

REPORT DOCUMENTATION PAGE			Form Approved OMB NO. 0704-0188		
The public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA, 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number. PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.					
1. REPORT DATE (DD-MM-YYYY) 22-12-2016		2. REPORT TYPE Final Report		3. DATES COVERED (From - To) 23-Sep-2013 - 22-Sep-2016	
4. TITLE AND SUBTITLE Final Report: Minimum Power Requirements and Optimal Rotor Design for Conventional and Compound Helicopters Using Higher Harmonic Blade Root Control			5a. CONTRACT NUMBER W911NF-13-1-0469		
			5b. GRANT NUMBER		
			5c. PROGRAM ELEMENT NUMBER 611102		
6. AUTHORS Kenneth C. Hall, Eli B. Giovanetti			5d. PROJECT NUMBER		
			5e. TASK NUMBER		
			5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAMES AND ADDRESSES Duke University C/O Office of Research Support 2200 W. Main St., Ste. 710 Durham, NC 27705 -4677			8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS (ES) U.S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211			10. SPONSOR/MONITOR'S ACRONYM(S) ARO		
			11. SPONSOR/MONITOR'S REPORT NUMBER(S) 63875-EG.11		
12. DISTRIBUTION AVAILABILITY STATEMENT Approved for Public Release; Distribution Unlimited					
13. SUPPLEMENTARY NOTES The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.					
14. ABSTRACT The goal of this research is to understand the fundamental limits of the performance of real helicopter rotors, and the benefits of higher harmonic control (HHC) for applications involving conventional helicopter configurations, coaxial counter rotating rotors, and compound helicopters. We seek to answer the following questions: If one has limited control inputs to the rotor, what is the optimal performance, how close is the constrained input optimum to the rubber rotor optimum, and what periodic control inputs and fixed rotor geometry are required to achieve the optimum? In this work, we develop numerical tools, based on a variational approach, for finding the optimal design.					
15. SUBJECT TERMS Helicopter aerodynamics, optimal propulsion, compound helicopters, hover aerodynamics, forward flight					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT UU	15. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON Kenneth C. Hall
a. REPORT UU	b. ABSTRACT UU	c. THIS PAGE UU			19b. TELEPHONE NUMBER 919-660-5328

Report Title

Final Report: Minimum Power Requirements and Optimal Rotor Design for Conventional and Compound Helicopters Using Higher Harmonic Blade Root Control

ABSTRACT

The goal of this research is to understand the fundamental limits of the performance of real helicopter rotors, and the benefits of higher harmonic control (HHC) for applications involving conventional helicopter configurations, coaxial counter rotating rotors, and compound helicopters. We seek to answer the following questions: If one has limited control inputs to the rotor, what is the optimal performance, how close is the constrained input optimum to the rubber rotor optimum, and what periodic control inputs and fixed rotor geometry are required to achieve the optimum? In this work, we develop numerical tools, based on a variational approach, for finding the optimal design and resulting performance of helicopter rotors in cruise and hover, and also rotors designed for efficient operation at multiple flight conditions. The aerodynamic models are based on a vortex-lattice model for forward flight, blade element momentum theory for hover, with C81 tables to model compressible and viscous losses. We show that for conventional and coaxial counterrotating rotors, using radially varying twist and chord distributions and higher harmonic blade pitch control can produce significant reductions in required power, especially at high advance ratios.

Enter List of papers submitted or published that acknowledge ARO support from the start of the project to the date of this printing. List the papers, including journal references, in the following categories:

(a) Papers published in peer-reviewed journals (N/A for none)

Received

Paper

08/19/2015	5.00	Eli B. Giovanetti, Kenneth C. Hall. Optimum Design of Compound Helicopters that Use Higher Harmonic Control, Journal of Aircraft, (8 2015): 1444. doi: 10.2514/1.C032941
------------	------	--

TOTAL: 1

Number of Papers published in peer-reviewed journals:

(b) Papers published in non-peer-reviewed journals (N/A for none)

Received

Paper

TOTAL:

Number of Papers published in non peer-reviewed journals:

(c) Presentations

Number of Presentations: 0.00

Non Peer-Reviewed Conference Proceeding publications (other than abstracts):

Received

Paper

08/30/2016 10.00 Giovanetti, Eli B., and Hall, Kenneth C.. Axisymmetric Potential Flow Model of Single or Coaxial Actuator Disks,
American Helicopter Society 72nd Annual Forum. 17-MAY-16, West Palm Beach, Florida. : ,

TOTAL: 1

Number of Non Peer-Reviewed Conference Proceeding publications (other than abstracts):

Peer-Reviewed Conference Proceeding publications (other than abstracts):

Received

Paper

12/02/2016 4.00 Eli B. Giovanetti, Kenneth C. Hall. Minimum Loss Load, Twist and Chord Distributions For Coaxial Helicopters in Hover,
American Helicopter Society 71st Annual Forum. 05-MAY-15, Virginia Beach, VA. : ,

12/08/2016 1.00 Eli B. Giovanetti, Kenneth C. Hall. Optimum Design of Compound Helicopters Using Higher Harmonic Control,
AHS 70th Annual Forum and Technology Display. 20-MAY-14, Montreal, Quebec. : ,

12/08/2016 3.00 Eli B. Giovanetti, Kenneth C. Hall. A Variational Approach to Multipoint Aerodynamic Optimization of Conventional and Coaxial Helicopter Rotors,
American Helicopter Society 71st Annual Forum. 05-MAY-15, Virginia Beach, VA. : ,

TOTAL: 3

Number of Peer-Reviewed Conference Proceeding publications (other than abstracts):

(d) Manuscripts

Received Paper

08/31/2015 7.00 Eli B. Giovanetti, Kenneth C. Hall. A Variational Approach to Multipoint Aerodynamic Optimization of Conventional and Coaxial Helicopter Rotors, Journal of the American Helicopter Society (05 2015)

TOTAL: 1

Number of Manuscripts:

Books

Received Book

TOTAL:

Received Book Chapter

TOTAL:

Patents Submitted

Patents Awarded

Awards

Eli B. Giovanetti, Robert L. Lichten Award South East Regional Winner, 2014

Graduate Students

<u>NAME</u>	<u>PERCENT SUPPORTED</u>	Discipline
Eli Giovanetti	0.00	
Matthew Tedesco	0.67	
FTE Equivalent:	0.67	
Total Number:	2	

Names of Post Doctorates

<u>NAME</u>	<u>PERCENT SUPPORTED</u>
FTE Equivalent:	
Total Number:	

Names of Faculty Supported

<u>NAME</u>	<u>PERCENT SUPPORTED</u>	National Academy Member
Kenneth C. Hall	0.13	No
FTE Equivalent:	0.13	
Total Number:	1	

Names of Under Graduate students supported

<u>NAME</u>	<u>PERCENT SUPPORTED</u>
FTE Equivalent:	
Total Number:	

Student Metrics

This section only applies to graduating undergraduates supported by this agreement in this reporting period

The number of undergraduates funded by this agreement who graduated during this period: 0.00

The number of undergraduates funded by this agreement who graduated during this period with a degree in science, mathematics, engineering, or technology fields:..... 0.00

The number of undergraduates funded by your agreement who graduated during this period and will continue to pursue a graduate or Ph.D. degree in science, mathematics, engineering, or technology fields:..... 0.00

Number of graduating undergraduates who achieved a 3.5 GPA to 4.0 (4.0 max scale):..... 0.00

Number of graduating undergraduates funded by a DoD funded Center of Excellence grant for Education, Research and Engineering:..... 0.00

The number of undergraduates funded by your agreement who graduated during this period and intend to work for the Department of Defense 0.00

The number of undergraduates funded by your agreement who graduated during this period and will receive scholarships or fellowships for further studies in science, mathematics, engineering or technology fields: 0.00

Names of Personnel receiving masters degrees

<u>NAME</u>	
Eli B. Giovanetti	
Total Number:	1

Names of personnel receiving PHDs

<u>NAME</u>	
Eli B. Giovanetti	
Total Number:	1

Names of other research staff

<u>NAME</u>	<u>PERCENT SUPPORTED</u>
FTE Equivalent:	
Total Number:	

Sub Contractors (DD882)

Inventions (DD882)

Scientific Progress

See Attachment

Technology Transfer

The PI met with Dr. Bob Ormiston and others in the US Army Aviation Development Directorate of the Aviation & Missile Research, Development & Engineering Center located at Ames Research Center in March of 2013 and August of 2014, and also at the AHS Annual Forums, to discuss ongoing work.

Minimum Power Requirements and Optimal Rotor Design for Conventional and Compound Helicopters Using Higher Harmonic Blade Root Control: Scientific Progress and Accomplishments

Kenneth C. Hall and Eli Giovanetti

Duke University

Department of Mechanical Engineering and Materials Science

Abstract

The following is an executive summary of the work performed under Army Research Office Grant W911NF1310469. Attached as appendices to this summary are four technical papers detailing the major technical findings.

Contents

Motivation and Goals	1
Background and Prior Work	2
Work Performed in this Investigation	2
Accomplishments	2
Main Findings	3
Compound Helicopters	3
Multipoint Design of Helicopter Rotors	4
Optimal Rotor Design for Hover	4
Aerodynamic Modeling of Hover	5
References	6
Appendix I: Optimum Design of Compound Helicopters that Use Higher Harmonic Control.	7
Appendix II: A Variational Approach to Multipoint Aerodynamic Optimization of Conventional and Coaxial Helicopter Rotors	18
Appendix III: Minimum Loss Load, Twist, and Chord Distributions For Coaxial Helicopters in Hover	32
Appendix IV: Axisymmetric Potential Flow Model of Single or Coaxial Actuator Disks	42

Motivation and Goals

The goal of this research is to develop preliminary design tools for helicopter rotors, and to understand the fundamental limits of performance of real helicopter rotors and the benefits of higher harmonic control (HHC). The work is a follow on to previous work of the PI and his colleagues, in which we determined the optimal performance of “rubber rotors,” that is, the circulation distribution that gives minimum power required for forward flight without regard to how (or even if) that circulation can be achieved.

In the present work, we seek to answer the following questions: (1) if one has only limited design and control inputs to the rotor (for example, blade planform, twist, and harmonic pitch control), what is the constrained optimal performance of the rotor; (2) how close is the constrained optimal performance to the rubber rotor optimum; and (3) what control inputs and rotor geometry are required to achieve the optimum?

Over a three-year period, we have developed a number of analyses to address these issues for both hover and forward flight; these analyses are discussed in detail reprints attached to this report as appendices (see Appendices I-IV). This work builds on ideas for optimally aerodynamic helicopters begun in the work of Moffitt and his colleagues [1, 2], Ormiston [3], Peters et al. [4], and Hall and Hall [5, 6, 7, 8], among others.

Background and Prior Work

Hall and Hall [8] recently developed the HH_Optimal_Rotor analysis and code, which uses a variational technique for computing the theoretical optimal aerodynamic performance of conventional and compound helicopters in trimmed flight. The optimal circulation distribution minimizes the sum of the induced and viscous power required to develop a prescribed lift and/or thrust, subject to constraints that the helicopter be trimmed in pitch and roll. The minimum total power circulation distribution problem is cast as a variational problem, which in turn is solved very efficiently using a vortex-lattice technique. Included in the analysis is the viscous profile power, which is estimated at each airfoil section using an experimentally or numerically computed drag polar. The resulting analysis – which is the viscous helicopter analog of Goldstein’s inviscid propeller theory – gives rigorous upper bounds on the performance of conventional and compound helicopters and may be used to predict the rotor/wing loadings that produce optimal performance. For a prescribed lift and thrust, HH_Optimal_Rotor computes the optimal loading distribution on the aerodynamic components and the resultant power. The analysis technique is described in detail in Ref. [8].

Of special note, Hall and Hall have shown that helicopters with either coaxial rotors or a wing and rotor in combination can substantially reduce power loss by producing a more efficient wake structure and by reducing the induced power associated with roll trim – at least for rubber rotors.

Building on this work, Hall and Giovanetti [9] developed a method for computing the optimal rotor design for conventional and compound helicopters in forward flight using conventional and higher harmonic control. The optimal design problem minimizes the sum of induced and viscous power required to develop a prescribed lift and/or thrust while maintaining vehicle trim. The optimization solves for a number of design variables, including blade geometry (radial blade twist and chord) and higher harmonic control of blade pitch. Results showed that for both conventional and counter-rotating coaxial rotors, using optimized twist and chord distributions together with higher harmonic blade pitch control substantially reduces required power, especially at high advance ratios.

Hall and Giovanetti considered only isolated conventional or coaxial rotor(s) and placed no constraints on the propulsive force generated by the rotor(s). Each configuration was analyzed at a shaft angle (rotor angle of attack) of $\alpha = -5^\circ$, representative of a realistic flight condition at high speed. Of course, in the operation of an actual rotorcraft, some propulsive force is required to counteract fuselage drag and maintain constant speed straight and level flight.

Work Performed in this Investigation

Accomplishments

During this reporting period, we have developed a vortex-lattice based variational method for computing the optimal aerodynamic performance of conventional and compound helicopter ro-

tors in high-speed forward flight. Included in the analysis is the option to use higher harmonic control to reduced required power. We have also developed hover models to use along side the vortex lattice model for multipoint design optimization of rotors. Specific accomplishments are enumerated below.

1. We extended our previous analysis to include the effect of compressibility drag. Accurately accounting for the large rise in the coefficient of drag at high Mach numbers allows us to assess the impact of this component of profile power loss on rotor design and performance. Compressible C81 tables are used to model the sectional lift and drag polars of the blade sections.
2. We developed an analysis for optimizing the performance of a conventional or coaxial rotor that is part of a compound helicopter system that uses an auxiliary propulsor (e.g., a propeller) to provide a portion of the forward thrust (propulsive force). The analysis finds the optimal rotor blade design including blade planform and twist distributions as well as higher harmonic control inputs for blade pitch. The analysis can also be used to find the optimal division of the forward thrust between the main rotor(s) and propulsor. See Appendix I for details of this analysis.
3. We developed a multipoint optimization that determines the blade geometry and root pitch inputs that maximize performance between two forward flight conditions – for example, between a high-speed and low-speed cruise condition. We applied this optimization to both conventional and coaxial rotors and showed that higher harmonic control can significantly reduce power in cruise. See Appendix II for details.
4. We developed a multipoint optimization that determines the blade geometry and root pitch inputs, including higher harmonic control, that maximize performance between *hover* and forward flight. See Appendix II for details.
5. We developed a blade element momentum theory of hover for conventional and coaxial counter rotating rotors (which is used in the multipoint optimizer). See Appendix III for details.
6. We developed and exercised a method for determining the (single point) optimal performance and rotor design of a coaxial rotor in hover, modeled both with and without the swirl component of induced velocity. See Appendix III for details.
7. We developed a coaxial hover model in which the wake is modeled with axisymmetric sheets of shed vorticity, whose position is found as part of the solution. The method can be used to compute both the wake contraction below a rotor and the induced inflow above a rotor. See Appendix IV for details.

Main Findings

In this section, we present some of the main findings of the present research. For more details, see the attached appendices.

Compound Helicopters

We examined the optimal performance of compound helicopters with co-axial rotors and an auxiliary propeller [10]. We analyzed configurations at an advance ratio of $\mu = 0.85$ over a range of rotor shaft angles (angle of attack). We found the following:

1. For compound helicopters, varying the prescribed propulsive force of the vehicle has a large effect on the optimal shaft angle. In turn, the shaft angle has an effect on the optimal rotor design. Additionally, the optimal shaft angle is highly dependent on the propeller design and efficiency.

2. For compound helicopters operating at high advance ratios, higher harmonic control can significantly reduce total vehicle power loss compared to conventional control. Higher harmonic control results in a more efficient rotor system, leading to an increasingly negative optimal shaft angle. This allows the rotors to provide a larger fraction of the vehicle's propulsive force. The use of higher harmonic control reduces both induced and viscous components of power in part by decreasing the circulation generated on the retreating side of each rotor.
3. Coaxial rotors can achieve efficient forward flight by operating with a high lateral lift offset equal to about one-half the rotor radius.

Multipoint Design of Helicopter Rotors

We performed multipoint optimization studies of conventional and coaxial rotors, optimizing the performance between two forward flight advance ratios, and also between a single forward flight advance ratio and hover [11]. We performed cruise/cruise and hover/cruise optimizations on representational conventional and coaxial rotors. Based on our research, we reached the following conclusions:

1. For a conventional four-bladed helicopter rotor optimized for advance ratios of $\mu = 0.2$ and $\mu = 0.5$, it is not possible to achieve optimal or near optimal performance at both flight conditions using a single rotor design. As a result, performance tradeoffs must be made in selecting a rotor design. Weighting the $\mu = 0.5$ power loss more heavily results in a moderate penalty in performance at $\mu = 0.2$; however, heavily weighting the $\mu = 0.2$ power loss results in a very large penalty in performance at $\mu = 0.5$, with a total power loss that is three times the single point optimum. Higher harmonic control provides a relatively small benefit at both advance ratios, and is marginally effective at improving the shape and location of the Pareto frontier.
2. For a conventional rotor optimized for hover and forward flight at $\mu = 0.5$, there is a large tradeoff in performance between the two flight conditions. The (single point optimal) hover and forward flight optimal rotor designs are dramatically different, and a single rotor design cannot achieve optimal or near optimal performance at both flight conditions. The multipoint optimal rotor blends characteristics from the two single point designs, resulting in balancing of performance between the two flight conditions. In such a case, the desired rotor shape should be carefully selected from designs along the Pareto efficiency curve based on mission requirements. Higher harmonic control once again marginally improves the location of the Pareto efficiency curve across all values of α .
3. For a coaxial rotor optimized for advance ratios of $\mu = 0.3$ and $\mu = 0.85$ s, the Pareto efficiency curve has a very sharp corner or "knee," indicating that near optimal performance can be achieved at both flight conditions with a single rotor design. Higher harmonic control significantly reduces power at the $\mu = 0.85$ flight condition, improving the Pareto efficiency curve while not significantly altering its general shape.
4. For a coaxial rotor optimized for hover and $\mu = 0.85$ flight, a substantial tradeoff in performance must again be made between the two flight conditions, as there is no sharp knee in the Pareto curve. However, the forward flight performance of the rotor is more sensitive to changes in rotor design away from the $\mu = 0.85$ optimal rotor, resulting in the multipoint optimum rotor design being dominated by the $\mu = 0.85$ optimal twist and chord distributions, even at small weighting towards this flight condition. Higher harmonic control is again beneficial for high-speed forward flight conditions, and substantially improves the location and shape of the Pareto frontier.

Optimal Rotor Design for Hover

Using blade element momentum (BEMT) theory coupled with McAlister's inflow theory[12], we found the optimum blade planform and twist distribution for helicopter rotors in hover [13]. The

BEMT theory can optionally include or exclude the influence of swirl in the flow. Profile losses and their effects on power and thrust were included using airfoil drag polars. The optimization approach was used to redesign a coaxial rotor similar in design to the Harrington Rotor 2. Based on that study, we found:

1. Optimal coaxial rotors designed with and without swirl modeling are nearly identical at radial stations outboard of $r/R = 0.3$. At values of r/R less than about 0.3, rotors optimized accounting for swirl have significantly different optimal twist and chord distributions than those optimized neglecting swirl.
2. Induced swirl has a larger effect on performance at higher thrust coefficients.
3. At disk loadings typical of those seen on helicopters, the effect of swirl on rotor design and performance predictions within the framework of combined blade element momentum theory is relatively small. Other model refinements, such as detailed wake modeling and improved mutual interference models, may have a larger impact on optimal rotor design and performance predictions than the inclusion of the swirl component of induced velocity.
4. Optimally designed coaxial rotors for hover have significantly different twist and chord distributions on the upper and lower rotors, resulting from the different inflow distributions acting on each rotor.
5. Using the optimal *upper* rotor blade design for *both* the upper and lower rotors leads to a relatively small decrease in performance, indicating that near optimal hover performance can be achieved with upper and lower rotors using the same blade design.

Aerodynamic Modeling of Hover

The cruise/hover multipoint optimization analysis requires a model of hover aerodynamics for coaxial rotors; we used the mutual interference model of McAlister et al. [12] together with blade element momentum theory. However, we were concerned that that model might not represent the metal interaction with sufficient fidelity. Thus we developed an axisymmetric vortex sheet model of the flow through an actuator disk [14]. We represent the continuous vortex sheet of the wake using discrete vortex rings with non-uniform spacing. We impose a no through flow boundary condition on the vortex sheet and determine its position via Newton iteration. The model can be used for rotors with multiple vortex sheets, allowing the analysis of non-uniformly loaded single and dual coaxial actuator disks. We investigated a variety of cases and arrived made the following conclusions:

1. A singularity occurs where the vortex sheet terminates at the edge of the actuator. This singularity is resolved through the formation of a 45° logarithmic spiral in hover, as originally hypothesized by Spalartdisk [15]. The spiral structure of the wake results in a non-uniform inflow, particularly near the edge of the disk, that includes reversed flow through the actuator disk.
2. In axial flight, the size of this spiral decreases, vanishing in the limit as the ratio of the climb speed to induced inflow velocity approaches infinity.
3. In the case of descending flight, the size of the spiral structure increases with descent rate until the wake enters the vortex ring state. In the vortex ring state, no converged solution can be found using the methods described here. Once the descent velocity is sufficiently high, the disk enters a windmill state, which also produces a spiral vortex sheet structure at the actuator disk tip with a spiral in the same direction as the climb and hover cases. In the windmill state, the vortex sheet convects in the direction of the free stream flow, compared to the low descent velocity cases where the vortex sheet convection and free stream are in opposite directions.
4. Coaxial actuator disks have a pronounced spiral wake structure on the lower disk, where the upwash induced by the upper disk acts as a descent velocity for the edge of the lower disk. The upper disk sees the opposite effect, with a net flow through the disk in the

traditional direction induced by the lower disk, suppressing the spiral structure and more closely resembling an actuator disk in climbing axial flight.

References

- [1] Moffitt, R. C., and Bissell, J. R., "Theory and Application of Optimum Airloads to Rotors in Hover and Forward Flight," American Helicopter Society 38th Annual Forum Proceedings, Anaheim, CA, May 4–7, 1982.
- [2] Moffitt, R. C., and Rivera, S., "An Implicit Multi-Point Lifting Line Optimizer for Twist and Chord," American Helicopter Society International Technical Specialists' Meeting on Rotorcraft Multidisciplinary Design Optimization, Atlanta, GA, April 27–28, 1993.
- [3] Ormiston, R. A., "Further Investigations of Helicopter Rotor Induced Power," American Helicopter Society 61st Annual Forum Proceedings, Grapevine, TX, June 1–3, 2005.
- [4] Peters, D.A., File, C.L., and Ormiston, R. A., "Rotor Performance and Optimum Power with Realistic Constraints by Finite-State Inflow," American Helicopter Society 67th Annual Forum Proceedings, Virginia Beach, Virginia, USA, 3–5 May 2011.
- [5] Hall, S. R., Yang, K. Y., and Hall, K. C., "Helicopter Rotor Lift Distributions for Minimum-Induced Power Loss," *Journal of Aircraft*, Vol. 31, (4), July–August 1994, pp. 837–845.
- [6] Hall, K. C., and Hall, S. R., "Minimum Induced Power Requirements for Flapping Flight," *Journal of Fluid Mechanics*, Vol. 323, September 1996, pp. 285–315.
- [7] Hall, K. C., Pigott, S. A., and Hall, S. R., "Power Requirements for Large-Amplitude Flapping Flight," *Journal of Aircraft*, Vol. 35, (3), May–June 1998, pp. 352–361.
- [8] Hall, K. C., and Hall, S. R., "A Variational Method for Computing the Optimal Aerodynamic Performance of Conventional and Compound Helicopters," *Journal of the American Helicopter Society*, Vol. 55, (4), October 2010.
- [9] Hall, K. C., and Giovanetti, E. B., "Minimum Power Requirements and Optimal Rotor Design for Conventional and Compound Helicopters Using Higher Harmonic Control," American Helicopter Society 69th Annual Forum, Phoenix, Arizona, May 21–23, 2013.
- [10] Giovanetti, E. B., Hall, K. C., "Optimum Design of Compound Helicopters that Use Higher Harmonic Control," *Journal of Aircraft*, Vol. 52, No. 5, September–October 2015,
- [11] Giovanetti, E. B., Hall, K. C., "A Variational Approach to Multipoint Aerodynamic Optimization of Conventional and Coaxial Helicopter Rotors," American Helicopter Society 71st Annual Forum, May 5–7, 2015, Virginia Beach, Virginia.
- [12] McAlister, K. W., Tung, C., Rand, O., Khromov, V., "Experimental and Numerical Study of a Model Coaxial Rotor," presented at the AHS 62nd Annual Forum, Phoenix, Arizona, May 2006.
- [13] Giovanetti, E. B., Hall, K. C., "Minimum Loss Load, Twist and Chord Distributions For Coaxial Helicopters in Hover," *Journal of the American Helicopter Society*, Vol. 62, 2017, pp. 012001
- [14] Giovanetti, E. B., and Hall, K. C., "Axisymmetric Potential Flow Model of Single or Coaxial Actuator Disks," American Helicopter Society 72nd Annual Forum, May 17–19, 2016, West Palm Beach, Florida.
- [15] Spalart, P. R., "On the Simple Actuator Disk," *Journal of Fluid Mechanics*, 494:399 - 405, 2003.

Appendix I: Optimum Design of Compound Helicopters that Use Higher Harmonic Control.

JOURNAL OF AIRCRAFT
Vol. 52, No. 5, September–October 2015

Optimum Design of Compound Helicopters that Use Higher Harmonic Control

Eli B. Giovanetti* and Kenneth C. Hall†
Duke University, Durham, North Carolina 27708

DOI: 10.2514/1.C032941

The optimal design of a compound helicopter comprised of counterrotating coaxial rotors, a propeller, and optionally a fixed wing is investigated. The blade geometry, azimuthal blade pitch inputs, optimal shaft angle (rotor angle of attack), and division of propulsive and lifting forces among the components that minimize the total power for a given flight condition are determined. The optimal design problem is cast as a variational statement that minimizes the sum of induced and viscous power losses for a prescribed lift, propulsive force, and vehicle trim condition. The rotor, propeller, and wing geometry and control inputs are related to the far-field circulation through a lifting-line model that accounts for experimentally or computationally determined nonlinear lift and drag polars. The variational statement is discretized using a vortex lattice wake, and the resulting nonlinear constrained optimization problem is solved via Newton iteration. Results show that varying the prescribed propulsive force of the system affects the optimal shaft angle and rotor design, that higher harmonic control reduces the total vehicle power loss (inefficiency) in high-speed flight by as much as 15%, and that imposing a maximum allowable lateral lift offset can greatly increase the power loss of the coaxial rotors.

Nomenclature

A	=	rotor disk area, πR^2
$\mathbf{A}$	=	matrix relating control inputs $\boldsymbol{\theta}$ to circulation $\boldsymbol{\Gamma}$
A_n	=	n th Fourier coefficient of blade pitch
B	=	number of blades
$\mathbf{B}$	=	matrix relating force $\mathbf{F}$ to circulation $\boldsymbol{\Gamma}$
B_n	=	n th Fourier coefficient of blade pitch
C_{Fx}	=	propulsive force coefficient, $F_x/\rho A \Omega^2 R^2$
C_L	=	lift coefficient, $L/\rho A \Omega^2 R^2$
C_P	=	power coefficient, $P/\rho A \Omega^3 R^3$
c	=	blade chord
c_d	=	airfoil sectional drag coefficient
$\mathbf{D}$	=	matrix relating moment $\mathbf{M}$ to circulation $\boldsymbol{\Gamma}$
E	=	kinetic energy per period in wake
$\mathbf{F}$	=	time-averaged aerodynamic force vector
$\mathbf{K}$	=	quadratic power matrix
L	=	vehicle lift
$\mathbf{M}$	=	time-averaged aerodynamic moment vector
N	=	number of harmonics in higher harmonic control
n	=	Fourier coefficient index
$\mathbf{n}$	=	unit vector normal to wake surface
P	=	rotor power loss
R	=	rotor radius
$\mathbf{R}$	=	vector function describing relationship between circulation $\boldsymbol{\Gamma}$ and control inputs $\boldsymbol{\theta}$
$\mathbf{R}_{\boldsymbol{\Gamma}}$	=	Jacobian of $\mathbf{R}$ with respect to $\boldsymbol{\Gamma}$
$\mathbf{R}_{\boldsymbol{\theta}}$	=	Jacobian of $\mathbf{R}$ with respect to $\boldsymbol{\theta}$
$\mathbf{r}$	=	moment arm from vehicle center to lifting surface
T	=	temporal period of wake
u	=	relative fluid velocity perpendicular to span
V	=	vehicle forward speed

W	=	wake sheet surface
$\mathbf{w}$	=	induced velocity in far wake
α_{shaft}	=	rotor disk angle of attack
Γ	=	circulation
θ_0	=	fixed blade twist distribution
$\boldsymbol{\theta}$	=	vector of control inputs
λ	=	equality constraint Lagrange multiplier
λ_F	=	Lagrange multiplier for force constraint
λ_M	=	Lagrange multiplier for moment constraint
λ_R	=	Lagrange multiplier for vector function $\mathbf{R}$
μ	=	advance ratio, $V/\Omega R$
ξ	=	Kelvin linear impulse
Π	=	power Lagrangian
ρ	=	fluid density
σ	=	rotor solidity, $Bc/\pi R$
σ_{TW}	=	thrust-weighted rotor solidity
σ_{TWM}	=	modified thrust-weighted rotor solidity
ϕ	=	velocity potential function
ψ	=	azimuthal angle
Ω	=	rotor rotational speed

Subscripts

R	=	prescribed value
prop	=	variable associated with propeller
rotor	=	variable associated with rotor system
v	=	viscous

1. Introduction

COMPOUND helicopters require less power than conventional helicopters to operate at high speeds [1–3], making them likely candidates for future high-speed vertical lift vehicles. In this paper, we investigate the optimal design of a compound helicopter comprising a propeller, a coaxial rotor system, and optionally a lifting wing. We apply the optimization method of Hall and Giovanetti [4] to determine the optimal blade geometry, the azimuthal blade pitch inputs, and the optimal division of propulsive forces and lift among the various components. We investigate the effect of the higher harmonic blade pitch control, rotor shaft angle, fuselage drag, constrained lift offset, use of a lifting wing, and inclination of the propeller on total vehicle power requirements.

A number of investigators have studied the performance of compound helicopters in high-speed forward flight. Orchard and Newman [3] investigated fundamental design features of compound helicopters using a wing, a single rotor, and a propulsor. They noted

Presented at the AHS International 70th Annual Forum & Technology Display, Montréal, Québec, Canada, 20–22 May 2014; received 23 April 2014; revision received 28 May 2015; accepted for publication 6 June 2015; published online 11 August 2015. Copyright © 2014 by American Helicopter Society International, Inc.. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission. Copies of this paper may be made for personal or internal use, on condition that the copier pay the \$10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 1533-3868/15 and \$10.00 in correspondence with the CCC.

*Graduate Research Assistant, Department of Mechanical Engineering and Materials Science. Student Member AIAA.

†Julian Francis Abele Professor of Mechanical Engineering and Materials Science. Fellow AIAA.

that the efficiency of the propulsor will have a large effect on whether autorotating the rotor at high speeds is more efficient than powering the rotor. Bagai [5] described the aerodynamic design of the Sikorsky X2 Technology Demonstrator main rotor blades. Bagai noted that the use of two per revolution (2/rev) harmonic control may prove beneficial in improving the aerodynamic efficiency of the coaxial rotors.

Ormiston [1], building on his previous work on the induced power of helicopter rotors [6–8], evaluated three configurations: a rotor using a wing, a rotor using auxiliary propulsion, and a rotor using both auxiliary propulsion and a wing. He determined that the case using both a wing and auxiliary propulsion had the best performance (i.e., the lowest required power). He also concluded that induced power losses significantly impact the aerodynamic efficiency of compound rotorcraft and are an important discriminator between high- and low-performance configurations. Johnson [9], Johnson et al. [10], Yeo and Johnson [11], and Russell and Johnson [12] all analyzed the performance, sizing, and design of compound configurations to satisfy specific mission requirements using the comprehensive code CAMRAD II. Johnson [9] concluded that a lift offset of 0.25 of the rotor radius is effective in reducing rotor induced and profile power in high-speed cruise.

Rand and Khromov [13] introduced a drag vs power chart as a semi-analytic tool for understanding the effect of the rotor, propeller, wing, and fuselage contributions to propulsive force, drag, and the overall power required for compound configurations. The authors concluded that the optimal configuration is dependent on each component's efficiency and that the optimal propulsive force balance occurs when the local rotor and propeller propulsive efficiencies, defined as the incremental change in propulsive force per unit change in required power, are equal.

In 2010, Hall and Hall [2] presented a variational method for computing the optimal circulation distribution that produces minimum power for conventional and compound helicopters in forward flight. In that work, the authors treat the rotor as a “rubber rotor”; that is, they determined the optimal circulation without regard to practical control inputs or blade geometry (chord and twist), providing a rigorous upper bound on the performance of a rotorcraft configuration. Building on this work, Hall and Giovanetti [4] developed a method for computing the optimal rotor design for conventional and compound helicopters in forward flight using conventional and higher harmonic control. The optimal design problem minimizes the sum of induced and viscous power required to develop a prescribed lift and/or thrust while maintaining vehicle trim. The optimization solves for a number of design variables, including the blade geometry (radial blade twist and chord) and higher harmonic control of the blade pitch. Results showed that for both conventional and counterrotating coaxial rotors using optimized twist and chord distributions together with higher harmonic blade pitch control substantially reduces required power, especially at high advance ratios.

Hall and Giovanetti [4] considered only isolated conventional or coaxial rotor(s) and placed no constraints on the propulsive force generated by the rotor(s). Each configuration was analyzed at a shaft angle of $\alpha_{\text{shaft}} = -5^\circ$; that is, the rotor was tilted slightly forward providing some propulsive force. Of course, some propulsive force is required to counteract fuselage drag and to maintain constant-speed straight and level flight. In this paper, we include a propeller in our helicopter model to provide a propulsive force and include a propulsive force constraint to balance a hypothetical fuselage drag. We apply an updated version of our optimization algorithm to the entire compound helicopter, allowing us to investigate minimum power requirements, the optimal shaft angle, and the optimal aerodynamic design of the compound helicopter as a system.

The results presented here demonstrate the importance of both the propeller and rotor efficiencies in determining the optimal division of the propulsive force between the rotor and propeller and the importance of the shaft angle on high-speed performance and the optimal rotor design. We show that higher harmonic control significantly reduces the total vehicle power at high advance ratios. We also show that limiting the maximum lateral lift offset of the coaxial rotors to 30% of the rotor radius or less results in a substantial increase in power and that the effectiveness of higher harmonic control to reduce

these rotor power losses is dependent on the shaft angle. Lastly, we investigate the use of a lifting wing, as well as a propeller that provides some lift in addition to thrust, and determine that each of these design features provides only marginal improvements in aerodynamic efficiency.

II. Aerodynamic Modeling of Rotor

The technical approach used in this paper follows [4] closely and is briefly summarized in this section. We wish to determine the rotor, propeller, and wing design variables and control inputs that produce the minimum required power in forward flight. We assume light loading and/or high advance ratios, so a prescribed wake is appropriate. We assume high aspect ratio lifting surfaces, so the rotor blade, propeller, and wing geometries and control inputs can be related to the circulation in the wake using the rotorcraft equivalent of lifting-line theory. Finally, we assume that the reduced frequency based on chord is small. Thus, the sectional lift and drag can be described using quasi steady two-dimensional lift and drag curves found from experiment or using a computational fluid dynamic analysis. Alternatively, C-81 tables can be used to account accurately for the effect of the Mach number on an airfoil's sectional lift and drag coefficients.

We calculate inviscid forces, moments, and induced power using a far-field approach. The forces and moments acting on the rotor/wing/propeller system are a result of the apparent linear and angular momentum (i.e., Kelvin linear and angular impulses) deposited in the wake of each lifting component. The Trefftz volume bounds one period of the flowfield in the far wake, bounded between two infinite parallel planes roughly transverse to the flight direction as shown in Fig. 1.

The far-field flow is assumed to be inviscid, incompressible, and irrotational, except for the trailing and shed vorticity in the wake. Note that the assumption of incompressible flow requires only that the induced velocities in the wake be small compared to the speed of sound, an assumption consistent with the light loading model. (The flow will in general be compressible in the near field of the rotor.)

As discussed in [2,4], using these assumptions, the time-averaged force on the rotor is equal to

$$\mathbf{F} = -\frac{\xi}{T} = \frac{\rho}{T} \iint_W \Gamma \mathbf{n} dA \quad (1)$$

where the integral is taken over one side of the wake denoted by W , Γ is the jump in potential (the circulation), $\mathbf{n}$ is the unit normal to the wake, and T is the temporal period of the wake, usually equal to $2\pi/\Omega B$ because only $1/B$ of a turn of the wake is required to achieve periodicity.

Likewise, the time-averaged moment acting on the system is equal to

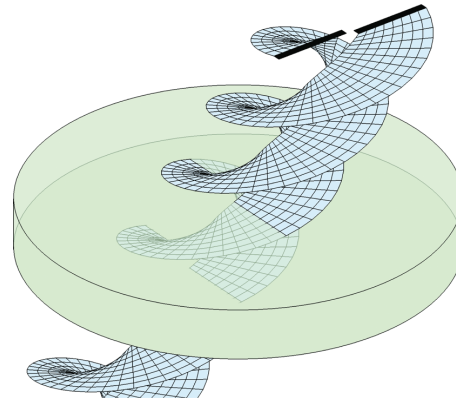


Fig. 1 Schematic of the prescribed wake showing one period of the far wake bounded by the Trefftz volume.

$$\mathbf{M} = \frac{\rho}{T} \iint_{\mathcal{W}} \Gamma \mathbf{r} \times \mathbf{n} \, dA \quad (2)$$

where $\mathbf{r}$ is the moment arm extending from the center of gravity of the aircraft to the element of the wake area at the time the wake is generated.

The induced power losses due to lift and thrust of a conventional helicopter rotor, or rotor/wing/propeller system for a compound helicopter, arise from the deposition of kinetic energy into the wake. The time-averaged induced power P_i , equal to the rate of kinetic energy production, is given by

$$P_i = -\frac{\rho}{2T} \iint_{\mathcal{W}} \Gamma \mathbf{w} \cdot \mathbf{n} \, dA \quad (3)$$

where $\mathbf{w}$ is the induced velocity.

The profile power P_v may be expressed as the work per cycle divided by the period, so that

$$P_v = \frac{1}{2T} \iint_{\mathcal{W}} \rho u^2 c_d \, dA \quad (4)$$

where u is the relative velocity of a given airfoil section normal to the span of the rotor.

III. Optimal Rotor Performance

In the design of a rotor, one may select a fixed blade twist distribution $\theta_0(r)$ and implement some limited set of harmonic blade pitch control inputs. For such a configuration, the blade twist as a function of radius r and azimuth ψ can be described as

$$\theta(r, \psi) = \theta_0(r) + A_0 + \sum_{n=1}^N A_n \cos(n\psi) + \sum_{n=1}^N B_n \sin(n\psi) \quad (5)$$

where A_n and B_n are the Fourier coefficients of the blade pitch and N is the number of harmonics in the higher harmonic control system. A conventional helicopter, with collective and cyclic control, corresponds to $N = 1$.

We denote the total vector of design variables and control inputs by $\boldsymbol{\theta}$. For a harmonic blade pitch control scheme as shown in Eq. (5), the vector $\boldsymbol{\theta}$ will be composed of $\theta_0(r)$ at a discrete set of radii and the Fourier coefficients A_0 , A_n , and B_n . Alternatively, $\theta_0(r)$ can be represented by a summation of shape functions in the radial direction, in which case the coefficients of the shape functions would be members of $\boldsymbol{\theta}$.

To compute the forces, moments, and power on a rotor, we must find the circulation in terms of the design variables and control inputs. In this study, we use a lifting-line approximation to compute the wash on the blades. Using our lightly loaded model, the induced wash is a linear function of the circulation. However, the resulting induced angles of attack on the rotor blade can be large, especially in and near the reverse flow region (large enough to require the use of a nonlinear lift curve) rendering the circulation a nonlinear function of the control inputs.

We express the optimal design of a rotor/wing/propeller system with a finite number of design variables and control inputs as a variational statement. We seek to find the circulation distribution that minimizes power loss subject to the constraints that the rotor generates a prescribed force $\mathbf{F}_R$, the rotor is trimmed in pitch and roll ($\mathbf{M}_R = 0$), and the circulation is realizable using our finite set of design variables and control inputs. Thus, we adjoin to the power these constraints using Lagrange multipliers to obtain the Lagrangian power

$$\Pi = \frac{1}{2} \Gamma^T \mathbf{K} \Gamma + P_v + \lambda_F^T \cdot (\mathbf{B} \Gamma - \mathbf{F}_R) + \lambda_M^T \cdot (\mathbf{D} \Gamma - \mathbf{M}_R) + \lambda_R^T \cdot \mathbf{R}(\Gamma, \boldsymbol{\theta}) \quad (6)$$

where the nonlinear lifting-line equations are given by the vector function

$$\mathbf{R}(\Gamma, \boldsymbol{\theta}) = 0 \quad (7)$$

In the present analysis, we use a nonlinear lifting-line analysis, but $\mathbf{R}$ could also describe some other numerical method for computing the circulation in terms of the design variables, e.g., a panel method.

Taking the variation of Eq. (6) and setting the result to zero for arbitrary variations in the circulation, control inputs, and Lagrange multipliers gives a set of linear equations for the circulation, controls, and Lagrange multipliers, together with Eq. (7). The linear equations are given by

$$\begin{bmatrix} \frac{1}{2}(\mathbf{K} + \mathbf{K}^T) & \mathbf{B}^T & \mathbf{D}^T & \mathbf{R}_\Gamma^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}_\theta^T \\ \mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{D} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Gamma \\ \lambda_F \\ \lambda_M \\ \lambda_R \end{bmatrix} = \begin{bmatrix} -P_{v\Gamma} \\ -P_{v\theta} \\ \mathbf{F}_R \\ \mathbf{M}_R \end{bmatrix} \quad (8)$$

where $\mathbf{R}_\Gamma$ and $\mathbf{R}_\theta$ are the Jacobians of $\mathbf{R}$ with respect to Γ and $\boldsymbol{\theta}$, respectively, and $\mathbf{P}_\Gamma$ and $\mathbf{P}_{v\theta}$ are the Jacobians of P_v with respect to Γ and $\boldsymbol{\theta}$, respectively.

The nonlinear relationship between the circulation and the rotor design variables and control inputs complicates the solution process a bit. We approximate the change in circulation with respect to a change in design variables as linear, allowing us to compute the Jacobians through closed-form expressions and solve Eq. (8) directly. Equation (7) is then solved at the updated set of design variables to compute the correct circulation, the Jacobians are recalculated, and the process is repeated until convergence. For more details on this approach, see [4].

In addition to the equality constraints on forces and moments, we also implement inequality constraints on the chord of each lifting component using mathematical programming via augmented Lagrangians, in order to prevent the optimization from taking advantage of negative or infinitesimal chords. We also implement an equality constraint on the thrust-weighted solidity of the rotor. Because we are performing a single point optimization at very high flight speeds and because of the high dynamic pressures present on the advancing side of each rotor and the natural roll trim of the coaxial system, the resulting optimal chord is often quite small, too small for the helicopter to hover. Constraining the thrust-weighted solidity ensures that the design is capable of hover and low-speed flight. However, because the thrust-weighted solidity heavily weights solidity at the blade tip and because the circulation goes to zero at the tip regardless of the chord, we constrain instead a modified thrust-weighted solidity defined by

$$\sigma_{\text{TWM}} \equiv \frac{3}{R^3} \int_0^R r^2 \sigma(r) \cdot \{1 - \exp[-(1 - r/R)/\epsilon]\} dr \quad (9)$$

where ϵ is a constant taken to be 0.10. The effect of the term in braces in the integrand is to remove the influence of the solidity of the tip. The shape of the tip is then more naturally selected to minimize power.

Using this method, the optimal rotor geometry and higher harmonic control of rotors, wings, or propellers in forward flight can be computed. Additionally, the resulting power loss for each particular configuration is computed, lending insight into the aerodynamic effects of propulsive force requirements, higher harmonic control, limiting the rotors' lift offset, and other design features.

IV. Computational Results

We consider first a vehicle modeled roughly after the Sikorsky X2 Technology Demonstrator (TD), with two coaxial counterrotating four-bladed rotors and a six-bladed pusher propeller to provide an auxiliary propulsive force. We analyze rotors similar in design intent to the X2 rotor system in high-speed cruise as described by Bagai [5]. The parameters for the coaxial rotor system are listed in Table 1. The blade geometries of the upper and lower rotors are constrained to be equal, although each rotor has independent root pitch inputs.

The propeller is roughly modeled after the X2 TD as well, with an rpm that is six times that of the rotors' and a radius equal to 25% of the

Table 1 Coaxial rotor parameters

Parameter	Value
Advance ratio, μ	0.85
Relative tip Mach number	0.90
Coefficient of lift, C_L	0.02324
Thrust-weighted solidity, σ_{TW}	0.1441
C_L/σ_{TW}	0.1613
Root cutout	10% of rotor radius
Vertical rotor separation	20% of rotor radius

rotor radius. Unlike the X2 TD, however, both the rotors and propeller use a NACA 0012 airfoil from the root to the tip.

To analyze a single advance ratio and shaft angle, we use a lattice of vortex rings containing 18 ring elements in the spanwise direction and 15 elements in the azimuthal direction for each rotor blade. Because of the 4/rev temporal periodicity of the problem, only one-quarter of a revolution of the wake is needed to achieve spatial periodicity of the wake. The spatially periodic lattice is replicated and extended 40 periods (10 complete turns of the rotor) downstream for the near field analysis and 40 periods upstream and 40 periods downstream for the far wake induced power analysis. The propeller is modeled using 8 elements in the spanwise direction and 18 elements in the azimuthal direction for each full turn of the propeller. Because of its higher rotation rate, one and one-half turns of the propeller are modeled for each quarter-turn of the wake.

A. Effect of Propulsive Force on Power

First, we will vary the propulsive force required by the vehicle in straight and level flight, which is equivalent to changing the value of the vehicle's fuselage drag. This allows us to study the effect that propulsive force requirements have on the optimal shaft angle and on optimal rotor design. We examine two cases for comparison: a high propulsive force case, corresponding to a total propulsive force coefficient of $C_{Fx} = F_x/\rho A \Omega^2 R^2 = 0.0058$, and a low propulsive force case, corresponding to $C_{Fx} = 0.0031$. The cases analyzed here each use conventional $N = 1$ control.

For each case, we evaluate the performance and rotor design over a range of rotor shaft angles. All rotor and propeller design variables are optimized at each individual shaft angle. Thus, each data point represents a unique rotor design that provides the best possible performance at that flight condition. Note that only the rotor system is operating at the given shaft angle. Unless otherwise noted, the propeller axis is aligned with the direction of flight.

B. Effect of Propulsive Force on Optimal Shaft Angle

Figure 2 shows the vehicle total power loss (i.e., total power minus useful thrust power) for the high and low propulsive force cases across a range of shaft angles, normalized by the square of the

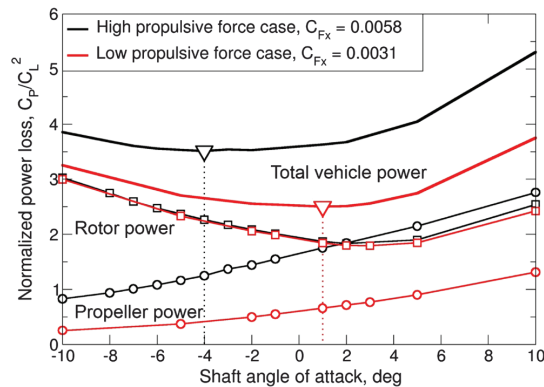


Fig. 2 Power loss of a coaxial rotor with a pusher propeller with varying levels of propulsive force.

coefficient of lift of the vehicle. Note that the total power loss referenced here is the same total power loss defined by Ormiston in [14] and used in previous papers [2,4] and represents the net power loss of the system, that is, the sum of the induced and profile losses of the rotor and propeller. The power loss is divided into two components: the propeller power loss and the rotor power loss. The total power loss of the vehicle is the sum of these two components. A negative shaft angle indicates that the rotor system is tipped forward into the flow, providing a positive propulsive force in the flight direction. Although not shown in the figure, the normalized useful power (thrust power coefficient divided by the coefficient of the lift squared) $C_{P_{useful}}/C_L^2$ is 9.13 for the high propulsive force case and 4.88 for the low propulsive force case. Thus, the power loss is a significant fraction of the total power.

As expected, the high propulsive force case has a higher overall power loss than the low propulsive force case across all shaft angles. The two cases, however, have significantly different optimal shaft angles. The low propulsive force case has a minimum total power at a shaft angle of $\alpha_{\text{shaft}} = 1$ deg, compared to $\alpha_{\text{shaft}} = -4$ deg for the high propulsive force case.

To investigate the reason for this difference in the optimal shaft angle, we examine a slightly simplified form of the variational statement. We seek to minimize the total power subject to the constraint that the required propulsive force (thrust) and the required lift are generated by the rotor/propeller system. Thus, we form the Lagrangian Π given by

$$\Pi = P_{\text{rotor}} + P_{\text{prop}} - \lambda_T(T_{\text{rotor}} + T_{\text{prop}} - T_R) - \lambda_L(L_{\text{rotor}} + L_{\text{prop}} - L_R) \quad (10)$$

where P is the power, T is the thrust, and L is the lift of either the rotor or propeller as denoted by subscripts. T_R and L_R are the required thrust and lift of the vehicle, respectively. Taking the variation of Eq. (10), making the assumption that the power and thrust of each component is independent of the other component, and setting the result to zero gives

$$\delta\Pi = 0 = \left(\frac{\partial P_{\text{rotor}}}{\partial T_{\text{rotor}}} - \lambda_T\right)\delta T_{\text{rotor}} - \left(\frac{\partial P_{\text{prop}}}{\partial T_{\text{prop}}} - \lambda_T\right)\delta T_{\text{prop}} - (T_{\text{rotor}} + T_{\text{prop}} - T_R)\delta\lambda_T - (L_{\text{rotor}} + L_{\text{prop}} - L_R)\delta\lambda_L \quad (11)$$

Equation (11) must be true for arbitrary variations in the rotor thrust, propeller thrust, and the Lagrange multipliers. Therefore, each of the four terms in parentheses must equal zero independently. For the first two terms in parentheses to satisfy this condition, we must have

$$\lambda_T = \frac{\partial P_{\text{rotor}}}{\partial T_{\text{rotor}}} = \frac{\partial P_{\text{prop}}}{\partial T_{\text{prop}}} \quad (12)$$

Equation (12) states that the minimum total vehicle power occurs when the change in power due to an incremental change in thrust is equal between the rotor and the propeller. This optimality condition was also derived graphically by Rand and Khromov [13].

Figure 3 shows the power loss as a function of the coefficient of the propulsive force for the rotor and propeller for both the high and low propulsive force cases. The optimality condition, Eq. (12), is satisfied at the shaft angle where the rotor and propeller power curves have the same slope. Note that the rotor power curves for the high and low propulsive force cases are very similar; the power depends primarily on the shaft angle and is only very weakly dependent on the propulsive force. The propeller power curves for the two cases are virtually identical; the propeller power is almost entirely dependent on the thrust it produces. For a given combined coefficient of thrust, the system will be optimal when the slopes of the power curves are equal. We see that for the low total propulsive force case it is more efficient to tilt the rotor aft so that the rotor generates almost no thrust. For the higher total propulsive force case, the optimum occurs when both the rotor and propeller generate significant thrust. In any event,

1448

GIOVANETTI AND HALL

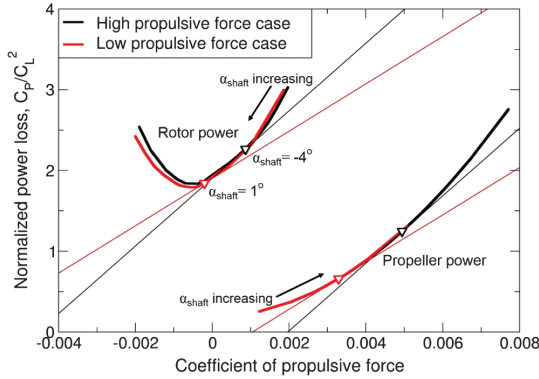


Fig. 3 Rotor and propeller power vs propulsive force.

we see that in both cases the largest share of thrust is generated by the propeller.

Figure 2 also reinforces an important conclusion reached by Rand and Khromov [13]. At high speed, the minimum total vehicle power is not necessarily achieved by operating at the minimum rotor power, which occurs in this case at $\alpha_{\text{shaft}} = 3.5$ deg, or by placing the rotor system in an autorotative state, which occurs at roughly $\alpha_{\text{shaft}} = 7.5$ deg. Rather, the minimum total power depends on the design and efficiency of both the propeller and rotor.

C. Effect of Propulsive Force on Optimal Rotor Design

Each of the previous design points represents a rotor and propeller optimized for one specific shaft angle. For each value of the total propulsive force, there is an optimal shaft angle and corresponding rotor design. In Fig. 4, we illustrate how the optimal rotor design varies with the total propulsive force. Figure 4 shows the optimal radial twist distribution and optimal planform for the high and low propulsive force cases, corresponding to an optimum shaft angle of $\alpha_{\text{shaft}} = -4$ deg and 1 deg, respectively. Note that the optimal planform and twist distributions for these two cases differ significantly, showing that the shaft angle and rotor geometry of an optimized vehicle are strongly influenced by the flight conditions (at least for a single point design).

D. Higher Harmonic Control

As demonstrated by Hall and Giovanetti [4], higher harmonic control is effective in reducing the total power loss in an isolated

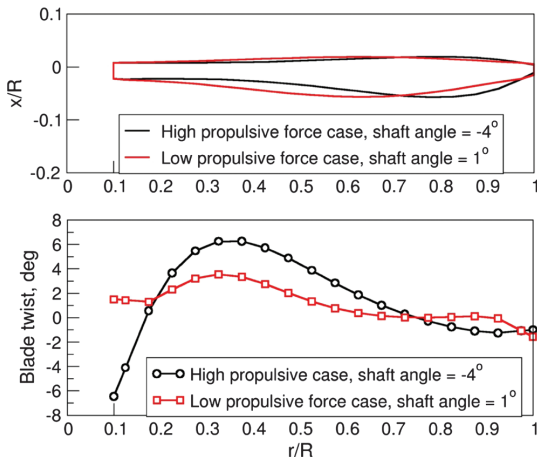


Fig. 4 Optimal rotor blade planform and twist at $\alpha_{\text{shaft}} = -4$ deg and $\alpha_{\text{shaft}} = 1$ deg.

coaxial rotor system, particularly at high advance ratios. Here, we consider the effect of higher harmonic control on the power and rotor design for a complete compound vehicle, including a propulsive force constraint.

Figure 5 shows the total vehicle power loss for the high propulsive force case using $N = 1$ control (conventional cyclic control) and $N = 3$ control (higher harmonic control). Using $N = 3$ control reduces the rotor power loss significantly, particularly at negative shaft angles, and results in an optimal shaft angle of $\alpha_{\text{shaft}} = -9$ deg. This decrease in the optimal shaft angle allows the rotors to provide a much larger fraction of the vehicle's overall propulsive force, approximately 33%, compared to 15% for the $N = 1$ control case at its optimal shaft angle of $\alpha_{\text{shaft}} = -4$ deg. Overall, $N = 3$ control reduces the total vehicle power loss by 15% over $N = 1$ control at each case's optimal shaft angle. While not shown in Fig. 5, $N = 2$ control results in an optimal shaft angle of $\alpha_{\text{shaft}} = -7$ deg and reduces power loss by 7.5% compared to $N = 1$ control. Harmonic control above $N = 3$ produces little additional power reduction.

Figure 6 shows the induced and viscous contributions to the total power loss for the $N = 1$ and $N = 3$ control cases. Higher harmonic control results in a lower total power loss by reducing both the induced and viscous components of power. Note that for shaft angles ranging from $\alpha_{\text{shaft}} = 1$ deg to $\alpha_{\text{shaft}} = 5$ deg higher harmonic control provides almost no advantage over conventional control, although these shaft angles are far from optimal in this configuration.

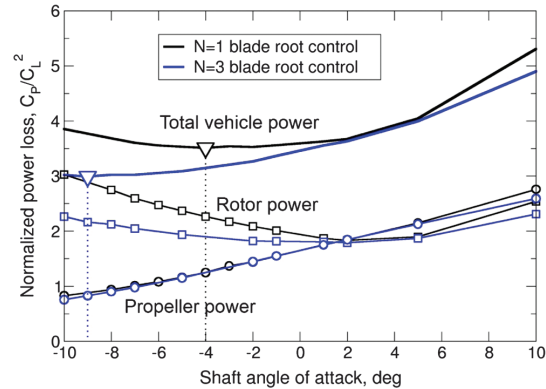


Fig. 5 Power loss of a coaxial rotor with a pusher propeller using $N = 1$ and $N = 3$ blade root control divided into propeller and rotor components.

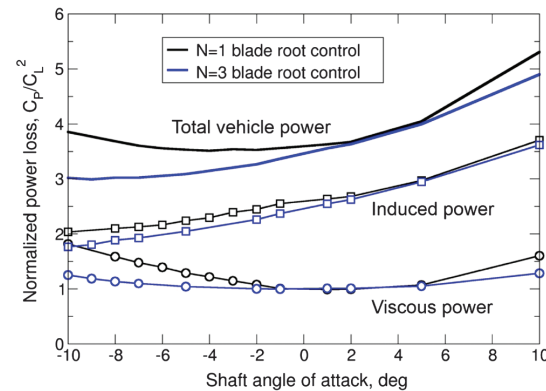


Fig. 6 Power loss of a coaxial rotor with a pusher propeller using $N = 1$ and $N = 3$ blade root control divided into induced and viscous components.

GIOVANETTI AND HALL

1449

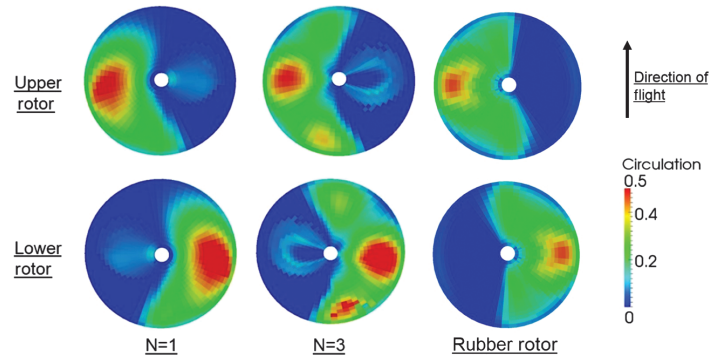


Fig. 7 Optimal normalized circulation distribution for $N = 1$ control, $N = 3$ control, and the rubber rotor solution. The upper rotor is spinning clockwise, and the lower rotor is spinning counterclockwise.

Figure 7 shows the optimal circulation distribution for the $N = 1$ and $N = 3$ cases. Also plotted is the circulation distribution for the optimal rubber rotor solution, which represents the absolute minimum power solution without regard to the rotor geometry or control inputs required to achieve this circulation. The $N = 1$ case generates a large positive circulation on the advancing side of each rotor, in addition to a moderate positive circulation on the retreating side of each rotor. The $N = 3$ case has multiple regions of concentrated circulation on the advancing side, one at $\psi = 90^\circ$, similar to the $N = 1$ case, but also smaller areas of concentrated circulation at $\psi = 0^\circ$ and $\psi = 180^\circ$. Additionally, the $N = 3$ case does not have the moderate positive circulation on the retreating side seen in the $N = 1$ case, instead having a region of low circulation that closely resembles the rubber rotor result.

Both cases have regions of high circulation, and therefore high lift, in a region just past the blade midspan on the advancing side. Thus, each rotor carries a large rolling moment and a lateral lift offset. The moment carried by each rotor is transmitted through the hub, resulting in a net vehicle rolling moment of zero. Figure 8 shows the blade pitch inputs as a function of azimuth for the two cases. The $N = 1$ azimuthal pitch input varies between 6 and 10 deg, while the $N = 3$ pitch input has a dramatic decrease in pitch around $\psi = 270^\circ$, corresponding to the region of near-zero circulation on the retreating side of each rotor. This decrease in pitch reduces circulation on the retreating side of the rotor, making the circulation distribution closer to the rubber rotor optimum, thereby reducing power.

Figure 9 shows the optimal radial twist and planform distribution for the $N = 1$ and $N = 3$ cases. Figure 10 shows an overhead view of the coaxial rotor system using the $N = 3$ optimal blade planform. The $N = 3$ blade has a wider chord near midspan, while the $N = 1$

blade has a wider chord in the outboard region. The radial twist distributions are fairly similar between the two cases, with the exception of the inner 20% of the blade, where the $N = 1$ blade has a large positive twist gradient.

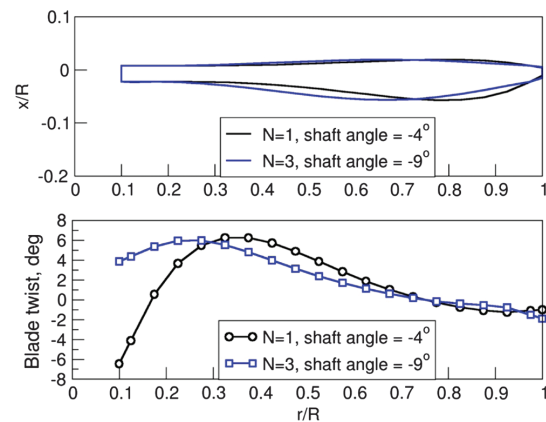


Fig. 9 Optimal rotor blade planform and twist for $N = 1$ and $N = 3$ blade root control.

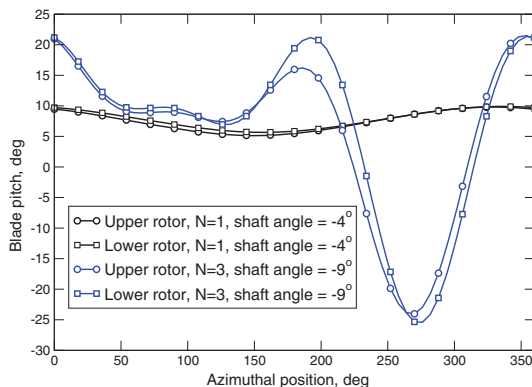


Fig. 8 Azimuthal pitch inputs using $N = 1$ and $N = 3$ blade root control.

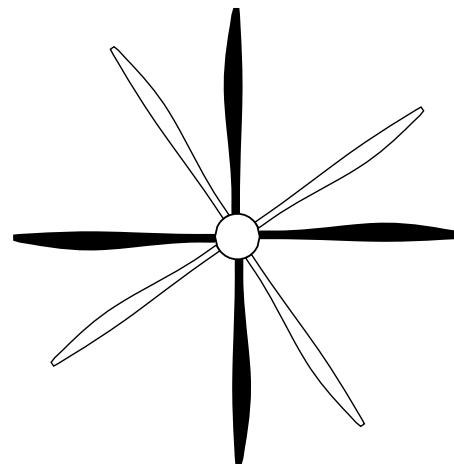


Fig. 10 Overhead view of the optimal blade planform for the $N = 3$ design.

E. Effect of Lift Offset

Lift offset (LOS) is defined as the lateral distance from the centerline of the aircraft to the center of the lift on a rotor. In nondimensional form, the lift offset is defined as $\Delta M_x/LR$, where ΔM_x is the difference in the rolling moment between the two rotors, R is the rotor radius, and L is the total lift. The $N = 1$ and $N = 3$ minimum power solutions from the previous section have nondimensional lift offsets of 0.54 and 0.49, respectively, indicating that the optimal solution offloads lift on the retreating side and generates the majority of the lift on the advancing side. In the design of an actual rotor, it may be necessary to constrain the rotor's lift offset to limit blade root and shaft bending moments for structural considerations. For example, according to Bagai [5], in the design of the X2 technology demonstrator, the lift offset was limited to 0.3 for structural and hub weight considerations. In this section, we constrain the lift offset to a maximum value of 0.3 to determine the effect that the reduced lift offset has on the rotor performance and design.

Figure 11 shows the normalized total power loss for $N = 1$ and $N = 3$ blade root harmonic control with the rotor system's lift offset constrained to a maximum value of 0.3. We observe that for both levels of harmonic control the lift offset constrained solution has significantly higher power loss than the unconstrained result (cf. Fig. 5). At the optimal shaft angle for each case, the $N = 1$ lift offset constrained case has a 40% higher total power loss than the unconstrained result, while the $N = 3$ lift offset constrained case has a 49% higher total power loss than the unconstrained result. This increase in power is due almost entirely to an increase in the rotor system's power. The propeller power, as expected, is essentially unaffected by the changes in the rotor design.

The decreased efficiency of the rotor system with the lift offset constraint shifts the optimal shaft angle to be more positive, i.e., to tilt the rotor system aft. The optimal shaft angle for the $N = 1$ case has shifted from $\alpha_{\text{shaft}} = -4^\circ$ deg with the lift offset unconstrained to $\alpha_{\text{shaft}} = 6^\circ$ deg with the lift offset constrained. For the $N = 3$ case, the optimal shaft angle has shifted from $\alpha_{\text{shaft}} = -9^\circ$ deg unconstrained to $\alpha_{\text{shaft}} = -4^\circ$ deg with the lift offset constrained.

The amount of power reduction achieved with higher harmonic control and the lift offset constrained to 0.3 is highly dependent on the shaft angle. At negative shaft angles, the $N = 1$ rotor has very high power losses, and use of $N = 3$ control reduces these power losses significantly. For example, at $\alpha_{\text{shaft}} = -4^\circ$ deg, the $N = 3$ case reduces the rotor system power loss by 43%, leading to a 34% reduction in total vehicle power loss compared to the $N = 1$ case. However, as the shaft angle increases, the rotor system is tilted aft and enters an increasingly autorotative state in which the benefits of higher harmonic control decrease. In fact, at $\alpha_{\text{shaft}} = 8^\circ$ deg, $N = 3$ control reduces the total vehicle power loss by less than 1%.

At the optimal shaft angle for each lift offset constrained case, $N = 3$ control reduces the total power loss by 11% compared to $N = 1$ control. Because the two cases have minimum powers at such

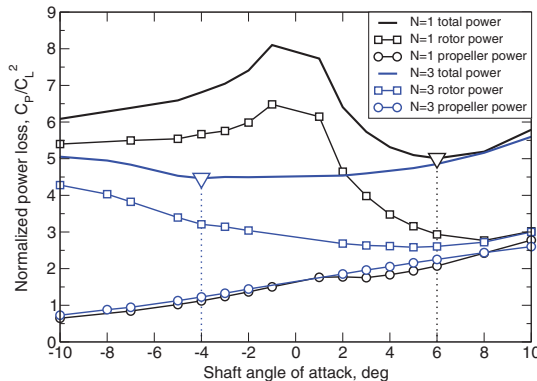


Fig. 11 Normalized power loss from the baseline compound constrained to a lateral lift offset of 0.3 using $N = 1$ and $N = 3$ control.

disparate shaft angles, the optimal blade designs for the two cases, shown in Fig. 12, differ significantly. The $N = 1$ planform locates much of the blade area on the inboard 50% of the blade, while for the $N = 3$ case, the chord distribution is nearly centered about the midspan point.

Finally, we investigate how higher harmonic control achieves such large power reductions at negative shaft angles. Figure 13 shows the optimal normalized lift distribution for $N = 1$ and $N = 3$ control for the cases with the lift offset constrained to 0.3. Also shown are the rubber rotor results with and without the offset constraint, along with the unconstrained $N = 1$ case, all at a shaft angle of $\alpha_{\text{shaft}} = -4^\circ$ deg. We see that the constrained higher harmonic control ($N = 3$) case comes much closer to reproducing the ideal constrained rubber rotor solution than does the conventional control ($N = 1$) case.

Figure 14 shows the optimal azimuthal pitch inputs for the $N = 1$ and $N = 3$ constrained lift offset cases, as well as the $N = 1$ unconstrained case, again at a shaft angle of $\alpha_{\text{shaft}} = -4^\circ$ deg. Note that for the unconstrained case the cyclic variation in the pitch is modest, about 4 deg. For the constrained conventional control, however, the cyclic control is quite large, varying more than 40 deg (of course the blade planform and twist distributions are also different). For the higher harmonic control case, the pitch control is similar to the conventional cyclic control case, except for the very large negative pitch on the retreating side of the rotor. Together with the blade geometry, this higher harmonic control drives the circulation, and hence the force distribution, closer to the constrained rubber rotor optimum.

These results show that constraining the maximum lift offset greatly increases the rotor and vehicle power losses and has a large influence on the optimal shaft angle and rotor design. Higher harmonic control significantly reduces power loss, and, as we have seen, this drives the optimal shaft angle to be more negative; i.e., the rotor generates a more significant share of the propulsive force. Because lift offset constrained rotors using conventional rotor control have higher losses, the optimal shaft angles tend to be positive (toward autorotation) with the propeller producing the majority of the propulsive force. As a result of the difference in optimal shaft angles, the optimal conventional ($N = 1$) and higher harmonic ($N = 3$) rotor designs differ quite substantially.

F. Use of Wing to Supplement Lift

We now add a lifting wing to our vehicle model, to determine its effect on total power requirements. The wing has a span equal to one rotor radius and is located one-half of a rotor radius below the lower rotor of the coaxial system. To ensure the optimization does not make use of excessively small chord values, the chord is constrained to be no smaller than 8% of the wing span. The optimization finds the fixed radial twist and chord distributions of the wing. Note also that in this

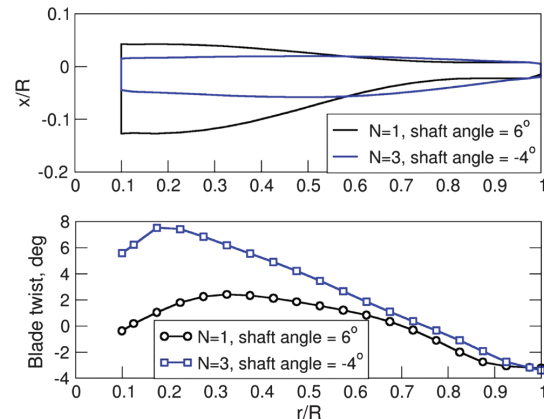


Fig. 12 Optimal rotor blade planform and twist for $N = 1$ and $N = 3$ blade root control with the lateral lift offset constrained to be 0.3.

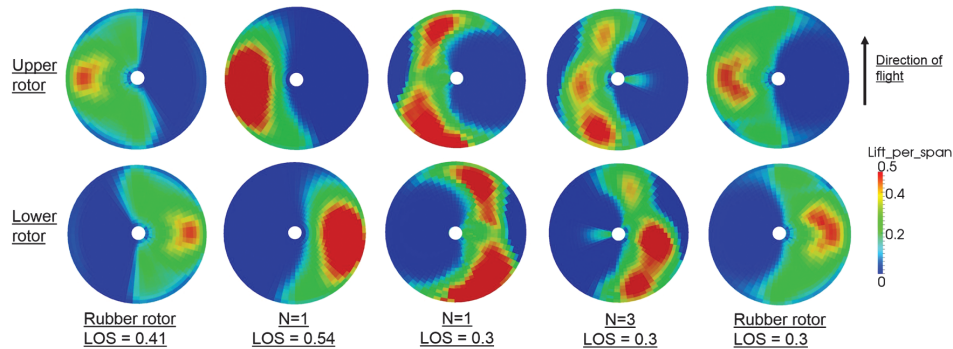


Fig. 13 Optimal normalized lift distribution for $N = 1$ and $N = 3$ control with the lateral lift offset constrained to a value of 0.3.

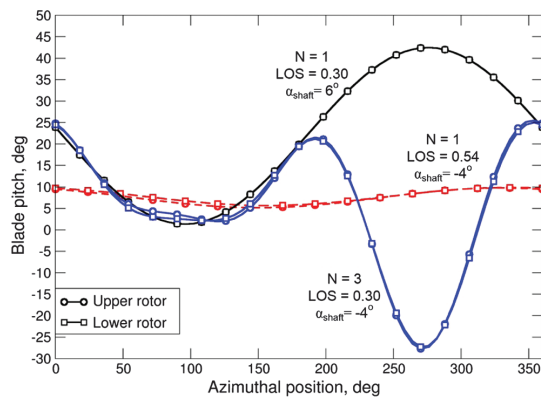


Fig. 14 Azimuthal pitch inputs for the baseline compound constrained to a lateral lift offset of 0.3 using $N = 1$ and $N = 3$ control and the baseline compound unconstrained in the lateral lift offset using $N = 1$ control.

analysis the power loss associated with the wing must be added to the fuselage drag to compute the total propulsive force required for the vehicle. The wing wake is modeled using 20 vortex-lattice ring elements in the spanwise direction and 15 elements in the flight direction per quarter-period of revolution of the rotors.

Figure 15 shows the total power for the winged coaxial configuration over a range of shaft angles, along with the baseline configuration that does not include a wing. The use of a wing reduces the total vehicle power loss by less than 1% and slightly increases the optimal shaft angle to $\alpha_{\text{shaft}} = -3^\circ$. While the presence of a wing decreases the rotor system's power by supplementing its lift, the additional wing power is almost equal to the power saved, resulting in a very similar but very slightly lower total vehicle power. This result is due to the high efficiency of the coaxial rotor and is in contrast to what a similar analysis shows for a single rotor with a wing (e.g., the Cheyenne helicopter), in which the use of a wing significantly reduces power at high advance ratios [2]. Although not plotted, the optimal wing lift share varies nearly linearly between 22% of the vehicle lift at a shaft angle of $\alpha_{\text{shaft}} = -10^\circ$ to a value of approximately 43% at a shaft angle of $\alpha_{\text{shaft}} = 10^\circ$. As the rotor tilts aft toward auto-rotation, the wing provides an increasing proportion of the vehicle's lift.

Figure 16 shows the wing twist distribution at the optimal shaft angle of $\alpha_{\text{shaft}} = -3^\circ$. While not plotted, the optimum wing chord is equal to the minimum chord constraint at all radial stations, resulting in a rectangular planform. Because the wing is so lightly loaded in all cases, it is advantageous to minimize the wing's surface area and viscous drag. Also note that the twist distribution is approximately symmetric, resulting in a nearly zero rolling moment on the wing. For a similar analysis using a single rotor in combination with a

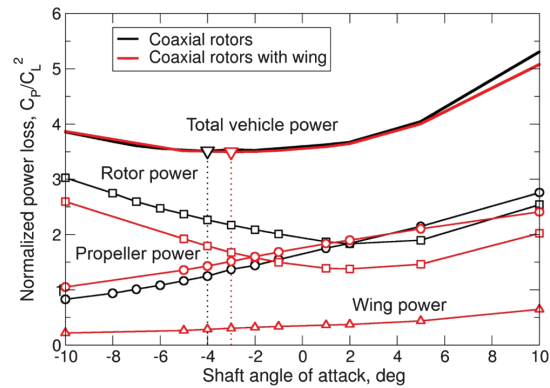


Fig. 15 Power loss of a coaxial rotor with a pusher propeller compared to a coaxial rotor with a pusher propeller and wing.

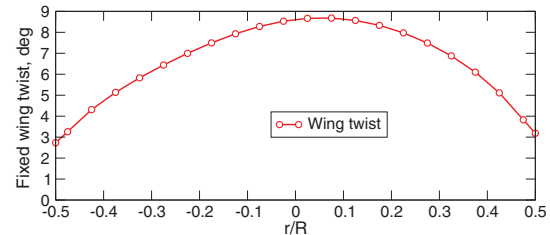


Fig. 16 Optimal wing twist distribution.

wing [2], the optimal wing is asymmetric, with increased values of the twist and chord on the retreating side of the rotor to generate more lift in this region and balance the rolling moment of the rotor. Because the coaxial rotor system is naturally in roll trim, the wing no longer has the primary function of providing roll trim. Rather, the wing simply supplements the rotor's lift and does not provide the dramatic power reductions present when used within a single rotor Cheyenne-style compound.

For the case considered here, the coaxial rotors are unconstrained in lift offset. If a wing were used in a case that also included a lift offset constraint, we might expect higher power savings, as the rotors are significantly less efficient, leading to potential power reductions by using the wing to offload the rotors.

G. Using Propeller to Supplement Lift

Another design feature available to compound helicopters is the ability to use the propeller's thrust to supplement the lift of the vehicle. A similar feature is sometimes used on the tail rotors of

1452

GIOVANETTI AND HALL

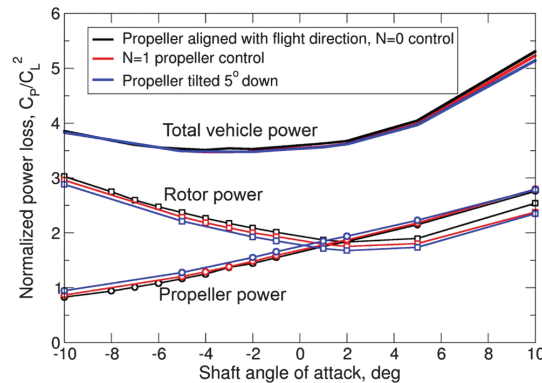


Fig. 17 Normalized power from the baseline compound, a compound using $N = 1$ control on the propeller, and a compound with the propeller angled 5 deg downwards.

conventional helicopters, which can be inclined slightly downward to provide additional lift. There are two ways to allow the propeller to provide lift: inclining the propeller downward by some angle, allowing a component of its thrust to act in the vertical direction, or providing the propeller $N = 1$ cyclic control, allowing it to preferentially generate more or less force on the upstroke or downstroke, producing a small vertical force.

Figure 17 shows the total power vs rotor shaft angle for three cases: a coaxial configuration with the propeller tilted 5 deg downward from horizontal, a coaxial configuration using $N = 1$ pitch control on the propeller, and the baseline coaxial configuration using $N = 0$ pitch control on the propeller and the propeller aligned with the freestream flow. A tilt angle of 5 deg is plotted because it provides the largest power reduction of the various tilt angles considered. Although all three cases have very similar overall power loss, the 5 deg tilted propeller reduces vehicle power loss by 1.5% compared to the baseline, while providing approximately 6% of the total vehicle lift. The $N = 1$ propeller reduces the vehicle power loss by approximately 1% and provides approximately 3% of the total vehicle lift. In both cases, the propeller offloads the rotor system, decreasing the rotor power while increasing the propeller power by a smaller amount.

V. Conclusions

In this paper, we have presented the results for the optimal aerodynamic design and performance of the rotors and propeller (and optionally a lifting wing) of a compound helicopter using counter-rotating coaxial rotors. We analyzed each configuration at an advance ratio of $\mu = 0.85$ over a range of rotor shaft angles. At each shaft angle, all design variables were optimized to determine the minimum power design for a given flight condition. We reached the following conclusions:

- 1) Varying the prescribed propulsive force of the vehicle has a large effect on the optimal shaft angle. In turn, the shaft angle has an effect on the optimal rotor design. Additionally, the optimal shaft angle is highly dependent on the propeller design and efficiency.
- 2) For the cases considered, higher harmonic control reduces total vehicle power loss by as much as 15% compared to conventional control at each configuration's optimal shaft angle. Higher harmonic control results in a more efficient rotor system, leading to an increasingly negative optimal shaft angle. This allows the rotors to provide a larger fraction of the vehicle's propulsive force. The use of higher harmonic control reduces both induced and viscous components of power in part by decreasing the circulation generated on the retreating side of each rotor.
- 3) Both the conventional and higher harmonic control cases achieve efficient forward flight by operating with a high lateral lift offset equal to about one-half the rotor radius.
- 4) Constraining the maximum lateral lift offset of a coaxial rotor system to a smaller value, as may be required in the design of an

actual rotor, significantly increases the rotor and vehicle power losses. For example, when the maximum lateral lift offset is limited to 30% of the rotor radius, the total vehicle power loss increases by over 40% for both conventional and higher harmonic control.

5) Higher harmonic control dramatically reduces rotor power losses at negative shaft angles when the maximum lateral lift offset is constrained. At high positive shaft angles, the benefit of higher harmonic control is greatly diminished.

6) Higher harmonic control results in a dramatically different optimal shaft angle from conventional control when the maximum lateral lift offset is constrained: $\alpha_{\text{shaft}} = -4$ deg for higher harmonic control vs $\alpha_{\text{shaft}} = 6$ deg for conventional control. For a constrained maximum lift offset of 0.3, higher harmonic control reduces total vehicle power loss by 11% compared to conventional control at each configuration's optimal shaft angle.

7) The use of a lifting wing to supplement the rotor lift has a very modest effect on aircraft efficiency, decreasing overall vehicle power loss by less than 1%. The optimal wing lift share varies between 22 and 43% depending on the shaft angle. A wing may be more effective at reducing power if the rotor system is also constrained in lift offset, a case that was not analyzed in this paper.

8) Tilting the propeller downward by 5 deg to supplement the rotor system's lift reduces the total vehicle power loss by 1.5%. Using conventional cyclic control on the propeller also allows the propeller to generate a small lift component, reducing vehicle power loss by 1.0%.

Acknowledgments

The first author is funded through a National Defense Science and Engineering Graduate Fellowship. This material is based upon work supported by the U.S. Army Research Office under award number W911NF-13-1-0469. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the view of the U.S. Army Research Office.

References

- [1] Ormiston, R. A., "Applications of the Induced Power Model and Performance to Conventional and Advanced Rotorcraft," *Proceedings of the American Helicopter Society Aeromechanics Specialists Conference*, American Helicopter Soc., Alexandria, VA, Jan. 2010.
- [2] Hall, K. C., and Hall, S. R., "A Variational Method for Computing the Optimal Aerodynamic Performance of Conventional and Compound Helicopters," *Journal of the American Helicopter Society*, Vol. 55, No. 4, 2010, pp. 42006-1-42006-16. doi:10.4050/JAHS.55.042006
- [3] Orchard, M., and Newman, S., "The Fundamental Configuration and Design of the Compound Helicopter," *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, Vol. 217, No. 6, 2013, pp. 297-315. doi:10.1243/095441003772538570
- [4] Hall, K. C., and Giovanetti, E. B., "Minimum Power Requirements and Optimal Rotor Design for Conventional and Compound Helicopters Using Higher Harmonic Control," *Proceedings of the American Helicopter Society 69th Annual Forum*, American Helicopter Soc., Alexandria, VA, May 2013.
- [5] Bagai, A., "Aerodynamic Design of the X2 Technology Demonstrator Main Rotor Blade," *Proceedings of the American Helicopter Society 64th Annual Forum*, American Helicopter Soc., Alexandria, VA, April-May 2008.
- [6] Ormiston, R. A., "Induced Power of the Helicopter Rotor," *Proceedings of the American Helicopter Society 60th Annual Forum*, American Helicopter Soc., Alexandria, VA, June 2004.
- [7] Ormiston, R. A., "Further Investigations of Helicopter Rotor Induced Power," *Proceedings of the American Helicopter Society 61st Annual Forum*, American Helicopter Soc., Alexandria, VA, June 2005.
- [8] Ormiston, R. A., "An Analytical Formulation for Lifting Rotor Induced Power," *Proceedings of the American Helicopter Society 65th Annual Forum*, American Helicopter Soc., Alexandria, VA, May 2009.
- [9] Johnson, W., "Influence of Lift Offset on Rotorcraft Performance," *Proceedings of the American Helicopter Society Future Vertical Lift Aircraft Design Conference*, American Helicopter Soc., Alexandria, VA, Jan. 2010.

- [10] Johnson, W., Moodie, A. M., and Yeo, H., "Design and Performance of Lift-Offset Rotorcraft for Short-Haul Missions," *Proceedings of the American Helicopter Society Aeromechanics Specialists Conference*, American Helicopter Soc., Alexandria, VA, Jan. 2012.
- [11] Yeo, H., and Johnson, W., "Optimum Design of a Compound Helicopter," *Journal of Aircraft*, Vol. 46, No. 4, 2009, pp. 1210–1221. doi:10.2514/1.40101
- [12] Russell, C., and Johnson, W., "Exploration of Configuration Options for a Large Civil Compound Helicopter," *Proceedings of the American Helicopter Society 69th Annual Forum*, American Helicopter Soc., Alexandria, VA, May 2013.
- [13] Rand, O., and Khromov, V., "Compound Helicopter: Insight and Optimization," *Proceedings of the American Helicopter Society 69th Annual Forum*, American Helicopter Soc., Alexandria, VA, May 2013.
- [14] Ormiston, R. A., "On the Definitions of Rotor and Rotorcraft Power and Performance," *Proceedings of the American Helicopter Society 69th Annual Forum*, American Helicopter Soc., Alexandria, VA, May 2013.

Appendix II: A Variational Approach to Multipoint Aerodynamic Optimization of Conventional and Coaxial Helicopter Rotors

A Variational Approach to Multipoint Aerodynamic Optimization of Conventional and Coaxial Helicopter Rotors

Eli B. Giovanetti

Graduate Research Assistant
Mechanical Engineering and Materials Science
Duke University
Durham, NC, U.S.

Kenneth C. Hall

Julian Francis Abele Professor of
Mechanical Engineering and Materials Science
Duke University
Durham, NC, U.S.

ABSTRACT

We present a variational approach to the multipoint aerodynamic design optimization of conventional and coaxial helicopter rotors. The optimal design problem is cast as a variational statement that minimizes the weighted sum of induced and viscous power losses between two flight conditions for prescribed vehicle trim constraints at each flight condition. The resulting nonlinear constrained optimization problem is solved via Newton iteration and may be used to map the Pareto frontier, i.e., the set of rotor designs (radial twist and chord distributions and harmonic blade pitch inputs) for which it is not possible to improve upon the performance in one flight conditions without degrading performance in the other. The two flight conditions can represent different advance ratios (including hover), disk loadings, altitude, or other conditions of interest. For forward flight computations, the rotor control inputs are related to the circulation on the blades (and in the wake) through a lifting-line/vortex-lattice method that accounts for nonlinear sectional lift and drag polars. For hovering flight, the rotor performance is analyzed using Blade Element Momentum Theory. We map the Pareto frontier for both a cruise/cruise and hover/cruise multipoint optimization, and show that significant tradeoffs must be made in designing a rotor to balance performance between two flight conditions, particularly hover and high speed forward flight. We also show that higher harmonic control is capable of reducing rotor power and improving the Pareto frontier, particularly for coaxial rotors. Finally, we present a number of rotor designs that best balance performance between two flight conditions.

NOTATION

A	= rotor disk area, πR^2	P	= rotor total power loss
$\mathbf{A}$	= matrix relating control inputs Θ to circulation Γ	Q	= torque
B	= number of blades	r	= radial position
$\mathbf{B}$	= matrix relating force $\mathbf{F}$ to circulation Γ	R	= rotor radius
C_L	= lift coefficient, $L/\rho A \Omega^2 R^2$	$\mathbf{R}$	= vector function describing the relationship between the circulation Γ and the control inputs θ
C_T	= thrust coefficient, $T/\rho A \Omega^2 R^2$	T	= thrust
C_Q	= torque coefficient, $Q/\rho A \Omega^2 R^3$	U	= axial inflow velocity
C_P	= power coefficient, $P/\rho A \Omega^3 R^3$	w	= axial induced wash velocity
c_d	= airfoil sectional drag coefficient	α	= weighting parameter
c_ℓ	= airfoil sectional lift coefficient	β	= weighting parameter, $1 - \alpha$
$c_{\ell\alpha}$	= lift curve slope	δ	= variational operator
c	= blade chord	Γ	= circulation
D	= drag	θ	= blade pitch angle
$\mathbf{D}$	= matrix relating moment $\mathbf{M}$ to circulation Γ	Θ	= vector of control inputs
F	= Prandtl tip loss factor	λ	= total axial inflow velocity, $U + w$
$\mathbf{F}$	= time-averaged aerodynamic force vector	λ_F	= Lagrange multiplier for force constraint
L	= lift	λ_M	= Lagrange multiplier for moment constraint
$\mathbf{K}$	= quadratic power matrix	λ_R	= Lagrange multiplier for vector function $\mathbf{R}$
$\mathbf{k}$	= linear power vector	λ_Q	= Lagrange multiplier for torque balance constraint
$\mathbf{M}$	= time-averaged aerodynamic moment vector	λ_T	= Lagrange multiplier for thrust constraint
N	= number of harmonics in higher harmonic control	μ	= advance ratio, $V/\Omega R$
		Π	= power Lagrangian
		σ	= rotor solidity, $Bc/\pi R$
		ϕ	= relative flow angle
		Ω	= rotor rotational speed

Presented at the AHS 71st Annual Forum, Virginia Beach, Virginia, May 5–7, 2015. Copyright © 2015 by the American Helicopter Society International, Inc. All rights reserved.

$\bar{(\)}$ = non-dimensional quantity

Subscripts:

R = prescribed value

v = viscous

tot = total

u = upper

ℓ = lower

c = contracted quantity

INTRODUCTION

Aerodynamic optimization of conventional and coaxial helicopter rotors in forward flight is an important issue in the efficient design of advanced helicopter configurations. Hall and Giovanetti (Refs. 1, 2) developed a technique for determining the optimal blade twist, chord distribution, and periodic blade pitch inputs for a conventional or coaxial rotor in high speed forward flight, using either conventional collective and cyclic or higher harmonic control. The method finds the rotor design subject to trim constraints that yields the minimum total power loss including both induced and profile (viscous and compressible) losses at a single advance ratio. In the design of a practical rotor, however, it may be advantageous to optimize rotor design to balance performance between two or more flight conditions, for example, between two different advance ratios or between a forward advance ratio and hover.

There have been a number previous investigations on the effect of rotor blade shape and geometry on helicopter rotor performance at multiple flight conditions. Johnson (Ref. 3), Yeo and Johnson (Refs. 5, 7), Johnson, Moodie, and Yeo (Ref. 4), and Moodie and Johnson (Ref. 6) have all sought to optimize various rotor parameters – including blade twist and chord distributions – of a compound helicopter configuration to balance hover and forward flight performance. In each of these studies, a rotor design that optimizes the performance between two conditions was typically determined using parametric variation informed by the sensitivities of a limited number of design variables describing blade chord and twist. The performance of each combination of twist variables was then examined graphically in a plot of forward flight performance versus hover performance.

Rand and Khromov (Ref. 8) describe a method for determining the Pareto frontier by first randomly varying important design parameters and plotting the hover versus cruise performance. (The Pareto frontier is the set of rotor designs for which it is impossible to improve upon the performance in one of the two flight conditions without degrading performance in the other (Ref. 9)). Random designs that are “non-dominated,” i.e., near the apparent Pareto frontier, were then further optimized via gradient based optimization techniques.

In this paper, we describe a method for performing a multipoint optimization, finding the Pareto frontier for two flight conditions. The problem is posed as a constrained optimization problem, and solved directly using Newton iteration. The

method we describe is capable of mapping the entire Pareto frontier directly for any arbitrary number of design variables, eliminating the need to evaluate the performance of a rotor for hundreds or thousands of random parametric variations. We apply the method to both conventional and coaxial rotors, for both hovering and forward flight conditions, and for both conventional (collective and cyclic) blade pitch control and higher harmonic control.

APPROACH: CRUISE/CRUISE MULTIPOINT OPTIMIZATION

We present here a brief summary of the single point optimization methodology before extending it to multipoint design. For more detail on the single point optimization approach, see Hall and Giovanetti (Ref. 1).

For a single point optimization, we seek to minimize the rotor power subject to constraints that the rotor produce a certain lift and/or thrust, and that the helicopter be in pitch and roll trim. The aerodynamic model we use is a lifting-line/vortex-lattice model, that is, the blades are modeled with a single line of vortex filaments with a lattice of quadrilateral vortex rings to model the circulation in a prescribed wake to model inviscid effects. Sectional drag polars modeled using C81 tables are used to account for profile losses, including both viscous and compressible losses.

Using this lift-line/vortex-lattice model, the unknown circulation strengths are related to the blade geometry and control inputs through a set of nonlinear equations of the form

$$\mathbf{R}(\mathbf{\Gamma}, \mathbf{\Theta}) \quad (1)$$

where $\mathbf{\Gamma}$ is the vector of circulation values at a number of discrete spanwise and azimuthal locations, and $\mathbf{\Theta}$ is the vector of rotor design variables and control inputs. The vector of design variables $\mathbf{\Theta}$ can include fixed blade twist and blade chord distributions along the blade span, and conventional or higher harmonic blade pitch control inputs.

The aerodynamic forces and moments acting on the rotor can be expressed in terms of the circulation, i.e.,

$$\mathbf{F} = \mathbf{B}\mathbf{\Gamma} \quad (2)$$

$$\mathbf{M} = \mathbf{D}\mathbf{\Gamma} \quad (3)$$

where $\mathbf{B}$ and $\mathbf{D}$ are matrices relating the circulation distribution to the forces and moments generated, respectively.

Finally, the total power loss for the rotor (the wasted power) is given by

$$P = \frac{1}{2} \mathbf{\Gamma}^T \mathbf{K} \mathbf{\Gamma} + P_v \quad (4)$$

The matrix $\mathbf{K}$ relates the circulation distribution $\mathbf{\Gamma}$ to the induced power P_i . P_v is the profile power loss, and accounts for both viscous and compressible contributions to the drag of the airfoil. Sectional drag coefficients are based on C81 data tables and are modified to account for the effect of both

local angle of attack and local Mach number. Unlike the induced power, which is quadratic in the circulation, the viscous power is a more complicated nonlinear function of the circulation, the rotor geometry, and the pitch inputs.

To cast the problem as a variational problem, we adjoin to the power (the cost function to be minimized) the force and moment constraints using Lagrange multipliers, that is,

$$\Pi = \frac{1}{2} \mathbf{\Gamma}^T \mathbf{K} \mathbf{\Gamma} + P_v + \boldsymbol{\lambda}_F^T \cdot (\mathbf{B} \mathbf{\Gamma} - \mathbf{F}_R) + \boldsymbol{\lambda}_M^T \cdot (\mathbf{D} \mathbf{\Gamma} - \mathbf{M}_R) + \boldsymbol{\lambda}_R^T \cdot \mathbf{R}(\mathbf{\Gamma}, \Theta) \quad (5)$$

where $\mathbf{M}_R$ and $\mathbf{F}_R$ are the prescribed forces and moments. To find the constrained optimum, one takes the variation of Eq. (5) and sets the result to zero for arbitrary variations in the unknown circulation, control inputs, and Lagrange multipliers. The result is a large set of nonlinear algebraic equations that are solved numerically for the optimum.

To perform a multipoint optimization, the process is similar except that now we consider two flight conditions denoted by the subscripts 1 and 2. These two flight conditions could represent different advance ratios, tip Mach numbers, disk loadings, elevation, or other conditions of interest. Of course, the blade geometries (chord and twist distributions) will be the same for the two flight conditions. The azimuthal variation of blade pitch, on the other hand, will necessarily be different for the two flight conditions to maintain flight trim. Thus, the total set of design variables describing the rotor geometry and control inputs for the two flight conditions are

$$\Theta = \begin{Bmatrix} \Theta_{\text{twist}} \\ \Theta_{\text{chord}} \\ \Theta_{\text{root1}} \\ \Theta_{\text{root2}} \end{Bmatrix} \quad (6)$$

where Θ_{twist} and Θ_{chord} are the design variables that describe the common fixed blade twist and chord distributions, respectively. Θ_{root1} and Θ_{root2} describe the root pitch inputs (conventional or higher harmonic control) at flight condition 1 and 2, respectively, which are independent and have no effect on rotor performance at the other flight condition.

Because the two flight conditions in general occur at different advance ratios, the wake geometries for the two conditions will be different, and separate vortex lattice grids must be used

for each. Therefore, the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{K}$, as well as the circulation vector $\mathbf{\Gamma}$, will differ for the two conditions, and are distinguished by the subscripts 1 or 2.

In the multipoint optimization, we minimize a weighted sum of the power loss for the two flight conditions, with a weight of α applied to flight condition 1 and β to flight condition 2, where $\beta = 1 - \alpha$. When $\alpha = 1$, for example, the optimization minimizes only the power at the first flight condition, while satisfying the trim constraints and evaluating performance at both flight conditions. By varying α between 0 and 1, we are able to find the Pareto frontier.

As before, we seek to minimize the (weighted) power loss, while satisfying the trim constraints at the two flight conditions. Thus we adjoin separate (unweighted) moment and force constraints for each flight condition to the power loss. The resulting Lagrangian power is given by

$$\begin{aligned} \Pi = & \alpha \left(\frac{1}{2} \mathbf{\Gamma}_1^T \mathbf{K}_1 \mathbf{\Gamma}_1 + P_{v1} \right) + \beta \left(\frac{1}{2} \mathbf{\Gamma}_2^T \mathbf{K}_2 \mathbf{\Gamma}_2 + P_{v2} \right) \\ & + \boldsymbol{\lambda}_{F1}^T \cdot (\mathbf{B}_1 \mathbf{\Gamma}_1 - \mathbf{F}_{R1}) + \boldsymbol{\lambda}_{F2}^T \cdot (\mathbf{B}_2 \mathbf{\Gamma}_2 - \mathbf{F}_{R2}) \\ & + \boldsymbol{\lambda}_{M1}^T \cdot (\mathbf{D}_1 \mathbf{\Gamma}_1 - \mathbf{M}_{R1}) + \boldsymbol{\lambda}_{M2}^T \cdot (\mathbf{D}_2 \mathbf{\Gamma}_2 - \mathbf{M}_{R2}) \\ & + \boldsymbol{\lambda}_{R1} \cdot \mathbf{R}_1(\mathbf{\Gamma}_1, \Theta) + \boldsymbol{\lambda}_{R2} \cdot \mathbf{R}_2(\mathbf{\Gamma}_2, \Theta) \end{aligned} \quad (7)$$

where

$$\mathbf{R}_1(\mathbf{\Gamma}_1, \Theta) = \mathbf{R}_2(\mathbf{\Gamma}_2, \Theta) = 0 \quad (8)$$

We can approximate the change in the circulation due to a change in design variables at each flight condition by

$$\begin{aligned} \mathbf{\Gamma}_1 & \approx \mathbf{\Gamma}_{01} + \mathbf{A}_1 \Delta \Theta \\ \mathbf{\Gamma}_2 & \approx \mathbf{\Gamma}_{02} + \mathbf{A}_2 \Delta \Theta \end{aligned} \quad (9)$$

where $\mathbf{\Gamma}_{01}$ and $\mathbf{\Gamma}_{02}$ are the circulation distributions at flight conditions 1 and 2 at the current estimate of the design variables Θ . The matrix $\mathbf{A}_1$ and $\mathbf{A}_2$ are obtained by linearizing Eq. (8).

Substituting Eq. (9) into Eq. (7) and setting the variation of Π to zero for arbitrary changes in the design variables and Lagrange multipliers gives the desired system of linearized equations for the multipoint optimization, i.e.,

$$\begin{bmatrix} \frac{1}{2} \alpha \mathbf{A}_1^T (\mathbf{K}_1 + \mathbf{K}_1^T) \mathbf{A}_1 + \frac{1}{2} \beta \mathbf{A}_2^T (\mathbf{K}_2 + \mathbf{K}_2^T) \mathbf{A}_2 & \mathbf{A}_1^T \mathbf{B}_1^T & \mathbf{A}_1^T \mathbf{D}_1^T & \mathbf{A}_2^T \mathbf{B}_2^T & \mathbf{A}_2^T \mathbf{D}_2^T \end{bmatrix} \begin{Bmatrix} \Delta \Theta \\ \boldsymbol{\lambda}_{F1} \\ \boldsymbol{\lambda}_{M1} \\ \boldsymbol{\lambda}_{F2} \\ \boldsymbol{\lambda}_{M2} \end{Bmatrix} \approx \begin{Bmatrix} -\alpha (\mathbf{A}_1^T \mathbf{K}_1 \mathbf{\Gamma}_{01} - \mathbf{k}_{v1}) - \beta (\mathbf{A}_2^T \mathbf{K}_2 \mathbf{\Gamma}_{02} - \mathbf{k}_{v2}) \\ \mathbf{F}_{R1} - \mathbf{B}_1 \mathbf{\Gamma}_{01} \\ \mathbf{M}_{R1} - \mathbf{D}_1 \mathbf{\Gamma}_{01} \\ \mathbf{F}_{R2} - \mathbf{B}_2 \mathbf{\Gamma}_{02} \\ \mathbf{M}_{R2} - \mathbf{D}_2 \mathbf{\Gamma}_{02} \end{Bmatrix} \quad (10)$$

The terms $\mathbf{k}_{v1}$ and $\mathbf{k}_{v2}$ are the gradients of the profile power terms P_{v1} and P_{v2} with respect to the change in design variables.

Equation (10) describes (approximately) a single Newton step in the direction of the constrained minimum. Because we use piecewise linear C81 tables to describe the profile drag, we omit the Hessian matrix terms associated with the profile power that appears on the left-hand side. Equation (10) is solved for the change in the design variables, $\Delta\Theta$, and the design variables Θ are then updated. We then update the circulation at each flight condition such that $\mathbf{R}_1(\Gamma_1, \Theta) = 0$ and $\mathbf{R}_2(\Gamma_2, \Theta) = 0$, i.e., we solve for Γ_1 and Γ_2 using the vortex lattice solver. Next, we update the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ at the new set of design variables, and continue iterating until the design variables have converged to a set that satisfies the trim constraints and minimizes the weighted sum of total power. Finally, we repeat this optimization procedure for a range of values of α between 0 and 1 to map out the Pareto frontier. Note that we also include an inequality constraint on the minimum allowable chord value (not shown in Eq. 10) to prevent unrealistically small or negative chord distributions. When activated, this constraint is implemented using Mathematical Programming via Augmented Lagrangians.

COMPUTATIONAL RESULTS: CRUISE/CRUISE MULTIPOINT OPTIMIZATION

In this section, we apply the multipoint optimization technique to two different rotor systems – a conventional four-bladed rotor and a coaxial rotor system comprised of an upper and lower four-bladed rotor. For each rotor system, we perform a multipoint optimization for two different forward advance ratios. (In a later section, we perform a multipoint optimization of these same rotors for a combination of hover and forward flight conditions.)

Conventional Rotor

We first perform an optimization analysis for a conventional rotor in forward flight, which is intended to roughly approximate the AH-64 Apache rotor. The coefficient of lift of the vehicle is prescribed to be $C_L = 0.00926$. The rotor has a 10% root cutout and operates at a shaft angle of -5° (tilted forward in the flight direction). We use a NACA0012 airfoil from root to tip and use C81 data tables to model the sectional coefficients of lift and drag based on the local angle of attack and local normal Mach number. Note that this analysis does not account for blade dynamics such as flapping or lead/lag motion (we assume a rigid rotor). Table 1 lists the parameters for the two design points analyzed: a (low speed) advance ratio of $\mu = 0.2$, and a (high speed) advance ratio of $\mu = 0.5$.

Using the techniques described in the previous section, we computed the optimal rotors along the Pareto frontier. For these calculations, we used a vortex-lattice mesh with 18 quadrilateral vortex ring elements in the spanwise direction

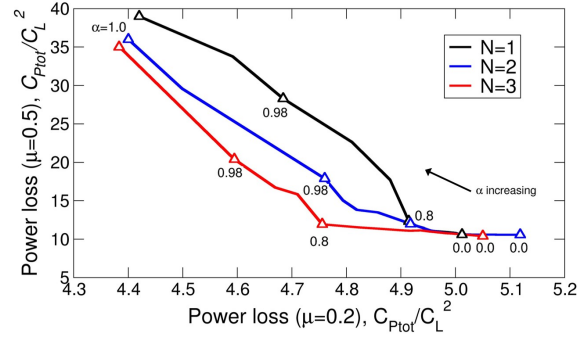


Fig. 1. Pareto frontiers for a conventional rotor using $N = 1$, $N = 2$, and $N = 3$ harmonic control.

and 15 elements in the azimuthal direction per quarter turn for each of the four blades for a total of 1080 vortex ring elements of unknown strength at each flight condition. Images of these vortex rings are repeated for multiple quarter turn revolutions of the rotor to form a complete wake extending 40 quarter revolutions from the primary lattice.

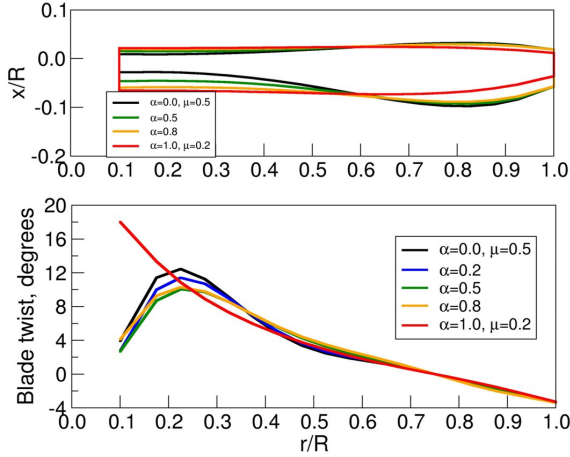
The computed power loss Pareto frontiers are shown in Figure 1 for varying levels of higher harmonic control. The notation $N = 1$ means harmonic blade pitch controls up to 1/rev are used (collective and cyclic); $N = 2$ means harmonics up to 2/rev control are used, and so on. Plotted is the total power loss at $\mu = 0.5$ versus the total power loss at $\mu = 0.2$ along the Pareto frontier. The value of the weighting parameter α decreases as these lines are traversed from left to right, shifting the weighting of the power loss from the $\mu = 0.2$ to the $\mu = 0.5$ flight condition.

We see in Figure 1 that there are significant performance tradeoffs to be made when designing a conventional rotor operating at these two disparate flight conditions. Consider first the case of conventional swashplate control ($N = 1$). The most efficient rotor at $\mu = 0.2$ ($\alpha = 1$) is highly inefficient at the high speed ($\mu = 0.5$) flight condition, with power losses nearly four times the losses for a rotor design to operate only at $\mu = 0.5$ ($\alpha = 0$). Selecting an intermediate value of α that more evenly weights the performance of the rotor at both flights conditions dramatically reduces the total power loss for the high speed condition, with an attendant modest increase in total power loss at the low speed condition. For example, using the rotor found for $\alpha = 0.8$ produces a rotor with near minimum power loss at the high speed flight condition ($\mu = 0.5$), with a power loss at the low speed condition ($\mu = 0.2$) only 11% greater than a single point design at $\mu = 0.2$.

Higher harmonic control further reduces the total power loss, pushing the Pareto frontier down and to the left. Note that adding additional degrees of freedom to the design space, in this case higher harmonics of blade pitch control, will always improve the Pareto frontier – or at worst leave it unchanged. Higher harmonic control is seen here to be particularly effective at moving the Pareto frontier for values of α in the range of 0.8 to 0.98. In particular, the $\alpha = 0.8$ solution is seen to move the “knee” or “corner” of the frontier well to the

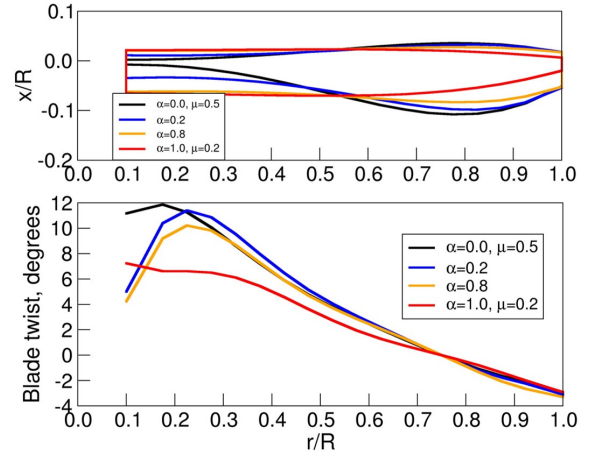
Table 1. Conventional Rotor Parameters.

	Hover	Flight Condition 1	Flight Condition 2
Advance ratio, μ	0.0	0.20	0.50
Relative tip Mach number	0.0	0.65	0.90
Coefficient of lift	0.00926	0.00926	0.00926
Shaft angle	0.0°	-5.0°	-5.0°

**Fig. 2. Optimal rotor designs for $N = 1$ control with varying values of α for a conventional rotor.**

left, producing a rotor with nearly minimum total power loss at the high speed flight condition, with only a small increase in power over the single point minimum at the low speed condition.

Figures 2 and 3 show the optimal rotor designs using $N = 1$ and $N = 3$ harmonic blade pitch control, respectively, for various values of α . For both levels of harmonic control, the $\alpha = 1.0$ blade (designed to minimize power at $\mu = 0.2$) has a monotonically decreasing twist distribution from root to tip. The rotors optimized using values of α below 1.0, thus including some weighting for the $\mu = 0.5$ flight condition total power loss, instead use a positive twist gradient on the inner 10% of the rotor. Additionally, the $\alpha = 1.0$ (low speed limit) planforms (for both $N = 1$ and $N = 3$) have a large chord on the inboard portion of the blade that tapers towards the tip. The $\alpha = 0.0$ (high speed limit) planforms have narrow chords at the root, with chords increasing to a maximum near a radius of 80% the tip radius. Especially for the $N = 3$ case, the chord at the root is unrealistically small. (In an earlier paper, the authors have applied root chord constraints to prevent this problem (Ref. 1).) For multipoint designs, e.g., $N = 3$ and $\alpha = 0.8$, the blade planform appears quite reasonable, with a nearly constant chord, with a slight bulge in the chord at a radius of 80% and a slight taper outboard.

**Fig. 3. Optimal rotor designs for $N = 3$ control with varying values of α for a conventional rotor.**

Coaxial Rotor

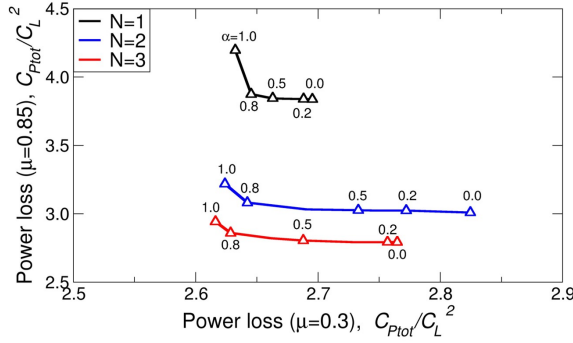
Next, we perform a multipoint optimization analysis for a slowed rotor coaxial system of the type found on the Sikorsky X2 Technology Demonstrator (X2 TD), with two coaxial counterrotating four bladed rotors. We optimize the performance between two design points: a low speed flight condition ($\mu = 0.30$), and the high speed design intent advance ratio ($\mu = 0.85$) as described by Bagai (Ref. 10). Each rotor has a root cutout of 10% and a vertical separation of 10% of the rotor diameter. Again, we neglect blade dynamics, a reasonable assumption for the rigid rotors used on the X2 TD and other coaxial designs lift offset rotors. Note that we allow the geometries of the upper and lower rotors to differ from one another.

Because of the difference in rotor speed for the two advance ratios, the nondimensional disk loading will vary between the two advance ratios. Additionally, the tip Mach number will vary based on rotor rotation rate and vehicle speed. Relevant parameters for the two design conditions were estimated using Figures 9 and 10 in a paper by Bagai (Ref. 10) and are shown in Table 2.

The power loss Pareto frontier is shown in Figure 4 for varying levels of harmonic control. Higher harmonic control of the azimuthal blade pitch ($N = 2$ and $N = 3$) dramatically reduces the power loss of optimal rotors at the high-speed flight condition ($\mu = 0.85$). While higher harmonic control does not significantly affect the shape of the Pareto frontier, it

Table 2. Coaxial Rotor Parameters.

	Hover	Flight Condition 1	Flight Condition 2
Advance ratio, μ	0.0	0.30	0.85
Relative tip Mach number	0.0	0.75	0.90
Coefficient of lift, C_L	0.0166	0.0166	0.0232
Shaft angle	0.0°	-5.0°	-5.0°

**Fig. 4. Pareto frontiers for a coaxial rotor using $N = 1$, $N = 2$, and $N = 3$ harmonic control.**

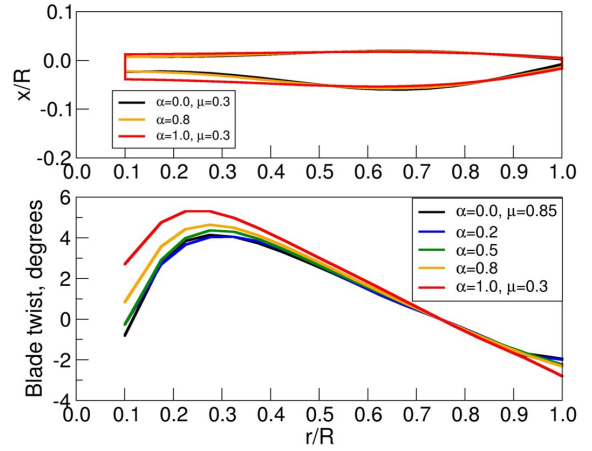
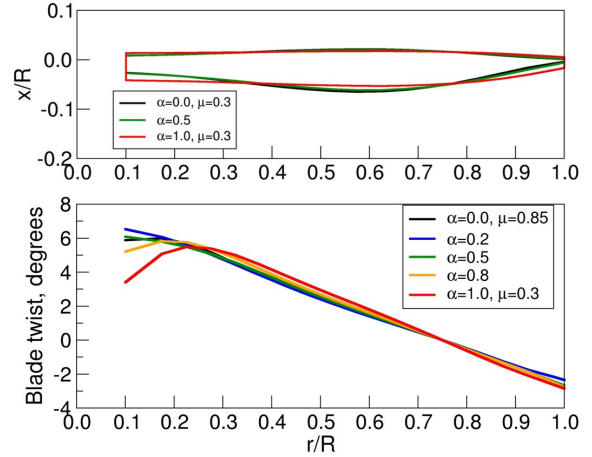
significantly reduces power at the high-speed flight condition, improving performance across all values of α . The reductions in power at the low-speed condition ($\mu = 0.3$) are more modest, and only for rotor designs heavily weighted for good performance at the low-speed condition ($\alpha \approx 1$).

Also, each of the Pareto frontiers has a fairly sharp “corner” or “knee,” which corresponds to a value of $\alpha \approx 0.8$. For this weighting, the performance of the multipoint optimized coaxial rotor at the two different advance ratios is nearly as good as the performance of rotors optimized separately at these two flight conditions. In other words, there is only a small penalty in performance for designing for the two flight conditions.

Figures 5 and 6 show the optimal rotor design for $N = 1$ and $N = 3$ harmonic control at varying values of α . Note that while there are some small changes in planform and twist distribution with varying α , in general each of the rotors looks very similar. The similarity of each of these rotors is expected based on Figure 4, as the multipoint design would likely be unable to achieve near-optimal performance at both flight conditions if the single point optimal rotors were significantly different. This is again in contrast to the conventional rotor analyzed in the previous section, in which the $\alpha = 0$ and $\alpha = 1$ rotors had significantly different twist and chord distributions that could not be successfully combined to achieve close to optimal performance at both flight conditions.

APPROACH: HOVER/CRUISE MULTIPOINT OPTIMIZATION

In this section, we describe a multipoint optimization algorithm for the design of a rotor operating at two flight con-

**Fig. 5. Optimal rotor designs for $N = 1$ control with varying values of α for a coaxial rotor.****Fig. 6. Optimal rotor designs for $N = 3$ control with varying values of α for a coaxial rotor.**

ditions, one in hover and the other in cruise. The hover flight condition represents an additional challenge, because the hover performance cannot be modeled using the (prescribed) rigid vortex-lattice wake analysis used for forward flight conditions.

In selecting a method of hover analysis, we seek an approach that is computationally inexpensive, extendable to coaxial rotors, and reasonably accurate. There have been a number of such models developed to compute the per-

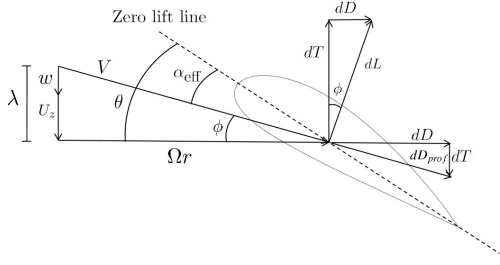


Fig. 7. Geometry of the flow at a blade section of a rotor, including only the axial component of velocity in both the inflow and induced wash.

formance of hovering conventional and coaxial helicopters. Leishman and Ananthan (Ref. 11) developed a combined Blade Element Momentum Theory (BEMT) for coaxial rotors in hover and axial flight, accounting for axial induced velocities, but not swirl. Juhasz (Ref. 14) et al. compared the results of BEMT, free vortex methods, and computational fluid dynamic techniques applied to a coaxial hovering rotor and found that BEMT accurately predicts the thrust and torque of each rotor. Because of its accuracy, simplicity, and computational efficiency, we use BEMT to analyze the hovering flight conditions for both conventional and coaxial rotors.

For this analysis, we use the Prandtl tip loss factor to account for the presence of a finite number of blades. We account for the induced velocities of the upper rotor on the inner portion of the lower rotor and assume the upper rotor and outer portion of the lower rotor are unaffected by the mutual interference of the rotors. However, a more refined accounting of the mutual interferences of the rotors can be included in this analysis, for example by using the influence coefficient method described by Rand and Khromov (Ref. 13).

We outline the fundamental equations and assumptions for BEMT as applied to coaxial rotors. More details can be found in Leishman and Ananthan (Ref. 11). The approach assumes that swirl losses in the flow are small and can therefore be ignored. The flow geometry for the no-swirl configuration at a given blade section is shown in Figure 7. V represents the total relative velocity at the blade, w the axial induced wash, U the axial velocity of the rotor (or inflow velocity), Ω is the rotational rate of the rotor, r the radial position, θ the blade pitch angle, ϕ the inflow angle, and α_{eff} the effective angle of attack.

The total axial inflow velocity at the rotor face is the sum of the axial flight velocity and the induced axial flow, i.e., $\lambda = w + U$. For convenience, we define the following normalized variables:

$$\bar{r} = \frac{r}{R} \quad \bar{U} = \frac{U}{\Omega R} \quad \bar{w} = \frac{w}{\Omega R} \quad \bar{V} = \frac{V}{\Omega R} \quad \bar{\lambda} = \frac{\lambda}{\Omega R} \quad (11)$$

The axial inflow is assumed to be small compared to the rotor velocity, so we make the small angle assumption $\cos \phi \approx 1$ so that $V \approx \Omega R$.

Next, we apply (inviscid) momentum theory to find the differential thrust coefficient dC_T and differential torque dC_Q at

each radial station of a rotor. For a single rotor, this gives

$$dC_T = 4F\bar{\lambda}\bar{w}\bar{r} d\bar{r} = 4F\bar{\lambda}(\bar{\lambda} - \bar{U})\bar{r} d\bar{r} \quad (12)$$

$$dC_Q = 4F\bar{\lambda}^2\bar{w}\bar{r} d\bar{r} = 4F\bar{\lambda}^2(\bar{\lambda} - \bar{U})\bar{r} d\bar{r} \quad (13)$$

where $F = F(\bar{\lambda}, \bar{r}, B)$ is the Prandtl tip loss factor, which accounts approximately for the tip losses associated with non-axisymmetric flow in rotors with a finite number of blades.

We also know from blade element theory that the differential thrust may be expressed in terms of the airfoil sectional properties and coefficient of lift, the relative inflow velocity, and the relative angle of attack. In terms of non-dimensional parameters,

$$dC_T = \frac{1}{2} \sigma C_{l\alpha} (\theta \bar{r}^2 - \bar{\lambda} \bar{r}) d\bar{r} \quad (14)$$

where $\sigma = Nc/\pi R$ is the rotor solidity and $C_{l\alpha}$ is the lift curve slope. Note that in computing the angle of attack, we have made the small angle assumption for ϕ so that $\tan \phi \approx \phi \approx \bar{\lambda}/\bar{r}$. In Eq. (14), we also assume that the component of airfoil sectional drag acting in the axial direction is small compared to the sectional lift, and can be ignored.

For a single rotor – or the upper rotor of a coaxial rotor system – we can equate the two expressions for differential thrust, Eqs. (12) and (14), giving a quadratic equation for the unknown inflow, $\bar{\lambda}$. This quadratic equation is solved to obtain the inflow at each radial station as a function of the local twist θ , solidity σ , and axial advance ratio $\bar{U}$, i.e.,

$$\bar{\lambda}(\bar{r}, \bar{U}) = \sqrt{\left(\frac{\sigma C_{l\alpha}}{16F} - \frac{\bar{U}}{2}\right)^2 + \frac{\sigma C_{l\alpha}}{8F} \theta \bar{r}} - \left(\frac{\sigma C_{l\alpha}}{16F} - \frac{\bar{U}}{2}\right) \quad (15)$$

Of course, the Prandtl tip loss factor F is itself a function of $\bar{\lambda}$, so Eq. (15) is solved iteratively by assuming an initial value for $\bar{\lambda}$, computing F , solving for $\bar{\lambda}$, and repeating until converged.

For a coaxial rotor, one must model the interaction of the upper rotor with the lower. Here we assume the only interaction between the two rotors is the impingement of the wake from the upper rotor on the lower rotor. Following Leishman et al. (Ref. 11), the inflow $\bar{\lambda}_\ell$ to the lower rotor is divided into two parts, the flow inside the contracted wake of the upper rotor, and the flow outside, that is $r < r_c$ and $r > r_c$, where r_c is the radius of the contracted wake from the upper rotor as it passes through the lower rotor.

Outside of the contracted wake, it is assumed there is no effect from the upper rotor, and Eq. (15) can be used to compute the inflow and the resulting differential thrust and power using Eqs. (12) and (14). Inside the contracted radius r_c , the cross-sectional area of the wake is reduced from an area of A to A_c . Thus, the velocity of the flow leaving the upper rotor accelerates, resulting in an axial inflow velocity of approximately $\bar{U}_\ell \approx (A/A_c)\bar{\lambda}_u$ at radii given approximately by $r_\ell \approx r_u \sqrt{A_c/A}$. The contraction ratio of the upper rotor's wake A_c/A is not computed in our analysis, but rather is prescribed a priori based on momentum theory or experimental data. As

in the case of the upper rotor, we equate Eqs. (12) and (14) and solve for the lower rotor's inflow distribution $\bar{\lambda}_\ell$, giving

$$\bar{\lambda}_\ell(r) = \sqrt{\left(\frac{\sigma C_{\ell\alpha}}{16F} - \frac{\frac{A}{A_c} \bar{\lambda}_u}{2}\right)^2 + \frac{\sigma C_{\ell\alpha}}{8F} \theta_\ell \bar{r}} - \left(\frac{\sigma C_{\ell\alpha}}{16F} - \frac{\frac{A}{A_c} \bar{\lambda}_u}{2}\right) \quad (16)$$

for $r \leq r_c$.

Once the inflow distribution for each rotor is known, the overall thrust coefficient of each rotor, C_T , is found by integrating the differential thrust over the span with the result

$$C_T = \int_0^1 4F \bar{\lambda} (\bar{\lambda} - \bar{U}) \bar{r} d\bar{r} \quad (17)$$

Similarly, the coefficient of power for each rotor is given by

$$C_P = \int_0^1 4F \bar{\lambda}^2 (\bar{\lambda} - \bar{U}) \bar{r} d\bar{r} + \frac{1}{2} \int_0^1 \sigma c_d \bar{r}^3 d\bar{r} \quad (18)$$

where the first and second integrals in Eq. (18) represent the induced and viscous components of power, respectively. One can use lookup tables, or some other approximation of the coefficient of drag as a function of local angle of attack and Mach number, to determine the value of c_d .

We first consider a single point design of a rotor system in hover. For a coaxial rotor in hover, we seek to minimize the total power subject to constraints that the upper and lower rotors produce equal and opposite torques, and that the total thrust be equal to a specified value. Thus, we adjoin to the power the thrust and power constraints using Lagrange multipliers, with the result

$$\Pi = P_{\text{tot}} + \lambda_T (T - T_R) + \lambda_Q (P_u - P_\ell) \quad (19)$$

The power and thrust are a function of the design variables, which include the twist and chord distributions the blades of the upper and lower rotors.

To solve the optimization problem, we start with some initial guess of the design variables Θ . We then find the second-order Taylor expansion of the power in terms of the design variables, i.e.,

$$P_{\text{tot}}(\Theta + \Delta\Theta) \approx P_{\text{tot}}(\Theta) + \Delta\Theta^T \frac{\partial P_{\text{tot}}(\Theta)}{\partial \Theta} + \frac{1}{2} \Delta\Theta^T \frac{\partial^2 P_{\text{tot}}(\Theta)}{\partial \Theta^2} \Delta\Theta \quad (20)$$

Similarly, the total thrust and the power of the upper and lower rotors can be approximated with the first-order Taylor expansions

$$T(\Theta + \Delta\Theta) \approx T(\Theta) + \Delta\Theta^T \frac{\partial T(\Theta)}{\partial \Theta} \quad (21)$$

$$P_u(\Theta + \Delta\Theta) \approx P_u(\Theta) + \Delta\Theta^T \frac{\partial P_u(\Theta)}{\partial \Theta} \quad (22)$$

$$P_\ell(\Theta + \Delta\Theta) \approx P_\ell(\Theta) + \Delta\Theta^T \frac{\partial P_\ell(\Theta)}{\partial \Theta} \quad (23)$$

Substituting these expansions into Eq. (19), and simplifying notation by noting that all values of P and T are those found at the current set of design variables Θ , e.g., $P = P(\Theta)$, gives

$$\begin{aligned} \Pi = & P_{\text{tot}} + \Delta\Theta^T \frac{\partial P_{\text{tot}}}{\partial \Theta} + \frac{1}{2} \Delta\Theta^T \frac{\partial^2 P_{\text{tot}}}{\partial \Theta^2} \Delta\Theta \\ & + \lambda_T \left(T(\Theta) + \Delta\Theta^T \frac{\partial T}{\partial \Theta} \right) \\ & + \lambda_Q \left(P_u + \Delta\Theta^T \frac{\partial P_u}{\partial \Theta} - P_\ell - \Delta\Theta^T \frac{\partial P_\ell}{\partial \Theta} \right) \end{aligned} \quad (24)$$

Next, to find an estimate of the constrained optimum design, we take the variation of this expression with respect to the change in design variables $\Delta\Theta$ and the Lagrange multipliers with the result

$$\begin{aligned} \delta\Pi = & 0 \\ = & \left(\frac{\partial P_{\text{tot}}}{\partial \Theta} + \frac{\partial^2 P_{\text{tot}}}{\partial \Theta^2} \Delta\Theta + \lambda_T \frac{\partial T}{\partial \Theta} - \lambda_Q \left[\frac{\partial P_u}{\partial \Theta} - \frac{\partial P_\ell}{\partial \Theta} \right] \right) \delta\Delta\Theta \\ & + \left(T + \Delta\Theta^T \frac{\partial T}{\partial \Theta} - T_R \right) \delta\lambda_T \\ & + \left(P_u + \Delta\Theta^T \frac{\partial P_u}{\partial \Theta} - P_\ell - \Delta\Theta^T \frac{\partial P_\ell}{\partial \Theta} \right) \delta\lambda_Q \end{aligned} \quad (25)$$

Equation (25) must be zero for arbitrary variations $\delta\Delta\Theta$, $\delta\lambda_T$, and $\delta\lambda_Q$. Therefore, the terms in parentheses must be identically zero. The result is a system of linear equations describing the change in the design variables, i.e.,

$$\begin{bmatrix} \frac{\partial^2 P_{\text{tot}}}{\partial \Theta^2} & \frac{\partial T}{\partial \Theta} & \frac{\partial(P_u - P_\ell)}{\partial \Theta} \\ \frac{\partial T}{\partial \Theta} & 0 & 0 \\ \frac{\partial(P_u - P_\ell)}{\partial \Theta} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\Theta \\ \lambda_T \\ \lambda_Q \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial P_{\text{tot}}}{\partial \Theta} \\ T_R - T \\ P_\ell - P_u \end{Bmatrix} \quad (26)$$

Equation (26) is solved for the unknown change in design variables and Lagrange multipliers, and the design variables are updated. The entire process is then repeated until a converged design is found, and is effectively a Newton iteration.

The algorithm described above is implemented in Fortran 95. The derivative terms in Eq. (26) are difficult to express analytically. Instead, we use the Tapenade automatic differentiation compiler (Ref. 15) in reverse (adjoint) mode to generate the Fortran code needed to compute the gradient terms, and the forward differentiation mode of Tapenade to generate the code to compute the Hessian matrix.

Having described the single point hover optimization problem, we next consider the multipoint optimization of a rotor operating at two flight conditions, one in hover and one in cruise. We seek to minimize the weighted sum of the total power loss at the two flight conditions, again subject to trim constraints. Thus, our Lagrangian power is given by

$$\begin{aligned} \Pi = & \alpha \left(\frac{1}{2} \mathbf{\Gamma}_1^T \mathbf{K}_1 \mathbf{\Gamma}_1 + P_{v1} \right) + \beta P_{\text{tot}} + \lambda_{R1} \cdot \mathbf{R}_1(\mathbf{\Gamma}_1, \Theta) \\ & + \lambda_{F1}^T \cdot (\mathbf{B}_1 \mathbf{\Gamma}_1 - \mathbf{F}_{R1}) + \lambda_{M1}^T \cdot (\mathbf{D}_1 \mathbf{\Gamma}_1 - \mathbf{M}_{R1}) \\ & + \lambda_T (T - T_R) + \lambda_Q (P_u - P_\ell) \end{aligned} \quad (27)$$

where the forward flight power is weighted by α and the hover power is weighted by β . The combination of the powers from the two disparate flight conditions (and, therefore, very different computational analysis techniques) in this way, as a simple sum of powers and constraints dependent on common design

variables, makes the formulation of the multipoint optimization relatively straightforward.

Next, taking the variation of Π and setting the result to zero gives

$$\begin{bmatrix} \frac{1}{2}\alpha\mathbf{A}_1^T(\mathbf{K}_1 + \mathbf{K}_1^T)\mathbf{A}_1 + \frac{\partial^2 P_{\text{tot}}}{\partial \Theta^2} & \mathbf{A}_1^T \mathbf{B}_1^T & \mathbf{A}_1^T \mathbf{D}_1^T & \frac{\partial T}{\partial \Theta} & \frac{\partial(P_u - P_\ell)}{\partial \Theta} \\ \mathbf{B}_1 \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{D}_1 \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\partial T}{\partial \Theta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\partial(P_u - P_\ell)}{\partial \Theta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta \Theta \\ \lambda_{F1} \\ \lambda_{M1} \\ \lambda_T \\ \lambda_Q \end{Bmatrix} = \begin{Bmatrix} -\alpha(\mathbf{A}_1^T \mathbf{K}_1 \boldsymbol{\Gamma}_{01} - \mathbf{k}_{v1}) - \beta \frac{\partial P_{\text{tot}}}{\partial \Theta} \\ \mathbf{F}_{R1} - \mathbf{B}_1 \boldsymbol{\Gamma}_{01} \\ \mathbf{M}_{R1} - \mathbf{D}_1 \boldsymbol{\Gamma}_{01} \\ T_R - T \\ P_\ell - P_u \end{Bmatrix} \quad (28)$$

Once we have solved the Eq. (28) system of linear equations, we update the total vector of design variables and recompute the thrust and power at the new set of design variables. We then recompute the gradients and Hessian and iterate until the design variables have converged.

COMPUTATIONAL RESULTS: HOVER/CRUISE MULTIPPOINT OPTIMIZATION

In this section, we apply the multipoint optimization technique to the same two rotor systems previously considered a conventional four-bladed rotor and a coaxial rotor system with four-bladed rotors – but now designed to operate in both hover and high-speed forward flight conditions.

Conventional Rotor

The conventional rotor parameters are given in Table 1. For the case considered here, the forward flight power loss (condition 2 in Table 1) is weighted by α ; the hover total power is weighted by β . For these calculations, we again used a vortex-lattice mesh with 18 quadrilateral vortex ring elements in the spanwise direction and 15 elements in the azimuthal direction per quarter turn for each of the four blades for a total of 1080 vortex ring elements of unknown strength for the forward flight condition, with images of these vortex rings extending 40 quarter revolutions from the primary lattice. For the hover power computations, we discretized the rotor using 18 spanwise stations to match the spanwise discretization for the forward flight mesh.

Figure 8 shows the Pareto frontiers for varying levels of harmonic control. Plotted is the inverse of the figure of merit (FOM) versus the high-speed cruise power. We see there is a large tradeoff in performance between hover performance and cruise performance. No single design can achieve performances close to the single point optimum at both flight conditions, even when using $N = 3$ harmonic control.

Note that the optimal figure of merit for the single point hovering flight limit ($\alpha = 0$) is the same regardless of the level

of harmonic control. Because of the azimuthal symmetry of the hover problem, 1/rev and higher harmonic control cannot reduce power. Higher harmonic control, however, does provide a benefit in the multipoint optimization by reducing power in forward flight, improving the Pareto frontier across all values of α . Note for all levels of harmonic control, there is a fairly sharp knee around $\alpha = 0.2$. These rotor designs have need minimum cruise power loss, with some moderate penalty in hover performance.

Figures 9 and 10 show the optimal conventional rotor designs using $N = 1$ and $N = 3$ harmonic control at varying values of α . Note that the planform and twist distributions of the optimal hovering rotor ($\alpha = 0.0$) and the optimal $\mu = 0.5$ forward flight rotor ($\alpha = 1.0$) are dramatically different. The optimal hovering rotor has the expected hyperbolic twist and chord distribution (modified slightly to maintain optimality with the inclusion of the Prandtl tip loss factor), while the optimal forward flight rotor has a positive twist on the inner 10% of the blade, as well as a very small inboard chord that increases to a peak chord at a radial position of about 80% span. Varying α tends to blend these two disparate blade designs, although certain features in the optimized hovering rotor, such as the very large twist and chord distributions on the

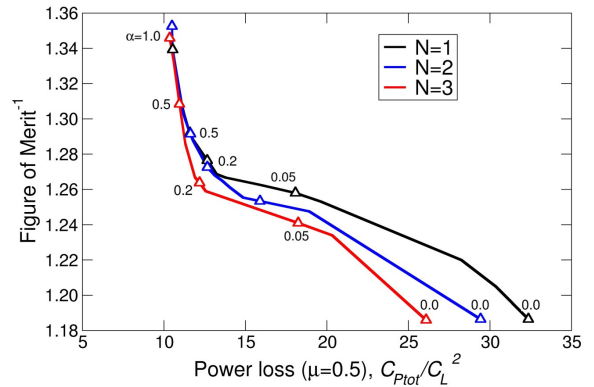


Fig. 8. Hover/cruise Pareto frontiers for a conventional rotor using $N = 1$, $N = 2$, and $N = 3$ harmonic control.

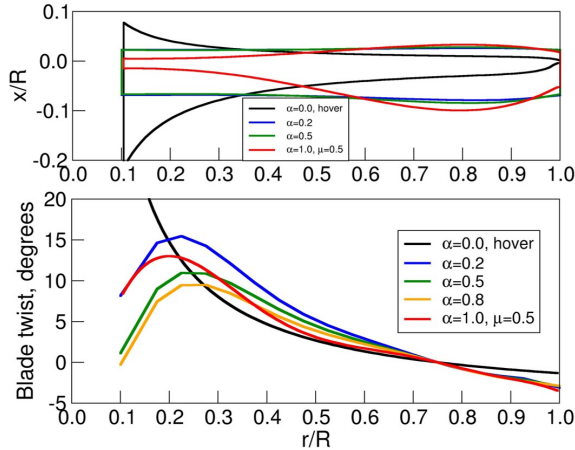


Fig. 9. Optimal hover/cruise multipoint rotor designs for $N = 1$ control with varying values of α for a conventional rotor.

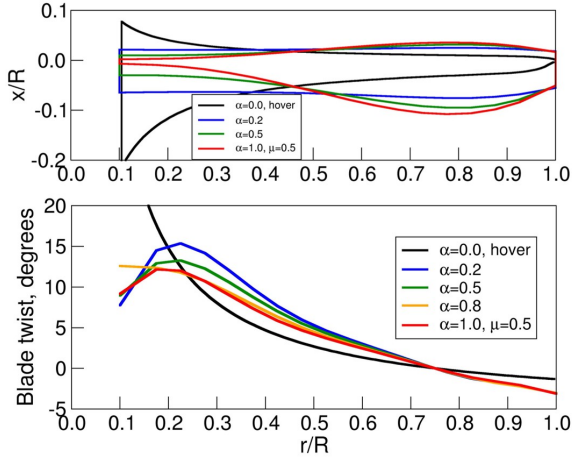


Fig. 10. Optimal hover/cruise multipoint rotor designs for $N = 3$ control with varying values of α for a conventional rotor.

inner portion of the blade, disappear almost immediately with increasing nonzero α .

Coaxial Rotor

We now analyze the coaxial rotor based on the Sikorsky X2 TD as previously described. The rotor parameters are given in Table 2. For the case considered here, the forward flight power loss (condition 2 in Table 2) is weighted by α ; the hover total power is weighted by β . In the hover analysis, we assume a wake contraction ratio of $r_c/R = 0.82$. The figure of merit for the hover condition is defined as $FOM = C_T^{3/2} / \sqrt{2} C_P$. (Note for co-axial rotors, there is not a single standard for measuring figure of merit. Leishman et al. (Ref. 12) present alternative definitions of the figure of merit.)

Figure 11 shows the computed Pareto frontiers for the coaxial hover/cruise multipoint optimization for varying lev-

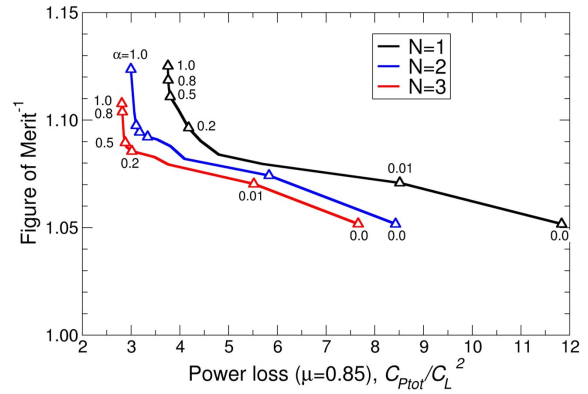


Fig. 11. Hover/cruise Pareto frontiers for a coaxial rotor using $N = 1$, $N = 2$, and $N = 3$ harmonic control.

els of harmonic control. There is no longer the sharp corner present in the cruise/cruise coaxial multipoint optimization; rather, there is no single rotor design that can achieve close to optimal performance between the two flight conditions. Again, as we expect, higher harmonic control does not provide a benefit rotors optimized only for hover ($\alpha = 0$). However, $N = 2$ and $N = 3$ control is effective at reducing the optimal hovering rotor's power loss in forward flight. In general, higher harmonic control provides large benefits to performance across the entire Pareto frontier.

It is also interesting to note that a rotor designed only for forward flight ($\alpha = 1$) results in an increase in hovering power of only about 5 to 7 percent above the cruise only optimum ($\alpha = 1$), depending on the level of harmonic control. This is in contrast to the high sensitivity of forward flight performance to rotor design, where the $\alpha = 0.0$ hovering rotor results in a three times increase in total power loss at $\mu = 0.85$. Also, we see that there is a knee in the Pareto frontier curves at values of α of about 0.2, producing rotors with good hover performance with nearly optimal cruise performance.

Figures 12 and 13 show the $N = 1$ and $N = 3$ optimal rotors at varying values of α . Note that in each of these cases, the twist and chord distribution on the upper and lower rotors are defined by independent sets of design variables; in other words, there is no constraint or requirement that they be equal. The hovering rotor uses significantly different twist and chord distributions on the upper and lower rotor. These differences arise because of the different inflows on the upper and lower rotors in hover. While beneficial in hover, the differences in upper and lower blade design quickly disappear with increased weighting towards the forward flight condition. The $\alpha = 0.2$ rotor shows some slight differences in upper and lower twist, but above this value, the twist of the upper and lower rotor are effectively equal. Furthermore, the planforms of the optimal rotors rapidly approach the $\alpha = 1.0$ design for values of α greater than about 0.2. This rapid convergence to the optimal forward flight rotor is due in part to the high sensitivity of forward flight performance to blade design.

Figure 14 shows the circulation distribution for the coaxial

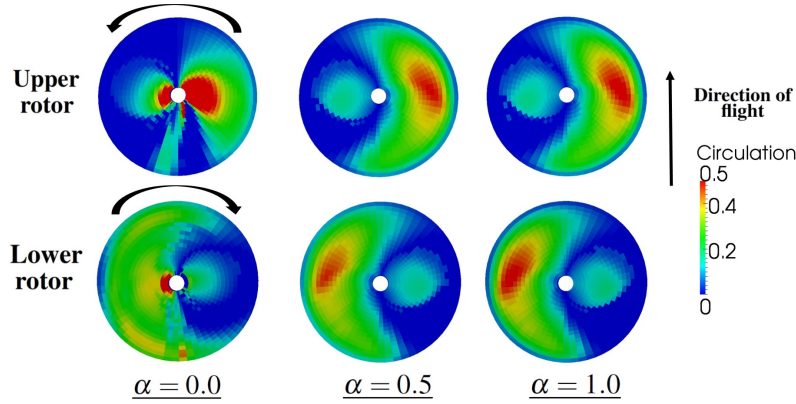


Fig. 14. Normalized circulation distributions for the optimal $N = 1$ coaxial rotor with varying values of α at $\mu = 0.85$.

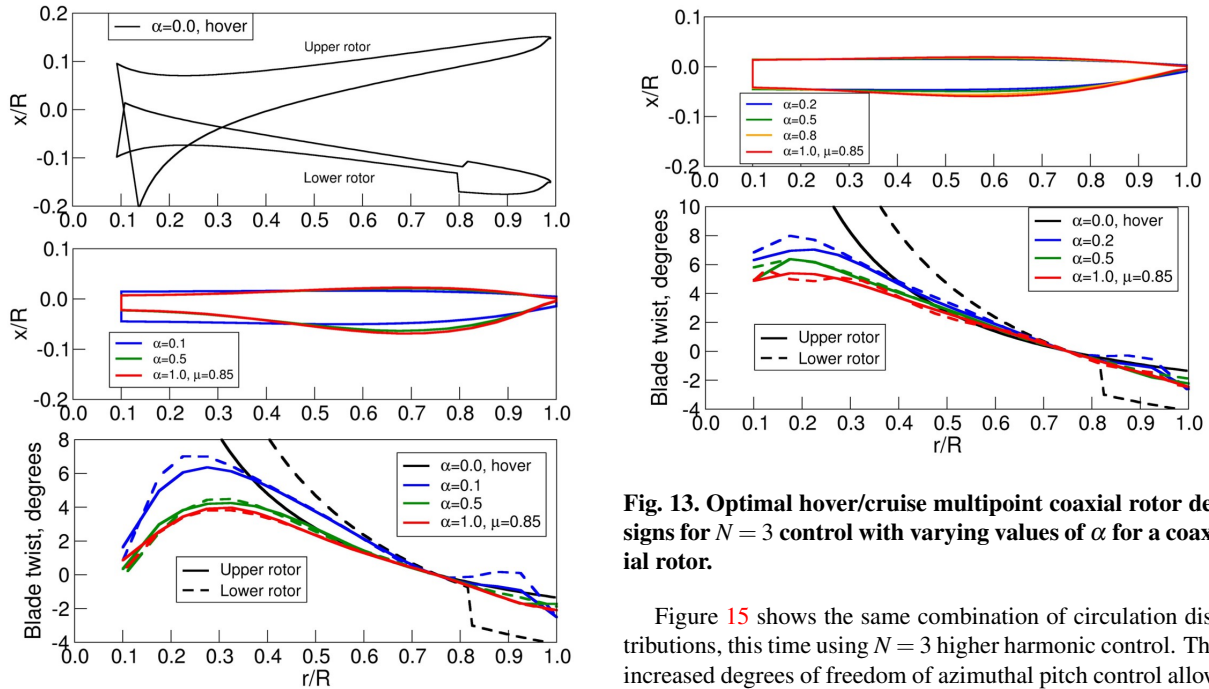


Fig. 12. Optimal hover/cruise multipoint rotor designs for $N = 1$ control with varying values of α for a coaxial rotor.

rotor using conventional control ($N = 1$) at $\mu = 0.85$ for three levels of α : the hovering optimal rotor ($\alpha = 0.0$), the $\mu = 0.85$ optimal rotor ($\alpha = 1.0$), and a rotor determined by multipoint optimization ($\alpha = 0.5$). We see that the optimal hovering rotor, with no consideration of the forward flight power loss in its design, generates regions of very high circulation on the inner portion of the blade on both the advancing and retreating sides, leading to high induced and profile power losses. The $\alpha = 0.5$ rotor, with equal weighting on both flight conditions, generates a circulation distribution that is very similar to – and nearly as efficient as – the $\alpha = 1.0$ rotor while also providing a modest benefit in hover efficiency.

Fig. 13. Optimal hover/cruise multipoint coaxial rotor designs for $N = 3$ control with varying values of α for a coaxial rotor.

Figure 15 shows the same combination of circulation distributions, this time using $N = 3$ higher harmonic control. The increased degrees of freedom of azimuthal pitch control allow the $\alpha = 1$ rotor to decrease the regions of high circulation in forward flight, particularly on the retreating side, resulting in a lower total power. As in the $N = 1$ case, the $\alpha = 0.5$ rotor nearly replicates the $\alpha = 1.0$ rotor forward flight optimal circulation distribution.

Finally, Figure 16 shows the radial distribution of thrust and total power loss for the $\alpha = 0.0$, $\alpha = 0.5$, and $\alpha = 1.0$ rotors in hover. Without the use of hyperbolic twist and chord distributions and the significantly different upper and lower twist and chord distributions, the $\alpha = 0.5$ and $\alpha = 1.0$ rotors are unable to match the optimal hovering rotor. In fact, the thrust and power of the $\alpha = 0.5$ rotor much more closely resemble the $\alpha = 1.0$ rotor, as the twist and chord required to match the hovering optimal's thrust and power distributions impose a large penalty on forward flight performance and are therefore eliminated even at small values of α .

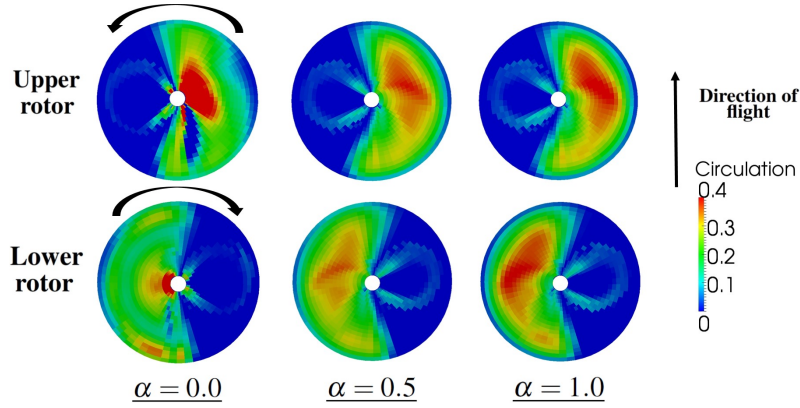


Fig. 15. Normalized circulation distributions for the optimal $N = 3$ coaxial rotor with varying values of α at $\mu = 0.85$.

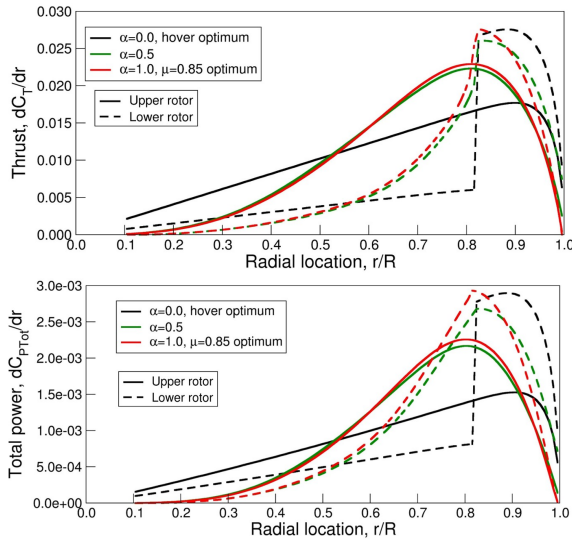


Fig. 16. Radial thrust and total power distributions for the optimal $N = 1$ coaxial rotor in hover with varying values of α .

CONCLUSIONS

In this paper, we have described a method of multipoint aerodynamic rotor optimization. The method can be applied to conventional or coaxial rotors, and can be used to optimize performance between any two advance ratios, or a single advance ratio and hover, by minimizing the weighted sum of the aerodynamic losses at two different flight conditions. We used the method to perform cruise/cruise optimization and hover/cruise optimizations on representational conventional and coaxial rotors. Based on the results presented in this paper, we reached the following conclusions:

1. For the conventional rotor optimized for the $\mu = 0.2$ and $\mu = 0.5$ flight conditions, it is not possible to achieve optimal or near optimal performance at both flight conditions using a single rotor design. As a result, perfor-

mance tradeoffs must be made in selecting a rotor design. Weighting the $\mu = 0.5$ power loss more heavily results in a moderate penalty in performance at $\mu = 0.2$; however, heavily weighting the $\mu = 0.2$ power loss results in a very large penalty in performance at $\mu = 0.5$, with a total power loss that is three times the single point optimum. As a result, the optimal rotor design quickly approaches the $\mu = 0.5$ optimum with increased values of α (i.e., increased weighting towards the $\mu = 0.5$ flight conditions). Higher harmonic control provides a relatively small benefit at both advance ratios, and is marginally effective at improving the shape and location of the Pareto frontier.

2. For the conventional rotor optimized for hover and forward flight at $\mu = 0.5$, there is a large tradeoff in performance between the two flight conditions. The (single point optimal) hover and forward flight optimal rotor designs are dramatically different, and a single rotor design cannot achieve optimal or near optimal performance at both flight conditions. As α is varied, the multipoint optimal rotor blends characteristics from the two single point designs, resulting in balancing of performance between the two flight conditions. In such a case, the desired rotor shape should be carefully selected from designs along the Pareto efficiency curve based on mission requirements. Higher harmonic control once again marginally improves the location of the Pareto efficiency curve across all values of α .
3. For the coaxial rotor optimized for the $\mu = 0.3$ and $\mu = 0.85$ flight conditions, the Pareto efficiency curve has a very sharp corner or “knee,” indicating that near optimal performance can be achieved at both flight conditions with a single rotor design. Higher harmonic control significantly reduces power at the $\mu = 0.85$ flight condition, improving the Pareto efficiency curve while not significantly altering its general shape.
4. For the coaxial rotor optimized for hover and $\mu = 0.85$ flight, a substantial tradeoff in performance must again be made between the two flight conditions, as there is

bro sharp knee in the Pareto curve. However, the forward flight performance of the rotor is more sensitive to changes in rotor design away from the $\mu = 0.85$ optimal rotor, resulting in the multipoint optimum rotor design being dominated by the $\mu = 0.85$ optimal twist and chord distributions, even at small weighting towards this flight condition. For example, certain design characteristics that are beneficial in hover, such as using different blade designs on the upper and lower rotor, impose a large penalty on forward flight performance and are quickly eliminated as the forward flight condition is more heavily weighted. Higher harmonic control substantially improves the location and shape of the Pareto frontier.

ACKNOWLEDGMENTS

This material is based upon work supported by the U.S. Army Research Office under Award No. W911NF-13-1-0469. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the view of the U.S. Army Research Office. Additionally, the first author is funded through a National Defense Science and Engineering Graduate Fellowship.

Author contact: Eli B. Giovanetti, ebgl1@duke.edu;
Kenneth C. Hall, kenneth.c.hall@duke.edu.

REFERENCES

- ¹Hall, K. C. and Giovanetti, E. B., "Minimum Power Requirements and Optimal Rotor Design for Conventional and Compound Helicopters Using Higher Harmonic Control," presented at the AHS 69th Annual Forum, May 21-23, 2013.
- ²Giovanetti, E. B. and Hall, K. C., "Optimum Design of Compound Helicopters Using Higher Harmonic Control," presented at the AHS 70th Annual Forum, May 20-22, 2014.
- ³Johnson, W., "Influence of Lift Offset on Rotorcraft Performance," AHS Aeromechanics Specialists' Conference, San Francisco, CA, January 20-22, 2008.
- ⁴Johnson, W., Moodie, A. M., and Yeo, H., "Design and Performance of Lift-Offset Rotorcraft for Short-Haul Missions," presented at the AHS Future Vertical Lift Aircraft Design Conference, San Francisco, California, January 18-20, 2012.
- ⁵Yeo, H. and Johnson, W., "Optimum Design of a Compound Helicopter," *Journal of Aircraft*, Vol. 46, No. 4, July - August 2009.
- ⁶Moodie, A. M. and Yeo, H., "Design of a Cruise-Efficient Compound Helicopter," *Journal of the American Helicopter Society*, 2012.
- ⁷Yeo, H. and Johnson, W., "Investigation of Maximum Blade Loading Capability of Lift-Offset Rotors," *Journal of the American Helicopter Society*, 2014.
- ⁸Rand, O. and Khromov, V., "Compound Helicopter: Insight and Optimization," presented at the AHS 69th Annual Forum, May 21-23, 2013.
- ⁹Steuer, R. E., "Multiple Criteria Optimization: Theory, Computation, and Application," New York: John Wiley and Sons Inc., 1986.
- ¹⁰Bagai, A., "Aerodynamic Design of the X2 Technology Demonstrator Main Rotor Blade" presented at the AHS 64th Annual Forum, April 29- May 1, 2008.
- ¹¹Leishman, J. G., Ananthan, S., "An Optimum Coaxial Rotor System for Axial Flight," *Journal of the American Helicopter Society*, October 2008.
- ¹²Leishman, J. G., Syal, M., "Figure of Merit Definition for Coaxial Rotors," *Journal of the American Helicopter Society*, June 2008.
- ¹³Rand, O., and Khromov, V., "Aerodynamic Optimization of Coaxial Rotor in Hover and Axial Flight," 27th International Congress of the Aeronautical Sciences, 2010.
- ¹⁴Juhasz, O., Syal, M., Celi, R., Khromov, V., Rand, O., Ruzicka, G., Strawn, R., "Comparison of Three Coaxial Aerodynamic Prediction Methods Including Validation with Model Test Data," *Journal of the American Helicopter Society*, Volume 59, 2014.
- ¹⁵Hascoet, L. and Pascual, V., "Tapenade 2.1 User's Guide," Project Tropics, INRIA Sophia-Antipolis, 2004.

Appendix III: Minimum Loss Load, Twist, and Chord Distributions For Coaxial Helicopters in Hover

Minimum Loss Load, Twist, and Chord Distributions for Coaxial Helicopters in Hover



Eli B. Giovanetti
Graduate Research Assistant



Kenneth C. Hall*
Julian Francis Abele Professor

Mechanical Engineering and Materials Science
Duke University, Durham, NC

This paper presents an approach for determining the optimal (minimum power) geometry of a hovering coaxial rotor using blade element momentum theory, including swirl. The analysis accounts for the presence of a finite number of blades using the Prandtl tip loss factor, the effect of profile drag using experimentally or computationally determined drag polars, and the mutual interference between the two rotors using an empirically determined influence coefficient method. Numerical results show that including the induced swirl in the model decreases the optimal figure of merit and that swirl has a larger impact at higher disk loadings. At the disk loadings typically found on helicopters, the effect of swirl is relatively small, particularly compared to mutual rotor interference or tip losses. Additionally, accounting for swirl affects the optimal rotor design near the blade root, at locations that would often be part of the root cutout of a realistic rotor.

Nomenclature

A	rotor disk area
A_c	cross-sectional area of contracted wake
C_P	rotor power coefficient, $P/\rho A \Omega^3 R^3$
C_T	rotor thrust coefficient, $T/\rho A \Omega^2 R^2$
c	airfoil chord
c_l, c_d	sectional coefficients of lift and drag
F	Prandtl tip loss factor
FM	figure of merit
i	radial station index
k	empirically determined parameter used in the mutual interference model
L	lift
N	number of rotor blades
n	iteration number
P	total power
P_i	induced power
R	rotor radius
R_{CO}	root cutout radius
r	radial coordinate
r_c	radius of contracted wake at lower rotor face
T	thrust
U	inflow velocity

V	airspeed relative to the airfoil
w	induced inflow, parallel to the lift vector
z	axial coordinate
θ	circumferential coordinate
ν_Q	Lagrange multiplier for torque constraint
ν_T	Lagrange multiplier for thrust constraint
Π	power Lagrangian
ρ	density
ϕ	relative flow angle
Ω	rotor rotational rate
$\bar{()}$	nondimensional quantity

Introduction

Compared to conventional helicopter rotors, a pair of axially separated coaxial rotors (of comparable radius and combined disk loading) generally require less power in both forward flight (Ref. 1) and hover (Ref. 2). In forward flight, the rapid rise in induced power with advance ratio is eliminated due to the more favorable wake structure generated by counterrotating rotors (Ref. 1). In hover, the figure of merit is improved because the wake from the upper rotor contracts before passing through the lower rotor. The lower rotor draws in additional air at the rotor tips, increasing the effective area of the rotor, and improving figure of merit (Ref. 2).

A number of researchers have studied coaxial rotor performance in hover both experimentally and using aerodynamic modeling. Coleman (Ref. 3) provides a comprehensive summary of coaxial rotor research. Nagashima and Nakanishi (Ref. 4) use a generalized momentum

*Corresponding author; e-mail: kenneth.c.hall@duke.edu.

Presented at the AHS 71st Annual Forum, Virginia Beach, VA, May 5–7, 2015.

Manuscript received June 2015; accepted August 2016.

E. B. GIOVANETTI

JOURNAL OF THE AMERICAN HELICOPTER SOCIETY

theory and free wake analysis to determine the optimal performance of a coaxial rotor in hover. The authors conclude that hover performance is largely dependent on axial spacing and blade pitch differences between the rotors and that certain mutual interference effects can be beneficial to performance.

More recently, Leishman and Ananthan (Ref. 5) formulated the combined blade element momentum theory (BEMT) for coaxial rotors in hover and axial flight, accounting for axial-induced velocities only. Results from that analysis were compared to experimental data from Harrington (Ref. 6) and showed good agreement. In addition, the authors, building upon the single rotor work of Gessow (Ref. 7), asserted that the optimal coaxial hovering rotor is that which produces a uniform disk loading and linear thrust per unit span for each rotor. To achieve torque balance, the slope (in the radial direction) of the thrust distributions will be different, with the upper rotor generating a larger share of the overall system thrust than the less efficient lower rotor. Juhasz et al. (Ref. 8) used BEMT to predict the performance of a model-scale coaxial rotor with highly twisted blades and compared results to a free vortex wake model and a computational fluid dynamics analysis. The authors concluded that BEMT could accurately predict the thrust and torque on the model coaxial rotor. Additionally, although BEMT may miss some fine details of the inflow distributions, for example, induced upwash near the outer edge of the rotor disk and circumferentially varying inflow, the circumferentially averaged inflow predictions are fairly accurately predicted.

Rand and Khromov (Ref. 9) used the calculus of variations and a BEMT analysis (also neglecting swirl velocities in the wake) to determine the optimal hovering coaxial rotor. Their results demonstrated that a linear thrust distribution on the lower rotor is *not* optimal, a result that differs from Leishman and Ananthan (Ref. 5) even when using identical assumptions about mutual rotor interference. Rand and Khromov (Ref. 9) also explore several empirical approximations for determining the mutual interference effects of the two rotors.

Syal and Leishman (Ref. 10) used both BEMT and a free vortex method (FVM) to formulate an optimization technique for a coaxial rotor system in hover using the method of feasible directions. The authors note, however, that the nonconvexity of the problem may limit the usefulness of formal optimization methods, and that extensive parametric studies may still be required.

The work in this paper builds upon the work of Leishman and Ananthan (Ref. 5), who formally laid out the coaxial BEMT theory without swirl, and Rand and Khromov (Ref. 9), who optimized the inflow and rotor design of coaxial rotors, to develop a variational approach for determining the optimal torque-balanced coaxial hovering rotor including swirl. The effects of the swirl component of induced velocity on performance (figure of merit), optimal induced inflow distributions, and optimal blade twist and chord distributions are investigated.

Technical Approach

Combined BEMT including swirl

A momentum theory formulation similar to that described by Leishman and Ananthan (Ref. 5) and Nagashima and Nakanishi (Ref. 4) is used to model coaxial rotors in hover. To incorporate losses due to swirl, and the effect on blade design of these losses, the momentum theory (including swirl) described by Modarres and Peters (Ref. 11) is used.

Consider a single rotor with rotational speed Ω operating, with some inflow with magnitude U , with axial and swirl components U_z and U_θ , respectively. Figure 1 shows the resultant relative velocity vector including the effects of induced inflow w and blade rotation Ωr . The sine and

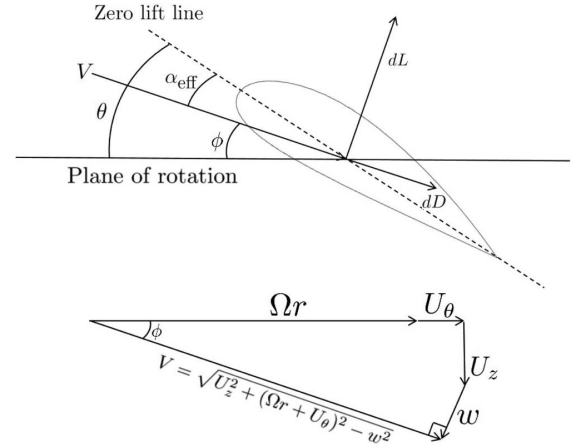


Fig. 1. Geometry of the flow at a blade section of a rotor, including the axial and swirl components of velocity.

cosine of the relative flow angle ϕ at a radial station r are given by

$$\sin \phi = \frac{U_z \sqrt{U_z^2 + (\Omega r + U_\theta)^2 - w^2} + w(\Omega r + U_\theta)}{U_z^2 + (\Omega r + U_\theta)^2} \quad (1)$$

$$\cos \phi = \frac{(\Omega r + U_\theta) \sqrt{U_z^2 + (\Omega r + U_\theta)^2 - w^2} - w U_z}{U_z^2 + (\Omega r + U_\theta)^2} \quad (2)$$

where V is the total local airspeed relative to the airfoil given by

$$V = \sqrt{U_z^2 + (\Omega r + U_\theta)^2 - w^2} \quad (3)$$

Following Ref. 11, a momentum balance is performed to obtain expressions for the thrust and power that account for both axial and swirl components of induced velocity. The lift per unit radius, dL/dr , acting perpendicular to the relative velocity is given by

$$\frac{dL}{dr} = 4\pi\rho F [U_z + w \cos \phi] w r \quad (4)$$

where ρ is the density of air and F is the Prandtl tip loss factor. This force can then be resolved into the axial and circumferential directions using Eqs. (1) and (2). Resolving the lift per span into axial and circumferential components, integrating along the span, and summing over the N blades, the thrust T and induced power P_i are found to be

$$T = \int_{R_{CO}}^R 4\pi\rho [U_z + w \cos \phi] (w \cos \phi) r dr \quad (5)$$

$$P_i = \int_{R_{CO}}^R 4\pi\rho [U_z + w \cos \phi] (w \sin \phi) (\Omega r + U_\theta) r dr \quad (6)$$

where R is the rotor radius and R_{CO} is the radius of the root cutout.

Figure 1 shows that the profile drag of the blade also contributes to both the thrust and torque acting on the rotor. The rotor profile drag per unit radius is expressed as

$$\frac{dD_{prof}}{dr} = \frac{1}{2} \rho N (U_z^2 + (\Omega r + U_\theta)^2 - w^2) c_d \quad (7)$$

where N is the number of blades and c_d is the sectional coefficient of drag. The profile drag force is resolved into its axial and circumferential directions again using Eqs. (1) and (2). Including the profile drag

contribution to the total thrust and power and nondimensionalizing gives

$$C_T = \int_{\bar{r}_{CO}}^1 \left[4(\bar{U}_z + \bar{w} \cos \phi) \bar{w} \cos \phi \bar{r} - \sin \phi \frac{N\bar{c}c_d}{2\pi} (\bar{U}_z^2 + (\bar{r} + \bar{U}_\theta)^2 - \bar{w}^2) \right] d\bar{r} \quad (8)$$

$$C_P = \int_{\bar{r}_{CO}}^1 \left[4(\bar{U}_z + \bar{w} \cos \phi) \bar{w} \sin \phi \bar{r} (\bar{r} + \bar{U}_\theta) + \cos \phi \frac{N\bar{c}c_d}{2\pi} (\bar{U}_z^2 + (\bar{r} + \bar{U}_\theta)^2 - \bar{w}^2) (\bar{r} + \bar{U}_\theta) \right] d\bar{r} \quad (9)$$

where all velocities have been normalized by ΩR and all lengths by R . Normalized quantities are expressed with an overbar, i.e., $w/\Omega R = \bar{w}$.

To extend this analysis to a coaxial rotor system, one must account for the mutual interaction of the two rotors. The outflow from the upper rotor contracts and is mapped onto the inner portion of the lower rotor, which has a total area of A_c , giving the inner portion of the lower rotor an inflow velocity of

$$\bar{U}_{\ell 1z} = \left(\frac{A}{A_c} \right) \bar{w}_{uz} \quad (10)$$

$$\bar{U}_{\ell 1\theta} = \left(\frac{A}{A_c} \right)^{3/2} \bar{w}_{u\theta} \quad (11)$$

where the subscripts u and $\ell 1$ denote variables associated with the upper rotor and inner portion of the lower rotor, respectively, and the subscript c denotes a wake-contracted quantity.

The influence of the lower rotor on the upper rotor is modeled by including a uniform axial inflow acting on the upper rotor that is induced by the lower rotor. To determine the magnitude of this inflow, the influence coefficient model developed by McAlister et al. (Ref. 12) is used. Using the Biot–Savart law and an empirical parameter k , the induced velocity due to a helical vortex filament trailing from the tip of a rotor in hover, measured at radial position $r = 0$ and axial position $\bar{z}$ (normalized by rotor radius), is given by

$$\frac{\bar{U}_z}{\bar{U}_z|_{\bar{z}=0}} = 1 + \left(\frac{|\bar{z}|}{\sqrt{1 + \bar{z}^2}} \right)^k \text{sign}(\bar{z}) \quad (12)$$

The exponent k is set to a value between 0.3 and 0.5 when analyzing points above the rotor and a value of 0.6 when analyzing points below the rotor. $\bar{U}_z|_{\bar{z}=0}$ is set equal to the average induced axial velocity on the lower rotor. There are no swirl velocities induced on the upper rotor due to the lower rotor. Equation (12) is used to approximate the radius r_c of the upper rotor's contracted wake as a function of vertical separation. For the influence of the upper rotor on the lower rotor, the outflow of the upper rotor was mapped onto the lower rotor taking into account the contraction of the wake.

Optimization approach

To determine the optimal induced velocity distribution, $\bar{w}$, on each rotor, a constrained optimization problem is formulated using the calculus of variations. The total power is minimized subject to two constraints, i.e., (1) that the rotor system generates a specified coefficient of lift and (2) that the torque on the rotors is equal and opposite (the rotors are in torque balance).

To solve this constrained optimization problem, at least conceptually, the two equality constraints are adjoined to the power to be minimized using Lagrange multipliers, v_T and v_Q , forming the Lagrangian power Π ,

$$\Pi = C_{Pu} + C_{Pl} - v_T(C_{Tu} + C_{Tl} - C_{T,\text{req}}) - v_Q(C_{Pu} - C_{Pl}) \quad (13)$$

The Euler–Lagrange equations are applied to Eq. (13) to find the induced inflow distribution that makes the Lagrangian power stationary, i.e.,

$$\frac{\partial \Pi}{\partial q_i} - \frac{d}{dr} \left(\frac{\partial \Pi}{\partial q_i'} \right) = 0, \quad i = 1, 2, 3 \quad (14)$$

where $q = (w, v_T, v_Q)$ and the prime indicates differentiation with respect to r . Because the Lagrangian power does not contain derivatives in the independent variable r , the Euler–Lagrange equations are simply given by

$$\frac{\partial \Pi}{\partial w} = 0 \quad (15)$$

$$\frac{\partial \Pi}{\partial v_T} = 0 \quad (16)$$

$$\frac{\partial \Pi}{\partial v_Q} = 0 \quad (17)$$

Application of the Euler–Lagrange equations results in a set of coupled integro-algebraic equations for the unknown induced inflow distribution (and the resulting load and torque distributions). In practice, solving the optimization problem is complicated by the fact that the inflow on the inner portion of the lower rotor is a function of the induced inflow on the upper rotor and the inflow on the upper rotor is a function of the induced inflow on the lower rotor. Furthermore, finding a closed form analytical solution for the optimal induced inflow on the two rotors that satisfies Eq. (17) is impractical if not impossible. Thus, in the present work, the optimal solution is computed numerically using an iterative ad hoc approach.

To start, the required thrusts for the upper and lower are initially set to be equal, so that $C_{Tu,\text{req}} = C_{Tl,\text{req}} = \frac{1}{2}C_{T,\text{req}}$. Each rotor is then optimized individually. Thus, the Lagrangian power for each rotor is formed, the calculus of variations is applied, and the optimal induced inflow is found. As an example, the Lagrangian power for the upper rotor is given by

$$\Pi_u = C_{Pu} - v_{Tu}(C_{Tu} - C_{Tu,\text{req}}) \quad (18)$$

which, when substituting in the expressions for thrust and power given in Eqs. (8) and (9), becomes

$$\begin{aligned} \Pi_u = \int_{\bar{r}_{CO}}^1 & \left[4(\bar{U}_z + \bar{w} \cos \phi) \bar{w} \sin \phi \bar{r} (\bar{r} + \bar{U}_\theta) \right. \\ & + \cos \phi \frac{N\bar{c}c_d}{2\pi} (\bar{U}_z^2 + (\bar{r} + \bar{U}_\theta)^2 - \bar{w}^2) (\bar{r} + \bar{U}_\theta) \\ & - v_{Tu} \left(4(\bar{U}_z + \bar{w} \cos \phi) \bar{w} \cos \phi \bar{r} \right. \\ & \left. \left. - \sin \phi \frac{N\bar{c}c_d}{2\pi} (\bar{U}_z^2 + (\bar{r} + \bar{U}_\theta)^2 - \bar{w}^2) - C_{Tu,\text{req}} \right) \right] d\bar{r} \quad (19) \end{aligned}$$

Here, the subscript u denoting the upper rotor has been omitted from most of the quantities in the integrand for simplicity. A similar expression applies to the lower rotor.

To put the Lagrangian power solely in terms of the induced inflow distribution $\bar{w}$, one can make use of Eqs. (1) and (2), which describe $\cos \phi$ and $\sin \phi$ in terms of the induced inflow, free stream inflow, and radial position. Substituting these expressions into Eq. (19) yields an expression for the Lagrangian power in terms of $\bar{w}$. Because of the profile power terms, however, the Lagrangian power is still a function of the sectional drag coefficient c_d and the chord $\bar{c}$, which are not necessarily known a priori. As a result, one must perform the optimization about some initial values of $\bar{c}$ and c_d and iterate, updating $\bar{c}$ and c_d until convergence.

Equation (15) states that at the stationary point of the Lagrangian, the derivative of the Eq. (19) integrand with respect to the upper rotor's

E. B. GIOVANETTI

JOURNAL OF THE AMERICAN HELICOPTER SOCIETY

induced axial inflow will be equal to zero. However, the analytical solution for the radial-induced inflow $\bar{w}$ that satisfies Eq. (15) is intractable. Instead of solving analytically, Eq. (15) is discretized and solved numerically using Newton's method to arrive at the optimal induced inflow distribution at each station i , that is, $\bar{w}_i$ is found such that $\partial \Pi / \partial \bar{w}_i = 0$. Using Newton's method gives the estimate of the induced inflow at the n th iteration,

$$\bar{w}_i^n = \bar{w}_i^{n-1} - \left. \frac{\partial \Pi / \partial \bar{w}_i}{\partial^2 \Pi / \partial \bar{w}_i^2} \right|^{n-1} \quad (20)$$

Solving Eq. (15) using the iterative scheme given in Eq. (20) gives the optimal induced inflow distribution for both axial- and swirl-induced inflows for any rotor with axial inflow distribution $\bar{U}_z$, swirl inflow distribution $\bar{U}_\theta$, chord distribution $\bar{c}$, and sectional coefficient of drag distribution c_d .

Once the optimal induced inflow distribution is known, the radial blade twist and chord distribution that generates the minimum induced loss inflow distribution with minimum profile losses is found. The lift per span on the rotor is given by blade element theory as

$$\frac{d\bar{L}}{d\bar{r}} = \left(\frac{N\rho\bar{c}c_l}{2} \right) (\bar{r}^2 + \bar{U}_z^2 - \bar{w}^2) \quad (21)$$

The sectional lift coefficient c_l is set equal to the value that corresponds to the airfoil section's maximum c_l/c_d . Assuming a linear lift curve, $c_l = c_{l\alpha}(\theta - \phi)$, the optimal blade twist is given by

$$\theta_{\text{opt}} = \phi + \frac{c_l}{c_{l\alpha}} \quad (22)$$

Equating the blade element statement of differential lift, Eq. (22), with the momentum balance statement from Eq. (4), yields the optimal chord value,

$$\bar{c}_{\text{opt}} = \frac{8F\pi (\bar{U}_z + \bar{w} \cos \phi) \bar{w} \bar{r}}{Nc_l ((\bar{r} + \bar{U}_\theta)^2 + \bar{U}_z^2 - \bar{w}^2)} \quad (23)$$

In Eqs. (22) and (23), the Prandtl tip loss factor F is incorporated into the optimal twist and chord equations. F is only a function of the number of blades, radial position, and induced inflow, all of which are known a priori. The rotor geometry computed using this process will generate the optimal induced inflow distributions accounting (approximately) for tip effects for rotors with a finite number of blades.

Strictly speaking, this process does not necessarily produce a minimum total power solution. Instead, the twist and chord distribution is found that minimizes profile power losses for a rotor that produces minimum induced loss inflow distribution, rather than minimizing the *sum* of the induced and profile powers. This method does provide a very close approximation to the actual minimum total power solution and has been used by several other authors, for example, Modarres and Peters (Ref. 11) and Rand and Khromov (Ref. 9).

In this work, an ad hoc numerical scheme is used to solve the optimality equations given above. To start, initial values are assumed for the chord distribution, the induced inflow on the upper rotor $\bar{U}_z$ caused by the lower rotor, and the thrust split between the two rotors. Typically, the upper and lower rotor thrusts are assumed to be equal and half the value of the total thrust required ($C_{T_u, \text{req}} = C_{T_l, \text{req}} = \frac{1}{2}C_{T, \text{req}}$). Note that, in general, a torque-balanced coaxial rotor system will have *unequal* thrusts on the two rotors.

Next, the optimal induced inflow distribution for the upper rotor at the current values of $\bar{U}_z$ is computed by solving Eq. (19) for each of the rotors separately using the iterative scheme in Eq. (20). The upper rotor's induced inflow onto the lower rotor is computed using Eqs. (10) and (11), and the mutual induced inflows are computed using Eq. (12). For this

computed optimal inflow, the optimal twist and chord distribution are found using Eqs. (22) and (23). The new chord value will affect both the torque balance (due to the new chord distribution having a different profile power loss) and the thrust generated by each rotor (due to the component of profile drag acting in the thrust direction). The resulting thrust and torque of the two rotors is then recomputed. In general, these new values of thrust and torque will not satisfy the thrust and torque balance constraints. Thus, the required thrust on each the two rotors is increased by an amount proportional to the deficit in total thrust. Similarly, the thrust on one rotor is increased and the value on the other decreased in proportion to the torque imbalance. Finally, the mutual inflow model is used to recompute the inflow to the upper rotor, and the process is repeated until convergence. At the completion of this process, the twist, chord, and induced inflow distributions have been computed that minimize total power at a given thrust coefficient while maintaining torque balance. Typically fewer than 10 outer iterations, which include inner Newton method iterations to compute the optimal inflow, are required.

Computational Results

Comparison of BEMT method with swirl to experimental results

In this section, numerical predictions of the performance of a coaxial rotor are presented using the coaxial BEMT analysis method, with and without swirl. The results are compared to the experimental results from Harrington's (Ref. 6) Rotor 2, to both determine the accuracy of the combined BEMT formulation and to illustrate the effect of modeling or neglecting swirl. For some of the numerical results presented here, a baseline rotor is used, which is similar to the Harrington Rotor 2, but uses a constant thickness airfoil section rather than a tapered section. The rotor system is composed of coaxial two-bladed counterrotating rigid rotors with zero coning. The rotors have a vertical separation of $\bar{z} = 0.16$, and the blades are untwisted with a uniform chord distribution of $\bar{c} = 0.12$. The lift curve is assumed to be linear with $c_{l\alpha} = 5.7$. The drag is described by a quadratic drag polar $c_d = c_{d0} + c_{d2}c_l^2$ with coefficients $c_{d0} = 0.011$ and $c_{d2} = 0.028$.

The influence of the upper rotor on the lower rotor operating within the upper rotor's contracted wake, and the influence of the lower rotor as a uniform inflow on the upper rotor, is modeled as described in the Technical Approach section. In the influence coefficient model, Eq. (12), the value of k below the rotor is assumed to be 0.6, resulting in a wake contraction ratio of $r_c/R = 0.866$. The Prandtl tip loss factor is included to approximate the effect of the finite number of blades on the induced downwash near the tip of each rotor. A root cutout is not included in this analysis.

For the analysis conducted here, the coaxial rotor performance is measured using the conventional figure of merit (FM) given by

$$FM = \frac{C_T^{3/2}}{\sqrt{2}C_P} \quad (24)$$

which is consistent with the figure of merit definition used by Coleman (Ref. 3). This conventional figure of merit can be converted to an alternative definition for coaxial rotors given by Leishman and Syal (Ref. 13) by multiplying it by a factor of $1/\sqrt{2}$.

Figure 2 shows the figure of merit versus the coefficient of thrust for the Harrington Rotor 2 experimental results and the baseline rotor results computed using the present combined BEMT analysis both with and without swirl. For each of the BEMT results, the shaft power of the two rotors is equal, i.e., the coaxial system is in torque balance, consistent with the Harrington experimental results. At least in terms

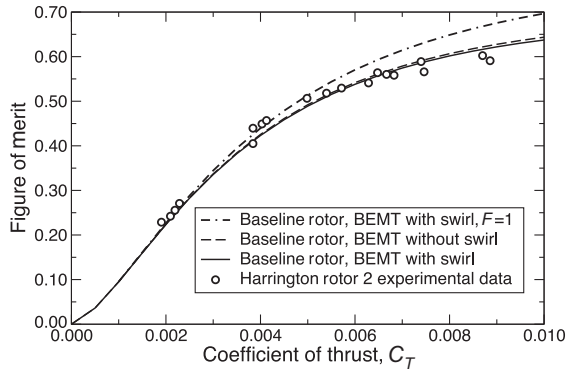


Fig. 2. Figure of merit versus coefficient of thrust for experimental results of the Harrington Rotor 2 (Ref. 6) compared to BEMT results, both with and without swirl.

of global performance predictions, the BEMT analysis is seen to be in good agreement with experimental results, consistent with the findings of Juhasz et al. (Ref. 8), and Leishman and Ananthan (Ref. 5). Including swirl in the analysis results in a decrease in figure of merit (for this particular rotor) of about 0.8% at a thrust coefficient of $C_T = 0.01$; the difference between the computed figure of merit with and without swirl increases with increasing thrust coefficient.

Optimized rotor results

Single rotor. In this section, results of rotor optimization are presented. For the first example, an inviscid version of the present optimization technique, including swirl, is applied to a single rotor and compared to two other well-known results—the ideal Betz inflow distribution (Ref. 14) for lightly loaded propellers and the optimal inflow distribution described by Glauert (Ref. 15) for rotors in hover. For a single rotor, the optimization process is significantly simplified, as there is no need for a mutual interference model or a constraint on torque balance. Additionally, because this is an inviscid optimization, profile losses and their effect on thrust and torque are not included. For the example considered here, tip corrections are not applied, i.e., $F = 1$.

The Betz lightly loaded optimal propeller inflow distribution (Ref. 14) has the parametric form

$$w(r) = w_0 \cos \phi \quad (25)$$

where w_0 is a constant. With this distribution of induced inflow, as described by Glauert (Ref. 15), “the vortex sheets, after an initial limited deformation, move axially backward as rigid screw surfaces.” However, this induced inflow distribution is close to, but not equal to the optimal inflow solution. In fact, slightly better performance can be achieved through a more complex induced inflow distribution found by setting the ratio of the change in blade element torque to a change in blade element thrust to be independent of the radial coordinate. Applying the optimization theory described in this paper to a single rotor, including the effects of swirl, the optimal solution described by Glauert (Ref. 15) is recovered.

Figure 3 shows the axial and swirl components of induced inflow for the optimum single rotor compared to the Betz induced inflow distributions at a thrust coefficient of $C_T = 0.004$. The two solutions differ mostly near the blade root, with the largest differences seen in the swirl components of the induced inflow.

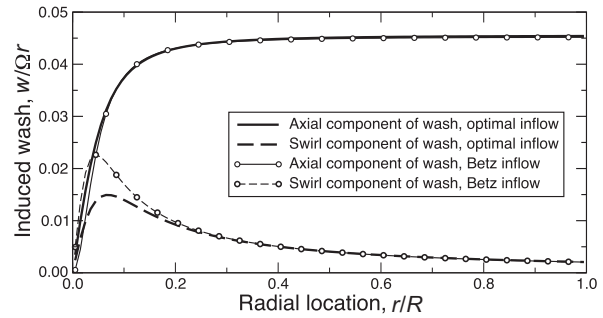


Fig. 3. Axial and swirl components of the induced inflow for the Betz inflow distribution compared to the optimal inflow distribution.

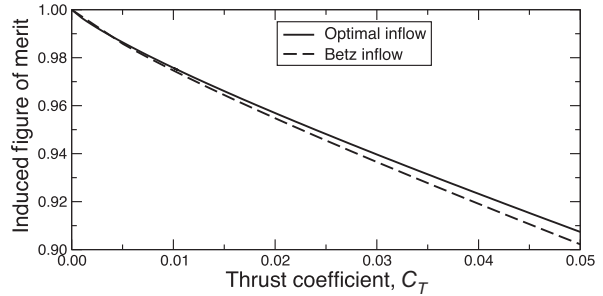


Fig. 4. Figure of merit versus the coefficient of thrust for the Betz inflow distribution compared to the optimal inflow distribution.

Figure 4 shows the induced figure of merit versus the coefficient of thrust for the optimal inflow distribution and the nearly optimal Betz inflow distribution. Glauert presents a similar figure (Figure 88 of Ref. 15); the optimization technique presented here replicates those results. The optimal inflow distribution has a slightly higher induced figure of merit than the Betz inflow distribution with an improvement of 0.57% in induced figure of merit at the very large thrust coefficient of $C_T = 0.05$. In summary, when applied to a single rotor with swirl, the optimization method presented here recovers the optimal single rotor–induced inflow distribution including swirl described by Glauert (Ref. 15).

Coaxial rotor. The present optimization technique is next applied to a coaxial rotor to determine the minimum loss coaxial rotor in hover. To quantify the effect of swirl on performance and on the optimal induced inflow distribution and rotor design, cases both with and without the swirl component of induced inflow are analyzed. In each instance, the shaft power of the two rotors is equal, i.e., the coaxial system is in torque balance. The Prandtl tip loss factor is included for each case, and the same mutual interference assumptions described in the preceding section comparing BEMT results to Harrington’s experimental results are used. For ease of comparison, the same airfoil characteristics, number of blades (two for each rotor), and vertical separation as the baseline rotor are used for the optimized rotors.

Figure 5 shows the convergence history of the optimization of the coaxial rotor. Shown is the computed figure of merit and upper and lower rotor powers as a function of iteration number. The figure of merit and the constraint that the power for the upper and lower rotors be equal are seen to converge in about eight iterations.

E. B. GIOVANNETTI

JOURNAL OF THE AMERICAN HELICOPTER SOCIETY

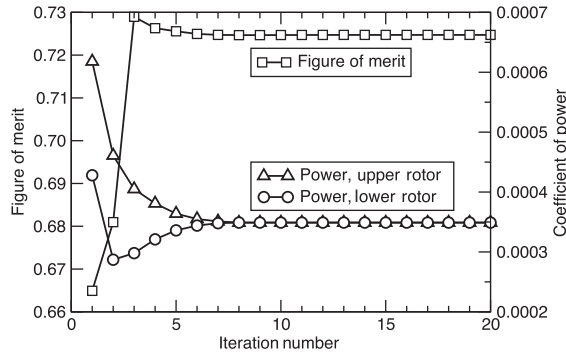


Fig. 5. Convergence of the figure of merit and torque constraints as a function of iteration number.

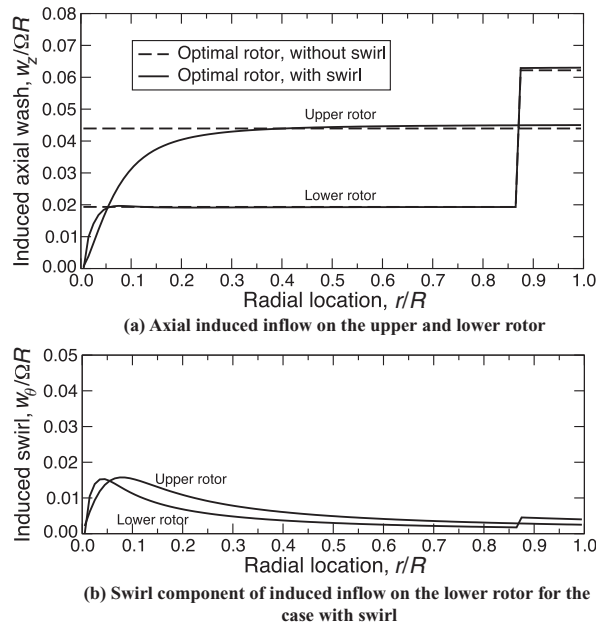


Fig. 6. Induced inflow at a thrust coefficient of $C_T = 0.008$ for optimally designed coaxial rotors, both accounting for and neglecting swirl.

Figure 6 shows the axial and swirl components of induced inflow for the optimal coaxial rotor at a thrust coefficient of $C_T = 0.008$. For both the upper and lower rotors, the swirl case is very similar to the nonswirl case, with the exception of radial stations near the blade root, where the axial-induced inflow goes to zero at a value of $r/R = 0$. For the upper rotor, the no-swirl case has a uniform induced inflow distribution, leading to the optimal performance originally reported by Gessow (Ref. 7).

Figure 7 shows the spanwise distributions of thrust, induced power, and profile power for the optimal rotors with and without swirl. The nonswirl optimal coaxial rotor generated using the present approach is in agreement with the optimization methodology of Rand and Khromov (Ref. 9) if applied using the same assumptions about mutual rotor interference and with tip losses neglected. Additionally, this rotor

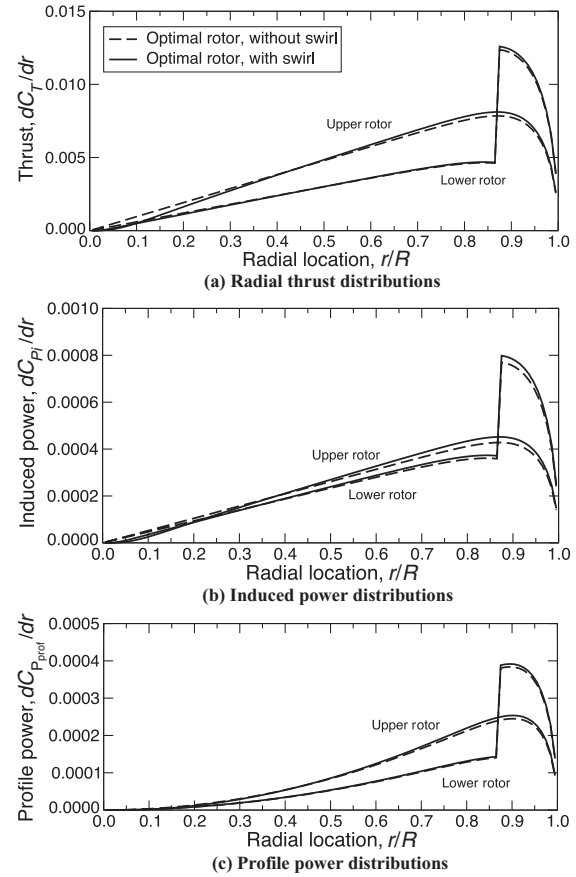


Fig. 7. Radial thrust, induced power, and profile power distributions at a thrust coefficient of $C_T = 0.008$ for optimally designed coaxial rotors, both accounting for and neglecting swirl.

gives better performance than the Leishman and Ananthan (Ref. 5) optimal hovering coaxial rotor, which uses a linear thrust distribution on each rotor.

For both the swirl and nonswirl cases, the inner portion of the lower rotor, which is operating in the contracted wake of the upper rotor, is significantly off-loaded, generating a smaller fraction of the lower rotor's total thrust. The outer portion of the lower rotor—which is outside the contracted wake of the upper rotor and is not influenced by the upper rotor's wash (based on the mutual interference model used here)—generates a much larger thrust and torque per unit span. The reason for this is that for minimum induced power, the axial velocity distribution in the combined wakes downstream of the rotors should be nearly uniform. The lower rotor draws in additional air at the tips that does not pass through the upper rotor. The increment in total pressure for this air must come entirely from the lower rotor. This is seen in Fig. 6. The induced axial inflow in the outer region of the lower blade is approximately equal to the sum of the axial-induced inflow of the upper and lower rotors in the inner region.

Figure 8 shows the radial twist and chord distributions for both the swirl and nonswirl optimal rotors. For the upper rotor, the nonswirl case shows the characteristic hyperbolic twist and chord distributions required to generate a uniform induced inflow distribution, as modified by the

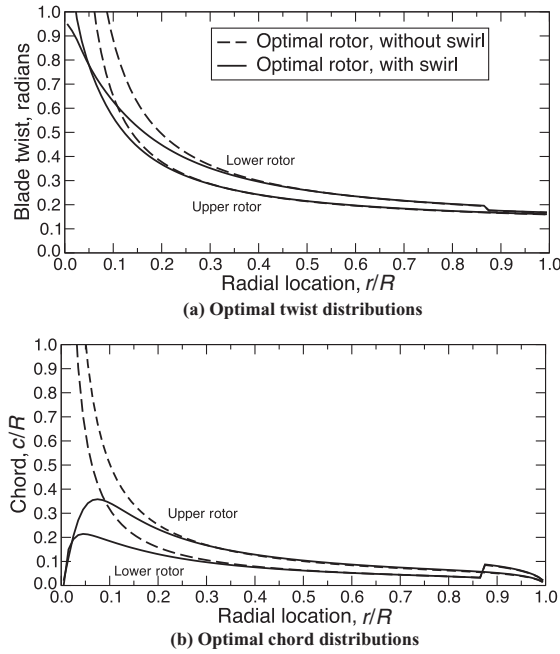


Fig. 8. Twist and chord distributions at a thrust coefficient of $C_T = 0.008$ for optimally designed coaxial rotors, both accounting for and neglecting swirl.

presence of tip losses. The lower rotor has a similar hyperbolic twist distribution with a breakpoint at an r_c/R of approximately 0.87, the radius outside of the contracted wake. The swirl case is nearly indistinguishable from the nonswirl case outboard of a radial position of r/R equal to 0.3. However, for values of r/R less than about 0.3 there are noticeable differences, as the chord of the swirl case upper and lower rotors goes to zero at an r/R equal to zero, whereas the chord of the nonswirl case upper and lower rotors continues to increase with decreasing r/R .

Figure 9 shows the upper and lower rotor planforms for the optimal coaxial rotor including the swirl component of induced inflow at $C_T = 0.008$. The optimal coaxial rotor uses very different blade planforms and twist distributions on the upper and lower rotors, a fact that is perhaps not surprising considering the significantly different inflow distributions of the two rotors. The difference in inflows between the upper and lower rotor also results in the upper rotor generating 56% of the total thrust to maintain a torque-balanced condition.

Figure 10 shows the figure of merit versus the thrust coefficient for the optimal coaxial rotor system (optimized at each thrust coefficient), both with and without the swirl component of induced inflow. The swirl case has a lower figure of merit across all thrust coefficients. The difference in the nonswirl and swirl case figure of merit increases with increasing thrust coefficient, i.e., the presence of swirl has a larger impact at higher loadings. At the high thrust coefficient of $C_T = 0.02$, the case accounting for swirl has a computed total figure of merit that is approximately 3.5% lower than the nonswirl case.

Next, the optimal rotors are analyzed at off design points to determine how they perform over a range of thrust coefficients that may be encountered in actual helicopter operation due to changes in vehicle weight and density altitude. Results are presented for two cases: the minimum induced power case, with only the twist optimized; and the minimum total power case, with both the twist and chord optimized.

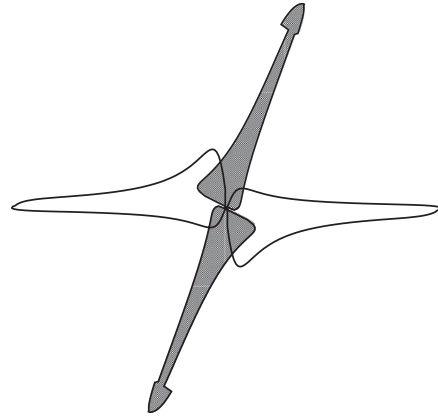


Fig. 9. Optimal blade planform for a hovering coaxial rotor at $C_T = 0.008$, based on combined BEMT including swirl. The upper rotor planform is shaded white, and the lower rotor is shaded gray.

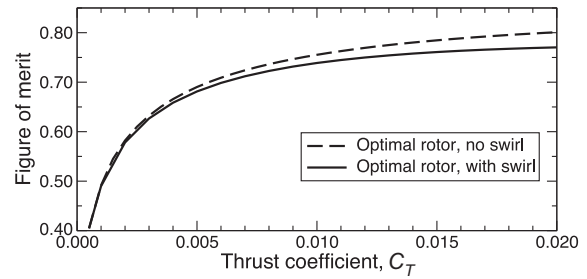


Fig. 10. Figure of merit versus the coefficient of thrust for the optimal coaxial rotor using BEMT, both with and without swirl.

Figure 11 shows the computed figure of merit versus thrust coefficient for a coaxial rotor with the twist distribution optimized and a uniform chord at $C_T = 0.008$ and $C_T = 0.012$. Also shown for comparison is the performance with the twist distribution optimized at every thrust coefficient, and the performance of the untwisted baseline rotor. All optimized cases shown are optimized using BEMT with swirl. The $C_T = 0.008$ and $C_T = 0.012$ rotors have a figure of merit very close to the rotors optimized anew at each thrust coefficient, even at off-design points. Additionally, these optimized rotors far outperform the untwisted baseline rotor, with a figure of merit 9.1% higher than the baseline baseline rotor at $C_T = 0.008$.

Figure 12 shows the figure of merit versus thrust coefficient for a coaxial rotor with both the twist and chord distributions (minimum total power) optimized at $C_T = 0.008$ and $C_T = 0.012$. Also shown is the figure of merit with the twist and chord distribution minimized at every thrust coefficient, and the figure of merit of the untwisted baseline rotor. Also plotted is the twist optimized case at every thrust coefficient, to show the performance improvement available using a nonuniform chord distribution. At off design points, the optimized twist and chord rotors have significantly degraded performance, in contrast to the twist only optimization. However, at their respective design points, the optimized twist and chord rotors far outperform the twist only optimization and the baseline rotor, with a figure of merit 20% higher than the baseline rotor at $C_T = 0.008$.

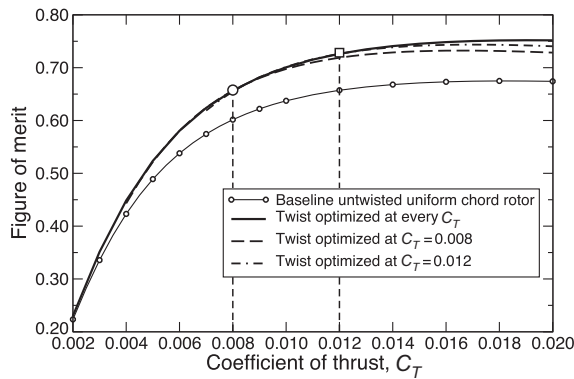


Fig. 11. Figure of merit versus the coefficient of thrust for the optimal $C_T = 0.008$ and $C_T = 0.012$ hovering coaxial rotors including swirl. Only the twist has been optimized in each case, i.e., only induced powers are minimized.

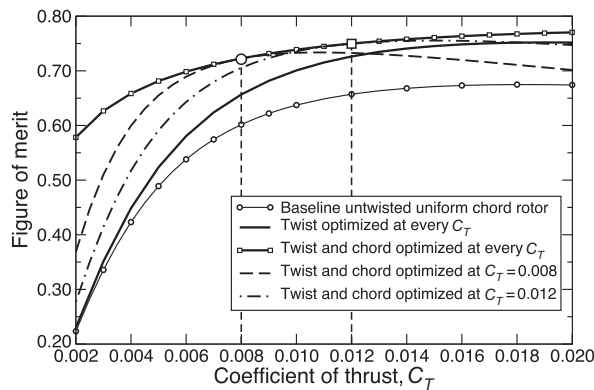


Fig. 12. Figure of merit versus the coefficient of thrust for optimal $C_T = 0.008$ and $C_T = 0.012$ hovering coaxial rotors including swirl. Both the chord and twist have been optimized, i.e., the sum of induced and profile power losses is minimized.

As previously observed, the optimal hovering coaxial rotor system has significantly different chord and twist distributions on the upper and lower rotors. Such a design, while optimal in hover, may not perform as well in forward flight. To quantify the benefit of using different blade designs on the upper and lower rotors, the performance of the optimal rotor is compared to a rotor using the upper blade design of the optimal on both rotors. In other words, the lower rotor is simply replaced with the upper rotor design, but oriented in the appropriate counterrotating direction. However, the upper and lower rotors use different amounts of collective control so that the two rotors can be torque balanced. Figure 13 shows the figure of merit versus the thrust coefficient for these two cases. Using different blade designs on the upper and lower rotors is seen to improve slightly the figure of merit, especially at higher thrust coefficients. For a thrust coefficient $C_T = 0.002$, there is a very small difference in performance between the rotor system with identical (mirrored) blades and the rotor system with different (optimal) upper and lower blades. For a thrust coefficient $C_T = 0.016$, the optimal rotor provides a figure of merit

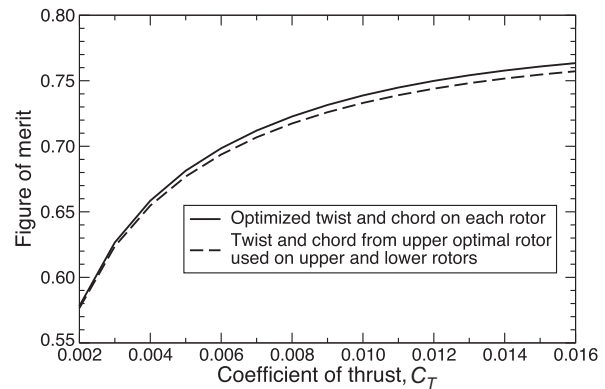


Fig. 13. Figure of merit versus the coefficient of thrust for the optimal hovering coaxial rotor including swirl at each thrust coefficient, compared to the performance of a coaxial rotor using the optimal upper rotor design for both the upper and lower rotor.

1.36% higher than the rotor with identical blades, a noticeable but relatively small improvement. Therefore, while using different blade twist and chord distributions on the upper and lower rotors is optimal, using rotor blades with similar upper and lower twist and chord distributions does not significantly reduce hover performance.

Conclusions

This paper presents a method for computing the optimal coaxial rotor design for hover using BEMT including the effect of the swirl component of induced inflow. The method determines the induced inflow distributions that minimize induced power and the twist and chord distributions that minimize profile power losses. The method models the effect of the contracted wake from the upper rotor on the axial and circumferential components of inflow acting on the inner portion of the lower rotor, and approximates the lower rotor's effect on the upper rotor as a uniform axial inflow dependent on the average induced inflow of the lower rotor and the vertical separation of the rotors. Tip effects accounting for a finite number of blades are included using the Prandtl tip correction factor. Profile losses and their effects on power and thrust are included using airfoil drag polars. This optimization approach was applied to a baseline coaxial rotor similar in design to the Harrington Rotor 2. Based on the work presented in this paper, the following conclusions are made:

- 1) The optimization methodology including swirl recovers the Glauert optimal induced inflow distribution for a single rotor, achieving a slightly higher induced figure of merit than the Betz inflow distribution.
- 2) The optimal coaxial rotors with and without swirl are nearly identical at radial stations outboard of $r/R = 0.3$. At values of r/R less than about 0.3, the rotors optimized accounting for swirl have significantly different optimal twist and chord distributions than those optimized neglecting swirl.
- 3) As expected, accounting for the effects of swirl has a larger effect on performance at higher thrust coefficients. The swirl case has an optimal figure of merit 3.5% lower than the no swirl case at a thrust coefficient C_T equal to 0.02.
- 4) At disk loadings typical of those seen on helicopters, the effect of swirl on rotor design and performance predictions within the framework of combined blade element momentum theory is relatively small. Other model refinements, such as detailed wake modeling and improved mutual

interference models, may have a larger impact on optimal rotor design and performance predictions than the inclusion of the swirl component of induced velocity.

5) Optimizing twist to minimize the induced power of the coaxial rotor studied here increases the figure of merit by 9% compared to a baseline untwisted rotor at a thrust coefficient C_T equal to 0.008. Optimizing both the twist and chord increases the figure of merit by 20% over the baseline rotor at a C_T of 0.008.

6) The optimal coaxial rotors have significantly different twist and chord distribution on the upper and lower rotors, resulting from the different inflow distributions acting on each rotor. However, using the optimal upper rotor blade design for both the upper and lower rotors (with a change in rotation direction for the lower rotor) leads to a relatively small decrease in performance, indicating that near optimal hover performance can be achieved with upper and lower rotors using the same blade design.

Acknowledgments

This material is based upon work supported by the U.S. Army Research Office under award no. W911NF-13-1-0469. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the view of the U.S. Army Research Office. Additionally, the first author was funded through a National Defense Science and Engineering Graduate Fellowship.

References

- ¹Hall, K. C., and Hall, S. R., "A Variational Method for Computing the Optimal Aerodynamic Performance of Conventional and Compound Helicopters," *Journal of the American Helicopter Society*, **55**, 042006 (2010).
- ²Johnson, W., "Influence of Lift Offset on Rotorcraft Performance," American Helicopter Society Aeromechanics Specialists' Conference, San Francisco, CA, January 23–25, 2008.
- ³Coleman, C. P., "A Survey of Theoretical and Experimental Coaxial Rotor Aerodynamic Research," NASA Technical Paper 3675, 1997.
- ⁴Nagashima, T., and Nakanishi, K., "Optimum Performance and Wake Geometry of a Co-axial Rotor in Hover," *Vertica*, Vol. 7, (3), 1983, pp. 225–239.
- ⁵Leishman, J. G., and Ananthan, S., "An Optimum Coaxial Rotor System for Axial Flight," *Journal of the American Helicopter Society*, Vol. 53, (4), 2008, pp. 366–381.
- ⁶Harrington, R. D., "Full-Scale-Tunnel Investigation of the Static-Thrust Performance of a Coaxial Helicopter Rotor," NACA Technical Note 2318, 1951.
- ⁷Gessow, A., "Effect of Rotor-Blade Twist and Plan-Form Taper on Helicopter Hovering Performance," NACA Technical Note 1542, 1948.
- ⁸Juhasz, O., Syal, M., Celi, R., Khromov, V., Rand, O., Ruzicka, G. C., and Strawn, R. C., "Comparison of Three Coaxial Aerodynamic Prediction Methods Including Validation with Model Test Data," *Journal of the American Helicopter Society*, **59**, 032006 (2014).
- ⁹Rand, O., and Khromov, V., "Aerodynamic Optimization of Coaxial Rotor in Hover and Axial Flight," 27th International Congress of the Aeronautical Sciences, Nice, France, September 19–24, 2010.
- ¹⁰Syal, M., and Leishman, J. G., "Aerodynamic Optimization Study of a Coaxial Rotor in Hovering Flight," *Journal of the American Helicopter Society*, **57**, 042003 (2012).
- ¹¹Modarres, R., and Peters, D. A., "Optimum Actuator-Disk Performance in Hover and Axial Flight by a Compact Momentum Theory with Swirl," *Journal of the American Helicopter Society*, **60**, 012003 (2015).
- ¹²McAlister, K. W., Tung, C., Rand, O., Khromov, V., and Wilson, J. S., "Experimental and Numerical Study of a Model Coaxial Rotor," American Helicopter Society 62nd Annual Forum Proceedings, Phoenix, AZ, May 9–11, 2006.
- ¹³Leishman, J. G., and Syal, M., "Figure of Merit Definition for Coaxial Rotors," *Journal of the American Helicopter Society*, Vol. 53, (3), 2008, pp. 290–300.
- ¹⁴Betz, A., "Schraubenpropeller mit geringstem Energieverlust. Mit einem Zusatz von L. Prandtl," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1919, pp. 193–217.
- ¹⁵Glauert, H., "Airplane Propellers," in *Aerodynamic Theory*, Vol. IV, Div. L, W.P. Durand, Ed., Julius Springer, Berlin, 1935.

Appendix IV: Axisymmetric Potential Flow Model of Single or Coaxial Actuator Disks

Axisymmetric Potential Flow Model of Single or Coaxial Actuator Disks

Eli B. Giovanetti

Graduate Research Assistant
Mechanical Engineering and Materials Science
Duke University
Durham, NC, U.S.

Kenneth C. Hall

Julian Francis Abele Professor of
Mechanical Engineering and Materials Science
Duke University
Durham, NC, U.S.

ABSTRACT

This paper presents a computationally efficient method for computing the axisymmetric three-dimensional flow field induced by the trailing vortex system of either a single actuator disk or a pair of closely spaced coaxial actuator disks. A rotor in axial flight or hover (or a pair of coaxial rotors) is modeled as an actuator disk (or disks); the associated wake is modeled using contracting cylindrical sheets of vorticity approximated by discrete vortex rings. The resulting system of vortex sheets is axisymmetric and aligned with flow streamlines. The location of the sheets of vorticity is found using Newton iteration. A singularity occurs where the outer vortex sheet vortex sheet terminates at the edge of the actuator disk. This singularity is resolved through the formation of a 45° logarithmic spiral in hover, which produces a non-uniform inflow at the face of the disk, particularly near the edge of the disk where the flow field is entirely reversed. Finally, the model is applied to coaxial actuator disks to quantify the mutual interference of coaxial rotors at various axial spacings, and when operating in either a torque-balanced or and equal circulation state.

NOMENCLATURE

ds	= Distance between successive vortex rings
m	= Constant used to set spacing of vortex rings
r	= Radius
P	= Rotor power
R	= Radius of actuator disk
$\mathbf{R}$	= Residual vector
$\mathbf{R}_\theta$	= Jacobian of the residual vector with respect to the wake position
T	= Rotor thrust
u	= x (radial) component of induced velocity
V_c	= Axial free stream velocity; positive indicates climbing flight, negative descending flight
V_i	= Induced velocity at the actuator disk
$\mathbf{V}$	= Induced velocity field
w	= z (axial) component of induced velocity
x	= Radial position of vortex sheet
z	= Axial position of vortex sheet
Γ	= Circulation
θ	= Vector of angles between a segment of the vortex sheet and the positive x axis
ρ	= Density

Subscripts:

P	= Coordinate at which velocity is measured
R	= Coordinate of sending vortex ring

Superscripts:

k	= Iteration number
-----	--------------------

INTRODUCTION

Momentum theory, also known as actuator disk theory, provides a simple but powerful tool for predicting the performance of propellers, windmills, and rotors. First developed by Rankine (Ref. 1) and extended by Froude (Ref. 2), momentum theory models the rotor as an axisymmetric actuator disk, a disk of infinitesimal thickness through which incompressible air flows, conserving mass, but producing a finite jump in pressure. In its simplest form, the induced flow and thrust loading is uniform over the disk. As originally observed by Lanchester (Ref. 3), however, this model presents inconsistencies, as the infinitely thin actuator disk cannot maintain a pressure jump at the edge without a singularity. In this paper, we develop a computationally efficient model for computing the flow field induced by the trailing vortex system of single or coaxial actuator disks that accurately accounts for this singularity at the edge of the disk. We apply this model to single or coaxial actuator disks to investigate the shape of the wake and the induced velocities at the actuator disks and elsewhere in the flow field.

A number of researchers have investigated the flow field induced by propellers in axial flight. Betz and Prandtl (Ref. 4) used vortex theory to compute the optimal induced velocity profile at the disk for a lightly loaded propeller with infinite blades, including the effects of swirl. Goldstein (Ref. 5) extended this work to determine the optimal induced velocity of a lightly loaded finite bladed propeller. Hough and Ordway (Ref. 6) used a vortex representation of the helical trailing

wake of a lightly loaded propeller similar to that of Goldstein, and computed the induced velocities at any point in the field using the Biot-Savart law. Hough and Ordway (Ref. 6) also Fourier analyzed these velocity fields in the circumferential direction, and then considered the zeroth harmonic, i.e., the mean component. The authors demonstrated that the steady component of the induced velocities for a finite bladed propeller are equivalent to the actuator disk (infinite blade) result.

Conway (Ref. 7) used an analytical approach to solve for the induced velocities of a propeller actuator disk of arbitrary radial loading. Conway (Ref. 8) later extended this approach to develop a semi-analytical nonlinear actuator disk model for arbitrary loading, taking the slipstream contraction into account. Conway did not, however, extend his model to the zero free stream (i.e., hover) case. Greenberg and Powers (Ref. 9) developed an iterative approach to determining the wake geometry and vorticity distribution on an actuator disk, and applied the model to the hovering actuator disk with both uniform and radially varying loading.

One problem with the seemingly simple actuator disk model is the way in which the vortex sheet is terminated at the edge of the disk. In a 1915 paper, Lanchester (Ref. 3) observed that momentum theory was inconsistent at the edges of the disk, as the infinitely thin actuator disk cannot maintain a pressure jump at the edge without a singularity. Lanchester predicted that as a result of the neglect of this singularity, the classical momentum theory would underestimate the induced velocity at the disk.

Conway (Ref. 8) addresses this apparent inconsistency, and observes that while a numerical scheme could be constructed to compute the details of the roll up of the vortex sheet at the outer edge of the disk, doing so would undermine the simplicity of the actuator disk model. Schmidt and Sparenberg (Ref. 10) were able to compute a numerical solution for an actuator disk of infinite radius (essentially an actuator strip) and zero axial velocity. The authors found that the vortex sheet exhibits a spiraling behavior near the outer edge of the disk, and were able to apply this spiral structure to a uniformly loaded actuator disk of finite radius. They found that streamlines pass through the disk more than once at the disk's edge as a result of the spiraling vortex sheet.

In 2005, Spalart (Ref. 11) revisited the actuator disk, making several interesting observations. As Conway, Lanchester, and others had previously observed, in the classical model a vortex sheet ends abruptly at the outer edge of the disk, which would result in an infinite velocity; Spalart therefore rules out an abrupt termination of the sheet as a viable solution. Instead, he postulates that in hover "it is most likely that the exact shape [of the vortex sheet], for the infinitely thin sharp-edged disk, is a volute wound around the disk's edge. The two dimensional Euler equations admit an exact solution with a vortex sheet of uniform density placed on a 45 degree spiral." He notes, however, that such a solution, either analytical or numerical, does not exist in the literature (although such a solution was in fact found previously by Schmidt and Sparenberg (Ref. 10)). Additionally, Spalart observes that the in-

duced velocity at the actuator disk is far from uniform (and is completely reversed at the edge), an observation that he notes conflicts with both common beliefs and one-dimensional arguments.

In this paper, we construct a steady axisymmetric potential flow model of the trailing wake of an actuator disk. To determine the geometry of the vortex sheet, we impose a boundary condition that the vortex sheet must be aligned with streamlines of the flow field at all points, i.e., that there is no flow through the sheet. We use Newton iteration to determine the shape of the vortex sheet that satisfies this boundary condition. We represent the continuous axisymmetric vortex sheet as a system of discrete vortex rings. For a uniformly loaded actuator disk in hover, we find that the wake at the outer edge of the actuator disk assumes the shape of a 45 degree logarithmic spiral, as hypothesized by Spalart (Ref. 11). Also in accordance with the findings of Spalart and the analysis of Schmidt and Sparenberg (Ref. 10), we find that the induced flow distribution is far from uniform over the disk, and is in fact reversed near the edge of the disk. We examine the shape of the wake, including its spiral edge structure, at a range of axial velocities, including negative axial velocities with the rotor acting as a windmill. We use the same wake modeling approach to investigate coaxial actuator disks, and examine the effect of vertical separation on the resulting flow field and rotor performance. Finally, we compare the mutual interference of the coaxial rotor predicted in our wake model to that computed using the simple influence coefficient model described in McAlister et al. (Ref. 12), and find that despite the large discrepancy in complexity, the two models produce reasonably similar results.

WAKE MODEL

As described by Conway in Reference (Ref. 7), for a uniformly loaded actuator disk, there are four vortex distributions that induce perturbations to the mean flow. The first is a contracting vortex tube consisting of discretized ring vortices, shown in Figure 1. These ring vortices are trailed from the edge of the actuator disk and extend to downstream infinity. This distribution is itself a vorticity conserving system and is solely responsible for all axial and radial induced velocities and for the contraction of the wake itself. In addition to the ring vortex system, there are three other vortex systems that combine to form their own vorticity conserving system: a constant strength hub vortex at the center of the disk extending to downstream infinity, a distribution of vorticity on the actuator disk itself, extending radially, and a distribution of vorticity on the streamtube's surface that is normal to the ring vortices and equal in strength to the hub vortex. The only non-zero component of velocity they induce is in the circumferential direction, i.e., the swirl component of induced velocity. The swirl component of induced velocity has a relatively small effect on rotor performance and design at the disk loadings typically seen on helicopters. As a result, we model only the axial and radial induced velocity fields and wake contraction, which can be accomplished by modeling exclusively the ring vortex system.

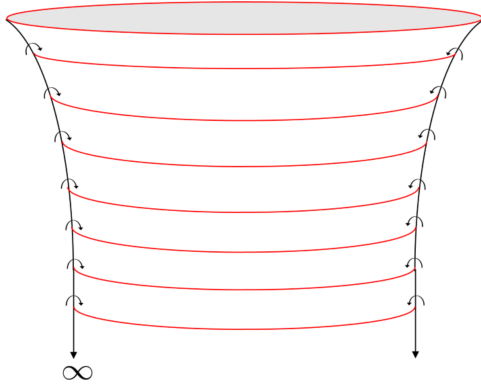


Fig. 1. The vortex ring system of a uniformly loaded actuator disk.

Ring Vortices

We compute the velocity field induced at a point in space P by a vortex line of differential length $d\mathbf{l}$ using the Biot-Savart law

$$\mathbf{V} = \frac{\Gamma}{-4\pi} \int_l \frac{\mathbf{r} \times d\mathbf{l}}{\|\mathbf{D}\|^3} \quad (1)$$

where $\mathbf{D}$ is a vector extending from P to the line segment $d\mathbf{l}$. Because of the axial symmetry of the problem, the induced flow field will include only axial and radial velocities. We can integrate Eq. (1) around the circumference of a unit strength vortex ring to yield expressions for the axial and radial velocities induced at some P , with coordinates denoted x_P, z_P . We denote the x and y positions of the vortex ring's intersection with the meridional plane as x_R, z_R , and denote the circumferential coordinate of the ring as ϕ_R , giving

$$u(x_P, z_P) = \int_{\phi=0}^{\phi=2\pi} \frac{(z_P - z_R) \cos(\phi_R) x_R d\phi_R}{4\pi \|\mathbf{D}\|^3} \quad (2)$$

$$w(x_P, z_P) = \int_{\phi=0}^{\phi=2\pi} \frac{[x_R - x_P \cos(\phi_R)] x_R d\phi_R}{4\pi \|\mathbf{D}\|^3} \quad (3)$$

where $u(x_P, z_P)$ and $w(x_P, z_P)$ are the induced radial and axial velocities, respectively, due to a unit strength vortex ring.

Contraction of the Wake

In our steady state wake model, we impose a boundary condition that the vortex sheet, as represented by discrete vortex rings, is aligned with the streamlines at all points. This no-through-flow boundary condition allows us to define the position of the vortex sheet. Of course, the problem can be challenging to solve, as the flow field is not known *a priori*, but is rather a function of the mean flow and the velocity induced by the vortex sheet itself, which in turn is a function of its position.

Because the problem is axisymmetric, we need only to satisfy the no through flow boundary condition at a single azimuthal slice of the disk. By defining the radial and axial position of each vortex ring at this single azimuthal location, we

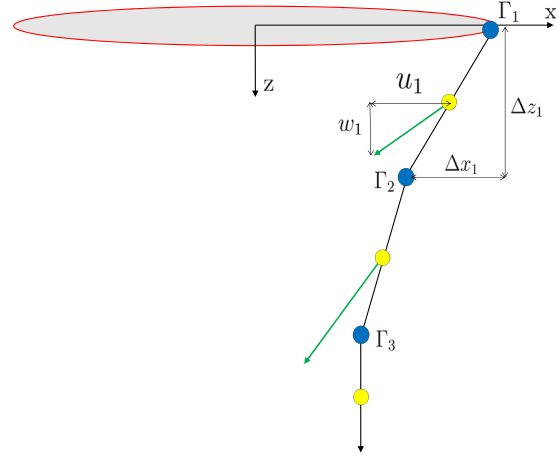


Fig. 2. An azimuthal cut of the axisymmetric vortex ring model, with collocation points shown as yellow circles, velocity vectors acting at the collocation points as green arrows, and the intersection of the vortex rings with the selected azimuthal plane as blue circles.

can fully define the wake. Within this azimuthal slice, we place collocation points equidistant between each successive vortex ring.

We compute the induced velocity at a given point in space by summing the influence of every vortex ring in the model on that point. To efficiently model the semi-infinite wake, which extends indefinitely in the positive z direction, we compute the sending influence of vortex rings until their influence on the contracting portion of the wake is sufficiently small. At every spatial location, the total flow velocity is then comprised of the sum of this induced velocity and the mean flow due to axial motion of the disk. In hover, the mean flow is of course zero. In axial flight, however, the flow can be in the positive z direction (ascending flight) or the negative z direction (descending flight). To determine if the no-through-flow boundary condition is satisfied, we compute the induced and total flow velocity at each collocation point. We denote the x and z components of flow velocity at a given collocation point i as u_i and w_i , respectively. Because the ring vortex system does not induce any circumferential velocities, in a given azimuthal plane, the y component of velocity is always zero.

We then need to define the slope of the sheet at each collocation point to assess the flow tangency boundary condition. To illustrate this process, Figure 2 shows a simplified schematic of a wake model consisting of only three vortex rings. Note that the first ring is placed at the edge of the disk. The first collocation point is then located midway between the first and second vortex rings. The straight line segment connecting the two neighboring ring vortices then defines the slope of the vortex sheet on which a given collocation point lies. As illustrated here, the first segment of the vortex sheet has slope $\Delta z_1 / \Delta x_1$.

We can express the boundary condition at the i^{th} collocation

tion point as

$$\frac{\Delta z_i}{\Delta x_i} = \frac{w_i}{u_i} \quad (4)$$

where w_i is the sum of the axial induced and climb velocity, u_i is the component of induced velocity in the radial direction. Δx_i and Δz_i are defined by

$$\Delta x_i = x_{i+1} - x_i \quad (5)$$

$$\Delta z_i = z_{i+1} - z_i \quad (6)$$

Newton Iteration to Determine Wake Position

Satisfying the no through flow condition is complicated slightly by the fact that the induced flow field changes with a change in any vortex ring's position, making the no-through-flow condition nonlinear. We use an iterative approach based on Newton iteration to determine the correct vortex ring positioning.

We use the no-through-flow boundary condition in Eq. (4) to construct a vector of residuals $\mathbf{R}$ that we wish to set to zero. The i^{th} entry of $\mathbf{R}$ is defined by

$$R_i(x_i, z_i) = \Delta x_i w_i - \Delta z_i u_i \quad (7)$$

We have reached a wake shape that satisfies the no through flow boundary condition when all of the entries of $\mathbf{R}$ are equal to zero, i.e., when

$$\mathbf{R}(\mathbf{x}, \mathbf{z}) = 0 \quad (8)$$

To efficiently parameterize the position of the wake, we define an angle θ_i that represents the angle between the i^{th} segment of the vortex sheet and the positive x axis. We then define ds_i as the distance between vortex ring i and ring $i+1$, as illustrated in Figure 3. Note that this value does not have to be uniform along the wake (i.e., the vortex rings can be non-uniformly spaced), but it is fixed from iteration to iteration. Therefore, we can fully define the shape of the wake using the angle θ_i at each vortex ring.

We use Newton iteration to determine the entries of the vector $\boldsymbol{\theta}$ such that

$$\mathbf{R}(\boldsymbol{\theta}) = 0 \quad (9)$$

At each iteration k , we update the vector of angles $\boldsymbol{\theta}$ using

$$\boldsymbol{\theta}^{k+1} = \boldsymbol{\theta}^k - \mathbf{R}(\boldsymbol{\theta}^k)^{-1} \mathbf{R}(\boldsymbol{\theta}^k) \quad (10)$$

where $\mathbf{R}(\boldsymbol{\theta}^k)_\theta$ is the Jacobian of the vector function $\mathbf{R}$ with respect to $\boldsymbol{\theta}$. It is difficult to compute the Jacobian of $\mathbf{R}(\boldsymbol{\theta})$ analytically. Instead, we use automatic differentiated code generated by the compiler Tapenade (Ref. 16) to compute the entries of the Jacobian $\mathbf{R}(\boldsymbol{\theta}^k)_\theta$.

Vortex Scaling

To satisfy Kelvin's circulation theorem, we must scale the strength of the discrete vortex rings as the shape of the wake changes throughout the iteration. First, it is often advantageous to use a non-uniform spacing of the vortex rings, meaning that the value of ds between each successive vortex ring

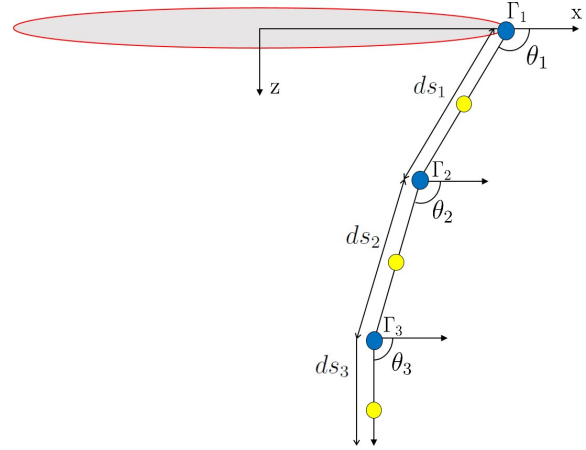


Fig. 3. Schematic showing the definition of the angle θ_i and length ds_i that fully define the position of each vortex ring.

is not constant. Typically, a more dense ring spacing near the disk is necessary to capture the details of the wake structure, while in the far field, a larger spacing is sufficient. To account for this, we scale the vortex rings' strengths by their spacing ds relative to some arbitrary reference length ds_{ref} with a known circulation Γ_{ref} , i.e.,

$$\Gamma_i = \Gamma_{\text{ref}} \frac{ds_i}{ds_{\text{ref}}} \quad (11)$$

In addition to scaling by the length of the wake each ring represents, we must consider the change in vortex density within this region. Each vortex ring represents a continuous sheet of vorticity that is convecting along the streamtube at the wake transport velocity. We must scale the strength of each vortex ring to account for this varying vortex density such that the quantity $\gamma_i V_i$ is constant throughout the sheet, where V_i is the component of total velocity parallel to a vortex segment ds_i and γ_i is the vortex density of the sheet. To account for this, we define the circulation for some reference ring with a known velocity V_{ref} and scale the circulation of each vortex ring using

$$\Gamma_i = \Gamma_{\text{ref}} \frac{V_{\text{ref}}}{V_i} \quad (12)$$

In total, we scale the circulation strength of each vortex ring by these two ratios, all relative to some reference ring. In practice, we often fix the circulation strength of a ring in the far wake and use this value as Γ_{ref} . The total scaling for each ring is then

$$\Gamma_i = \Gamma_{\text{ref}} \frac{ds_i}{ds_{\text{ref}}} \frac{V_{\text{ref}}}{V_i} \quad (13)$$

Iterative Solution Algorithm

We update the circulation strength of each vortex ring using Eq. (13) after each Newton iteration step. The Jacobian $\mathbf{R}(\boldsymbol{\theta})_\theta$ does not account for the change in circulation that will occur with a change in wake shape and convection velocity.

While we could formulate a computation of the Jacobian that accounts for these secondary effects, it is sufficient to simply update the circulation after each Newton step. Although this leads to a decrease in the accuracy of the Jacobian and the resulting Newton step, the changes in circulation from one iteration to the next are generally small and become progressively smaller as the iteration reaches convergence, i.e., as $\mathbf{R}(\boldsymbol{\theta}) \rightarrow 0$. Therefore, neglecting the change in circulation in the computation of the Jacobian has a negligible impact on the ability of the Newton iteration to locate a solution.

Multiple Vortex Sheet Model

Analyzing a system with multiple vortex sheets, such as in the case of coaxial actuator disks or a single actuator disk that does not have a uniform bound circulation and trails additional vortex sheets at locations inboard of the edge of the disk, uses fundamentally the same algorithm and wake definition described previously. Multiple vortex sheets are discretized into rings, and the positions of each of these rings that satisfy the no-through-flow boundary condition are still determined using Newton iteration. The induced velocity at a given collocation point is dependent on all vortex sheets. Additional care must be taken to ensure that the separate vortex sheets do not cross, which results in large induced velocities, poorly conditioned Jacobians, and chaotic behavior leading to divergence. In cases such as these (e.g., a very close radial spacing), a significant under-relaxation factor is used to prevent crossing of the vortex sheets.

Figure 4 shows the same axisymmetric vortex sheet model for coaxial disks. Points associated with the upper disk's vortex sheet are labeled as $(1,x)$, while those associated with the lower are labeled as $(2,x)$. Note that in the coaxial case, the upper disk's vortex sheet passes through the lower actuator disk. We have the option of maintaining the same circulation strength for rings that have passed through the lower disk (subject to the correct circulation scaling of Eq. 13) or of changing the strength of these ring vortices to reflect a change in the bound circulation of the lower disk. This allows us to model a non-uniform circulation distribution on the lower disk without introducing a third vortex sheet. We use this method to more easily approximate the minimum power thrust distributions determined by Rand and Khromov (Ref. 13) and Giovanetti and Hall (Ref. 14), where the lower rotor generates a much larger thrust at points outside of the upper rotor's contracted wake.

COMPUTATIONAL RESULTS

Single Actuator Disk in Hover or Axial Ascent

We first apply our axisymmetric vortex sheet model of the wake to a single actuator disk at varying rates of axial ascent (climb). In axial ascent, the free stream flow has a positive value and acts in the same direction as the induced flow (like a propeller). In this section, we investigate the structure of the wake, particularly near the edge of the disk, and see how

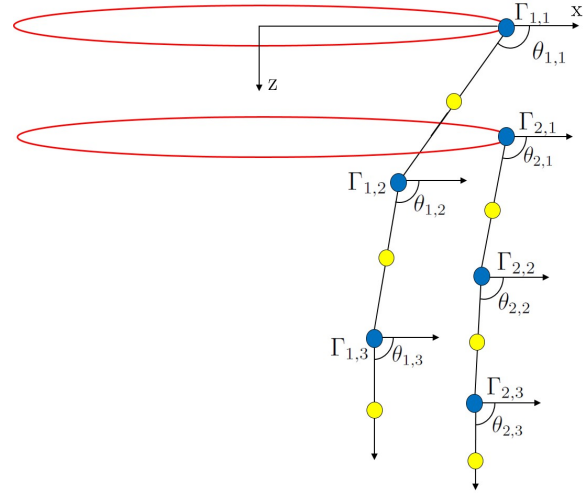


Fig. 4. Schematic of the vortex ring structure of coaxial actuator disks. Yellow circles represent collocation points. Blue circles represent the location of the vortex rings within this azimuthal plane.

this structure evolves with different rates of axial ascent. We express the rate of climb as the ratio of the climb velocity V_c to the induced velocity at the center of the actuator disk, $V_i(x=0)$. A value of $V_c/V_i = 0$ corresponds to the hover case. For the cases shown here, we use a logarithmic spacing of the ring vortices to allow for small spacing near the edge of the disk where finer detail is needed, and larger spacing downstream where the wake has a slower rate of contraction and a larger spacing is sufficient. We set the entries of the vector of ring spacings ds using

$$ds_i = \ln\left(1.0 + \frac{i}{m}\right) \quad (14)$$

where the index i starts at a value of 1 at the rim of the actuator disk and the variable m is a constant that can be varied to increase or decrease the spacing for a given case. For the single actuator disk hover case shown here, we set $m = 10,000$ and use 250 rings in the model, while for the axial flight cases, we set $m = 2,000$ and use 115 rings.

Figure 5 shows the computed shape of the wake in the vicinity of the rim of the disk for several different climb velocities, including hover. Figure 6 shows the same result for a larger domain, allowing us to observe the final contraction of each case. Note that because our model is axisymmetric, we need only look at a given azimuthal slice to visualize the wake. Our results appear to directly support the hypothesis of Spalart (Ref. 11) that the exact shape of the vortex sheet near the edge of the actuator disk in hover is a 45° logarithmic spiral. Figure 7 shows a closeup of the hovering case near the edge of the disk. Drawn on the graph are lines tangent to the wake sheet at the radial coordinate $x = 1$, the x location of the center of the spiral. These tangent lines have a slope of -1 , demonstrating graphically that the wake structure of the hovering rotor very closely approximates a 45° logarithmic

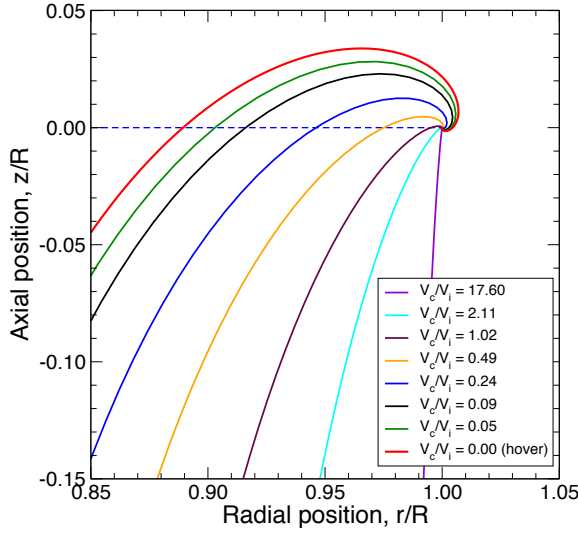


Fig. 5. The shape of the wake near the edge of the actuator disk with decreasing climb velocity. The blue dashed line indicates the location of the actuator disk.

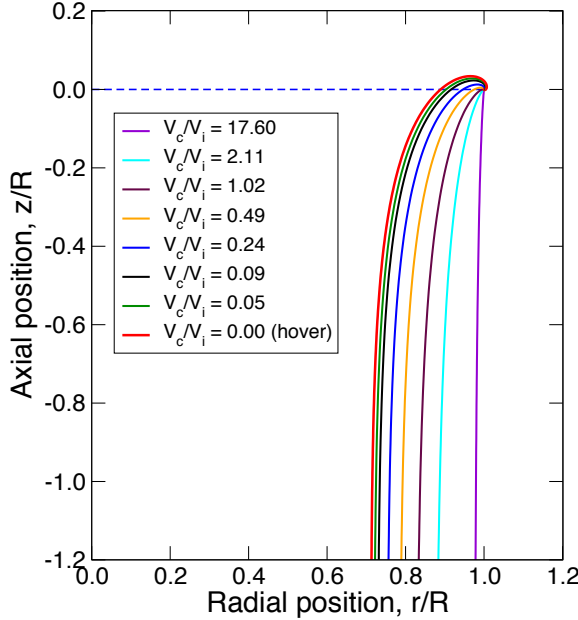


Fig. 6. The shape of the wake with decreasing climb velocity. The blue dashed line indicates the location of the actuator disk.

spiral. Interestingly, despite the spiral structure at the edge of the disk and the resultant non-uniform induced inflow, the final wake contraction for the hover case shown in Figure 6 still approaches the value predicted by a one-dimensional momentum balance of $R_c/R = 1/\sqrt{2} \approx 0.7071$.

Additionally, our findings are in agreement with another of Spalart's assertions, i.e., the extent of the spiral wake shrinks in axial flight but is still present. Figure 5 shows that at a climb ratio of $V_c/V_i = 2.11$, the extent of the spiral has shrunk such

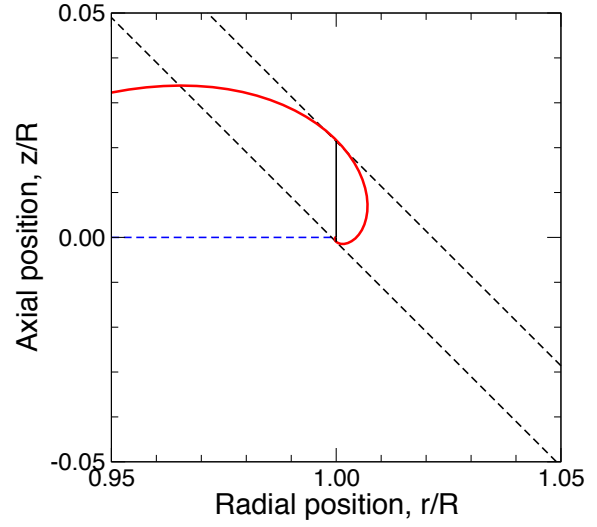


Fig. 7. The shape of the wake near the edge of the actuator disk for the hovering case. Note that the tangent lines are drawn at 45° , indicating that the wake has formed a logarithmic spiral of 45° pitch.

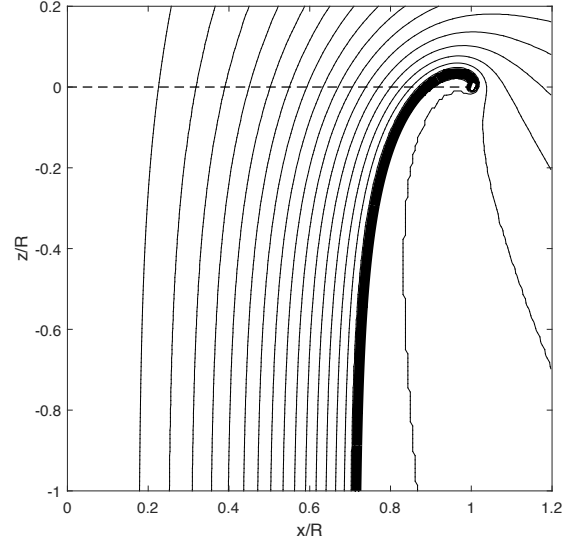


Fig. 8. Streamlines for the hovering actuator disk.

that it is no longer visible with our vortex ring discretization. However, at climb velocities lower than this, there is still a very clear spiral structure in which the vortex sheet goes above the actuator disk before passing back through it.

Figure 8 shows streamlines for the hovering case. Note that the actuator disk is drawing in flow from all points outside of the streamtube, including areas below and outside of the disk. This is in contrast to the notional actuator disk often depicted where the streamlines have a funnel shape, indicating that the disk is only drawing in flow from above.

Figure 9 shows the axial and radial induced velocities at the face of the actuator disk. In these plots, a downwash on

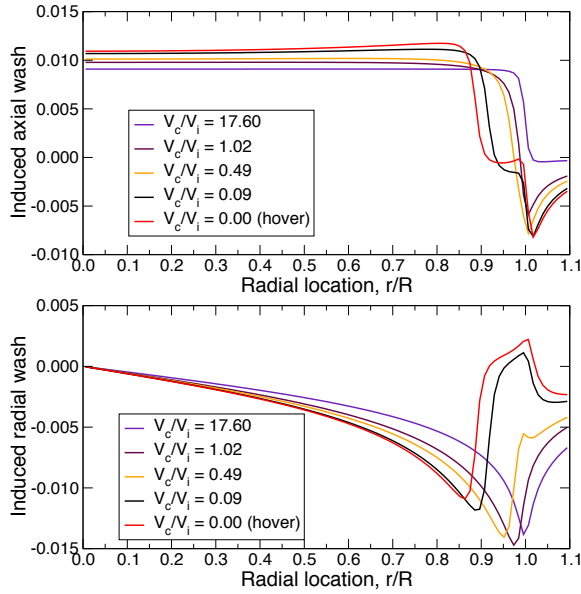


Fig. 9. Top: Radial distribution of induced axial velocity for varying climb velocities. Bottom: Radial distribution of induced radial velocity for varying climb velocities.

the disk is taken as positive. Note that we have extended the radial coordinate to a value of 1.1, beyond the edge of the actuator disk, to characterize the flow in this region. For the high climb velocity case of $V_c/V_i = 17.60$, we see that the axial induced velocity distribution is nearly uniform, with a sharp drop off at the edge of the disk to a value of approximately zero beyond the edge of the disk. This nearly uniform inflow is consistent with the classical actuator disk assumptions. As the climb velocity decreases, however, the inflow becomes less uniform. The most striking example is hover. There is a sharp drop in induced inflow at $r/R = 0.87$, which is where the spiraling vortex sheet passes through the disk. At points outboard of this intersection, the induced velocity is slightly negative, which is the result of the reversed flow at the disk's edge. The highly non-uniform inflow distribution over the actuator disk has been observed in numerical and analytical solutions by Conway (Ref. 8), Greenberg and Powers (Ref. 9), and Schmidt and Sparenberg (Ref. 10), and was a central topic of Spalart's analysis of the actuator disk in (Ref. 11).

The radial velocities, shown in the bottom plot of Figure 9, go to zero at the center of the actuator disk and have a large negative value at the edge of the disk, indicating radial inflow. At values of V_c/V_i at or below 0.09, we observe a significant change in the radial velocity distribution. This is due to the formation of a larger spiral at the edge of the disk, resulting in a radial outflow as fluid passes through the disk before circling around the tip of the spiral and being ingested by the disk.

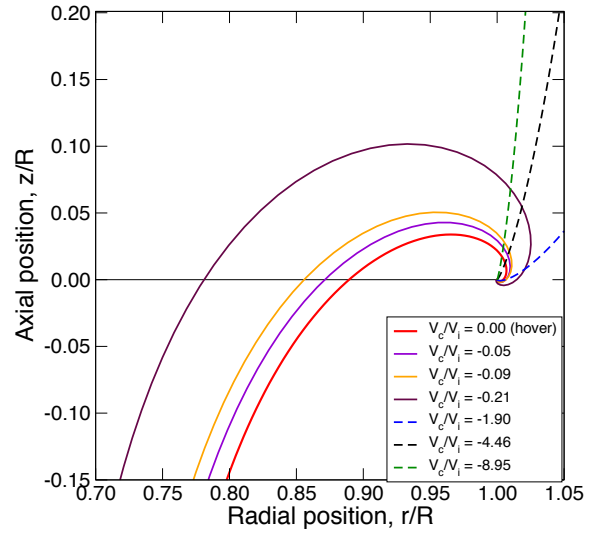


Fig. 10. The shape of the wake near the edge of the actuator disk with various negative climb velocities, i.e., in descending flight. The dashed lines indicate descent velocities at which the rotor has fully transitioned into the windmill state.

Single Actuator Disk in Descent

Figure 10 shows the shape of the vortex sheet for a range of negative axial velocities, i.e., for cases where the disk is descending and the induced wash is in the opposite direction of the free stream. Figure 11 shows a zoomed out view of the same cases. We see that for larger magnitude negative axial velocities (decreasing V_c/V_i) the spiral at the edge of the actuator disk continues to expand. However, at values of V_c/V_i between -0.2 and -1.9 , the iterative method fails to converge. This flow condition is commonly referred to as the vortex ring state. Due to the similarity in magnitudes between the induced velocity at the disk and the free stream velocity, and the fact that the two velocities are in opposite directions, this flow state is characterized by significant recirculatory flow and sometimes turbulent flow conditions. At $V_c/V_i < -1.9$, the descent velocity is large enough to fully reverse the shape of the streamtube, and the iterative approach is once again able to reach a converged shape for the wake. At these large negative values of V_c/V_i , the rotor is still generating an upward thrust, however, it is functioning in a windmill state, slowing the flow and removing energy from the free stream resulting in an expansion of the streamtube. In this condition, we see that the spiral at the disk's edge, although small, has the same orientation as in the hover and climbing axial flight cases.

Coaxial Actuator Disks

In this section, we investigate the wake shape and induced flow field of coaxial actuator disks in hover. We first compare cases with equal bound circulation on the two disks, which results in unequal torques between the two rotors due to the difference in mutually induced inflows. We then investigate

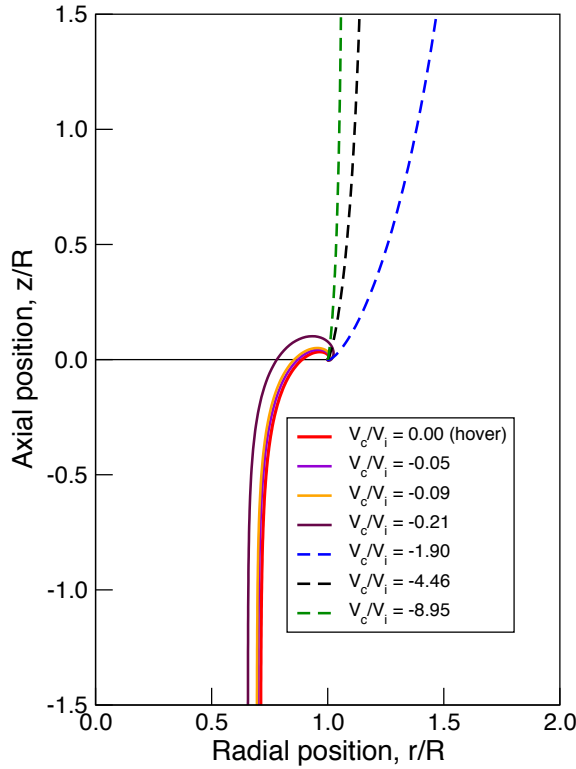


Fig. 11. The shape of the wake with negative climb velocity, i.e., in descending flight. The dashed lines indicate descent velocities at which the rotor has fully transitioned into the windmill state.

cases where the lower disk trails a sheet of negative vorticity at its intersection with the upper disk's vortex sheet, leading to an increase in thrust on the outer portion of the lower disk. The net thrust distribution for this case very closely resembles the optimal coaxial rotor determined by Giovanetti and Hall (Ref. 14). We then compare the mutual interference of the two rotors predicted by our axisymmetric vortex sheet model to that predicted by the simple influence coefficient model developed by McAlister et al. (Ref. 12).

To analyze coaxial disks, we use a logarithmic spacing parameter $m = 800$, defined in Eq. (14), for each vortex sheet. We also include a breakpoint in the wake, below which the ring spacing transitions from logarithmic to linear, a technique that was found to speed up convergence of the Newton iteration. The number of rings used varies with the vertical separation of the disks, but was between 65 and 100 per vortex sheet for every case.

Rotors with equal circulation. Figure 12 shows the induced wash distributions for coaxial actuator disks with an axial separation of $z/R = 0.3$ and equal circulation on the disks. We see from the top plot of the figure that the upper rotor has a relatively constant induced wash distribution, with a gradual reduction in induced wash towards the tip. This is in contrast to the single actuator disk in hover, shown in Figure 9, where

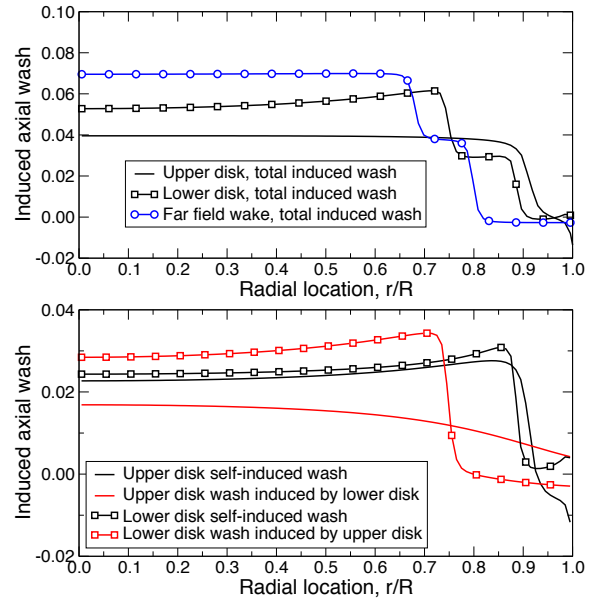


Fig. 12. Top: Radial distribution of induced axial velocity for non-torque-balanced (equal circulation) coaxial actuator disks with vertical separation of $z/R = 0.3$. Bottom: Radial distribution of induced axial velocity split into contributions from each of the two disks' vortex sheets.

the induced wash is negative outboard of $r/R = 0.87$. The cause of this is well illustrated by the streamlines shown in Figure 13. The outer portion of the upper disk is subjected to a sizable wash caused by the lower disk drawing in flow. As a result, the edge spiral is significantly less pronounced, just as the edge spiral of a disk in axial flight is less pronounced (cf. Figure 5). The result is a more uniform induced wash distribution. The outer portion of the lower actuator disk sees the opposite effect: it is subjected to a small upward velocity induced by the upper rotor drawing in flow from its sides. This is equivalent to the lower disk operating in a descending flight condition. The result is a more pronounced edge spiral and small or negative induced wash outboard of where this spiral passes through the actuator disk. In the far field, we observe that the velocity is not uniform, as the portion of the wake that passes through both actuator disks, which extends from the hub to $r/R \approx 0.6$, has a higher velocity than the outer portion that only passes through the lower disk.

The bottom plot of Figure 12 divides each disk's induced wash into components resulting from the upper disk's vortex sheet and the lower disk's vortex sheet. In other words, this plot quantifies which elements of the inflow are self induced and which are a result of the interference of the other disk. We see that the lower disk induces a smooth wash distribution on the upper disk that decreases somewhat towards the tip. The upper disk induces a large positive wash over the lower disk within its contracted wake, and a small negative induced wash over the outer portion of the lower disk.

For comparison, Figure 14 shows streamlines for coaxial actuator disks of equal circulation with a (larger) vertical sep-

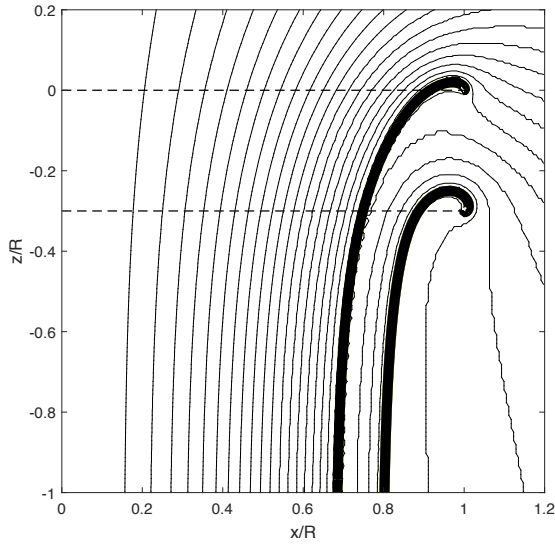


Fig. 13. Streamlines of non-torque-balanced (equal circulation) coaxial actuator disks with a vertical separation of $z/R = 0.3$.

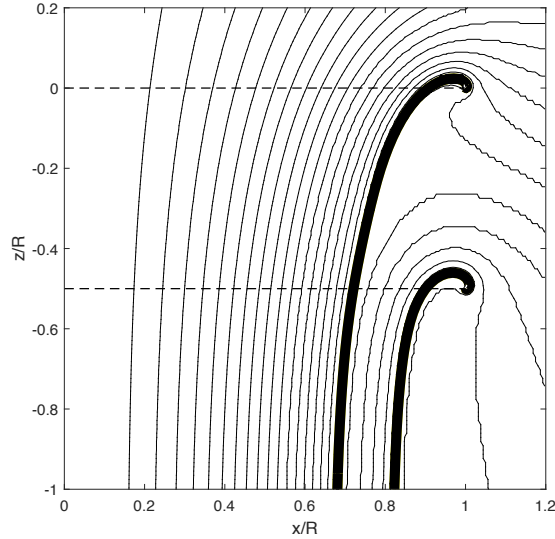


Fig. 14. Streamlines of non-torque-balanced (equal circulation) coaxial actuator disks with a vertical separation of $z/R = 0.5$.

aration of $z/R = 0.5$. Because of the larger vertical separation, the upper disk's wake has contracted to a smaller radius before impinging upon the lower disk.

Rotors in torque balance. In this section we investigate coaxial actuators disks that have equal torque, thereby more closely approximating the actual operating state of a hovering coaxial rotor with no anti-torque device. Additionally, we introduce a change in vorticity to the upper disk's vortex sheet as it passes through the lower disk to model a change in the

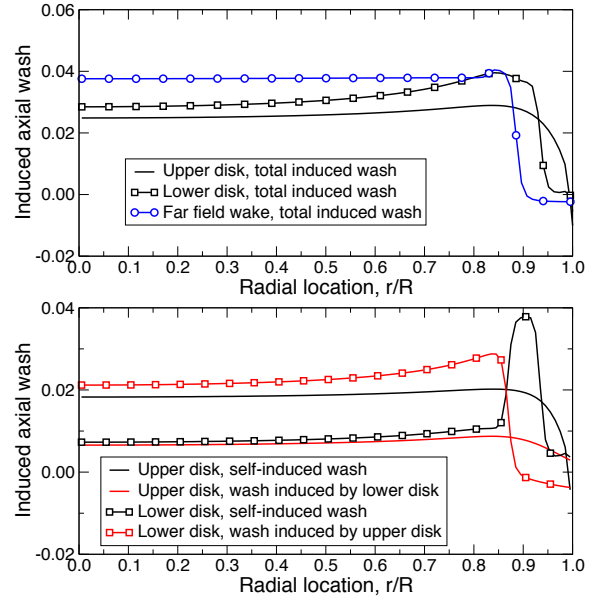


Fig. 15. Top: Radial distribution of induced axial velocity for torque-balanced actuator disks with vertical separation of $z/R = 0.16$. Bottom: Radial distribution of induced axial velocity split into contributions from each of the two disks' vortex sheets.

bound circulation of the lower disk. We set this change in vorticity to be negative, thereby causing an increase in the thrust generated by the lower rotor outboard of this trailed vortex and more closely resembling the optimal coaxial rotor thrust distributions of (Ref. 14). We compute the differential induced power of a given actuator disk using

$$dP_i = V_i dT \quad (15)$$

Figure 15 shows the axial induced wash distribution for torque balanced actuator disks with a vertical separation of $z/R = 0.16$, the same vertical separation of the well known Harrington experimental rotor (Ref. 15). Note that the far field induced wash is now uniform within the slipstream, in contrast to the non-torque-balanced case shown in Figure 12 where there was a non-uniform far field flow. In the bottom plot of Figure 15, we see a large increase in the self-induced velocity of the lower rotor at $r/R = 0.85$. This is caused by the change in vorticity of the upper disk's vortex sheet as it passes through the lower disk.

Figures 16 shows the streamlines for a torque balanced case with a vertical separation of $z/R = 0.16$. Not surprisingly, we observe a very slight expansion in the upper disk's vortex sheet as it passes through the lower disk and changes vorticity.

Finally, Figure 17 compares the mutual interference predicted by our axisymmetric vortex sheet model to that predicted by the influence coefficient model developed by McAlister et al. (Ref. 12). Compared to the present actuator disk model, the influence coefficient model underpredicts the wash

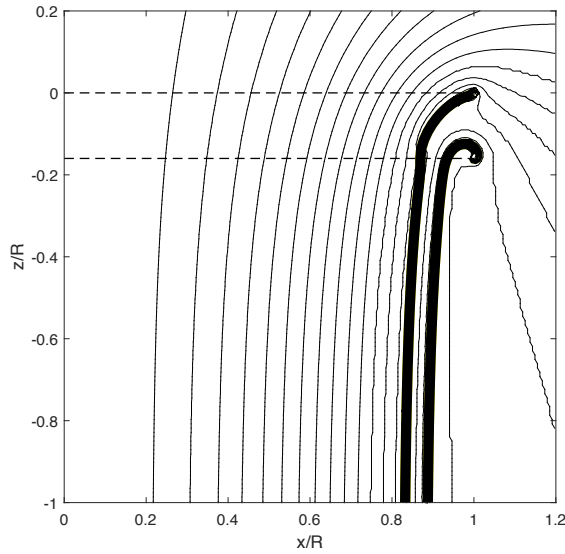


Fig. 16. Streamlines of torque balanced coaxial actuator disks with a vertical separation of $z/R = 0.16$.

on the upper rotor induced by the lower rotor, and does not capture the increased induced flow near the edge of the disk. The influence of the upper rotor on the lower rotor is qualitatively similar between the two models. Inside of the contracted streamtube, the predicted induced velocities are in reasonably good agreement. Outside of the contracted streamtube, the influence coefficient model assumes zero influence on the lower rotor, while the actuator disk model predicts a small negative induced velocity drawn up through the lower rotor by the upper rotor. As a final observation, the predicted wake contraction as a function of vertical separation by the influence coefficient model is in fair agreement with our actuator disk model.

CONCLUSIONS

We have developed a axisymmetric vortex sheet model of the flow through an actuator disk. We represent the continuous vortex sheet of the wake using discrete vortex rings with non-uniform spacing. We impose a no through flow boundary condition on the vortex sheet and determine its position via Newton iteration. The model can be used for rotors with multiple vortex sheets, allowing the analysis of non-uniformly loaded single and dual coaxial actuator disks.

In this paper, we investigated a variety of cases and arrived at the following conclusions:

1. The singularity that occurs where the vortex sheet terminates at the edge of the actuator disk is resolved through the formation of a 45° logarithmic spiral in hover, as originally hypothesized by Spalart (Ref. 11). The spiral structure of the wake results in a non-uniform inflow, particularly near the edge of the disk, that includes reversed flow through the actuator disk.

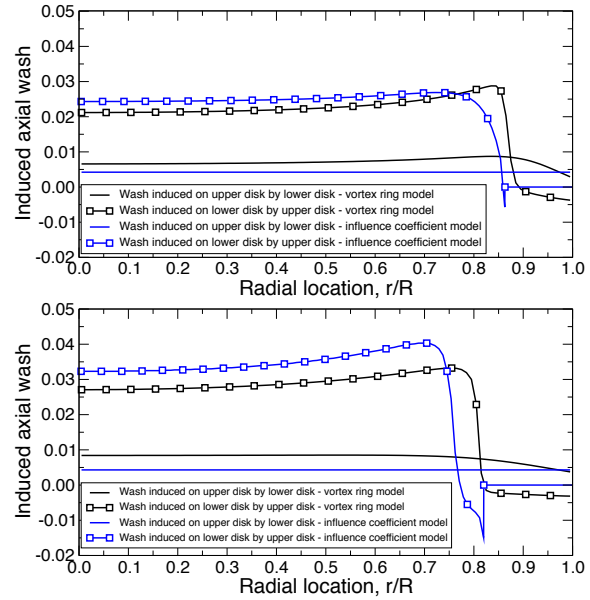


Fig. 17. Top: Comparison of the mutual interference predicted by the axisymmetric potential flow axisymmetric vortex sheet model with those predicted by the influence coefficient model of McAlister (Ref. 12) for a vertical separation $z/R = 0.16$. Bottom: Same comparison for a vertical separation of $z/R = 0.3$.

2. In axial flight, the size of this spiral decreases, although it is always theoretically present, vanishing in the limit where V_c/V_i approaches infinity.
3. In the case of descending flight, the size of the spiral structure increases with descent rate until the wake enters the vortex ring state. In the vortex ring state, no converged solution can be found using the methods described here. Once the descent velocity is sufficiently high, the disk enters a windmill state, which also produces a spiral vortex sheet structure at the actuator disk tip with a spiral in the same direction as the climb and hover cases. In the windmill state, the vortex sheet convects in the direction of the free stream flow, compared to the low descent velocity cases where the vortex sheet convection and free stream are in opposite directions.
4. Coaxial actuator disks have a pronounced spiral structure on the lower disk, where the upwash induced by the upper disk acts as a descent velocity for the edge of the lower disk. The upper disk sees the opposite effect, with a net flow through the disk in the traditional direction induced by the lower disk, suppressing the spiral structure and more closely resembling an actuator disk in climbing axial flight.
5. For the non-torque-balanced (equal circulation) coaxial actuator disks, the field flow in the far wake is not uniform, but rather has different velocities in the flow that passes through both actuator disks and the flow that enters the slipstream through the lower disk only. For both

torque-balanced coaxial cases analyzed here, the far field velocities are nearly uniform in the far wake.

6. The mutual interference effects predicted by the present model agree reasonably well with the predictions made using the influence coefficient model of McAlister et al. (Ref. 12), with two notable differences. First, the influence of the lower rotor on the upper rotor is under-predicted by the McAlister model compared to the axisymmetric vortex sheet model. Second, the influence of the upper rotor on the outer portion of the lower rotor is approximated as zero in the McAlister model, while the vortex ring model predicts a small upwash through the lower disk in this region.

ACKNOWLEDGMENTS

This material is based upon work supported by the U.S. Army Research Office under Award No. W911NF-13-1-0469. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the view of the U.S. Army Research Office. Additionally, the first author was funded through a National Defense Science and Engineering Graduate Fellowship.

Author contact: Eli B. Giovanetti, ebg11@duke.edu;
Kenneth C. Hall, kenneth.c.hall@duke.edu.

REFERENCES

- ¹Rankine, W.J.M., "On the Mechanical Principles of the Action of Propellers," RINA, 1865.
- ²Froude, R.E., "On the Part Played in Propulsion by Differences of the Fluid Pressure," Transactions of the Institute of Naval Architects, 30: 390 - 405, 1889.
- ³Lanchester, F.W., "A Contribution to the Theory of Propulsion and the Screw Propeller," Journal of the American Society for Naval Engineers, 27(2):509-510, 1915.
- ⁴Betz, A., "Schraubenpropeller mit geringstem Energieverlust," *Göttinger Nachrichten*, p. 193, 1919.
- ⁵Goldstein, S., "On the vortex theory of screw propellers," Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, Vol. 123, No. 792, Apr. 6, 1929, pp. 440-465.
- ⁶Hough, G.R., and Ordway, D.E., "The Generalized Actuator Disk," Technical Report, DTIC Document, 1964.
- ⁷Conway, J.T., "Analytical Solutions for the Actuator Disk with Variable Radial Distribution of Load," Journal of Fluid Mechanicals, 297: 327 - 355, 1995.
- ⁸Conway, J.T., "Exact Actuator Disk Solutions for Non-Uniform Heavy Loading and Slipstream Contraction," Journal of Fluid Mechanicals, 365: 235 - 267, 1998.
- ⁹Greenberg, M.D., and Powers, S.R., "Nonlinear Actuator Disk Theory and Flow Field Calculations, Including Nonuniform Loading," Technical Report, 1970.
- ¹⁰Schmidt, G.H., and Sparenberg, J.A., "On the Edge Singularity of an Actuator Disk with Large Constant Normal Load," Journal of Ship Research, 21(2), 1977.
- ¹¹Spalart, P.R., "On the Simple Actuator Disk," Journal of Fluid Mechanics, 494:399 - 405, 2003.
- ¹²McAlister, K.W., Tung, C., Rand, O., Khromov, V., "Experimental and Numerical Study of a Model Coaxial Rotor," presented at the AHS 62nd Annual Forum, Phoenix, Arizona, May 2006.
- ¹³Rand, O., and Khromov, V., "Aerodynamic Optimization of Coaxial Rotor in Hover and Axial Flight," 27th International Congress of the Aeronautical Sciences, 2010.
- ¹⁴Giovanetti, E.B., and Hall, K.C., "Minimum Loss Load, Twist, and Chord Distributions for Coaxial Helicopters in Hover," Presented at the AHS 71st Annual Forum, Virginia Beach, Virginia, May 2015.
- ¹⁵Harrington, R. D., "Full-Scale Tunnel Investigation of the Static Thrust Performance of a Coaxial Helicopter Rotor," NACA Technical note 2318, 1951.
- ¹⁶Hascoet, L. and Pascual, V., "Tapenade 2.1 User's Guide," Project Tropics, INRIA Sophia-Antipolis, 2004.