ARL-TR-8492 • Sep 2018

US Army Research Laboratory

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ARL-TR-8492 • SEP 2018

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REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED (From - To)**
September 2018	Technical Report	March 2018–May 2018

4. TITLE AND SUBTITLE
Consistent Electrostatics of Crystalline Conductors

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Michael Grinfeld, Pavel Grinfeld, and Steven B Segletes

5d. PROJECT NUMBER
AH80

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
US Army Research Laboratory
ATTN: RDRL-WMP-C
Aberdeen Proving Ground, MD 21005-5066

8. PERFORMING ORGANIZATION REPORT NUMBER
ARL-TR-8492

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES
authors' email: <michael.greenfield4.civ@mail.mil>, <pg77@drexel.edu>, <steven.b.segletes.civ@mail.mil>.

14. ABSTRACT
In a recent report, we drew attention to the paradoxical inconsistency of classical electrostatics with the model of a crystalline conductor. There are different ways to avoid this inconsistency. Mostly, they rely on physical *ad hoc* assumptions associated with the introduction of additional constants and models of charged liquids. In this report, we suggest another approach, which does not require the introduction of any additional physical constants. The proposed theory includes classical electrostatics as a special case.

15. SUBJECT TERMS
electrostatics, lattice, charge density, conductors

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			Steven B Segletes
Unclassified	Unclassified	Unclassified	UU	21	**19b. TELEPHONE NUMBER (Include area code)** 410-278-6010

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

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Acknowledgments

The authors would like to express gratitude to Dr Constantine Fountzoulas for his helpful review of this manuscript. The authors are additionally grateful to Ms Carol Johnson for her diligent effort in editing this report.

1. Introduction

Electromagnetism is in the background of various military applications, including the problem of protection against shaped charges, which Russian adversaries call "cumulative" jets.[1]

One of the many problems faced in the computer implementation of physical models is associated with the fact that certain physical fields experience discontinuities across boundaries and interfaces. Some fields (the models thereof) even become infinitely large when approaching such interfaces. Theorists have accumulated various sophisticated mathematical tools for handling these problems when treating the problems analytically. The problem of infinite charge density in the vicinity of boundaries in metals is further aggravated when using numerical modeling, since the singularities in the underlining physical theories are in violent discord with the inability of finite discretizations to handle sharp gradients. One of the methods for handling this violent discord is based on the replacement of the classical physical theories with modified ones that allow for the avoidance of infinite charge density at the boundaries. We suggested one approach of this sort in a prior report[2] and continue developing the approach in this one.

Let us recall the paradoxical inconsistency of classical electrostatics first noted in our prior report.[2] According to classical electrostatics, all excess electric charges, positive or negative, concentrate on the conductor's boundary with a finite 2-D density. In mathematical physics, these sort of boundaries are known as "simple layers" of charges.[3,4] This surface density can be either positive or negative, depending on the total excess charge of the conductor. What is the physical meaning of this finite 2-D density of electric charge? Actually, this concept implies that the associated 3-D density of the electric charge is infinite. Of course, this infinity, on its own, is an essential inconsistency of classical electrostatics. However, in many cases, this particular limitation is not very important and classical electrostatics provides researchers with reasonable results.[3,4] It is not, however, this inconsistency that we discussed in our prior work.[2]

The paradoxical inconsistency that we introduced[2] is different. In fact, the density of the easily movable negative charges can grow and reach quite large values (for example, on the surface of a body), in accordance with classical electrostatics. If the positive charges are easily movable and compressible also, like in the case of ionized gas, the same can be said about the gas of positive ions. In this sense, classical electrostatics can be applied to a gaseous plasma. The situation, however, changes dramatically when the positive charges belong to a (near rigid) lattice. In this case, the maximum positive charge density is basically fixed, and we cannot expect that it might assume infinite values, even in an idealized sense. In other words, there cannot be *surfaces* (or *interfaces*) with finite positive charges—excess positive charge can only manifest in *regions* (*i.e.*, *domains*). Thus, we deal in this case with the asymmetry of how to treat the situations of negative *versus* positive excess charge. This very asymmetry dictates the necessity to revise classical electrostatics.

To address this inconsistency, we have to reformulate classical electrostatics in such a way that nowhere does the positive 3-D charge density exceeding the value q_+ appear.

2. A Simple Consistent Model of Electrostatics to Avoid the Paradoxical Inconsistency

We propose the following model of a crystalline rigid conductor. The model is shown schematically in Fig. 1.*

The key suggestion is the following: the total domain Ω of the conductor is split into two subdomains, Ω_n and Ω_a. Generally speaking, both subdomains may comprise several separate regions. In Fig. 1, the subdomains of Ω_n are shown in gray, whereas the subdomains of Ω_a are shown in red. We assume that, in Ω_n, the macroscopic density of both positive and negative charges (q_+ and q_-, respectively) are equal to each other. Thus, the net macroscopic charge density q vanishes in Ω_n:

$$q = q_+ + q_- = 0 \qquad \text{within } \Omega_n. \tag{1}$$

Within the subdomain Ω_a, our assumption is different. We just assume that the density of negative charges q_- vanishes; in other words, we arrive at the following relationships:

$$q_- = 0, \quad q = q_+ \qquad \text{within } \Omega_a. \tag{2}$$

*In this and the figures to follow (though its justification remains to be assumed or derived later in this report), one may follow the heuristic rule-of-thumb that dark- and light-gray denote electrically neutral domains and interfaces, respectively, red denotes a domain void of negative charges, while blue denotes an interface comprised solely of mobile negative charges and characterized by a surface charge density.

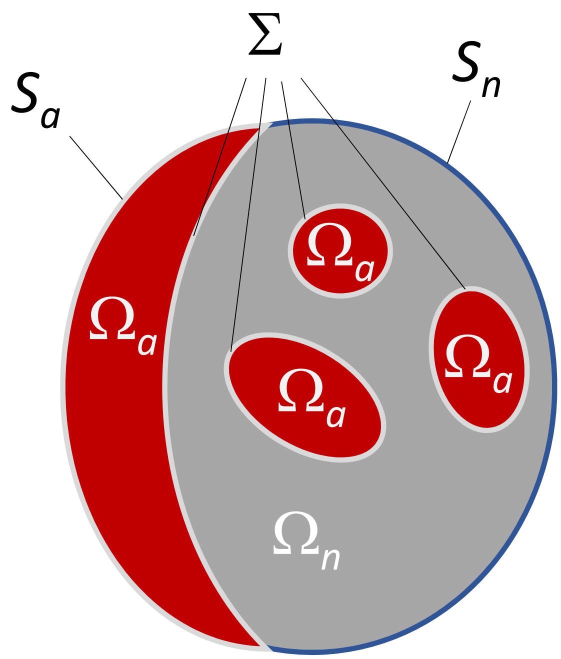

Fig. 1 The cross-sectional (cutaway) geometry of a conductor

Thus, by the construction, the positive charge density exceeds the value q_+ nowhere.

Also, from Fig. 1, we point out that, between the two Ω domains and between each domain and the vacuum surround, we postulate zero-thickness interface layers. These interfaces are denoted Σ, S_a, and S_n and are further described and analyzed later in this report.

Let us formulate the system of equations in the mathematical form of a boundary value problem. In view of Eqs. 1 and 2, the electrostatics bulk equations inside the domains Ω_n and Ω_a read

$$\nabla^i \nabla_i \varphi = 0 \tag{3}$$

and

$$\nabla^i \nabla_i \varphi = -4\pi q_+ \quad , \tag{4}$$

respectively.

Outside the external boundary S (the union of S_n and S_a), we also have to use the Poisson equation

$$\nabla^i \nabla_i \varphi = -4\pi q_{\text{ext}} \quad , \tag{5}$$

where q_{ext} is a given distribution of the external sources of the electrostatic field.

Within the domain Ω_n (with the mobile negative charges potentially on the adjoining interface), the electrostatic potential is assumed constant everywhere:

$$\varphi = \varphi_0 = \text{const.} \qquad \text{inside } \Omega_n \quad , \tag{6}$$

as always in classical electrostatics. Equation 6 automatically satisfies the electrostatics Eq. 3; thus, there is no need to deal with the Laplace equation (Eq. 3) inside the domain Ω_n.

Let us now discuss the boundary conditions for our bulk equations of electrostatics. Per Fig. 1, *a priori*, there are three sorts of zero-thickness interfaces. First, there are interfaces S_n between the domain Ω_n and vacuum. Secondly, there are interfaces S_a between the domain Ω_a and vacuum. Lastly, there are interfaces Σ between the domains Ω_n and Ω_a. For all three types of interfaces, the boundary conditions are different:

1. We begin with the S_n interfaces. Those are the traditional interfaces of classical electrostatics, possessing the boundary conditions

$$\begin{aligned} \left[\varphi\right]_{-}^{+} &= 0 \\ N^i \nabla_i \varphi\big|_{S_n} &= -4\pi\sigma_{S_n} \end{aligned} \tag{7}$$

   where N^i is the interface's outward normal and σ_{S_n} is the 2-D density of the surface charges. The former of the boundary conditions given as Eq. 7 reflects the continuity of the electrostatic potential; the latter is the consequence of the Gauss law applied to the interface with the finite 2-D density of electric charges (the so-called "simple layer of charges"). It is essential that this density is negative since only negative charges are mobile and are able to generate the unlimited 3-D density of the electric charge (as the boundary layer near S_n approaches zero thickness).

2. We proceed with the S_a interfaces. Since, by the postulated models, there are no surface charges at these interfaces, we suggest using the standard boundary

conditions of continuity of electrostatic potential and its first derivatives:

$$\begin{aligned} \left[\varphi\right]_-^+ &= 0 \\ \left[\nabla_i \varphi\right]_-^+ N^i &= 0 \end{aligned} \quad . \tag{8}$$

3. Finally, we consider the Σ-interfaces, separating the Ω_n and Ω_a domains. Across those interfaces, we are still using the electrostatics boundary conditions

$$\begin{aligned} \left[\varphi\right]_-^+ &= 0 \\ \left[\nabla_i \varphi\right]_-^+ N^i &= -4\pi\sigma_{an} \end{aligned} \tag{9}$$

Thus, *a priori*, we assume that there can be 2-D accumulations of the negative charges. This is a quite plausible assumption for the Σ-interfaces since there is a source of negative mobile charges from the domain Ω_n.

There is, however, a significant difference between the S_n and S_a interfaces, on the one hand, and the Σ interfaces, on the other hand. The difference is the following: the location of S_n and S_a are known up-front, from the conductor geometry, whereas the location of the Σ interfaces are not. In order to determine the location of the Σ interfaces, we need one more equation. This additional equation can be chosen based on various principles. We postulate this additional condition in the simplest form, rejecting the possibility of charge accumulation on Σ interfaces:

$$\sigma_{an} = 0 \quad . \tag{10}$$

Finally, we need a charge balance equation for the mobile negative charges. Let their total charge be equal to Q_-. These charges are located within the domain Ω_n with the volumetric density $q_- = q_+$ and at the interface S_n with the surface density σ_{S_n}. Thus, we arrive at the charge balance relationship:

$$\int_{\Omega_n} d\Omega\, q_- + \int_{S_n} dS\, \sigma_{S_n} = Q_- \quad . \tag{11}$$

As a side note, the assumption given by Eq. 10 can be substantiated on the basis of a rigorous mathematical analysis if we accept the principle of minimum electrostatic energy.[5,6]

3. 1-D Solutions for a Flat Layer

Consider a flat 1-D system, which carries the total number of Q mobile charges per unit cross section. We explore the distributions of charges that are symmetric with respect to the plane $z = 0$. Since both S_a and Σ interfaces accumulate zero charge and possess zero thickness, their visible presence is omitted in the figures that follow.

3.1 Configuration 1

If the net bounded charge Q_+, which includes the positive charges of the lattice ions and negative charge of the bounded negative charges, is less than $|Q_-|$, we get the standard configuration of classical electrostatics, shown in Fig. 2.

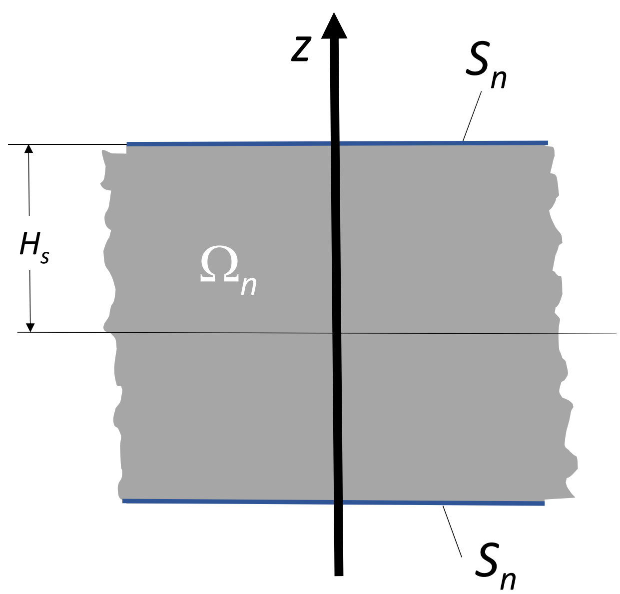

Fig. 2 The "standard" configuration of classical electrostatics

According to our terminology, there is no Ω_a domain, which is free of the mobile negative charges. The whole domain between the boundaries appears to be the neutral domain Ω_n, having the S_n-type interfaces with the negative surface charge density equal to

$$\sigma_{S_n} = \frac{Q_+ - |Q_-|}{2} \quad . \tag{12}$$

3.2 Configuration 2

Consider a flat 1-D system presented in Fig. 3.

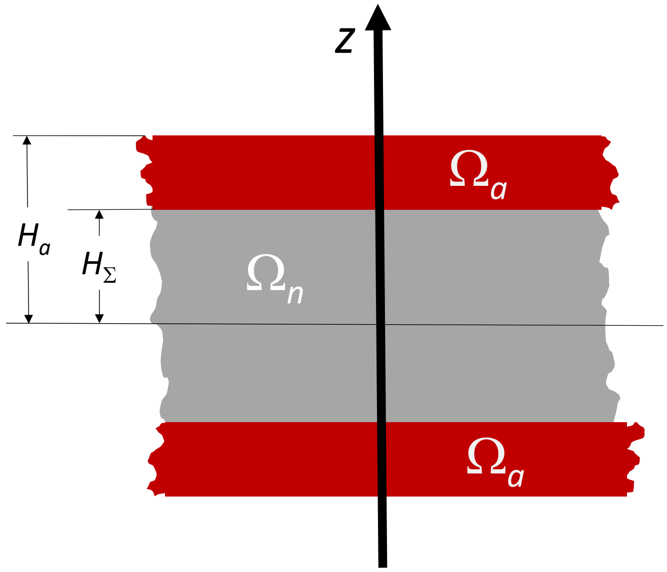

Fig. 3 A configuration with finite Ω_a domains (red) at the external boundaries

Within the gray Ω_n domain, the 1-D Laplace equation reads

$$\frac{d^2\varphi}{dz^2} = 0 \qquad \text{for } |z| \le H_\Sigma \quad . \tag{13}$$

Within the red Ω_a domain, the 1-D Laplace equation reads

$$\frac{d^2\varphi}{dz^2} = -4\pi q_+ \qquad \text{for } H_\Sigma < |z| \le H_a \quad . \tag{14}$$

Outside the plate, in the absence of the outside charges, we arrive at the Laplace equation again:

$$\frac{d^2\varphi}{dz^2} = 0 \qquad \text{for } |z| > H_a \quad . \tag{15}$$

Boundary conditions at both interfaces H_Σ and H_a are the same; namely,

$$\begin{aligned} \left[\varphi\right]_-^+ &= 0 \\ \left[\frac{d\varphi}{dz}\right]_-^+ &= 0 \end{aligned} \quad . \tag{16}$$

At $z = 0$, we choose the following boundary conditions:

$$\begin{aligned} \varphi(0) &= 0 \\ \left.\frac{d\varphi}{dz}\right|_0 &= 0 \end{aligned} \quad . \tag{17}$$

The charge balance equation (Eq. 11) is

$$(2H_\Sigma)q_- = Q_- \quad . \tag{18}$$

In view of the bulk Laplace equation (Eq. 13) and the boundary conditions (Eq. 17), the electrostatic potential φ vanishes everywhere inside the gray Ω_n domain:

$$\varphi(z) = 0 \qquad \text{for } |z| \leq H_\Sigma \quad . \tag{19}$$

Now, combining Eqs. 1 and 18, we arrive at the following relationship for the thickness of the neutral domain:

$$H_\Sigma = \frac{1}{2q_+}\,|Q_-| \quad . \tag{20}$$

The general solution of the Poisson equation (Eq. 14) inside the red Ω_a domain reads

$$\varphi(z) = -2\pi q_+ z^2 + C_1 z + C_2 \qquad \text{for } H_\Sigma < |z| \leq H_a \quad , \tag{21}$$

where C_1 and C_2 are the constants that should be determined from the boundary conditions (Eq. 16) and solution (Eq. 19) within the gray Ω_n domain, which gives us two linear equations

$$\begin{aligned} -2\pi q_+ H_\Sigma^2 + C_1 H_\Sigma + C_2 &= 0 \\ -4\pi q_+ H_\Sigma + C_1 &= 0 \end{aligned} \tag{22}$$

with the solution

$$\begin{aligned} C_1 &= 4\pi q_+ H_\Sigma \\ C_2 &= -2\pi q_+ H_\Sigma^2 \end{aligned} \quad . \tag{23}$$

Using Eq. 23, we can rewrite the general solution Eq. 21 as follows

$$\varphi(z) = -2\pi q_+ \left(z - H_\Sigma\right)^2 \qquad \text{for } H_\Sigma < |z| \leq H_a \quad . \tag{24}$$

The solution given by Eq. 24 implies

$$\begin{aligned} \varphi(H_a) &= -2\pi q_+ \left(H_a - H_\Sigma\right)^2 \\ \left.\frac{d\varphi}{dz}\right|_{H_a} &= -4\pi q_+ \left(H_a - H_\Sigma\right) \end{aligned} \quad . \tag{25}$$

The general solution of the Laplace equation (Eq. 15) within the vacuum domain has the following general solution

$$\varphi(z) = \left.\varphi\right|_{z=H_a} + \left(z - H_a\right) \left.\frac{d\varphi}{dz}\right|_{z=H_a} \qquad \text{for } |z| \geq H_a \quad . \tag{26}$$

Using Eq. 25, we can rewrite the general solution (Eq. 26) as follows:

$$\varphi(z) = -2\pi q_+ \left(H_a - H_\Sigma\right)^2 - 4\pi q_+ \left(z - H_a\right)\left(H_a - H_\Sigma\right) \qquad \text{for } |z| \geq H_a \quad . \tag{27}$$

3.3 Configuration 3

Consider now the same flat 1-D system but with a different arrangement of the domains, presented in Fig. 4. For this configuration, we arrive at the system

$$\frac{d^2\varphi}{dz^2} = -4\pi q_+ \qquad \text{for } |z| \leq H_\Sigma \quad , \tag{28}$$

$$\varphi = \text{const} \qquad \text{for } H_\Sigma < |z| \leq H_S \quad , \tag{29}$$

$$\frac{d^2\varphi}{dz^2} = 0 \qquad \text{for } |z| > H_S \quad . \tag{30}$$

The system of bulk equations, Eqs. 28–30, should be considered with the following boundary conditions:

1. at $z = 0$, we choose the following boundary conditions:

$$\begin{aligned} \varphi(0) &= 0 \\ \left.\frac{d\varphi}{dz}\right|_0 &= 0 \end{aligned} \tag{31}$$

2. at $z = H_\Sigma$, we use the following boundary conditions; namely, continuity of

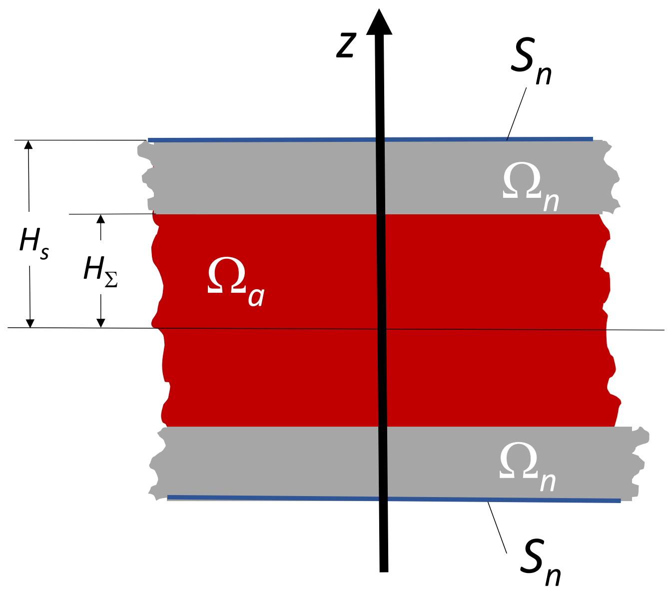

Fig. 4 A configuration with a finite Ω_a domain (red) in the middle of the body

φ and its derivative across the Σ interface:

$$\begin{aligned} [\varphi]_-^+ &= 0 \\ \left[\frac{d\varphi}{dz}\right]_-^+ &= 0 \end{aligned} \tag{32}$$

3. at $z = H_S$, the surface changes can accumulate and we choose the boundary conditions

$$\begin{aligned} [\varphi]_-^+ &= 0 \\ N^i \nabla_i \phi\big|_{S_n} &= -4\pi\sigma_{S_n} \end{aligned} \tag{33}$$

The charge balance equation (Eq. 11) reads

$$(H_S - H_\Sigma)q_- + \sigma_{S_n} = \frac{Q_-}{2} \quad . \tag{34}$$

We then obtain

$$\varphi = -2\pi q_+ z^2 \qquad \text{for } |z| \le H_\Sigma \quad , \tag{35}$$

$$\varphi = -2\pi q_+ H_\Sigma^2 \quad \text{for } H_\Sigma < |z| \le H_S \quad . \tag{36}$$

According to the solutions given by Eq. 36, φ is constant in the domain Ω_n, and so $d\varphi/dz = 0$ when approaching the interface at $z = H_\Sigma$ from above. In contrast, the parabolic form of Eq. 35 for the central Ω_a domain would lead us to the conclusion that $d\varphi/dz = -4\pi q_+ H_\Sigma$ when approaching the interface at $z = H_\Sigma$ from below. These two solutions violate the interface boundary condition for $d\varphi/dz$ at $z = H_\Sigma$, given by Eq. 32, which calls for continuity of the derivative of φ across the Σ interface.

Therefore, one must conclude that the hypothetical configuration 3 represents a nonphysical configuration of the electric charge distribution.

4. Conclusion

In an earlier report,[2] we demonstrated the paradoxical inconsistency of classical electrostatics for the models permitting only finite density of positive charges. To handle this paradox, classical electrostatics should be significantly modified. There are several possibilities of reasonable modifications. Basically, they lead to the introduction of additional physical mechanisms and additional material constants.

Introduction of additional material constants and effects is the precursor of significant changes in the technical complexity of the associated boundary value problems. The growth of the complexity can entail the disappearance of the keynote feature of classical electrostatics: this feature is the possibility of analytical exploration of the interesting physical problems.

To minimize the complexification of classical electrostatics, we suggested another approach, which is *not* associated with the introduction of any additional material constants. After formulation of the novel boundary value problem, we demonstrated how it can be solved analytically in the simplest instructive 1-D cases of the flat plate.

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