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VERIFICATION ANALYSIS OF ARMSTRONG'S STOCHASTIC SALVO EQUATIONS USING DATA FARMING

by

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June 2018

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13. ABSTRACT (maximum 200 words)

Models such as Hughes' deterministic salvo equations are used by countries around the world to assist in determining the numbers, capabilities, and employment strategies of their naval warships. Armstrong extended Hughes' model to create a stochastic salvo model (SSM). Armstrong also evaluated his key assumptions by comparing his closed-form solutions against simulation as a more realistic alternative. This thesis performs a more comprehensive comparison of the SSM versus simulation, utilizing sophisticated design of experiments. Statistical models and case studies are used to identify which combinations of model inputs cause the largest biases (or differences) between the simulation and the SSM. The results show that for independent missiles the SSM closely matches the simulation throughout the region explored. The bias increases when the missiles are correlated or the force levels are large. This is particularly noticeable when estimating the probabilities of zero loss or annihilation. The bias also depends critically on whether the forces are in an overkill, intermediate, or over defense situation. The SSM, our R simulation, and a prototype characteristic function evaluator of the binomial stochastic salvo model are all implemented in a "Shiny" application. This facilitates exploration of the various models within a single user-friendly interface.

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VERIFICATION ANALYSIS OF ARMSTRONG'S STOCHASTIC SALVO EQUATIONS USING DATA FARMING

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ABSTRACT

Models such as Hughes' deterministic salvo equations are used by countries around the world to assist in determining the numbers, capabilities, and employment strategies of their naval warships. Armstrong extended Hughes' model to create a stochastic salvo model (SSM). Armstrong also evaluated his key assumptions by comparing his closed-form solutions against simulation as a more realistic alternative. This thesis performs a more comprehensive comparison of the SSM versus simulation, utilizing sophisticated design of experiments. Statistical models and case studies are used to identify which combinations of model inputs cause the largest biases (or differences) between the simulation and the SSM. The results show that for independent missiles the SSM closely matches the simulation throughout the region explored. The bias increases when the missiles are correlated or the force levels are large. This is particularly noticeable when estimating the probabilities of zero loss or annihilation. The bias also depends critically on whether the forces are in an overkill, intermediate, or over defense situation. The SSM, our R simulation, and a prototype characteristic function evaluator of the binomial stochastic salvo model are all implemented in a "Shiny" application. This facilitates exploration of the various models within a single user-friendly interface.

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LIST OF ACRONYMS AND ABBREVIATIONS

ASCM	Anti-Ship Cruise Missiles
CDF	Cumulative Distribution Function
CIWS	Close-In Weapon System
DOE	Design of Experiments
FER	Fractional Exchange Ratio
M&S	Modeling and Simulation
NOB	Nearly Orthogonal and Balanced
NOLH	Nearly Orthogonal Latin Hypercube
PDF	Probability Density Function
SEED	Simulation Experiments and Efficient Designs
SAM	Surface-to-Air Missile
SSM	Stochastic Salvo Model

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EXECUTIVE SUMMARY

In 1995, Wayne Hughes formulated a maritime missile combat model based on his observations and experience as a naval officer. Hughes' salvo model successfully delivers critical insights for naval missile warfare. It provides a useful reference for decision makers to evaluate the impact of warship capabilities such as offensive power, defensive power, staying power, as well as the size of the force. Each salvo exchange consists of two behaviors, offensive and defensive, and the appropriate strategy may depend heavily on whether the goal is to fight for sea control or sea denial. Defenders will generally try to lower the attacker's accuracy and intercept incoming missiles to achieve their goals. Attackers seeking sea control will generally try to use overwhelming force to eliminate the defenders while preserving their own forces. The discrete pulses of weapon exchanges that comprise modern maritime combat radically change the nature of engagement when compared to Lanchester's equations.

Armstrong's stochastic salvo model (SSM) extends Hughes' salvo model by incorporating randomness, while remaining solvable in closed-form, by using analytically tractable approximations to the probability distributions. We implemented a simulation model of Armstrong's SSM that directly models the probability distributions, and consequently is not subject to either truncation or approximation biases. Therefore, we view the simulation as a source of "ground truth" for evaluating the SSM's performance.

We observe experimentally that when one side has overwhelming power and the other side has low force levels the standard deviation of loss is small, but it increases when facing credible resistance from the adversary. A key finding is that force dominance by either side is associated with low standard deviations for the loss of the disadvantaged force—in other words, their losses are strongly consistent. When neither side is dominant, the standard deviation of losses is relatively high for both sides. Armstrong's SSM also shows that neither party is guaranteed a victory unless they have overwhelming power.

Based on our research, Armstrong's model suffers bias in predicting losses due to truncation of the distributions used in his model. Additionally, in some scenarios the approximation is poor and contributes to the bias. As a result, the bias is hard to predict. This research uses sophisticated designs of experiments (DOE) to quantify the bias over a broad range of SSM input factors. The experimental results show that for offensive missiles that are mutually independent the SSM closely matches the simulation results throughout the region explored. The differences between the two models increase when the missiles' targeting abilities are correlated and the forces are large, especially in estimating the probabilities of zero loss or annihilation.

Our results, using simulation as a baseline, show that Armstrong's model performs quite well at correctly predicting both the mean and the standard deviation of losses when all missiles behave independently. However, the probability of zero loss can be underestimated or overestimated for both independent and correlated missiles. With independent missiles, only overestimation is observed for the probability of total loss, while both over- and underestimation are observed with correlated missiles. It is noteworthy that the standard deviation of loss yields both overestimated and underestimated results relative to the simulation, which stands in contrast to Armstrong's own experimentally based conclusion of overestimated results.

An Excel Crystal Ball add-in simulation and Armstrong's SSM can be easily implemented and studied using DOE. However, by implementing the model in R we can leverage its computational power and the user-friendly "Shiny" interface to create a fast and accurate simulation model. Moreover, using R allowed us to implement a prototype analytic model solver based on characteristic functions. This approach needs more work, but preliminary findings indicate that it yields outcomes that are much closer to the simulation than Armstrong's SSM can achieve.

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I am thankful that my country selected me for this precious opportunity to obtain my master's degree from the Naval Postgraduate School (NPS). The wonderful experience, such as solving practical problems and sharing so many wonderful thoughts with classmates, will never be forgotten. It has been a pleasure and no doubt we will see each other in some places of world.

I would also like to express my sincerest appreciation for my advisor, Professor Thomas W. Lucas, and second reader, Professor Paul J. Sanchez, for vast valuable guidance and inspiration. I am especially grateful to Professor Lucas. I took two classes with him and he always said, "Learning should be fun," and would tell us interesting stories. Since we are operations research officers, I strongly believe profound differences will be made when we start to change our thoughts, no matter how small or trivial.

Also, it has been my privilege and honor to study with and benefit from many dedicated operations research professors. Overall, I had a great time at NPS.

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I. INTRODUCTION

A. BACKGROUND

Many countries are eager to upgrade their navy based on their national objectives. With advances in technology, especially in sophisticated naval weapon systems, naval warfare has become more complicated than ever before. Where should nations make investments in their surface forces? For example, should they invest in large numbers of less capable ships or go with fewer numbers of ships that are more capable? How should countries design surface warships with respect to their offensive and defensive capabilities?

To inform naval surface ship design, Hughes (1995) developed a simple salvo model of naval surface missile combat. Most any naval officer can understand Hughes' model. However, since it is a deterministic model, Hughes' model does not provide information on variability. This is a limitation, because warfare is inherently stochastic (Ancker 1995).

Using Hughes' existing deterministic salvo model as a starting point, Armstrong (2005) proposed a closed-form stochastic salvo model (SSM). That is, the model itself can be solved analytically to estimate the mean, standard deviation, and distributions of key outcomes, such as losses. This provides model users with a broader understanding of possible outcomes.

Armstrong's SSM contains many assumptions and approximations, such as the number of well-targeted missiles follows a binomial distribution. The binomial results from an assumption that each missile fired has the same effectiveness and is independent of all the others. To simplify calculations, a continuous normal distribution is then used to approximate the discrete binomial distribution. To assess the effects of some of his critical assumptions and approximations, Armstrong (2011) compared the results of his closed-form solution to what a more realistic Monte Carlo simulation yields. The simulation solves the model without requiring the simplifications and approximations needed to make the SSM solvable in closed form. The differences between Armstrong's SSM and 50,000 replications of the simulation were calculated and studied at 486 design points using a full factorial design. The 50,000 replications results in an upper bound on the standard error of probability estimates of 0.0022. Armstrong concluded that the SSM generally matches the simulation well, even when the force sizes and missile salvos are small and the damage inflicted is not normally distributed. However, he finds that the model's accuracy suffers when there is positive correlation among offensive missiles.

B. THESIS OBJECTIVE

The objective of this thesis is to analyze Armstrong's SSM using a more comprehensive design of experiments. The thesis expands on Armstrong's (2011) SSM verification by employing a sophisticated design of experiment (DOE). This allows us to efficiently sample many more design points and fit meta-models that quantify the assumptions and approximations' effects on the results. This provides information on where Armstrong's model is most reliable and guidance on how to improve upon it.

C. RESEARCH QUESTIONS

This thesis extends Armstrong's (2011) verification analysis by employing data farming to address the following questions:

- Can we replicate Armstrong's (2011) verification experiments?
- Can we quantify the bias by fitting regression models of it for different force levels and correlation?
- With many more design points in our experiments, will we see the same behaviors?
- What are the reasons for underestimating or overestimating results from Armstrong's closed-form stochastic salvo model when compared to a simulation?
- Can we estimate (and therefore adjust for) the model biases (i.e., the difference between the closed-form solution and the simulation) as a function of the model inputs?
- Can we improve and manage model accuracy when positive correlation exists?

D. SCOPE OF THESIS

This thesis efficiently explores the differences between Armstrong's SSM and simulation using data farming to determine how robust the model is and find potential methods to improve it when bias exists. Data farming uses computing clusters and advanced design of experiments to "grow" computational data that can be analyzed by statistical methods.

E. THESIS FLOW

The rest of this thesis flows as follows. Chapter II contains a thorough description of Hughes' deterministic salvo model and Armstrong's stochastic salvo model. This includes explanations of both models' parameters and assumptions. In addition, Armstrong's (2011) verification experiments are replicated using the statistical programing language R (https://www.r-project.org). Chapter III describes the naval scenario that we use for the computations in this thesis as well has our design of experiments. Chapter IV presents the results of our experimentation and contains an analysis of the differences between Armstrong's SSM and the simulation using regression and partition metamodels. Chapter V summarizes this thesis, its conclusions, and recommends directions for future research.

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II. BACKGROUND AND LITERATURE REVIEW

This chapter introduces Hughes' deterministic salvo model and Armstrong's stochastic salvo model (SSM). We focus on replication of Armstrong's (2011) verification experiments in which he assesses how various modeling assumptions and approximations he makes compare to a simulation. The experiments reveal biases in Armstrong's closed form SSM, especially with correlated missiles. The biases increase with force strength and correlation.

A. HUGHES' SALVO MODEL (DETERMINISTIC)

Hughes' deterministic salvo model is a simple, aggregate representation of key attributes of missile combat between warships. The purpose of the model is focus on conventional anti-ship cruise missiles (ASCM) and surface-to-air missiles (SAM) exchange results, see (Hughes 1995). Users can utilize the model to analyze the military value of warship capabilities such as offensive power, defensive power, staying power, and force levels. A dilemma for naval architects is how to determine the best combination of these attributes that should be in a modern warship.

Schulte's (1994) thesis researched naval combat from 1967 to 1991 in terms of ASCMs. He found empirically that the probability of a hit on defenseless targets is 0.913, whereas a probability of a hit on defendable targets is 0.684 and defended targets is down to 0.264. The results provide critical evidence of historical effectiveness in naval combat. Hughes' primary measure of effectiveness when comparing forces in his model is the fractional exchange ratio (FER).

Hughes developed his model to provide descriptive insights for naval missile warfare, much as Lanchester's equations or stochastic duel models have for ground combat. Hughes' objective was to have a model of similar simplicity, but one that uses pulses or salvos of combat, incorporates defensive missiles, and explicitly includes staying power. These are some of the key aspects that he felt differentiate naval missile combat from other forms of battle, in particular ground warfare.

The formation of Hughes' model sets up two forces against each other, sides A and B. Each force is homogeneous in that all their warships have the same characteristics. The forces change in strength, (i.e., loss of ships), from a simultaneous exchange of missiles (i.e., a salvo). For consistency, we introduce the Hughes' deterministic salvo model using Armstrong's SSM notation and definitions.

$$\Delta A = -\frac{\beta B - yA}{w}, \quad 0 \le -\Delta A \le A$$
$$\Delta B = -\frac{\alpha A - zB}{x}, \quad 0 \le -\Delta B \le B \tag{1}$$

where

ΔA = number of units in force A put out of action from B's salvo

ΔB = number of units in force B put out of action from A's salvo

A = number of units in force A

B = number of units in force B

α = number of well-aimed missiles fired by each A unit

β = number of well-aimed missiles fired by each B unit

y = number of well-aimed missiles intercepted by each A

z = number of well-aimed missiles intercepted by each B

w = number of hits by B's missiles needed to put one A out of action

x = number of hits by A's missiles needed to put one B out of action.

As an illustration, let us consider an example from Hughes (1995).

Number of units:	A=2, B=6
Striking power:	α = 24, β =6
Defensive power:	y = 16, z = 1
Staying power:	w = 2, x = 1

In this situation, A has substantially more capable ships and B has superior numbers. Evaluating the losses from Equation (1) without the constraints yields

$$\Delta A = -\frac{36-32}{2} = -2, \quad \Delta B = -\frac{48-6}{1} = -42.$$

From *B's* perspective, its defenses can be overwhelmed by *A* seven times over. However, in the end, *B* has just enough power to eliminate *A* despite *A*'s dramatic advantages in both offensive and defensive power. This is an example of what Hughes calls instability—in such cases the side with greater combat potential is not guaranteed victory. Related examples include World War II torpedo bombers attacking aircraft carriers or Taiwanese corvette deployments to provide credible asymmetric threats. This example's insight drives warship designers, operational officers, and budget decision makers to rethink naval surface warfare.

Additional insights that Hughes gleaned from his model include that instability occurs when offensive power is large relative to defensive power and staying power. It is worth noting that offensive power is cheaper than defensive power and dramatically more affordable than staying power. Hughes also finds that "numerical superiority is the force attribute that is consistently most advantageous" (Hughes 1995).

A critical feature of Hughes' deterministic salvo model is that it is simple to use. Consequently, it enables naval officers to readily analyze potential force-on-force missile exchanges without advanced training.

B. ARMSTRONG'S STOCHASTIC SALVO MODEL (CLOSED-FORM)

Armstrong (2005) formulated a stochastic extension of Hughes' deterministic salvo model to allow for closed-form calculations of stochastic aspects such as the *standard deviation of loss* and the probability of annihilation. For ease of comparison, we use Armstrong's notation throughout this thesis. Armstrong's stochastic salvo model (SSM) is based on Hughes' deterministic salvo model and incorporates variability by employing α, z, and v (where the mean of damage $v = 1/x$) as random variables in calculating *B*'s losses. A similar approach is used to calculate *A*'s losses. Since the model is symmetric, the calculations in this thesis focus on *B*'s losses. Assuming that the probability of success for each missile is independent, Armstrong models α and z as independent binomial random variables with probabilities p_α and p_z (the probabilities of being well-targeted) and sample sizes n_α and n_z, respectively. Unless it is intercepted, a missile is well-targeted if it will hit its target. He models the damage v as a normal random variable. Since the difference between two independent binomials has no convenient closed-form, Armstrong approximates the discrete binomials and their difference using normal random variables. Net_{AB} in the SSM replaces the difference in binomial distributions with a normal distribution. In the model development, the calculation of the remaining force (B_1) after a salvo is

$$B_1 = B - \sum_{k=0}^{NetAB} v_k = B - \int_0^{NetAB} v\,dt,\ 0 \le B_1 \le B, \tag{2}$$

where

$$\sum_{k=0}^{NetAB} v_k = \int_0^{NetAB} v\,dt,$$

calculates the compound losses assuming homogeneity for α and z as.

$$Net_{AB} \equiv Off_A - Def_B$$

$$Off_A \equiv A\alpha = \sum_{i=1}^{A} \alpha = \int_0^{A} \alpha\,dt$$

$$Def_B \equiv Bz = \sum_{j=1}^{B} z = \int_0^{B} zdt$$

$$Dmg \equiv v \sim N(\mu_v,\ \sigma_v).$$

Off_A is the total offensive missiles fired from side *A*, Def_B is the total defensive missiles fired from side *B*, and Net_{AB} is the offensive missiles from side *A* that *B's* defenses are unable to intercept. Table 1 summarizes Armstrong's model notation.

Table 1. Armstrong Salvo Model Notation. Adapted from Armstrong (2005).

SYMBOL	DESCRIPTION
A, B	Beginning force strength
α_i, β_i	Offensive power per unit
w, x	Staying power per unit
y_i, z_i	Defensive power per unit
u_k, v_k	# Ships lost per hit
n_α, n_b	Max # of offensive missiles per ship per salvo
p_α, p_b	P(an offensive missile is successfully well-targeted)
Off_A, Off_B	Total # of well-targeted offensive missiles per salvo
n_y, n_z	Max # of defensive missiles per ship per salvo
p_y, p_z	P(defensive missile successfully intercepts)
Def_A, Def_B	Total # of accurate defensive missiles per salvo
Net_{BA}, Net_{AB}	Total # of non-intercepted offensive missiles
μ_u, μ_v	*1/w*, *1/x* : mean loss of ships
σ_u, σ_v	Standard deviation of loss per hit
μ_α, μ_β	Mean offensive power
A_1, B_1	Actual surviving force strength after one salvo
A_1^*, B_1^*	Nominal surviving force strength after one salvo
F_{NetBA}, G_{NetAB}	CDF of nominal # of non-intercepted offensive missiles
f_{NetBA}, g_{NetAB}	PDF of nominal # of non-intercepted offensive missiles
$F_{A_1^*}, G_{B_1^*}$	CDF of nominal surviving force strength
$f_{A_1^*}, g_{B_1^*}$	PDF of nominal surviving force strength

When each offensive missile side A fires has independent probability p_α of successfully targeting side B, the offensive power (α) from each warship follows a binomial distribution with mean $\mu_\alpha = n_\alpha p_\alpha$ and variance $\sigma_\alpha^2 = n_\alpha p_\alpha(1-p_\alpha)$ from Armstrong's (2005) formulation. The random number of successfully aimed missiles, Off_A, for side A has $E[Off_A] = A\mu_\alpha = An_\alpha p_\alpha$ and $Var[Off_A] = A\sigma_\alpha^2 = An_\alpha p_\alpha(1-p_\alpha)$. Armstrong uses this mean and variance to approximate the number of well-targeted missiles using a normal distribution.

The next step is to model the defensive missiles from side B in the same manner as the offensive missiles. A binomial distribution is used for firing defensive missiles, with n_z the number fired and p_z the probability of successful intercept. This distribution has a mean of $\mu_z = n_z p_z$ and a variance of $\sigma_z^2 = n_z p_z(1-p_z)$ for each ship. As for the normal distribution approximation, the total effective defensive missiles fired Def_B has $E[Def_A] = B\mu_z = Bn_z p_z$ and $Var[Def_A] = B\sigma_z^2 = Bn_z p_z(1-p_z)$. The loss function is referred to as *nominal* remaining force $Net_{AB} = Off_A - Def_B$. In this normal approximation, Net_{AB} can be negative when defensive power exceeds offensive power, whereas in reality the number of losses must be in the range [0, B].

Armstrong's damage function has a normal distribution with mean μ_v and variance σ_v^2 . The staying power of a ship is defined as one over the mean damage per missile hit, i.e., $1/\mu_v$. This is the number of hits it takes to knock a ship out of action.

Armstrong (2005) derives the nominal mean and variance of surviving B forces after the first salvo exchange as

$$\mu_{B_1^*} = B - E[Net_{AB}] \cdot E[v] = B - (A \cdot \mu_\alpha - B \cdot \mu_z) \cdot \mu_v \qquad (3)$$

$$\begin{aligned}\sigma_{B_1^*}^2 &= (A \cdot \mu_\alpha - B \cdot \mu_z) \cdot \sigma_v^2 + (A \cdot \sigma_\alpha^2 + B \cdot \sigma_z^2) \cdot \mu_v^2 \\ &-2\sigma_v^2 \cdot (A \cdot \mu_\alpha - B \cdot \mu_z) \cdot G_{NetAB}(0) \\ &+2\sigma_v^2 \cdot (A \cdot \sigma_\alpha^2 + B \cdot \sigma_z^2) \cdot g_{NetAB}(0).\end{aligned} \qquad (4)$$

Armstrong (2005) details the following example.

Ships: $A = B = 6$,

Probability of offensive missiles and defensive missiles: $p_\alpha = p_z = 0.5$,

Maximum number of offensive missiles per ship $n_\alpha = 8$,

Maximum number of defensive missiles per ship $n_z = 4$,

Evaluating the nominal equations, i.e., equations (3) and (4), yields $\mu_\alpha = 8 \cdot (0.5) = 4$ and; the nominal number of hits is $E[Net_{AB}] = 6 \times 4 - 6 \times 2 = 12$ missiles; and the variance is $Var[Net_{AB}] = 12 + 6 = 18$ missiles^2 , resulting in a standard deviation of 4.24.

Let $Dmg_v \sim N(\mu_v = 0.5,\ \sigma_v = 0.2)$ and combine Net_{AB} and Dmg_v into a composite normal distribution. In this example, the nominal surviving force is $B_1^* \sim N(\mu = 0,\ \sigma = 2.23)$. Figure 1 depicts Armstrong's compound loss distribution. In reality, *B*'s losses are bounded above by the number of ships (i.e., the maximum loss is six) and below by zero.

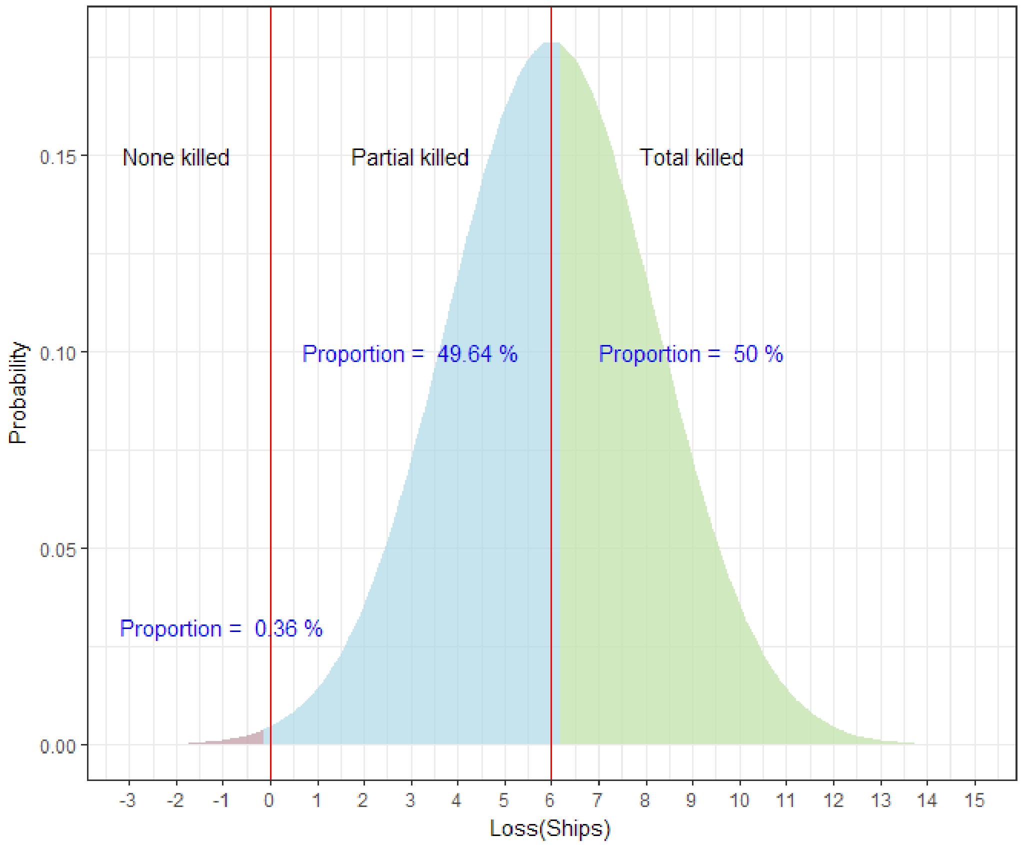


Figure 1. Remaining Force Distribution

The next step is to calculate the *actual* surviving force by truncating the tails for infeasible outcomes,

$$E[ActualLoss] = E[ActualLoss \mid 0 < Loss < B] \cdot P[0 < Loss < 6] \\ + \; 0 \cdot P[Loss \leq 0] \; + \; 6 \cdot P[Loss \geq 6].$$

Let $\mu^*_{B_1}$, $\sigma^2_{B_1}$, $g_{B^*_1}(t)$, and $G_{B^*_1}(t)$ be the mean, variance, probability density function (PDF), and cumulative distribution function (CDF) of the *nominal* surviving force strength. Armstrong (2005) derives the mean *actual* surviving force strength as

$$EB_1 = E[B_1] = \int_{0+\mu_v/2}^{B-\mu_v/2} t \cdot g_{B^*_1}(t)dt + B \cdot (1 - G_{B^*_1}(B - \mu_v / 2)), \qquad (5)$$

which can alternatively be expressed for calculation purposes as

$$E[B_1] = \mu_{B_1^*} \cdot [G_{B_1^*}(B - \mu_v / 2) - G_{B_1^*}(0 + \mu_v / 2)]$$
$$-\sigma^2_{B_1^*} \cdot [g_{B_1^*}(B - \mu_v / 2) - g_{B_1^*}(0 + \mu_v / 2)] + B \cdot (1 - G_{B_1^*}(B - \mu_v / 2)). \quad (6)$$

The Armstrong's (2005) calculation of variance of actual surviving force strength is

$$VB_1 = Var[B_1] = \int_{0+\mu_v/2}^{B-\mu_v/2} t^2 \cdot g_{B_1^*}(t)dt$$
$$+B^2 \cdot (1 - G_{B_1^*}(B - \mu_v / 2)) - (EB_1)^2, \quad (7)$$

which can be calculated as

$$(\mu^2_{B_1^*} + \sigma^2_{B_1^*}) \cdot [G_{B_1^*}(B - \mu_v / 2) - G_{B_1^*}(0 + \mu_v / 2)]$$
$$+B^2 \cdot (1 - G_{B_1^*}(B - \mu_v / 2)) - (EB_1)^2$$
$$-\sigma^2_{B_1^*} \cdot [(B - \mu_v / 2 + \mu_{B_1^*}) \cdot g_{B_1^*}(B - \mu_v / 2) \quad (8)$$
$$-(0 + \mu_v / 2 + \mu_{B_1^*}) \cdot g_{B_1^*}(0 + \mu_v / 2)].$$

The probability of side *A* winning (i.e., *A* having remaining ships while *B* is annihilated) is

$$P(A_{win}) = [1 - F_{A_1^*}(0 + \frac{\mu_u}{2})] \cdot [G_{B_1^*}(0 + \frac{\mu_v}{2})]. \quad (9)$$

Armstrong (2005) analyzes the following situation in detail:

Ships: $A = 6$, $B = 1$ to 15

Number of offensive surface-to-surface missiles n_α, $n_b = 8$

Number of defensive surface-to-air missiles (SAMs) n_y, $n_z = 3$

Probability that offensive missile is well-targeted p_α, $p_b = 0.68$

Probability of successful defensive missile p_y, $p_z = 0.68$

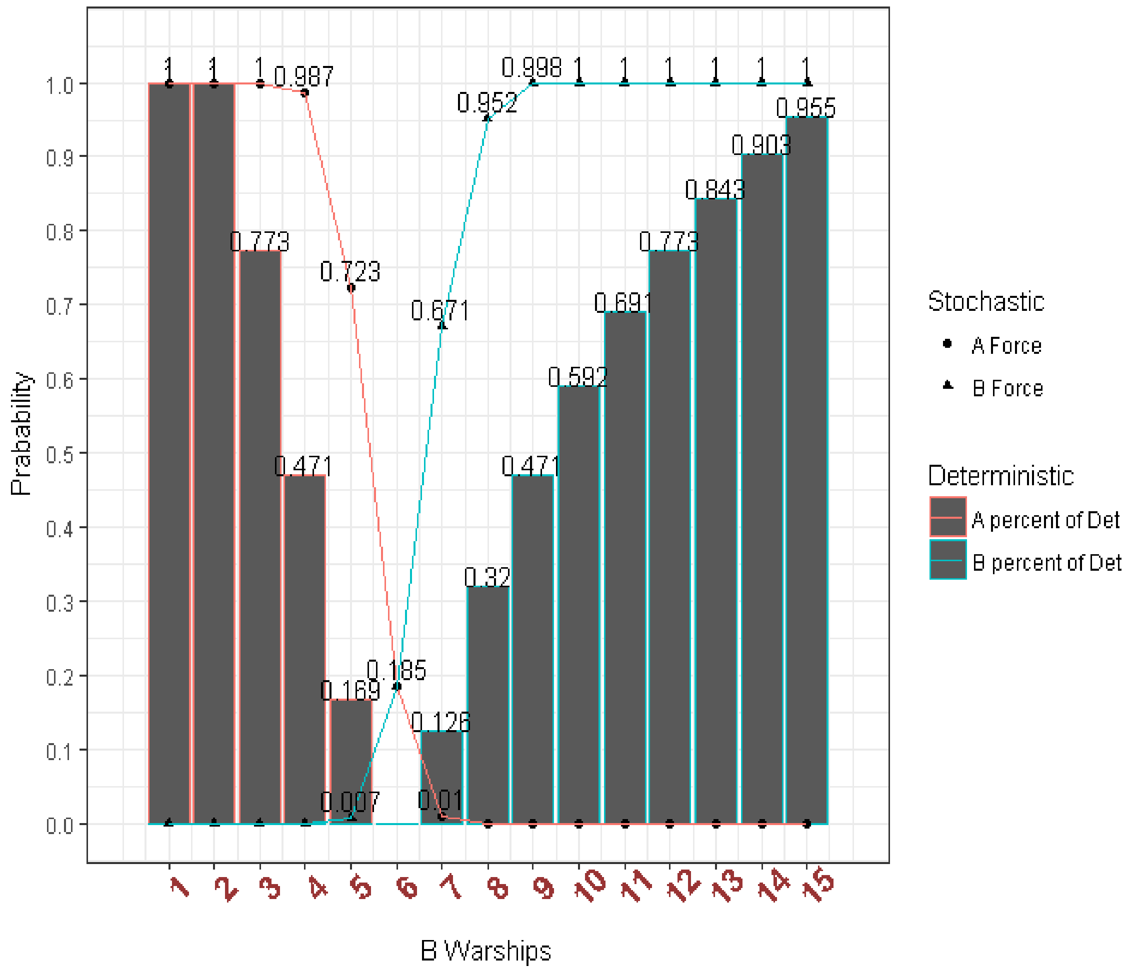
$$Damage \sim N(\mu_u = \mu_v = \frac{1}{3}\ ,\ \sigma^2_u = \sigma^2_v = \left(\frac{1}{7.5}\right)^2).$$

In order to display the proportion of each surviving force strength in Figure 2, Hughes's deterministic model results are transformed as follows to deliver more informative insight for later discussion:

$$0 \le \frac{\Delta A}{A} \le 1,\ 0 \le \frac{\Delta B}{B} \le 1.$$

Figure 2 plots the probability of victory for each side as *B* ranges from 1 to 15 using Armstrong's SSM and also contains the proportion of remaining forces in Hughes' deterministic salvo model. A related graph appears in Armstrong (2005).

We see from the graph that the stochastic model allows for either side to win in some situations, even when it is outnumbered. In contrast, with Hughes' deterministic salvo model one side wins in every situation other than the 100% chance of mutual destruction when $A = B = 6$. It is noteworthy that when one side dominates the other (in this case when *B* is near its extremes), the models produce similar results. It is also noticeable that the shape of side *A's* victory is not symmetric with side *B's* victory. If side *B* wants to achieve sea control, it needs to amass more forces to have an overwhelming advantage to increase its victory probability. Following Armstrong (2005), sea control has come to be defined as annihilating the other force while maintaining some of yours.



The bars outlined in red are the proportion of *A* survivors, the bars outlined in blue are *B* survivors in Hughes' deterministic salvo model. The lines are the probabilities of victory for *A* (circles) and *B* (triangles) for Armstrong's SSM.

Figure 2. Probability of Victory for *A* and *B* as Force Strength *B* is Varied

Figure 3 shows how the standard deviations of ship losses varies with *B*. This too is similar to a plot from Armstrong (2005). One insight readily apparent is that when one side dominates the other, the standard deviation drops for the force at a disadvantage. When neither side is dominant, the standard deviation is relatively high for both sides.

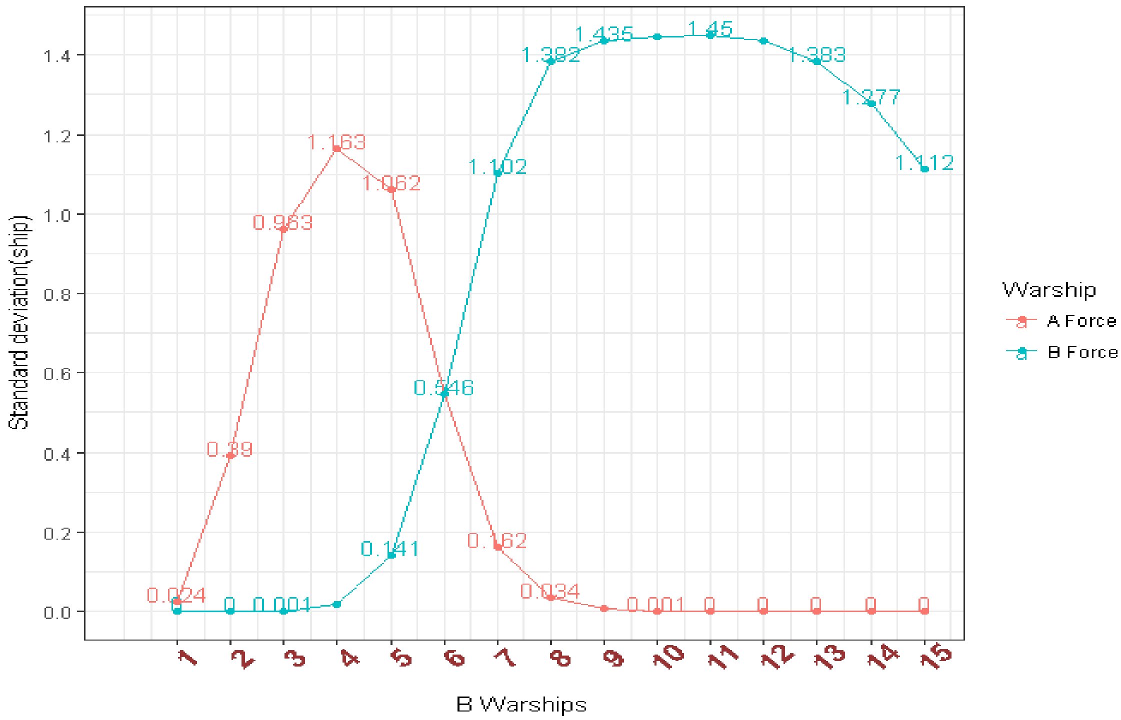


Figure 3. Standard Deviation of Survivors as Force Strength *B* Is Varied

By bringing Figure 2 and Figure 3's results together, when side *A* starts to take damage in Hughes' deterministic salvo model, then side *A's* standard deviation increases. After peaking, *A's* standard deviation decreases as *B*'s force level increases, eventually going to zero as *A* faces almost certain annihilation. In addition, side *B* almost always wins the battle when its number of ships exceeds seven. However, the *standard deviation of loss* for side *B* does not drop significantly for even up to 15 ships.

In Armstrong's (2005) study, he concludes "a navy's preferences for risk (variability) and armament (offensive versus defensive) will depend on not only its mission objectives but also on whether it expects to fight from a position of strength or of weakness."

C. ARMSTRONG'S VERIFICATION STUDY

Armstrong's SSM makes several simplifying assumptions that make it solvable in closed form. In particular, normal random variables are used in place of binomial random variables and the damage inflicted per hit is also assumed to be normally distributed. To assess these assumptions, Armstrong (2011) compared his SSM to what is obtained from a simulation using random draws from binomial distributions over a broad range of parameters and other damage functions. This section replicates Armstrong's (2011) SSM verification experiments using the statistical programming language *R* instead of the Excel Crystal Ball add-in he used.

Armstrong also modified his model by allowing for positive correlation among the offensive missiles. With the advent of sophisticated electronic warfare and defensive counter measures such as chaff, multiple offensive missiles can be jointly distracted or led astray. In such cases, there is positive correlation between offensive missiles' probabilities of hit that might affect the composite distribution of stochastic behavior.

To address these issues, Armstrong's study focuses on small samples since Armstrong states that the fit of the model will improve with larger samples. The test scenarios start from 18-on-18 ships with 18 offensive missiles against 18 and 9 defensive missiles. The next pair of scenarios uses one third as many ships—the missile configuration is 6-on-6 and 6-on-3 per salvo. The last two scenarios decrease the force levels by a factor of three as well, with 2-on-2 and 2-on-1 per salvo.

The probabilities p_α and p_z for missiles are varied across three values: 0.5, 0.67, and 0.83, which represent offensive and defensive missile combinations. The experiments study the impact of correlation on offensive missile behaviors at three levels: 0 (independent), 0.2, and 0.4.

The damage function uses three types of probability distributions, all parameterized to have the same mean and standard deviation so that any observed differences can be attributed to the shape of the distribution rather than its moments. The first is a normal distribution with $\mu_v = 0.33,\ \sigma_v = 0.11$. The next is a uniform distribution with a range of 0.1395 to 0.5205. The third is a triangular distribution with $a = 0.1744,\ b = 0.6411,\ \text{and}\ c = 0.1744$.

Armstrong's full factorial experiment has 486 design points, as summarized in Table 2. For each of these design points, Armstrong simulated 50,000 battles to estimate the simulation model's outputs, such as expected damage, probability of no casualties, etc.

Table 2. Configuration of Armstrong's (2011) Experiments

Variables	Values	Number of Designs
Ships $A=B$	18, 6, and 2	3
Number of offensive missile n_α	1 and 0.5	2
Number of defensive missile n_z	1	1
Probability of offensive missile p_α	0.5, 0.67, and 0.83	3
Probability of defensive missile p_z	0.5, 0.67, and 0.83	3
Offensive missile positive correlated	0, 0.2, and 0.4	3
Types of damage function	normal distribution uniform distribution triangular distribution	3
Full factorial design points		486

(1) Independent Monte Carlo Simulation Set-up

The numbers of well-targeted offensive and defensive missiles are modeled as binomials in the simulation. The number of missiles that successfully hit ships is used to generate the random damage each side suffers. These random quantities are easily generated using R's built-in packages.

(2) Correlated Monte Carlo Simulation Set-up

To generate offensive missiles outcomes that are pairwise correlated, we start from multivariate normal distribution. In our simulation, we generate a vector Y of normally distributed random variables with mean vector M and variance-covariance matrix Σ. We can derive matrix L such that $LL^T = \Sigma$ using Cholesky decomposition (Gentle 2009). The vector Y is then calculated as

$$Y = LZ + M, \tag{10}$$

where Z is a vector of identical and independent distributed $N(0,1)$ random variables. In practice, this is accomplished using R's built-in functions. The following example illustrates the concept:

Ships $A = 3$

Number of offensive surface-to-surface missiles $n_{\alpha} = 1$

Probability that offensive missile is well-targeted $p_{\alpha} = 0.67$

Offensive missile pairwise positive correlation = 0.4.

In Figure 4, we use R to generate 10,000 replications of three univariate normal random variables all with pairwise correlations of 0.4.

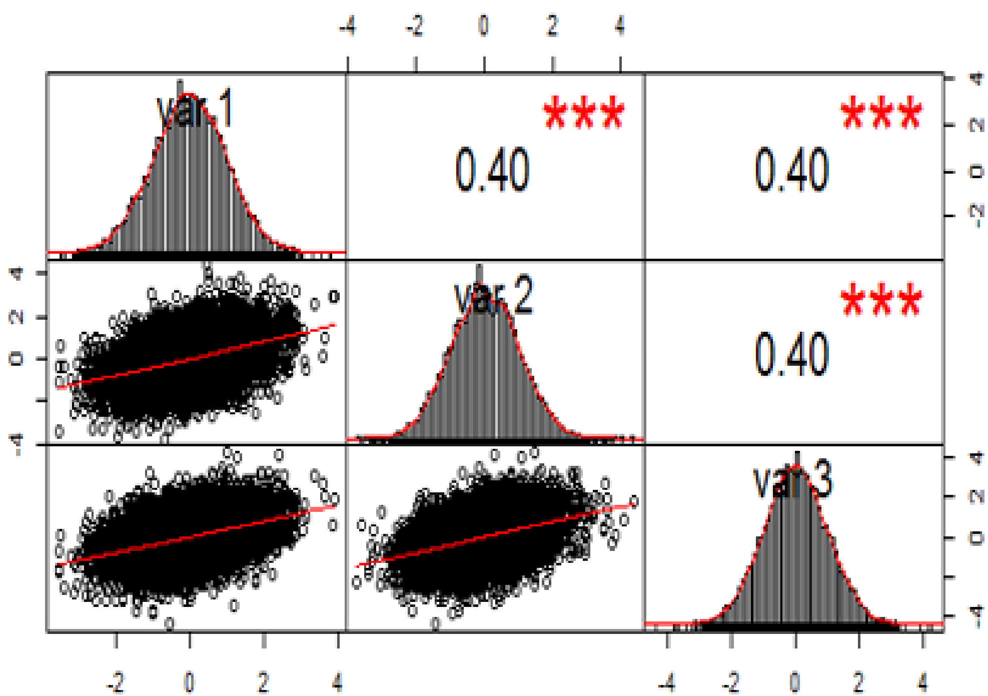


Figure 4. Modeling Dependence Example

In the next step, each univariate normal random variable from Y is mapped to a Bernoulli random variable by applying the normal CDF to find its corresponding probability, which is in turn compared to the Bernoulli's success probability. Note that applying non-linear transforms in general is not correlation preserving—and we have applied two. Empirical testing shows this correlation is not preserved from the normal random variables to the Bernoulli random variables. In our example, the pairwise correlation between the normal random variables is 0.4, but the resulting Bernoulli random variables have pairwise correlation of approximately 0.24. Using this approach produces results that are statistically indistinguishable from Armstrong's. Hence, we use this method throughout the remainder of this research.

(3) Preliminary Experiment

An illustrative example from Armstrong (2011) in his notation has

Ships: $A = 6$, $B = 6$

Number of offensive surface-to-surface missiles $n_\alpha = 1$

Number of defensive SAMs $n_z = 0.5$

Probability that offensive missile is well-targeted $p_\alpha = 0.67$

Probability of successful defensive missile $p_z = 0.67$

$Damage \sim N(0.33,\ 0.11^2)$

Number of trials per design point = 50,000

Note: Armstrong's and Hughes' models are aggregated, thus three missiles distributed across six ships is expressed as.5 missiles per ship.

Armstrong finds that the compound loss distribution has a *mean loss* of 0.679 ships and a *standard deviation of loss* of 0.467. We replicated this example in R and obtained statistically identical results. However, we also notice the heavy tail shown in Figure 5. In order to clarify the issue, Figure 6 shows the identical mean and standard deviation which

Excel produced with more bins in its histogram. Note that only 49,759 of the 50,000 samples are displayed in Figure 6 due to visual truncation.

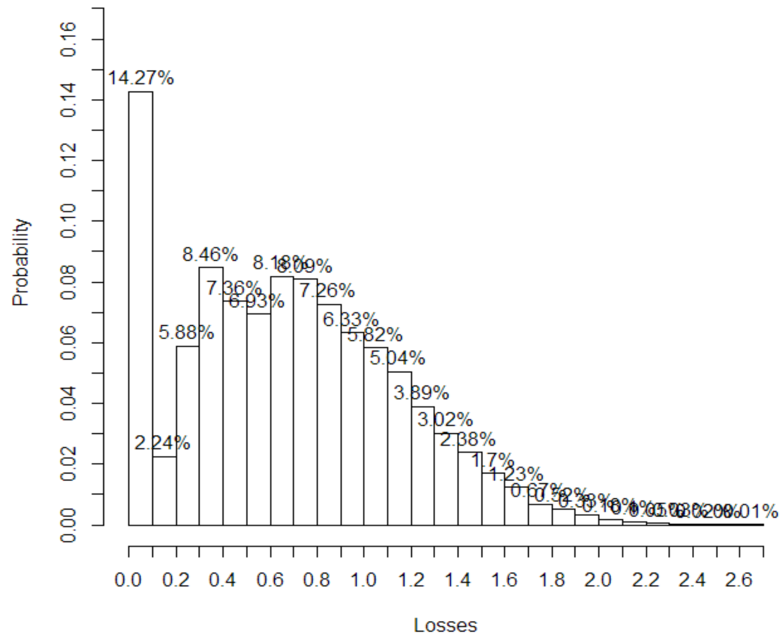


Figure 5. Distribution of Losses in the Monte Carlo Simulation in R

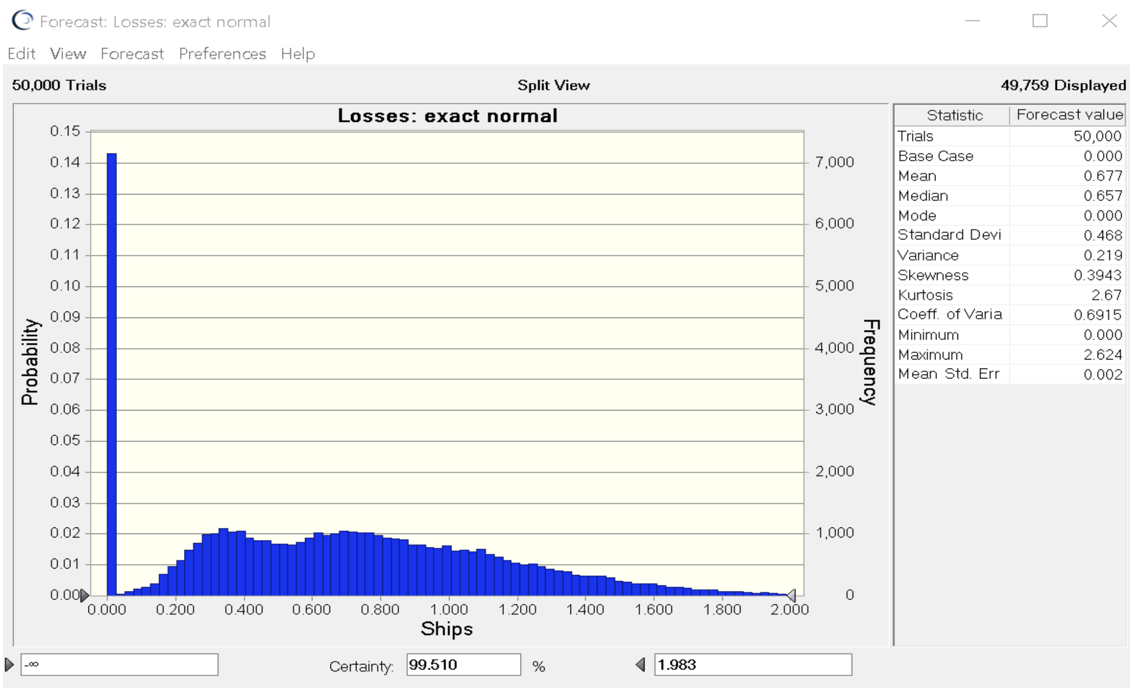


Figure 6. Distribution of Losses using Crystal Ball in Excel

We repeat the simulation with the same parameters, but add a correlation between offensive missiles of 0.2. The Crystal Ball simulation now yields a mean loss of 0.698 and a standard deviation of 0.517 versus 0.697 and 0.516, respectively, from the simulation in R. This is a difference of 0.001 with a pooled standard error of 0.0033, so the two model's outcomes are statistically indistinguishable. When the correlation is 0.4 in the same scenario, the Crystal Ball simulation returns a mean loss of 0.721 and a standard deviation of 0.559, whereas we got 0.723 and 0.557, respectively, in R. When the number of offensive missiles is up to 12, the probability is 0.67, the correlation is 0.4, and without any defensive missiles, the Crystal Ball simulation produces a mean number of 8.037 offensive missile hits with a standard deviation of 3.219, comparable to 8.011 and 3.17, respectively, in our R simulation. We conclude that the R simulation successfully replicates Armstrong's Crystal Ball simulation.

(4) Original SSM and Simulation Comparison

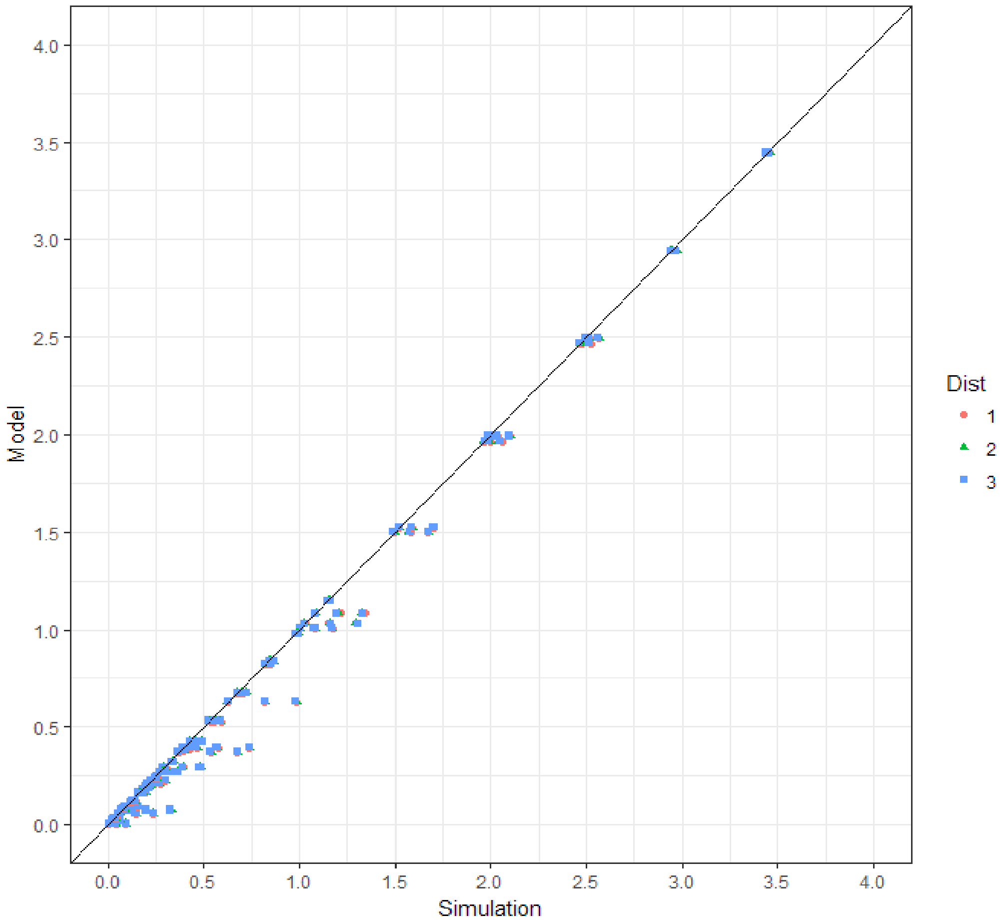
Armstrong's original SSM was explained earlier Another version, which he calls his modified model, is explained below. Over the same 486 experiments that Armstrong

(2011) performed, Figures 7 through 10 show that our results are virtually identical to Armstrong's original SSM. Because of the approximations in Armstrong's SSM, there are systematic differences (biases) between the closed-form solution and the simulation. We see that the SSM often matches or is close to what the simulation produces. When there are differences, the SSM tends to underestimate relative to the simulation's outcomes, as it does for *mean loss*, *95^{th} percentile of loss*, and *standard deviation of loss*. The exception to this is with the *probability of zero loss*, where the SSM can overestimate or underestimate relative to the simulation.

In summary, Armstrong (2011) showed the accuracy of his original SSM by using the experiments described above to generate plots similar to Figures 7 through 10. He found that the shape of the damage function distributions does not have any significant impact on the results. When offensive missiles are independent his model provides an exceptionally good fit, but adding correlation yields biased outcomes.

(5) Original Model Experiments

Figure 7 shows Armstrong's analytic original SSM's *mean loss* against the simulation's *mean loss*. The diagonal reference line indicates a perfect fit between the model and simulation. The bias is defined as the difference between the simulation and Armstrong's closed-form SSM model. In the scatter plot, the model usually produces results similar to those of the simulation. When the results differ, the model is underestimating the *mean loss*. It is also noticeable that the three types of damage function do not have much influence on the model, as was addressed in Armstrong (2011).



In the original model, damage function denote Dist 1 to 3 refer to normal, uniform, and triangle distribution.

Figure 7. Mean Loss for 486 Model

The scatter plot of *probability of zero loss* shows some bias in Figure 8. Armstrong's SSM implements an approximating normal distribution to represent the compound loss distribution. Some of the differences between the points and the reference line are around 20%, which is a substantial error. The model sometimes overestimates and sometimes underestimates the simulation's result.

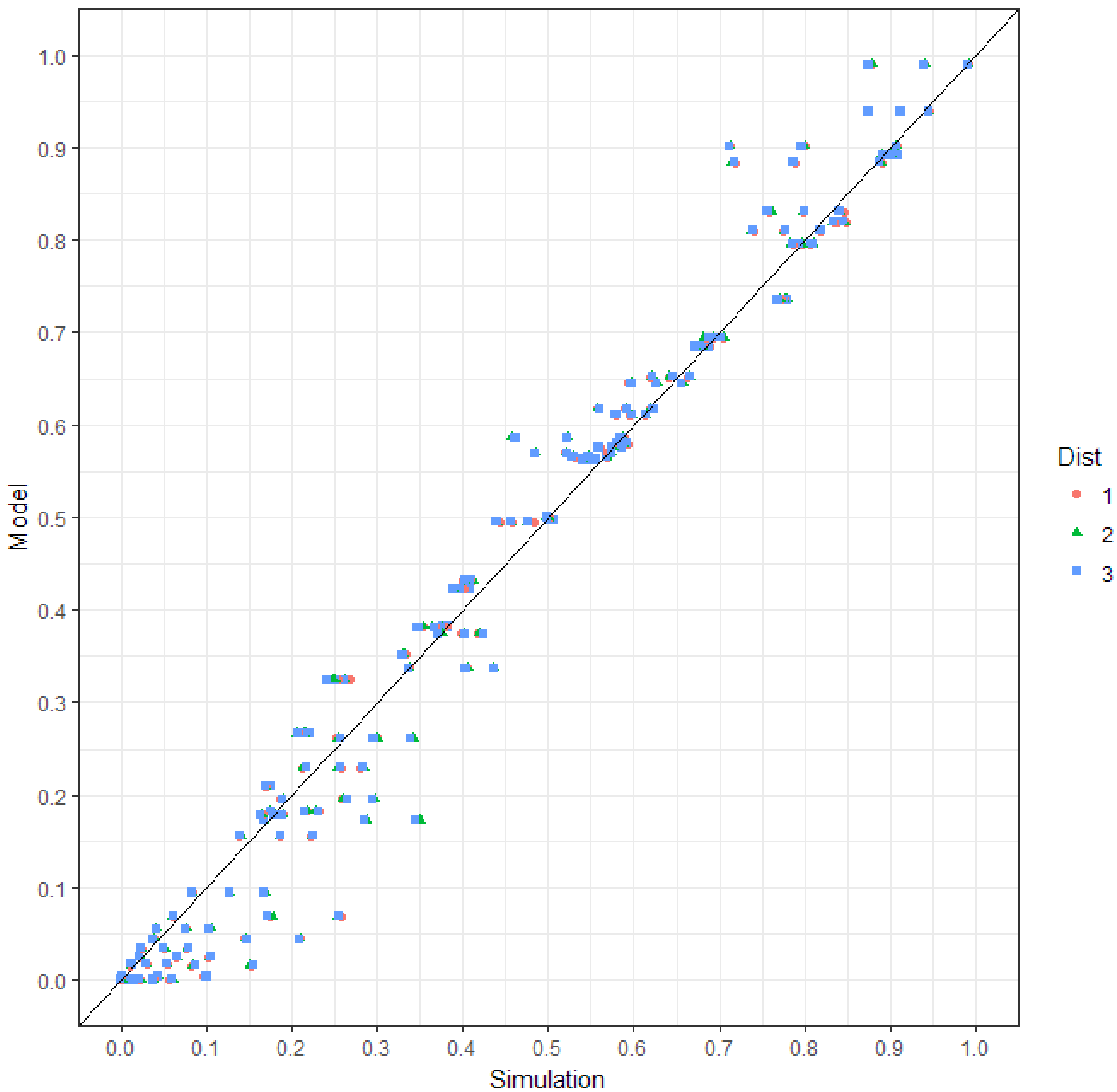


Figure 8. Probability of Zero Loss

Figure 9 depicts the *95^th^ percentile of loss* for both the simulation and the SSM. The model tends to underestimate the *95^th^ percentile of loss* in comparison to the simulation.

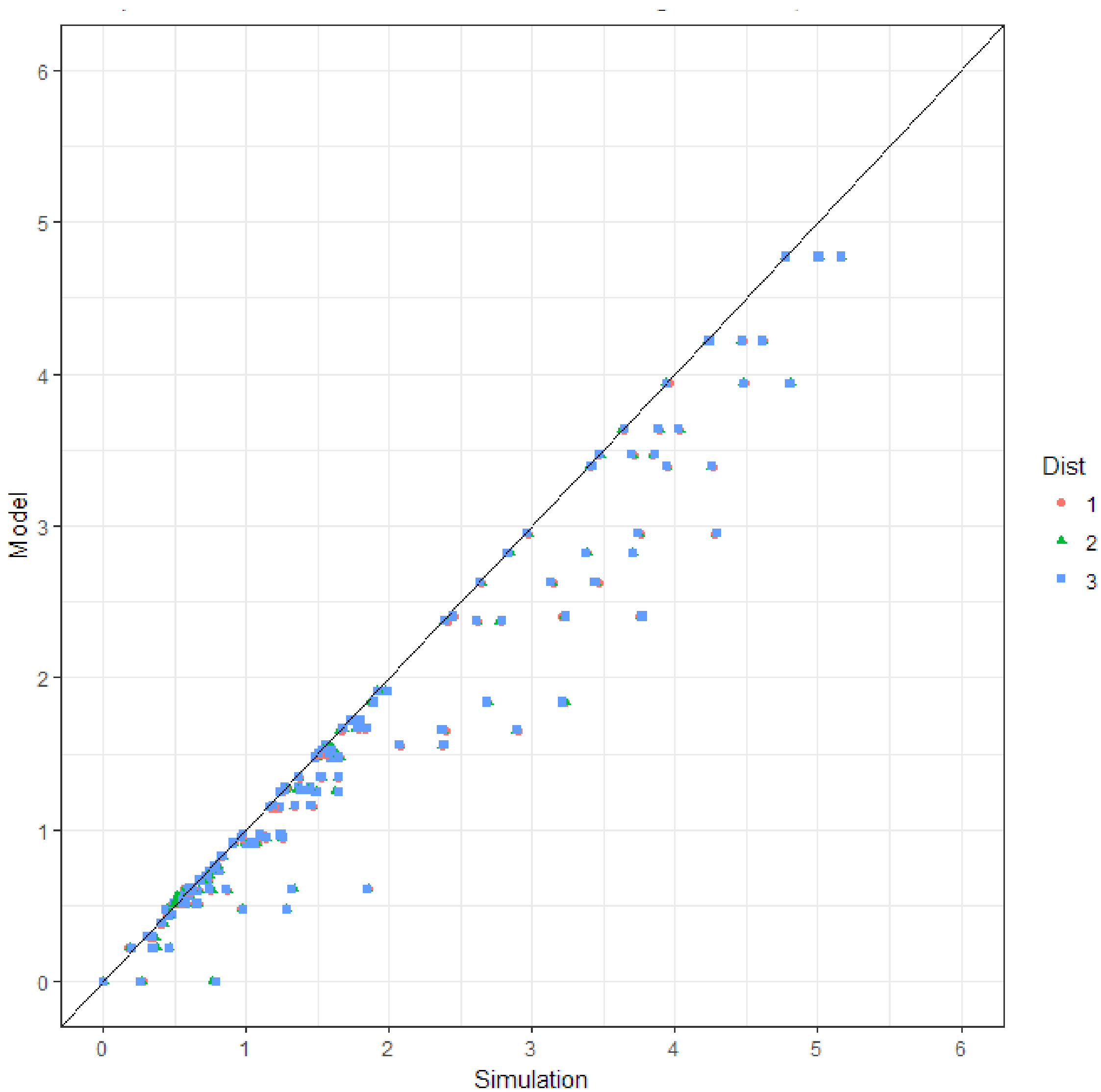


Figure 9. 95th Percentile of Loss

The *standard deviation of loss* also has a trend similar to the *95th percentile of loss*, with all deviations indicating underestimation in Figure 10.

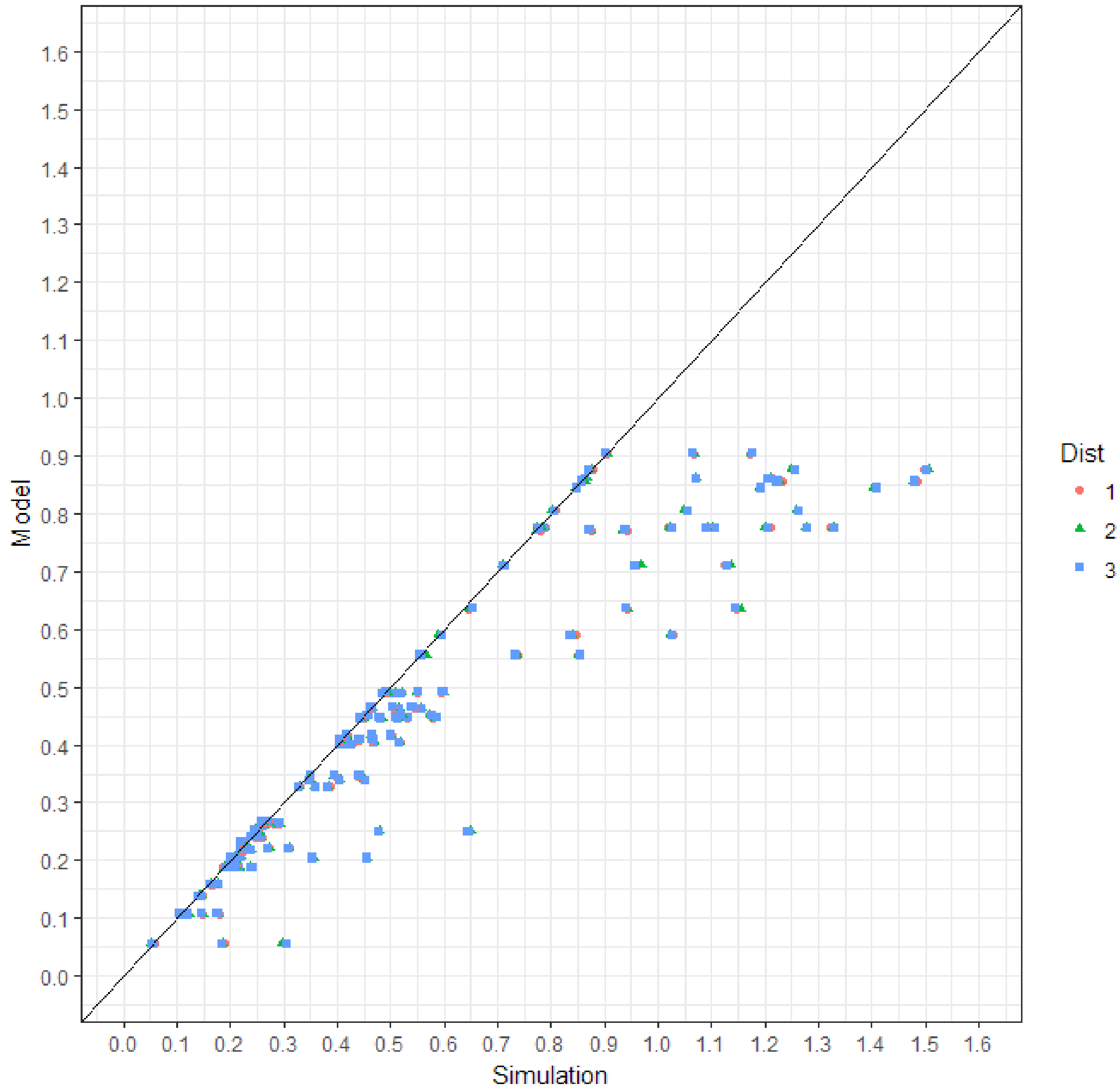


Figure 10. Standard Deviation of Loss

What values of the scenario are causing the largest differences between Armstrong's SSM and the simulation? To explore this, Figure 11 plots the error (or bias) in standard deviation against the force level and correlation. For small force levels and zero correlation (i.e., independence), the SSM is very close to the simulation. However, we notice the standard deviation increases when force level increases, especially at higher correlations. The magnitude of errors increases as force level increases regardless of correlation. This contrasts with Armstrong's expectation that large force levels would improve the fit. Note the extreme non-homogeneity of variance for error.

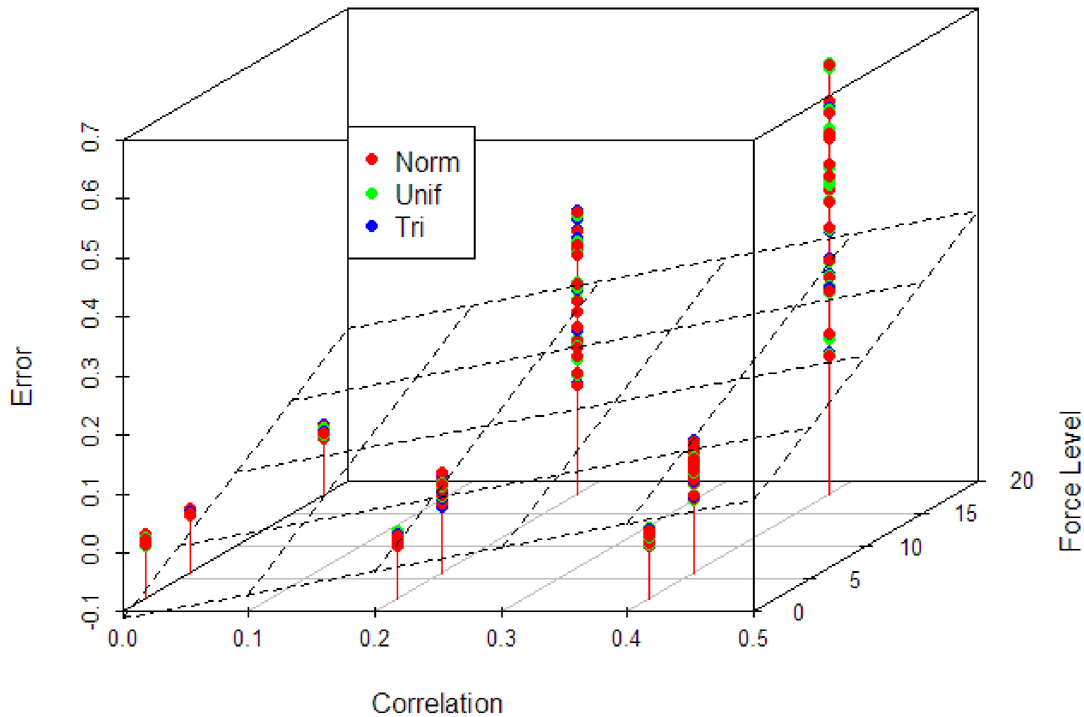


Figure 11. Plot of Regression Fit of Standard Deviation Error in Original Model

(6) Development of Modified Model

Armstrong's SSM model assumes missile behaviors are unrelated—that is, they are independent. However, when defenders use counter measures such as chaff to confuse the guidance and targeting systems, it would likely induce positive correlation among missile behaviors and outcomes. It seems reasonable that adding offensive missile correlation would provide a potentially useful extension of the original model.

To do this, Armstrong (2011) developed a modified SSM. He denotes $p_\alpha = q_\alpha r_\alpha$, where q_α represents an independent random influence that an individual missile attack is successful. Likewise, r_α represents a common influence on all of the offensive missiles and models a positive correlation. The correlation equation between two missiles is now calculated as

$$Cor[missile_i, missile_j] = (q_\alpha - r_\alpha \cdot q_\alpha) / (1 - r_\alpha \cdot q_\alpha), \tag{11}$$

while the expected number and variance of offensive missiles are

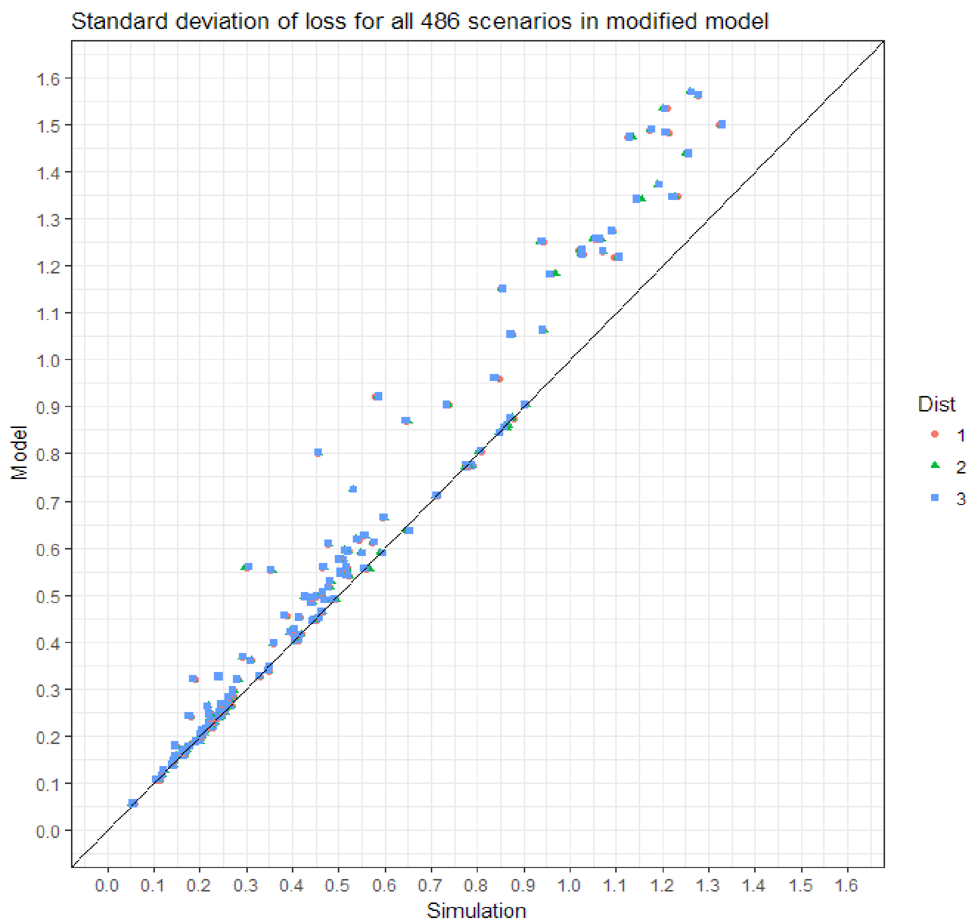
$$E[Offensive_A] = n_\alpha \cdot q_\alpha \cdot r_\alpha, \tag{12}$$

and

$$Var[Offensive_A] = n_\alpha \cdot q_\alpha \cdot r_\alpha (1 - q_\alpha + n_\alpha \cdot q_\alpha - n_\alpha \cdot q_\alpha \cdot r_a). \tag{13}$$

(7) Modified Model Experiments

Figure 12 repeats the results of Armstrong's experiments on the *standard deviation of loss* with the modified model. When the missiles are independent (correlation is zero), the results mirror the original model and fall on or near the reference line. However, the modified model overestimates the *standard deviation of loss* when there are significant levels of correlation. This is the reverse of what is seen with the original model.



the modified model tended to overestimate the simulation.

Figure 12. Standard Deviation of Loss in Modified Model

Figure 13 shows that the modified SSM tends to overestimate the *standard deviation of loss* when force level and correlation increase. We also see nonhomogeneous variance and an apparent interaction between force level and correlation.

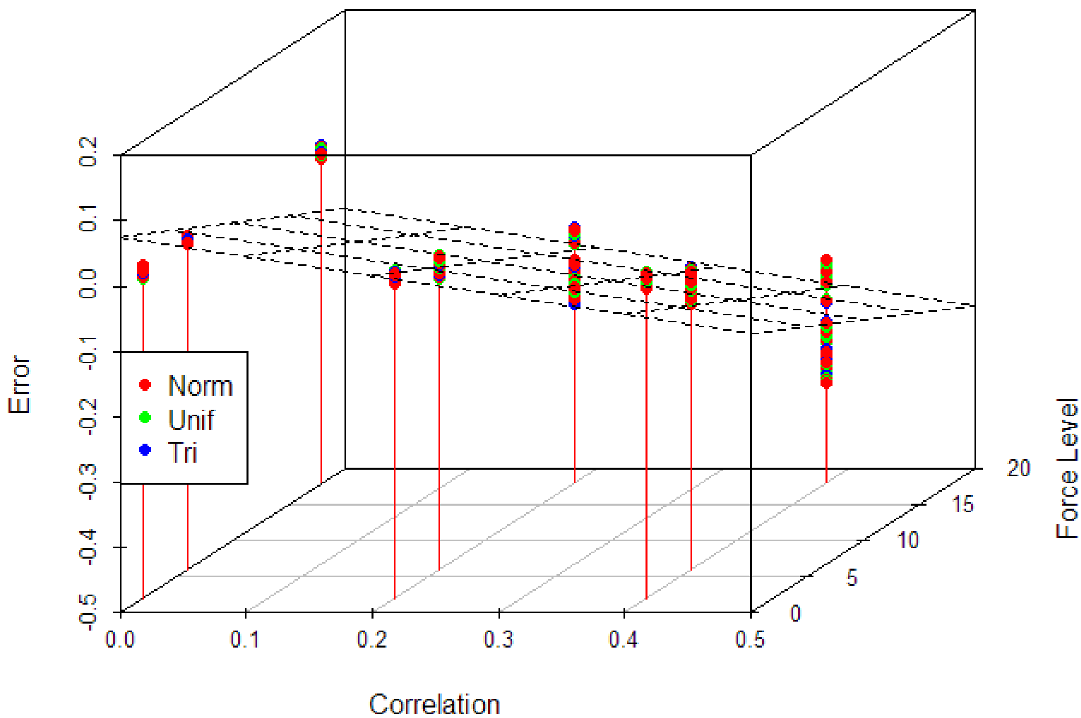


Figure 13. Plot of Regression Fit of Standard Deviation Error in Modified Model

D. CHAPTER SUMMARY

When summarizing the 486 experiments, we see that the independent scenario fits the model well. However, there are significant biases in estimates of the *standard deviation of loss* in the correlated missiles scenarios. This issue is caused in part by the truncated normal distribution in order to get results from the compound loss distribution. The next chapter describes a new set of experiments that uses many more design points while expanding the ranges of the factors that are explored. The results are used to build models of the bias as a function of the input factors. This provides guidance on how to improve the SSM.

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III. EXPERIMENTAL DESIGN

A. BACKGROUND SCENARIO

In this chapter, we expand on Armstrong's experiments by efficiently using more design points over an extended range. From the previous chapter, we can roughly identify some trends, but it is difficult to quantify the results. Without a better design of experiments, Armstrong's 486 design points might not fully explain the nature of the stochastic salvo model (SSM) and its differences from the simulation. In order to generate a sufficient and well-designed experiment, the Naval Postgraduate School's Simulation Experiments and Efficient Design (SEED) Center for Data Farming (see http://harvest.nps.edu) provides resources that enable users to build sophisticated designs, such as the 5,120 design point balanced, space-filling, and nearly orthogonal design used for this analysis.

B. ASSUMPTIONS

In previous experiments, the performance of Armstrong's SSM was consistent with baseline results obtained by simulation, especially in the independent scenario. However, one of Armstrong's experiment's limitations is that it restricted the factors to limited ranges to avoid overkill scenarios. This is because Armstrong's stochastic salvo model's actual loss equations (see equations (5) and (7)) use an approximation to achieve the calculation of actual loss. Due to the truncation of the compound loss distribution, the model's actual loss equation may introduce a bias when estimating remaining force.

The nominal equations (see equations (3) and (4)) calculate the surviving force without any adjustment for overkill or underkill. However, the compound loss equation is applied when an integer number of missiles successfully hit the ship. When defensive power is greater than offensive power, equation (2) will yield negative damage values rather than zero. One objective of this extended study is to assess the ability of the nominal equation and actual loss equation to correctly estimate the mean and standard deviation of ship losses.

Another concern that should be addressed up front is that the insights that can be obtained from the experiments depend critically on the design. When experiments are poorly designed, they preclude our ability to analyze some types of factor effects. For example, a design that has only two levels per factor cannot identify nonlinear relationships, potentially resulting in serious lack-of-fit and poor predictive power. Moreover, a poor design can result in effects of interest being confounded (that is, mathematically indistinguishable). To avoid such issues, we seek designs that are space-filling and orthogonal. The Nearly Orthogonal and Balanced (NOB) design spreadsheet, available at the Seed Center's website, provides the requisite properties and was used to generate the design used for our extended analysis.

A pitfall with running a large experiment is that performing large numbers of calculations in Microsoft Excel—as Armstrong did—can be very time consuming. Recall that Armstrong ran his simulation 50,000 times for each design point. The 50,000 replications ensure that the standard error for estimates of probabilities is bounded by 0.0022. To run our experiments more efficiently, we leveraged R's computational power by using the `apply` function to vectorize the calculations.

C. DESIGN POINTS

The design of experiments in this chapter is similar, though more extensive, than the range that Armstrong used to verify his SSM for the number of ships, missiles, and damage functions. However, we vary the probabilities of missile success and correlation values from .5 to nearly one. When side *A* attacks against side *B*, 10 factors are used to compare side *B's* actual loss with Armstrong's SSM. The binomial distribution has an integer number of successes, and the number of ships varies from two to 18, so another part of our design that differs from Armstrong is that n_α and n_z are varied discretely between 1 and 2 in order to calculate the number of successful missiles. This is more reasonable because missiles should be an integer and we still can cover all the range of Armstrong's experiments. The r_α and r_z are shared probabilities, and can be found in Table 3. Again, these were varied from .5 to near one for correlated missile scenarios. Having both equal to one corresponds to having independent missile outcomes.

Table 3. Summary of NOB Design of Experiments

Variables	Values	Decimal
Ships: *A*	2 to 18	0
Ships: *B*	2 to 18	0
Number of offensive missile n_α	1 to 2	0
Number of defensive missile n_z	1 to 2	0
Probability of offensive missile (Individual) q_α	0.5 to 0.999	3
Probability of offensive missile (Shared) r_α	0.5 to 0.999	3
Probability of defensive missile (Individual) q_z	0.5 to 0.999	3
Probability of defensive missile (Shared) r_z	0.5 to 0.999	3
Damage function (Mean) μ_v	0.22 to 0.44	3
Damage function (SD) σ_v	0.01 to 0.22	3
Total stacked NOB design points		5,120

The Nearly Orthogonal Latin Hypercube (NOLH) is a design that is both space-filling and nearly orthogonal (Cioppa and Lucas 2007). NOLHs have broad applicability and allow users to analyze metamodels with diverse data mining skills. However, they are constructed for continuous variables and their orthogonality properties can degrade if rounding is used for factors with a limited and discrete number of levels. This could be a problem for our design, where n_α and n_z only have two levels. However, Vieira et al. (2013) addressed this issue by constructing an extension of the NOLH concept that maintains near orthogonality for a mix of discrete and continuous factors while maintaining nearly balanced sampling for discrete factors. The resulting designs were designated as

Nearly Orthogonal-and-Balanced (NOB) designs. They are available in spreadsheet form from the SEED Center website.

The NOB spreadsheet uses $n = 512$ design points for up to 300 factors, of which up to 100 can be continuous. To improve upon our space-filling and near orthogonality properties, as well as to provide more degrees of freedom when fitting meta models, we permuted the assignments of factors to the spreadsheet columns and generated 10 stacks of the resulting designs. This yields 5,120 design points that are nearly balanced over all discrete inputs and uniformly distributed over the continuous factors. Moreover, the columns in the design matrix are nearly orthogonal, which minimizes estimation issues arising from collinearity. Figure 14 shows that the pairwise correlations in the table are all close to zero, i.e., the inputs are nearly orthogonal. We also see the design's space-filling properties—there are no large white spaces (i.e., unsampled areas) in the pairwise plots for the continuous factors.

Multivariate

Correlations

	A	B	na	qa	ra	nz	qz	rz	mu.v	sd.v
A	1.0000	-0.0038	-0.0009	-0.0017	0.0008	-0.0011	-0.0016	-0.0027	0.0009	0.0016
B	-0.0038	1.0000	-0.0025	0.0019	-0.0021	0.0013	0.0019	0.0011	-0.0010	-0.0041
na	-0.0009	-0.0025	1.0000	-0.0034	-0.0057	0.0027	0.0042	-0.0075	0.0024	-0.0046
qa	-0.0017	0.0019	-0.0034	1.0000	-0.0047	0.0027	-0.0014	0.0007	0.0008	0.0048
ra	0.0008	-0.0021	-0.0057	-0.0047	1.0000	0.0033	0.0030	0.0001	-0.0032	0.0007
nz	-0.0011	0.0013	0.0027	0.0027	0.0033	1.0000	-0.0092	-0.0025	-0.0020	-0.0082
qz	-0.0016	0.0019	0.0042	-0.0014	0.0030	-0.0092	1.0000	0.0034	0.0078	-0.0002
rz	-0.0027	0.0011	-0.0075	0.0007	0.0001	-0.0025	0.0034	1.0000	-0.0052	-0.0024
mu.v	0.0009	-0.0010	0.0024	0.0008	-0.0032	-0.0020	0.0078	-0.0052	1.0000	0.0018
sd.v	0.0016	-0.0041	-0.0046	0.0048	0.0007	-0.0082	-0.0002	-0.0024	0.0018	1.0000

Scatterplot Matrix

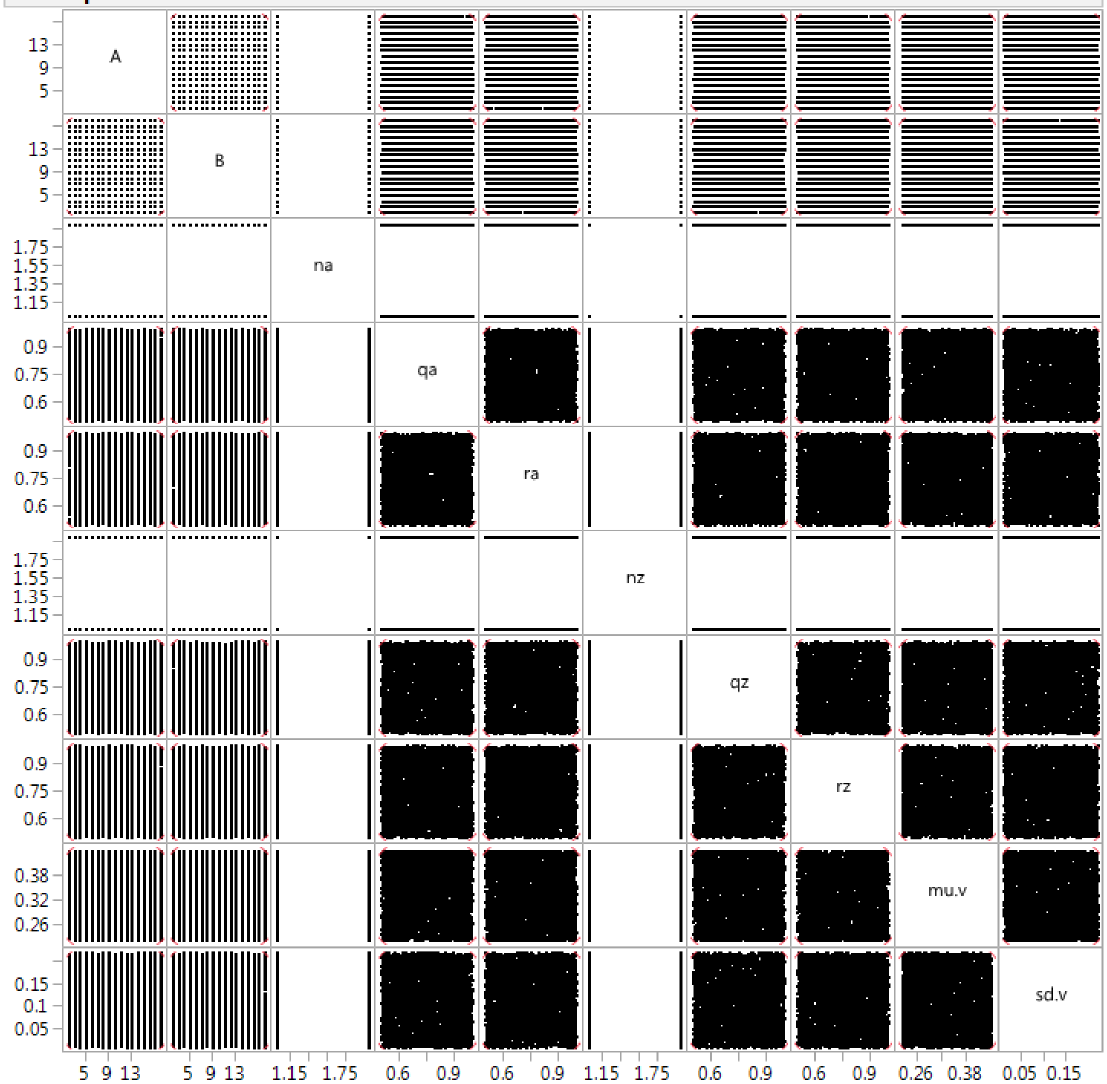


Figure 14. Correlation Matrix for Stacked NOB Design

D. TIME REQUIRED TO RUN THE EXPERIMENT

In the Crystal Ball simulation of the Armstrong preliminary experiment scenario with 6-on-6 ships, computing time was around 70 seconds to complete 50,000 replications of a single scenario on a Microsoft Surface Pro 4 with an i7 processor and 8GB of memory. A naïve projection is that the computing time for 5,120 design points would be slightly less than 100 hours. That is almost certainly an underestimate, however, since scenarios with higher force levels must do more calculations and can be expected to require more time.

Our first implementation of the simulation in R ran all trials of the Monte Carlo simulation sequentially, taking no advantage of the fact that most modern CPUs have multiple cores. All functions initially used R's default settings. The total computing time for the independent case took 7.74 hours on one core of the specified machine. R's ability to pre-generate 50,000 random variables at once improved the run time significantly. R is a vector-oriented language, and rows corresponded to simulation trials. By using the `apply` family of functions to conduct calculations in a row-wise fashion, the computing time was reduced to 1.36 hours to perform 50,000 simulation replications at each of the 5,120 design points. For future studies with larger designs, we can improve run time even further using parallelization.

IV. RESULTS AND ANALYSIS

A. SUMMARY OF RESEARCH

In Armstrong's stochastic salvo model (SSM), the analytic solution obtained using approximations of normal distributions with continuity correction is calculated analytically. Our results using simulation as a baseline show that Armstrong's model performs quite well at correctly predicting both the *mean loss* and the *standard deviation of loss* in the independent scenario. However, larger biases exist in predicting the *probability of zero loss* and *probability of total loss*. Underestimation and overestimation are observed for *probability of zero loss* in both the independent and correlation scenarios. For *probability of total loss,* only overestimation is observed in the independent scenario, while both over- and underestimation are observed in the correlated scenario. We use least squares regression and partition trees to quantify these biases and identify which factors are most influential and whether design points where the biases are the largest have commonalities.

B. NOB DESIGN EXPERIMENTS IN INDEPENDENT MISSILES SCENARIO

In this section, we use a stacked NOB design with 5,120 design points to compare how well Armstrong's SSM matches the simulation with independent offensive missiles for the four response variables. In addition to looking at many more design points, our experiments extend the ranges of some of the factors. In these runs the shared probability for offensive and defensive missiles (i.e., r_α and r_z) are fixed at one, which implies independence. Therefore, for these experiments we explore eight factors (see Table 4). For each design point the number of replications is 50,000, as in Armstrong's experiments.

Table 4. Independent Missiles Design in Stacked NOB

Variables	Values	Decimal
Ships: A	2 to 18	0
Ships: B	2 to 18	0
Number of offensive missile n_α	1 to 2	0
Number of defensive missile n_z	1 to 2	0
Probability of offensive missile (Individual) q_α	0.5 to 0.999	3
Probability of offensive missile (Shared) r_α	1	0
Probability of defensive missile (Individual) q_z	0.5 to 0.999	3
Probability of defensive missile (Shared) r_z	1	0
Damage function (Mean) μ_v	0.22 to 0.44	3
Damage function (SD) σ_v	0.01 to 0.22	3
Total stacked NOB design points		5,120

Armstrong's SSM calculates the remaining force as a truncated normal distribution to approximate both the underlying binomial structure and the fact that losses cannot be negative. Using a truncated normal allows us to analytically assess many measures, such as *mean loss*, *standard deviation of loss*, the *probability of zero loss*, the *probability of total loss*, and more. In evaluating these quantities via simulation, we are not using distributional approximation, but we need to correctly adjust our calculations to eliminate situations where ship losses go negative or are greater than the actual number of ships, as mentioned in Chapter 2. To account for this, Equation (2) is rewritten as

$$Losses = \max(0,\ \min(B,\ \sum_{k=0}^{NetAB} v_k)).$$

The salvo equations naturally can be classified into three categories depending on the *nominal mean loss* (i.e., the loss distribution before truncation). We classify the results of *nominal mean loss* greater than the initial force strength (*B*) as an *Over kill* scenario. In such a situation, all defending ships would be lost in the deterministic salvo equations. When the *nominal mean loss* is strictly between zero and the initial force, the case is defined as an *Intermediate* scenario. *Over defense* happens when *nominal mean loss* is negative, which indicates defensive power is superior to offensive power. In the deterministic salvo model, an O*ver defense* scenario results in zero losses. These categories are encapsulated in a factor we define as *Compare*, which turns out to be a strong classifier of outcomes.

In Figure 15, we find Armstrong's SSM predicts *mean loss* quite well. Over all 5,120 design points, the difference in *mean loss* between the SSM and the simulation is never more than 0.03. To put this in perspective, predicted losses in the model range from zero to over ten. The result also validates Armstrong's findings in which all the points essentially fell on his reference line.

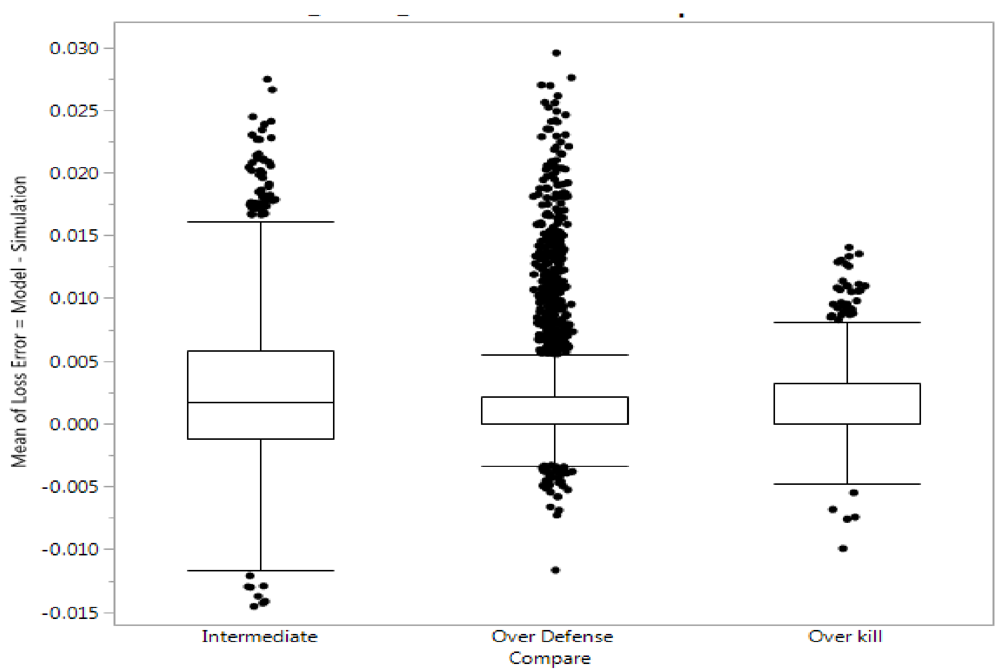


Figure 15. Difference of Mean in Actual Losses for SSM vs Simulation in Independent Scenario

Figure 16 shows that the difference of the *standard deviation of loss* between the SSM and simulation is also small for the vast majority of design points. For perspective, the mean standard deviation over the design points is 0.364, with a maximum of 1.62. We observe that outliers here are wider than in Figure 15, but the differences are small. We conclude that the three scenario types (i.e., *Intermediate*, *Over defense*, and *Over kill*) do not significantly impact biases in estimating *mean loss* and *standard deviation of loss*.

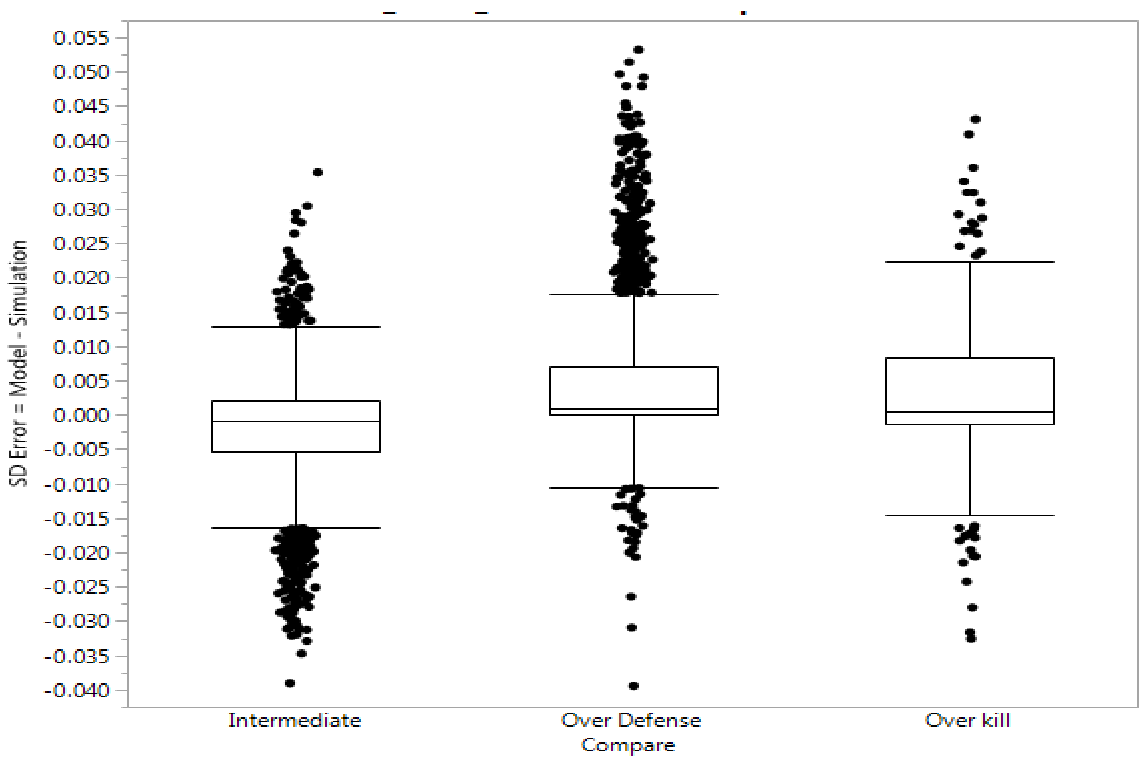


Figure 16. Difference of Standard Deviation of Actual Losses for SSM vs Simulation in Independent Scenario

The *probability of zero loss* shows more bias in the *Intermediate* and *Over defense* scenarios, see Figure 17. We observe that the magnitude of the bias (i.e., |model – simulation|) can be up to 0.1 (or 10%). To see which situations cause the largest lack of fit, we examine some extreme observations (see Table 5). The extreme points of Scenarios 1 and 2 represent an *Intermediate* scenario and Scenario 3 indicates an *Over defense* situation. Both Scenario 1 and Scenario 2 have three offensive missiles and two defensive missiles, hence the normal approximation to the binomial is stressed due to the small number of missiles. We observe that the *mean loss*, *standard deviation of loss*, and *probability of total loss* all fit well in these three scenarios. The total number of design points with bias above 1% is 842 out of 5,120, or 16.45% of all the independent NOB experiments.

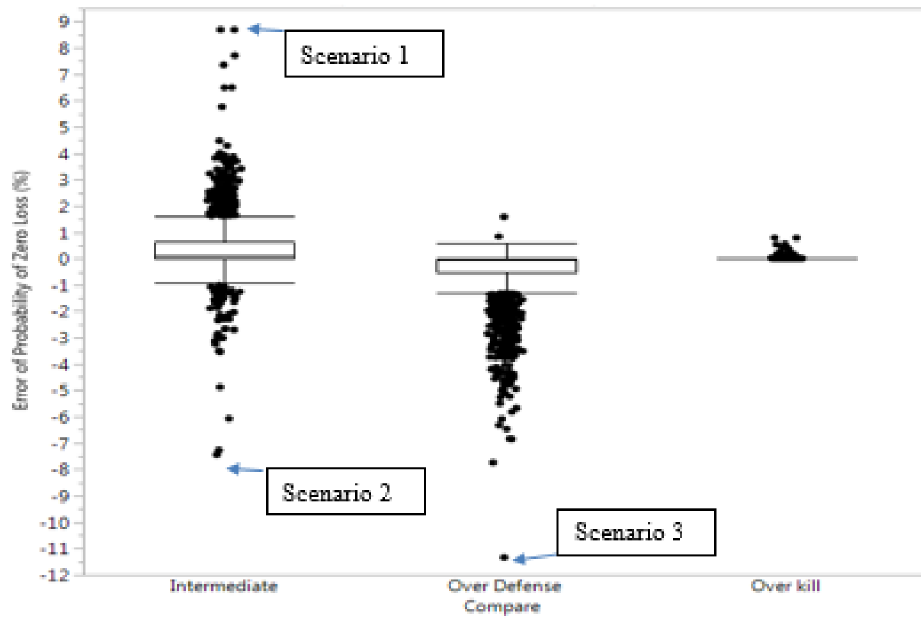


Figure 17. Percentage Difference in Probability of Zero Loss for SSM vs Simulation in Independent Scenario

Table 5. *Probability of Zero Loss* Details for Scenario 1, 2, and 3 from Figure 17

Variables	Scenario 1	Scenario 2	Scenario 3
Ships: *A*	3	3	9
Ships: *B*	2	2	9
Number of offensive missile n_α	1	1	1
Number of defensive missile n_z	1	1	1
Probability of offensive missile (Individual) q_α	0.985	0.644	0.916
Probability of offensive missile (Shared) r_α	1	1	1
Probability of defensive missile (Individual) q_z	0.663	0.936	0.964
Probability of defensive missile (Shared) r_z	1	1	1
Damage function (Mean) μ_v	0.312	0.229	0.228
Damage function (SD) σ_v	0.16	0.216	0.205
Model's *mean loss*	**0.5072**	**0.1077**	**0.0688**
Model's *standard deviation of loss*	**0.2979**	**0.1696**	**0.142**
Model's *probability of zero loss*	**0.119**	**0.643**	**0.762**
Model's *probability of total loss*	**0**	**0**	**0**
Simulation's *mean loss*	**0.5083**	**0.0878**	**0.0393**
Simulation's *standard deviation of loss*	**0.2983**	**0.1759**	**0.1269**
Simulation's *probability of zero loss*	**0.033**	**0.717**	**0.875**
Simulation's *probability of total loss*	**0**	**0**	**0**

Figure 18 shows that the model tends to overestimate the probability of total loss in the *Intermediate* and *Over kill* conditions. The bias can be as high as 0.20 (or 20%). It is reasonable that *Over defense* should have a nearly perfect fit because the *probability of total loss* is zero in most cases. In this situation, Armstrong's model does not present as well in terms of remaining force distribution as it does for *mean loss* and the *standard deviation of loss.* Table 6 shows the extreme points in *Intermediate* and *Over kill* in Scenario 1 and Scenario 2. The total number of design points with bias above 1% is 431 out of 5,120 design points, or 8.41% of all the independent NOB experiments, which indicates Armstrong's SSM mostly predicts *probability of total loss* well.

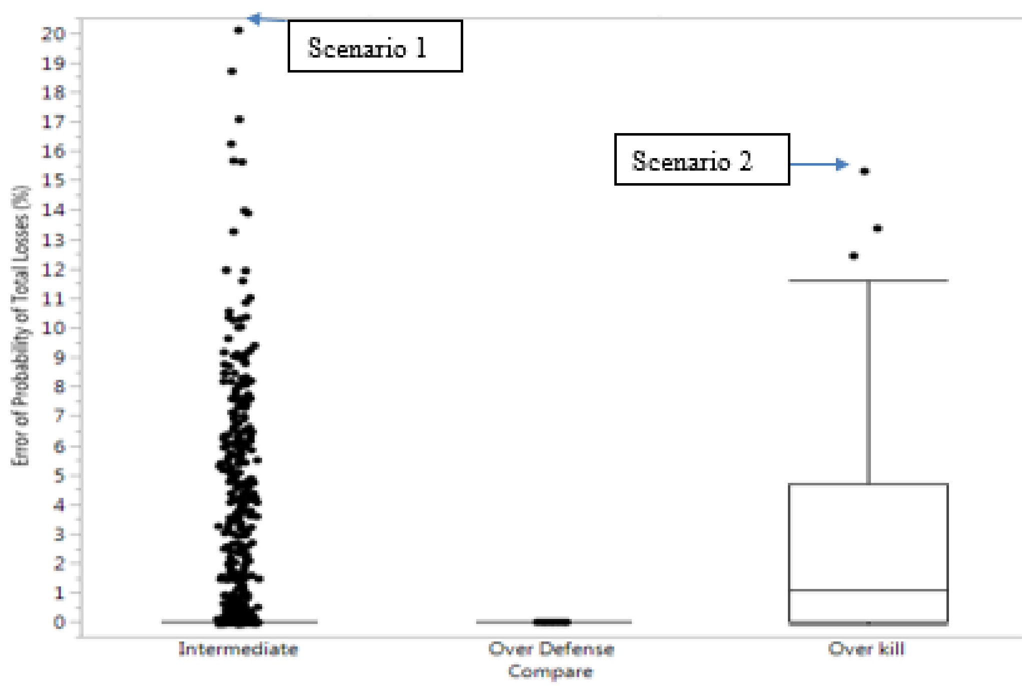


Figure 18. Percentage Difference of Probability of Total Loss for SSM vs Simulation in Independent Scenario

Table 6. *Probability of Total Loss* Details for Scenario 1 and 2 from Figure 18

Variables	**Scenario 1**	**Scenario 2**
Ships: *A*	5	7
Ships: *B*	3	2
Number of offensive missile n_α	2	1
Number of defensive missile n_z	1	1
Probability of offensive missile (Individual) q_α	0.983	0.96
Probability of offensive missile (Shared) r_α	1	1
Probability of defensive missile (Individual) q_z	0.983	0.74
Probability of defensive missile (Shared) r_z	1	1
Damage function (Mean) μ_v	0.412	0.399
Damage function (SD) σ_v	0.135	0.181
Model's *mean loss*	**2.7642**	**1.8465**
Model's *standard deviation of loss*	**0.3014**	**0.28008**
Model's *probability of zero loss*	**0**	**0**
Model's *probability of total loss*	**0.541**	**0.7097**
Simulation's *mean loss*	**2.7432**	**1.8324**
Simulation's *standard deviation of loss*	**0.292**	**0.2671**
Simulation's *probability of zero loss*	**0**	**0**
Simulation's *probability of total loss*	**0.339**	**0.557**

From previous descriptive analysis, the three types of scenarios have a significant impact on the model in terms of remaining force distribution. Figure 19 contains a partition tree of the difference between the model and simulation for *probability of total loss* using JMP Pro 13 (https://www.jmp.com). We split seven times and the partition tree can provide an R-square of only 0.3. Further splits do not provide significant improvement in fitting the model. We conclude that the partition tree does not capture much of the overall variability.

The top splits all involve the force sizes and the type of scenario. In particular, the first split is on *B*, the number of defending ships. The intuition is that side *B* dominates the other factors because it is the key to prevent total loss. The bias is relatively large when *B* is small (2 or 3) and *A* is large (5 or more). When *B* is large (4 or more), the *Over kill* situation has the largest mean bias.

When we split more, we also notice side *A*'s force level is important. This is consistent with Hughes' (1995) observation that numerical superiority plays a critical role in maritime missile battles. After considering total factor contributions, we conclude that the level of the defending force (*B*) is the most influential factor in the partition model over the ranges explored.

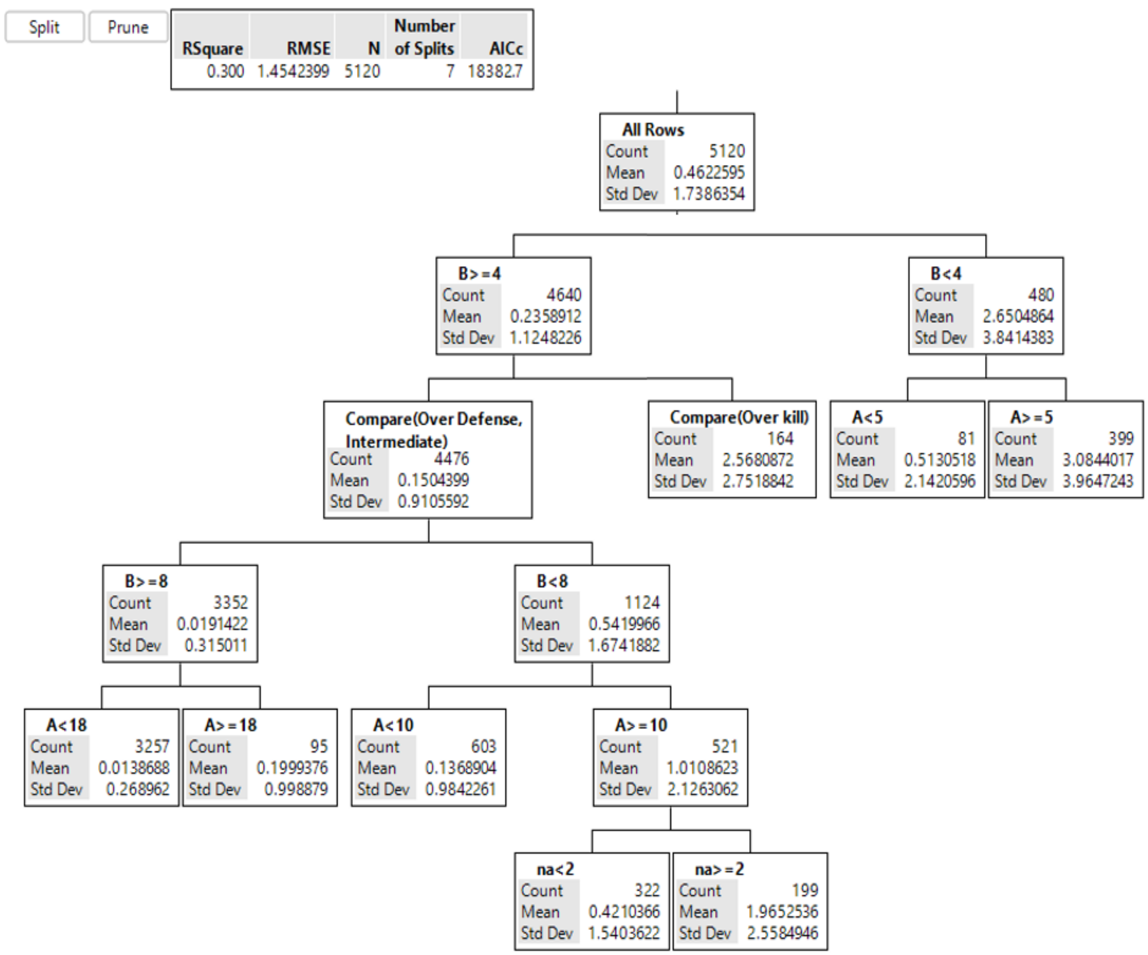


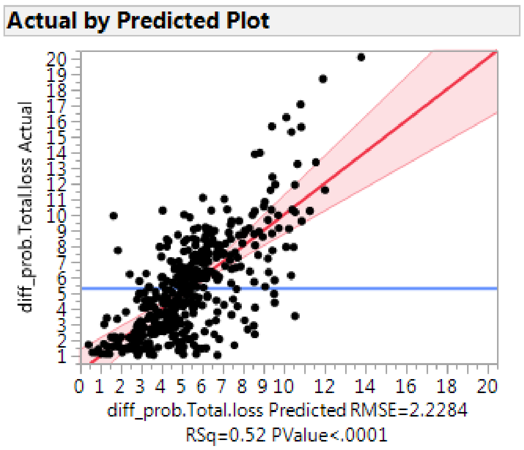
Figure 19. Partition Tree of Difference in Probability of Total Loss of Model and Simulation in Independent Scenario

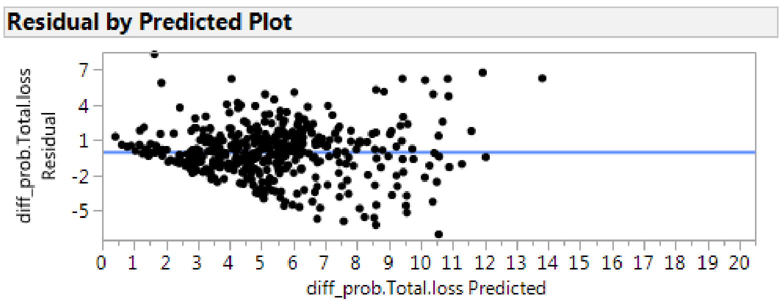
Stepwise regression provides another method to see which inputs predict large differences between the SSM and simulation. We formulate a model with all continuous factors entered as polynomials up to degree two, and two-way interactions between all factors. A generic quadratic multivariate regression can be expressed as

$$Y = \beta_0 + \sum_{i=1}^{k} \beta_i X_i + \sum_{i=1}^{k} \beta_{i,i} (\mathrm{X}_i - \bar{X}_i)^2 + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \beta_{i,j} (\mathrm{X}_i - \bar{X}_i)(\mathrm{X}_j - \bar{X}_j) + \varepsilon \,. \quad (14)$$

Model terms are then selected via minimum Bayesian information criterion (BIC). Note that JMP automatically centers each factor around its average to eliminate polynomial and interaction collinearities, improving estimation.

The first regression model we fit used the difference in *probability of total loss* between SSM and simulation for our response variable. All factors used in the design are candidates to be used as independent variables in model fitting. Note that lack of normality does not affect the unbiasedness of least square estimates. We find there are high leverage points in this model and decide to fit a second model focused on only the ill-fitting points. We accomplish this by filtering out the design points where the magnitude of the difference in *probability of total loss* is less than 0.01, leaving 431 out of the 5,120 design points. Our best model fit has an R-square of 0.52. The Residual by Predicted Plot shows that the data have non-constant variance. Figure 20 shows that force levels and three scenario conditions dominate the other factors, similar to our findings with the partition tree, and it is noteworthy that they do so via factor interactions.





Sorted Parameter Estimates

Term	Estimate	Std Error	t Ratio	Prob>|t|
(B-4.16473)*Compare[Intermediate]	-3.107455	0.201433	-15.43	<.0001*
(A-12.7146)*Compare[Intermediate]	1.1349982	0.077972	14.56	<.0001*
(na-1.6891)*Compare[Intermediate]	8.4029949	0.600621	13.99	<.0001*
(qa-0.76547)*Compare[Intermediate]	19.183976	1.407172	13.63	<.0001*
(mu.v-0.3471)*Compare[Intermediate]	33.770379	2.668267	12.66	<.0001*
(nz-1.42459)*Compare[Intermediate]	-2.641296	0.278111	-9.50	<.0001*
(qz-0.73646)*Compare[Intermediate]	-4.883367	0.819723	-5.96	<.0001*
(B-4.16473)*(na-1.6891)	1.4055123	0.275914	5.09	<.0001*
B	-0.919306	0.283783	-3.24	0.0013*
(A-12.7146)*(qz-0.73646)	-0.623902	0.195479	-3.19	0.0015*
(B-4.16473)*(B-4.16473)	0.0995656	0.033238	3.00	0.0029*
mu.v	7.972035	2.914257	2.74	0.0065*
(na-1.6891)*(mu.v-0.3471)	-8.223349	4.122683	-1.99	0.0467*
nz	-0.427899	0.295341	-1.45	0.1481
na	1.1891101	0.838235	1.42	0.1568
qa	2.1675442	1.594016	1.36	0.1746
Compare[Intermediate]	0.1429087	0.197019	0.73	0.4686
qz	0.4215005	0.848171	0.50	0.6195
A	-0.020823	0.094601	-0.22	0.8259

Figure 20. Difference of Probability of Total Loss Regression in Independent NOB Experiments

Interaction profiles (Figure 21) confirm that the *Compare* categorical variable radically affects the impact of all other factors in determining the response. For example, in an *Over kill* scenario, a larger size of *A* brings down the bias in *probability of total loss*, whereas in an *Intermediate* scenario bias increases with *A*. We see the opposite effects in the interaction between *Compare* and *B*. Looking at the set of factors in its entirety, we note that increasing offensive factors decreases bias in an *Over kill* situation, while increasing defensive factors decreases bias in an *Intermediate* situation.

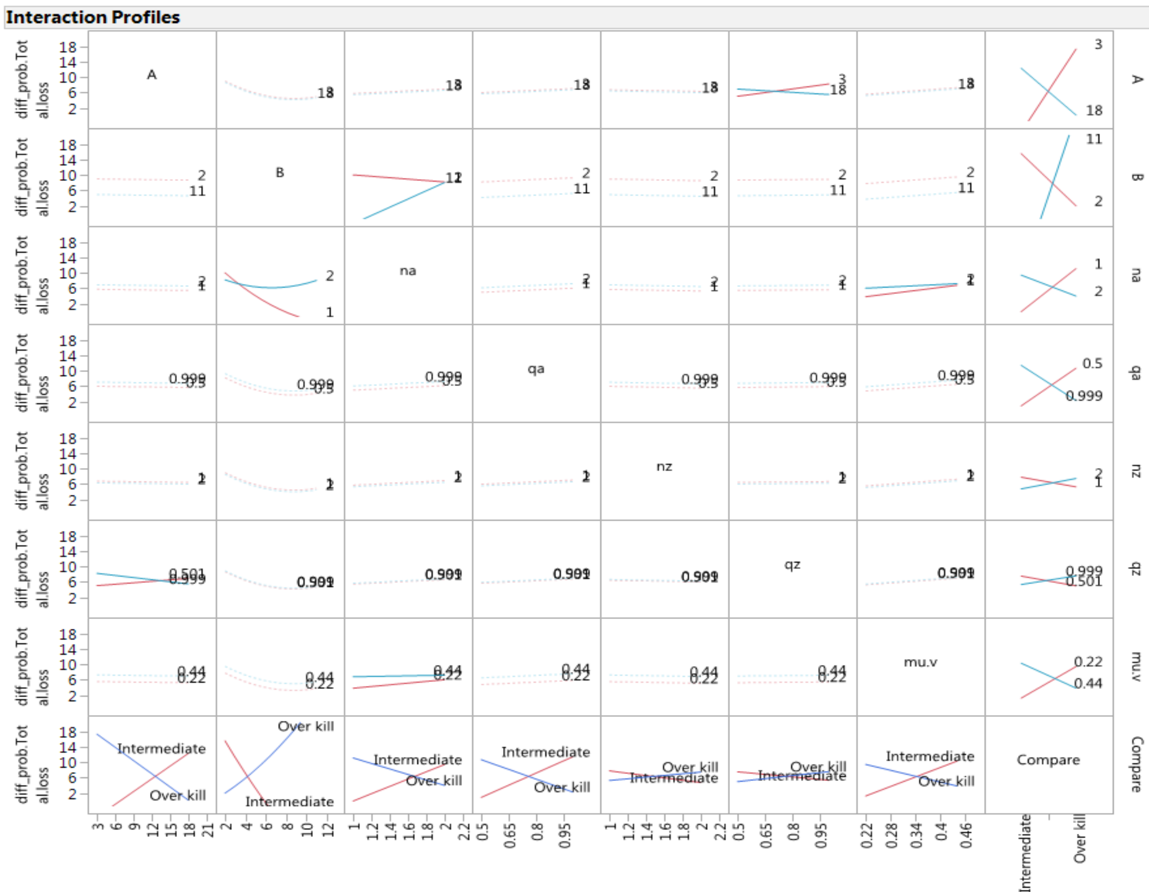


Figure 21. Difference of Probability of Total Losses Interaction Profiles in Independent NOB Experiments

C. NOB DESIGN EXPERIMENTS IN CORRELATED MISSILES SCENARIO

Armstrong (2011) found that significant positive correlations between offensive missiles results in the SSM not fitting as well. An approach he took to address this is to increase the variance based on the correlation. Armstrong calls this the modified model. Let's look at an example to see what impact this can have. Consider the case of A = 18, n_α

= 1, p_α = 0.83, and correlation equals 0.4. From equations (11) and (13), we get $q_\alpha = 0.898$ and $r_\alpha = 0.9242762$. The simulation variance for offensive missiles is 11.921, whereas for Armstrong's modified model it is 19.81043 and for Armstrong's original model it is 2.5398. Recall Armstrong's stochastic model uses the normal approximation to a binomial distribution. When he formulates the model with correlation using the normal approximation, the bias of variance will increase with both the magnitude of the correlation and the force levels.

To further illustrate, our simulation uses the multivariate normal distribution to draw the random variables and we get results that are statistically identical to Armstrong's Crystal Ball simulations.

In Armstrong's experiments, he never has more than two offensive missiles per ship. Thus, we are interested in pairwise correlations. In the bivariate normal case there are four possible outcomes given Z_1 and Z_2: (1,1), (1,0), (0,1), and (0,0) correspond to both missiles hit, two cases where one missile hits, and both missiles miss, respectively. It is relatively easy to find the probability that both hit and both miss, and the remaining probability is thus that one hit. In our scenario with $q_\alpha = 0.898$, $r_\alpha = 0.9242762$, we can use R's `mvtnorm` package to compute the probabilities. This yields 1.66 and 0.3433815 as the mean and variance of number of missile hits, in statistical agreement with the simulation. Since Armstrong's model calculates values of 1.66 and 0.3950799, we can see that Armstrong's model yields biased results for variance. Additionally, when there are more than two correlated missiles we cannot readily find an analytical solution, but simulation will continue to yield reliable results.

To run our experiments with correlated missiles, we use the same methodology as before based on a stacked NOB design containing 5,120 design points. The offensive missile shared probability r_α ranges from 0.5 to 0.999, bringing our design up to nine factors. Table 7 displays the correlated offensive missile design configuration.

Table 7. Correlated Missiles Design in Stacked NOB

Variables	Values	Decimal
Ships: *A*	2 to 18	0
Ships: *B*	2 to 18	0
Number of offensive missile n_α	1 to 2	0
Number of defensive missile n_z	1 to 2	0
Probability of offensive missile (Individual) q_α	0.5 to 0.999	3
Probability of offensive missile (Shared) r_α	0.5 to 0.999	3
Probability of defensive missile (Individual) q_z	0.5 to 0.999	3
Probability of defensive missile (Shared) r_z	1	0
Damage function (Mean) μ_v	0.22 to 0.44	3
Damage function (SD) σ_v	0.01 to 0.22	3
Total stacked NOB design points		5,120

Figure 22 shows more variation than we observed in the independent scenario in Armstrong's SSM. Recall that in Armstrong's 486 experiments, the *mean of losses* did not fit well in the bottom-left of Figure 8. In these correlated missile experiments, we see biases of magnitude approaching one ship in *mean loss* for the *Intermediate* cases. For the independent scenario, all the biases were .03 or less.

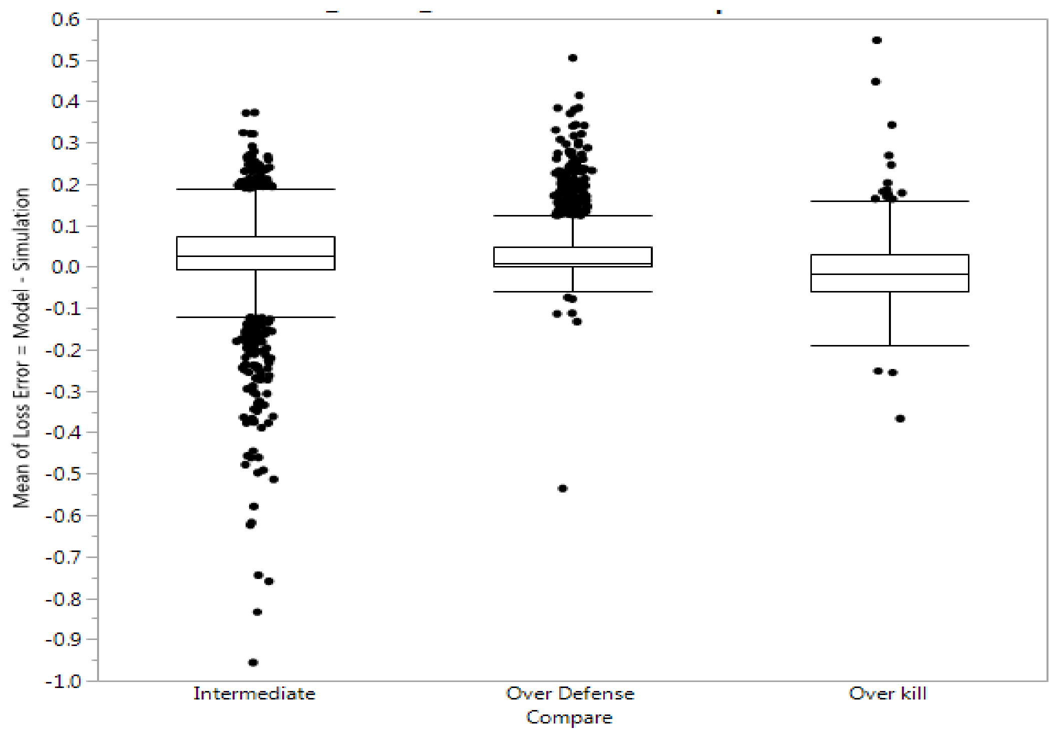


Figure 22. Difference of Mean of Actual Losses in SSM vs Simulation in Correlated Scenario

In some *Over defense* scenarios, when the simulation standard deviation is zero, Armstrong's model yields a negative variance. In our 5,120 experiments, we have 88 data points that are missing because of this.

The *standard deviation of actual losses* shows more interesting results. Figure 23 shows the magnitude of model overestimation and underestimation relative to the simulation, by type of battle. We see that the SSM dramatically overestimates the *standard deviation of actual loss*, especially in the *Intermediate* case.

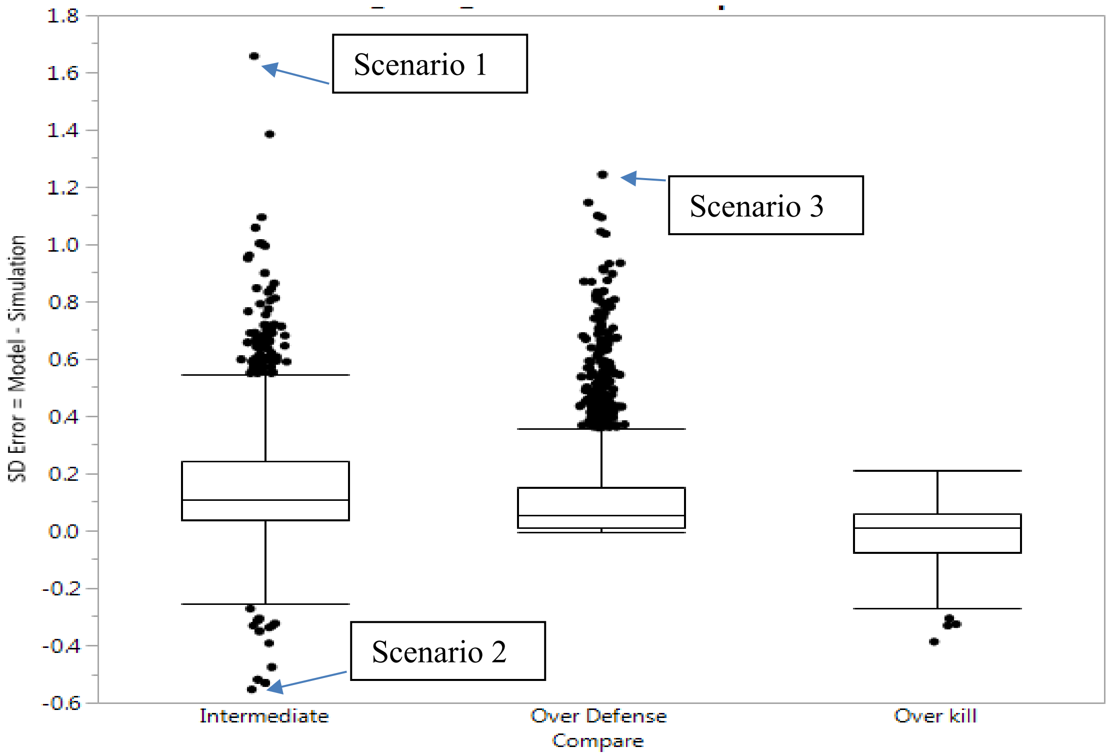


Figure 23. Difference of Standard Deviation of Actual Losses in SSM vs Simulation in Correlated Scenario

In Table 8, Scenarios 1 and 2 show that there can be an extreme bias, either positive or negative, in *standard deviation of actual loss* in an *Intermediate* scenario. Scenario 3 is a large bias in an *Over defense* situation. These three scenarios involve high values for force levels *A* and *B*.

We also notice there are several "not a number" (NaN) outcomes in the *Over defense* scenario because Armstrong's SSM can produce negative variances in Equation (15). Scenario 4 shows that when defensive power dominates offensive power, the SSM may fail to produce *standard deviation of actual loss*. Note that Scenario 4 does not show in Figure 23.

Table 8. Standard Deviation Details for Scenarios 1, 2, and 3 from Figure 23 as Well as Scenario 4, Where Armstrong's SSM Produces Negative Variance

Variables	Scenario 1	Scenario 2	Scenario 3	Scenario 4
Ships: *A*	18	17	16	3
Ships: *B*	16	9	15	14
Number of offensive missile n_α	2	2	2	2
Number of defensive missile n_z	2	1	2	2
Probability of offensive missile (Individual) q_α	0.99	0.962	0.973	0.732
Probability of offensive missile (Shared) r_α	0.8	0.52	0.692	0.793
Correlation	0.95192	0.9239	0.91735	0.36118
Probability of defensive missile (Individual) q_z	0.769	0.829	0.889	0.947
Probability of defensive missile (Shared) r_z	1	1	1	1
Damage function (Mean) μ_v	0.381	0.4	0.322	0.37
Damage function (SD) σ_v	0.147	0.081	0.065	0.105
Model's *mean loss*	**3.022**	**4.1552**	**1.1533**	**0**
Model's *standard deviation of loss*	**3.697**	**3.6238**	**2.1601**	**NaN**
Model's *probability of zero loss*	**0.407**	**0.2908**	**0.651**	**1**
Model's *probability of total loss*	**0.0048**	**0.2241**	**0.0002**	**0**
Simulation's *mean loss*	**3.1255**	**4.3619**	**0.8867**	0
Simulation's *standard deviation of loss*	**2.041**	**4.1778**	**0.9169**	**0**
Simulation's *probability of zero loss*	**0.236**	**0.4165**	**0.439**	**1**
Simulation's *probability of total loss*	**0**	**0.3666**	**0**	**0**

The difference of *probability of zero loss* shows wide variation, especially in *Over defense* scenarios. We select some extreme points from the three scenarios and observe that they are all high correlation cases, with correlations very close to one (see Table 9)

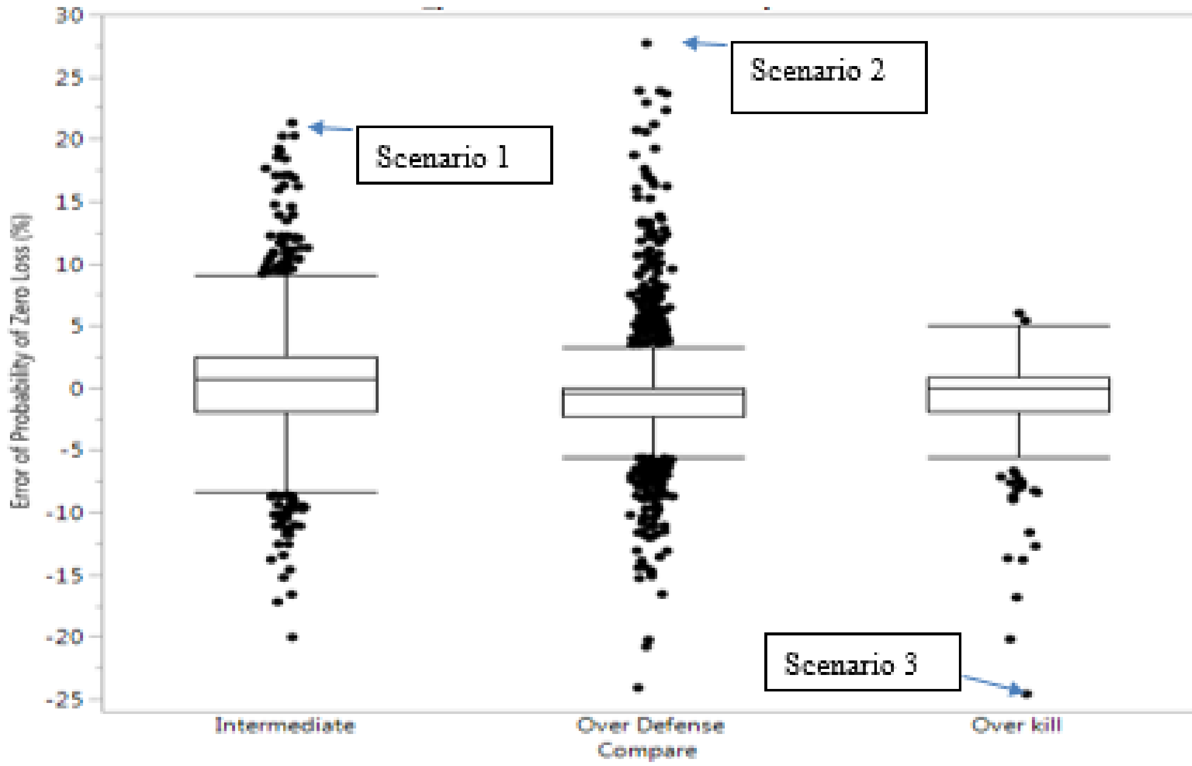


Figure 24. Percentage Difference in Probability of Zero Loss in SSM vs Simulation in Correlated Scenario

Table 9. *Probability of Zero Loss* Details for Scenario 1, 2, and 3 from Figure 24

Variables	Scenario 1	Scenario 2	Scenario 3
Ships: *A*	15	14	18
Ships: *B*	18	13	2
Number of offensive missile n_α	1	1	2
Number of defensive missile n_z	1	1	1
Probability of offensive missile (Individual) q_α	0.996	0.989	0.999
Probability of offensive missile (Shared) r_α	0.864	0.694	0.574
Correlation	0.9713	0.9649	0.9977
Probability of defensive missile (Individual) q_z	0.66	0.941	0.816
Probability of defensive missile (Shared) r_z	1	1	1
Damage function (Mean) μ_v	0.396	0.423	0.399
Damage function (SD) σ_v	0.171	0.16	0.192
Model's *mean loss*	**1.0967**	**0.6282**	**1.6412**
Model's *standard deviation of loss*	**1.4297**	**1.227**	**0.7368**
Model's *probability of zero loss*	**0.462**	**0.684**	**0.151**
Model's *probability of total loss*	**0**	**0**	**0.791**
Simulation's *mean loss*	**1.0219**	**0.4401**	**1.1934**
Simulation's *standard deviation of loss*	**0.8842**	**0.4862**	**0.9767**
Simulation's *probability of zero loss*	**0.249**	**0.408**	**0.3969**
Simulation's *probability of total loss*	**0**	**0**	**0.5909**

Figure 25 also displays poor prediction in terms of remaining force distribution, with some biases reaching a magnitude of 0.25 (i.e., 25%). From Table 10, which summarizes the three extreme points, we see that Armstrong's SSM suffers significant bias in such high correlation scenarios. As before, we filter out data with bias below 1% to focus on the poor fits. There are 838 out of 5,120 design points with a bias greater than 1% (i.e., a probability of 0.01).

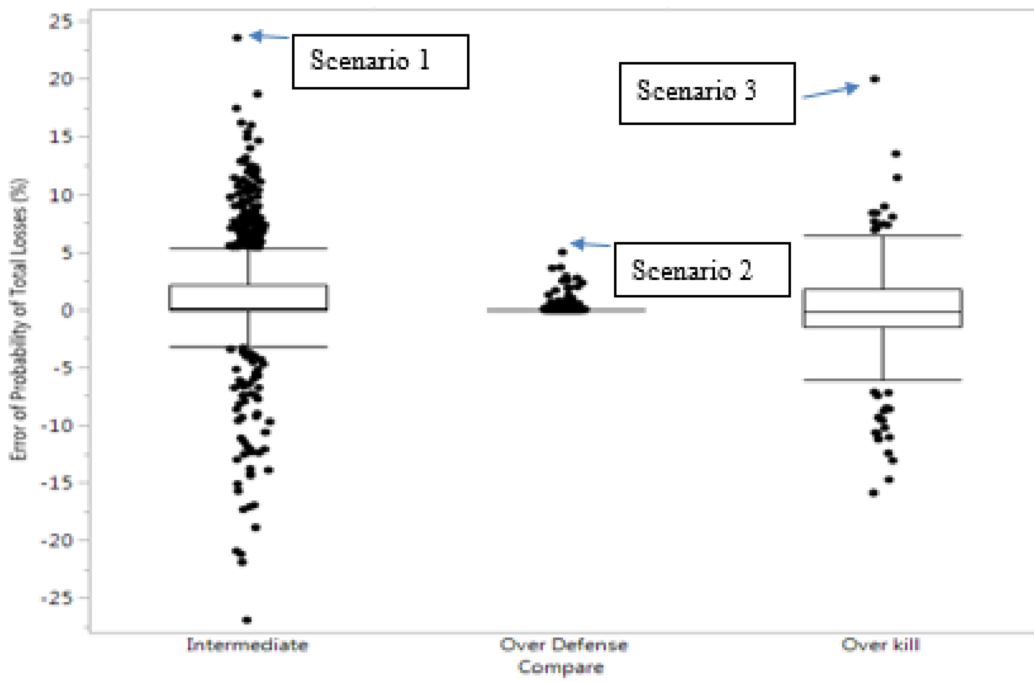


Figure 25. Percentage Difference in Probability of Total Loss in SSM vs Simulation in Correlated Scenario

Table 10. *Probability of Total Loss* Details for Scenario 1, 2, and 3 from Figure 25

Variables	Scenario 1	Scenario 2	Scenario 3
Ships: *A*	18	6	18
Ships: *B*	10	2	2
Number of offensive missile n_α	2	1	2
Number of defensive missile n_z	1	2	1
Probability of offensive missile (Individual) q_α	0.977	0.958	0.999
Probability of offensive missile (Shared) r_α	0.739	0.531	0.574
Correlation	0.91726	0.9145	0.9977
Probability of defensive missile (Individual) q_z	0.996	0.865	0.816
Probability of defensive missile (Shared) r_z	1	1	1
Damage function (Mean) μ_v	0.367	0.406	0.399
Damage function (SD) σ_v	0.035	0.153	0.192
Model's *mean loss*	**5.5477**	**0.3879**	**1.6412**
Model's *standard deviation of loss*	**3.7358**	**0.6058**	**0.7368**
Model's *probability of zero loss*	**0.158**	**0.618**	**0.151**
Model's *probability of total loss*	**0.244**	**0.055**	**0.791**
Simulation's *mean loss*	**6.5039**	**0.4275**	**1.1934**
Simulation's *standard deviation of loss*	**4.0082**	**0.5356**	**0.9767**
Simulation's *probability of zero loss*	**0.218**	**0.539**	**0.396**
Simulation's *probability of total loss*	**0.009**	**0.005**	**0.591**

Figure 26 displays the partition tree for the difference between the model and simulation for the *probability of total loss* for all 5,120 of the correlated observations. Similar to the independent experiment, we split seven times. Here, the R-square only gets to 0.12, and further splitting does not yield significant improvement. The highest mean bias leaf occurs in the *Intermediate* cases for small numbers of defending ships *B*. We were unable to find further splits to explain the variability circled in red, which corresponds to the right-hand (*Compare(Intermediate)*) branch of the partition tree.

The partition tree illustrates that what little explanatory power there is resides in the defending side *B*'s force level and *Compare (Over kill, Over defense, Intermediate)* factors. Surprisingly, the shared probability of paired missile, r_a, does not play a significant role.

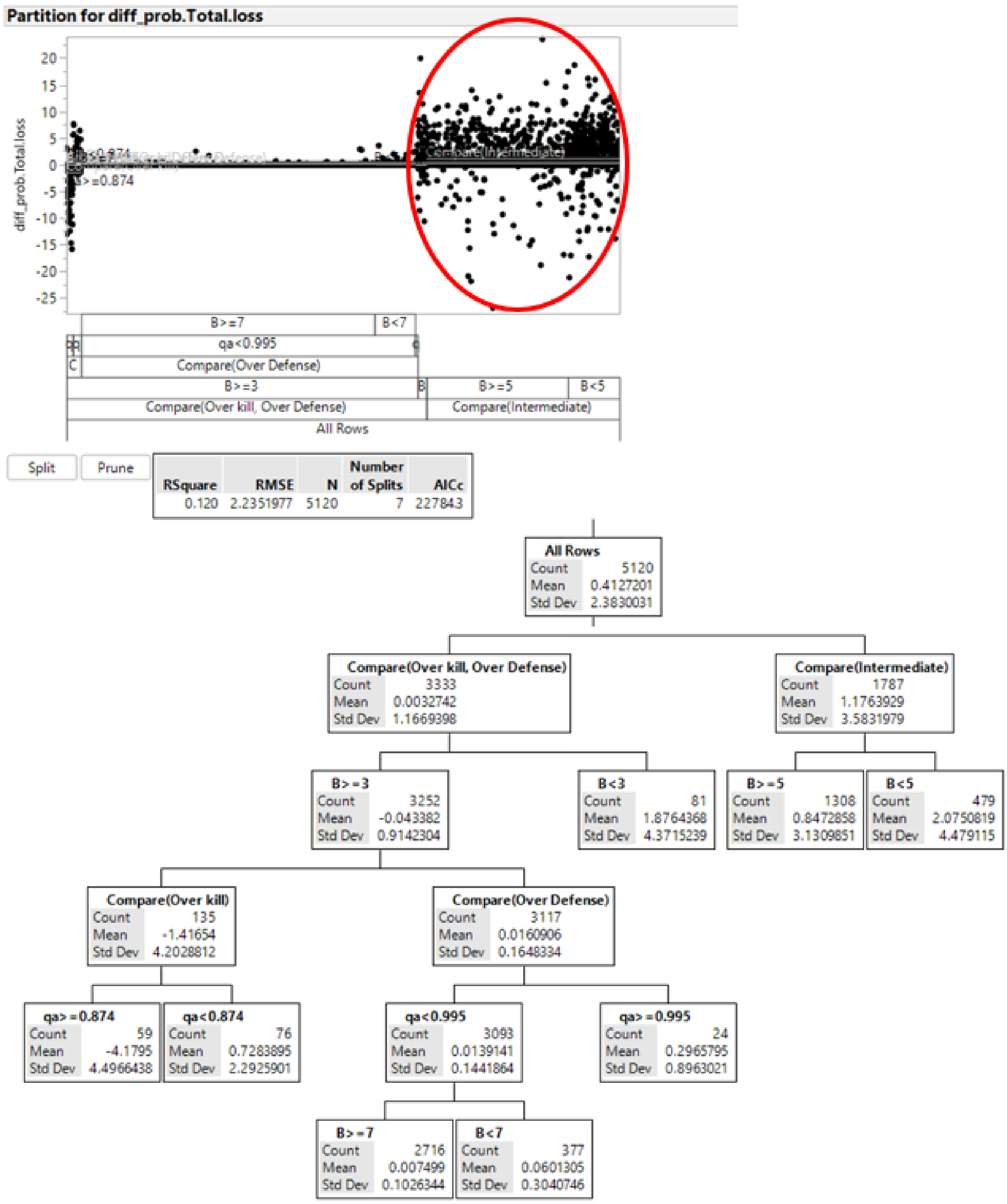
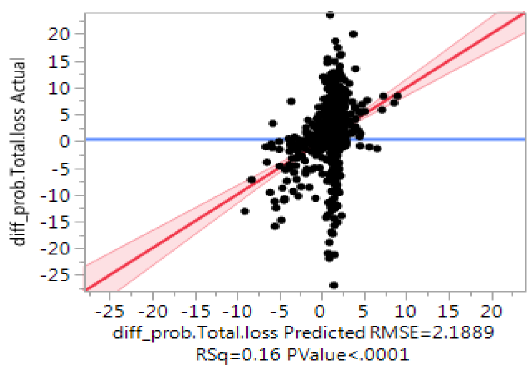
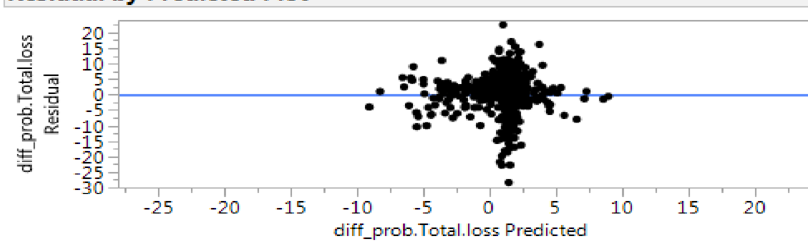


Figure 26. Partition Tree of Difference in Probability of Total Loss in SSM and Simulation in Correlated Scenario

Using the same regression methodology previously described for the 838 ill-fitting observations, a stepwise fit is attempted on all factors up to 2nd degree polynomial and two-way interactions. Figure 27 depicts the poor fit. The R-square has a value of 0.16, and the residuals show no obvious patterns to exploit for improvement. As with the partition tree, *B* force levels and *Compare* are the dominant factors. It is noteworthy that the *Compare* factor has a strong interaction with all other factors. In the independent scenario, *Intermediate* conditions play the critical interaction role among all factors.



Residual by Predicted Plot



Sorted Parameter Estimates

Term	Estimate	Std Error	t Ratio	Prob>|t|
B	-0.66326	0.049432	-13.42	<.0001*
(B-10)*Compare{Over kill-Over Defense}[-1]	0.6570605	0.049764	13.20	<.0001*
Compare{Over kill-Over Defense}[0]	4.4483463	0.39902	11.15	<.0001*
Compare{Over kill-Over Defense}[-1]	3.923772	0.397398	9.87	<.0001*
(B-10)*Compare{Over kill-Over Defense}[0]	0.4828792	0.050085	9.64	<.0001*
(qa-0.7495)*Compare{Over kill-Over Defense}[-1]	4.1016042	0.492789	8.32	<.0001*
qa	-3.8301	0.463153	-8.27	<.0001*
(ra-0.7495)*Compare{Over kill-Over Defense}[0]	4.0647729	0.497945	8.16	<.0001*
A	0.1470235	0.02082	7.06	<.0001*
ra	-3.160633	0.451424	-7.00	<.0001*
(ra-0.7495)*Compare{Over kill-Over Defense}[-1]	3.218477	0.481706	6.68	<.0001*
(A-10)*Compare{Over kill-Over Defense}[-1]	-0.142414	0.021486	-6.63	<.0001*
(qa-0.7495)*(ra-0.7495)	8.322509	1.510276	5.51	<.0001*
(qa-0.7495)*Compare{Over kill-Over Defense}[0]	2.7929621	0.509967	5.48	<.0001*
(A-10)*Compare{Over kill-Over Defense}[0]	-0.057942	0.022406	-2.59	0.0097*

Figure 27. Regression Model of Difference of Probability of Total Loss in Correlated NOB Experiments

From the Interaction Profiles (Figure 28), we see that the strongest interaction is between *B* and *Compare*. The interactions are not as strong as they are for the independent case, where we observed sign reversals on the factor effects based on the *Compare* classification.

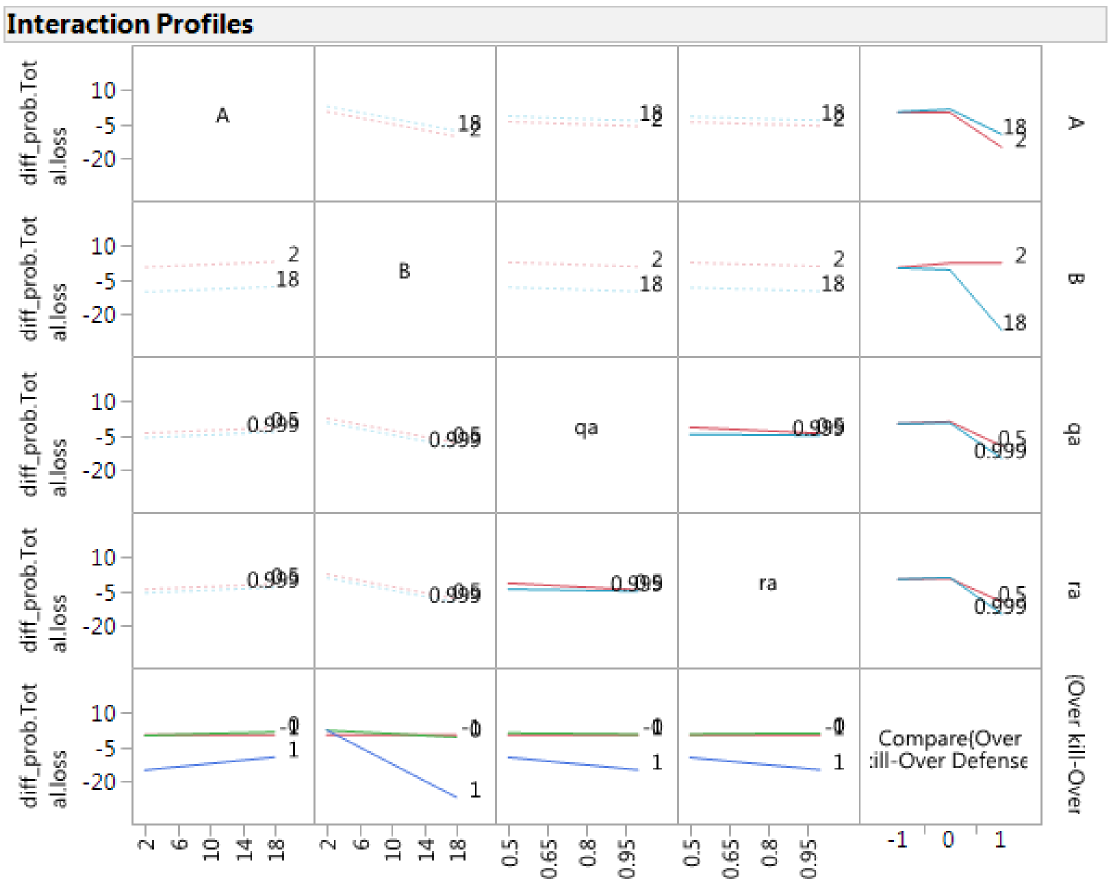


Figure 28. Difference of Probability of Total Loss Interaction Profiles in Correlated NOB Experiments

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V. SUMMARY AND CONCLUSION

Keep your model as simple as possible, but no simpler.

—Albert Einstein

Sanchez (2007) pointed out that it is impossible to build a model that can answer all questions that could be posed, since that would require a model containing the entire world's state space. In order to be useful, a model must simplify our view of the world and be focused on a particular set of questions. Hughes' salvo model does this elegantly to permit naval officers to gain insight about maritime combat. We might argue that Hughes' deterministic salvo model is too simple for some purposes, as discussed in Lucas and McGunnigle (2002).

Hughes' salvo model has two components in each exchange, offense and defense. It is more dynamic than Lanchester's model, which only has one component. If one side is sophisticated and aggressive, they can emerge as the "leader" in the salvo. As an attacker, based on tactics, ASCM waypoints can be used to increase the probability of hits. As a defender, firing SAMs or other countermeasures in response to incoming offensive missiles, combined with evasive maneuvering, can improve the ship's chances of survival. It follows that Hughes' salvo model yields much richer insights for modern surface warfare than Lanchester's equations.

A key simplifying assumption of Hughes' salvo model is that it treats all elements deterministically, including inherently random quantities such as the number of missiles that strike a target and the damage inflicted. However, warfare is intrinsically stochastic and models of it should usually reflect this truth (Lucas 2000). Armstrong's stochastic salvo model (SSM) extends Hughes' model to provide probabilistic information, and hence greater insight, such as the probability of victory and the margin of uncertainty. However, it does this at a cost of additional complexity. Despite this, the SSM is still simple enough to be solved in closed form. This simplicity is obtained by introducing several assumptions, such as the normal approximation to the binomial.

The SSM tells us that neither side is guaranteed a victory unless it has overwhelming superiority. In any competition, the attacker always wants to increase his offensive power, while the defender always wants to lower offensive power or damage and increase defensive power. In studying the competition between two sides, the SSM delivers more critical insights, such as probability of victory and significant impact on attributes. When we look at historical battles, it makes us strongly believe the model should include stochastic behavior.

Armstrong pursued an analytical modeling approach, based on using a normal approximation to the binomial distribution to achieve closed-form solutions. Nonetheless, in some scenarios the approximation is poor and results in bias. Moreover, this bias is hard to predict. This research used sophisticated DOE to quantify the bias over a broad range of input factors. The experimental results show that for independent offensive missiles the SSM closely matches the simulation throughout the region explored. The difference increases when the missiles are correlated and the forces are large, especially in estimating the probabilities of zero loss or annihilation. The difference also depends critically on whether the forces are in an overkill, intermediate, or over-defense situation.

An alternative is to computationally evaluate the model without invoking the simplifying assumptions by using Monte Carlo simulation. The results obtained from this approach are both intuitive and robust, since they are based on the exact formulation of the model and do not involve approximation. When combined with designed experimentation, computer models can be used to derive input/output metamodels with excellent predictive power. Moore’s law has made this approach even more appealing over time, as models with greater complexity can be solved with increasing rapidity due to hardware improvements. We can also leverage the rapid growth in software for modeling and analysis, which allows us to build and evaluate models with greater complexity and fewer assumptions, but which is nevertheless easier to control and interact with. An example is provided in Figure 29, which illustrates how the Shiny interface to R was used to create an interactive and friendly model interface for users, with more functionality and faster computing time than Excel. The Shiny application facilitates exploration of Armstrong’s

SSM, our R simulation, and our prototype characteristic function evaluator of the salvo model in a single framework.

Figure 29. Salvo Model Application with Simulation in Shiny

Lu, Ren, and Liu (2016) proposed using characteristic functions to provide analytic solutions for independent scenarios in terms of probability of total loss. However, their results do not agree with simulation results. This led us to the work of Shevchenko (2010), who evaluated compound Poisson distributions using characteristic functions. We have adapted his approach to create a prototype analytic model solver based on characteristic functions in R and added it to the Shiny application. This research is ongoing, but preliminary findings indicate that it yields outcomes that are much closer to the simulation than Armstrong's SSM can achieve.

APPENDIX. R MODEL

```
### -------------- INTERMEDIATE CALCULATIONS  -----------------####

  # Average Offensive Power
  off.A <- A*na*qa*ra
  off.B <- B*nb*qb*rb

  # Variance Offensive Power
  if(Modified){
    off.A.var <- A*na*qa*ra*(1-qa+A*na*qa*(1-ra))
    off.B.var <- B*nb*qb*rb*(1-qb+B*nb*qb*(1-rb))
  }else{
    off.A.var <- A*na*pa*(1-pa)
    off.B.var <- B*nb*pb*(1-pb)
  }


  # SD Offensive Power
  off.A.sd <- sqrt(off.A.var)
  off.B.sd <- sqrt(off.B.var)

  # Average Defensive Power
  def.A <- A*ny*qy*ry
  def.B <- B*nz*qz*rz

  # Variance Defensive Power
  if(Modified){
    def.A.var <- A*ny*qy*ry*(1-qy+A*ny*qy*(1-ry))
    def.B.var <- B*nz*qz*rz*(1-qz+B*nz*qz*(1-rz))
  }else{
    def.A.var <- A*ny*py*(1-py)
    def.B.var <- B*nz*pz*(1-pz)
  }

  # SD Defensive Power
  def.A.sd <- sqrt(def.A.var)
  def.B.sd <- sqrt(def.B.var)

  # Expected Nominal Hits
  NetBA <- off.B - def.A
  NetAB <- off.A - def.B

  # Var Nominal Hits
  NetBA.var <- off.B.var + def.A.var
  NetAB.var <- off.A.var + def.B.var

  # SD Nominal Hits
  NetBA.sd <- sqrt(NetBA.var)
  NetAB.sd <- sqrt(NetAB.var)

 ### -------------- END INTERMEDIATE CALCULATIONS ----------------- ###
```

```
### -------------- PROPOSITION 1 -----------------

# Calcualte Nominal Surviving Strengths EA1*, EB1*
EA1.star <- A - NetBA*mu.u
EB1.star <- B - NetAB*mu.v

VA1.star<-(NetBA*sd.u^2)+(NetBA.var*mu.u^2)-
2*(sd.u^2)*NetBA*pnorm(0,NetBA,NetBA.sd)+2*(sd.u^2)*NetBA.var*dnorm(0,N
etBA,NetBA.sd)

VB1.star<-(NetAB*sd.v^2)+(NetAB.var*mu.v^2)-
2*(sd.v^2)*NetAB*pnorm(0,NetAB,NetAB.sd)+2*(sd.v^2)*NetAB.var*dnorm(0,N
etAB,NetAB.sd)

VA1star.sd <- sqrt(VA1.star)
VB1star.sd <- sqrt(VB1.star)

### -------------- END PROPOSITION 1 ----------------- ###

### -------------- PROPOSITION 2 -----------------####
 # CALCULATE ACTUAL SURVIVING STRENGTHS EA1, EB1
EA1<-EA1.STAR*(PNORM(A-MU.U/2,   EA1.STAR,   VA1STAR.SD)-PNORM(0+MU.U/2,
EA1.STAR,    VA1STAR.SD))-VA1.STAR*(DNORM(A-MU.U/2,EA1.STAR,VA1STAR.SD)-
DNORM(0+MU.U/2,EA1.STAR,    A1STAR.SD))+A*(1-PNORM(A-MU.U/2,    EA1.STAR,
VA1STAR.SD))

EB1<-EB1.STAR*(PNORM(B-MU.V/2,   EB1.STAR,   VB1STAR.SD)-PNORM(0+MU.V/2,
EB1.STAR,  VB1STAR.SD))-  VB1.STAR*(DNORM(B-MU.V/2,EB1.STAR,VB1STAR.SD)-
DNORM(0+MU.V/2,  EB1.STAR,  VB1STAR.SD))+B*(1-PNORM(B-MU.V/2,  EB1.STAR,
VB1STAR.SD))

VA1  <-  (EA1.STAR^2+VA1.STAR)*(PNORM(A-MU.U/2,  EA1.STAR,  VA1STAR.SD)-
PNORM(0+MU.U/2,         EA1.STAR,         VA1STAR.SD))+(A^2)*(1-PNORM(A-
MU.U/2,EA1.STAR,VA1STAR.SD))-(EA1^2)-VA1.STAR*((A-
(MU.U/2)+EA1.STAR)*DNORM(A-MU.U/2,EA1.STAR,VA1STAR.SD)
-(0+(MU.U/2)+EA1.STAR)*DNORM(0+MU.U/2,EA1.STAR,VA1STAR.SD))

VB1  <-  (EB1.STAR^2+VB1.STAR)*(PNORM(B-MU.V/2,  EB1.STAR,  VB1STAR.SD)-
PNORM(0+MU.V/2,EB1.STAR,VB1STAR.SD))+(B^2)*(1-PNORM(B-
MU.V/2,EB1.STAR,VB1STAR.SD))-(EB1^2)-VB1.STAR*((B-
(MU.V/2)+EB1.STAR)*DNORM(B-U.V/2,EB1.STAR,VB1STAR.SD)
-(0+(MU.V/2)+EB1.STAR)*DNORM(0+MU.V/2,EB1.STAR,VB1STAR.SD))

VA1.SD <- SQRT(VA1 )

VB1.SD <- SQRT(VB1 )

### -------------- END PROPOSITION 2 ----------------- ###
```

```
### -------------- PROPOSITION 3 -----------------####

  # CALCULATE CONDITIONAL SURVIVING FORCES
ECA1 <- EA1/(1-PNORM(0+MU.U/2,  EA1.STAR, VA1STAR.SD))

ECB1 <- EB1/(1-PNORM(0+MU.V/2,  EB1.STAR, VB1STAR.SD))

VCA1 <- ((VA1 + (EA1^2))/(1-PNORM(0+MU.U/2,  EA1.STAR, VA1STAR.SD))) -
(ECA1^2)

VCB1 <- ((VB1 + (EB1^2))/(1-PNORM(0+MU.V/2,  EB1.STAR, VB1STAR.SD))) -
(ECB1^2)

  # SOLVE FOR LETHALITY

P.ANOTSURVIVE <- PNORM(0 + MU.U/2,EA1.STAR,VA1STAR.SD)

P.ASOMESURVIVE<-PNORM(A-(MU.U/2),EA1.STAR,VA1STAR.SD)-
PNORM(0+MU.U/2,EA1.STAR,VA1STAR.SD)

P.AALLSURVIVE <- 1 - PNORM(A-(MU.U/2),EA1.STAR,VA1STAR.SD)
P.BNOTSURVIVE <- PNORM(0 + MU.V/2,EB1.STAR,VB1STAR.SD)

P.BSOMESURVIVE<-PNORM(B-(MU.V/2),EB1.STAR,VB1STAR.SD)-
PNORM(0+MU.V/2,EB1.STAR,VB1STAR.SD)

P.BALLSURVIVE <- 1 - PNORM(B-(MU.V/2),EB1.STAR,VB1STAR.SD)

### -------------- END OF PROPOSITION 3 ----------------- ###

### -------------- PROPOSITION 4 ----------------- ###
P.AWINS<-  (1-PNORM(0+MU.U/2,  EA1.STAR,  VA1STAR.SD))*(PNORM(0+MU.V/2,
EB1.STAR, VB1STAR.SD))

P.BWINS<-  (PNORM(0+MU.U/2,  EA1.STAR,  VA1STAR.SD))*(1-PNORM(0+MU.V/2,
EB1.STAR, VB1STAR.SD))

P.BOTHSURVIVES<-(1-PNORM(0+MU.U/2,EA1.STAR,VA1STAR.SD))*(1-
PNORM(0+MU.V/2,  EB1.STAR, VB1STAR.SD))

P.BOTHDEAD<-  (PNORM(0+MU.U/2,  EA1.STAR,  VA1STAR.SD))*(PNORM(0+MU.V/2,
EB1.STAR, VB1STAR.SD))

### -------------- END OF PROPOSITION 4 ----------------- ###
```

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