b)^\xi, x_{min})) = 0 = \mathcal{P}^{\rightarrow\xi}(X^\xi \in (x_{max}, (1 + b)^\xi])$ and $\mathcal{P}^{\rightarrow\xi}(X^\xi \in [x_{min}, x_{max}]) = 1$, then we must have $\mathcal{P}^{\rightarrow\xi}(X^\xi \in [(1 - b)^\xi, (1 + b)^\xi]) = 1$; i.e., the specification of a being anything other than 1 is inconsistent in this case and is not allowed. We note that by our construction of each PDF Type, $f_{X^\xi}^\alpha$, $\alpha = A, B, C, D$, we have, as required,

$$\int_{-\infty}^{\infty} f_{X^\xi}^\alpha(x) dx = \int_{x_{min}}^{x_{max}} f_{X^\xi}^\alpha(x) dx = 1; \quad (10)$$

and if $S \subseteq \Omega^{\rightarrow}$ is any (measurable) subset then, whenever $f_{X^\xi} = f_{X^\xi}^\alpha$, we have

$$\mathcal{P}^{\rightarrow\xi}(x \in S) = \int_S f_{X^\xi}^\alpha(x) dx. \quad (11)$$

The four PDF types, for any $\xi \in \mathcal{V}_X$ and all $x \in \mathcal{V}_{X^*}$ are as follows:

Type A: If $(1 - b)^\xi > x_{min}$ and $(1 + b)^\xi < x_{max}$ then $f_{X^\xi} = f_{X^\xi}^A$ where

$$f_{X^\xi}^A(x) = \begin{cases} \{2\phi_X(\xi)(1 - a)/[(1 - b)^\xi - x_{min}]^2\}(x - x_{min}) & \text{if } x_{min} \leq x < (1 - b)^\xi \\ a/2b\xi & \text{if } (1 - b)^\xi \leq x \leq (1 + b)^\xi \\ \{2[1 - \phi_X(\xi)](1 - a)/[x_{max} - (1 + b)^\xi]^2\}(x_{max} - x) & \text{if } (1 + b)^\xi < x \leq x_{max}; \end{cases} \quad (12)$$

Type B: If $(1 - b)^\xi > x_{min}$ and $(1 + b)^\xi \geq x_{max}$ then $f_{X^\xi} = f_{X^\xi}^B$ where

$$f_{X^\xi}^B(x) = \begin{cases} \{2(1 - a)/[(1 - b)^\xi - x_{min}]^2\}(x - x_{min}) & \text{if } x_{min} \leq x < (1 - b)^\xi \\ a/[x_{max} - (1 - b)^\xi] & \text{if } (1 - b)^\xi \leq x \leq x_{max}; \end{cases} \quad (13)$$

Type C: If $(1 - b)^\xi \leq x_{min}$ and $(1 + b)^\xi < x_{max}$ then $f_{X^\xi} = f_{X^\xi}^C$ where

$$f_{X^\xi}^C(x) = \begin{cases} a/[(1 + b)^\xi - x_{min}] & \text{if } x_{min} \leq x \leq (1 + b)^\xi \\ \{2(1 - a)/[x_{max} - (1 + b)^\xi]^2\}(x_{max} - x) & \text{if } (1 + b)^\xi < x \leq x_{max}; \end{cases} \quad (14)$$

Type D: If $(1 - b)^\xi \leq x_{min}$ and $(1 + b)^\xi \geq x_{max}$ then $f_{X^\xi} = f_{X^\xi}^D$ where

$$f_{x^\xi}^D(x) = 1/(x_{\max} - x_{\min}) \quad \text{if } x_{\min} \leq x \leq x_{\max}. \quad (15)$$

We also denote the individual segments of each $f_{x^\xi}^\alpha$, $\alpha = A, B, C, D$, by an additional superscript, i.e., as $f_{x^\xi}^{\alpha, n}$. Using the notation

$$x^\# \equiv x/(1 - b) \quad \text{and} \quad x^b \equiv x/(1 + b) \quad (16)$$

then for any $x \in \mathcal{V}_{x^*}$ we have

$$\begin{aligned} f_{x^\xi}^{A,1}(x) &= \{[2\phi_x(\xi)(1 - a)/(1 - b)^2]/(\xi - x_{\min}^\#)^2\}(x - x_{\min}), \\ f_{x^\xi}^{A,2}(x) &= a/2b\xi, \\ f_{x^\xi}^{A,3}(x) &= \{[2(1 - \phi_x(\xi))(1 - a)/(1 + b)^2]/(\xi - x_{\max}^b)^2\}(x_{\max} - x); \\ f_{x^\xi}^{B,1}(x) &= \{[2(1 - a)/(1 - b)^2]/(\xi - x_{\min}^\#)^2\}(x - x_{\min}), \\ f_{x^\xi}^{B,2}(x) &= -[a/(1 - b)]/(\xi - x_{\max}^\#); \\ f_{x^\xi}^{C,1}(x) &= [a/(1 + b)]/(\xi - x_{\min}^b), \\ f_{x^\xi}^{C,2}(x) &= \{[2(1 - a)/(1 + b)^2]/(\xi - x_{\max}^b)^2\}(x_{\max} - x); \\ f_{x^\xi}^{D,1}(x) &= 1/(x_{\max} - x_{\min}). \end{aligned} \quad (17)$$

Further, using the notation

$$x^- \equiv (1 - b)x \quad \text{and} \quad x^+ \equiv (1 + b)x \quad (18)$$

for any $x \in \mathcal{V}_{x^*}$ (for which, with respect to Eq. (16), we have $x^\# > x^+$ and $x^b > x^-$ since $0 < 1 - b^2 < 1$) then we write Eqs. (12) – (16) more succinctly—for any $\xi \in \mathcal{V}_x$ and all $x \in \mathcal{V}_{x^*}$ —as

Type A: If $\xi^- > x_{\min}$ and $\xi^+ < x_{\max}$ then for $\xi \in \mathcal{V}_x$

$$f_{x^\xi}^A(x) = \begin{cases} f_{x^\xi}^{A,1}(x) & \text{if } x < \xi^- \\ f_{x^\xi}^{A,2}(x) & \text{if } \xi^- \leq x \leq \xi^+ \\ f_{x^\xi}^{A,3}(x) & \text{if } \xi^+ < x; \end{cases} \quad (19)$$

Type B: If $\xi^- > x_{min}$ and $\xi^+ \geq x_{max}$ then for $\xi \in \mathcal{V}_X$

$$f_{X^\xi}^B(x) = \begin{cases} f_{X^\xi}^{B,1}(x) & \text{if } x < \xi^- \\ f_{X^\xi}^{B,2}(x) & \text{if } \xi^- \leq x; \end{cases} \quad (20)$$

Type C: If $\xi^- \leq x_{min}$ and $\xi^+ < x_{max}$ then for $\xi \in \mathcal{V}_X$

$$f_{X^\xi}^C(x) = \begin{cases} f_{X^\xi}^{C,1}(x) & \text{if } x \leq \xi^+ \\ f_{X^\xi}^{C,2}(x) & \text{if } \xi^+ < x; \end{cases} \quad (21)$$

Type D: If $\xi^- \leq x_{min}$ and $\xi^+ \geq x_{max}^b$ then for $\xi \in \mathcal{V}_X$

$$f_{X^\xi}^D(x) = f_{X^\xi}^{D,1}(x) \text{ if } x_{min} \leq \xi \leq x_{max}. \quad (22)$$

For any given $\xi \in \mathcal{V}_X$, one and only one of the four possible types may be taken on by f_{X^ξ} , this type depending solely upon the (fixed) values of x_{min} , x_{max} , and b . We may demonstrate this by collectively using the constraining relations given above in the first line of the specification of each Type to make explicit the association between any $\xi \in \mathcal{V}_X$ and the Type of f_{X^ξ} . To this end, it is useful to distinguish two cases, namely,

$$\text{Case I: } x_{min}^\# \geq x_{max}^b \quad (23a)$$

and

$$\text{Case II: } x_{min}^\# < x_{max}^b. \quad (23b)$$

Now the set of all ξ that satisfy the constraints for the Type A PDF, which set we denote by Ξ_A , is given by

$$\begin{aligned} \Xi_A &= \{\xi \in \mathcal{V}_X \mid \xi > x_{min}^\# \ \& \ \xi < x_{max}^b \ \& \ x_{min} \leq \xi \leq x_{max} \ \& \ \text{Case I}\} \\ &\cup \{\xi \in \mathcal{V}_X \mid \xi > x_{min}^\# \ \& \ \xi < x_{max}^b \ \& \ x_{min} \leq \xi \leq x_{max} \ \& \ \text{Case II}\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{min}^\# < \xi < x_{max}^b \ \& \ \text{Case II}\} \end{aligned} \quad (24)$$

since the first set in the above union is empty. Similarly, for the Type B PDF we have

$$\begin{aligned}
\Xi_B &= \{\xi \in \mathcal{V}_X \mid \xi > x_{min}^\# \ \& \ \xi \geq x_{max}^b \ \& \ x_{min} \leq \xi \leq x_{max} \ \& \ \text{Case I}\} \\
&\cup \{\xi \in \mathcal{V}_X \mid \xi > x_{min}^\# \ \& \ \xi \geq x_{max}^b \ \& \ x_{min} \leq \xi \leq x_{max} \ \& \ \text{Case II}\} \\
&= \{\xi \in \mathcal{V}_X \mid x_{min}^\# < \xi \leq x_{max} \ \& \ \text{Case I}\} \\
&\cup \{\xi \in \mathcal{V}_X \mid x_{max}^b \leq \xi \leq x_{max} \ \& \ \text{Case II}\}
\end{aligned} \tag{25}$$

and for the Type C PDF we have

$$\begin{aligned}
\Xi_C &= \{\xi \in \mathcal{V}_X \mid \xi \leq x_{min}^\# \ \& \ \xi < x_{max}^b \ \& \ x_{min} \leq \xi \leq x_{max} \ \& \ \text{Case I}\} \\
&\cup \{\xi \in \mathcal{V}_X \mid \xi \leq x_{min}^\# \ \& \ \xi < x_{max}^b \ \& \ x_{min} \leq \xi \leq x_{max} \ \& \ \text{Case II}\} \\
&= \{\xi \in \mathcal{V}_X \mid x_{min} \leq \xi < x_{max}^b \ \& \ \text{Case I}\} \\
&\cup \{\xi \in \mathcal{V}_X \mid x_{min} \leq \xi \leq x_{min}^\# \ \& \ \text{Case II}\}
\end{aligned} \tag{26}$$

while for the Type D PDF we have

$$\begin{aligned}
\Xi_D &= \{\xi \in \mathcal{V}_X \mid \xi \leq x_{min}^\# \ \& \ \xi \geq x_{max}^b \ \& \ x_{min} \leq \xi \leq x_{max} \ \& \ \text{Case I}\} \\
&\cup \{\xi \in \mathcal{V}_X \mid \xi \leq x_{min}^\# \ \& \ \xi \geq x_{max}^b \ \& \ x_{min} \leq \xi \leq x_{max} \ \& \ \text{Case II}\} \\
&= \{\xi \in \mathcal{V}_X \mid \max[x_{min}, x_{max}^b] \leq \xi \leq \min[x_{max}, x_{min}^\#] \ \& \ \text{Case I}\}.
\end{aligned} \tag{27}$$

Collecting the results of Eqs. (24) – (27) on the basis of case we find:

For **Case I** $[x_{min}^\# \geq x_{max}^b]$: If $\xi \in \mathcal{V}_X$ then

$$f_{X^\xi}(x) = \begin{cases} f_{X^\xi}^C(x) & \text{if } x_{min} \leq \xi < x_{max}^b \\ f_{X^\xi}^D(x) & \text{if } \max[x_{min}, x_{max}^b] \leq \xi \leq \min[x_{min}^\#, x_{max}] \\ f_{X^\xi}^B(x) & \text{if } x_{min}^\# < \xi \leq x_{max}; \end{cases} \tag{28}$$

For **Case II** $[x_{min}^\# < x_{max}^b]$: If $\xi \in \mathcal{V}_X$ then

$$f_{X^\xi}(x) = \begin{cases} f_{X^\xi}^C(x) & \text{if } x_{min} \leq \xi \leq x_{min}^\# \\ f_{X^\xi}^A(x) & \text{if } x_{min}^\# < \xi < x_{max}^b \\ f_{X^\xi}^B(x) & \text{if } x_{max}^b \leq \xi \leq x_{max}. \end{cases} \tag{29}$$

It is also useful to make explicit the four *subcases* of Case I that are engendered by the four possible conditions allowed by Eq. (28-second line), these subcases corresponding to the two possible choices of $\max[x_{min}, x_{max}^b]$ combined with the two possible choices of $\min[x_{max}, x_{min}^\#]$. And while it is possible to do so quickly in a straightforward manner, we will nevertheless now briefly digress to introduce a slightly more complicated method for

doing so, the merit in this method being that it is easily generalizable to accommodating the several additional instances in this paper in which expansion to subcases will be required.

To this end, for $j = 1, \dots, J_{max}$, denote any position-ordered pair of real-valued parameters, say $\pi_1^{(j)}$ and $\pi_2^{(j)}$, by $\pi^{(j)} \equiv \langle \pi_1^{(j)}, \pi_2^{(j)} \rangle$ and let

$$\theta(\pi^{(j)}) = \begin{cases} - & \text{if } \pi_1^{(j)} < \pi_2^{(j)} \\ + & \text{if } \pi_1^{(j)} \geq \pi_2^{(j)}. \end{cases} \quad (30)$$

Then there are no possibilities for the value-ordering of $\pi_1^{(j)}$ and $\pi_2^{(j)}$ that are not represented by $\theta(\pi^{(j)})$ (i.e., the classification *via* θ of the value-ordering of the pair $\langle \pi_1^{(j)}, \pi_2^{(j)} \rangle$ is exhaustive). Further, $\theta(\pi^{(j)}) = "+"$ iff both $\min[\pi_1^{(j)}, \pi_2^{(j)}] = \pi_2^{(j)}$ & $\max[\pi_1^{(j)}, \pi_2^{(j)}] = \pi_1^{(j)}$, while if $\theta(\pi^{(j)}) = "-"$ then both $\min[\pi_1^{(j)}, \pi_2^{(j)}] = \pi_1^{(j)}$ & $\max[\pi_1^{(j)}, \pi_2^{(j)}] = \pi_2^{(j)}$ (not iff). In other words, we may unambiguously infer $\min[\pi_1^{(j)}, \pi_2^{(j)}]$ and $\max[\pi_1^{(j)}, \pi_2^{(j)}]$ in all possible occurrences of $\pi^{(j)} = \langle \pi_1^{(j)}, \pi_2^{(j)} \rangle$ from the value of $\theta(\pi^{(j)})$. Thus, from any of the $2^{J_{max}}$ sequences $\langle \theta(\pi^{(j)}) \rangle_{j=1}^{J_{max}} \in \{-, +\}^{J_{max}}$ associated with the sequence $\langle \pi^{(j)} \rangle_{j=1}^{J_{max}}$ of J_{max} such position-ordered pairs of real-valued parameters, we may unambiguously infer the sequence $\langle \text{extremum}[\pi_1^{(j)}, \pi_2^{(j)}] \rangle_{j=1}^{J_{max}}$ where "extremum" may be either "min" or "max".

Returning now from our digression to the four-subcase endeavor motivated by Eq. (28), and using the above, we take $\pi^{(1)} = \langle x_{min}, x_{max}^b \rangle$ and $\pi^{(2)} = \langle x_{min}^\#, x_{max} \rangle$ (hence $J_{max} = 2$) and consider the four (exhaustive and mutually exclusive) possibilities for $\langle \theta(\pi^{(1)}), \theta(\pi^{(2)}) \rangle$, namely,

$$\langle \theta(\pi^{(1)}), \theta(\pi^{(2)}) \rangle = \begin{array}{llll} \text{(i) } \langle -, - \rangle & \text{(ii) } \langle -, + \rangle & \text{(iii) } \langle +, - \rangle & \text{(iv) } \langle +, + \rangle. \end{array} \quad (31)$$

These possibilities are tantamount to

$$\begin{aligned}
\text{(i)} \quad & x_{\min} < x_{\max}^b \quad \& \quad x_{\min}^{\#} < x_{\max} \quad \text{and} \quad \max[x_{\min}, x_{\max}^b] = x_{\max}^b \quad \& \quad \min[x_{\min}^{\#}, x_{\max}] = x_{\min}^{\#} \\
\text{(ii)} \quad & x_{\min} < x_{\max}^b \quad \& \quad x_{\min}^{\#} \geq x_{\max} \quad \text{and} \quad \max[x_{\min}, x_{\max}^b] = x_{\max}^b \quad \& \quad \min[x_{\min}^{\#}, x_{\max}] = x_{\max} \\
\text{(iii)} \quad & x_{\min} \geq x_{\max}^b \quad \& \quad x_{\min}^{\#} < x_{\max} \quad \text{and} \quad \max[x_{\min}, x_{\max}^b] = x_{\min} \quad \& \quad \min[x_{\min}^{\#}, x_{\max}] = x_{\min}^{\#} \\
\text{(iv)} \quad & x_{\min} \geq x_{\max}^b \quad \& \quad x_{\min}^{\#} \geq x_{\max} \quad \text{and} \quad \max[x_{\min}, x_{\max}^b] = x_{\min} \quad \& \quad \min[x_{\min}^{\#}, x_{\max}] = x_{\max}.
\end{aligned} \tag{32}$$

We note, however, that (iii) cannot occur (in Case I) because the two conditions therein imply that $x_{\max}^b \leq x_{\min} < (1-b)x_{\max}$ so that $x_{\max}/(1-b^2) < x_{\max}$ which is a contradiction of our stipulation that $0 < b < 1$. We label the remaining three possibilities as Cases Ia, Ib, and Ic, respectively. Thus, using Eq. (28) under the conditions (i), (ii), and (iv) of Eq. (32) (and taking into account Eq. (23a)) we find:

Case Ia [$x_{\min} < x_{\max}^b \leq x_{\min}^{\#} < x_{\max}$]: If $\xi \in \mathcal{V}_X$ then

$$f_{X^{\xi}}(x) = \begin{cases} f_{X^{\xi}}^C(x) & \text{if } x_{\min} \leq \xi < x_{\max}^b \\ f_{X^{\xi}}^D(x) & \text{if } x_{\max}^b \leq \xi \leq x_{\min}^{\#} \\ f_{X^{\xi}}^B(x) & \text{if } x_{\min}^{\#} < \xi \leq x_{\max}; \end{cases} \tag{33}$$

Case Ib [$x_{\min} < x_{\max}^b \leq x_{\min}^{\#}$ & $x_{\min}^{\#} \geq x_{\max}$]: If $\xi \in \mathcal{V}_X$ then

$$f_{X^{\xi}}(x) = \begin{cases} f_{X^{\xi}}^C(x) & \text{if } x_{\min} \leq \xi < x_{\max}^b \\ f_{X^{\xi}}^D(x) & \text{if } x_{\max}^b \leq \xi \leq x_{\max} \end{cases} \tag{34}$$

since $\{\xi \mid x_{\min}^{\#} < \xi \leq x_{\max}\} = \emptyset$; and

Case Ic [$x_{\min} \geq x_{\max}^b$ & $x_{\max}^b \leq x_{\min}^{\#}$ & $x_{\min}^{\#} \geq x_{\max}$]: If $\xi \in \mathcal{V}_X$ then

$$f_{X^{\xi}}(x) = f_{X^{\xi}}^D(x) \quad \text{if } x_{\min} \leq \xi \leq x_{\max} \tag{35}$$

since $\{\xi \mid x_{\min} \leq \xi < x_{\max}^b\} = \emptyset = \{\xi \mid x_{\min}^{\#} < \xi \leq x_{\max}\}$.

We note also that for Case II there are no subcases since all three intervals for ξ in Eq. (29) are manifestly non-void.

We observe that there are two separate but related aspects to the influence upon Eq. (28) of the possibilities delineated in Eq. (31) for $\langle \theta(\pi^{(1)}), \theta(\pi^{(2)}) \rangle$. The first aspect provides

for the removal of the ambiguity of the max and min terms from Eq. (28) but, in doing so, thereby necessitates three separate versions of Case I, which versions we have here labeled as Cases Ia, Ib, and Ic. In this regard, these cases differ for two reasons: (1) because the max term specifying the left endpoint of the interval of validity for $f_{X^\xi}^D$ differs from case to case, as does the min term specifying the right endpoint of that interval; and (2) because the interval of validity for $f_{X^\xi}^B$, or the intervals of validity for $f_{X^\xi}^B$ and $f_{X^\xi}^C$, may be void. The second aspect provides for the restrictions, for each of the three cases, of the relationships between the parameters x_{min} , x_{max} , x_{max}^b , and $x_{min}^\#$ that must be satisfied in order for each particular case to be valid. We will observe this pattern of required additional subcases, with the accompanying above two-faceted influence of the analogues of Eq. (31), several times in the rest of this paper.

In summary, for any given $\xi \in \mathcal{V}_X$, one and only one of the four possible PDF Types $f_{X^\xi}^\alpha$ may be taken on by f_{X^ξ} as determined solely by the (fixed) values of x_{min} , x_{max} , and b . In particular, there are two main cases that govern this correspondence between ξ and $f_{X^\xi}^\alpha$, denoted here as Case I and Case II and, further, Case I may be partitioned into three subcases. We then have the following case structure:

- (ia) **Case Ia:** $x_{min} < x_{max}^b \leq x_{min}^\# < x_{max}$ with f_{X^ξ} given as per Eq. (33);
- (ib) **Case Ib:** $x_{min} < x_{max}^b \leq x_{min}^\#$ & $x_{min}^\# \geq x_{max}$ with f_{X^ξ} given as per Eqs. (34);
- (ic) **Case Ic:** $x_{min} \geq x_{max}^b$ & $x_{max}^b \leq x_{min}^\#$ & $x_{min}^\# \geq x_{max}$ with f_{X^ξ} given as per Eq. (35);
- (ii) **Case II:** $x_{min} < x_{min}^\# < x_{max}^b < x_{max}$ with f_{X^ξ} given as per Eq. (29).

The four cases in Eq. (36) may be interpreted as describing the four possibilities for the variation of f_{X^ξ} as ξ increases continuously from ξ_{min} to ξ_{max} , i.e., as describing the *family* $\{f_{X^\xi}\}_{\xi \in \mathcal{V}_X}$ of PDF's. (Note that in case $\mathcal{V}_{X^*} \setminus \mathcal{V}_X \neq \emptyset$ then all of the above results for ξ still hold for any $\xi' \in \mathcal{V}_{X^*} \setminus \mathcal{V}_X$ despite the fact that for such ξ' , $\xi' \notin \mathcal{V}_X$.) Thus there are four

varieties of such PDF sets $\{f_{X^\xi}\}_{\xi \in \mathcal{V}_X}$. For Case Ia, there are three PDF Types that appear in succession as ξ increases continuously from ξ_{min} to ξ_{max} , namely $f_{X^\xi}^C$ then $f_{X^\xi}^D$ then $f_{X^\xi}^B$. The transition from $f_{X^\xi}^C$ to $f_{X^\xi}^D$ occurs at the critical ξ -value $\xi = x_{max}^b$ and the transition from $f_{X^\xi}^D$ to $f_{X^\xi}^B$ occurs at the critical ξ -value $\xi = x_{min}^\#$. For Case Ib, there are two PDF Types that appear in succession as ξ increases, namely $f_{X^\xi}^C$ then $f_{X^\xi}^D$. The transition from $f_{X^\xi}^C$ to $f_{X^\xi}^D$ occurs at the critical ξ -value $\xi = x_{max}^b$. For Case Ic, there is only one PDF Type that appears “in succession” as ξ increases, namely $f_{X^\xi}^D$. Finally, for Case II, there are three PDF Types that appear in succession as ξ increases continuously from ξ_{min} to ξ_{max} , namely $f_{X^\xi}^C$ then $f_{X^\xi}^A$ then $f_{X^\xi}^B$. The transition from $f_{X^\xi}^C$ to $f_{X^\xi}^A$ occurs at the critical ξ -value $\xi = x_{min}^\#$ and the transition from $f_{X^\xi}^A$ to $f_{X^\xi}^B$ occurs at the critical ξ -value $\xi = x_{max}^b$.

We reiterate that the uncertainty in X , when the value of X is ξ , is specified by the PDF f_{X^ξ} , which in turn is in fact precisely one of the four PDF's, $f_{X^\xi}^\alpha$, $\alpha = A, B, C, D$, with that one correct PDF completely determined by that value of ξ as prescribed by Eq. (36) (via Eqs. (29) and (33) – (35)) and with the x -dependence of that PDF $f_{X^\xi}^\alpha$ given by the appropriate one of Eqs. (19) – (22). That is, the PDF f_{X^ξ} , for each *fixed* $\xi \in \mathcal{V}_X$ is a function of variable x having domain $\mathcal{V}_{X^*}$. More generally, we regard $f_{X^\xi}(x)$ as a function of two variables, $\langle \xi, x \rangle \in \mathcal{V}_X \times \mathcal{V}_{X^*}$; indeed, in Section II.4 we will need to elucidate the behavior of $f_{X^\xi}(x)$ as a function of $\xi \in \mathcal{V}_X$ for each fixed $x \in \mathcal{V}_{X^*}$.

II.3 Code- n Input Variable Uncertainty

Until this point in our development we have specified a *single value* (fixed but arbitrary within its allowed range) of the input variable $\mathcal{W}_n$ to Code- n , namely $\omega_{n,s} \in \mathcal{V}_{\mathcal{W}_n}$. We have done so to emphasize that even a single value of $\mathcal{W}_n$ leads to all possible values of real-world quantity X_n^* represented in Code- n by output variable X_n , with X_n^* also represented

in $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow}, \mathcal{P}_n^{\rightarrow \xi n; \omega_n})$ by random variable $X_n^{\xi n; \omega_n}$. In particular, the input value $\omega_{n,\$}$ results in the output value $\xi_{1; \omega_{n,\$}} = g_n(\omega_{n,\$}) \in \mathcal{V}_{X_n}$ of X_n , with X_n basically smeared at $\xi_{1; \omega_{n,\$}}$ into $\mathcal{V}_{X_n^*}$ as governed by $f_{X_n^{\xi n; \omega_{n,\$}}}(x_n)$ as based upon a_n and b_n .

We now want to consider the situation in which the input value $\omega_{n,\$}$ of $\mathcal{W}_n$ has some uncertainty associated with it because of the uncertainty in its actual real-world value so that, in general, $\omega_{n,\$}$ must be smeared into some (non-singleton) subset of the set $\mathcal{V}_{W_n^*}$ of all possible values of real-world quantity W_n^* , the latter being represented in Code- n by input variable $\mathcal{W}_n$. Note that the input value of $\mathcal{W}_n$ is still the single value $\omega_{n,\$}$ but the smearing of $\mathcal{W}_n$ at $\omega_{n,\$}$ dictates consideration of values $w_n \neq \omega_{n,\$}$, $w_n \in \mathcal{V}_{W_n^*} = \mathcal{V}_{\mathcal{W}_n}$ (which consideration is done externally to Code- n via our UQ methodology).

To formalize this process we now introduce—in analogy with the probability space $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow}, \mathcal{P}_n^{\rightarrow \xi n})$ introduced to accommodate uncertainty in the value of X_n —a probability space to accommodate uncertainty in the value of $\mathcal{W}_n$. This space has outcome set $\Omega_n^{\leftarrow} = \mathcal{V}_{W_n^*}$ (and σ -algebra, $\mathcal{A}_n^{\leftarrow}$, of events in $\Omega_n^{\leftarrow}$ taken to be the Lebesgue-measurable subsets of $\mathcal{V}_{W_n^*}$). We represent W_n^* in our probability framework by a random variable, W_n , given by $W_n(w_n) = w_n$ for $w_n \in \Omega_n^{\leftarrow}$. The specification of the probability measure in $(\Omega_n^{\leftarrow}, \mathcal{A}_n^{\leftarrow})$, say $\mathcal{P}_n^{\leftarrow}$, may be accomplished by specifying a PDF, f_{W_n} , for W_n that will be ascribed to $\mathcal{W}_n$ as well in order to define the uncertainty of $\mathcal{W}_n$. Since this procedure requires a given (but arbitrary within $\mathcal{V}_{\mathcal{W}_n}$) value, ω_n , for input variable $\mathcal{W}_n$ then the PDF for W_n must depend upon ω_n ; we acknowledge this ω_n -dependence by writing $W_n^{\omega_n}$ in lieu of merely W_n and therefore writing as well $f_{W_n^{\omega_n}}$ in lieu of f_{W_n} , and $\mathcal{P}_n^{\leftarrow \omega_n}$ in lieu of $\mathcal{P}_n^{\leftarrow}$. Note that we are in fact considering a *family* of probability spaces, $\{(\Omega_n^{\leftarrow}, \mathcal{A}_n^{\leftarrow}, \mathcal{P}_n^{\leftarrow \omega_n})\}_{\omega_n \in \mathcal{V}_{\mathcal{W}_n}}$ where $W_n^{\omega_n}$ is a random variable in $(\Omega_n^{\leftarrow}, \mathcal{A}_n^{\leftarrow}, \mathcal{P}_n^{\leftarrow \omega_n})$ with $W_n^{\omega_n} = w_n$ for $w_n \in \Omega_n^{\leftarrow}$; further, each $W_n^{\omega_n}$ is a

continuous random variable since each possesses a PDF $f_{W_n^{\omega_n}}$. In what follows we fix ω_n (to be $\omega_{n,s}$). Also note that this PDF $f_{W_n^{\omega_n}}(w_n)$ for $w_n \in \mathcal{V}_{W_n^*}$ expresses the uncertainty distribution for the value of input variable $\mathcal{W}_n$ (source-(1)) rather than the uncertainty distribution for the modeling goodness/numerical uncertainty (source-(2+3)) of Code- n , that latter distribution being given for Code- n as previously, based upon a_n and b_n .

Next we introduce a probability space that allows for the joint (random) observation of $w_n \in \mathcal{V}_{W_n^*}$ and $x_n \in \mathcal{V}_{X_n^*}$. The outcome set for this space is $\Omega_{nn} \equiv \Omega_n^{\leftarrow} \times \Omega_n^{\rightarrow}$ with the members of Ω_{nn} being the ordered pairs $\langle w_n, x_n \rangle$ of possible real-world values of real-world quantities W_n^* and X_n^* respectively. The extension of $W_n^{\omega_n}$ on $(\Omega_n^{\leftarrow}, \mathcal{A}_n^{\leftarrow})$ to $(\Omega_{nn}, \mathcal{A}_{nn})$, namely $\hat{W}_n^{\omega_n}$, is straightforward and is given by $\hat{W}_n^{\omega_n}(\langle w_n, x_n \rangle) = w_n$. On the other hand, the random variable $X_n^{\xi_n}$ on $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow})$, involving merely a single $\xi_n = \xi_{n;\omega_n} = g_n(\omega_n)$, is no longer sufficient: For in order to compute the probability that the real-world quantity X_n^* has value $x_n \in \mathcal{V}_{X_n^*}$ we must consider, rather than merely the probability that random variable $X_n^{g_n(\omega_n)}$ has value x_n (which random variable is basically smeared at $g_n(\omega_n)$ via a_n and b_n), instead a (weighted) superposition, over all members of the *collection* of random variables $\mathfrak{X}_n \equiv \{X_n^{g_n(w_n)} \mid w_n \in \mathcal{V}_{W_n^*}\}$, of the probability, for each $w_n \in \mathcal{V}_{W_n^*}$, that $X_n^{g_n(w_n)}$ has value x_n ; and for this we will need, rather, the random variable $\hat{X}_n$ in $(\Omega_{nn}, \mathcal{A}_{nn})$ given by $\hat{X}_n(\langle w_n, x_n \rangle) = x_n$. (Note that since we have taken $\mathcal{V}_{W_n^*} = \mathcal{V}_{W_n}$ [Eq.(3)] then $\mathcal{V}_{W_n^*} = \text{dom } g_1$ and so $g_n(w_n)$ is well-defined for all $w_n \in \mathcal{V}_{W_n^*}$.) The specification of the probability assignment in Ω_{nn} (or, more precisely, in $(\Omega_{nn}, \mathcal{A}_{nn})$), say $\mathcal{P}_{nn}^{\omega_n, g_n}$, may be accomplished by assigning a joint PDF $f_{\langle \hat{W}_n^{\omega_n}, \hat{X}_n \rangle}$ to the 2-vector random variable $\langle \hat{W}_n^{\omega_n}, \hat{X}_n \rangle$. In general, in terms of conditional and marginal densities, $f_{\hat{X}_n | \hat{W}_n^{\omega_n}}(x_n | w_n)$ and $f_{\hat{W}_n^{\omega_n}}(w_n)$ respectively, we have

$$f_{\langle \hat{W}_n^{\omega_n}, \hat{X}_n \rangle}(w_n, x_n) = f_{\hat{X}_n | \hat{W}_n^{\omega_n}}(x_n | w_n) f_{\hat{W}_n^{\omega_n}}(w_n). \quad (37)$$

But since for any (measurable) $S \subseteq \mathcal{V}_{W_n^*}$ we want

$\mathcal{P}_{nn}^{\omega_n, g_n}(\hat{W}_n^{\omega_n} \in S) = \mathcal{P}_n^{\leftarrow \omega_n}(W_n^{\omega_n} \in S)$, i.e., $\int_S \hat{f}_{\hat{W}_n^{\omega_n}}(w_n) dw_n = \int_S f_{W_n^{\omega_n}}(w_n) dw_n$, then we take

$$\hat{f}_{\hat{W}_n^{\omega_n}}(w_n) = f_{W_n^{\omega_n}}(w_n) \text{ for all } w_n \in \mathcal{V}_{W_n^*}; \quad (38)$$

and since, further, for any (measurable) $T \subseteq \mathcal{V}_{X_n^*}$ and $w_n \in \mathcal{V}_{W_n^*}$ we want to have

$\mathcal{P}_{nn}^{\omega_n, g_n}(\hat{X}_n \in T | \hat{W}_n^{\omega_n} = w_n) = \mathcal{P}_n^{\rightarrow g_n(w_n)}(X_n^{g_n(w_n)} \in T)$, i.e., $\int_T \hat{f}_{\hat{X}_n | \hat{W}_n^{\omega_n}}(x_n | w_n) dx_n =$

$\int_T f_{X_n^{g_n(w_n)}}(x_n) dx_n$, then we take

$$\hat{f}_{\hat{X}_n | \hat{W}_n^{\omega_n}}(x_n | w_n) = f_{X_n^{g_n(w_n)}}(x_n) \text{ for all } x_n \in \mathcal{V}_{X_n^*} \text{ and } w_n \in \mathcal{V}_{W_n^*}. \quad (39)$$

Thus Eq. (37) becomes

$$\hat{f}_{\langle \hat{W}_n^{\omega_n}, \hat{X}_n \rangle}(w_n, x_n) = f_{W_n^{\omega_n}}(w_n) f_{X_n^{g_n(w_n)}}(x_n) \text{ for all } x_n \in \mathcal{V}_{X_n^*} \text{ and } w_n \in \mathcal{V}_{W_n^*}. \quad (40)$$

We then have the marginal PDF for $\hat{X}_n$ given by, for all $x_n \in \mathcal{V}_{X_n^*}$ and $w_n \in \mathcal{V}_{W_n^*}$,

$$\begin{aligned} f_{\hat{X}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n) &\equiv \hat{f}_{\hat{X}_n}(x_n) \\ &= \int_{w_{n, \min}}^{w_{n, \max}} \hat{f}_{\langle \hat{W}_n^{\omega_n}, \hat{X}_n \rangle}(w_n, x_n) dw_n \\ &= \int_{w_{n, \min}}^{w_{n, \max}} f_{W_n^{\omega_n}}(w_n) f_{X_n^{g_n(w_n)}}(x_n) dw_n. \end{aligned} \quad (41)$$

Note that, by Eqs. (35) and (10) (with $\xi = g_n(w_n)$ in the latter),

$$\begin{aligned} \int_{x_{n, \min}}^{x_{n, \max}} \hat{f}_{\hat{X}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n) dx_n \\ &= \int_{w_{n, \min}}^{w_{n, \max}} f_{W_n^{\omega_n}}(w_n) \int_{x_{n, \min}}^{x_{n, \max}} f_{X_n^{g_n(w_n)}}(x_n) dx_n dw_n \\ &= \int_{w_{1, \min}}^{w_{1, \max}} f_{W_n^{\omega_n}}(w_n) \cdot 1 dw_n = 1 \end{aligned} \quad (42)$$

as required.

When $f_{W_n^{\omega_n}}(w_n) = \delta(w_n - \omega_{n, \$})$ for some $\omega_{n, \$} \in \mathcal{V}_{W_n^*}$ (i.e., is sharp) then, via Eq. (41),

$f_{\hat{X}_n}(\omega_{n,\$}, f_{W_n^{\omega_{n,\$}}}, g_n; x_n) = f_{\hat{X}_n}(\omega_n, \delta(w_n - \omega_{n,\$}), g_n; x_n) = f_{X_n^{g_n(\omega_{n,\$})}}(x_n) = f_{X_n^{\xi_n; \omega_{n,\$}}}^\alpha(x_n)$ where $\xi_n; \omega_{n,\$} = g_n(\omega_{n,\$})$ and $\alpha \in \{A, B, C, D\}$. Thus, the PDF $f_{\hat{X}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n)$ of Eq. (41) specifies the *compound* smearing of Code- n output variable X_n when PDF $f_{W_n^{\omega_n}}$ is non-sharp and specifies as well the *basic* smearing of X_n when $f_{W_n^{\omega_n}}(w_n) = \delta(w_n - \omega_{n,\$})$. Finally, using $f_{\hat{X}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n)$ we have for any (measurable) set $A \in \vec{\mathcal{A}}_n$ that

$$\mathcal{P}_{nn}^{\omega_n, g_n}(\hat{X}_n \in A) = \int_A f_{\hat{X}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n) dx_n. \quad (43)$$

This then quantifies the uncertainty in X_n , hence (by definition) in $\mathcal{X}_n$, when Code- n input variable $\mathcal{W}_n$ has value ω_n and Code- n output variable X_n has value $\xi_n; \omega_n = g_n(\omega_n)$.

II.4 Evaluation of the Integral $\int_{w_n, \text{min}}^{w_n, \text{max}} f_{W_n^{\omega_n}}(w_n) f_{X_n^{g_n(w_n)}}(x_n) dw_n$ of Eq. (41)

We next want to evaluate the second integral in Eq. (41) for $f_{\hat{X}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n)$ for each $x_n \in \mathcal{V}_{X_n^*}$ (if $x_n \in \mathbb{R} \setminus \mathcal{V}_{X_n^*}$ then $f_{\hat{X}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n) = 0$ since $f_{X_n^{g_n(w_n)}}(x_n) = 0$ for such x_n —see the first paragraph in Section II.2). Once again, for simplicity of notation we will suppress all subscripts and superscripts “ n ”.

Firstly, an explicit expression for the factor (PDF) $f_{W_n^{\omega_n}}(w)$ in the integrand is provided as an item of Code- n input data—either externally (by the user) in case $n = 1$, or by the smeared output PDF of Code- $(n - 1)$ when $n > 1$. Secondly, concerning the other factor in the integrand, (PDF) $f_{X_n^{g(w)}}(x)$, there are four possibilities for it, namely, $f_{X_n^{g(w)}}^\alpha$, $\alpha = A, B, C, D$, with $f_{X_n^{g(w)}}$ taking on one or another of those possible forms depending upon the values of $x_{\min}, x_{\max}, b$, and $g(w) \in \mathcal{V}_X$ (and as well upon a , as in Section II.2), with $g(w)$ varying continuously and inclusively (but not necessarily monotonically) between $\xi_{\min}^g$ and $\xi_{\max}^g$ (i.e., in $\mathcal{V}_X$) as the value of the integration variable w in Eq. (41) increases

monotonically and continuously from w_{min} to w_{max} . In particular, from Eqs. (29) and (33) – (35) we have

Case Ia [$x_{min} < x_{max}^b \leq x_{min}^\# < x_{max}$]:

$$f_{xg(w)}(x) = \begin{cases} f_{xg(w)}^C(x) & \text{if } x_{min} \leq g(w) < x_{max}^b \\ f_{xg(w)}^D(x) & \text{if } x_{max}^b \leq g(w) \leq x_{min}^\# \\ f_{xg(w)}^B(x) & \text{if } x_{min}^\# < g(w) \leq x_{max} \end{cases} \quad (44)$$

Case Ib [$x_{min} < x_{max}^b \leq x_{min}^\#$ & $x_{min}^\# \geq x_{max}$]:

$$f_{xg(w)}(x) = \begin{cases} f_{xg(w)}^C(x) & \text{if } x_{min} \leq g(w) < x_{max}^b \\ f_{xg(w)}^D(x) & \text{if } x_{max}^b \leq g(w) \leq x_{max}, \end{cases} \quad (45)$$

Case Ic [$x_{min} \geq x_{max}^b$ & $x_{max}^b \leq x_{min}^\#$ & $x_{min}^\# \geq x_{max}$]:

$$f_{xg(w)}(x) = f_{xg(w)}^D(x) \quad \text{if } x_{min} \leq g(w) \leq x_{max}, \quad (46)$$

Case II [$x_{min}^\# < x_{max}^b$]:

$$f_{xg(w)}(x) = \begin{cases} f_{xg(w)}^C(x) & \text{if } x_{min} \leq g(w) \leq x_{min}^\# \\ f_{xg(w)}^A(x) & \text{if } x_{min}^\# < g(w) < x_{max}^b \\ f_{xg(w)}^B(x) & \text{if } x_{max}^b \leq g(w) \leq x_{max}, \end{cases} \quad (47)$$

where, from Eqs. (19) – (22), we have for any $g(w) \in \mathcal{V}_x$ and all $x \in \mathcal{V}_{x^*}$ (with notation as in Eq. (18))

Type A: If $g(w)^- > x_{min}$ and $g(w)^+ < x_{max}$ then

$$f_{xg(w)}^A(x) = \begin{cases} f_{xg(w)}^{A,1}(x) & \text{if } x < g(w)^- \\ f_{xg(w)}^{A,2}(x) & \text{if } g(w)^- \leq x \leq g(w)^+ \\ f_{xg(w)}^{A,3}(x) & \text{if } g(w)^+ < x, \end{cases} \quad (48)$$

Type B: If $g(w)^- > x_{min}$ and $g(w)^+ \geq x_{max}$ then

$$f_{x^{g(w)}}^B(x) = \begin{cases} f_{x^{g(w)}}^{B,1}(x) & \text{if } x < g(w)^- \\ f_{x^{g(w)}}^{B,2}(x) & \text{if } g(w)^- \leq x, \end{cases} \quad (49)$$

Type C: If $g(w)^- \leq x_{min}$ and $g(w)^+ < x_{max}$ then

$$f_{x^{g(w)}}^C(x) = \begin{cases} f_{x^{g(w)}}^{C,1}(x) & \text{if } x \leq g(w)^+ \\ f_{x^{g(w)}}^{C,2}(x) & \text{if } g(w)^+ < x, \end{cases} \quad (50)$$

Type D: If $g(w)^- \leq x_{min}$ and $\xi \geq g(w)^+ \geq x_{max}$ then

$$f_{x^{g(w)}}^D(x) = f_{x^{g(w)}}^{D,1}(x) \quad \text{if } x_{min} \leq g(w) \leq x_{max}, \quad (51)$$

with, from Eq. (17),

$$\begin{aligned} f_{x^{g(w)}}^{A,1}(x) &= \{[2\phi_X(g(w))(1-a)/(1-b)^2]/(g(w) - x_{min}^\#)^2\}(x - x_{min}), \\ f_{x^{g(w)}}^{A,2}(x) &= a/2bg(w), \\ f_{x^{g(w)}}^{A,3}(x) &= \{[2(1 - \phi_X(g(w)))(1-a)/(1+b)^2]/(g(w) - x_{max})^2\}(x_{max} - x); \\ f_{x^{g(w)}}^{B,1}(x) &= \{[2(1-a)/(1-b)^2]/(g(w) - x_{min}^\#)^2\}(x - x_{min}), \\ f_{x^{g(w)}}^{B,2}(x) &= -[a/(1-b)]/(g(w) - x_{max}^\#); \\ f_{x^{g(w)}}^{C,1}(x) &= [a/(1+b)]/(g(w) - x_{min}), \\ f_{x^{g(w)}}^{C,2}(x) &= \{[2(1-a)/(1+b)^2]/(g(w) - x_{max})^2\}(x_{max} - x); \\ f_{x^{g(w)}}^{D,1}(x) &= 1/(x_{max} - x_{min}). \end{aligned} \quad (52)$$

In analogy with the final paragraph Section II.2, we point out that the PDF $f_{x^{g(w)}}(x)$, as specified for each $g(w) \in \mathcal{V}_X$ —i.e., for each $w \in \mathcal{V}_{W^*}$ —via Eqs. (44) – (51), is a function of *variable* x having domain $\mathcal{V}_{X^*}$. These expressions for $f_{x^{g(w)}}(x)$ are, however, of limited use to us—as they stand—for evaluation of the integral of Eq. (41) because in order to evaluate that integral we must specify $f_{x^{g(w)}}(x)$ rather, for each *fixed* $x \in \mathcal{V}_{X^*}$, as a function of *variable* $w \in \mathcal{V}_{W^*}$; and we do that specification next.

II.4.1 PDF's $f_{x_n^{\xi_n}}(x_n)$ for Fixed $x_n \in \mathcal{V}_{x_n^*}$ as a Function of Variable $\xi_n \in \mathcal{V}_{x_n}$

We begin by *fixing* any $x \in \mathcal{V}_{x^*}$ and rewriting Eqs. (48) – (51) in a form better suited for investigating the behavior of the PDF's therein as w varies in $\mathcal{V}_{w^*}$. In connection with these rewritten equations note that $g(w) = x$ iff $w \in g^{-1}[\{x\}] = \{w' \in \mathcal{V}_{w^*} \mid g(w') = x\}$ and, in particular, if $x \in \mathcal{V}_{x^*} \setminus \mathcal{V}_x$ then $g^{-1}[\{x\}] = \emptyset$. We have:

Type A:

If $g(w) > x_{\min}^{\#}$ (iff $w \in g^{-1}[(x_{\min}^{\#}, x_{\max}]]$) and $g(w) < x_{\max}^b$ (iff $w \in g^{-1}[[x_{\min}, x_{\max}^b))$) then

$$f_{x^{g(w)}}^A(x) = \begin{cases} f_{x^{g(w)}}^{A,3}(x) & \text{if } g(w) < x^b \quad (\text{iff } w \in g^{-1}[[x_{\min}, x^b)) \\ f_{x^{g(w)}}^{A,2}(x) & \text{if } x^b \leq g(w) \leq x^{\#} \quad (\text{iff } w \in g^{-1}[[x^b, x^{\#}]] \\ f_{x^{g(w)}}^{A,1}(x) & \text{if } x^{\#} < g(w) \quad (\text{iff } w \in g^{-1}[(x^{\#}, x_{\max}]]), \end{cases} \quad (53)$$

Type B:

If $g(w) > x_{\min}^{\#}$ (iff $w \in g^{-1}[(x_{\min}^{\#}, x_{\max}]]$) and $g(w) \geq x_{\max}^b$ (iff $w \in g^{-1}[[x_{\max}^b, x_{\max}]]$) then

$$f_{x^{g(w)}}^B(x) = \begin{cases} f_{x^{g(w)}}^{B,2}(x) & \text{if } g(w) \leq x^{\#} \quad (\text{iff } w \in g^{-1}[[x_{\min}, x^{\#}]] \\ f_{x^{g(w)}}^{B,1}(x) & \text{if } x^{\#} < g(w) \quad (\text{iff } w \in g^{-1}[(x^{\#}, x_{\max}]]), \end{cases} \quad (54)$$

Type C:

If $g(w) \leq x_{\min}^{\#}$ (iff $w \in g^{-1}[[x_{\min}, x_{\min}^{\#}]]$) and $g(w) < x_{\max}^b$ (iff $w \in g^{-1}[[x_{\min}, x_{\max}^b))$) then

$$f_{x^{g(w)}}^C(x) = \begin{cases} f_{x^{g(w)}}^{C,2}(x) & \text{if } g(w) < x^b \quad (\text{iff } w \in g^{-1}[[x_{\min}, x^b)) \\ f_{x^{g(w)}}^{C,1}(x) & \text{if } x^b \leq g(w) \quad (\text{iff } w \in g^{-1}[[x^b, x_{\max}]]), \end{cases} \quad (55)$$

Type D:

If $g(w) \leq x_{\min}^{\#}$ (iff $w \in g^{-1}[[x_{\min}, x_{\min}^{\#}]]$) and $g(w) \geq x_{\max}^b$ (iff $w \in g^{-1}[[x_{\max}^b, x_{\max}]]$) then

$$f_{x^{g(w)}}^D(x) = f_{x^{g(w)}}^{D,1}(x) \quad \text{if } x_{\min} \leq g(w) \leq x_{\max}. \quad (56)$$

To proceed, we now revert—for simplicity of notation—to using $f_{x^{\xi}}(x)$ in lieu of $f_{x^{g(w)}}(x)$ (since each $g(w)$ is in fact equal to some ξ , namely $\xi = \xi_w$); thus we consider variation of $f_{x^{\xi}}(x)$ with $\xi \in \mathcal{V}_x$ for each *fixed* $x \in \mathcal{V}_{x^*}$. Later, after we have completed our analysis in

terms of $f_{x^\xi}(x)$, we will then restate our important results therefrom in terms of $f_{x^{g(w)}}(x)$. In terms of $f_{x^\xi}(x)$ we have, from Eqs. (53) – (56) (or, more directly, from Eqs. (19) – (22)):

Type A: If $\xi > x_{min}^\#$ and $\xi < x_{max}^b$ then for $\xi \in \mathcal{V}_x$

$$f_{x^\xi}^A(x) = \begin{cases} f_{x^\xi}^{A,3}(x) & \text{if } \xi < x^b \\ f_{x^\xi}^{A,2}(x) & \text{if } x^b \leq \xi \leq x^\# \\ f_{x^\xi}^{A,1}(x) & \text{if } x^\# < \xi; \end{cases} \quad (57)$$

Type B: If $\xi > x_{min}^\#$ and $\xi \geq x_{max}^b$ then for $\xi \in \mathcal{V}_x$

$$f_{x^\xi}^B(x) = \begin{cases} f_{x^\xi}^{B,2}(x) & \text{if } \xi \leq x^\# \\ f_{x^\xi}^{B,1}(x) & \text{if } x^\# < \xi; \end{cases} \quad (58)$$

Type C: If $\xi \leq x_{min}^\#$ and $\xi < x_{max}^b$ then for $\xi \in \mathcal{V}_x$

$$f_{x^\xi}^C(x) = \begin{cases} f_{x^\xi}^{C,2}(x) & \text{if } \xi < x^b \\ f_{x^\xi}^{C,1}(x) & \text{if } x^b \leq \xi; \end{cases} \quad (59)$$

Type D: If $\xi \leq x_{min}^\#$ and $\xi \geq x_{max}^b$ then for $\xi \in \mathcal{V}_x$

$$f_{x^\xi}^D(x) = f_{x^\xi}^{D,1}(x) \text{ if } x_{min} \leq \xi \leq x_{max}. \quad (60)$$

We next proceed, in analogy with the derivation of the sets Ξ_A , Ξ_B , Ξ_C , and Ξ_D of Eqs. (24) – (27), to derive—for each of Cases I and II (including as well each individual subcase thereof) and for all PDF segments $(f_{x^\xi}^{\alpha, n})$ in Eqs. (57) – (60) comprised by each of these subcases—the sets of ξ 's that result from enforcing all constraints on ξ thereby required and under the condition that a fixed $x \in \mathcal{V}_{x^*}$ is specified; this will lead to results analogous to those of Eqs. (29) and (33) – (35) but given instead in terms of the eight PDF segments rather than in terms of the four PDF (main) Types $(f_{x^\xi}^\alpha)$.

We consider first Case I and its subcases. For Case Ia, as specified via Eq. (33) and comprising $f_{X^\xi}^C$ then $f_{X^\xi}^D$ then $f_{X^\xi}^B$, the conditions $x_{\min} < x_{\max}^b \leq x_{\min}^\# < x_{\max}$ of Eq. (36ia) prevail. We then have—by referring to Eqs. (33-first line) and (59)—that the set of all $\xi \in \mathcal{V}_X$ satisfying the constraints for PDF segment $f_{X^\xi}^{C,1}$ in Case Ia, which set we denote by $\Xi_{C,1}^{\text{Ia}}(x)$, is given by

$$\begin{aligned}\Xi_{C,1}^{\text{Ia}}(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi < x_{\max}^b \ \& \ \xi \leq x_{\min}^\# \ \& \ \xi < x_{\max}^b \ \& \ \xi \geq x^b \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x^b] \leq \xi < \min[x_{\max}, x_{\min}^\#, x_{\max}^b]\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x^b] \leq \xi < x_{\max}^b\};\end{aligned}\tag{61}$$

likewise, we have for PDF segment $f_{X^\xi}^{C,2}$ in Case Ia that

$$\begin{aligned}\Xi_{C,2}^{\text{Ia}}(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi < x_{\max}^b \ \& \ \xi \leq x_{\min}^\# \ \& \ \xi < x_{\max}^b \ \& \ \xi < x^b \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \mid x_{\min} \leq \xi < \min[x_{\max}, x_{\min}^\#, x_{\max}^b, x^b]\} \\ &= \{\xi \mid x_{\min} \leq \xi < x^b\}.\end{aligned}\tag{62}$$

Further, from Eqs. (27-second line) and (60) we have for PDF segment $f_{X^\xi}^{D,1}$ in Case Ia that

$$\begin{aligned}\Xi_{D,1}^{\text{Ia}}(x) &= \{\xi \in \mathcal{V}_X \mid x_{\max}^b \leq \xi \leq x_{\min}^\# \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x_{\max}^b] \leq \xi \leq \min[x_{\max}, x_{\min}^\#]\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\max}^b \leq \xi \leq x_{\min}^\#\};\end{aligned}\tag{63}$$

and from Eqs. (33-third line) and (58) we have for PDF segments $f_{X^\xi}^{B,1}$ and $f_{X^\xi}^{B,2}$ in Case Ia that

$$\begin{aligned}\Xi_{B,1}^{\text{Ia}}(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min}^\# < \xi \leq x_{\max} \ \& \ \xi > x_{\min}^\# \ \& \ \xi \geq x_{\max}^b \ \& \ \xi > x^\# \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x_{\min}^\#, x_{\max}^b, x^\#] < \xi \leq \min[x_{\max}, x^\#]\} \\ &= \{\xi \in \mathcal{V}_X \mid x^\# < \xi \leq x_{\max}\}\end{aligned}\tag{64}$$

and

$$\begin{aligned}\Xi_{B,2}^{\text{Ia}}(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min}^\# < \xi \leq x_{\max} \ \& \ \xi > x_{\min}^\# \ \& \ \xi \geq x_{\max}^b \ \& \ \xi \leq x^\# \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x_{\min}^\#, x_{\max}^b] < \xi \leq \min[x_{\max}, x^\#]\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\min}^\# < \xi \leq \min[x_{\max}, x^\#]\}.\end{aligned}\tag{65}$$

Next, for Case Ib, as specified via Eq. (34) and comprising $f_{X^\xi}^C$ then $f_{X^\xi}^D$, the conditions $x_{\min} < x_{\max}^b \leq x_{\min}^\#$ & $x_{\min}^\# \geq x_{\max}$ of Eq. (36ib) prevail. We then have—by referring to Eqs. (34-first line) and (59)—that the set of all $\xi \in \mathcal{V}_X$ satisfying the constraints for PDF segments $f_{X^\xi}^{C,1}$ and $f_{X^\xi}^{C,2}$ in Case Ib are given by

$$\begin{aligned}\Xi_{C,1}^{\text{Ib}}(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi < x_{\max}^b \text{ \& \> } \xi < \min[x_{\min}^\#, x_{\max}^b] \text{ \& \> } \xi \geq x^b \text{ \& \> } x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x^b] \leq \xi < \min[x_{\max}, x_{\min}^\#, x_{\max}^b]\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x^b] \leq \xi < x_{\max}^b\}\end{aligned}\quad (66)$$

and

$$\begin{aligned}\Xi_{C,2}^{\text{Ib}}(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi < x_{\max}^b \text{ \& \> } \xi < \min[x_{\min}^\#, x_{\max}^b] \text{ \& \> } \xi < x^b \text{ \& \> } x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi < \min[x_{\max}, x_{\min}^\#, x_{\max}^b, x^b]\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi < x^b\};\end{aligned}\quad (67)$$

further, from Eqs. (34-second line) and (60) we have for PDF segment $f_{X^\xi}^{D,1}$ in Case Ib

$$\begin{aligned}\Xi_{D,1}^{\text{Ib}}(x) &= \{\xi \in \mathcal{V}_X \mid x_{\max}^b \leq \xi \leq x_{\max} \text{ \& \> } x_{\max}^b \leq \xi \leq x_{\min}^\# \text{ \& \> } x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x_{\max}^b] \leq \xi \leq \min[x_{\max}, x_{\min}^\#]\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\max}^b \leq \xi \leq x_{\max}\}.\end{aligned}\quad (68)$$

Lastly, for Case Ic as specified via Eq. (35) and comprising only $f_{X^\xi}^D$, the conditions

$x_{\min} \geq x_{\max}^b$ & $x_{\max}^b \leq x_{\min}^\#$ & $x_{\min}^\# \geq x_{\max}$ of Eq. (36ic) prevail. Then from Eqs. (35) and (60)

we have for PDF segment $f_{X^\xi}^{D,1}(x)$ in Case Ic

$$\begin{aligned}\Xi_{D,1}^{\text{Ic}}(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi \leq x_{\max} \text{ \& \> } x_{\max}^b \leq \xi \leq x_{\min}^\#\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x_{\max}^b] \leq \xi \leq \min[x_{\max}, x_{\min}^\#]\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi \leq x_{\max}\}.\end{aligned}\quad (69)$$

The above expressions given in Eqs. (57) – (69) lead, via Eqs. (33) – (35), respectively, to the following expressions. Given any $x \in \mathcal{V}_{X^*}$ then (with superscript “x” appended to the Case Ia designator):

Case ^xIa [$x_{min} < x_{max}^b \leq x_{min}^\# < x_{max}$]: If $\xi \in \mathcal{V}_x$ then

$$f_{x^\xi}(x) = \begin{cases} f_{x^\xi}^{C,2}(x) & \text{if } x_{min} \leq \xi < x^b \\ f_{x^\xi}^{C,1}(x) & \text{if } \max[x_{min}, x^b] \leq \xi < x_{max}^b \\ f_{x^\xi}^{D,1}(x) & \text{if } x_{max}^b \leq \xi \leq x_{min}^\# \\ f_{x^\xi}^{B,2}(x) & \text{if } x_{min}^\# < \xi \leq \min[x^\#, x_{max}] \\ f_{x^\xi}^{B,1}(x) & \text{if } x^\# < \xi \leq x_{max}; \end{cases} \quad (70)$$

Case ^xIb [$x_{min} < x_{max}^b \leq x_{min}^\#$ & $x_{min}^\# \geq x_{max}$]: If $\xi \in \mathcal{V}_x$ then

$$f_{x^\xi}(x) = \begin{cases} f_{x^\xi}^{C,2}(x) & \text{if } x_{min} \leq \xi < x^b \\ f_{x^\xi}^{C,1}(x) & \text{if } \max[x_{min}, x^b] \leq \xi < x_{max}^b \\ f_{x^\xi}^{D,1}(x) & \text{if } x_{max}^b \leq \xi \leq x_{max}; \end{cases} \quad (71)$$

Case ^xIc [$x_{min} \geq x_{max}^b$ & $x_{max}^b \leq x_{min}^\#$ & $x_{min}^\# \geq x_{max}$]: If $\xi \in \mathcal{V}_x$ then

$$f_{x^\xi}(x) = f_{x^\xi}^{D,1}(x) \quad \text{if } x_{min} \leq \xi \leq x_{max}. \quad (72)$$

We point out that the difference between Case ^xIa as specified above and Case Ia as specified in Eq. (33) is that the former depends upon the specification of some (fixed but arbitrary) $x \in \mathcal{V}_{x^*}$ for the definition of its associated $f_{x^\xi}(x)$ while the latter does not (but the Case criterion, namely, $x_{min} < x_{max}^b \leq x_{min}^\# < x_{max}$, is the same for both). The same comment applies to Cases ^xIb and ^xIc relative to Cases Ib and Ic, respectively, the latter two as specified in Eqs. (34) and (35), respectively. When referring to any of Cases ^xIa, ^xIb, or ^xIc generically we will write Case ^xI. The same comments apply as well to Case II vs. the shortly-to-be-introduced Case ^xII (and its subcases).

As was the case for Eq. (28) of Case I, it is useful here to make explicit the four subcases of Case ^xIa that are engendered by the four possible conditions allowed by Eq. (70-second line) in conjunction with Eq. (70-fourth line). Using the methodology discussed

previously, we take $\pi^{(1)} = \langle x_{min}, x^b \rangle$ and $\pi^{(2)} = \langle x^\#, x_{max} \rangle$ (hence $J_{max} = 2$) and consider the four (exhaustive and mutually exclusive) possibilities for $\langle \emptyset(\pi^{(1)}), \emptyset(\pi^{(2)}) \rangle$, namely,

$$\langle \emptyset(\pi^{(1)}), \emptyset(\pi^{(2)}) \rangle = \begin{array}{llll} \text{(i)} \langle -, - \rangle & \text{(ii)} \langle -, + \rangle & \text{(iii)} \langle +, - \rangle & \text{(iv)} \langle +, + \rangle. \end{array} \quad (73)$$

These possibilities are tantamount to

$$\begin{array}{ll} \text{(i)} & x_{min} < x^b \quad \& \quad x^\# < x_{max} \quad \text{and} \quad \max[x_{min}, x^b] = x^b \quad \& \quad \min[x^\#, x_{max}] = x^\# \\ \text{(ii)} & x_{min} < x^b \quad \& \quad x^\# \geq x_{max} \quad \text{and} \quad \max[x_{min}, x^b] = x^b \quad \& \quad \min[x^\#, x_{max}] = x_{max} \\ \text{(iii)} & x_{min} \geq x^b \quad \& \quad x^\# < x_{max} \quad \text{and} \quad \max[x_{min}, x^b] = x_{min} \quad \& \quad \min[x^\#, x_{max}] = x^\# \\ \text{(iv)} & x_{min} \geq x^b \quad \& \quad x^\# \geq x_{max} \quad \text{and} \quad \max[x_{min}, x^b] = x_{min} \quad \& \quad \min[x^\#, x_{max}] = x_{max}. \end{array} \quad (74)$$

We note, however, that (i) cannot occur (in Case ^xI) because the two conditions therein imply that $x_{min} < x^b = x/(1+b) = [(1-b)x/(1-b)]/(1+b) = [(1-b)/(1+b)]/x^\# < [(1-b)/(1+b)]x_{max}$ so that $x_{min}/(1-b) < x_{max}/(1+b)$, i.e., $x^\#_{min} < x^b_{max}$, which is a contradiction of the Case I condition of Eq. (23a).

We now examine the restrictions, for each of (ii), (iii), and (iv) individually, that the first two relationships on the LHS of each of the corresponding lines Eq. (74) dictate must be satisfied by x in order for each of these three subcases to be valid. We note that $x^\#_{min} \geq x^b_{max}$ iff $x^+_{min} \geq x^-_{max}$; hence, alternative criteria for Cases I and II may be written as

$$\text{Case I: } x^+_{min} \geq x^-_{max} \quad (75a)$$

and

$$\text{Case II: } x^+_{min} < x^-_{max}. \quad (75b)$$

Now for (ii) we have $\max[x^+_{min}, x^-_{max}] < x \leq x_{max}$; thus for (ii) we have the x -constraint

$$x^+_{min} < x \leq x_{max}. \quad (76)$$

Similarly, for (iii) we have $x_{min} \leq x < \min[x^-_{max}, x^+_{min}]$, i.e., the x -constraint

$$x_{min} \leq x < x^-_{max} \quad (77)$$

and for (iv) we have the x -constraint

$$x^-_{max} \leq x \leq x^+_{min}. \quad (78)$$

At this point the x -constraints required by the four possibilities for $\langle \emptyset(\pi^{(1)}), \emptyset(\pi^{(2)}) \rangle$ in Eq. (67)—which x -constraints appear on the LHS's Eq. (74)—have been determined, with only three of these four being non-self-contradictory and given by Eqs. (76) – (78). We label as Cases $^x\text{IaR1}$, $^x\text{IaR2}$, and $^x\text{IaR3}$ the three possibilities (iii), (iv), and (ii), respectively (in order of increasing x -range [^xR]).

What remains is to write the three versions of Eq. (70) associated with the remaining three possibilities for $\langle \emptyset(\pi^{(1)}), \emptyset(\pi^{(2)}) \rangle$ in Eq. (73)—these versions being dictated under conditions (iii), (iv), and (ii) (i.e., under Cases $^x\text{IaR1}$, $^x\text{IaR2}$, and $^x\text{IaR3}$) by the last two relationships on the RHS's of each of the corresponding lines Eq. (74). We find:

Case $^x\text{IaR1}$ [Eq. (36ia) & $x_{\min} \leq x < x_{\max}^-$]: If $\xi \in \mathcal{V}_x$ then $f_{x^\xi}(x) = f_{x^\xi, \{1\}}(x)$ (79)
for

$$f_{x^\xi, \{1\}}(x) = \begin{cases} f_{x^\xi}^{C,1}(x) & \text{if } x_{\min} \leq \xi < x_{\max}^b \\ f_{x^\xi}^{D,1}(x) & \text{if } x_{\max}^b \leq \xi \leq x_{\min}^\# \\ f_{x^\xi}^{B,2}(x) & \text{if } x_{\min}^\# < \xi \leq x^\# \quad [\emptyset \text{ if } x = x_{\min}] \\ f_{x^\xi}^{B,1}(x) & \text{if } x^\# < \xi \leq x_{\max}; \end{cases} \quad (80)$$

Case $^x\text{IaR2}$ [Eq. (36ia) & $x_{\max}^- \leq x \leq x_{\min}^+$]: If $\xi \in \mathcal{V}_x$ then $f_{x^\xi}(x) = f_{x^\xi, \{2\}}(x)$ (81)
for

$$f_{x^\xi, \{2\}}(x) = \begin{cases} f_{x^\xi}^{C,1}(x) & \text{if } x_{\min} \leq \xi < x_{\max}^b \\ f_{x^\xi}^{D,1}(x) & \text{if } x_{\max}^b \leq \xi \leq x_{\min}^\# \\ f_{x^\xi}^{B,2}(x) & \text{if } x_{\min}^\# < \xi \leq x_{\max}; \end{cases} \quad (82)$$

Case $^x\text{IaR3}$ [Eq. (36ia) & $x_{\min}^+ < x \leq x_{\max}^+$]: If $\xi \in \mathcal{V}_x$ then $f_{x^\xi}(x) = f_{x^\xi, \{3\}}(x)$ (83)
for

$$f_{x^\xi, \{3\}}(x) = \begin{cases} f_{x^\xi}^{C,2}(x) & \text{if } x_{\min} \leq \xi < x^b \\ f_{x^\xi}^{C,1}(x) & \text{if } x^b \leq \xi < x_{\max}^b \quad [\emptyset \text{ if } x = x_{\max}] \\ f_{x^\xi}^{D,1}(x) & \text{if } x_{\max}^b \leq \xi \leq x_{\min}^\# \\ f_{x^\xi}^{B,2}(x) & \text{if } x_{\min}^\# < \xi \leq x_{\max}. \end{cases} \quad (84)$$

Having expanded Eq. (70) as above, we observe that we must make explicit the two subcases of Case ^xIb that are engendered by the two possible conditions allowed by Eq. (71) and we do so now, labeling them ^xIbR1 and ^xIbR2. Here we take $\pi^{(1)} = \langle x_{min}, x^b \rangle$ (hence $J_{max} = 1$) with $\langle \emptyset(\pi^{(1)}) \rangle$ given by (i) $\langle + \rangle$ or (ii) $\langle - \rangle$, these being tantamount, respectively, to

$$(i) \ x_{min} \geq x^b \text{ (iff } x \leq x_{min}^+) \text{ and } \max[x_{min}, x^b] = x_{min} \quad (85)$$

$$(ii) \ x_{min} < x^b \text{ (iff } x > x_{min}^+) \text{ and } \max[x_{min}, x^b] = x^b.$$

Now using Eq. (71) under the conditions (i) and (ii) of Eqs. (85) (and taking into account Eqs. (23a) and (36ib)) we have:

Case ^xIbR1 [Eq. (36ib) & $x_{min} \leq x \leq x_{min}^+$]: If $\xi \in \mathcal{V}_x$ then $f_{x^\xi}(x) = f_{x^\xi, \{4\}}(x)$ (86)
for

$$f_{x^\xi, \{4\}}(x) = \begin{cases} f_{x^\xi}^{C,1}(x) & \text{if } x_{min} \leq \xi < x_{max}^b \\ f_{x^\xi}^{D,1}(x) & \text{if } x_{max}^b \leq \xi \leq x_{max}; \end{cases} \quad (87)$$

Case ^xIbR2 [Eq. (36ib) & $x_{min}^+ < x \leq x_{max}$]: If $\xi \in \mathcal{V}_x$ then $f_{x^\xi}(x) = f_{x^\xi, \{5\}}(x)$ (88)
for

$$f_{x^\xi, \{5\}}(x) = \begin{cases} f_{x^\xi}^{C,2}(x) & \text{if } x_{min} \leq \xi < x^b \\ f_{x^\xi}^{C,1}(x) & \text{if } x^b \leq \xi < x_{max}^b \quad [\emptyset \text{ if } x = x_{max}] \\ f_{x^\xi}^{D,1}(x) & \text{if } x_{max}^b \leq \xi \leq x_{max}. \end{cases} \quad (89)$$

Finally, Case ^xIc remains as given in Eq. (72), and having only one x-range, but for consistency of notation (and enhanced information content) we have:

Case ^xIcR1 [Eq. (36ic)]: If $\xi \in \mathcal{V}_x$ then $f_{x^\xi}(x) = f_{x^\xi, \{6\}}(x)$ (90)

for

$$f_{x^\xi, \{6\}}(x) = f_{x^\xi}^{D,1}(x) \quad \text{if } x_{min} \leq \xi \leq x_{max}. \quad (91)$$

This concludes Case ^xI.

We next consider Case ^xII, which case turns out to be somewhat more complex than Case ^xI. For Case II, as specified via Eq. (29) and comprising $f_{x^\xi}^C$ then $f_{x^\xi}^A$ then $f_{x^\xi}^B$, the conditions $x_{max} > x_{max}^b > x_{min}^\# > x_{min}$ of Eq. (36ii) prevail. We have from Eqs. (29-first line)

and (59) that the set of all ξ that satisfy the constraints for PDF segment $f_{X^\xi}^{C,1}$ in Case II, which set we denote by $\Xi_{C,1}^\Pi(x)$, is given by

$$\begin{aligned}\Xi_{C,1}^\Pi(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi \leq x_{\min}^\# \ \& \ \xi \leq x_{\min}^\# \ \& \ \xi < x_{\max}^b \ \& \ \xi \geq x^b \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi \leq x_{\min}^\# \ \& \ \xi \geq x^b \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x^b] \leq \xi \leq \min[x_{\max}, x_{\min}^\#]\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x^b] \leq \xi \leq x_{\min}^\#\};\end{aligned}\tag{92}$$

likewise, for PDF segment $f_{X^\xi}^{C,2}$ in Case II, we have that

$$\begin{aligned}\Xi_{C,2}^\Pi(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi \leq x_{\min}^\# \ \& \ \xi \leq x_{\min}^\# \ \& \ \xi < x_{\max}^b \ \& \ \xi < x^b \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi \leq x_{\min}^\# \ \& \ \xi < x^b \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi < \min[x_{\max}, x_{\min}^\#, x^b]\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\min} \leq \xi < \min[x_{\min}^\#, x^b]\}.\end{aligned}\tag{93}$$

Further, from Eqs. (29-second line) and (57), for PDF segments $f_{X^\xi}^{A,1}$, $f_{X^\xi}^{A,2}$ and $f_{X^\xi}^{A,3}$ in Case II we have

$$\begin{aligned}\Xi_{A,1}^\Pi(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min}^\# < \xi < x_{\max}^b \ \& \ \xi > x/(1-b_1) \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x_{\min}^\#, x^b] < \xi < \min[x_{\max}, x_{\max}^b]\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\min}^\# < \xi < x_{\max}^b\}\end{aligned}\tag{94}$$

and

$$\begin{aligned}\Xi_{A,2}^\Pi(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min}^\# < \xi < x_{\max}^b \ \& \ x^b \leq \xi \leq x_{\min}^\# \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x_{\min}^\#, x^b] < \xi < \min[x_{\max}, x_{\max}^b, x_{\min}^\#]\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}^\#, x^b] < \xi < \min[x_{\max}^b, x_{\min}^\#]\}\end{aligned}\tag{95}$$

and

$$\begin{aligned}\Xi_{A,3}^\Pi(x) &= \{\xi \in \mathcal{V}_X \mid x_{\min}^\# < \xi < x_{\max}^b \ \& \ \xi < x^b \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\ &= \{\xi \in \mathcal{V}_X \mid \max[x_{\min}, x_{\min}^\#] < \xi < \min[x_{\max}, x_{\max}^b]\} \\ &= \{\xi \in \mathcal{V}_X \mid x_{\min}^\# < \xi < x_{\max}^b\}.\end{aligned}\tag{96}$$

Finally, from Eqs. (29-third line) and (58) we have for PDF segments $f_{X^\xi}^{B,1}$ and $f_{X^\xi}^{B,2}$ in Case II that

$$\begin{aligned}
\Xi_{B,1}^{\text{II}}(x) &= \{\xi \in \mathcal{V}_x \mid x_{\max}^b \leq \xi \leq x_{\max} \ \& \ \xi > x_{\min}^\# \ \& \ \xi \geq x_{\max}^b \ \& \ \xi > x^\# \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\
&= \{\xi \in \mathcal{V}_x \mid x_{\max}^b \leq \xi \leq x_{\max} \ \& \ \xi > x^\# \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\
&= \{\xi \in \mathcal{V}_x \mid \max[x_{\min}, x_{\max}^b, x^\#] < \xi \leq x_{\max}\} \\
&= \{\xi \in \mathcal{V}_x \mid \max[x_{\max}^b, x^\#] < \xi \leq x_{\max}\}
\end{aligned} \tag{97}$$

and

$$\begin{aligned}
\Xi_{B,2}^{\text{II}}(x) &= \{\xi \in \mathcal{V}_x \mid x_{\max}^b \leq \xi \leq x_{\max} \ \& \ \xi > x_{\min}^\# \ \& \ \xi \geq x_{\max}^b \ \& \ \xi \leq x^\# \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\
&= \{\xi \in \mathcal{V}_x \mid x_{\max}^b \leq \xi \leq x_{\max} \ \& \ \xi \leq x^\# \ \& \ x_{\min} \leq \xi \leq x_{\max}\} \\
&= \{\xi \in \mathcal{V}_x \mid \max[x_{\min}, x_{\max}^b] \leq \xi \leq \min[x_{\max}, x^\#]\} \\
&= \{\xi \in \mathcal{V}_x \mid x_{\max}^b \leq \xi \leq \min[x_{\max}, x^\#]\}.
\end{aligned} \tag{98}$$

The above expressions given in Eqs. (92) – (98) lead to the following. Given any $x \in \mathcal{V}_{x^*}$ then (with superscript “x” appended to the Case II designator):

Case ^xII [Eq. (36ii) & $x_{\min} < x_{\min}^\# < x_{\max}^b < x_{\max}$]: If $\xi \in \mathcal{V}_x$ then

$$f_{x^\xi}(x) = \begin{cases} f_{x^\xi}^{C,2}(x) & \text{if } x_{\min} \leq \xi < \min[x_{\min}^\#, x^b] \\ f_{x^\xi}^{C,1}(x) & \text{if } \max[x_{\min}, x^b] \leq \xi \leq x_{\min}^\# \\ f_{x^\xi}^{A,3}(x) & \text{if } x_{\min}^\# < \xi < x^b \\ f_{x^\xi}^{A,2}(x) & \text{if } \max[x_{\min}^\#, x^b] < \xi < \min[x^\#, x_{\max}^b] \\ f_{x^\xi}^{A,1}(x) & \text{if } x^\# < \xi < x_{\max}^b \\ f_{x^\xi}^{B,2}(x) & \text{if } x_{\max}^b \leq \xi \leq \min[x^\#, x_{\max}] \\ f_{x^\xi}^{B,1}(x) & \text{if } \max[x^\#, x_{\max}^b] < \xi \leq x_{\max}. \end{cases} \tag{99}$$

Analogously to the subcases of ^xI, it is useful to make explicit the subcases of Case ^xII that are engendered by the various possible conditions allowed by Eq. (95) via all but its third and fifth lines. Our procedure here will be identical to that used in Case ^xI. We firstly observe that there are only four *distinct* pairs that appear as arguments of “min” or “max” in Eq. (95) and we then take for these four pairs

$$\pi^{(1)} = \langle x_{\min}^{\#}, x^b \rangle \quad \pi^{(2)} = \langle x^{\#}, x_{\max}^b \rangle \quad \pi^{(3)} = \langle x_{\min}, x^b \rangle \quad \pi^{(4)} = \langle x^{\#}, x_{\max} \rangle \quad (100)$$

(hence $J_{\max} = 4$) and consider the sixteen (exhaustive and mutually exclusive) possibilities

$$\langle \emptyset(\pi^{(1)}), \emptyset(\pi^{(2)}), \emptyset(\pi^{(3)}), \emptyset(\pi^{(4)}) \rangle =$$

$$\begin{aligned} & \text{(i)} \langle -, -, -, - \rangle \quad \text{(ii)} \langle -, -, -, + \rangle \quad \text{(iii)} \langle -, -, +, - \rangle \quad \text{(iv)} \langle -, -, +, + \rangle \\ & \text{(v)} \langle -, +, -, - \rangle \quad \text{(vi)} \langle -, +, -, + \rangle \quad \text{(vii)} \langle -, +, +, - \rangle \quad \text{(viii)} \langle -, +, +, + \rangle \\ & \text{(ix)} \langle +, -, -, - \rangle \quad \text{(x)} \langle +, -, -, + \rangle \quad \text{(xi)} \langle +, -, +, - \rangle \quad \text{(xii)} \langle +, -, +, + \rangle \\ & \text{(xiii)} \langle +, +, -, - \rangle \quad \text{(xiv)} \langle +, +, -, + \rangle \quad \text{(xv)} \langle +, +, +, - \rangle \quad \text{(xvi)} \langle +, +, +, + \rangle. \end{aligned} \quad (101)$$

Analogously to our procedure in Eq. (74), we record these possibilities as being tantamount to the following (where, for the present, we explicitly write only for the possibility of Eq. (101i) the associated max and min values to be used in Eq. (99)):

$$\begin{aligned} & \text{(i)} \quad x_{\min}^{\#} < x^b \quad \& \quad x^{\#} < x_{\max}^b \quad \& \quad x_{\min} < x^b \quad \& \quad x^{\#} < x_{\max} \quad \text{and} \\ & \quad \min[x_{\min}^{\#}, x^b] = x_{\min}^{\#} \quad \& \quad \max[x_{\min}^{\#}, x^b] = x^b \quad \& \quad \min[x^{\#}, x_{\max}^b] = x^{\#} \\ & \quad \& \quad \max[x^{\#}, x_{\max}^b] = x_{\max}^b \quad \& \quad \max[x_{\min}, x^b] = x^b \quad \& \quad \min[x^{\#}, x_{\max}] = x^{\#} \\ & \text{(ii)} \quad x_{\min}^{\#} < x^b \quad \& \quad x^{\#} < x_{\max}^b \quad \& \quad x_{\min} < x^b \quad \& \quad x^{\#} \geq x_{\max} \\ & \text{(iii)} \quad x_{\min}^{\#} < x^b \quad \& \quad x^{\#} < x_{\max}^b \quad \& \quad x_{\min} \geq x^b \quad \& \quad x^{\#} < x_{\max} \\ & \text{(iv)} \quad x_{\min}^{\#} < x^b \quad \& \quad x^{\#} < x_{\max}^b \quad \& \quad x_{\min} \geq x^b \quad \& \quad x^{\#} \geq x_{\max} \\ & \text{(v)} \quad x_{\min}^{\#} < x^b \quad \& \quad x^{\#} \geq x_{\max}^b \quad \& \quad x_{\min} < x^b \quad \& \quad x^{\#} < x_{\max} \\ & \text{(vi)} \quad x_{\min}^{\#} < x^b \quad \& \quad x^{\#} \geq x_{\max}^b \quad \& \quad x_{\min} < x^b \quad \& \quad x^{\#} \geq x_{\max} \\ & \text{(vii)} \quad x_{\min}^{\#} < x^b \quad \& \quad x^{\#} \geq x_{\max}^b \quad \& \quad x_{\min} \geq x^b \quad \& \quad x^{\#} < x_{\max} \\ & \text{(viii)} \quad x_{\min}^{\#} < x^b \quad \& \quad x^{\#} \geq x_{\max}^b \quad \& \quad x_{\min} \geq x^b \quad \& \quad x^{\#} \geq x_{\max} \\ & \text{(ix)} \quad x_{\min}^{\#} \geq x^b \quad \& \quad x^{\#} < x_{\max}^b \quad \& \quad x_{\min} < x^b \quad \& \quad x^{\#} < x_{\max} \\ & \text{(x)} \quad x_{\min}^{\#} \geq x^b \quad \& \quad x^{\#} < x_{\max}^b \quad \& \quad x_{\min} < x^b \quad \& \quad x^{\#} \geq x_{\max} \\ & \text{(xi)} \quad x_{\min}^{\#} \geq x^b \quad \& \quad x^{\#} < x_{\max}^b \quad \& \quad x_{\min} \geq x^b \quad \& \quad x^{\#} < x_{\max} \\ & \text{(xii)} \quad x_{\min}^{\#} \geq x^b \quad \& \quad x^{\#} < x_{\max}^b \quad \& \quad x_{\min} \geq x^b \quad \& \quad x^{\#} \geq x_{\max} \\ & \text{(xiii)} \quad x_{\min}^{\#} \geq x^b \quad \& \quad x^{\#} \geq x_{\max}^b \quad \& \quad x_{\min} < x^b \quad \& \quad x^{\#} < x_{\max} \end{aligned} \quad (102)$$

$$\begin{aligned}
\text{(xiv)} \quad & x_{\min}^{\#} \geq x^b \quad \& \quad x^{\#} \geq x_{\max}^b \quad \& \quad x_{\min} < x^b \quad \& \quad x^{\#} \geq x_{\max} \\
\text{(xv)} \quad & x_{\min}^{\#} \geq x^b \quad \& \quad x^{\#} \geq x_{\max}^b \quad \& \quad x_{\min} \geq x^b \quad \& \quad x^{\#} < x_{\max} \quad (102, \text{cont'd}) \\
\text{(xvi)} \quad & x_{\min}^{\#} \geq x^b \quad \& \quad x^{\#} \geq x_{\max}^b \quad \& \quad x_{\min} \geq x^b \quad \& \quad x^{\#} \geq x_{\max}.
\end{aligned}$$

Similarly to the earlier subcases of Case ^xI (see Eqs. (32) and (74)), some of the possibilities (i) through (xvi) above are precluded by the inconsistency of their required conditions in the present Case ^xII . We now examine this issue. Firstly, because of the inconsistency of $x_{\min}^{\#} < x^b$ and $x_{\min} \geq x^b$ (i.e., $\mathcal{O}(\pi^{(1)}) = -$ and $\mathcal{O}(\pi^{(3)}) = +$), to wit, the contradiction $x_{\min}^{\#} < x^b \leq x_{\min}$, then possibilities (iii), (iv), (vii) and (viii) are to be eliminated. Secondly, because of the inconsistency of $x^{\#} < x_{\max}^b$ and $x^{\#} \geq x_{\max}$ (i.e., $\mathcal{O}(\pi^{(2)}) = -$ and $\mathcal{O}(\pi^{(4)}) = +$), to wit, the contradiction $x_{\max} \leq x^{\#} < x_{\max}^b$, then possibilities (ii), (iv) (again), (x) and (xii) are to be eliminated. Thus remaining are the nine possibilities are (i), (v), (vi), (ix), (xi) and (xiii) – (xvi). For convenience we renumber these remaining nine by (i)' through (ix)', respectively:

$$\begin{aligned}
& \text{(i)} \rightarrow \text{(i)'} & \text{(ix)} \rightarrow \text{(iv)'} & \text{(xiv)} \rightarrow \text{(vii)'} \\
& \text{(v)} \rightarrow \text{(ii)'} & \text{(xi)} \rightarrow \text{(v)'} & \text{(xv)} \rightarrow \text{(viii)'} \\
& \text{(vi)} \rightarrow \text{(iii)'} & \text{(xiii)} \rightarrow \text{(vi)'} & \text{(xvi)} \rightarrow \text{(ix)'}.
\end{aligned} \tag{103}$$

We now write, for these nine remaining possibilities, the explicit min and max terms for them in analogy with Eq. (102i); and we then use these to record the nine versions of Eq. (99) corresponding to these nine cases, denoting the resulting nine functions by $f_{x^{\#}, [i]}'$ through $f_{x^{\#}, [ix]}'$. We have:

$$\begin{aligned}
\text{(i)'} \quad & \min[x_{\min}^{\#}, x^b] = x_{\min}^{\#} \quad \& \quad \max[x_{\min}^{\#}, x^b] = x^b \quad \& \quad \min[x^{\#}, x_{\max}^b] = x^{\#} \\
& \& \quad \max[x^{\#}, x_{\max}^b] = x_{\max}^b \quad \& \quad \max[x_{\min}, x^b] = x^b \quad \& \quad \min[x^{\#}, x_{\max}] = x^{\#} \quad (104) \\
\text{(ii)'} \quad & \min[x_{\min}^{\#}, x^b] = x_{\min}^{\#} \quad \& \quad \max[x_{\min}^{\#}, x^b] = x^b \quad \& \quad \min[x^{\#}, x_{\max}^b] = x_{\max}^b \\
& \& \quad \max[x^{\#}, x_{\max}^b] = x^{\#} \quad \& \quad \max[x_{\min}, x^b] = x^b \quad \& \quad \min[x^{\#}, x_{\max}] = x^{\#} \\
\text{(iii)'} \quad & \min[x_{\min}^{\#}, x^b] = x_{\min}^{\#} \quad \& \quad \max[x_{\min}^{\#}, x^b] = x^b \quad \& \quad \min[x^{\#}, x_{\max}^b] = x_{\max}^b \\
& \& \quad \max[x^{\#}, x_{\max}^b] = x^{\#} \quad \& \quad \max[x_{\min}, x^b] = x^b \quad \& \quad \min[x^{\#}, x_{\max}] = x_{\max}
\end{aligned}$$

$$\begin{aligned}
\text{(iv)'} \quad & \min[x_{\min}^{\#}, x^b] = x^b \ \& \ \max[x_{\min}^{\#}, x^b] = x_{\min}^{\#} \ \& \ \min[x^{\#}, x_{\max}^b] = x^{\#} \\
& \& \ \max[x^{\#}, x_{\max}^b] = x_{\max}^b \ \& \ \max[x_{\min}, x^b] = x^b \ \& \ \min[x^{\#}, x_{\max}] = x^{\#} \\
\text{(v)'} \quad & \min[x_{\min}^{\#}, x^b] = x^b \ \& \ \max[x_{\min}^{\#}, x^b] = x_{\min}^{\#} \ \& \ \min[x^{\#}, x_{\max}^b] = x^{\#} \\
& \& \ \max[x^{\#}, x_{\max}^b] = x_{\max}^b \ \& \ \max[x_{\min}, x^b] = x_{\min} \ \& \ \min[x^{\#}, x_{\max}] = x^{\#} \\
\text{(vi)'} \quad & \min[x_{\min}^{\#}, x^b] = x^b \ \& \ \max[x_{\min}^{\#}, x^b] = x_{\min}^{\#} \ \& \ \min[x^{\#}, x_{\max}^b] = x_{\max}^b \ (104, \text{cont'd}) \\
& \& \ \max[x^{\#}, x_{\max}^b] = x^{\#} \ \& \ \max[x_{\min}, x^b] = x^b \ \& \ \min[x^{\#}, x_{\max}] = x^{\#} \\
\text{(vii)'} \quad & \min[x_{\min}^{\#}, x^b] = x^b \ \& \ \max[x_{\min}^{\#}, x^b] = x_{\min}^{\#} \ \& \ \min[x^{\#}, x_{\max}^b] = x_{\max}^b \\
& \& \ \max[x^{\#}, x_{\max}^b] = x^{\#} \ \& \ \max[x_{\min}, x^b] = x^b \ \& \ \min[x^{\#}, x_{\max}] = x_{\max} \\
\text{(viii)'} \quad & \min[x_{\min}^{\#}, x^b] = x^b \ \& \ \max[x_{\min}^{\#}, x^b] = x_{\min}^{\#} \ \& \ \min[x^{\#}, x_{\max}^b] = x_{\max}^b \\
& \& \ \max[x^{\#}, x_{\max}^b] = x^{\#} \ \& \ \max[x_{\min}, x^b] = x_{\min} \ \& \ \min[x^{\#}, x_{\max}] = x^{\#} \\
\text{(ix)'} \quad & \min[x_{\min}^{\#}, x^b] = x^b \ \& \ \max[x_{\min}^{\#}, x^b] = x_{\min}^{\#} \ \& \ \min[x^{\#}, x_{\max}^b] = x_{\max}^b \\
& \& \ \max[x^{\#}, x_{\max}^b] = x^{\#} \ \& \ \max[x_{\min}, x^b] = x_{\min} \ \& \ \min[x^{\#}, x_{\max}] = x_{\max}.
\end{aligned}$$

For the nine versions of Eq. (99) corresponding to these nine cases we then have (noting that those ranges for $\xi \in \mathcal{V}_X$ that are in fact void have been deleted, thus resulting in differing numbers of components from case to case):

$$\text{(i)'} \quad f_{X^{\xi}, \{i\}}(x) = \begin{cases} f_{X^{\xi}}^{C,2}(x) & \text{if } x_{\min} \leq \xi \leq x_{\min}^{\#} \\ f_{X^{\xi}}^{A,3}(x) & \text{if } x_{\min}^{\#} < \xi < x^b \\ f_{X^{\xi}}^{A,2}(x) & \text{if } x^b \leq \xi \leq x^{\#} \\ f_{X^{\xi}}^{A,1}(x) & \text{if } x^{\#} < \xi < x_{\max}^b \\ f_{X^{\xi}}^{B,1}(x) & \text{if } x_{\max}^b \leq \xi \leq x_{\max}; \end{cases} \quad (105)$$

$$\text{(ii)'} \quad f_{X^{\xi}, \{ii\}}(x) = \begin{cases} f_{X^{\xi}}^{C,2}(x) & \text{if } x_{\min} \leq \xi \leq x_{\min}^{\#} \\ f_{X^{\xi}}^{A,3}(x) & \text{if } x_{\min}^{\#} < \xi < x^b \\ f_{X^{\xi}}^{A,2}(x) & \text{if } x^b \leq \xi < x_{\max}^b \\ f_{X^{\xi}}^{B,2}(x) & \text{if } x_{\max}^b \leq \xi \leq x^{\#} \\ f_{X^{\xi}}^{B,1}(x) & \text{if } x^{\#} < \xi \leq x_{\max}; \end{cases} \quad (106)$$

(iii)'

$$f_{X^\xi, \{iii\}'}(x) = \begin{cases} f_{X^\xi}^{C,2}(x) & \text{if } x_{min} \leq \xi \leq x_{min}^\# \\ f_{X^\xi}^{A,3}(x) & \text{if } x_{min}^\# < \xi < x^b \\ f_{X^\xi}^{A,2}(x) & \text{if } x^b \leq \xi < x_{max}^b \\ f_{X^\xi}^{B,2}(x) & \text{if } x_{max}^b \leq \xi \leq x_{max}; \end{cases} \quad (107)$$

(iv)'

$$f_{X^\xi, \{iv\}'}(x) = \begin{cases} f_{X^\xi}^{C,2}(x) & \text{if } x_{min} \leq \xi < x^b \\ f_{X^\xi}^{C,1}(x) & \text{if } x^b \leq \xi \leq x_{min}^\# \\ f_{X^\xi}^{A,2}(x) & \text{if } x_{min}^\# < \xi \leq x^\# \\ f_{X^\xi}^{A,1}(x) & \text{if } x^\# < \xi < x_{max}^b \\ f_{X^\xi}^{B,1}(x) & \text{if } x_{max}^b \leq \xi \leq x_{max}; \end{cases} \quad (108)$$

(v)'

$$f_{X^\xi, \{v\}'}(x) = \begin{cases} f_{X^\xi}^{C,1}(x) & \text{if } x_{min} \leq \xi \leq x_{min}^\# \\ f_{X^\xi}^{A,2}(x) & \text{if } x_{min}^\# < \xi \leq x^\# \\ f_{X^\xi}^{A,1}(x) & \text{if } x^\# < \xi < x_{max}^b \\ f_{X^\xi}^{B,1}(x) & \text{if } x_{max}^b \leq \xi \leq x_{max}; \end{cases} \quad (109)$$

(vi)'

$$f_{X^\xi, \{vi\}'}(x) = \begin{cases} f_{X^\xi}^{C,2}(x) & \text{if } x_{min} \leq \xi < x^b \\ f_{X^\xi}^{C,1}(x) & \text{if } x^b \leq \xi \leq x_{min}^\# \\ f_{X^\xi}^{A,2}(x) & \text{if } x_{min}^\# < \xi < x_{max}^b \\ f_{X^\xi}^{B,2}(x) & \text{if } x_{max}^b \leq \xi \leq x^\# \\ f_{X^\xi}^{B,1}(x) & \text{if } x^\# < \xi \leq x_{max}; \end{cases} \quad (110)$$

(vii)'

$$f_{X^\xi, \{vii\}'}(x) = \begin{cases} f_{X^\xi}^{C,2}(x) & \text{if } x_{min} \leq \xi < x^b \\ f_{X^\xi}^{C,1}(x) & \text{if } x^b \leq \xi \leq x_{min}^\# \\ f_{X^\xi}^{A,2}(x) & \text{if } x_{min}^\# < \xi < x_{max}^b \\ f_{X^\xi}^{B,2}(x) & \text{if } x_{max}^b \leq \xi \leq x_{max}; \end{cases} \quad (111)$$

(viii)'

$$f_{X^\xi, \{viii\}'}(x) = \begin{cases} f_{X^\xi}^{C,1}(x) & \text{if } x_{min} \leq \xi \leq x_{min}^\# \\ f_{X^\xi}^{A,2}(x) & \text{if } x_{min}^\# < \xi < x_{max}^b \\ f_{X^\xi}^{B,2}(x) & \text{if } x_{max}^b \leq \xi \leq x^\# \\ f_{X^\xi}^{B,1}(x) & \text{if } x^\# < \xi \leq x_{max}; \end{cases} \quad (112)$$

(ix)'

$$f_{X^\xi, \{ix\}'}(x) = \begin{cases} f_{X^\xi}^{C,1}(x) & \text{if } x_{min} \leq \xi \leq x_{min}^\# \\ f_{X^\xi}^{A,2}(x) & \text{if } x_{min}^\# < \xi < x_{max}^b \\ f_{X^\xi}^{B,2}(x) & \text{if } x_{max}^b \leq \xi \leq x_{max}. \end{cases} \quad (113)$$

Having found the nine versions, $f_{X^\xi, \{i\}'}$ through $f_{X^\xi, \{ix\}'}$ of Eq. (99) for the nine cases (i)' through (ix)' of Eqs. (105) – (113) respectively, we now must examine the x -constraints for each of these versions that dictate, for each of these, its x -range of validity, where the x -constraints for each version are given by the four relationships on each of the corresponding lines of Eq. (104) (using the correspondences in numbering from Eq. (103)). For this purpose, we introduce the notation

$$x^{-b} \equiv [(1 - b)/(1 + b)]x = (x^-)^b = (x^b)^- \quad (114)$$

$$x^{+\#} \equiv [(1 + b)/(1 - b)]x = (x^+)^{\#} = (x^{\#})^+$$

for any $x \in \mathcal{V}_{x^*}$; note that $x^{-b} < x^{+\#}$. Consider firstly (i)' in Eq. (104). From it we find from that

$$\max[x_{min}^{+\#}, x_{min}^+] < x < \min[x_{max}^{-b}, x_{max}^-], \quad (115a)$$

i.e.,

$$x_{min}^{+\#} < x < x_{max}^{-b}. \quad (115b)$$

We will in fact write Eq. (115a) more completely as

$$\max[x_{min}^{+\#}\{<\}, x_{min}^{+}\{<\}] < x < \min[x_{max}^{-b}\{<\}, x_{max}^{-}\{<\}] \quad (115c)$$

to indicate, for example, that the constraint $x_{min}^{+\#} < x$ on x in Eq. (102i) involves “<” rather than “≤”; when the constraint on x involves “≤” rather than “<” (e.g., $x_{max}^{-b} \leq x$ in Eq. (102v) then “{≤}” will appear directly after the parameter occurring in the applicable analog of Eq. (115c). (This distinction is important because, were $\max[x_{min}^{+\#}\{≤\}, x_{min}^{+}\{<\}] = x_{min}^{+\#}$ then Eq. (115b) *would* become instead $x_{min}^{+\#} \leq x < x_{max}^{-b}$.) Similar reasoning yields the following x -constraints for each of the nine subcases:

- (i)' $x_{min}^{+\#} < x < x_{max}^{-b}$
- (ii)' $\max[x_{min}^{+\#}\{<\}, x_{max}^{-b}\{≤\}, x_{min}^{+}\{<\}] < x < x_{max}^{-}\{<\}$ so
 $\max[x_{min}^{+\#}\{<\}, x_{max}^{-b}\{≤\}] < x < x_{max}^{-}$
- (iii)' $\max[x_{min}^{+\#}\{<\}, x_{max}^{-b}\{≤\}, x_{min}^{+}\{<\}, x_{max}^{-}\{≤\}] < x \leq x_{max}$ so
 $\max[x_{min}^{+\#}\{<\}, x_{max}^{-}\{≤\}] < x \leq x_{max}$
- (iv)' $x_{min}^{+}\{<\} < x < \min[x_{min}^{+\#}\{≤\}, x_{max}^{-b}\{<\}, x_{max}^{-}\{<\}]$ so
 $x_{min}^{+} < x < \min[x_{min}^{+\#}\{≤\}, x_{max}^{-b}\{<\}]$
- (v)' $x_{min} \leq x < \min[x_{min}^{+\#}\{≤\}, x_{max}^{-b}\{<\}, x_{min}^{+}\{≤\}, x_{max}^{-}\{<\}]$ so
 $x_{min} \leq x < \min[x_{max}^{-b}\{<\}, x_{min}^{+}\{≤\}] \quad (116)$
- (vi)' $\max[x_{max}^{-b}\{≤\}, x_{min}^{+}\{<\}] < x < \min[x_{min}^{+\#}\{≤\}, x_{max}^{-}\{<\}]$ so
 $\max[x_{max}^{-b}\{≤\}, x_{min}^{+}\{<\}] < x < \min[x_{min}^{+\#}\{≤\}, x_{max}^{-}\{<\}]$
- (vii)' $\max[x_{max}^{-b}\{≤\}, x_{min}^{+}\{<\}, x_{max}^{-}\{≤\}] < x < x_{min}^{+\#}\{≤\}$ so
 $x_{max}^{-} \leq x \leq x_{min}^{+\#}$
- (viii)' $x_{max}^{-b}\{≤\} \leq x < \min[x_{min}^{+\#}\{≤\}, x_{min}^{+}\{≤\}, x_{max}^{-}\{<\}]$ so
 $x_{max}^{-b} \leq x \leq x_{min}^{+}$
- (ix)' $\max[x_{max}^{-b}\{≤\}, x_{max}^{-}\{≤\}] \leq x \leq \min[x_{max}^{-b}\{≤\}, x_{min}^{+}\{≤\}]$ so
 $x_{max}^{-} \leq x \leq x_{min}^{+}$

It is immediately apparent from Eq. (75b), which specifies the Case ^xII restriction, that possibility (ix)' is precluded (i.e., the set of all x satisfying (ix)' and Eq. (75b) is empty), leaving the eight cases (i)' – (viii)' remaining. Concerning these remaining eight, we note the occurrence of *subcases in the x-constraints* engendered by items (ii)' - (vi)': two subcases for each of (ii)' – (v)' and four subcases for (vi)'—this circumstance did not occur in Case ^xI. We observe that there are in fact only three *distinct* pairs that appear as arguments of “min” or “max” in Eq. (116), namely $\langle x_{min}^{+\#}, x_{max}^{-b} \rangle$, $\langle x_{min}^{+\#}, x_{max}^{-} \rangle$, and $\langle x_{max}^{-b}, x_{min}^{+} \rangle$. And although we are not yet concerned with the simultaneous occurrence of these three pairs, we will nevertheless write our x -constraint results in terms of these ordered pairs, namely,

$$\pi^{(1)} = \langle x_{min}^{+\#}, x_{max}^{-b} \rangle \quad \pi^{(2)} = \langle x_{min}^{+\#}, x_{max}^{-} \rangle \quad \pi^{(3)} = \langle x_{max}^{-b}, x_{min}^{+} \rangle \quad (117)$$

with (recall) $\Theta(\pi^{(j)}) = -$ if $\pi_1^{(j)} < \pi_2^{(j)}$ and $\Theta(\pi^{(j)}) = +$ if $\pi_1^{(j)} \geq \pi_2^{(j)}$. As indicated above, we only need give results for items (ii)' - (vi)' (with the x -constraints for $f_{x^{\xi}, [i]}'$, $f_{x^{\xi}, [vii]}'$, and $f_{x^{\xi}, [viii]}'$ being given in Eq. (116)). Again from Eq. (116) we have the following x -constraints on $f_{x^{\xi}, [ii]}'$ through $f_{x^{\xi}, [vi]}'$:

$$\begin{aligned} \text{(ii)'} \quad & x_{max}^{-b} \leq x < x_{max}^{-} \quad \text{if } \Theta(\pi^{(1)}) = - & x_{min}^{+\#} < x < x_{max}^{-} \quad \text{if } \Theta(\pi^{(1)}) = + \\ \text{(iii)'} \quad & x_{max}^{-} \leq x \leq x_{max} \quad \text{if } \Theta(\pi^{(2)}) = - & x_{min}^{+\#} < x \leq x_{max} \quad \text{if } \Theta(\pi^{(2)}) = + \\ \text{(iv)'} \quad & x_{min}^{+} < x \leq x_{min}^{+\#} \quad \text{if } \Theta(\pi^{(1)}) = - & x_{min}^{+} < x < x_{max}^{-b} \quad \text{if } \Theta(\pi^{(1)}) = + \\ \text{(v)'} \quad & x_{min} \leq x < x_{max}^{-b} \quad \text{if } \Theta(\pi^{(3)}) = - & x_{min} \leq x \leq x_{min}^{+} \quad \text{if } \Theta(\pi^{(3)}) = + \\ \text{(vi)'} \quad & x_{max}^{-b} \leq x \leq x_{min}^{+\#} \quad \text{if } \langle \Theta(\pi^{(2)}), \Theta(\pi^{(3)}) \rangle = \langle -, + \rangle \\ & x_{min}^{+} < x < x_{max}^{-} \quad \text{if } \langle \Theta(\pi^{(2)}), \Theta(\pi^{(3)}) \rangle = \langle +, - \rangle \\ & x_{max}^{-b} \leq x < x_{max}^{-} \quad \text{if } \langle \Theta(\pi^{(2)}), \Theta(\pi^{(3)}) \rangle = \langle +, + \rangle \end{aligned} \quad (118)$$

We note that the possibility $\langle \Theta(\pi^{(2)}), \Theta(\pi^{(3)}) \rangle = \langle -, - \rangle$ cannot occur because of the inconsistency of $x_{min}^{+\#} < x_{max}^{-}$ and $x_{max}^{-b} < x_{min}^{+}$, to wit, the contradiction $x_{min} \cdot x_{max} =$

$$x_{min}^{+\#} \cdot x_{max}^{-b} < x_{max}^{-} \cdot x_{min}^{+} = (1 - b^2) x_{min} \cdot x_{max}.$$

At this point we could append the x -constraints of Eq. (118) to their respective versions of $f_{x^\varepsilon, \{j\}}$, $j = (i)', \dots, (viii)'$ and be done, but we wish to accomplish one thing more, as follows. We recall that in Case ^xIa , as given by its multi-subcase specification of f_{x^ε} in Eq. (70), we used the x -constraints dictated by its three possible versions—as specified in Eq. (74)—to organize these three versions of f_{x^ε} into Cases $^x\text{IaR1}$, $^x\text{IaR2}$, and $^x\text{IaR3}$ according to their x -ranges of validity, namely, $x_{\min} \leq x < x_{\max}^-$, $x_{\max}^- \leq x \leq x_{\min}^+$, and $x_{\min}^+ < x \leq x_{\max}^+$, respectively, and thereby a valid version of the Case ^xIa f_{x^ε} was readily available explicitly for every $x \in [x_{\min}, x_{\max}]$. We now wish to do the same here in the present Case ^xII : to organize the eight possible versions, $f_{x^\varepsilon, \{j\}}$, of f_{x^ε} with their accompanying x -constraints for validity, into all possible sub-collections thereof with each such sub-collection having the property that, for every $x \in [x_{\min}, x_{\max}]$, the sub-collection contains one and only one member, say $^x f_{x^\varepsilon, \{j\}}$, that is valid for that x . A necessary condition for the members of any such sub-collection is that the conditions that determined their x -ranges of validity be mutually consistent. (This condition is, however, not sufficient because the members' x -ranges of validity may either overlap or not exhaust all of $[x_{\min}, x_{\max}]$. It turns out that neither of these two conditions will prevail in the sub-collections that we will present below.)

To explore the above necessary condition for sub-collections, we perform—for the three pairs $\pi^{(1)} = \langle x_{\min}^{+\#}, x_{\max}^- \rangle$, $\pi^{(2)} = \langle x_{\min}^{+\#}, x_{\max}^- \rangle$, and $\pi^{(3)} = \langle x_{\max}^-, x_{\min}^+ \rangle$ of Eq. (117), used to elucidate the x -ranges of validity of the $f_{x^\varepsilon, \{j\}}$'s—the procedure performed here several times previously and consider the eight (exhaustive and mutually exclusive) possibilities (hence $J_{\max} = 3$)

$$\begin{aligned}
& \langle \emptyset(\pi^{(1)}), \emptyset(\pi^{(2)}), \emptyset(\pi^{(3)}) \rangle = \\
& \begin{array}{llll}
(\alpha) \langle -, -, - \rangle & (\beta) \langle -, -, + \rangle & (\gamma) \langle -, +, - \rangle & (\delta) \langle -, +, + \rangle \\
(\varepsilon) \langle +, -, - \rangle & (\zeta) \langle +, -, + \rangle & (\eta) \langle +, +, - \rangle & (\theta) \langle +, +, + \rangle.
\end{array}
\end{aligned} \tag{119}$$

These possibilities are tantamount to the following:

$$\begin{aligned}
(\alpha) \quad & x_{min}^{+\#} < x_{max}^{-b} \quad \& \quad x_{min}^{+\#} < x_{max}^{-} \quad \& \quad x_{max}^{-b} < x_{min}^{+} \\
(\beta) \quad & x_{min}^{+\#} < x_{max}^{-b} \quad \& \quad x_{min}^{+\#} < x_{max}^{-} \quad \& \quad x_{max}^{-b} \geq x_{min}^{+} \\
(\gamma) \quad & x_{min}^{+\#} < x_{max}^{-b} \quad \& \quad x_{min}^{+\#} \geq x_{max}^{-} \quad \& \quad x_{max}^{-b} < x_{min}^{+} \\
(\delta) \quad & x_{min}^{+\#} < x_{max}^{-b} \quad \& \quad x_{min}^{+\#} \geq x_{max}^{-} \quad \& \quad x_{max}^{-b} \geq x_{min}^{+} \\
(\varepsilon) \quad & x_{min}^{+\#} \geq x_{max}^{-b} \quad \& \quad x_{min}^{+\#} < x_{max}^{-} \quad \& \quad x_{max}^{-b} < x_{min}^{+} \\
(\zeta) \quad & x_{min}^{+\#} \geq x_{max}^{-b} \quad \& \quad x_{min}^{+\#} < x_{max}^{-} \quad \& \quad x_{max}^{-b} \geq x_{min}^{+} \\
(\eta) \quad & x_{min}^{+\#} \geq x_{max}^{-b} \quad \& \quad x_{min}^{+\#} \geq x_{max}^{-} \quad \& \quad x_{max}^{-b} < x_{min}^{+} \\
(\theta) \quad & x_{min}^{+\#} \geq x_{max}^{-b} \quad \& \quad x_{min}^{+\#} \geq x_{max}^{-} \quad \& \quad x_{max}^{-b} \geq x_{min}^{+}.
\end{aligned} \tag{120}$$

Similarly to our earlier use of this procedure, some of the possibilities (α) through (θ) above are precluded by the inconsistency of their required conditions in the present Case ^xII. Firstly, because of the inconsistency of $x_{min}^{+\#} < x_{max}^{-}$ and $x_{max}^{-b} < x_{min}^{+}$, to wit, the contradiction $x_{min} \cdot x_{max} = x_{min}^{+\#} \cdot x_{max}^{-b} < x_{max}^{-} \cdot x_{min}^{+} = (1 - b^2) x_{min} \cdot x_{max}$, then possibilities (α) and (ε) are to be eliminated. Secondly, because of the inconsistency of $x_{min}^{+\#} < x_{max}^{-b}$ and $x_{max}^{-b} < x_{min}^{+}$, to wit, the contradiction $x_{max}^{-b} < x_{min}^{+} < x_{min}^{+\#} < x_{max}^{-b}$, then possibilities (α) (again) and (γ) are to be eliminated. Lastly, because of the inconsistency of $x_{min}^{+\#} < x_{max}^{-b}$ and $x_{min}^{+\#} \geq x_{max}^{-}$, to wit, the contradiction $x_{min}^{+\#} \geq x_{max}^{-} > x_{max}^{-b} > x_{min}^{+\#}$, then possibilities (γ) (again) and (δ) are to be eliminated. Thus remaining are the four possibilities (β), (ζ), (η), and (θ). The conditions satisfied in these four instances as given by Eq. (120) may be rewritten in a form similar to that of Eq. (36), to wit:

$$\begin{aligned}
\text{For } (\zeta): \quad & x_{min}^{+} \leq x_{max}^{-b} \leq x_{min}^{+\#} < x_{max}^{-} & [\langle \emptyset(\pi^{(1)}), \emptyset(\pi^{(2)}), \emptyset(\pi^{(3)}) \rangle = \langle +, -, + \rangle] \\
\text{For } (\theta): \quad & x_{min}^{+} \leq x_{max}^{-b} \leq x_{min}^{+\#} \quad \& \quad x_{min}^{+\#} \geq x_{max}^{-} & [\langle \emptyset(\pi^{(1)}), \emptyset(\pi^{(2)}), \emptyset(\pi^{(3)}) \rangle = \langle +, +, + \rangle] \\
\text{For } (\eta): \quad & x_{min}^{+} > x_{max}^{-b} \quad \& \quad x_{max}^{-b} \leq x_{min}^{+\#} \quad \& \quad x_{min}^{+\#} \geq x_{max}^{-} & [\langle \emptyset(\pi^{(1)}), \emptyset(\pi^{(2)}), \emptyset(\pi^{(3)}) \rangle = \langle +, +, - \rangle] \\
\text{For } (\beta): \quad & x_{max}^{-} > x_{max}^{-b} > x_{min}^{+\#} > x_{min}^{+} & [\langle \emptyset(\pi^{(1)}), \emptyset(\pi^{(2)}), \emptyset(\pi^{(3)}) \rangle = \langle -, -, + \rangle].
\end{aligned} \tag{121}$$

We thus now identify four subcases of Case ^xII corresponding, respectively, to (ζ), (θ), (η), and (β), namely:

$$\begin{aligned}
 \text{(iia) Case } ^x\text{IIa: } & x_{min}^+ \leq x_{max}^{-b} \leq x_{min}^{+\#} < x_{max}^- \\
 \text{(iib) Case } ^x\text{IIb: } & x_{min}^+ \leq x_{max}^{-b} \leq x_{min}^{+\#} \text{ \& } x_{min}^{+\#} \geq x_{max}^- \\
 \text{(iic) Case } ^x\text{IIc: } & x_{min}^+ > x_{max}^{-b} \text{ \& } x_{max}^{-b} \leq x_{min}^{+\#} \text{ \& } x_{min}^{+\#} \geq x_{max}^- \\
 \text{(iid) Case } ^x\text{IIId: } & x_{max}^- > x_{max}^{-b} > x_{min}^{+\#} > x_{min}^+.
 \end{aligned} \tag{122}$$

We may now coalesce the results of Eqs. (116), (118) and (121) to specify the required sub-collections. For example, for Case ^xIIa, with $\langle \theta(\pi^{(1)}), \theta(\pi^{(2)}), \theta(\pi^{(3)}) \rangle = \langle +, -, + \rangle$, we find from Eqs. (116i'), (116vii') and (116viii'), and Eqs(118ii') - (118vi'), that:

$$f_{x^\zeta}(x) = \left[\begin{array}{ll} f_{x^\zeta, \{i\}}(x) & \text{if } x_{min}^{+\#} < x < x_{max}^{-b} \\ f_{x^\zeta, \{ii\}}(x) & \text{if } x_{min}^{+\#} < x < x_{max}^- \\ f_{x^\zeta, \{iii\}}(x) & \text{if } x_{max}^- \leq x \leq x_{max} \\ f_{x^\zeta, \{iv\}}(x) & \text{if } x_{min}^+ < x < x_{max}^{-b} \\ f_{x^\zeta, \{v\}}(x) & \text{if } x_{min} \leq x \leq x_{min}^+ \\ f_{x^\zeta, \{vi\}}(x) & \text{if } x_{max}^{-b} \leq x \leq x_{min}^{+\#} \\ f_{x^\zeta, \{vii\}}(x) & \text{if } x_{max}^- \leq x \leq x_{min}^{+\#} \\ f_{x^\zeta, \{viii\}}(x) & \text{if } x_{max}^{-b} < x < x_{min}^+. \end{array} \right. \tag{123}$$

But the x-intervals of validity indicated in Eq. (123) for $f_{x^\zeta, \{i\}}(x)$, $f_{x^\zeta, \{vii\}}(x)$, and $f_{x^\zeta, \{viii\}}(x)$ are all void because of the Case ^xIIa condition in Eq. (122ia); further, the remaining five mutually disjoint x-intervals of validity, with union $[x_{min}, x_{max}]$, occur in increasing order (as ordered, for example, by the value of the interval left endpoint) as belonging to $f_{x^\zeta, \{v\}}$,

$f_{x^\zeta, \{iv\}}$, $f_{x^\zeta, \{vi\}}$, $f_{x^\zeta, \{iii\}}$, and $f_{x^\zeta, \{ii\}}$. Hence we have for Case ^xIIa:

For **Case ^xI Ia** [$x_{min} < x_{min}^{\#} < x_{max}^b < x_{max}$ & $x_{min}^+ \leq x_{max}^{-b} \leq x_{min}^{+\#} < x_{max}^-$]:

$$\begin{aligned}
\text{Case } ^x\text{I IaR1 } [x_{min} \leq x \leq x_{min}^+]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{v\}}(x) \\
\text{Case } ^x\text{I IaR2 } [x_{min}^+ < x < x_{max}^{-b}]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{iv\}}(x) \\
\text{Case } ^x\text{I IaR3 } [x_{max}^{-b} \leq x \leq x_{min}^{+\#}]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{vi\}}(x) \\
\text{Case } ^x\text{I IaR4 } [x_{min}^{+\#} < x < x_{max}^-]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{iii\}}(x) \\
\text{Case } ^x\text{I IaR5 } [x_{max}^- \leq x \leq x_{max}]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{iii\}}(x).
\end{aligned} \tag{124}$$

Now in this manner we may further combine Eqs. (90), (92) and (95) to find:

For **Case ^xI Ib** [$x_{min} < x_{min}^{\#} < x_{max}^b < x_{max}$ & $x_{min}^+ \leq x_{max}^{-b} \leq x_{min}^{+\#}$ & $x_{min}^{+\#} \geq x_{max}^-$]:

$$\begin{aligned}
\text{Case } ^x\text{I IbR1 } [x_{min} \leq x \leq x_{min}^+]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{v\}}(x) \\
\text{Case } ^x\text{I IbR2 } [x_{min}^+ < x < x_{max}^{-b}]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{iv\}}(x) \\
\text{Case } ^x\text{I IbR3 } [x_{max}^{-b} \leq x < x_{max}^-]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{vi\}}(x) \\
\text{Case } ^x\text{I IbR4 } [x_{max}^- \leq x \leq x_{min}^{+\#}]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{vii\}}(x) \\
\text{Case } ^x\text{I IbR5 } [x_{min}^{+\#} < x \leq x_{max}]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{iii\}}(x);
\end{aligned} \tag{125}$$

For **Case ^xI Ic** [$x_{min} < x_{min}^{\#} < x_{max}^b < x_{max}$ & $x_{min}^+ > x_{max}^{-b}$ & $x_{max}^{-b} \leq x_{min}^{+\#}$ & $x_{min}^{+\#} \geq x_{max}^-$]:

$$\begin{aligned}
\text{Case } ^x\text{I IcR1 } [x_{min} \leq x < x_{max}^{-b}]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{v\}}(x) \\
\text{Case } ^x\text{I IcR2 } [x_{max}^{-b} \leq x \leq x_{min}^+]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{viii\}}(x) \\
\text{Case } ^x\text{I IcR3 } [x_{min}^+ < x < x_{max}^-]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{vi\}}(x) \\
\text{Case } ^x\text{I IcR4 } [x_{max}^- \leq x \leq x_{min}^{+\#}]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{vii\}}(x) \\
\text{Case } ^x\text{I IcR5 } [x_{min}^{+\#} < x \leq x_{max}]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{iii\}}(x);
\end{aligned} \tag{126}$$

For **Case ^xI Id** [$x_{min} < x_{min}^{\#} < x_{max}^b < x_{max}$ & $x_{max}^- > x_{max}^{-b} > x_{min}^{+\#} > x_{min}^+$]:

$$\begin{aligned}
\text{Case } ^x\text{I IdR1 } [x_{min} \leq x \leq x_{min}^+]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{v\}}(x) \\
\text{Case } ^x\text{I IdR2 } [x_{min}^+ < x \leq x_{min}^{+\#}]: \quad & f_{X^\xi}(x) = f_{X^\xi, \{iv\}}(x)
\end{aligned}$$

$$\text{Case } ^x\text{IIdR3 } [x_{min}^{+\#} < x < x_{max}^{-b}]: \quad f_{x^\xi}(x) = f_{x^\xi, \{i\}}(x) \quad (127)$$

$$\text{Case } ^x\text{IIdR4 } [x_{max}^{-b} \leq x < x_{max}^-]: \quad f_{x^\xi}(x) = f_{x^\xi, \{iii\}}(x)$$

$$\text{Case } ^x\text{IIdR5 } [x_{max}^- \leq x \leq x_{max}]: \quad f_{x^\xi}(x) = f_{x^\xi, \{iii\}}(x).$$

We also collect here for convenience the Case ^xI results:

For **Case ^xIa** $[x_{min} < x_{max}^b \leq x_{min}^\# < x_{max}]$:

$$\text{Case } ^x\text{IaR1 } [x_{min} \leq x < x_{max}^-]: \quad f_{x^\xi}(x) = f_{x^\xi, \{1\}}(x) \quad [\text{Eq. (79)}]$$

$$\text{Case } ^x\text{IaR2 } [x_{max}^- \leq x \leq x_{min}^+]: \quad f_{x^\xi}(x) = f_{x^\xi, \{2\}}(x) \quad [\text{Eq. (81)}] \quad (128)$$

$$\text{Case } ^x\text{IaR3 } [x_{min}^+ < x \leq x_{max}]: \quad f_{x^\xi}(x) = f_{x^\xi, \{3\}}(x); \quad [\text{Eq. (83)}]$$

For **Case ^xIb** $[x_{min} < x_{max}^b \leq x_{min}^\# \text{ \& } x_{min}^\# \geq x_{max}]$:

$$\text{Case } ^x\text{IbR1 } [x_{min} \leq x \leq x_{min}^+]: \quad f_{x^\xi}(x) = f_{x^\xi, \{4\}}(x) \quad [\text{Eq. (86)}] \quad (129)$$

$$\text{Case } ^x\text{IbR2 } [x_{min}^+ < x \leq x_{max}]: \quad f_{x^\xi}(x) = f_{x^\xi, \{5\}}(x); \quad [\text{Eq. (88)}]$$

For **Case ^xIc** $[x_{min} \geq x_{max}^b \text{ \& } x_{max}^b \leq x_{min}^\# \text{ \& } x_{min}^\# \geq x_{max}]$:

$$\text{Case } ^x\text{IcR1 } [x_{min} \leq x \leq x_{max}]: \quad f_{x^\xi}(x) = f_{x^\xi, \{6\}}(x). \quad [\text{Eq. (90)}] \quad (130)$$

The equations above for Cases ^xI and ^xII merely express the piecewise-defined function $f_{x^\xi}(x)$ of the two variables ξ, x for their respective cases. For example, in Fig. 2 we illustrate the “patchwork” domain of $f_{x^\xi}(x)$ for Case ^xIa .

In summary, for any given $x \in \mathcal{V}_{x^*}$, the determination of $f_{x^\xi}(x)$ for all $\xi \in \mathcal{V}_x$ requires a family consisting of one through five distinct PDF segments, as per Eqs. (105) – (113), with the particular such family as—specified *via* Eqs. (124) – (130)—based solely upon the (fixed) values of x_{min} , x_{max} , and b as well as upon the specific value of x . There are two main cases that govern this variation, denoted by Case ^xI and Case ^xII . Case ^xI may be partitioned into three subcases, namely ^xIa , ^xIb , and ^xIc , these differing from each other only by their specific requirements on x_{min} , x_{max} , and b , with Case ^xII differing from these

as well only by its specific requirements on x_{min} , x_{max} , and b , with all constraints as specified in Eq. (36). Further, (sub)Case ^xIa itself has three sub-subcases differing from each other only by their x -ranges of validity, while (sub)Case ^xIb itself has two sub-subcases, again differing from each other only by their x -ranges of validity, and (sub)Case ^xIc has no sub-subcases. Case ^xII , in addition to its requirements on x_{min} , x_{max} , and b as per Eq. (36), may be partitioned into four subcases, namely ^xIIa , ^xIIb , ^xIIc , and $^xII d$, with these differing from each other only by *yet additional specific requirements* on x_{min} , x_{max} , and b_1 , as given by Eq. (122). Further, each of these subcases itself has five sub-subcases, with the five sub-subcases of each given subcase differing from each other only by their x -ranges of validity,. All these results are written out immediately above in the paragraph preceding this present one.

This concludes our development of the results that specify the applicable PDF segments of $f_{x^\xi}^\alpha(x)$, $\alpha = A, B, C, D$, for each *fixed* $x \in \mathcal{V}_{x^*}$, as functions of *variable* $\xi \in \mathcal{V}_x$. We now restate the most important results of this analysis—Eqs. (17), (80), (82), (84), (87), (89), (91), (105) – (111), and (124) – (130)—in terms of $f_{x^{g(w)}}(x)$ (i.e., ξ is being supplanted by $g(w) = \xi$), specifying $f_{x^{g(w)}}(x)$ as a function of variable $w \in \mathcal{V}_{w^*}$ for each fixed $x \in \mathcal{V}_{x^*}$. Firstly, note that *the results of Eq. (17) have already been previously restated for our present purpose in Eq. (52)*. Before re-writing the remaining results, however, it will be convenient in what follows to unify the numbering of the functions in the set $\{f_{x^\xi, \{j\}}(x)\}_{j=1}^6$, $\{f_{x^\xi, \{n\}}(y)\}_{n=i}^{viii}$ of Eqs. (80), (82), (84), (87), (89), (91), (105) – (111) and write for that set rather $\{f_{x^\xi}^{[\nu]}(x)\}_{\nu=1}^{14}$, with the ν index being assigned to the elements $f_{x^\xi}^{[\nu]}(x)$ in the same order in which the functions appear in the former set; i.e., $f_{x^\xi}^{[1]}(x) = f_{x^\xi, \{1\}}(x)$, ..., $f_{x^\xi}^{[6]}(x) = f_{x^\xi, \{6\}}(x)$, $f_{x^\xi}^{[7]}(x) = f_{x^\xi, \{i\}}(x)$, ..., $f_{x^\xi}^{[14]}(x) = f_{x^\xi, \{viii\}}(x)$. For convenience, we write the correspondence between the Roman numeral indices and the ν indices here:

$$\begin{array}{lll}
 (i)' \rightarrow 7 & (iv)' \rightarrow 10 & (vi)' \rightarrow 12 \\
 (ii)' \rightarrow 8 & & (vii)' \rightarrow 13 \\
 (iii)' \rightarrow 9 & (v)' \rightarrow 11 & (viii)' \rightarrow 14.
 \end{array} \tag{131}$$

Thus we have *in toto* the functions $f_{x^{\xi}}^{[v]}(x)$ for $v = 1, \dots, 14$. The re-stated results referred to previously, employing the $f_{x^{g(w)}}^{[v]}(x)$ in lieu of the $f_{x^{\xi}}^{[v]}(x)$, are then as follows for every $x \in \mathcal{V}_{x^*}$:

For Eqs. (80), (82), (84), (87), (89), (91), (105) – (111) we have:

$$f_{x^{g(w)}}^{[1]}(x) = \begin{cases} f_{x^{g(w)}}^{C,1}(x) & \text{if } w \in g^{-1}[x_{min}^b, x_{max}^b)] \\ f_{x^{g(w)}}^{D,1}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{min}^{\#}]] \\ f_{x^{g(w)}}^{B,2}(x) & \text{if } w \in g^{-1}[x_{min}^{\#}, x_{max}^{\#}]] \\ f_{x^{g(w)}}^{B,1}(x) & \text{if } w \in g^{-1}[x_{max}^{\#}, x_{max}]]; \end{cases} \quad (132)$$

$$f_{x^{g(w)}}^{[2]}(x) = \begin{cases} f_{x^{g(w)}}^{C,1}(x) & \text{if } w \in g^{-1}[x_{min}^b, x_{max}^b)] \\ f_{x^{g(w)}}^{D,1}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{min}^{\#}]] \\ f_{x^{g(w)}}^{B,2}(x) & \text{if } w \in g^{-1}[x_{min}^{\#}, x_{max}]]; \end{cases} \quad (133)$$

$$f_{x^{g(w)}}^{[3]}(x) = \begin{cases} f_{x^{g(w)}}^{C,2}(x) & \text{if } w \in g^{-1}[x_{min}, x^b)] \\ f_{x^{g(w)}}^{C,1}(x) & \text{if } w \in g^{-1}[x^b, x_{max}^b)] \\ f_{x^{g(w)}}^{D,1}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{min}^{\#}]] \\ f_{x^{g(w)}}^{B,2}(x) & \text{if } w \in g^{-1}[x_{min}^{\#}, x_{max}]]; \end{cases} \quad (134)$$

$$f_{x^{g(w)}}^{[4]}(x) = \begin{cases} f_{x^{g(w)}}^{C,1}(x) & \text{if } w \in g^{-1}[x_{min}^b, x_{max}^b)] \\ f_{x^{g(w)}}^{D,1}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{max}]]; \end{cases} \quad (135)$$

$$f_{x^{g(w)}}^{[5]}(x) = \begin{cases} f_{x^{g(w)}}^{C,2}(x) & \text{if } w \in g^{-1}[x_{min}, x^b)] \\ f_{x^{g(w)}}^{C,1}(x) & \text{if } w \in g^{-1}[x^b, x_{max}^b)] \\ f_{x^{g(w)}}^{D,1}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{max}]]; \end{cases} \quad (136)$$

$$f_{xg(w)}^{[6]}(x) = f_{xg(w)}^{D,1}(x) \quad \text{if } w \in g^{-1}[x_{min}, x_{max}]; \quad (137)$$

$$f_{xg(w)}^{[7]}(x) = \begin{cases} f_{xg(w)}^{C,2}(x) & \text{if } w \in g^{-1}[x_{min}, x_{min}^{\#}] \\ f_{xg(w)}^{A,3}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x^b) \\ f_{xg(w)}^{A,2}(x) & \text{if } w \in g^{-1}[x^b, x^{\#}] \\ f_{xg(w)}^{A,1}(x) & \text{if } w \in g^{-1}(x^{\#}, x_{max}^b) \\ f_{xg(w)}^{B,1}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{max}]; \end{cases} \quad (138)$$

$$f_{xg(w)}^{[8]}(x) = \begin{cases} f_{xg(w)}^{C,2}(x) & \text{if } w \in g^{-1}[x_{min}, x_{min}^{\#}] \\ f_{xg(w)}^{A,3}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x^b) \\ f_{xg(w)}^{A,2}(x) & \text{if } w \in g^{-1}[x^b, x_{max}^b] \\ f_{xg(w)}^{B,2}(x) & \text{if } w \in g^{-1}[x_{max}^b, x^{\#}] \\ f_{xg(w)}^{B,1}(x) & \text{if } w \in g^{-1}(x^{\#}, x_{max}]; \end{cases} \quad (139)$$

$$f_{xg(w)}^{[9]}(x) = \begin{cases} f_{xg(w)}^{C,2}(x) & \text{if } w \in g^{-1}[x_{min}, x_{min}^{\#}] \\ f_{xg(w)}^{A,3}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x^b) \\ f_{xg(w)}^{A,2}(x) & \text{if } w \in g^{-1}[x^b, x_{max}^b] \\ f_{xg(w)}^{B,2}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{max}]; \end{cases} \quad (140)$$

$$f_{xg(w)}^{[10]}(x) = \begin{cases} f_{xg(w)}^{C,2}(x) & \text{if } w \in g^{-1}[x_{min}, x^b] \\ f_{xg(w)}^{C,1}(x) & \text{if } w \in g^{-1}[x^b, x_{min}^{\#}] \\ f_{xg(w)}^{A,2}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x^{\#}) \\ f_{xg(w)}^{A,1}(x) & \text{if } w \in g^{-1}(x^{\#}, x_{max}^b) \\ f_{xg(w)}^{B,1}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{max}]; \end{cases} \quad (141)$$

$$f_{xg(w)}^{[11]}(x) = \begin{cases} f_{xg(w)}^{C,1}(x) & \text{if } w \in g^{-1}[x_{min}, x_{min}^{\#}] \\ f_{xg(w)}^{A,2}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x_{min}^{\#}] \\ f_{xg(w)}^{A,1}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x_{max}^b) \\ f_{xg(w)}^{B,1}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{max}]; \end{cases} \quad (142)$$

$$f_{xg(w)}^{[12]}(x) = \begin{cases} f_{xg(w)}^{C,2}(x) & \text{if } w \in g^{-1}[x_{min}, x_{min}^b) \\ f_{xg(w)}^{C,1}(x) & \text{if } w \in g^{-1}[x_{min}^b, x_{min}^{\#}] \\ f_{xg(w)}^{A,2}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x_{max}^b) \\ f_{xg(w)}^{B,2}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{min}^{\#}] \\ f_{xg(w)}^{B,1}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x_{max}]; \end{cases} \quad (143)$$

$$f_{xg(w)}^{[13]}(x) = \begin{cases} f_{xg(w)}^{C,2}(x) & \text{if } w \in g^{-1}[x_{min}, x_{min}^b) \\ f_{xg(w)}^{C,1}(x) & \text{if } w \in g^{-1}[x_{min}^b, x_{min}^{\#}] \\ f_{xg(w)}^{A,2}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x_{max}^b) \\ f_{xg(w)}^{B,2}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{max}]; \end{cases} \quad (144)$$

$$f_{xg(w)}^{[14]}(x) = \begin{cases} f_{xg(w)}^{C,1}(x) & \text{if } w \in g^{-1}[x_{min}, x_{min}^{\#}] \\ f_{xg(w)}^{A,2}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x_{max}^b) \\ f_{xg(w)}^{B,2}(x) & \text{if } w \in g^{-1}[x_{max}^b, x_{min}^{\#}] \\ f_{xg(w)}^{B,1}(x) & \text{if } w \in g^{-1}(x_{min}^{\#}, x_{max}]. \end{cases} \quad (145)$$

And for Eqs. (124) – (130) we have:

Case ^xIa [$x_{min} < x_{max}^b \leq x_{min}^\# < x_{max}$]:

$$\text{Case } ^x\text{IaR1 } [x_{min} \leq x < x_{max}^-]: f_{xg(w)}(x) = f_{xg(w)}^{[1]}(x) \quad (146)$$

$$\text{Case } ^x\text{IaR2 } [x_{max}^- \leq x \leq x_{min}^+]: f_{xg(w)}(x) = f_{xg(w)}^{[2]}(x) \quad (147)$$

$$\text{Case } ^x\text{IaR3 } [x_{min}^+ < x \leq x_{max}]: f_{xg(w)}(x) = f_{xg(w)}^{[3]}(x); \quad (148)$$

Case ^xIb [$x_{min} < x_{max}^b \leq x_{min}^\#$ & $x_{min}^\# \geq x_{max}$]:

$$\text{Case } ^x\text{IbR1 } [x_{min} \leq x \leq x_{min}^+]: f_{xg(w)}(x) = f_{xg(w)}^{[4]}(x) \quad (149)$$

$$\text{Case } ^x\text{IbR2 } [x_{min}^+ < x \leq x_{max}]: f_{xg(w)}(x) = f_{xg(w)}^{[5]}(x); \quad (150)$$

Case ^xIc [$x_{min} \geq x_{max}^b$ & $x_{max}^b \leq x_{min}^\#$ & $x_{min}^\# \geq x_{max}$]:

$$\text{Case } ^x\text{IcR1 } [x_{min} \leq x \leq x_{max}]: f_{xg(w)}(x) = f_{xg(w)}^{[6]}(x); \quad (151)$$

Case ^xIla [$x_{min} < x_{min}^\# < x_{max}^b < x_{max}$ & $x_{min}^+ \leq x_{max}^{-b} \leq x_{min}^{+\#} < x_{max}^-$]:

$$\text{Case } ^x\text{IlaR1 } [x_{min} \leq x \leq x_{min}^+]: f_{xg(w)}(x) = f_{xg(w)}^{[11]}(x) \quad (152)$$

$$\text{Case } ^x\text{IlaR2 } [x_{min}^+ < x < x_{max}^{-b}]: f_{xg(w)}(x) = f_{xg(w)}^{[10]}(x) \quad (153)$$

$$\text{Case } ^x\text{IlaR3 } [x_{max}^{-b} \leq x \leq x_{min}^{+\#}]: f_{xg(w)}(x) = f_{xg(w)}^{[12]}(x) \quad (154)$$

$$\text{Case } ^x\text{IlaR4 } [x_{min}^{+\#} < x < x_{max}^-]: f_{xg(w)}(x) = f_{xg(w)}^{[8]}(x) \quad (155)$$

$$\text{Case } ^x\text{IlaR5 } [x_{max}^- \leq x \leq x_{max}]: f_{xg(w)}(x) = f_{xg(w)}^{[9]}(x); \quad (156)$$

Case ^xIib [$x_{min} < x_{min}^\# < x_{max}^b < x_{max}$ & $x_{min}^+ \leq x_{max}^{-b} \leq x_{min}^{+\#}$ & $x_{min}^{+\#} \geq x_{max}^-$]:

$$\text{Case } ^x\text{IibR1 } [x_{min} \leq x \leq x_{min}^+]: f_{xg(w)}(x) = f_{xg(w)}^{[11]}(x) \quad (157)$$

$$\text{Case } ^x\text{IibR2 } [x_{min}^+ < x < x_{max}^{-b}]: f_{xg(w)}(x) = f_{xg(w)}^{[10]}(x) \quad (158)$$

$$\text{Case } ^x\text{IibR3 } [x_{max}^{-b} \leq x < x_{max}^-]: f_{xg(w)}(x) = f_{xg(w)}^{[12]}(x) \quad (159)$$

$$\text{Case } ^x\text{IibR4 } [x_{max}^- \leq x \leq x_{min}^{+\#}]: f_{xg(w)}(x) = f_{xg(w)}^{[13]}(x) \quad (160)$$

$$\text{Case } ^x\text{IibR5 } [x_{min}^{+\#} < x \leq x_{max}]: f_{xg(w)}(x) = f_{xg(w)}^{[9]}(x); \quad (161)$$

Case x_{IIC} [$x_{min} < x_{min}^{\#} < x_{max}^b < x_{max}$ & $x_{min}^+ > x_{max}^{-b}$ & $x_{max}^{-b} \leq x_{min}^{+\#}$ & $x_{min}^{+\#} \geq x_{max}^-$]:

$$\text{Case } x_{IICR1} [x_{min} \leq x < x_{max}^{-b}]: f_{xg(w)}(x) = f_{xg(w)}^{[11]}(x) \quad (162)$$

$$\text{Case } x_{IICR2} [x_{max}^{-b} \leq x \leq x_{min}^+]: f_{xg(w)}(x) = f_{xg(w)}^{[14]}(x) \quad (163)$$

$$\text{Case } x_{IICR3} [x_{min}^+ < x < x_{max}^-]: f_{xg(w)}(x) = f_{xg(w)}^{[12]}(x) \quad (164)$$

$$\text{Case } x_{IICR4} [x_{max}^- \leq x \leq x_{min}^{+\#}]: f_{xg(w)}(x) = f_{xg(w)}^{[13]}(x) \quad (165)$$

$$\text{Case } x_{IICR5} [x_{min}^{+\#} < x \leq x_{max}]: f_{xg(w)}(x) = f_{xg(w)}^{[9]}(x); \quad (166)$$

Case x_{IID} [$x_{min} < x_{min}^{\#} < x_{max}^b < x_{max}$ & $x_{max}^- > x_{max}^{-b} > x_{min}^{+\#} > x_{min}^+$]:

$$\text{Case } x_{IIDR1} [x_{min} \leq x \leq x_{min}^+]: f_{xg(w)}(x) = f_{xg(w)}^{[11]}(x) \quad (167)$$

$$\text{Case } x_{IIDR2} [x_{min}^+ < x \leq x_{min}^{+\#}]: f_{xg(w)}(x) = f_{xg(w)}^{[10]}(x) \quad (168)$$

$$\text{Case } x_{IIDR3} [x_{min}^{+\#} < x < x_{max}^{-b}]: f_{xg(w)}(x) = f_{xg(w)}^{[7]}(x) \quad (169)$$

$$\text{Case } x_{IIDR4} [x_{max}^{-b} \leq x < x_{max}^-]: f_{xg(w)}(x) = f_{xg(w)}^{[8]}(x) \quad (170)$$

$$\text{Case } x_{IIDR5} [x_{max}^- \leq x \leq x_{max}]: f_{xg(w)}(x) = f_{xg(w)}^{[9]}(x). \quad (171)$$

We have now completed the explicit specification of the factor $f_{xg(w)}(x)$ (more fully, of the factor $f_{x_n g_n(w_n)}(x_n)$) that appears in the second integral of Eq. (41) for PDF $f_{\hat{x}_n}(\omega_n, f_{w_n \omega_n}, g_n; x_n)$. We next want to make some observations concerning the integration interval $[w_{min}, w_{max}]$ in Eq. (41); also, we put a further restriction on the function g (more fully, on the functions g_n). We first introduce some further notation to be used in this regard. We denote the individual PDF segments of each $f_{xg(w)}^{[v]}(x)$, $v = 1, \dots, 14$, by an additional superscript thus: $f_{xg(w)}^{[v], k}(x)$. For example, we write, from Eq. (132), $f_{xg(w)}^{[1], 1}(x) = f_{xg(w)}^{C, 1}(x)$ and $f_{xg(w)}^{[1], 2}(x) = f_{xg(w)}^{D, 1}(x)$. We thus also write

$$f_{xg(w)}^{[v], k}(x) = f_{xg(w)}^{\alpha', k'}(x) \quad (172)$$

where $\alpha' = \alpha'(v, k)$ and $k' = k'(v, k)$ so that, for example, $\alpha'(1, 1) = C$ and $k'(1, 1) = 1$.

Consider then the behavior of the function $f_{x^{g(\bullet)}}^{[v]}(x)$ on $[w_{min}, w_{max}]$. Firstly, $f_{x^{g(\bullet)}}^{[v]}(x)$ is defined—via one of Eqs. (132) – (145)—by employing one to five of the PDF segments $f_{x^{g(w)}}^{\alpha', k'}(x)$, with that number of segments—denoted here by $\#(v)$ —depending upon the value of v . From Eqs. (132) – (145) we have

$$\#(v) = \begin{cases} 5 & \text{if } v(y) = 7, 8, 10, 12 \\ 4 & \text{if } v(y) = 1, 3, 9, 11, 13, 14 \\ 3 & \text{if } v(y) = 2, 5 \\ 2 & \text{if } v(y) = 4 \\ 1 & \text{if } v(y) = 6. \end{cases} \quad (173)$$

Secondly, the domains of each of the segments of each $f_{x^{g(\bullet)}}^{[v]}(x)$ are of the form $g^{-1}[\mathcal{H}(x_L, x_R)]$, where $\mathcal{H}(x_L, x_R)$ is some open, closed or semi-closed interval in $[x_{min}, x_{max}]$, with $\mathcal{H}(x_L, x_R)$ having left endpoint x_L and right endpoint x_R . We label the domains of the segments of $f_{x^{g(\bullet)}}^{[v]}(x)$ by $g^{-1}[\mathcal{H}_k^{[v]}(x)]$, $k = 1, \dots, \#(v)$, with k increasing according to the top-to-bottom order indicated in the applicable equation of Eqs. (132) – (145) for $f_{x^{g(\bullet)}}^{[v]}(x)$ (e.g., if $v = 1$ then Eq. (132) is the applicable one). In Table I (based upon Eqs. (132) – (145)) we give the $\mathcal{H}_k^{[v]}(x)$'s and their associated $f_{x^{g(w)}}^{\alpha'(v, k), k'(v, k)}(x)$'s for $v = 1, \dots, 14$. We note that it is possible that $\mathcal{H}_k^{[v]}(x)$ is void for some v, k and x so that $g^{-1}[\mathcal{H}_k^{[v]}(x)]$ is likewise void. For example, when $v = 1, k = 3$, and $x = x_{min}$ (as may occur in Case $^y\text{IaR1}$) then $\mathcal{H}_3^{[1]}(x_{min}) = (x_{min}^\#, x_{min}^\#] = \emptyset$. It is also possible that $g^{-1}[\mathcal{H}_k^{[v]}(x)]$ is void even if $\mathcal{H}_k^{[v]}(x)$ is not. This situation can arise as a consequence of our allowing $\mathcal{V}_x \subseteq \mathcal{V}_{x^*}$: for if $x_{min} < \xi_{min}^g$ then $g^{-1}[x_{min}, \xi_{min}^g] = \emptyset$.

Now while for each $k = 1, \dots, \#(v)$, the set $g^{-1}[\mathcal{H}_k^{[v]}(x)]$ is a union of mutually disjoint subintervals of $[w_{min}, w_{max}]$ —since g is continuous on its domain $[w_{min}, w_{max}]$ and $\mathcal{H}_k^{[v]}(x)$ is an interval in $[x_{min}, x_{max}]$ —we will *further assume* that each such nonvoid $g^{-1}[\mathcal{H}_k^{[v]}(x)]$ is a *finite* union of mutually disjoint nonvoid intervals; from a *practical* point of view we see no

loss in generality by making this assumption. (As examples, the function g^β defined on $[w_{min}, w_{max}]$ by $g^\beta(w) = (3/2) + \sin(w)$ satisfies this finiteness condition while the function g^γ , defined on $[w_{min}, w_{max}]$ by $g^\gamma(w) = (3/2) + (w - w_{min})\sin[1/(w - w_{min})]$ if $w > w_{min}$ and $g^\gamma(w) = 0$ if $w = w_{min}$, is continuous yet does not satisfy this condition since $(g^\gamma)^{-1}[[3/2, 3/2]]$ is an infinite union of intervals). So each nonvoid $g^{-1}[\mathcal{H}_k^{[v]}(x)]$ is itself a finite union of mutually disjoint nonvoid open, closed or semi-closed intervals in $[w_{min}, w_{max}]$ and, further, $\{g^{-1}[\mathcal{H}_k^{[v]}(x)]\}_{k=1}^{\#(v)}$ is a finite partition of $[w_{min}, w_{max}]$ (we allow $\emptyset$ to be a member of the partition); we will refer to this condition on g as “ g has a finitely partitionable domain”. We denote by $\#(v, k, g; x)$ the total number of non-void disjoint subintervals comprised by nonvoid $g^{-1}[\mathcal{H}_k^{[v]}(x)]$ and we denote the w -coordinates of the left and right endpoints of these non-void disjoint subintervals comprised by $g^{-1}[\mathcal{H}_k^{[v]}(x)]$ as

$$w_{1, \mathcal{L}}^{[v], k}(g; x) < \dots < w_{\#(v, k, g; x), \mathcal{L}}^{[v], k}(g; x) \quad \& \quad w_{1, \mathcal{R}}^{[v], k}(g; x) < \dots < w_{\#(v, k, g; x), \mathcal{R}}^{[v], k}(g; x),$$

$$k = 1, \dots, \#(v) \quad (174a)$$

with

$$w_{\ell, \mathcal{R}}^{[v], k}(g; x) \leq w_{\ell+1, \mathcal{L}}^{[v], k}(g; x), \quad \ell = 1, \dots, \#(v, k, g; x) - 1. \quad (174b)$$

In the above, $w_{1, \mathcal{L}}^{[v], k}(g; x) = w_{min}$ for precisely one k and $w_{\#(v, k, g; x), \mathcal{R}}^{[v], k}(g; x) = w_{max}$ for precisely one (in general differing) k ; further, when $g^{-1}[\mathcal{H}_k^{[v]}(x)] = \emptyset$ then $\#(v, k, g; x) = 0$. Denoting

$$I_\ell^{[v], k}(g; x) = [w_{\ell, \mathcal{L}}^{[v], k}(g; x), w_{\ell, \mathcal{R}}^{[v], k}(g; x)] \subseteq [w_{min}, w_{max}],$$

$$k = 1, \dots, \#(v), \ell = 1, \dots, \#(v, k, g; x), \quad (175)$$

then the set of intervals $\{I_\ell^{[v], k}(g; x)\}_{\ell=1}^{\#(v, k, g; x)}_{k=1}^{\#(v)}$ also forms a partition of $[w_{min}, w_{max}]$, i.e.,

$$[w_{min}, w_{max}] = \bigcup_{k=1}^{\#(v)} \bigcup_{\ell=1}^{\#(v, k, g; x)} I_\ell^{[v], k}(g; x), \quad (176)$$

and

$$f_{\chi g(w)}^{[v]}(x) = f_{\chi g(w)}^{[v], k}(x) = f_{\chi g(w)}^{\alpha'(v, k), k'(v, k)}(x) \quad \text{if } w \in I_\ell^{[v], k}(g; x),$$

$$k = 1, \dots, \#(v), \ell = 1, \dots, \#(v, k, g; x). \quad (177)$$

Strictly speaking, this last equality may not hold true at the endpoints $w_{\ell, \mathcal{L}}^{[v], k}(g; x)$ and

$w_{\ell, \mathbb{R}}^{[v], k}(g; x)$ of $I_{\ell}^{[v], k}(g; x)$; but since $f_{xg(w)}^{[v], k}(x)$ will be used only as a factor in integrands then such discrepancies at the (finite number of) interval endpoints will not affect the values of the integrals thereof; we may likewise omit—as we have done—the endpoint w_{max} and consider only $[w_{min}, w_{max})$.

We are now prepared to fully evaluate the integral in Eq. (41) for PDF $f_{\hat{x}_n}(\omega_n, f_{w_n \omega_n}, g_n; x_n)$ by applying the results of this section; we do that next.

II.4.2 Evaluation of the Integral

The integral to be evaluated in Eq. (41) is $\int_{w_{n, min}}^{w_{n, max}} f_{w_n \omega_n}(w_n) f_{x_n g_n(w_n)}(x_n) dw_n$ for all

$x_n \in \mathcal{V}_{x_n^*}$. Continuing our practice of suppressing all subscripts and superscripts “ n ”, we

then write this simply as $\int_{w_{min}}^{w_{max}} f_{w \omega}(w) f_{xg(w)}(x) dw$. Now according to the results in the

previous section, Eqs. (146) – (171) tell us that $f_{xg(w)}(x) = f_{xg(w)}^{[v]}(x)$ for some $v = 1, \dots, 14$, with the particular value of v dictated by the value of $x \in \mathcal{V}_{x^*}$; we acknowledge the x -dependence of v by writing $v(x)$ in lieu of simply v and therefore write the integral as

$$\mathcal{J}^{v(x)}(f_{w \omega}, g; x) \equiv \int_{w_{min}}^{w_{max}} f_{w \omega}(w) f_{xg(w)}^{[v(x)]}(x) dw. \quad (178)$$

Further, from Eqs. (175) - (177), we have for this integral

$$\mathcal{J}^{v(x)}(f_{w \omega}, g; x) = \sum_{k=1}^{\#(v(x))} \sum_{\ell=1}^{\#(v(x), k, g; x)} \int_{w_{\ell}^{[v(x)], k}(g; x)}^{w_{\ell}^{[v(x)], k}(g; x)} f_{w \omega}(w) f_{xg(w)}^{\alpha'(v(x), k), k'(v(x), k)}(x) dw \quad (179)$$

where $\sum_{\ell=1}^{\#(v(x), k, g; x)}$ is not performed when $\#(v(x), k, g; x) = 0$.

The *general* evaluation of $\mathcal{J}^{v(x)}(f_{w \omega}, g; x)$ is now complete; hence so is the general evaluation of the integral in Eq. (41) since $f_{\hat{x}_n}(\omega_n, f_{w_n \omega_n}, g_n; x_n) = \mathcal{J}^{v(x_n)}(f_{w_n \omega_n}, g_n; x_n)$.

II.4.3 Computational Considerations

The general result for $f_{\hat{x}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n)$ given in Eq. (179) notwithstanding, it is nevertheless desirable to further express that integral in a form useful for computation. We first note that there are eight possibilities for $f_{x^{g(w)}}^{\alpha'(v, k), k'(v, k)}(x)$, namely $f_{x^{g(w)}}^{A,1}(x)$, $f_{x^{g(w)}}^{A,2}(x)$, $f_{x^{g(w)}}^{A,3}(x)$, $f_{x^{g(w)}}^{B,1}(x)$, $f_{x^{g(w)}}^{B,2}(x)$, $f_{x^{g(w)}}^{C,1}(x)$, $f_{x^{g(w)}}^{C,2}(x)$, $f_{x^{g(w)}}^{D,1}(x)$. However, a perusal of Eq. (52) reveals that there are only four differing integral forms, along with one integral family, that appear in these eight integrals, namely, the integrals

$$\mathcal{J}_1[f_{W^\omega}; \lambda_1, \lambda_2] = \int_{\lambda_1}^{\lambda_2} dw f_{W^\omega}(w) \quad \text{for } \langle \alpha', k' \rangle = \langle D, 1 \rangle$$

$$\mathcal{J}_2[f_{W^\omega}, g; \lambda_1, \lambda_2] = \int_{\lambda_1}^{\lambda_2} dw f_{W^\omega}(w)/g(w) \quad \text{for } \langle \alpha', k' \rangle = \langle A, 2 \rangle \quad (180a)$$

$$\mathcal{J}_3[f_{W^\omega}, g; E; \lambda_1, \lambda_2] = \int_{\lambda_1}^{\lambda_2} dw f_{W^\omega}(w)/[g(w) - E] \quad \text{for } \langle \alpha', k' \rangle = \langle B, 2 \rangle, \langle C, 1 \rangle$$

$$\mathcal{J}_4[f_{W^\omega}, g; E; \lambda_1, \lambda_2] = \int_{\lambda_1}^{\lambda_2} dw f_{W^\omega}(w)/[g(w) - E]^2 \quad \text{for } \langle \alpha', k' \rangle = \langle A, 3 \rangle, \langle B, 1 \rangle, \langle C, 2 \rangle$$

and the integral family

$$\mathcal{J}_5[f_{W^\omega}, g, \phi_X; E; \lambda_1, \lambda_2] = \int_{\lambda_1}^{\lambda_2} dw f_{W^\omega}(w) \{ \phi_X(g(w))/[g(w) - E]^2 \} \quad \text{for } \langle \alpha', k' \rangle = \langle A, 1 \rangle, \langle A, 3 \rangle, \quad (180b)$$

where, from Eq. (52), E is the constant

$$E = \begin{cases} x_{min}^b & \text{if } \langle \alpha', k' \rangle = \langle C, 1 \rangle, \\ x_{min}^\# & \text{if } \langle \alpha', k' \rangle = \langle A, 1 \rangle \text{ or } \langle B, 1 \rangle \\ x_{max}^b & \text{if } \langle \alpha', k' \rangle = \langle A, 3 \rangle \text{ or } \langle C, 2 \rangle \\ x_{max}^\# & \text{if } \langle \alpha', k' \rangle = \langle B, 2 \rangle \end{cases} \quad (181)$$

and, from Eq. (179), λ_1 is one of the $w_{\ell, \mathcal{L}}^{[v(x)], k}(g; x)$'s while λ_2 is then $w_{\ell, \mathcal{R}}^{[v(x)], k}(g; x)$'s. For some g 's meeting our requirements (strictly positive and continuous on a finitely partitionable domain) $[w_{min}, w_{max}]$, with $w_{min} > 0$, taken together with some f_{W^ω} 's and some ϕ_X 's, the integrals in Eqs.(180) can be evaluated analytically (e.g., $g(w) = w^n$, $n = 1, 2, \dots$; $f_{W^\omega}(w) = 1/(w_{max} - w_{min})$; and $\phi_X(g(w)) = \text{constant}$). For other g 's, f_{W^ω} 's and ϕ_X 's these definite integrals must be evaluated numerically.

Concerning ϕ_X , it is worthwhile to point out that an attractive, non-trivial choice for it is available that allows $\mathcal{J}_5$ to be evaluated analytically when $g(w) = w^n$ and, e.g., $f_{W^\omega}(w) = 1/(w_{\max} - w_{\min})$. This choice assigns the probability fractions to the left and right wings of the PDF $f_{X^{g(w)}}^A$ based upon the relative lengths of their bases, these lengths being, respectively, $(1 - b)g(w) - x_{\min} > 0$ and $x_{\max} - (1 + b)g(w) > 0$, according to

$$\begin{aligned} \phi_X(g(w); b, x_{\min}, x_{\max}) \\ &= [(1 - b)g(w) - x_{\min}] / \{[(1 - b)g(w) - x_{\min}] + [x_{\max} - (1 + b)g(w)]\} \\ &= -[(1 - b)/2b][g(w) - x_{\min}^\#] / \{g(w) - [(x_{\max} - x_{\min})/2b]\}. \end{aligned} \quad (182)$$

It is important at this point to explore whether the four integrals $\mathcal{J}_2$ through $\mathcal{J}_5$ appearing in Eqs. (180) are proper (i.e., if the integrands are bounded on their integration domains). Since the PDF f_{W^ω} is bounded on $[w_{\min}, w_{\max}]$, it is sufficient to consider only the g -bearing denominators of the integrands. To this end, firstly, consider the factor $1/g(w)$ in the integral $\mathcal{J}_2[f_{W^\omega}, g; \lambda_1, \lambda_2]$; since $1/g$ is bounded in $[\lambda_1, \lambda_2]$ —above by $1/x_{\min} < \infty$ and below by $1/x_{\max} > 0$ —then $\mathcal{J}_2[f_{W^\omega}, g; \lambda_1, \lambda_2]$ is proper. Consider next the factor $1/[g(w) - E]$ in the three integrals $\mathcal{J}_3[f_{W^\omega}, g; E; \lambda_1, \lambda_2]$ through $\mathcal{J}_5[f_{W^\omega}, g, \phi_X; E; \lambda_1, \lambda_2]$ (the presence of ϕ_X in the integral $\mathcal{J}_5$ does not affect our arguments since $0 < \phi_X < 1$). The integration domain for these is $[\lambda_1, \lambda_2] \subseteq [w_{\min}, w_{\max}]$, with $g(w) \in [x_{\min}, x_{\max}]$. Now two of the possible values for E , namely $x_{\min}^b$ and $x_{\max}^\#$, are such that $0 < x_{\min}^b < x_{\min}$ and $\infty > x_{\max}^\# > x_{\max}$ so that $1/[g(w) - E]$ is bounded above and below in $[\lambda_1, \lambda_2]$ for these two values of E and the integrals are proper. For example, for $E = x_{\min}^b$ we have $x_{\max} > x_{\min} > x_{\min}^b$ so $\infty > x_{\max} - x_{\min}^b > 0$, and for all $w \in [w_{\min}, w_{\max}]$ we have $\infty > x_{\max} - x_{\min}^b \geq g(w) - x_{\min}^b \geq x_{\min} - x_{\min}^b > 0$ so $0 < 1/(x_{\max} - x_{\min}^b) \leq 1/[g(w) - x_{\min}^b] \leq 1/(x_{\min} - x_{\min}^b) < \infty$; the argument for $E = x_{\max}^\#$ is analogous. For the remaining two possible values of E , namely $x_{\min}^\#$ and $x_{\max}^b$, the required reasoning—provided below—is more delicate.

Consider, firstly, $E = x_{\min}^\#$. This value for E occurs only when (from Eq. (181))

$f_{xg(w)} = f_{xg(w)}^{A,1}(x)$ or $f_{xg(w)} = f_{xg(w)}^{B,1}(x)$ so that the term $1/[g(w) - x_{min}^\#]$ (actually, squared) appears in the integrand only for these two PDF segments; further, the only integrals involved here are $\mathcal{J}_4[f_{w^\omega}, g; E; \lambda_1, \lambda_2]$ and $\mathcal{J}_5[f_{w^\omega}, g, \phi_x; E; \lambda_1, \lambda_2]$. Now $f_{xg(w)}^{B,1}(x)$ appears as an integrand, with the limits of the w -integrations for it lying in $g^{-1}[(x^\#, x_{max}^\#)]$, only in $f_{xg(w)}^{[1]}(x)$, $f_{xg(w)}^{[8]}(x)$, $f_{xg(w)}^{[12]}(x)$, and $f_{xg(w)}^{[14]}(x)$ (call this the first $f_{xg(w)}^{B,1}(x)$ group); and $f_{xg(w)}^{B,1}(x)$ appears as an integrand, with the limits of the w -integrations for it lying in $g^{-1}[(x_{max}^b, x_{max}^\#)]$, only in $f_{xg(w)}^{[7]}(x)$, $f_{xg(w)}^{[10]}(x)$, and $f_{xg(w)}^{[11]}(x)$ (call this the second $f_{xg(w)}^{B,1}(x)$ group); and $f_{xg(w)}^{B,1}(x)$ appears in no other $f_{xg(w)}^{[v]}(x)$'s. Also, $f_{xg(w)}^{A,1}(x)$ appears as an integrand only in $f_{xg(w)}^{[7]}(x)$, $f_{xg(w)}^{[10]}(x)$, and $f_{xg(w)}^{[11]}(x)$ —along with $f_{xg(w)}^{B,1}(x)$ —where the limits of the w -integrations for it lie in $g^{-1}[(x^\#, x_{max}^b)]$; and $f_{xg(w)}^{A,1}(x)$ appears in no other $f_{xg(w)}^{[v]}(x)$'s. So we must establish the boundedness of $1/[g(w) - x_{min}^\#]$ in the disjoint union of intervals $g^{-1}[(x^\#, x_{max}^\#)]$, and in the disjoint union of intervals $g^{-1}[(x_{max}^b, x_{max}^\#)]$, and in the disjoint union of intervals $g^{-1}[(x^\#, x_{max}^b)]$. But it is sufficient, in order to establish the boundedness of $1/[g(w) - x_{min}^\#]$ in all three of these unions, rather to do so for the union $g^{-1}[(x^\#, x_{max}^\#)]$ alone, as follows: Firstly, since $x_{max}^b < x_{max}^\#$, then $g^{-1}[(x^\#, x_{max}^b)] \subseteq g^{-1}[(x^\#, x_{max}^\#)]$ so that all the $f_{xg(w)}^{A,1}(x)$ appearances would be taken care of; further, it is easy to check (from Eqs. (138), (141), and (142)) for all members of the second $f_{xg(w)}^{B,1}(x)$ group that we have $x_{max}^b > x^\#$ so that also $g^{-1}[(x_{max}^b, x_{max}^\#)] \subseteq g^{-1}[(x^\#, x_{max}^\#)]$ and all the second group $f_{xg(w)}^{B,1}(x)$ appearances would be taken care of. We thus must show that $1/[g(w) - x_{min}^\#]$ is bounded in $g^{-1}[(x^\#, x_{max}^\#)]$ for any x allowable in any of the Cases that use at least one of the above seven $f_{xg(w)}^{[v]}(x)$'s. Those Cases are, from Eqs. (146) – (171), $^x\text{IaR1}$, $^x\text{IIaR1-R4}$, $^x\text{IIbR1-R3}$, $^x\text{IIcR1-R3}$, and $^x\text{IIDR1-R4}$. The value $x = x_{min}$ —if admitted by a Case—is problematic (see below) and we return to it shortly. Of the just-listed Cases, only $^x\text{IaR1}$, $^x\text{IIaR1}$, $^x\text{IIbR1}$, $^x\text{IIcR1}$, and $^x\text{IIDR1}$ admit $x = x_{min}$ while the remaining ones admit only $x > x_{min}$. We first dispense with $x > x_{min}$. Now

when $x > x_{min}$ then $x^\# > x_{min}^\#$ so $\varepsilon \equiv x^\# - x_{min}^\# > 0$; hence if $w \in g^{-1}[(x^\#, x_{max}^\#)]$ then $g(w) > x^\#$ and $g(w) - x_{min}^\# > x^\# - x_{min}^\# = \varepsilon > 0$ so $0 < 1/[g(w) - x_{min}^\#] \leq 1/\varepsilon$ and $1/[g(w) - x_{min}^\#]$ is bounded in $g^{-1}[(x^\#, x_{max}^\#)]$ and we are done with $x > x_{min}$. Next, when $x = x_{min}$ and $w \in g^{-1}[x^\#]$, we have $g(w) = x^\# = x_{min}^\#$ so that $1/[g(w) - x_{min}^\#] = \infty$; and indeed *the integrals* $\mathcal{J}_4[f_{W^\omega}, g; E; \lambda_1, \lambda_2]$ and $\mathcal{J}_5[f_{W^\omega}, g, \phi_X; E; \lambda_1, \lambda_2]$ may then be improper as they stand. (We consider $w \in g^{-1}[x^\#]$ even though $x^\# \notin (x^\#, x_{max}^\#]$ since $g^{-1}[x^\#]$ may provide interval endpoints for the integrations over the disjoint union of intervals comprising $g^{-1}[(x^\#, x_{max}^\#)]$.) To circumvent this problem we observe from Eq. (52) that in fact when $x = x_{min}$ then $f_{X^{g(w)}}^{A,1}(x_{min}) = 0 = f_{X^{g(w)}}^{B,1}(x_{min})$ since the factors $(x - x_{min})$ therein become zero (i.e., $\lim_{g(w) \rightarrow x_{min}^\#} [(x_{min} - x_{min})/(g(w) - x_{min}^\#)] = 0$). Hence in this case the two integrals $\mathcal{J}_4[f_{W^\omega}, g; E; \lambda_1, \lambda_2]$ and $\mathcal{J}_5[f_{W^\omega}, g, \phi_X; E; \lambda_1, \lambda_2]$, when they have integrands in which $f_{X^{g(w)}}^{A,1}(x_{min})$ or $f_{X^{g(w)}}^{B,1}(x_{min})$ are factors, are zero (even if they are not improper) and need not be further evaluated—indeed, should not be evaluated in a numerical computation because they may appear improper: For though they may converge, a numerical computation of the integrals may fail since it will involve the quotient 0/0 at the lower limit $g(w) = x_{min}^\#$.

Consider, lastly, $E = x_{max}^b$, which occurs only when (from Eq. (181)) $f_{X^{g(w)}} = f_{X^{g(w)}}^{A,3}(x)$ or $f_{X^{g(w)}} = f_{X^{g(w)}}^{C,2}(x)$ so that the term $1/[g(w) - x_{max}^b]$ (actually, squared) appears in the integrand only for these two PDF segments; further, once again the only integrals involved here are $\mathcal{J}_4[f_{W^\omega}, g; E; \lambda_1, \lambda_2]$ and $\mathcal{J}_5[f_{W^\omega}, g, \phi_X; E; \lambda_1, \lambda_2]$. The arguments presented in the previous paragraph hold essentially *verbatim* in this case as well except for the appropriate changes made to the entities involved. In order to record these changes, we repeat, for completeness, the above paragraph appropriately modified. So then, $f_{X^{g(w)}}^{C,2}(x)$ appears in the integrand, with limits of the w -integrations lying in $g^{-1}[x_{min}, x^b]$ only in $f_{X^{g(w)}}^{[3]}(x)$,

$f_{xg(w)}^{[5]}(x)$, $f_{xg(w)}^{[10]}(x)$, $f_{xg(w)}^{[12]}(x)$, and $f_{xg(w)}^{[13]}(x)$ (call this the first $f_{xg(w)}^{C,2}(x)$ group); and $f_{xg(w)}^{C,2}(x)$ appears as an integrand, with the limits of the w -integrations for it lying in $g^{-1}[[x_{min}, x_{min}^{\#}]]$ only in $f_{xg(w)}^{[7]}(x)$, $f_{xg(w)}^{[8]}(x)$, and $f_{xg(w)}^{[9]}(x)$ (call this the second $f_{xg(w)}^{C,2}(x)$ group); and $f_{xg(w)}^{C,2}(x)$ appears in no other $f_{xg(w)}^{[v]}(x)$'s. Also, $f_{xg(w)}^{A,3}(x)$ appears as an integrand only in $f_{xg(w)}^{[7]}(x)$, $f_{xg(w)}^{[8]}(x)$, and $f_{xg(w)}^{[9]}(x)$ —along with $f_{xg(w)}^{C,2}(x)$ —where the limits of the w -integrations for it lie in $g^{-1}[(x_{min}^{\#}, x^b)]$; and $f_{xg(w)}^{A,3}(x)$ appears in no other $f_{xg(w)}^{[v]}(x)$'s. So we must establish the boundedness of $1/[g(w) - x_{max}^b]$ in the disjoint union of intervals $g^{-1}[[x_{min}, x^b]]$, and in the disjoint union of intervals $g^{-1}[[x_{min}, x_{min}^{\#}]]$, and in the disjoint union of intervals $g^{-1}[(x_{min}^{\#}, x^b)]$. But it is sufficient, in order to establish the boundedness of $1/[g(w) - x_{max}^b]$ in all three intervals, rather to do so for $g^{-1}[(x_{min}^{\#}, x^b)]$ alone, as follows: Firstly, since $x_{min}^{\#} > x_{min}$, then $g^{-1}[(x_{min}^{\#}, x^b)] \subseteq g^{-1}[[x_{min}, x^b]]$ so that all the $f_{xg(w)}^{A,3}(x)$ appearances would be taken care of; further, it is easy to check (from Eqs. (138) – (140)) for all members of the second $f_{xg(w)}^{C,2}(x)$ group that we have $x_{min}^{\#} < x^b$ so that also $g^{-1}[[x_{min}, x_{min}^{\#}]] \subseteq g^{-1}[[x_{min}, x^b]]$ and all the second group $f_{xg(w)}^{C,2}(x)$ appearances would be taken care of. We must thus show that $1/[g(w) - x_{max}^b]$ is bounded in $g^{-1}[[x_{min}, x^b]]$ for any x allowable in any of the Cases that use at least one of the above eight $f_{xg(w)}^{[v]}(x)$'s. Those Cases are, from Eqs. (146) – (171), $^x\text{IaR3}$, $^x\text{IbR2}$, $^x\text{IIaR2-R5}$, $^x\text{IIbR2-R5}$, $^x\text{IIcR3-R5}$, and $^x\text{IIDR2-R5}$. The value $x = x_{max}$ —if admitted by a Case—is problematic (see below) and we return to it shortly. Of the just-listed Cases, only $^x\text{IaR3}$, $^x\text{IbR2}$, $^x\text{IIaR5}$, $^x\text{IIbR5}$, $^x\text{IIcR5}$, and $^x\text{IIDR5}$ admit $x = x_{max}$ while the remaining ones admit only $x < x_{max}$. We first dispense with $x < x_{max}$. Now when $x < x_{max}$ then $x^b < x_{max}^b$ so $-\varepsilon' \equiv x^b - x_{max}^b < 0$; hence if $w \in g^{-1}[[x_{min}, x^b]]$ then $g(w) < x^b$ and $g(w) - x_{max}^b < x^b - x_{max}^b = -\varepsilon' < 0$ so $0 > 1/[g(w) - x_{max}^b] \geq -(1/\varepsilon')$ and $1/[g(w) - x_{max}^b]$ is bounded in $g^{-1}[[x_{min}, x^b]]$ and we are done with $x < x_{max}$. Next, when

$x = x_{\max}$ and $w \in g^{-1}[x^b]$ we have $g(w) = x^b = x_{\max}^b$ so that $1/[g(w) - x_{\max}^b] = \infty$; and indeed the integrals $\mathcal{J}_4[f_{W^\omega}, g; E; \lambda_1, \lambda_2]$ and $\mathcal{J}_5[f_{W^\omega}, g, \phi_X; E; \lambda_1, \lambda_2]$ may then be improper as they stand. (We consider $w \in g^{-1}[x^b]$ even though $x^b \notin [x_{\min}, x^b)$ since $g^{-1}[x^b]$ may provide interval endpoints for the integrations over the disjoint union of intervals comprising $g^{-1}([x_{\min}, x^b])$.) To circumvent this problem we observe from Eq. (52) that in fact when $x = x_{\max}$ then $f_{X^{g(w)}}^{A,3}(x_{\max}) = 0 = f_{X^{g(w)}}^{C,2}(x_{\max})$ since the factors $(x_{\max} - x)$ therein become zero (i.e., $\lim_{g(w) \rightarrow x_{\max}^b} [(x_{\max} - x_{\max})/(g(w) - x_{\max}^b)] = 0$). Hence in this case the two integrals $\mathcal{J}_4[f_{W^\omega}, g; E; \lambda_1, \lambda_2]$ and $\mathcal{J}_5[f_{W^\omega}, g, \phi_X; E; \lambda_1, \lambda_2]$, when they have integrands in which $f_{X^{g(w)}}^{A,3}(x)$ or $f_{X^{g(w)}}^{C,2}(x)$ are factors, are zero (even if they are not improper) and need not be further evaluated—indeed, should not be evaluated in a numerical computation because they may appear improper: For though they may converge, a numerical computation of the integrals may fail since it will involve the quotient 0/0 at the upper limit $g(w) = x_{\max}^b$.

Our discussion of the properness of the integrals $\mathcal{J}_2$ through $\mathcal{J}_5$ is now complete.

Returning now to the computation of $\int_{\lambda_1}^{\lambda_2} f_{W^\omega}(w) f_{X^{g(w)}}^{\alpha'(v(x), k), k'(v(x), k)}(x) dw$ in Eq. (179), it is convenient to define two auxiliary quantities. Firstly, for $0 < a \leq 1$ and $0 < b < 1$, we denote

$$\begin{aligned} K^{A,1}(a, b) &= 2(1-a)/(1-b)^2 & K^{A,2}(a, b) &= a/2b & K^{A,3}(a, b) &= 2(1-a)/(1+b)^2 \\ K^{B,1}(a, b) &= 2(1-a)/(1-b)^2 & & & K^{B,2}(a, b) &= -a/(1-b) \\ K^{C,1}(a, b) &= a/(1+b) & K^{D,1}(a, b) &= 1 & K^{C,2}(a, b) &= 2(1-a)/(1+b)^2; \end{aligned} \quad (183)$$

note that $K^{B,1}(a, b) = K^{A,1}(a, b)$ and $K^{C,2}(a, b) = K^{A,3}(a, b)$. And, secondly, we denote

$$\begin{aligned}
S^{\alpha', k'}[f_{W^\omega}, g; \phi_X; \lambda_1, \lambda_2; x] = & \left[\begin{array}{ll}
\mathcal{J}_5[f_{W^\omega}, g, \phi_X; x_{min}^\#; \lambda_1, \lambda_2](x - x_{min}) & \text{if } \langle \alpha', k' \rangle = \langle A, 1 \rangle \text{ \& } x > x_{min} \\
0 & \text{if } \langle \alpha', k' \rangle = \langle A, 1 \rangle \text{ \& } x = x_{min} \\
\mathcal{J}_2[f_{W^\omega}, g; \lambda_1, \lambda_2] & \text{if } \langle \alpha', k' \rangle = \langle A, 2 \rangle \\
\{\mathcal{J}_4[f_{W^\omega}, g; x_{max}^b; \lambda_1, \lambda_2] - \mathcal{J}_5[f_{W^\omega}, g; x_{max}^b; \phi_X; \lambda_1, \lambda_2]\}(x_{max} - x) & \text{if } \langle \alpha', k' \rangle = \langle A, 3 \rangle \text{ \& } x < x_{max} \\
0 & \text{if } \langle \alpha', k' \rangle = \langle A, 3 \rangle \text{ \& } x = x_{max} \\
\mathcal{J}_4[f_{W^\omega}, g; x_{min}^\#; \lambda_1, \lambda_2](x - x_{min}) & \text{if } \langle \alpha', k' \rangle = \langle B, 1 \rangle \text{ \& } x > x_{min} \\
0 & \text{if } \langle \alpha', k' \rangle = \langle B, 1 \rangle \text{ \& } x = x_{min} \\
\mathcal{J}_3[f_{W^\omega}, g; x_{max}^\#; \lambda_1, \lambda_2] & \text{if } \langle \alpha', k' \rangle = \langle B, 2 \rangle \\
\mathcal{J}_3[f_{W^\omega}, g; x_{min}^b; \lambda_1, \lambda_2] & \text{if } \langle \alpha', k' \rangle = \langle C, 1 \rangle \\
\mathcal{J}_4[f_{W^\omega}, g, x_{max}^b; \lambda_1, \lambda_2](x_{max} - x) & \text{if } \langle \alpha', k' \rangle = \langle C, 2 \rangle \text{ \& } x < x_{max} \\
0 & \text{if } \langle \alpha', k' \rangle = \langle C, 2 \rangle \text{ \& } x = x_{max} \\
\mathcal{J}_1[f_{W^\omega}; \lambda_1, \lambda_2]/(x_{max} - x_{min}) & \text{if } \langle \alpha', k' \rangle = \langle D, 1 \rangle
\end{array} \right. \quad (184)
\end{aligned}$$

Using the above definitions, we may then represent the integrals $\int_{\lambda_1}^{\lambda_2} f_{W^\omega}(w) f_{Xg(w)}^{\alpha', k'}(x) dw$ as

$$\int_{\lambda_1}^{\lambda_2} f_{W^\omega}(w) f_{Xg(w)}^{\alpha', k'}(x) dw = K^{\alpha', k'}(a, b) S^{\alpha', k'}[f_{W^\omega}, g; \phi_X; \lambda_1, \lambda_2; x]. \quad (185)$$

We now return to Eq. (179) in order to replace the integrals therein with expressions of the form appearing on the RHS of Eq. (185). We then have

$$\begin{aligned}
\mathcal{J}^{v(x)}(f_{W^\omega}, g; x) &= \sum_{k=1}^{\#(v(x))} \sum_{\ell=1}^{\#(v(x), k, g; x)} \{K^{\alpha'(v(x), k), k'(v(x), k)}(a, b) \\
&\quad \bullet S^{\alpha'(v(x), k), k'(v(x), k)}[f_{W^\omega}, g; \phi_X; w_{\ell, \mathcal{X}}^{[v(x)], k}(g; x), w_{\ell, \mathcal{X}}^{[v(x)], k}(g; x); x]\} \\
&= f_{\hat{X}}^{\alpha'}(\omega, f_{W^\omega}, g; x).
\end{aligned} \quad (186)$$

Finally, reinserting the index “ n ” into the above result we find for Eq. (41) that

$$\begin{aligned}
& f_{\hat{X}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n) \\
&= \mathcal{J}^{v(x_n)}(f_{W_n^{\omega_n}}, g_n; x_n) \\
&= \sum_{k=1}^{\#(v(x_n))} \sum_{\ell=1}^{\#(v(x_n), k, g_n; x_n)} \{K^{\alpha'(v(x_n), k), k'(v(x_n), k)}(a_n, b_n) \\
&\quad \bullet \mathcal{S}^{\alpha'(v(x_n), k), k'(v(x_n), k)}[f_{W_n^{\omega_n}}, g_n; \phi_{X_n}; w_{\ell, \mathcal{X}}^{[v(x_n)], k}(g_n; x_n), w_{\ell, \mathcal{X}}^{[v(x_n)], k}(g_n; x_n); x_n]\}.
\end{aligned} \tag{187}$$

II.6 Code- n Having More Than One Output Variable

When Code- n has more than one output variable, say $X_n^{(m)}$, $m = 1, \dots, M$, each of which depends solely upon single input variable $\mathcal{W}_n$ (e.g., $X_n^{(1)} = \mathcal{W}_n$ and $X_n^{(2)} = \log \mathcal{W}_n$ with $\xi_{n; \omega_n}^{(1)} = g_n^{(1)}(\omega_n) = \omega_n$ and $\xi_{n; \omega_n}^{(2)} = g_n^{(2)}(\omega_n) = \log \omega_n$) then the above development may be applied to each $X_n^{(m)}$ separately.

III. Code Chains

In this section we treat code chains, beginning by considering any two-code chain and applying our single-code results of the previous section to each of the two codes in the chain; we then extend these results to any arbitrary finite-length code chain.

III.1 A Two-Code Chain

We designate the two codes in the two-code chain as Code-1 and Code-2, with Code-2 succeeding Code-1, and with each code having only one input variable and one output variable. Both codes are completely characterized for our purposes by the results in Section II, with $n = 1$ and $n = 2$ there.

The central idea here is that the value $\xi_{1; \omega_1} = g_1(\omega_1)$ of Code-1 output variable X_1 —resulting from value ω_1 of Code-1 input variable $\mathcal{W}_1$ —serves dually as the value of Code-2 input variable $\mathcal{W}_2$ by being passed to Code-2 as the value of the latter's input. If we denote by $\omega_{2; \xi_1} \in \mathcal{V}_{\mathcal{W}_2}$ the value of Code-2 input variable $\mathcal{W}_2$ that is passed to Code-2 by Code-1 when the value of the latter's output variable X_1 is $\xi_1 \in \mathcal{V}_{X_1}$, then we have

$$\omega_{2; \xi_1, \omega_1} = \xi_{1; \omega_1} = g_1(\omega_1). \tag{188}$$

Note that the input value of $\mathcal{W}_2$ is still the single value $\omega_{2;\xi_1;\omega_1}$ but the smearing of $\mathcal{W}_2$ at $\omega_{2;\xi_1;\omega_1}$ —which is to be given precisely by the (compoundly) smeared PDF of $\mathcal{X}_1$ at ξ_1 —dictates consideration of values $w_2 \neq \omega_{2;\xi_1;\omega_1}$, $w_2 \in \mathcal{V}_{W_2^*} = \mathcal{V}_{\mathcal{W}_2}$ (the equality *via* Eq. (3)) as virtual values of code input variable $\mathcal{W}_2$. Thus, if in general we denote by $w_{2;x_1} \in \mathcal{V}_{W_2^*}$ the smearing value of Code-2 input variable $\mathcal{W}_2$ that is passed to Code-2 by Code-1 when the smearing value of the latter's output variable $\mathcal{X}_1$ is $x_1 \in \mathcal{V}_{X_1^*}$, then we may characterize the invariance of this passed value as

$$w_{2;x_1} = x_1. \quad (189)$$

The relationship in Eq. (189) may conveniently thought of as a mapping, $\mathcal{J}_{12}$, between $\mathcal{V}_{X_1^*}$ and $\mathcal{V}_{W_2^*}$, namely, $\mathcal{J}_{12}: \mathcal{V}_{X_1^*} \rightarrow \mathcal{V}_{W_2^*}$ given by $\mathcal{J}_{12}(x_1) = x_1$ (note that $w_{2;x_1} = x_1 = \mathcal{J}_{12}(x_1)$). Furthermore, since the Code-1-associated real-world output variable X_1^* and the Code-2-associated real-world input variable W_2^* represent one and the same physical quantity in the code chain modeling then we take

$$\mathcal{V}_{W_2^*} = \mathcal{V}_{X_1^*}. \quad (190)$$

Then $\mathcal{J}_{12}^{-1}: \mathcal{V}_{W_2^*} \rightarrow \mathcal{V}_{X_1^*}$ is also a function and $\mathcal{J}_{12}^{-1}(w_{2;x_1}) = w_{2;x_1} = x_1$. As an extension of Eq. (190) we note that combining it with Eqs. (6) with $n = 1$ and Eq. (3) with $n = 2$ yields

$$\mathcal{V}_{X_1} \subseteq \mathcal{V}_{X_1^*} = \mathcal{V}_{W_2^*} = \mathcal{V}_{\mathcal{W}_2}. \quad (191)$$

Also, since $\text{rng } g_1 = \mathcal{V}_{X_1}$ and $\text{dom } g_2 = \mathcal{V}_{\mathcal{W}_2}$ then we may use Eqs. (189) to write

$$\xi_{2;\omega_2;\xi_1;\omega_1} = g_2(\omega_{2;\xi_1;\omega_1}) = (g_2 \circ g_1)(\omega_1) \quad (192)$$

for the ξ_2 -value of the Code-2 output variable $\mathcal{X}_2$ when the value of the Code-1 input variable $\mathcal{W}_1$ is $\omega_1 \in \mathcal{V}_{\mathcal{W}_1}$.

Now the single output value $\xi_{1;\omega_1}$ of $\mathcal{X}_1$, as a result of being produced from Code-1 and therefore subject to Code-1 uncertainty (compound) smearing arising from $f_{W_1^{\omega_1}}$, as well as from the basic smearing of $\mathcal{X}_1$ (both as developed in Section II), is to be properly represented in uncertainty considerations by its smearing PDF, namely, by $f_{\hat{\mathcal{X}}_1}(\omega_1, f_{W_1^{\omega_1}}, g_1; x_1)$, $x_1 \in \mathcal{V}_{\mathcal{X}_1^*}$, of Eq. (41) with $n = 1$. That is, the Code-2 input variable $\mathcal{W}_2$ is uncertain and this input uncertainty, as specified by the PDF $f_{W_2^{\omega_2}}(w_2)$, is, for all $w_2 \in \mathcal{V}_{W_2^*} = \mathcal{V}_{\mathcal{X}_1^*}$, precisely

$$\begin{aligned} f_{W_2^{\omega_2}}(w_2) &= f_{\hat{\mathcal{X}}_1}(\omega_1, f_{W_1^{\omega_1}}, g_1; \mathcal{J}_{12}^{-1}(w_2)) \\ &= f_{\hat{\mathcal{X}}_1}(\omega_1, f_{W_1^{\omega_1}}, g_1; w_2). \end{aligned} \quad (193)$$

Note that, using Eq. (42) with $n = 1$, along with Eq. (190), we have

$$\begin{aligned} \int_{w_{2, \min}}^{w_{2, \max}} f_{W_2^{\omega_2}}(w_2) dw_2 &= \int_{x_{1, \min}}^{x_{1, \max}} f_{W_2^{\omega_2}}(w_2) dw_2 \\ &= \int_{x_{1, \min}}^{x_{1, \max}} f_{\hat{\mathcal{X}}_1}(\omega_1, f_{W_1^{\omega_1}}, g_1; w_2) dw_2 \\ &= 1 \end{aligned} \quad (194)$$

as required. Now, using Eq (193) in Eq. (41) with $n = 2$ and then again using Eq. (41) with $n = 1$, we find for all $x_2 \in \mathcal{V}_{\mathcal{X}_2^*}$ that

$$\begin{aligned} f_{\hat{\mathcal{X}}_2}(\omega_2, f_{W_2^{\omega_2}}, g_2; x_2) &= \int_{w_{2, \min}}^{w_{2, \max}} f_{W_2^{\omega_2}}(w_2) f_{\mathcal{X}_2^{\omega_2}}(x_2) dw_2 \\ &= \int_{w_{2, \min}}^{w_{2, \max}} f_{\hat{\mathcal{X}}_1}(\omega_1, f_{W_1^{\omega_1}}, g_1; w_2) f_{\mathcal{X}_2^{\omega_2}}(x_2) dw_2 \\ &= \int_{w_{2, \min}}^{w_{2, \max}} \int_{w_{1, \min}}^{w_{1, \max}} f_{W_1^{\omega_1}}(w_1) f_{\mathcal{X}_1^{\omega_1}}(w_2) f_{\mathcal{X}_2^{\omega_2}}(x_2) dw_1 dw_2 \\ &\equiv f_{\hat{\mathcal{X}}_2}(\omega_1, f_{W_1^{\omega_1}}, g_1, g_2; x_2) \end{aligned} \quad (195)$$

where the last line in Eq. (195) is our suggestive notation for the compoundly smeared PDF for the value x_2 of Code-2 output variable $\mathcal{X}_2$ when the value of the Code-1 input variable $\mathcal{W}_1$ is ω_1 , with $\mathcal{W}_1$ having uncertainty PDF given by $f_{\mathcal{W}_1^{\omega_1}}$.

Using similar techniques, we find for a three-code chain that the (compoundly smeared) PDF for the value x_3 of Code-3 output variable $\mathcal{X}_3$ when the value of the Code-1 input variable $\mathcal{W}_1$ is ω_1 , with $\mathcal{W}_1$ having uncertainty PDF given by $f_{\mathcal{W}_1^{\omega_1}}$, is given, for all

$x_3 \in \mathcal{V}_{\mathcal{X}_3^*}$, by

$$\begin{aligned} & f_{\hat{\mathcal{X}}_3}(\omega_1, f_{\mathcal{W}_1^{\omega_1}}, g_1, g_2, g_3; x_3) \\ &= \int_{w_{3, \min}}^{w_{3, \max}} \int_{w_{2, \min}}^{w_{2, \max}} \int_{w_{1, \min}}^{w_{1, \max}} f_{\mathcal{W}_1^{\omega_1}}(w_1) f_{\mathcal{X}_1^{g_1(w_1)}}(w_2) f_{\mathcal{X}_2^{g_2(w_2)}}(w_3) f_{\mathcal{X}_3^{g_3(w_3)}}(x_3) dw_1 dw_2 dw_3. \end{aligned} \quad (196)$$

In the next section we derive an expression for $f_{\hat{\mathcal{X}}_N}(\omega_1, f_{\mathcal{W}_1^{\omega_1}}, g_1, g_2, \dots, g_N; x_N) =$

$f_{\hat{\mathcal{X}}_N}(\omega_N, f_{\mathcal{W}_N^{\omega_N}}, g_N; x_N)$ for any $N < \infty$.

III.2 Arbitrary Finite-Length Code Chains

In this section we extend our UQ propagation formulation from two-code chains to chains having an arbitrary finite number, $N < \infty$, of component codes (N -code chains).

We label the N codes in the chain by Code-1, ..., Code- N , in order of occurrence in the chain, with each code having only one input variable and one output variable and being completely characterized for our purposes by the results in Section II, with $n = 1, \dots, N$ there. Once again, the central idea here is that the value $\xi_{n; \omega_n} = g_n(\omega_n)$ of Code- n output variable $\mathcal{X}_n$ —resulting from value ω_n of Code- n input variable $\mathcal{W}_n$ —serves dually as the value of Code- $(n + 1)$ input variable $\mathcal{W}_{n+1}$ by being passed to Code- $(n + 1)$ as the value of the latter's input. If we denote by $\omega_{n+1; \xi_n} \in \mathcal{V}_{\mathcal{W}_{n+1}}$ the value of Code- $(n + 1)$ input variable $\mathcal{W}_{n+1}$ that is passed to Code- $(n + 1)$ by Code- n when the value of the latter's output variable $\mathcal{X}_n$ is $\xi_n \in \mathcal{V}_{\mathcal{X}_n}$, then we have

$$\omega_{n+1; \xi_n} = \xi_{n; \omega_n} = g_n(\omega_n). \quad (197)$$

Note that the input value of $\mathcal{W}'_{n+1}$ is still the single value $\omega_{n+1; \xi_n; \omega_n}$ but the smearing of $\mathcal{W}'_{n+1}$ at $\omega_{n+1; \xi_n; \omega_n}$ —which is to be given precisely by the (compoundly) smeared PDF of $\mathcal{X}_n$ at ξ_n —dictates consideration of values $w_{n+1} \neq \omega_{n+1; \xi_n; \omega_n}$, $w_{n+1} \in \mathcal{V}_{W_{n+1}^*} = \mathcal{V}_{\mathcal{W}_{n+1}}$ (the equality via Eq. (3)) as virtual values of code input variable $\mathcal{W}'_{n+1}$. Thus, if in general we denote by $w_{n+1; \mathcal{X}_n} \in \mathcal{V}_{W_{n+1}^*}$ the smearing value of Code- $(n+1)$ input variable $\mathcal{W}'_{n+1}$ that is passed to Code- $(n+1)$ by Code- n when the smearing value of the latter's output variable $\mathcal{X}_n$ is $x_n \in \mathcal{V}_{X_n^*}$, then we may characterize the invariance of this passed value as

$$w_{n+1; \mathcal{X}_n} = x_n. \quad (198)$$

The relationship in Eq. (198) may conveniently thought of as a mapping, $\mathcal{J}_{n, n+1}$, between $\mathcal{V}_{X_n^*}$ and $\mathcal{V}_{W_{n+1}^*}$, namely, $\mathcal{J}_{n, n+1}: \mathcal{V}_{X_n^*} \rightarrow \mathcal{V}_{W_{n+1}^*}$ given by $\mathcal{J}_{n, n+1}(x_n) = w_{n+1; \mathcal{X}_n} = x_n$ (note that $w_{n+1; \mathcal{X}_n} = x_n = \mathcal{J}_{n, n+1}(x_n)$). Furthermore, since the Code- n -associated real-world output variable X_n^* and the Code- $(n+1)$ -associated real-world input variable W_{n+1}^* represent one and the same physical quantity in the code chain modeling then we take

$$\mathcal{V}_{W_{n+1}^*} = \mathcal{V}_{X_n^*}. \quad (199)$$

Then $\mathcal{J}_{n, n+1}^{-1}: \mathcal{V}_{W_{n+1}^*} \rightarrow \mathcal{V}_{X_n^*}$ is also a function and $\mathcal{J}_{n, n+1}^{-1}(w_{n+1; \mathcal{X}_n}) = w_{n+1; \mathcal{X}_n} = x_n$. As an extension of Eq. (199) we note that combining it with Eqs. (6) and Eq. (3) with $n \rightarrow n+1$ yields

$$\mathcal{V}_{\mathcal{X}_n} \subseteq \mathcal{V}_{X_n^*} = \mathcal{V}_{W_{n+1}^*} = \mathcal{V}_{\mathcal{W}_{n+1}}. \quad (200)$$

Also, since $\text{rng } g_n = \mathcal{V}_{\mathcal{X}_n}$ and $\text{dom } g_{n+1} = \mathcal{V}_{\mathcal{W}_{n+1}}$ then we may use Eqs. (197) to write

$$\xi_{n+1; \omega_{n+1; \xi_n; \omega_n}} = g_{n+1}(\omega_{n+1; \xi_n; \omega_n}) = (g_{n+1} \circ g_n)(\omega_n) \quad (201)$$

for the ξ_{n+1} -value of the Code- $(n+1)$ output variable $\mathcal{X}_{n+1}$ when the value of the Code- n input variable $\mathcal{W}_n$ is ω_n ; then by repeated application of Eq. (201) for $n = N-1, \dots, 1$, we find

$$\xi_N = (g_N \circ g_{N-1} \circ g_{N-2} \circ \dots \circ g_1)(\omega_1) \quad (202)$$

for the value ξ_N of Code- N output variable χ_N when the value of Code-1 input variable ω_1 is $\omega_1 \in \mathcal{V}_{\omega_1}$.

Now the single output value $\xi_{n; \omega_n}$ of χ_n , as a result of being produced from Code- n and therefore subject to Code- n uncertainty (compound) smearing arising from $f_{W_n^{\omega_n}}$, as well as from the basic smearing of χ_n (both as developed in Section II), is to be properly represented in uncertainty considerations by its smearing PDF, namely, by

$f_{\hat{\chi}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n)$, $x_n \in \mathcal{V}_{\chi_n^*}$, of Eq. (41). That is, the Code- $(n+1)$ input variable ω_{n+1} is uncertain and this input uncertainty, as specified by the PDF $f_{W_{n+1}^{\omega_{n+1}}}(w_{n+1})$, is, for all $w_{n+1} \in \mathcal{V}_{W_{n+1}^*} = \mathcal{V}_{\chi_n^*}$, precisely

$$\begin{aligned} f_{W_{n+1}^{\omega_{n+1}}}(w_{n+1}) &= f_{\hat{\chi}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; \mathcal{J}_{n, n+1}^{-1}(w_{n+1})) \\ &= f_{\hat{\chi}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; w_{n+1}). \end{aligned} \quad (203)$$

Note that, using Eq. (42) with $n \rightarrow n+1$, along with Eq. (183), we have

$$\begin{aligned} \int_{w_{n+1, \min}}^{w_{n+1, \max}} f_{W_{n+1}^{\omega_{n+1}}}(w_{n+1}) dw_{n+1} &= \int_{x_{n, \min}}^{x_{n, \max}} f_{W_{n+1}^{\omega_{n+1}}}(w_{n+1}) dw_{n+1} \\ &= \int_{x_{n, \min}}^{x_{n, \max}} f_{\hat{\chi}_n}(\omega_n, f_{W_n^{\omega_n}}, g_n; w_{n+1}) dw_{n+1} \\ &= 1 \end{aligned} \quad (204)$$

as required. Now, using Eq (203) in Eq. (41) with $n \rightarrow n+1$ we find for all $x_{n+1} \in \mathcal{V}_{\chi_{n+1}^*}$ that

$$\begin{aligned}
& f_{\hat{x}_{n+1}}(\omega_{n+1}, f_{w_{n+1}^{\omega_{n+1}}}, g_{n+1}; x_{n+1}) \\
&= \int_{w_{n+1, \min}}^{w_{n+1, \max}} f_{w_{n+1}^{\omega_{n+1}}}(w_{n+1}) f_{x_{n+1}^{g_{n+1}(w_{n+1})}}(x_{n+1}) dw_{n+1} \\
&= \int_{w_{n+1, \min}}^{w_{n+1, \max}} f_{\hat{x}_n}(\omega_n, f_{w_n^{\omega_n}}, g_n; w_{n+1}) f_{x_{n+1}^{g_{n+1}(w_{n+1})}}(x_{n+1}) dw_{n+1}.
\end{aligned} \tag{205}$$

In particular, for $n = N - 1$ we have for all $x_N \in \mathcal{V}_{X_N}^*$

$$\begin{aligned}
& f_{\hat{x}_N}(\omega_N, f_{w_N^{\omega_N}}, g_N; x_N) \\
&= \int_{w_N, \min}^{w_N, \max} f_{\hat{x}_{N-1}}(\omega_{N-1}, f_{w_{N-1}^{\omega_{N-1}}}, g_{N-1}; w_N) f_{x_N^{g_N(w_N)}}(x_N) dw_N.
\end{aligned} \tag{206}$$

Now by alternatively writing $f_{\hat{x}_{N-1}}(\omega_{N-1}, f_{w_{N-1}^{\omega_{N-1}}}, g_{N-1}; w_N)$ in the integral in Eq. (206) by using Eq. (205) (the second integral therein) with $n = N - 2$, we have

$$\begin{aligned}
& f_{\hat{x}_{N-1}}(\omega_{N-1}, f_{w_{N-1}^{\omega_{N-1}}}, g_{N-1}; w_N) \\
&= \int_{w_{N-1, \min}}^{w_{N-1, \max}} f_{\hat{x}_{N-2}}(\omega_{N-2}, f_{w_{N-2}^{\omega_{N-2}}}, g_{N-2}; w_{N-1}) f_{x_{N-1}^{g_{N-1}(w_{N-1})}}(w_N) dw_{N-1}
\end{aligned} \tag{207}$$

so that, upon replacing $f_{\hat{x}_{N-1}}(\omega_{N-1}, f_{w_{N-1}^{\omega_{N-1}}}, g_{N-1}; w_N)$ in Eq. (206) via Eq. (207), we have

$$\begin{aligned}
& f_{\hat{x}_N}(\omega_N, f_{w_N^{\omega_N}}, g_N; x_N) \\
&= \int_{w_N, \min}^{w_N, \max} dw_N \int_{w_{N-1, \min}}^{w_{N-1, \max}} dw_{N-1} \{ f_{\hat{x}_{N-2}}(\omega_{N-2}, f_{w_{N-2}^{\omega_{N-2}}}, g_{N-2}; w_{N-1}) \\
&\quad \times f_{x_{N-1}^{g_{N-1}(w_{N-1})}}(w_N) f_{x_N^{g_N(w_N)}}(x_N) \}.
\end{aligned} \tag{208}$$

Then further replacing $f_{\hat{x}_{N-2}}(\omega_{N-2}, f_{w_{N-2}^{\omega_{N-2}}}, g_{N-2}; w_{N-1})$ in Eq. (208) by repeatedly using

Eq. (205) until $n = 1$ we arrive at

$$\begin{aligned}
& f_{\hat{\chi}_N}(\omega_N, f_{W_N^{\omega_N}}, g_N; x_N) \\
&= \int_{w_{N, \min}}^{w_{N, \max}} dw_N \int_{w_{N-1, \min}}^{w_{N-1, \max}} dw_{N-1} \int_{w_{N-2, \min}}^{w_{N-2, \max}} dw_{N-2} \cdots \int_{w_{1, \min}}^{w_{1, \max}} dw_1 \{ \bullet \}, \\
&\quad \{ \bullet \} = f_{W_1^{\omega_1}}(w_1) f_{X_1^{g_1(w_1)}}(w_2) f_{X_2^{g_2(w_2)}}(w_3) \cdots f_{X_{N-1}^{g_{N-1}(w_{N-1})}}(w_N) f_{X_N^{g_N(w_N)}}(x_N). \\
&\equiv f_{\hat{\chi}_N}(\omega_1, f_{W_1^{\omega_1}}, g_1, \dots, g_N; x_N)
\end{aligned} \tag{209}$$

where the last line in Eq. (209) is our suggestive notation for the compoundly smeared PDF for the value x_N of Code- N output variable χ_N when the value of the Code-1 input variable $\mathcal{W}_1$ is ω_1 , with $\mathcal{W}_1$ having uncertainty PDF given by $f_{W_1^{\omega_1}}$.

IV. Stochastic Phenomena Within Code- n

In this section, we develop a formalism to treat the presence within Code- n itself of a particular type of stochastic model, with the stochastic considerations comprised by that model differing from those arising from the uncertainty quantification considerations that we have been addressing heretofore. This additional Code- n model stochasticity arises, for example, when Code- n models the so-called *probability-of-effect* (P_e) phenomenon associated with the effect of high-power electromagnetic radiation (HPEM) upon digital electronics (where n is typically 1 or 2). We first develop a general formalism to treat this additional Code- n stochastic modeling and then illustrate the application of this formalism to P_e .

IV.1 Preliminaries

Until this point in our developments, we have been concerned with the situation in which the particular value $\xi_{n,\$} = \xi_{n;\omega_{n,\$}} \in \mathcal{V}_{\chi_n}$ of stand-alone Code- n output variable χ_n —that value resulting from particular value $\omega_{n,\$} \in \mathcal{V}_{\mathcal{W}_n}$ of Code- n input variable $\mathcal{W}_n$ —is “always” (i.e., for every Code- n run) given *via* $\xi_{n,\$} = g_n(\omega_{n,\$})$ for some *single*, specified function g_n . In the current section we extend our development to encompass the case in which the value $\xi_{n,\$}$ resulting from some particular given value $\omega_{n,\$}$ is given **not** by one and the same single function g_n for all Code- n runs but rather is given, in each individual Code- n

run, by some (run-dependent) member $g_{n;\mu_n}$ of a family $\mathcal{G}_n = \{g_{n;\mu_n}\}_{\mu_n \in S_n}$ of such functions (each $g_{n;\mu_n}$ constrained as per g_n [as given Section II]) *via*

$$\xi_{n,s} = \xi_{n,s}^{(\mu_n)} = g_{n;\mu_n}(\omega_{n,s}), \quad (210)$$

with the particular $g_{n;\mu_n}$ employed in any individual Code- n run being chosen stochastically from $\mathcal{G}_n$ and with $g_{n;\mu_n}$ —hence also $\xi_{n,s} = \xi_{n,s}^{(\mu_n)}$ —therefore varying in general from run to run. *Even when the value of ω_n remains unchanged at some particular $\omega_{n,s}$ in all of those runs, the value $\xi_{n,s} = \xi_{n,s}^{(\mu_n)}$ of output variable X_n differs in each of those runs, with $\xi_{n,s}^{(\mu_n)}$ thus being stochastically determined as well since $g_{n;\mu_n}$ is so determined.*

We point out that the index μ_n above is written in boldface to indicate that it is in general a multi-component entity: $\mu_n = \langle \mu_n^{[1]}, \dots, \mu_n^{[M_n]} \rangle$, with $\mu_n^{[q]} \in \mathbb{R}$ for

$q = 1, \dots, M_n$. Also, the associated index set $S_n \subseteq \mathbb{R}^{M_n}$ is chosen as appropriate to the specific situation under consideration and may be only countable.

The introduction of family $\mathcal{G}_n$ necessitates that modifications be made to the probability space $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow}, \mathcal{P}_n^{\rightarrow \xi_n})$ and to the random variable $X_n^{\xi_n}$ therein representing the Code- n real-word output variable X_n^* when the Code- n input is sharp—both of these unmodified entities appearing in Section II.1—as well to the probability space $(\Omega_{nn}, \mathcal{A}_{nn}, \mathcal{P}_{nn}^{\circ, g_n})$ and to the random variable $\hat{X}_n$ therein representing the Code- n output variable X_n when the Code- n input is non-sharp—both of these unmodified entities appearing in Section II.3. In what follows, we present the details of these modifications. We are ultimately interested in the required modifications in the latter of these two cases (because the P_e application is an example of the latter case) but—as in Section II—we want to first treat the former. As in all of our earlier sections, we still take $\omega_{n,s} \in \mathcal{V}_{\omega_n}$ to be fixed, whether the Code- n input be sharp or not.

IV.2 Probability Spaces

In this section we present the details or the required probability space modifications mentioned above as well as present a new probability space for the index set S_n .

IV.2.1 Probability Space for S_n

The stochastic behavior of the selection of $g_{n;\mu_n} \in G_n$ is captured by introducing a probability space $\Omega_{n,S_n} \equiv (S_n, \mathcal{A}_{S_n}, \mathcal{P}_{S_n})$ with $\mathcal{A}_{S_n} \subseteq 2^{\mathbb{R}^{M_n}}$ and with $\mathcal{P}_{S_n}(E)$ assigned for every event $E \in \mathcal{A}_{S_n}$. We assume *for convenience* that $\mathcal{P}_{S_n}(E)$ is specified by PDF $f_{S_n}: \mathbb{R}^{M_n} \rightarrow \mathbb{R}$ (i.e., by a PDF $f_{I_{S_n}}$ for the identity random variable I_{S_n} on $(S_n, \mathcal{A}_{S_n})$ to $\mathbb{R}^{M_n}$, with $f_{S_n} \equiv f_{I_{S_n}}$) so that

$$\begin{aligned} \mathcal{P}_{S_n}(E) &= \mathcal{P}_{S_n}(I_{S_n} \in E) \\ &= \int_E f_{I_{S_n}}(\mu_n) d\mu_n = \int_E f_{S_n}(\mu_n) d\mu_n = \int_E f_{S_n}(\mu_n^{[1]}, \dots, \mu_n^{[M_n]}) d\mu_n^{[1]} \dots d\mu_n^{[M_n]} \end{aligned} \quad (211)$$

with $\mathcal{P}_{S_n}(S_n) = 1$; note that

$$f_{S_n} = 0 \text{ in } \mathbb{R}^{M_n} \setminus S_n. \quad (212)$$

IV.2.2 Probability Space for Code- n Standing Alone and f_{W_n, ω_n} Sharp

For our discussion, it is useful to consider, in addition to G_n , the set of values $\Xi_{G_n}[\omega_n] \equiv \{g_{n;\mu_n}(\omega_n)\}_{\mu_n \in S_n} = \{\xi_n^{(\mu_n)}\}_{\mu_n \in S_n}$ (where we have dropped the label “\$”). To proceed we observe, firstly, that the outcome set $\Omega_n^{\rightarrow} = \mathcal{V}_{X_n^*}$ for the previous Code- n -standing-alone probability space still serves as the appropriate outcome set for every $\mu_n \in S_n$. The previous random variable $X_n^{\xi_n}$ on $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow})$, having PDF $f_{X_n^{\xi_n}}$, now becomes a family of random variables $\{X_n^{\xi_n^{(\mu_n)}}\}_{\mu_n \in S_n}$, with $X_n^{\xi_n^{(\mu_n)}}$ on $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow})$ having PDF $f_{X_n^{\xi_n^{(\mu_n)}}}$. The previous probability measure $\mathcal{P}_n^{\rightarrow \xi_n}$ on $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow})$ now becomes a family of probability measures $\{\mathcal{P}_n^{\rightarrow \xi_n^{(\mu_n)}}\}_{\mu_n \in S_n}$ with $\mathcal{P}_n^{\rightarrow \xi_n^{(\mu_n)}}$ on $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow})$ given by $\mathcal{P}_n^{\rightarrow \xi_n^{(\mu_n)}}(T) = \int_T f_{X_n^{\xi_n^{(\mu_n)}}}(x_n) dx_n$ for every $T \in$

$\mathcal{A}_n^{\rightarrow}$, where each PDF $f_{X_n^{\xi_n}(\mu_n)}$ is one of the four $f_{X_n^{\xi_n}(\mu_n)}$'s of Eqs. (12) – (15) (with ξ_n there $\rightarrow \xi_n(\mu_n)$ here) for some $\alpha \in \{A, B, C, D\}$; also note that

$$f_{X_n^{\xi_n}(\mu_n)}(x_n) = 0 \text{ in } \mathbb{R} \setminus \mathcal{V}_{X_n^*}. \quad (213)$$

Next, to accommodate in addition the stochastic $\mu_n \in S_n$ we first form outcome set

$$\Omega_{n, S_n}^{\rightarrow} = S_n \times \Omega_n^{\rightarrow} \quad (214)$$

containing ordered pairs $\langle \mu_n, x_n \rangle \in \mathbb{R}^{M_n+1}$ (where σ -algebra $\mathcal{A}_{n, S_n}^{\rightarrow} \subseteq 2^{\mathbb{R}^{M_n+1}}$ is that

generated by $\mathcal{A}_{S_n} \times \mathcal{A}_n^{\rightarrow}$), with random variable $X_{n, S_n}^{\xi_n}$ given on $(\Omega_{n, S_n}^{\rightarrow}, \mathcal{A}_{n, S_n}^{\rightarrow})$ by

$X_{n, S_n}^{\xi_n}(\mu_n, x_n) = x_n$ and having PDF $f_{X_{n, S_n}^{\xi_n}} : \mathbb{R} \rightarrow \mathbb{R}$, and with additional random variable $\mathcal{M}_{n, S_n}$

on $(\Omega_{n, S_n}^{\rightarrow}, \mathcal{A}_{n, S_n}^{\rightarrow})$ given by $\mathcal{M}_{n, S_n}(\mu_n, x_n) = \mu_n$ and having (assumed) PDF $f_{\mathcal{M}_{n, S_n}} : \mathbb{R}^{M_n} \rightarrow \mathbb{R}$.

For the 2-vector random variable $\langle \mathcal{M}_{n, S_n}, X_{n, S_n}^{\xi_n} \rangle$ we then have

$$f_{\langle \mathcal{M}_{n, S_n}, X_{n, S_n}^{\xi_n} \rangle}(\mu_n, x_n) = f_{X_{n, S_n}^{\xi_n} | \mathcal{M}_{n, S_n}}(x_n | \mu_n) f_{\mathcal{M}_{n, S_n}}(\mu_n) \quad (215)$$

with $f_{\langle \mathcal{M}_{n, S_n}, X_{n, S_n}^{\xi_n} \rangle}(\mu_n, x_n) = 0$ in $\mathbb{R}^{M_n+1} \setminus \Omega_{n, S_n}^{\rightarrow}$. Now, denoting the probability measure on

$(\Omega_{n, S_n}^{\rightarrow}, \mathcal{A}_{n, S_n}^{\rightarrow})$ by $\mathcal{P}_{n, S_n}^{\xi_n}$ we observe that since for any $\mu_n \in S_n$ and $T \in \mathcal{A}_n^{\rightarrow}$ we want to have

$$\mathcal{P}_{n, S_n}^{\xi_n}(X_{n, S_n}^{\xi_n} \in T | \mathcal{M}_{n, S_n} = \mu_n) = \mathcal{P}_n^{\xi_n(\mu_n)}(X_n^{\xi_n(\mu_n)} \in T), \text{ i.e., } \int_T f_{X_{n, S_n}^{\xi_n} | \mathcal{M}_{n, S_n}}(x_n | \mu_n) dx_n =$$

$\int_T f_{X_n^{\xi_n}(\mu_n)}(x_n) dx_n$, then we take

$$f_{X_{n, S_n}^{\xi_n} | \mathcal{M}_{n, S_n}}(x_n | \mu_n) = f_{X_n^{\xi_n}(\mu_n)}(x_n), \quad x_n \in \mathcal{V}_{X_n^*}; \quad (216)$$

note that, by Eq. (213), $f_{X_{n, S_n}^{\xi_n} | \mathcal{M}_{n, S_n}}(\bullet | \mu_n) = 0$ in $\mathbb{R} \setminus \mathcal{V}_{X_n^*}$. Further, since for any $E \in \mathcal{A}_{S_n}$ we

want $\mathcal{P}_{n, S_n}^{\xi_n}(\mathcal{M}_{n, S_n} \in E) = \mathcal{P}_{S_n}(E)$, i.e., we want

$$\int_E f_{S_n}(\mu_n) d\mu_n = \int_E \int_{x_{n, \min}}^{x_{n, \max}} f_{\langle \mathcal{M}_{n, S_n}, X_{n, S_n}^{\xi_n} \rangle}(\mu_n, x_n) dx_n d\mu_n$$

$$= \int_E \int_{x_{n, \min}}^{x_{n, \max}} f_{X_{n, S_n}^{\xi_n} | \mathcal{M}_{n, S_n}}(x_n | \mu_n) f_{\mathcal{M}_{n, S_n}}(\mu_n) dx_n d\mu_n \quad (217)$$

$$= \int_E f_{\mathcal{M}_{n, S_n}}(\mu_n) \left[\int_{x_{n, \min}}^{x_{n, \max}} f_{X_n^{\xi_n}(\mu_n)}(x_n) dx_n \right] d\mu_n = \int_E f_{\mathcal{M}_{n, S_n}}(\mu_n) \cdot 1 d\mu_n$$

—where in the above we have used Eq. (216) and have used Eq. (10) with $X_n^{\xi_n} = X_n^{\xi_n(\mu_n)}$

(since $f_{X_n^{\xi_n}(\mu_n)} = f_{X_n^{\xi_n}(\mu_n)}^\alpha$ for some α)—then we take

$$f_{\mathcal{M}_{n, S_n}}(\mu_n) = f_{S_n}(\mu_n), \quad \mu_n \in S_n, \quad (218)$$

with $f_{\mathcal{M}_{n, S_n}}(\mu_n) = 0$ in $\mathbb{R}^{M_n} \setminus S_n$ via Eq. (212). Thus Eq. (215) becomes

$$f_{\langle \mathcal{M}_{n, S_n}, X_{n, S_n}^{\xi_n} \rangle}(\mu_n, x_n) = f_{X_n^{\xi_n}(\mu_n)}(x_n) f_{S_n}(\mu_n), \quad \langle \mu_n, x_n \rangle \in \Omega_{n, S_n}^{\rightarrow} \quad (219)$$

(with $f_{\langle \mathcal{M}_{n, S_n}, X_{n, S_n}^{\xi_n} \rangle}(\mu_n, x_n) = 0$ in $\mathbb{R}^{M_n+1} \setminus \Omega_{n, S_n}^{\rightarrow}$ as above). The marginal PDF for $X_{n, S_n}^{\xi_n}$ is

given, for all $x_n \in \mathcal{V}_{X_n^*}$, by

$$f_{X_{n, S_n}^{\xi_n}}(\Xi_{\mathcal{G}_n}; x_n) \equiv \int_{S_n} f_{\langle \mathcal{M}_{n, S_n}, X_{n, S_n}^{\xi_n} \rangle}(\mu_n, x_n) d\mu_n = \int_{S_n} f_{S_n}(\mu_n) f_{X_n^{\xi_n}(\mu_n)}(x_n) d\mu_n, \quad (220)$$

namely, as a μ_n -weighted superposition of all the $f_{X_n^{\xi_n}(\mu_n)}(x_n)$'s. Also, as required,

$$\begin{aligned} \int_{x_{n, \min}}^{x_{n, \max}} f_{X_{n, S_n}^{\xi_n}}(\Xi_{\mathcal{G}_n}; x_n) dx_n &= \int_{x_{n, \min}}^{x_{n, \max}} \int_{S_n} f_{S_n}(\mu_n) f_{X_n^{\xi_n}(\mu_n)}(x_n) d\mu_n dx_n \\ &= \int_{S_n} f_{S_n}(\mu_n) \int_{x_{n, \min}}^{x_{n, \max}} f_{X_n^{\xi_n}(\mu_n)}(x_n) dx_n d\mu_n = \int_{S_n} f_{S_n}(\mu_n) \cdot 1 d\mu_n = 1. \end{aligned} \quad (221)$$

The (compoundly smeared [via $f_{S_n}(\mu_n)$ rather than via $f_{W_n^{\omega_n}}(w_n)$]) PDF $f_{X_{n, S_n}^{\xi_n}}(\Xi_{\mathcal{G}_n}; x_n)$ of Eq.

(220) for output variable X_n supplants the (basically smeared) PDF $f_{X_n^{\xi_n}}(x_n)$ of Section II.2

for output variable X_n when index set S_n —hence also the set $\Xi_{\mathcal{G}_n}$ —is not a singleton *but*

when $f_{W_n^{\omega_n}}$ is sharp (see, in particular, the comments following Eq. (42)). Otherwise, when

S_n is a singleton (and $f_{W_n^{\omega_n}}$ is sharp), say $S_n = \{0\}^{M_n}$ —hence if $\mu_n \in S_n$ then $\mu_n = \langle 0, 0, \dots, 0 \rangle \equiv \mu_{n,0}$ (an M_n -tuple)—then $\Xi_{\mathcal{G}_n} = \{\xi_n^{\mu_{n,0}}\}$ and $f_{S_n}(\mu_n) = \delta(\mu_n - \mu_{n,0})$ so from Eq. (220) we have

$$f_{X_n, S_n}^{\xi_n}(\Xi_{\mathcal{G}_n}; x_n) = \int_{S_n} \delta(\mu_n - \mu_{n,0}) f_{X_n, S_n}^{\xi_n^{\mu_{n,0}}}(x_n) d\mu_n = f_{X_n, S_n}^{\xi_n^{\mu_{n,0}}}(x_n) = f_{X_n, g_n; \mu_{n,0}}^{\alpha}(\omega_{n,0})(x_n) \quad (222)$$

where $\xi_n^{\mu_{n,0}} = g_n; \mu_{n,0}(\omega_{n,0})$ and $\alpha \in \{A, B, C, D\}$. In this singleton context we may also write

$$f_{X_n, S_n}^{\xi_n}(\Xi_{\mathcal{G}_n}; x_n) = f_{X_n, S_n}^{\xi_n}(\{\xi_n^{\mu_{n,0}}\}; x_n) = f_{X_n, g_n; \mu_{n,0}}^{\alpha}(\omega_{n,0})(x_n). \quad (223)$$

IV.2.3 Probability Space for Code- n Standing Alone and $f_{W_n^{\omega_n}}$ Non-Sharp

The outcome set for the applicable probability space here becomes

$$\Omega_{nn, S_n} = \Omega_n^{\leftarrow} \times \Omega_{n, S_n}^{\rightarrow} \quad (224)$$

containing ordered triples $\langle w_n, \mu_n, x_n \rangle$, with random variables $\hat{W}_{n, S_n}^{\omega_n}$, $\hat{\mathcal{M}}_{n, S_n}$ and $\hat{X}_{n, S_n}$ on

$(\Omega_{nn, S_n}, \mathcal{A}_{nn, S_n})$ being given by $\hat{W}_{n, S_n}^{\omega_n}(w_n, \mu_n, x_n) = w_n$, $\hat{\mathcal{M}}_{n, S_n}(w_n, \mu_n, x_n) = \mu_n$ and

$\hat{X}_{n, S_n}(w_n, \mu_n, x_n) = x_n$. For 3-vector random variable $\langle \hat{W}_{n, S_n}^{\omega_n}, \hat{\mathcal{M}}_{n, S_n}, \hat{X}_{n, S_n} \rangle$ we have

$$\begin{aligned} f_{\langle \hat{W}_{n, S_n}^{\omega_n}, \hat{\mathcal{M}}_{n, S_n}, \hat{X}_{n, S_n} \rangle}(w_n, \mu_n, x_n) \\ = f_{\hat{X}_{n, S_n} | \langle \hat{W}_{n, S_n}^{\omega_n}, \hat{\mathcal{M}}_{n, S_n} \rangle}(x_n | w_n, \mu_n) f_{\hat{\mathcal{M}}_{n, S_n} | \hat{W}_{n, S_n}^{\omega_n}}(\mu_n | w_n) f_{\hat{W}_{n, S_n}^{\omega_n}}(w_n) \end{aligned} \quad (225)$$

and we take (see Eqs. (37) - (40), (215) and (219)), for all $w_n \in \mathcal{V}_{W_n^*}$, $\mu_n \in S_n$, and

$x_n \in \mathcal{V}_{X_n^*}$,

$$\begin{aligned} f_{\hat{X}_{n, S_n} | \langle \hat{W}_{n, S_n}^{\omega_n}, \hat{\mathcal{M}}_{n, S_n} \rangle}(x_n | w_n, \mu_n) &= f_{X_n, g_n; \mu_n}(x_n), \\ f_{\hat{\mathcal{M}}_{n, S_n} | \hat{W}_{n, S_n}^{\omega_n}}(\mu_n | w_n) &= f_{\hat{\mathcal{M}}_{n, S_n}}(\mu_n) = f_{S_n}(\mu_n), \\ f_{\hat{W}_{n, S_n}^{\omega_n}}(w_n) &= f_{W_n^{\omega_n}}(w_n), \end{aligned} \quad (226)$$

where the first equality in the second line of Eq. (226) holds because we consider random variables $\hat{W}_{n,S_n}^{\omega_n}$ and $\hat{\mathcal{M}}_{n,S_n}$ to be independent. Then the marginal PDF for $\hat{X}_{n,S_n}$ is given, for all $x_n \in \mathcal{V}_{X_n^*}$, by

$$\begin{aligned} f_{\hat{X}_{n,S_n}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) &\equiv f_{\hat{X}_{n,S_n}}^{\wedge}(x_n) \\ &= \int_{S_n} \int_{w_{n, \min}}^{w_{n, \max}} f_{\langle \hat{W}_{n,S_n}^{\omega_n}, \hat{\mathcal{M}}_{n,S_n}, \hat{X}_{n,S_n} \rangle}(w_n, \mu_n, x_n) dw_n d\mu_n \\ &= \int_{S_n} f_{S_n}(\mu_n) \int_{w_{n, \min}}^{w_{n, \max}} f_{W_n^{\omega_n}}(w_n) f_{X_n^{g_n, \mu_n}(w_n)}(x_n) dw_n d\mu_n. \end{aligned} \quad (227)$$

Now by Eq. (41),

$$f_{\hat{X}_n}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) = \int_{w_{n, \min}}^{w_{n, \max}} f_{W_n^{\omega_n}}(w_n) f_{X_n^{g_n, \mu_n}(w_n)}(x_n) dw_n \quad (228)$$

so Eq. (227) becomes

$$f_{\hat{X}_{n,S_n}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) = \int_{S_n} f_{S_n}(\mu_n) f_{\hat{X}_n}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) d\mu_n \quad (229)$$

for all $x_n \in \mathcal{V}_{X_n^*}$, namely, a μ_n -weighted superposition of all the $f_{\hat{X}_n}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n)$'s (as expected). Note that $f_{X_n^{g_n, \mu_n}(w_n)}(x_n) = 0$ in $\mathbb{R} \setminus \mathcal{V}_{X_n^*}$ so that as well

$$f_{\hat{X}_n}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) = 0 \quad (\text{all } \mu_n \in S_n) \text{ \& } f_{\hat{X}_{n,S_n}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) = 0, \quad x_n \in \mathbb{R} \setminus \mathcal{V}_{X_n^*}. \quad (230)$$

The required $f_{\hat{X}_n}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n)$'s in Eq. (229) may be obtained by using Eq. (41) for each $\mu_n \in S_n$. Using Eqs. (42) we have, for all $\mu_n \in S_n$,

$$\int_{x_{n, \min}}^{x_{n, \max}} f_{\hat{X}_n}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) dx_n = 1 \quad (231)$$

so that

$$\begin{aligned} &\int_{x_{n, \min}}^{x_{n, \max}} f_{\hat{X}_{n,S_n}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) dx_n \\ &= \int_{x_{n, \min}}^{x_{n, \max}} \int_{S_n} f_{S_n}(\mu_n) f_{\hat{X}_n}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) d\mu_n dx_n \\ &= \int_{S_n} f_{S_n}(\mu_n) \int_{x_{n, \min}}^{x_{n, \max}} f_{\hat{X}_n}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) dx_n d\mu_n \\ &= \int_{S_n} f_{S_n}(\mu_n) \cdot 1 d\mu_n = 1 \end{aligned} \quad (232)$$

as well.

The PDF $f_{\chi_n, S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n)$ of Eq. (229) for output variable χ_n supplants the PDF $f_{\chi_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, g_n; x_n)$ of Section II.3 for output variable χ_n when index set S_n is not a singleton and when $f_{W_n^{\omega_n}}$ is in general non-sharp. Otherwise, when S_n is a singleton, $S_n = \{0\}^{M_n}$ (and $f_{W_n^{\omega_n}}$ may be non-sharp), then from Eq. (229) with $f_{S_n}(\mu_n) = \delta(\mu_n - \mu_{n,0})$ we have

$$f_{\chi_n, S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) = f_{\chi_n, S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, \{g_{n; \mu_{n,0}}\}; x_n) = f_{\chi_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, g_{n; \mu_{n,0}}; x_n).$$

IV.3 The Parameter Pair $\langle a_n, b_n \rangle$

In this section we introduce an additional allowed possibility—which possibility is a departure from what we have presented in this paper until now—for the Code- n parameter pair $\langle a_n, b_n \rangle$. This possibility is of importance here because it captures the situation encountered in the P_e modeling referred to in the introductory paragraph of this section (IV). It may be convenient in this regard to think of Code- n as performing only one task: taking input ω_n and producing output $\xi_n^{(\mu_n)} = g_{n; \mu_n}(\omega_n)$ for any and all possible $\mu_n \in S_n$. In this point of view, then, a_n and b_n need be assigned only to quantify the “goodness” of *that* Code- n computation.

IV.3.1 $0 < a_n \leq 1$ and $0 < b_n < 1$

As indicated above, the $f_{\chi_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, g_{n; \mu_n}; x_n)$'s in Eq. (228) required to compute $f_{\chi_n, S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n)$ in Eq. (229) may be obtained using Eq. (41)—or the more computationally friendly Eq. (187)—for each $\mu_n \in S_n$, with the usual a_n and b_n playing a role in Eq. (228) via the factor $f_{\chi_n}^{g_{n; \mu_n}(w_n)}(x_n)$ appearing in its integrand.

IV.3.2 $a_n \rightarrow 1$ and $b_n \rightarrow 0$

We now introduce the additional possibility referred to above for the assignment of a_n and b_n to Code- n . Essentially, we take the limit $\langle a_n, b_n \rangle \rightarrow \langle 1, 0 \rangle$, i.e., we posit that (see Eq. (8))

$$(\forall b_n > 0)\{\text{Prob}[(1 - b_n)\xi_{n;\omega_n} \leq x_{n;w_n} \mid w_n = \omega_n \leq (1 + b_n)\xi_{n;\omega_n}] = 1\}; \quad (233)$$

in this case, a_n and b_n need not be assigned to Code- n at all. This possibility corresponds to the situation in which one is “certain” that the functional form of each $g_{n;\mu_n}$ ($\mu_n \in S_n$) correctly models the physical situation under consideration in Code- n but one is unsure only as to which particular value of $\mu_n \in S_n$ is in fact the correct one for the problem at hand. That is, for each particular $g_{n;\mu_n}$ there is no Code- n modeling uncertainty (source- (2)) whatsoever—the Code- n modeling uncertainty arises entirely from the uncertainty as to which particular $g_{n;\mu_n}$ is in fact the correct one. Thus, in particular, there is—for each $g_{n;\mu_n}$ —no smearing at all of Code- n output variable χ_n at output value $\xi_{n;\omega_n}^{(\mu_n)} = g_{n;\mu_n}(\omega_n) \in \mathcal{V}_{\chi_n}$ (where the value of input variable $\mathcal{W}_n$ is $\omega_n \in \mathcal{V}_{w_n}$); i.e., when $w_n = \omega_n$ then the set of possible real-world values $x_n \in \mathcal{V}_{x_n}^*$ of real-world variable X_n^* represented in Code- n by χ_n is precisely the singleton $\{g_{n;\mu_n}(\omega_n)\}$, $g_{n;\mu_n}(\omega_n) \in \mathcal{V}_{\chi_n}$. (We assume here for the experiment modeled by Code- n , as we assumed in the Introduction to this paper, a single experimental value $x_{n;w_n}$ of X_n^* obtained by averaging over “sufficiently many” repeated trials of the experiment using input value $w_n = \omega_n$ of W_n^* , with that single obtained experimental value being $x_{n;w_n} = g_{n;\mu_n}(\omega_n)$.) We consider firstly just a single $g_{n;\mu_n}$ (i.e., $\#(S_n) = 1$); we then consider once again the general $\mathcal{G}_n = \{g_{n;\mu_n}\}_{\mu_n \in S_n}$.

As indicated above, in the present case, the random variable X_n on $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow})$ that represents X_n^* —that random variable heretofore, but no longer, being the basically smeared $X_n^{g_{n;\mu_n}(\omega_n)}$ —is not basically smeared. Indeed, this replacement random variable on $(\Omega_n^{\rightarrow}, \mathcal{A}_n^{\rightarrow})$ must have PDF $f(x_n) = \delta(x_n - g_{n;\mu_n}(\omega_n))$ for $x_n \in \mathcal{V}_{\chi_n}$ (in fact therefore for all $x_n \in \mathbb{R}$). Labeling this new random variable representing X_n^* as $X_{n,\delta}^{g_{n;\mu_n}(\omega_n)}$ we then have, for all $\omega_n \in \mathcal{V}_{w_n}$ and $x_n \in \mathcal{V}_{\chi_n}^*$,

$$f_{X_{n,\delta}^{g_{n;\mu_n}(\omega_n)}}(x_n) = \delta(x_n - g_{n;\mu_n}(\omega_n)). \quad (234)$$

Now we observe that the reasoning leading from Eq. (37) to Eq. (41) using $X_n^{g_n(\omega_n)}$ there also applies when using $X_{n,\delta}^{g_n;\mu_n(\omega_n)}$ in lieu of $X_n^{g_n(\omega_n)}$. Then *via* the “ δ -analogue” of Eq. (41), we have, for $x_n \in \mathcal{V}_{X_n^*}$,

$$\begin{aligned} f_{X_{n,\delta}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) \\ = \int_{w_{n, \min}}^{w_{n, \max}} f_{W_n^{\omega_n}}(w_n) f_{X_{n,\delta}^{g_n;\mu_n(w_n)}}(x_n) dw_n \\ = \int_{w_{n, \min}}^{w_{n, \max}} f_{W_n^{\omega_n}}(w_n) \delta(x_n - g_{n;\mu_n}(w_n)) dw_n. \end{aligned} \quad (235)$$

We next need to evaluate the integral in Eq. (235). For our application to *probability-of-effect* we consider only the case for which $g'_{n;\mu_n}$ exists and is continuous on its domain $\mathcal{V}_{w_n}$ and is never zero there; hence $g'_{n;\mu_n} > 0$ on $\mathcal{V}_{w_n}$, or $g'_{n;\mu_n} < 0$ on $\mathcal{V}_{w_n}$, so $g_{n;\mu_n}$ is 1-1 there and we write $g_{n;\mu_n}^{-1}(\{x_n\}) = g_{n;\mu_n}^{-1}(x_n)$ whenever $x_n \in \mathcal{V}_{X_n^*}$, where

$$\mathcal{V}_{X_n^*} \equiv [\xi_{n,\min}^{g_{n;\mu_n}}, \xi_{n,\max}^{g_{n;\mu_n}}]; \quad (236)$$

if $x_n \notin \mathcal{V}_{X_n^*}$, so that $x_n \in \mathcal{V}_{X_n^*} \setminus \mathcal{V}_{X_n^*}$, then $g_{n;\mu_n}^{-1}(\{x_n\}) = \emptyset$ and we cannot write $g_{n;\mu_n}^{-1}(x_n)$ for such x_n . To proceed, we seek an alternative form for the $\delta(x_n - g_{n;\mu_n}(w_n))$ factor in Eq.

(235). Making the substitution $t = g_{n;\mu_n}(w_n)$ there we find for $x_n \in \mathcal{V}_{X_n^*}$ that

$$\begin{aligned} f_{X_{n,\delta}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) \\ = \int_{g_{n;\mu_n}(w_{n, \min})}^{g_{n;\mu_n}(w_{n, \max})} [1/g'_{n;\mu_n}(g_{n;\mu_n}^{-1}(t))] f_{W_n^{\omega_n}}(g_{n;\mu_n}^{-1}(t)) \delta(x_n - t) dt \\ = \begin{cases} f_{W_n^{\omega_n}}(g_{n;\mu_n}^{-1}(x_n)) / |g'_{n;\mu_n}(g_{n;\mu_n}^{-1}(x_n))| & \text{if } x_n \in \mathcal{V}_{X_n^*} \\ 0 & \text{if } x_n \in \mathcal{V}_{X_n^*} \setminus \mathcal{V}_{X_n^*} \end{cases} \end{aligned} \quad (237)$$

since, using Eqs. (3), (7) and (236), if $g'_{n;\mu_n} > 0$ then $g_{n;\mu_n}(w_{n, \min}) = g_{n;\mu_n}(\omega_{n, \min}) = \xi_{n,\min}^{g_{n;\mu_n}}$

and $g_{n;\mu_n}(w_{n, \max}) = g_{n;\mu_n}(\omega_{n, \max}) = \xi_{n,\max}^{g_{n;\mu_n}}$, while if $g'_{n;\mu_n} < 0$ then $g_{n;\mu_n}(w_{n, \min}) = \xi_{n,\max}^{g_{n;\mu_n}}$

and since, using Eqs. (3), (7) and (236), if $g'_{n;\mu_n} > 0$ then $g_{n;\mu_n}(w_{n,min}) = g_{n;\mu_n}(\omega_{n,min}) = \xi_{n,min}^{g_{n;\mu_n}}$ and $g_{n;\mu_n}(w_{n,max}) = g_{n;\mu_n}(\omega_{n,max}) = \xi_{n,max}^{g_{n;\mu_n}}$, while if $g'_{n;\mu_n} < 0$ then $g_{n;\mu_n}(w_{n,min}) = \xi_{n,max}^{g_{n;\mu_n}}$ and $g_{n;\mu_n}(w_{n,max}) = \xi_{n,min}^{g_{n;\mu_n}}$. Note that by Eq. (237)

$$\begin{aligned}
& \int_{x_{n,min}}^{x_{n,max}} f_{X_{n,\delta}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) dx_n \\
&= \int_{\xi_{n,min}^{g_{n;\mu_n}}}^{\xi_{n,max}^{g_{n;\mu_n}}} f_{X_{n,\delta}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) dx_n \\
&= \int_{\xi_{n,min}^{g_{n;\mu_n}}}^{\xi_{n,max}^{g_{n;\mu_n}}} [1/|g'_{n;\mu_n}(g_{n;\mu_n}^{-1}(x_n))|] f_{W_n^{\omega_n}}(g_{n;\mu_n}^{-1}(x_n)) dx_n \\
&= \int_{g_{n;\mu_n}^{-1}(\xi_{n,min}^{g_{n;\mu_n}})}^{g_{n;\mu_n}^{-1}(\xi_{n,max}^{g_{n;\mu_n}})} [g'_{n;\mu_n}(z)/|g'_{n;\mu_n}(v)|] f_{W_n^{\omega_n}}(z) dz = 1
\end{aligned} \tag{238}$$

where the last line in this equation follows from the substitution $z = g_{n;\mu_n}^{-1}(x_n)$ and the observation that the lower and upper limits in the final integral are simply $w_{n,min}$ and $w_{n,max}$. The PDF $f_{X_{n,\delta}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n)$ of Eq. (237) for output variable X_n and *single function* $g_n = g_{n;\mu_n}$ supplants the PDF $f_{X_n}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n)$ of Eq. (41) (with g_n in the latter equation given here by $g_{n;\mu_n}$) for output variable X_n when $\langle a_n, b_n \rangle \rightarrow \langle 1, 0 \rangle$, whether $f_{W_n^{\omega_n}}$ is sharp or non-sharp.

Consider finally family $\mathcal{G}_n$ rather than simply the single function $g_{n;\mu_n}$. Using $f_{X_{n,\delta}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n)$ of Eq. (237) in the δ -analogue of Eq. (229) gives $f_{X_{n,\delta}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, \mathcal{G}_n; x_n)$ for $x_n \in \mathcal{V}_{X_n^*}$, namely,

$$\begin{aligned}
& f_{X_{n,\delta}, S_n}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, \mathcal{G}_n; x_n) \\
&= \int_{S_n} f_{S_n}(\mu_n) f_{X_{n,\delta}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, g_{n;\mu_n}; x_n) d\mu_n \\
&= \int_{\mathcal{D}_{S_n}(x_n)} [f_{W_n^{\omega_n}}(g_{n;\mu_n}^{-1}(x_n))/|g'_{n;\mu_n}(g_{n;\mu_n}^{-1}(x_n))|] f_{S_n}(\mu_n) d\mu_n
\end{aligned} \tag{239}$$

where

$$\mathcal{D}_{S_n}(x_n) \equiv \{ \mu_n \in S_n \mid \psi_{x_n; \mu_n} \ni x_n \}. \quad (240)$$

The PDF $f_{x_n, \delta, S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n)$ of Eq. (239) for output variable x_n supplants the PDF $f_{x_n, \delta}^\wedge(\omega_n, f_{W_n^{\omega_n}}, g_{n; \mu_n}; x_n)$ of Eq. (237) for output variable x_n when S_n is not a singleton and $\langle a_n, b_n \rangle \rightarrow \langle 1, 0 \rangle$ (whether $f_{W_n^{\omega_n}}$ is sharp or non-sharp).

To summarize our above several statements about supplanting the PDF

$f_{x_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, g_{n; \mu_n}; x_n)$ of Eq. (41) for output variable x_n we reiterate the following:

(1) If $0 < a_n \leq 1$ and $0 < b_n < 1$ then

(a) if using only one $g_{n; \mu_n}$ then $f_{x_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, g_{n; \mu_n}; x_n)$ of Eq. (41) applies

(b) if using a family G_n with $\#(S_n) > 1$ then $f_{x_n, S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n)$ of Eq. (229) applies in lieu of Eq. (41)

(2) If $\langle a_n, b_n \rangle \rightarrow \langle 1, 0 \rangle$ then

(a) if using only one $g_{n; \mu_n}$ then $f_{x_n, \delta}^\wedge(\omega_n, f_{W_n^{\omega_n}}, g_{n; \mu_n}; x_n)$ of Eq. (237) applies in lieu of Eq. (41)

(b) if using a family G_n with $\#(S_n) > 1$ then $f_{x_n, \delta, S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n)$ of Eq. (239) applies in lieu of Eq. (41).

All four cases above accommodate a general (i.e., sharp or non-sharp) $f_{W_n^{\omega_n}}$.

IV.4 Application to Probability-of-Effect

In this section we illustrate the application of the above formulation, for $a_n \rightarrow 1$ & $b_n \rightarrow 0$, to a family $G_n = \{g_{n; \mu_n}\}_{\mu_n \in S_n}$ that arises from the modeling of probability-of-effect; this is an instance of Case (2)(b) of the previous paragraph. (The formulation in Section IV.3.1 for Case (1)(b) is a straightforward extension of the use of Eq. (41) and/or Eq. (187) and we give no further discussion of it here.)

For this application we take

$$\begin{aligned} \mu_n &= \langle \mu_n^{[1]}, \mu_n^{[2]} \rangle \equiv \langle \alpha, \beta \rangle \\ \langle \alpha, \beta \rangle &\in \mathbb{R} \times \mathbb{R} \setminus \{0\} = S_n, \end{aligned} \quad (241)$$

so that $G_n = \{g_{n, \langle \alpha, \beta \rangle}\}_{\langle \alpha, \beta \rangle \in \mathbb{R} \times \mathbb{R} \setminus \{0\}}$, and

$$g_{n, \langle \alpha, \beta \rangle}(\omega_n) = 1/[1 + \exp(-(\alpha + \beta \ln \omega_n))] = 1/[1 + e^{-\alpha} \omega_n^{-\beta}] \quad (242)$$

with $\text{dom } g_{n, \langle \alpha, \beta \rangle} = [\omega_{n, \min}, \omega_{n, \max}]$ for some fixed but arbitrary $0 < \omega_{n, \min} < \omega_{n, \max} < \infty$;

$g_{n, \langle \alpha, \beta \rangle}$ is a *logistic function*. (In practice, the logistic function as used in the P_e application is usually written as

$g_{n, \langle a_0, a_1 \rangle}(\omega_n) = 1/[1 + \exp(-(a_0 + a_1 \log_{10} \omega_n))]$ so

$$\alpha \equiv a_0 \quad \text{and} \quad \beta \equiv a_1/\ln(10) \quad (243)$$

since $a_1 \log_{10} \omega = a_1 \ln \omega / \ln(10)$.) Further, for $\omega_n \in [\omega_{n, \min}, \omega_{n, \max}]$,

$$g'_{n, \langle \alpha, \beta \rangle}(\omega_n) = \beta e^{-\alpha} \omega_n^{-(\beta + 1)} / [1 + e^{-\alpha} \omega_n^{-\beta}]^2 \quad (244)$$

so when $\beta > 0$ then $g'_{n, \langle \alpha, \beta \rangle} > 0$ and when $\beta < 0$ then $g'_{n, \langle \alpha, \beta \rangle} < 0$. Note that $g_{n, \langle \alpha, \beta \rangle}$ is 1-

1 onto $[\xi_{n, \min}^{g_{n, \langle \alpha, \beta \rangle}}, \xi_{n, \max}^{g_{n, \langle \alpha, \beta \rangle}}]$ where, if $\beta > 0$ then

$$\xi_{n, \min}^{g_{n, \langle \alpha, \beta \rangle}} = g_{n, \langle \alpha, \beta \rangle}(\omega_{n, \min}) \quad \text{and} \quad \xi_{n, \max}^{g_{n, \langle \alpha, \beta \rangle}} = g_{n, \langle \alpha, \beta \rangle}(\omega_{n, \max}), \quad (245)$$

while if $\beta < 0$ then

$$\xi_{n, \min}^{g_{n, \langle \alpha, \beta \rangle}} = g_{n, \langle \alpha, \beta \rangle}(\omega_{n, \max}) \quad \text{and} \quad \xi_{n, \max}^{g_{n, \langle \alpha, \beta \rangle}} = g_{n, \langle \alpha, \beta \rangle}(\omega_{n, \min}). \quad (246)$$

Also,

$$g_{n, \langle \alpha, \beta \rangle}^{-1}(x_n) = e^{-\alpha/\beta} [x_n / (1 - x_n)]^{(1/\beta)}, \quad x_n \in [\xi_{n, \min}^{g_{n, \langle \alpha, \beta \rangle}}, \xi_{n, \max}^{g_{n, \langle \alpha, \beta \rangle}}]. \quad (247)$$

We also point out that, for the current application to P_e , the Code- n output variable x_n is a probability (of effect) so that all values x_n of X_n^* lie in the interval $[0, 1]$. We thus have $x_{n, \max} = 1$; but we may not take $x_{n, \min} = 0$ because we have stipulated in this paper that (for technical reasons) $x_{n, \min} > 0$ (see Eq. (5b)). We thus take $x_{n, \min}$ to be some fixed but arbitrary very small positive value and may then observe, if of interest, what happens to our results when $x_{n, \min}$ limits to 0^+ . Finally, we must specify a PDF $f_{S_n}(\alpha, \beta)$ giving the probability density for $\langle \alpha, \beta \rangle \in S_n$; but we postpone doing so until later when we present an example of computing $f_{X_{n, \delta, S_n}}^{\wedge}(\omega_n, f_{W_n}^{\omega_n}, G_n; x_n)$ analytically. (In practice, $f_{S_n}(\alpha, \beta)$ is

determined, based upon experimental data, using Bayesian methods [2].) Now from Eq. (237) we have

$$f_{X_n, \delta}^{\wedge}(\omega_n, f_{W_n}^{\omega_n}, g_{n, <\alpha, \beta>; x_n) = \begin{cases} f_{W_n}^{\omega_n}(g_{n, <\alpha, \beta>}^{-1}(x_n))/|g'_{n, <\alpha, \beta>}(g_{n, <\alpha, \beta>}^{-1}(x_n))| & \text{if } x_n \in \mathcal{V}_{X_n; <\alpha, \beta>} \\ 0 & \text{if } x_n \in \mathcal{V}_{X_n^*} \setminus \mathcal{V}_{X_n; <\alpha, \beta>} \end{cases} \quad (248)$$

and from Eqs. (239) - (241) we have, for every $x_n \in \mathcal{V}_{X_n^*}$,

$$\begin{aligned} f_{X_n, \delta, S_n}^{\wedge}(\omega_n, f_{W_n}^{\omega_n}, G_n; x_n) &= \int \int_{\mathbb{R}^2} f_{S_n}(\alpha, \beta) f_{X_n, \delta}^{\wedge}(\omega_n, f_{W_n}^{\omega_n}, g_{n, <\alpha, \beta>; x_n) d\alpha d\beta \\ &= \int_{\mathcal{D}_{S_n}(x_n)} [f_{W_n}^{\omega_n}(g_{n, <\alpha, \beta>}^{-1}(x_n))/|g'_{n, <\alpha, \beta>}(g_{n, <\alpha, \beta>}^{-1}(x_n))|] f_{S_n}(\alpha, \beta) d\alpha d\beta \end{aligned} \quad (249)$$

where

$$\mathcal{D}_S(x_n) \equiv \{ <\alpha, \beta> \in S_n \mid \mathcal{V}_{X_n; <\alpha, \beta>} \ni x_n \}. \quad (250)$$

Given $x_n \in \mathcal{V}_{X_n^*}$, the set $\mathcal{D}_S(x_n)$ can be determined from the condition (see Eq. (236)),

$\xi_{n, \min}^{g_{n, <\alpha, \beta>}} \leq x_n \leq \xi_{n, \max}^{g_{n, <\alpha, \beta>}}$ as follows. Supposing firstly that $\beta > 0$ then we require, from Eq.

(245), that $g_{n, <\alpha, \beta>}(w_{n, \min}) \leq x_n \leq g_{n, <\alpha, \beta>}(w_{n, \max})$; this leads in a straightforward manner to

$$\mathcal{D}_{S_n}(x_n) = \{ <\alpha, \beta> \in S_n \mid \ln(x_n/(1-x_n)) - \beta \ln \omega_{n, \max} \leq \alpha \leq \ln(x_n/(1-x_n)) - \beta \ln \omega_{n, \min} \}, \quad \beta > 0. \quad (251)$$

Similarly, when $\beta < 0$ then we require, from Eq. (246), that $g_{n, <\alpha, \beta>}(w_{n, \max}) \leq x_n \leq$

$g_{n, <\alpha, \beta>}(w_{n, \min})$ and this leads to

$$\mathcal{D}_{S_n}(x_n) = \{ <\alpha, \beta> \in S_n \mid \ln(x_n/(1-x_n)) - \beta \ln \omega_{n, \min} \leq \alpha \leq \ln(x_n/(1-x_n)) - \beta \ln \omega_{n, \max} \}, \quad \beta < 0. \quad (252)$$

Denoting, for $x_n \in \mathcal{V}_{X_n^*}$,

$$\Lambda_{\min}(x_n; \beta) = \ln(x_n/(1-x_n)) - \beta \ln \omega_{n, \min} \quad (253a)$$

and

$$\Lambda_{\max}(x_n; \beta) = \ln(x_n/(1-x_n)) - \beta \ln \omega_{n, \max}, \quad (253b)$$

with $\Lambda_{\max}(x_n; \beta) < \Lambda_{\min}(x_n; \beta)$ if $\beta > 0$ and $\Lambda_{\min}(x_n; \beta) < \Lambda_{\max}(x_n; \beta)$ if $\beta < 0$, we see that $\mathcal{D}_{S_n}(x_n)$ is the region in the β - α plane lying between the two non-parallel straight lines $\alpha = \Lambda_{\min}(x_n; \beta)$ and $\alpha = \Lambda_{\max}(x_n; \beta)$; these lines have slopes $-\ln \omega_{n, \min}$ and $-\ln \omega_{n, \max}$, respectively, with $-\ln \omega_{n, \min} > -\ln \omega_{n, \max}$, and intersect at their common α -intercept ($\beta = 0$) given by $\alpha = \ln(x_n/(1-x_n))$. Also, noting that

$$1/|g'_{n, <\alpha, \beta>}(g_{n, <\alpha, \beta>}^{-1}(x_n))| = |\beta|^{-1} e^{-\alpha/\beta} [x_n/(1-x_n)]^{(1/\beta)} / [x_n(1-x_n)] \quad (254)$$

we then have from Eq. (249) that, for every $x_n \in \mathcal{V}_{X_n^*}$,

$$\begin{aligned} f_{X_{n, \delta, S_n}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, \mathcal{G}_n; x_n) \\ = \int_{-\infty}^0 \int_{\Lambda_{\min}(x_n; \beta)}^{\Lambda_{\max}(x_n; \beta)} \{f_{W_n^{\omega_n}}(g_{n, <\alpha, \beta>}^{-1}(x_n)) |\beta|^{-1} e^{-\alpha/\beta} [x_n/(1-x_n)]^{(1/\beta)} / [x_n(1-x_n)]\} f_{S_n}(\alpha, \beta) d\alpha d\beta \\ + \int_0^{\infty} \int_{\Lambda_{\min}(x_n; \beta)}^{\Lambda_{\max}(x_n; \beta)} \{f_{W_n^{\omega_n}}(e^{-\alpha/\beta} [x_n/(1-x_n)]^{(1/\beta)}) |\beta|^{-1} e^{-\alpha/\beta} x_n^{(1/\beta)-1} / (1-x_n)^{(1/\beta)+1}\} f_{S_n}(\alpha, \beta) d\alpha d\beta \\ = [\int_{-\infty}^0 d\beta + \int_0^{\infty} d\beta] \int_{\Lambda_{\min}(x_n; \beta)}^{\Lambda_{\max}(x_n; \beta)} \{ \bullet \} \\ \{ \bullet \} = \{f_{W_n^{\omega_n}}(e^{-\alpha/\beta} [x_n/(1-x_n)]^{(1/\beta)}) \beta^{-1} e^{-\alpha/\beta} x_n^{(1/\beta)-1} / (1-x_n)^{(1/\beta)+1}\} f_{S_n}(\alpha, \beta) d\alpha. \end{aligned} \quad (255)$$

This result is the most general one given the particular family $\mathcal{G}_n = \{g_{n, <\alpha, \beta>}\}_{<\alpha, \beta> \in \mathbb{R} \times \mathbb{R} \setminus \{0\}}$.

To particularize, we need to specify $f_{W_n^{\omega_n}}$ and f_{S_n} and we do so in the example that follows.

The example we give of computing $f_{X_{n, \delta, S_n}}^{\wedge}(\omega_n, f_{W_n^{\omega_n}}, \mathcal{G}_n; x_n)$ via Eq. (255) is one that

admits—by choosing appropriate $f_{W_n^{\omega_n}}$ and f_{S_n} —an analytical solution. Firstly, in order to

keep the required integrations in Eq. (255) as simple as possible for our analytical example, we choose $f_{W_n^{\omega_n}}$ to be uniform in $\mathcal{V}_{W_n^*} = \mathcal{V}_{W_n} = [\omega_{n, \min}, \omega_{n, \max}] = [w_{n, \min}, w_{n, \max}]$ (see

Eq. (3); then $f_{W_n^{\omega_n}}(w_n) = 1/(\omega_{n, \max} - \omega_{n, \min})$ for all $w_n \in [w_{n, \min}, w_{n, \max}]$ so, likewise,

$f_{W_n^{\omega_n}}(g_{n, <\alpha, \beta>}^{-1}(x_n)) = 1/(\omega_{n, \max} - \omega_{n, \min})$ for all $<\alpha, \beta> \in \mathcal{D}_{S_n}(x_n)$ and $x_n \in \mathcal{V}_{X_n; <\alpha, \beta>}$. Equation

(255) thus becomes, for $x_n \in \mathcal{V}_{X_n^*}$,

$$f_{X_{n,\delta},S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) = (\Delta\omega_n)^{-1} \left[\int_{-\infty}^0 d\beta + \int_0^\infty d\beta \right] \int_{\Lambda_{min}(x_n; \beta)}^{\Lambda_{max}(x_n; \beta)} \{ \beta^{-1} e^{-\alpha/\beta} x_n^{(1/\beta)-1} / (1-x_n)^{(1/\beta)+1} \} f_{S_n}(\alpha, \beta) d\alpha. \quad (256)$$

where we have written $1/(\omega_{n, max} - \omega_{n, min}) = \Delta\omega_n$. Secondly, we have not (yet) been successful in identifying a non-trivial choice for $f_{S_n}(\alpha, \beta)$ that makes the analytical evaluation of the integral in Eq. (256) doable. But the (trivial) choice

$$f_{S_n}(\alpha, \beta) = \delta(\langle \alpha, \beta \rangle - \langle \alpha_0, \beta_0 \rangle), \quad \langle \alpha_0, \beta_0 \rangle \in S_n = \mathbb{R} \times \mathbb{R} \setminus \{0\} \quad (257)$$

allows us to analytically evaluate the integral in Eq. (256); for definiteness we take

$\beta_0 > 0$. We then have, for $x_n \in \mathcal{V}_{X_n^*}$,

$$f_{X_{n,\delta},S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) = \begin{cases} (\Delta\omega_n)^{-1} \beta_0^{-1} e^{-\alpha_0/\beta_0} x_n^{(1/\beta_0)-1} / (1-x_n)^{(1/\beta_0)+1} & \text{if } \langle \alpha_0, \beta_0 \rangle \in \mathcal{D}_{S_n}(x_n) \\ 0 & \text{if } \langle \alpha_0, \beta_0 \rangle \notin \mathcal{D}_{S_n}(x_n). \end{cases} \quad (258)$$

It is useful to verify for the PDF $f_{X_{n,\delta},S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n)$ of Eq. (258) that

$$\int_{x_{n, min}}^{x_{n, max}} f_{X_{n,\delta},S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) dx_n = 1 \text{ as required. To do so, we firstly recall, from Eq.}$$

(250), that $\langle \alpha_0, \beta_0 \rangle \in \mathcal{D}_{S_n}(x_n)$ iff $x_n \in \mathcal{V}_{X_n; \langle \alpha_0, \beta_0 \rangle}$; then by Eq. (258) we have

$$\int_{x_{n, min}}^{x_{n, max}} f_{X_{n,\delta},S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) dx_n = \int_{\mathcal{V}_{X_n; \langle \alpha_0, \beta_0 \rangle}} f_{X_{n,\delta},S_n}^\wedge(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) dx_n. \quad (259)$$

Next, since we have taken $\beta_0 > 0$ then $g_{n, \langle \alpha, \beta \rangle}$ is strictly increasing on $\mathcal{V}_{W_n}$ so that, by Eq. (245),

$$\xi_{n, min}^{g_{n, \langle \alpha_0, \beta_0 \rangle}} = g_{n, \langle \alpha_0, \beta_0 \rangle}(\omega_{n, min}) = 1 / [1 + e^{-\alpha_0} \omega_{n, min}^{-\beta_0}] \quad (260a)$$

and

$$\xi_{n, max}^{g_{n, \langle \alpha_0, \beta_0 \rangle}} = g_{n, \langle \alpha_0, \beta_0 \rangle}(\omega_{n, max}) = 1 / [1 + e^{-\alpha_0} \omega_{n, max}^{-\beta_0}]; \quad (260b)$$

hence from Eq. (236) we have

$$\mathbf{v}_{x_n; <\alpha_0, \beta_0>} = [1/[1 + e^{-\alpha_0} \omega_n^{-\beta_0, \min}], 1/[1 + e^{-\alpha_0} \omega_n^{-\beta_0, \max}]]. \quad (261)$$

Now using Eqs. (258) and (261) then Eq. (259) becomes

$$\begin{aligned} & \int_{x_{n, \min}}^{x_{n, \max}} f_{\hat{X}_{n, \delta, S_n}}(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) dx_n \\ &= (\Delta \omega_n)^{-1} \beta_0^{-1} e^{-\alpha_0/\beta_0} \int \frac{1/[1 + e^{-\alpha_0} \omega_n^{-\beta_0, \max}]}{1/[1 + e^{-\alpha_0} \omega_n^{-\beta_0, \min}]} x_n^{(1/\beta_0) - 1} / (1 - x_n)^{(1/\beta_0) + 1} dx_n. \end{aligned} \quad (262)$$

Once again, analytic evaluation of the integral in the above equation is not available in general (i.e., for arbitrary $\beta_0 > 0$) but is available for several values of β_0 . As examples, we choose three such β_0 's, namely $\beta_0 = 1/2, 1, 2$; also, the value of α_0 in these examples may be—for given β_0 —any value in the interval $[\Lambda_{\min}(x_n; \beta_0), \Lambda_{\max}(x_n; \beta_0)]$. For $\beta_0 = 1/2$ we have

$$\begin{aligned} & \int_{x_{n, \min}}^{x_{n, \max}} f_{\hat{X}_{n, \delta, S_n}}(\omega_n, f_{W_n^{\omega_n}}, G_n; x_n) dx_n \\ &= (\Delta \omega_n)^{-1} 2e^{-2\alpha_0} \int \frac{1/[1 + e^{-\alpha_0} \omega_n^{-1/2, \max}]}{1/[1 + e^{-\alpha_0} \omega_n^{-1/2, \min}]} x_n / (1 - x_n)^3 dx_n \end{aligned} \quad (263)$$

with $\int dx_n x_n / (1 - x_n)^3 = (x_n - 1/2) / (1 - x_n)^2$ and it is straightforward to calculate that the RHS of Eq. (263) is equal to 1. Similarly, for $\beta_0 = 1$ we have the required antiderivative being $\int dx_n / (1 - x_n)^2 = 1 / (1 - x_n)$ and for $\beta_0 = 2$ we have the required antiderivative being $\int dx_n x_n^{-1/2} / (1 - x_n)^{3/2} = 2[x_n / (1 - x_n)]^{1/2}$ and in both these additional cases the value of the RHS of the equation corresponding to Eq. (263) for $\beta_0 = 1$ and $\beta_0 = 2$ is easily shown to be 1.

V. Conclusion

We have presented a novel approach to quantifying uncertainties in the values of the output variables of computer codes used in numerical simulations of scientific and other processes of interest where these uncertainties are a result of (1) uncertainty in the values of the code input variables, (2) uncertainty in the form used in the code modeling, and (3) numerical “uncertainty” arising from the need to perform arithmetic computations via discretization of the often continuum model equations. We applied our uncertainty

quantification (UQ) methodology to both a single, stand-alone code as well as to an N -member code chain in which the j^{th} code in the chain produces a value of an output variable having uncertainty quantified in our sense and which output variable value subsequently serves as an input variable value for the $(j+1)^{\text{th}}$ code in the chain, which $(j+1)^{\text{th}}$ code then itself produces a value of an output variable also having uncertainty quantified in our sense. The uncertainties in all input and output variables are represented in our formulation by probability density functions (PDF's). Our entire formulation is analytic and requires (1) two "uncertainty constants" for each code in the chain, (2) knowledge of (or estimates of) real-world upper and lower bounds for all variables of interest, and (3) the functional relationships between all input and output variables of interest (which may be obtained, perhaps computationally, in the absence of uncertainty considerations).

References

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- [2] Timothy J. Clarke, *Bayesian Analysis for High Power Microwave Effects Data*, AFRL-DE-PS-TR-2005-1012-Vol.II, Air Force Research Laboratory, Directed Energy Directorate, Kirtland AFB, New Mexico 87117-5776 (2005)

Table I.

The $f_{xg(w)}^{[v(x)], k}(x)$'s and their domains $g^{-1}[\mathcal{H}_k^{[v(x)]}(x)] \subseteq [w_{\min}, w_{\max}]$ (from Eqs. (132) – (145))

	$v(x) = 1$ [#(1) = 4]		$v(x) = 2$ [#(2) = 3]		$v(x) = 3$ [#(3) = 4]		$v(x) = 4$ [#(4) = 2]	
	$\mathcal{H}_k^{[1]}(x)$	$f_{xg(w)}^{[1], k}(x)$	$\mathcal{H}_k^{[2]}(x)$	$f_{xg(w)}^{[2], k}(x)$	$\mathcal{H}_k^{[3]}(x)$	$f_{xg(w)}^{[3], k}(x)$	$\mathcal{H}_k^{[4]}(x)$	$f_{xg(w)}^{[4], k}(x)$
$k = 1$	$[x_{\min}, x_{\max}^b]$	$f_{xg(w)}^{C,1}(x)$	$[x_{\min}, x_{\max}^b]$	$f_{xg(w)}^{C,1}(x)$	$[x_{\min}, x^b]$	$f_{xg(w)}^{C,2}(x)$	$[x_{\min}, x_{\max}^b]$	$f_{xg(w)}^{C,1}(x)$
$k = 2$	$[x_{\max}^b, x_{\min}^{\#}]$	$f_{xg(w)}^{D,1}(x)$	$[x_{\max}^b, x_{\min}^{\#}]$	$f_{xg(w)}^{D,1}(x)$	$[x^b, x_{\max}^b]$	$f_{xg(w)}^{C,1}(x)$	$[x_{\max}^b, x_{\max}]$	$f_{xg(w)}^{D,1}(x)$
$k = 3$	$(x_{\min}^{\#}, x^{\#})$	$f_{xg(w)}^{B,2}(x)$	$(x_{\min}^{\#}, x_{\max}]$	$f_{xg(w)}^{B,2}(x)$	$[x_{\max}^b, x_{\min}^{\#}]$	$f_{xg(w)}^{D,1}(x)$	--	--
$k = 4$	$(x^{\#}, x_{\max}]$	$f_{xg(w)}^{B,1}(x)$	--	--	$(x_{\min}^{\#}, x_{\max}]$	$f_{xg(w)}^{B,2}(x)$	--	--
$k = 5$	--	--	--	--	--	--	--	--

	$v(x) = 5$ [#(5) = 3]		$v(x) = 6$ [#(6) = 1]		$v(x) = 7$ [#(7) = 5]		$v(x) = 8$ [#(8) = 5]	
	$\mathcal{H}_k^{[5]}(x)$	$f_{xg(w)}^{[5], k}(x)$	$\mathcal{H}_k^{[6]}(x)$	$f_{xg(w)}^{[6], k}(x)$	$\mathcal{H}_k^{[7]}(x)$	$f_{xg(w)}^{[7], k}(x)$	$\mathcal{H}_k^{[8]}(x)$	$f_{xg(w)}^{[8], k}(x)$
$k = 1$	$[x_{\min}, x^b]$	$f_{xg(w)}^{C,2}(x)$	$[x_{\min}, x_{\max}]$	$f_{xg(w)}^{D,1}(x)$	$[x_{\min}, x_{\min}^{\#}]$	$f_{xg(w)}^{C,2}(x)$	$[x_{\min}, x_{\min}^{\#}]$	$f_{xg(w)}^{C,2}(x)$
$k = 2$	$[x^b, x_{\max}^b]$	$f_{xg(w)}^{C,1}(x)$	--	--	$(x_{\min}^{\#}, x^b)$	$f_{xg(w)}^{A,3}(x)$	$(x_{\min}^{\#}, x^b)$	$f_{xg(w)}^{A,3}(x)$
$k = 3$	$[x_{\max}^b, x_{\max}]$	$f_{xg(w)}^{D,1}(x)$	--	--	$[x^b, x^{\#}]$	$f_{xg(w)}^{A,2}(x)$	$[x^b, x_{\max}^b]$	$f_{xg(w)}^{A,2}(x)$
$k = 4$	--	--	--	--	$(x^{\#}, x_{\max}^b)$	$f_{xg(w)}^{A,1}(x)$	$[x_{\max}^b, x^{\#}]$	$f_{xg(w)}^{B,2}(x)$
$k = 5$	--	--	--	--	$[x_{\max}^b, x_{\max}]$	$f_{xg(w)}^{B,1}(x)$	$(x^{\#}, x_{\max}]$	$f_{xg(w)}^{B,1}(x)$

Table I (cont'd)

	$v(x) = 9$ [#(9) = 4]		$v(x) = 10$ [#(10) = 5]		$v(x) = 11$ [#(11) = 4]		$v(x) = 12$ [#(12) = 5]	
	$\mathcal{H}_k^{[9]}(x)$	$f_{x^{g(w)}}^{[9], k}(x)$	$\mathcal{H}_k^{[10]}(x)$	$f_{x^{g(w)}}^{[10], k}(x)$	$\mathcal{H}_k^{[11]}(x)$	$f_{x^{g(w)}}^{[11], k}(x)$	$\mathcal{H}_k^{[12]}(x)$	$f_{x^{g(w)}}^{[12], k}(x)$
$k = 1$	$[x_{min}, x_{min}^{\#}]$	$f_{x^{g(w)}}^{C, 2}(x)$	$[x_{min}, x^b]$	$f_{x^{g(w)}}^{C, 2}(x)$	$[x_{min}, x_{min}^{\#}]$	$f_{x^{g(w)}}^{C, 1}(x)$	$[x_{min}, x^b]$	$f_{x^{g(w)}}^{C, 2}(x)$
$k = 2$	$(x_{min}^{\#}, x^b)$	$f_{x^{g(w)}}^{A, 3}(x)$	$[x^b, x_{min}^{\#}]$	$f_{x^{g(w)}}^{C, 1}(x)$	$(x_{min}^{\#}, x^{\#})$	$f_{x^{g(w)}}^{A, 2}(x)$	$[x^b, x_{min}^{\#}]$	$f_{x^{g(w)}}^{C, 1}(x)$
$k = 3$	$[x^b, x_{max}^b]$	$f_{x^{g(w)}}^{A, 2}(x)$	$(x_{min}^{\#}, x^{\#})$	$f_{x^{g(w)}}^{A, 2}(x)$	$(x^{\#}, x_{max}^b)$	$f_{x^{g(w)}}^{A, 1}(x)$	$(x_{min}^{\#}, x_{max}^b)$	$f_{x^{g(w)}}^{A, 2}(x)$
$k = 4$	$[x_{max}^b, x_{max}]$	$f_{x^{g(w)}}^{B, 2}(x)$	$(x^{\#}, x_{max}^b)$	$f_{x^{g(w)}}^{A, 1}(x)$	$[x_{max}^b, x_{max}]$	$f_{x^{g(w)}}^{B, 1}(x)$	$[x_{max}^b, x^{\#}]$	$f_{x^{g(w)}}^{B, 2}(x)$
$k = 5$	--	--	$[x_{max}^b, x_{max}]$	$f_{x^{g(w)}}^{B, 1}(x)$	--	--	$(x^{\#}, x_{max})$	$f_{x^{g(w)}}^{B, 1}(x)$

	$v(x) = 13$ [#(13) = 4]		$v(x) = 14$ [#(14) = 4]	
	$\mathcal{H}_k^{[13]}(x)$	$f_{x^{g(w)}}^{[13], k}(x)$	$\mathcal{H}_k^{[14]}(x)$	$f_{x^{g(w)}}^{[14], k}(x)$
$k = 1$	$[x_{min}, x^b]$	$f_{x^{g(w)}}^{C, 2}(x)$	$[x_{min}, x_{min}^{\#}]$	$f_{x^{g(w)}}^{C, 1}(x)$
$k = 2$	$[x^b, x_{min}^{\#}]$	$f_{x^{g(w)}}^{C, 1}(x)$	$(x_{min}^{\#}, x_{max}^b)$	$f_{x^{g(w)}}^{A, 2}(x)$
$k = 3$	$(x_{min}^{\#}, x_{max}^b)$	$f_{x^{g(w)}}^{A, 2}(x)$	$[x_{max}^b, x^{\#}]$	$f_{x^{g(w)}}^{B, 2}(x)$
$k = 4$	$[x_{max}^b, x_{max}]$	$f_{x^{g(w)}}^{B, 2}(x)$	$(x^{\#}, x_{max})$	$f_{x^{g(w)}}^{B, 1}(x)$
$k = 5$	--	--	--	--

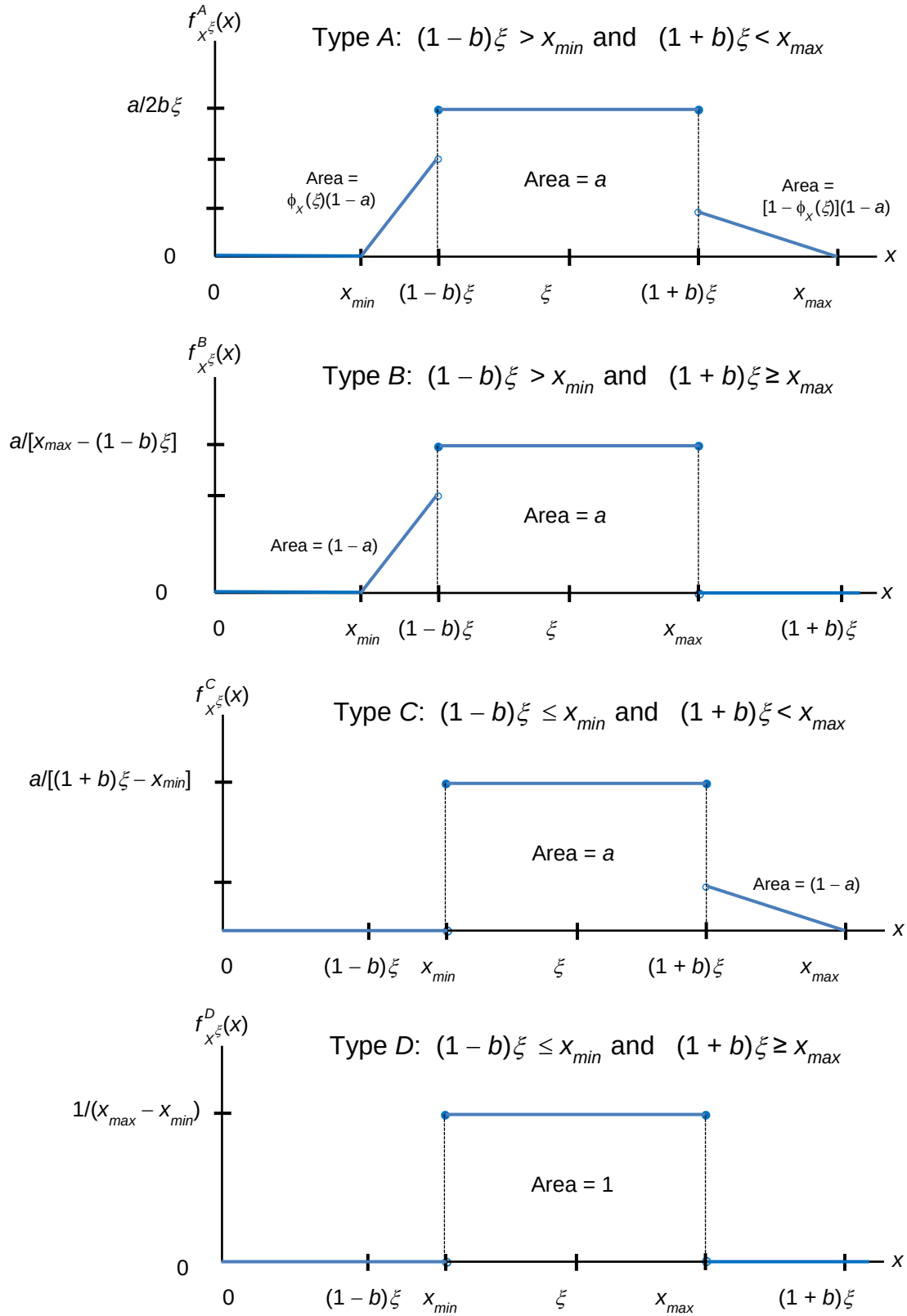


Figure 1. The PDF's $f_{X^\xi}^\alpha$, $\alpha = A, B, C, D$, for any fixed $\xi \in [x_{min}, x_{max}]$

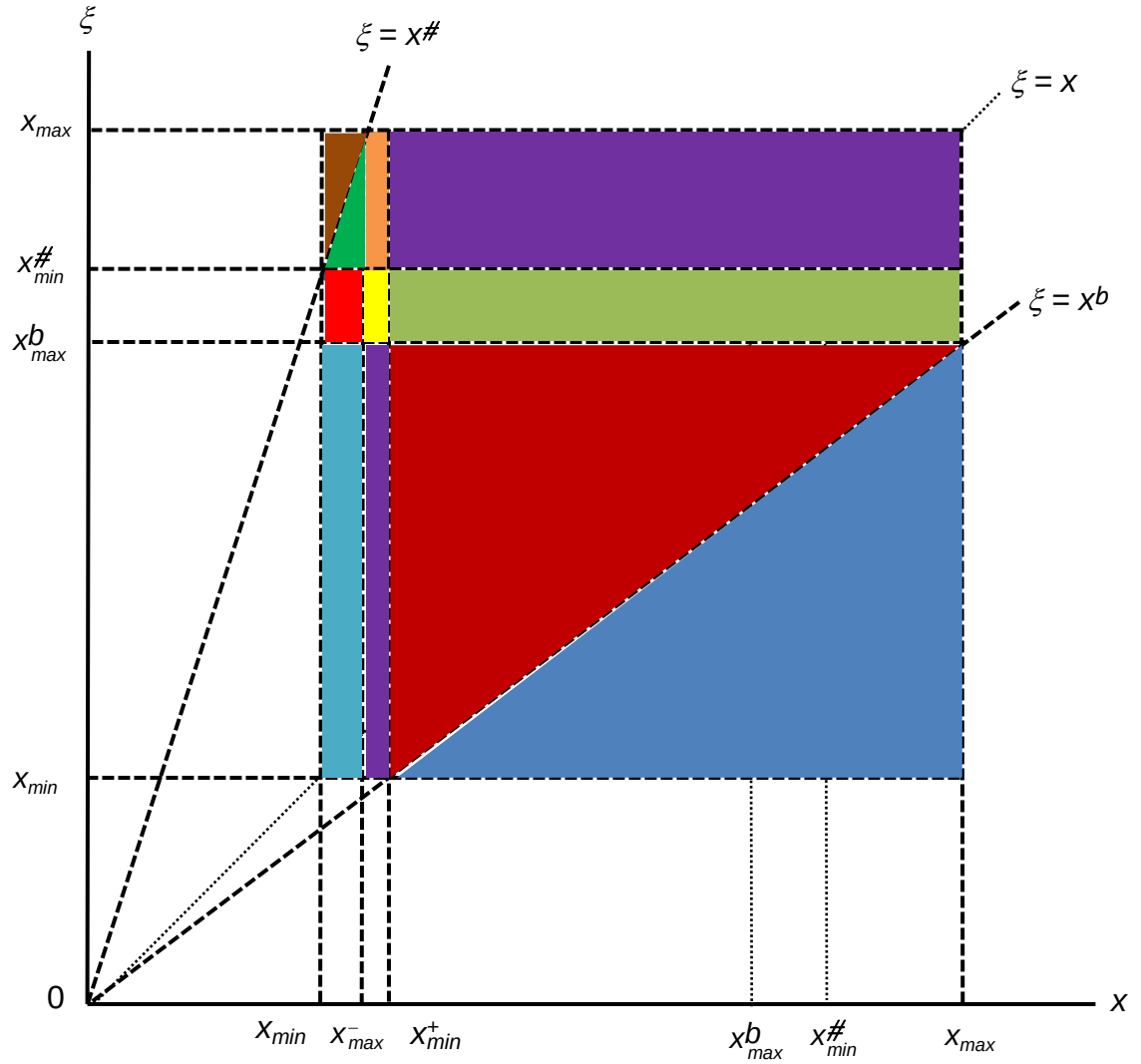


Figure 2. The “patchwork” domain of function of $f_{x^{\xi}}(x)$ for Case xIa . The explicit functional forms of $f_{x^{\xi}}(x)$ in the “patches” are given by Eq. (128) *via* Eqs. (79), (81), and (83), the latter three themselves using Eq. (17).

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