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THE NORMAL TO THE WALL PRESSURE GRADIENT FOR BLASIUS AND FALKNER-SKAN BOUNDARY LAYER FLOW

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14. ABSTRACT
The theoretical description of forced laminar flow over a flat plate is revisited. In many texts, an order of magnitude argument is incorrectly used to claim that the momentum equation normal to the wall reduces to just the pressure gradient term equal to zero. We use a Falkner-Skan analysis to discern the true nature of the normal to the wall momentum conservation equation. The Falkner-Skan solution obtained from the *x*-momentum equation (parallel to the wall) is used to calculate the normal to the wall *y*-pressure gradient using the *y*-momentum equation. The normal pressure gradient for the Blasius, as well as a more realistic Falkner-Skan solution, are examined.

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laminar boundary layer, normal pressure gradient, *y*-momentum equation, Falkner-Skan flow, Blasius flow

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1. SUMMARY

The theoretical description of forced laminar flow over a flat plate is revisited. In many texts, an order of magnitude argument is used to claim that the momentum equation normal to the wall reduces to just the pressure gradient term equal to zero. This is not a correct statement of conservation of momentum. A momentum balance must exist even if the pressure gradient is zero which is not even the case here. We use a Falkner-Skan analysis to discern the true nature of the normal to the wall momentum conservation equation. Inherent in this momentum equation is that the flow in the normal direction is due to a pressure gradient. The pressure gradient, in turn, is formed by fast-moving inlet flow encountering slow-moving boundary layer flow. The Falkner-Skan solution obtained from the x-momentum equation (parallel to the wall) is used to calculate the normal to the wall y-pressure gradient using the y-momentum equation. The y-pressure gradient for the Blasius, as well as a more realistic Falkner-Skan solution, are examined. It is pointed out that any theory of aerodynamic drag necessarily requires an understanding of the behavior of the normal pressure gradient in the boundary layer region and this work is a positive step in that direction.

2. INTRODUCTION

Forced laminar flow over a flat plate is one of the most basic boundary layer theoretical descriptions. It is used in every fluid flow textbook as part of an introduction to boundary layer flow and similarity solutions. For the flow depicted in Figure 1, conservation of momentum, energy, and mass govern the behavior of the fluid flow including the formation of the boundary layer due to a no-slip wall boundary. Ludwig Prandtl[1] used an order of magnitude argument approach to point out that the full Navier-Stokes momentum equations could be considerably simplified for this case. The derivation of the various Prandtl conservation equations that govern the boundary layer fluid flow is covered in many texts including Schlichting's[2] *Boundary Layer Theory*.

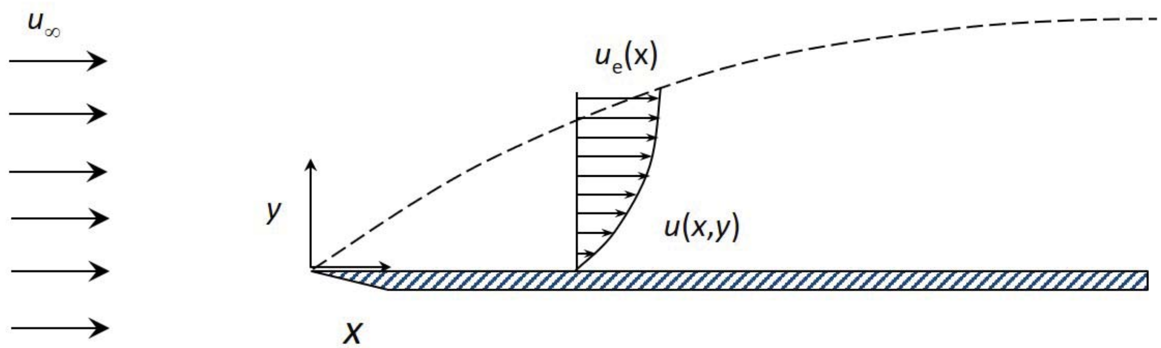

Figure 1: Schematic depicting Laminar Flow over a Flat Plate

For an isothermal 2D incompressible laminar boundary layer flow, it can be demonstrated that the *x*-component of the Prandtl boundary layer momentum balance (parallel to the wall) is given by

$$u(x,y)\frac{\partial u(x.y)}{\partial x} + v(x,y)\frac{\partial u(x,y)}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u(x,y)}{\partial y^2} \quad , \tag{1}$$

where ρ is the density, ν is the kinematic viscosity, *P* is the pressure, *u*(*x*,*y*) is the velocity in the *x*-direction, and *v*(*x*,*y*) is the velocity in the *y*-direction. The *y*-component of the Prandtl boundary layer momentum balance (normal to the wall) is given by

$$u(x,y)\frac{\partial v(x,y)}{\partial x} + v(x,y)\frac{\partial v(x,y)}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\frac{\partial^2 v(x,y)}{\partial y^2} \quad , \tag{2}$$

and the mass conservation equation is given by

$$\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y} = 0 \quad . \tag{3}$$

For isothermal forced laminar flow over a flat plate depicted in Figure 1, it turns out that the velocities $u(x,y)$ and $v(x,y)$ can be determined using a similarity solution method employing just the x-momentum and mass conservation equations. This is the route taken by Blasius[3] and Falkner and Skan[4]. With the solution in hand, the thinking has been that there is little need for the y-momentum equation. This has led many to actually try to dismiss the y-momentum equation entirely. It is very common to find texts indicating that the y-momentum equation reduces to just the pressure gradient term equal to zero (see, for example, White[5] or Cengel and Cimbala[6]). That is, many claim that Eq. 2 reduces to

$$\frac{\partial P}{\partial y} \cong 0 \quad . \tag{4}$$

The intent, apparently, is to make the point that y-momentum equation does not have a direct role in finding flow solutions. However, Eq. 4 is not a correct statement of momentum conservation. The y-momentum may be small compared to the x-momentum but it definitely exists since $v(x,y)$ must be nonzero. Momentum must still be conserved. To state that the y-momentum for the boundary layer situation reduces to Eq. 4 is simply wrong.

What has been lost in this widespread dismissal of the y-momentum equation is an understanding of the nature of the pressure gradient in the y-direction. As fast-moving inlet flow encounters slow-moving boundary layer flow, a pressure imbalance is created in the boundary layer region. The flow in the y-direction is due to the pressure gradient formed by this pressure imbalance. The y-momentum equation provides a way to determine the pressure gradient in the y-direction. Herein, we explore the nature of this pressure gradient using a Falkner-Skan analysis. The Falkner-Skan solution obtained from the x-momentum equation is used to calculate the y-pressure gradient using the y-momentum equation. The pressure gradients for the Blasius, as well as a more realistic Falkner-Skan solution, are examined. It is pointed out that any theory of aerodynamic drag necessarily requires an understanding of the behavior of the normal pressure gradient in the boundary layer region and this work is a positive step in that direction.

3. FALKNER-SKAN SIMILARITY SOLUTIONS

Beginning with the flow depicted in Figure 1, our Falkner-Skan analysis starts by partially nondimensionalizing the flow governing equations given by Eqs. 1-3. Since we are interested in similarity solutions, we start by choosing the similar thickness and velocity scaling parameters. For 2D wall-bounded flows, Schlichting[2] defines velocity profile similarity as the case where two velocity profiles taken at different stations along the plate differ only by simple scaling parameters in *y*, the normal direction to the wall and *u*(*x*,*y*), the velocity parallel to the wall in the flow direction. The velocity profile at a point *x* along the plate is defined as the velocity *u*(*x*,*y*) for all *y*-values. Weyburne[7] recently proved that for any 2D wall-bounded boundary layer flow that shows similar behavior of the velocity profile, the similar velocity scale must be the velocity at the boundary layer edge $u_e(x)$ and that the similar length scale must be the displacement thickness $\delta_1(x)$. Hence, to put the boundary layer equations into dimensionless form, we take the scaled *y*-parameter as

$$\eta = \frac{y}{\delta_1(x)} \quad , \tag{5}$$

and the velocity scaling parameter as $u_e(x)$.

We seek flow solutions to Eqs. 1-3 for which the displacement thickness $\delta_1(x)$ and the velocity at the boundary layer edge $u_e(x)$ are simple power functions of *x* in accordance with the well-known Faukner-Skan[4] solutions. Thus, assume that $\delta_1(x)$ and $u_e(x)$ are well approximated as

$$\delta_1(x) = b(x-x_0)^n \quad \text{and} \quad u_e(x) = a(x-x_0)^m \quad , \tag{6}$$

where x_0, a, b, *m*, and *n* are constants.

As a first step, we introduce a stream function so that we can combine the mass conservation equation and the momentum equations. Underlying the stream function approach is a critical assumption to the whole theoretical development and that is that the velocities *u*(*x*,*y*) and *v*(*x*,*y*) can be decomposed into a product of *x*-functionals and scaled *y*-functionals. Thus, assume that a stream function $\psi(x,y)$ exists such that

$$\frac{\psi(x,y)}{\delta_1(x)u_e(x)} = f(\eta) \quad , \tag{7}$$

where $f(\eta)$ is a dimensionless function that only depends on the scaled *y*-position, and that the stream function satisfies the conditions

$$u(x,y) = \frac{\partial\psi(x,y)}{\partial y}, \quad \text{and} \quad v(x,y) = -\frac{\partial\psi(x,y)}{\partial x} \quad . \tag{8}$$

Using the stream function together with Eqs. 5 and 6 (see Appendix A) means that

$$v(x,y) = -\frac{\partial \psi}{\partial x} = -(m+n)\frac{\delta_1 u_e}{x-x_0} f + n\frac{\delta_1 u_e}{x-x_0}\eta f' \quad , \tag{9}$$

and

$$u(x,y) = \frac{\partial \psi}{\partial y} = u_e f' \quad , \tag{10}$$

where the prime indicates differentiation with respect to η.

4. TRANSFORMED MOMENTUM EQUATIONS

The velocities given by given by Eqs. 9 and 10 can now be substituted into the momentum equations (Eqs. 1 and 2). The term by term transformation is relegated to Appendix A and B. The important result obtained from the transformed *x*-component of the momentum balance (Appendix A) is that Eq. 1 reduces to the well-known Falkner-Skan[4] result

$$f''' + \alpha\, ff'' + \beta\left(1 - f'^2\right) \;=\; 0 \quad , \tag{11}$$

where

$$\beta = \frac{ab^2}{\nu} m \quad , \text{and} \quad \alpha = \frac{ab^2}{\nu}\left(\frac{1+m}{2}\right) \quad . \tag{12}$$

This reduction was done in part by the using the fact that similarity requires that

$$m + 2n - 1 \;=\; 0 \quad , \tag{13}$$

(see Appendix A).

Our real interest herein is the *y*-momentum equation. The derivation in Appendix B results in the transformed *y*-momentum equation (Eq. B6) given by

$$\begin{aligned} \frac{(x-x_0)^2}{\rho u_e^2 \delta_1}\frac{dP}{dy} \;=\; & -\frac{\nu}{2ab^2}(3m-1)\, f'' + \frac{\nu}{2ab^2}(1-m)\eta\, f''' - \frac{1}{4}(m+1)^2\, ff' + \\ & + \frac{1}{4}(m-1)^2 \eta\, f'^2 \; - \frac{1}{4}(m+1)(m-1)\eta\, ff'' \quad . \end{aligned} \tag{14}$$

where we have used the fact that similarity requires that $m + 2n - 1 = 0$ (Eq. 13). In order to calculate this dimensionless pressure gradient, the first step is to calculate the *f* and its derivatives by solving Eq. 11 with the appropriate boundary conditions. In order to see the true nature of the reduced pressure gradient, we consider two examples that have aerodynamic drag implications.

4.1 Blasius Flat Plate

The Blasius[3] condition assumes the velocity above the boundary layer edge is constant. If the velocity u_e is constant (Eq. 6) then this means *m*=0 and $a = u_\infty$ where u_∞ is the steady state incoming velocity. Since $m + 2n - 1 = 0$ then this means that *n*=1/2. For this case, the transformed *x*-component of the momentum balance (Eq. 11) becomes

$$f''' + \frac{u_\infty b^2}{2\nu}\, ff'' \;=\; 0 \quad , \tag{15}$$

and the transformed *y*-component of the momentum balance (Eq. 14) becomes

$$\frac{x^2}{\rho u_\infty^2 \delta_1}\frac{dP}{dy} = \frac{\nu}{2ab^2}\eta f''' + \frac{\nu}{2ab^2} f'' - \frac{1}{4} ff' + \frac{1}{4}\eta f'^2 + \frac{1}{4}\eta ff'' \quad . \tag{16}$$

There are as yet no restraints on *b* so for convenience we choose

$$b^2 \propto \frac{\nu}{a} \quad , \tag{17}$$

which means *x*-component of the momentum balance (Eq. 15) becomes

$$f''' + \frac{ff''}{2} = 0 \quad , \tag{18}$$

and the *y*-component of the momentum balance (Eq. 16) becomes

$$\frac{(x-x_0)^2}{\rho u_\infty^2 \delta_1}\frac{dP}{dy} = \frac{1}{2}\eta f''' + \frac{1}{2} f'' - \frac{1}{4} ff' + \frac{1}{4}\eta f'^2 + \frac{1}{4}\eta ff'' \quad . \tag{19}$$

The velocity $v(x,y)$ for Blasius flow (Eq. 9) becomes

$$\frac{x-x_0}{u_\infty \delta_1} v(x,y) = -\frac{1}{2} f + \frac{1}{2}\eta f' \quad , \tag{20}$$

and velocity $u(x,y)$ becomes

$$\frac{u(x,y)}{u_\infty} = f' \quad . \tag{21}$$

Eq. 18 is easily solved with the appropriate boundary conditions.[3] With *f* and its derivatives in hand, the reduced velocities and reduced pressure gradient are calculated and plotted in Figure 2. As far as we know, this is the first time the scaled *y*-pressure gradient for the Blasius flow

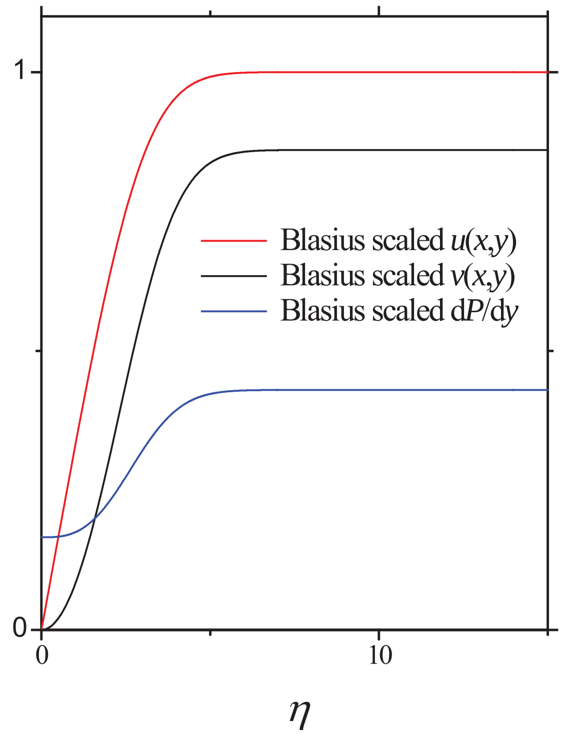

Figure 2: Scaled Velocities and *y*-pressure Gradient for Blasius Flow

solution have been presented. The pressure gradient asymptotes to a constant at large *y*-values given by

$$\frac{(x-x_0)^2}{\rho u_\infty^2 \delta_1}\frac{dP}{dy} \xrightarrow{y\to\infty} 0.4302 \quad . \tag{22}$$

4.2 Falkner-Skan Flow for Small *m*-values

A more realistic model for laminar flow over a flat plate depicted in Figure 1 is the Falkner-Skan flow with small *m*-values. If we assume for convenience that

$$b^2 \propto \frac{\nu}{a} \tag{23}$$

then the scaled velocities (Eqs. 9 and 10) and scaled pressure gradient (Eq. 14) can be obtained by solving Eq. 11 for a given *m* value. (Note that instead of Eq. 23, we could have made b^2 proportional to $2\nu/(a(m+1))$ in which case we recover the often seen $\beta = 2m/(m+1)$ and $\alpha = 1$ parameter set used in some texts). In Figures 3 and 4 we plot two laminar flow datasets[8,9] from the literature for which we hand fitted the Falkner-Skan velocity profile by adjusting the *m*-values. Also shown is the Blasius flow solution. The Falkner-Skan flow solution are a better fit to the experimental profiles.

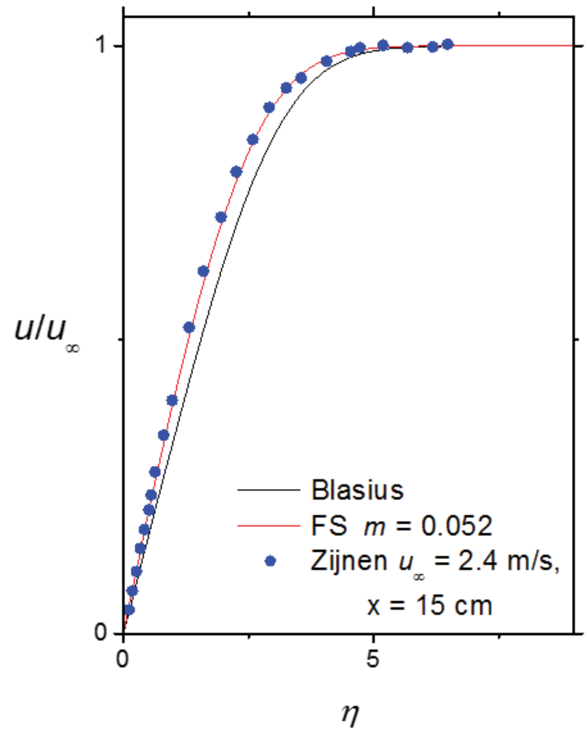

Figure 3: Blasius and Falkner-Skan Profiles compared to Laminar Data from Zijen[8]

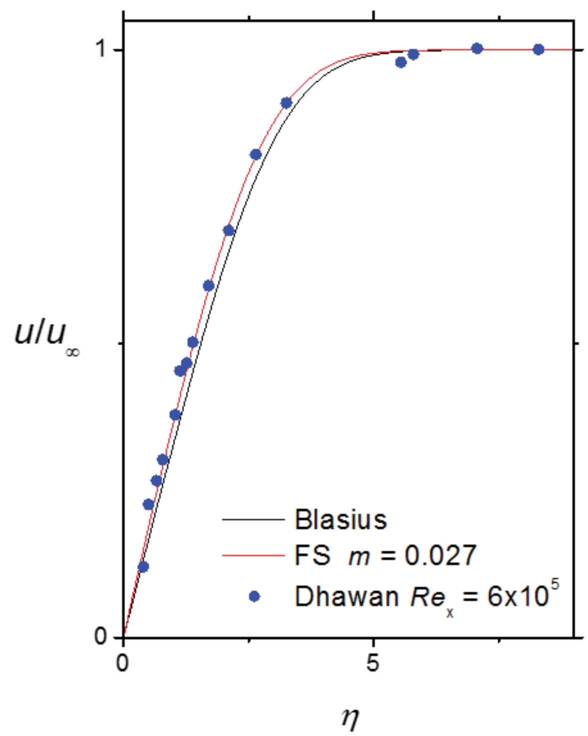

Figure 4: Blasius and Falkner-Skan Profiles compared to Laminar Data from Dhawan[9]

The two scaled velocities and the scaled pressure gradient are displayed in Figures 5 and 6 for the *m*-values from Figures 3 and 4. Notice that both the vertical velocity $v(x,y)$ and the pressure gradient dP/dy are finite in the boundary layer region and then change in a linear-like fashion to actually take on non-physical realizable values for large *y*-values. What is also noticeable is that the zero crossing point crossing changes for the velocity $v(x,y)$ with the *m*-value.

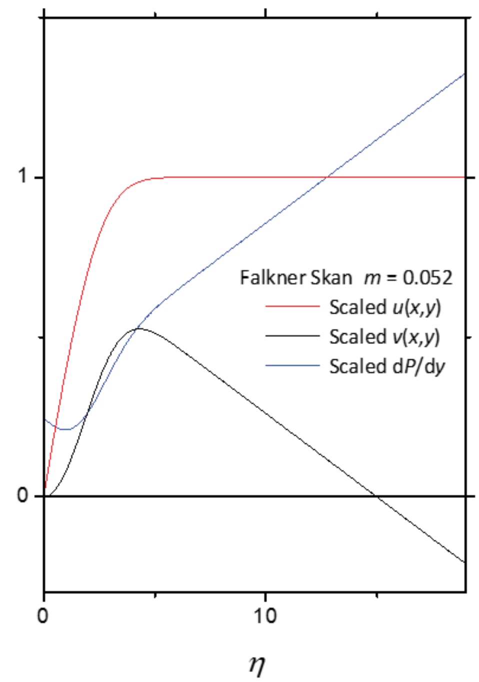

Figure 5: Falkner-Skan Velocities and Pressure Gradient for Data from Zijen[8]

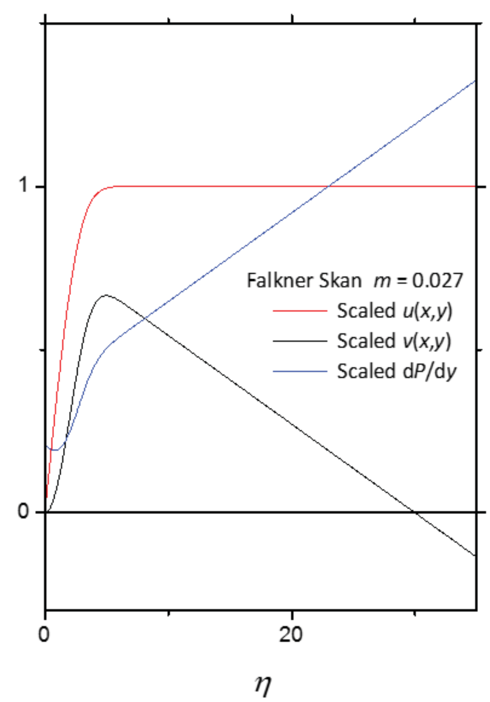

Figure 6: Falkner-Skan Velocities and Pressure Gradient for Data from Dhawan[9]

In Figure 7 we show a plot of this zero crossing η-value, which we designate η_{zp}, versus the *m*-value for the *v*(*x*,*y*) velocity (the line is added as a visual aide).

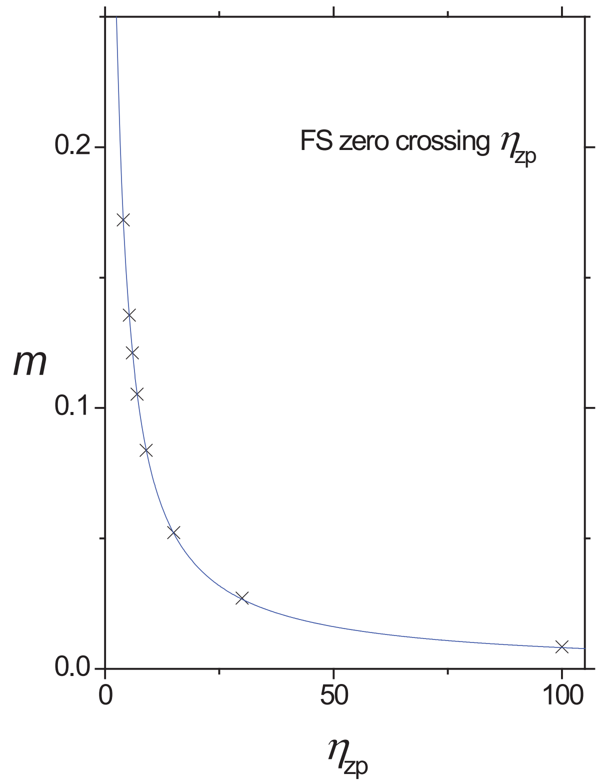

Figure 7: Plot of Zero Crossing Point of *v*(*x*,*y*) as a Function of the Falkner-Skan *m*-value

5. DISCUSSION

As we mentioned in the Introduction, it is very common to find texts indicating that the *y*-momentum equation reduces to just the *y*-pressure gradient term equal to zero. Not only is this not a correct statement of momentum conservation, but it has hidden away an important aspect of boundary layer theory. And that has to do with the pressure gradient in the direction normal to the wall. The *y*-momentum equation clarifies how the normal pressure gradient in boundary layer flow develops. From a theoretical standpoint, this new understanding of the behavior of the normal pressure gradient needs no further justification. On the practical side, we note that the Blasius and Falkner-Skan type velocity profiles are routinely used to model air flow around an airfoil using panel methods such as Xfoil.[10] An aerodynamic lift/drag theory needs to explain the velocity and pressure fields around the airfoil. Obviously, the *y*-pressure gradient in the boundary layer plays an important role in this and for the first time we now know what the complete Blasisus and the Falkner-Skan pressure gradient fields look like. Whether or not the new results prove useful in airfoil optimization remains to be seen.

In comparing the Blasius (Figure 2) and the small *m*-value Falkner-Skan solutions (Figures 3-4), one sees that the predicted $u(x,y)$ velocities for the two cases are not dramatically different. Where the real differences become noticeable is for the $v(x,y)$ velocities and the pressure gradient dP/dy (Figures 2, 5, and 6). Hence, the question of which flow is a better depiction of flow over a flat plate at zero incidence angle is of some importance. Although the Blasius solution is often associated with flow over a flat plate at zero incidence angle, it is NOT the flow depicted in Figure 1. The Blasius flow is actually a flow that must be artificially generated by the addition of some external pressure source. For example, consider the Nikuradse[11] experimental data fits that appear in Schlichting's book.[2] That data would seem to support the Blasius model for flow on a flat plate. However, Schlichting made the following comments in regards to Nikuradse's data: "It was found that the formation of the boundary layer is greatly influenced by the shape of the leading edge as well as by the very small pressure gradient which may exist in the external flow. J. Nikuradse introduced careful corrections for these possible effects when he carried out his measurements on a plate in a stream of air." In other words, the Nikuradse data was actually corrected for the presence of the pressure gradient. In fact, this is common in wind tunnel testing since the Blasius condition is often used to verify tunnel performance.[12,13] In the wind tunnel, the Blasius type flows are generated by actively countering the naturally occurring pressure gradient. This can be done by slightly tilting the plate[12] or manipulating a trailing edge flap,[13] for example. Unfortunately, the uncorrected data (true zero incidence flow) has never been made available. This means that we are unable to verify that the true theoretical model for flow over a flat plate depicted in Figure 1 is the small *m*-value Falkner-Skan flow, and not the Blasius flow. The data we presented in Figures 3 and 4 is by no means proof. More work needs to be done to verify that the true behavior of fluid flow over a plate at zero-degree inclination is in fact the small *m*-value Falkner-Skan flow.

6. CONCLUSION

A Falkner-Skan analysis is used to discern the true nature of the normal to the wall momentum conservation equation for boundary layer flow on a flat plate. The normal pressure gradient is calculated using the normal to the wall *y*-momentum equation for the Blasius as well as a more realistic Falkner-Skan flow. It is pointed out that the understanding the pressure gradient for boundary layer flow along a flat plate is a necessary step to understand flow along a wing.

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7. REFERENCES

[1]L. Prandtl, “Über Flüssigkeitsbewegung bei sehr kleiner Reibung,” Verhandlungen des Dritten Internationalen Mathematiker-Kongresses in Heidelberg 1904.

[2]H. Schlichting, *Boundary Layer Theory*, 7th edition, McGraw-Hill, New York, 1979.

[3]H. Blasius, “Grenzschichten in Flüssigkeiten mit kleiner Reibung,“ Zeitschrift für Mathematik und Physik, **56**, 1(1908).

[4]V. Falkner and S. Skan, Aero. Res. Coun. Rep. and Mem. no 1314, 1930.

[5]F. White, Viscous Fluid Flow, 2nd edition, McGraw-Hill, 1991.

[6]Y. Cengel and J. Cimbala, Fluid Mechanics: Fundamentals and Applications, 2nd edition, TBS, 2009.

[7]D. Weyburne, “Similarity of the Velocity Profile,” AF Tech report AFRL-RY-WP-TR-2014-0227, ADA609962, www.DTIC.mil, 2014.

[8]B. van der Higge Zijnen, “Measurements of the velocity distribution in the boundary layer along a plane surface,” Thesis Delft, 1924. Data taken from M. Inkman and S. Sohrab, WSEAS Conference, Cantabria Spain, 2008.

[9]S. Dhawan, “Direct measurements of skin friction,” NACA TN 2567 (1953).

[10]M. Drela, “XFOIL: An Analysis and Design System for Low Reynolds Number Airfoils,” In: Mueller T.J. (eds) *Low Reynolds Number Aerodynamics. Lecture Notes in Engineering*, **54**, Springer, Berlin, Heidelberg, 1989.

[11]J. Nikuradse, Laminare Reibungsschichten an der längsangetrömten Platte. Monograph, Zentrale f. wiss. Berichtswesen, Berlin, 1942.

[12]J. Jovanović, B. Frohnapfel, E. Škaljić, and M. Jovanović, “Persistence of the Laminar Regime in a Flat Plate Boundary Layer at very High Reynolds Number,” Thermal Science, **10**, 63(2006).

[13]N. Patten, T. M. Young, and P. Griffin, “Design and Characteristics of New Test Facility for Flat Plate Boundary Layer Research,” Proc. World Acad. Sci., Eng. and Tech., **58**, 366(2008).

APPENDIX A: FS X-MOMENTUM EQUATION

The x-component of the Prandtl momentum equations is the key equation for solving the flow governing equations. The first step is to figure out the scaled velocities. To do that, we use the stream function (Eqs. 6-8) together with Eqs. 5 and 6 which means that

$$
\begin{aligned}
v(x,y) &= -\frac{\partial \psi}{\partial x} = -\frac{d\left\{b(x-x_0)^n\, a(x-x_0)^m\right\}}{dx} f - b(x-x_0)^n\, a(x-x_0)^m \frac{\partial f}{\partial x} \\
v(x,y) &= -ab\frac{d(x-x_0)^{m+n}}{dx} f - ab(x-x_0)^{m+n}\left(\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}\right) \\
v(x,y) &= -ab(m+n)(x-x_0)^{m+n-1} f - ab(x-x_0)^{m+n}\left(-\frac{n\eta}{x-x_0} f'\right) \\
v(x,y) &= -ab(m+n)(x-x_0)^{m+n-1} f + abn(x-x_0)^{m+n-1}\eta f' \\
v(x,y) &= -(m+n)\frac{\delta_1 u_e}{x-x_0} f + n\frac{\delta_1 u_e}{x-x_0}\eta f' \quad ,
\end{aligned}
\tag{A1}
$$

where the prime indicates differentiation with respect to η and where we have used the fact that

$$
\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left\{\frac{y}{\delta_1(x)}\right\} = y\frac{d\{1/\delta_1\}}{dx} = y\frac{d\left\{1/b(x-x_0)^n\right\}}{dx} = -\frac{yn(x-x_0)^{-n-1}}{b} = -\frac{n\eta}{x-x_0}
\tag{A2}
$$

The velocity $u(x,y)$ becomes

$$
\begin{aligned}
u(x,y) &= \frac{\partial \psi}{\partial y} = \frac{\partial\{\ \delta_1(x)u_e(x) f(\eta)\}}{\partial y} = \delta_1 u_e \frac{\partial f}{\partial y} = \delta_1 u_e \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} = \delta_1 u_e f' \frac{1}{\delta_1} \\
u(x,y) &= a(x-x_0)^m f' = u_e f' \quad ,
\end{aligned}
\tag{A3}
$$

where we have used the fact that

$$
\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial y}\left\{\frac{y}{\delta_1(x)}\right\} = \frac{1}{\delta_1} = \frac{1}{b}(x-x_0)^{-n} \quad .
\tag{A4}
$$

Substituting the reduced velocities given by Eqs. A1 and A3 into the Prandtl x-component of the momentum equation (Eq. 1); starting on the left-hand side, we have

$$
\begin{aligned}
u\frac{\partial u}{\partial x} &= u_e f' \frac{\partial\{u_e f'\}}{\partial x} = a(x-x_0)^m f' \frac{\partial\{a(x-x_0)^m f'\}}{\partial x} \\
&= a(x-x_0)^m f' \left\{ \frac{\partial a(x-x_0)^m}{\partial x} f' + a(x-x_0)^m \frac{\partial f'}{\partial x} \right\} \\
&= a(x-x_0)^m f' \left\{ am(x-x_0)^{m-1} f' + ax^m \left(\frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial x} \right) \right\} \\
&= a^2 m x^{2m-1} f'^2 + a^2 x^{2m} f' \left(-\frac{n\eta}{x-x_0} f'' \right) \\
&= a^2 m (x-x_0)^{2m-1} f'^2 - a^2 n (x-x_0)^{2m-1} \eta f f'' \quad .
\end{aligned}
\tag{A5}
$$

The next term in Eq. 1 we need to nondimensionalize is

$$
\begin{aligned}
v\frac{\partial u}{\partial y} &= \left\{ -ab(m+n)(x-x_0)^{m+n-1} f + abn(x-x_0)^{m+n-1} \eta f' \right\} a(x-x_0)^m \frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial y} \\
&= \left\{ -ab(m+n)(x-x_0)^{m+n-1} f + abn(x-x_0)^{m+n-1} \eta f' \right\} a(x-x_0)^m f'' \frac{1}{b}(x-x_0)^{-n} \\
&= \left\{ -ab(m+n)(x-x_0)^{m+n-1} f + abn(x-x_0)^{m+n-1} \eta f' \right\} \frac{a}{b}(x-x_0)^{m-n} f'' \\
&= -a^2(m+n)(x-x_0)^{2m-1} ff'' + a^2 n (x-x_0)^{2m-1} \eta f f'' \quad .
\end{aligned}
\tag{A6}
$$

Adding these first two terms together we have

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= a^2 m (x-x_0)^{2m-1} f'^2 - a^2 n (x-x_0)^{2m-1} \eta f f'' \\
&\quad - a^2(m+n)(x-x_0)^{2m-1} ff'' + a^2 n (x-x_0)^{2m-1} \eta f f'' \\
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= a^2 m (x-x_0)^{2m-1} f'^2 - a^2(m+n)(x-x_0)^{2m-1} ff'' \quad .
\end{aligned}
\tag{A7}
$$

The next step is to transform the viscous component in Eq. 1 given by

$$\nu \frac{\partial^2 u}{\partial y^2} = \nu \frac{\partial \left\{ \frac{\partial \left\{ a(x-x_0)^m f' \right\}}{\partial \eta} \frac{\partial \eta}{\partial y} \right\}}{\partial \eta} \frac{\partial \eta}{\partial y}$$
$$= \nu \frac{\partial \left\{ \frac{\partial \left\{ a(x-x_0)^m f' \right\}}{\partial \eta} \frac{1}{b}(x-x_0)^{-n} \right\}}{\partial \eta} \frac{1}{b}(x-x_0)^{-n}$$
$$= \nu \frac{a(x-x_0)^{m-2n}}{b^2} f''' \quad . \tag{A8}$$

We assume that the pressure in the boundary layer is given by the Bernoulli equation which means that

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u_e \frac{du_e}{dx} = ax^m \frac{d\left\{ a(x-x_0)^m \right\}}{dx} = a^2 m (x-x_0)^{2m-1} \quad . \tag{A9}$$

Combining the transformed terms of Eq. 1, we get the reduced Prandtl *x*-component of the momentum balance as

$$a^2 m(x-x_0)^{2m-1} f'^2 - a^2(m+n)(x-x_0)^{2m-1} ff'' = a^2 m(x-x_0)^{2m-1} + \nu \frac{a(x-x_0)^{m-2n}}{b^2} f''' \tag{A10}$$
$$\nu \frac{a}{b^2}(x-x_0)^{m-2n} f''' + a^2(m+n)(x-x_0)^{2m-1} ff'' + a^2 m(x-x_0)^{2m-1}(1-f'^2) = 0 \quad .$$

For velocity profile similarity, we must have the *x*-dependent portion of each term change in the same way as we move along the plate. Equivalently, if we divide through by one of the *x*-terms then the resulting *x*-dependent ratios must be constant. Dividing Eq. A8 through by $\nu(a/b^2)(x-x_0)^{m-2n}$, we end up with two *x*-dependent parameters. Let

$$\beta = \frac{a^2 m(x-x_0)^{2m-1}}{\nu \frac{a}{b^2}(x-x_0)^{m-2n}} = \frac{ab^2}{\nu} m(x-x_0)^{m+2n-1} \quad . \tag{A11}$$

and

$$\alpha = \frac{a^2(m+n)(x-x_0)^{2m-1}}{\nu \frac{a}{b^2}(x-x_0)^{m-2n}} = \frac{ab^2}{\nu}(m+n)(x-x_0)^{m+2n-1} \quad . \tag{A12}$$

Similarity requires α and β must be constant (no direct *x*-dependence) which means we must have

$$m + 2n - 1 = 0 \quad . \tag{A13}$$

This reduces the transformed *x*-component of the momentum balance (Eq. A10) to the well-known Falkner-Skan[3] equation given by

$$f''' + \alpha f f'' + \beta\left(1 - f'^2\right) = 0 \quad . \tag{A14}$$

APPENDIX B: FS *Y*-MOMENTUM EQUATION

The *y*-component of the Prandtl momentum equations is of primary importance herein. Substituting the reduced velocities given by Eqs. A1 and A3 into the Prandtl *y*-component of the momentum equation (Eq. 2); starting on the left-hand side, we have

$$u\frac{\partial v}{\partial x} = ax^m f'\frac{\partial\left\{-ab(m+n)(x-x_0)^{m+n-1} f + abn(x-x_0)^{m+n-1}\eta f'\right\}}{\partial x}$$

$$= ax^m f'\left\{\begin{array}{l}-ab(m+n)(m+n-1)(x-x_0)^{m+n-2} f - ab(m+n)(x-x_0)^{m+n-1}\dfrac{\partial f}{\partial\eta}\dfrac{\partial\eta}{\partial x}\\ +abn(m+n-1)(x-x_0)^{m+n-2}\eta f' + abn(x-x_0)^{m+n-1}\dfrac{\partial\{\eta f'\}}{\partial\eta}\dfrac{\partial\eta}{\partial x}\end{array}\right\} \qquad \text{(B1a)}$$

$$= ax^m f'\left\{\begin{array}{l}-ab(m+n)(m+n-1)(x-x_0)^{m+n-2} f - ab(m+n)(x-x_0)^{m+n-1} f'\left\{-\dfrac{n\eta}{x}\right\}\\ +abn(m+n-1)(x-x_0)^{m+n-2}\eta f' + abn(x-x_0)^{m+n-1}\{f' + \eta f''\}\left\{-\dfrac{n\eta}{x-x_0}\right\}\end{array}\right\}$$

$$u\frac{\partial v}{\partial x} = ax^m f'\left\{\begin{array}{l}-ab(m+n)(m+n-1)(x-x_0)^{m+n-2} f + abn(m+n)(x-x_0)^{m+n-2}\eta f'\\ +abn(m+n-1)(x-x_0)^{m+n-2}\eta f' - abn^2(x-x_0)^{m+n-2}\{\eta f' + \eta^2 f''\}\end{array}\right\},$$

which can be further simplified to

$$u\frac{\partial v}{\partial x} = ax^m f'\left\{\begin{array}{l}-ab(m+n)(m+n-1)(x-x_0)^{m+n-2} f + abn(m+n)(x-x_0)^{m+n-2}\eta f'\\ +abn(m+n-1)(x-x_0)^{m+n-2}\eta f' - abn^2(x-x_0)^{m+n-2}\{\eta f' + \eta^2 f''\}\end{array}\right\}$$

$$u\frac{\partial v}{\partial x} = -a^2b(m+n)(m+n-1)(x-x_0)^{2m+n-2} ff' + a^2bn(m+n)(x-x_0)^{2m+n-2}\eta f'^2$$
$$+ a^2bn(m+n-1)(x-x_0)^{2m+n-2}\eta f'^2 - a^2bn^2(x-x_0)^{2m+n-2}\{\eta f'^2 + \eta^2 ff''\} \qquad \text{(B1b)}$$

$$u\frac{\partial v}{\partial x} = -a^2b(m+n)(m+n-1)(x-x_0)^{2m+n-2} ff' +$$
$$+ a^2bn(2m+n-1)(x-x_0)^{2m+n-2}\eta f'^2 - a^2bn^2(x-x_0)^{2m+n-2}\eta^2 ff'' \quad .$$

The next term is

$$
\begin{aligned}
v\frac{\partial v}{\partial y} &= v\frac{\partial\left\{-ab(m+n)(x-x_0)^{m+n-1}\, f \;\; + abn(x-x_0)^{m+n-1}\eta\, f'\right\}}{\partial y} \\
&= v\left\{-ab(m+n)(x-x_0)^{m+n-1}\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial y} + abn(x-x_0)^{m+n-1}\frac{\partial\eta\, f'}{\partial\eta}\frac{\partial\eta}{\partial y}\right\} \\
&= v\left\{\begin{array}{l} -ab(m+n)(x-x_0)^{m+n-1}\, f'\dfrac{1}{b}(x-x_0)^{-n} + \\ abn(x-x_0)^{m+n-1}\left\{f'+\eta f''\right\}\dfrac{1}{b}(x-x_0)^{-n} \end{array}\right\} \\
&= \left\{-ab(m+n)(x-x_0)^{m+n-1}\, f \;\; + abn(x-x_0)^{m+n-1}\eta\, f'\right\} \\
&\quad \left\{-a(m+n)(x-x_0)^{m-1}\, f' \;\; + an(x-x_0)^{m-1}\left\{f'+\eta f''\right\}\right\} \\
&= a^2b(m+n)^2(x-x_0)^{2m+n-2}\, ff' \; - a^2bn(m+n)(x-x_0)^{2m+n-2}\left\{ff'+\eta\, ff''\right\} \quad + \\
&\qquad - a^2bn(m+n)(x-x_0)^{2m+n-2}\eta\, f'^2 + a^2bn^2(x-x_0)^{2m+n-2}\left\{\eta\, f'^2+\eta^2 ff''\right\} \\
&= a^2b(m^2+2mn+n^2)(x-x_0)^{2m+n-2}\, ff' \; - a^2b(mn+n^2)(x-x_0)^{2m+n-2}\, ff' \\
&\quad - a^2bn(m+n)(x-x_0)^{2m+n-2}\eta\, ff'' \quad - a^2b(mn+n^2)(x-x_0)^{2m+n-2}\eta\, f'^2 \\
&\qquad + a^2bn^2(x-x_0)^{2m+n-2}\eta\, f'^2 + a^2bn^2(x-x_0)^{2m+n-2}\eta^2 ff'' \\
&= a^2b(m^2+mn)(x-x_0)^{2m+n-2}\, ff' \; - a^2bn(m+n)(x-x_0)^{2m+n-2}\eta\, ff'' \\
&\quad - a^2bmn(x-x_0)^{2m+n-2}\eta\, f'^2 + a^2bn^2(x-x_0)^{2m+n-2}\eta^2 ff'' \quad .
\end{aligned}
\tag{B2}
$$

The second derivative term is given by

$$\nu \frac{\partial^2 v}{\partial y^2} = \nu \frac{\partial^2 \left\{ -ab(m+n)(x-x_0)^{m+n-1} f \;\; + abn(x-x_0)^{m+n-1} \eta f' \right\}}{\partial y^2} \tag{B3}$$

$$= -\nu ab(m+n)(x-x_0)^{m+n-1} \frac{\partial^2 f}{\partial y^2} + \nu abn(x-x_0)^{m+n-1} \frac{\partial^2 \{\eta f'\}}{\partial y^2}$$

$$= -\nu ab(m+n)(x-x_0)^{m+n-1} \frac{\partial}{\partial y} \left\{ \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} \right\} + \nu abn(x-x_0)^{m+n-1} \frac{\partial}{\partial y} \left\{ \frac{\partial \eta f'}{\partial \eta} \frac{\partial \eta}{\partial y} \right\}$$

$$= \quad -\nu ab(m+n)(x-x_0)^{m+n-1} \frac{\partial}{\partial y} \left\{ \frac{1}{b} (x-x_0)^{-n} \; f' \right\} +$$

$$\nu abn(x-x_0)^{m+n-1} \frac{\partial}{\partial y} \left\{ \frac{1}{b} (x-x_0)^{-n} \left(f' + \eta f'' \right) \right\}$$

$$= \quad -\nu a(m+n)(x-x_0)^{m-1} \frac{\partial}{\partial y} \left\{ f' \right\} + \nu an(x-x_0)^{m-1} \frac{\partial}{\partial y} \left\{ \left(f' + \eta f'' \right) \right\}$$

$$= \quad -\nu a(m+n)(x-x_0)^{m-1} \frac{\partial f'}{\partial \eta} \frac{1}{b} (x-x_0)^{-n} +$$

$$\nu an(x-x_0)^{m-1} \frac{\partial \left(f' + \eta f'' \right)}{\partial \eta} \frac{1}{b} (x-x_0)^{-n}$$

$$= \quad -\frac{\nu a}{b}(m+n)(x-x_0)^{m-n-1} f'' + \frac{\nu a}{b} n(x-x_0)^{m-n-1} \left\{ f'' + f'' + \eta f''' \right\}$$

$$= \quad -\frac{\nu a}{b}(m-n)(x-x_0)^{m-n-1} f'' + \frac{\nu a}{b} n(x-x_0)^{m-n-1} \eta f''' \quad .$$

Combining the terms, we have

$$-a^2 b(m+n)(m+n-1)(x-x_0)^{2m+n-2} ff' \; +$$

$$+ a^2 bn(2m+n-1)(x-x_0)^{2m+n-2} \eta f'^2 \; \cancel{-a^2 bn^2 (x-x_0)^{2m+n-2} \eta^2 f f''} \quad + \tag{B4}$$

$$a^2 b(m^2 + mn)(x-x_0)^{2m+n-2} \; ff' \; - a^2 bn(m+n)(x-x_0)^{2m+n-2} \eta ff''$$

$$- a^2 bmn(x-x_0)^{2m+n-2} \eta f'^2 \; \cancel{+a^2 bn^2 (x-x_0)^{2m+n-2} \eta^2 f f''}$$

$$= \quad -\frac{\nu a}{b}(m-n)(x-x_0)^{m-n-1} f'' + \frac{\nu a}{b} n(x-x_0)^{m-n-1} \eta f''' - \frac{1}{\rho} \frac{dP}{dy} \quad .$$

Dividing through by $a^2 b(x-x_0)^{2m+n-2}$, the *y*-momentum equation becomes

$$\begin{aligned}
&-(m+n)(m+n-1)\, ff' \;+ n(2m+n-1)\eta\, f'^2 \quad + \\
&(m^2+mn)\, ff' \;- n(m+n)\eta\, ff'' - mn\eta\, f'^2 \\
&= \; -\frac{\nu}{ab^2}(m-n)(x-x_0)^{-m-2n+1} f'' + \frac{\nu}{ab^2} n(x-x_0)^{-m-2n+1}\eta\, f''' - \frac{(x-x_0)^{-2m-n+2}}{\rho a^2 b}\frac{dP}{dy} \quad .
\end{aligned} \tag{B5}$$

Since we must have $m+2n-1=0$, then

$$\begin{aligned}
\frac{(x-x_0)^2}{\rho u_e^2 \delta_1}\frac{dP}{dy} \;=\; & -\frac{\nu}{2ab^2}(3m-1)\, f'' + \frac{\nu}{2ab^2}(1-m)\eta\, f''' - \frac{1}{4}(m+1)^2\, ff' + \\
& + \frac{1}{4}(m-1)^2 \eta\, f'^2 \;\; - \frac{1}{4}(m+1)(m-1)\eta\, ff'' \quad .
\end{aligned} \tag{B6}$$

The scaled pressure gradient term can therefore be calculated by first solving the standard Falkner-Skan equation (Eq. A14) and then substituting the result into Eq. B6.