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as of 30-Oct-2018

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Major Goals: Finite element predictions of missile penetration into concrete and rock after impact has long been unsatisfactory. The standard finite models with constitutive models rate dependent cracking damage, viscoelasticity and crack growth rate based on activation energy concept have always greatly overestimated the depth of penetration into concrete mass and the exist velocity of missile penetrating through a wall.

Computational researcher would deny these statements but closer look reveals they have typically introduced artificial dissipation or damping in an empirical way (or based on some non-physical tricks such as artificial increase of strength limit, assuming a physically unjustified phenomenon called "dynamic overstress" in which the material strength is increased for strain rate > 1 , the increase reaching an order of magnitude at strain rates $4.E4$ /s to $6.E4$ /s.

Another, supposedly fancier approach favored by computational researchers has been to increase the strength, by an order of magnitude, by updating the material model with some non-physical computer algorithm, such as a genetic algorithm.

In spite of what many computer analysts claimed, and despite their realistically looking beautiful movies, such remedies have been a fiction. They could not predict, or could not be trusted to predict, impact situations very different from those used in the non-physical genetic algorithm or in intuitive empirical "dynamic overstress" enhancement.

Thus the major goal of this proposal was to develop a physically based formulation putting the so-called "dynamic overstress" on a realistic foundation with predictive capability. Secondly, it was to combine it with the microplane model, which is a physically realistic nonlinear constitutive model for cracking damage. So far, the wavecodes in national laboratories used plasticity models, such as Johnson-Cook's, which have not been realistic since concrete is not plastic except under enormous confining pressure, and could not capture the softening damage..

It was proposed to explain and model the "dynamic overstress" by modeling of the fine fragmentation, or comminution of concrete, which is known to always accompany impact. This proposed goal has been achieved.

The overall goal was to develop, for projectile penetration, a model that is actually predictive. In this regard, it was decided conduct a critical analysis of peridynamics, which is a model that emulates molecular dynamics by postulating particle skipping interactions and has been touted as a cure-all for dynamic failure of concrete and rock.

The goal was not only military application (bunkers, hardened structures) but also terrorist attack leading to similar problems. Although impact of meteorites and space collision were not the stated goal, the theory obtained is also extensible and ramifiable to such problems.

Accomplishments: To capture the phenomenon of "dynamic overstress", i.e., the apparent increase of strength of concrete at very high strain rates ($1/s$ to $1.E6$ /s) experienced in projectile impact and penetration, a new theory with partial analogy to turbulence has been developed. The increase is attributed to fine fragmentation, or

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comminution, of concrete driven by the release of local kinetic energy of the shear strain rate field of forming fragments of finite size (see attached journal articles [5,6,7]). In projectile impact problems, this kinetic energy exceeds the strain energy of the fragments by orders of magnitude.

The new theory gives the particle sizes, and combination with Schuhmann's law for rock blasting, give the particle size distribution. It yields the additional local kinetic energy density, ΔK , and show it is proportional to the deviatoric strain rate square [2,3,4]. To match test results, ΔK must be dissipated during finite element simulations of impact.

In the initial work, ΔK was dissipated by an artificial equivalent viscosity, not empirical but predicted by the theory [7, 3,4]. Subsequently it was found that dissipation by upscaling material tensile strength is equally effective, more physical and, importantly, can be logically combined with the microplane constitutive model M7 for fracturing damage in concrete [2]. This model, previously developed by the writer, formulated the damage constitutive law not in terms of tensors, but in terms of stress and strain vector acting on various planes, called "microplanes", within the material microstructure. This makes it possible to capture the orientation dependence of micro-fracturing and energy dissipation. The stress tensor is constructed from the stress vectors on all the microplanes by a variational principle (this is done at each integration point of each finite element at each time step) [2].

For users, it is important that the model can be implemented as a user's subroutine in commercial softwares such as Abaqus, LS-Dyna or Ansys. But it is essential to eliminate all the artificial dampings that are, overtly and covertly, implemented in these commercial softwares (to prevent wild oscillations of output and make the response look smooth and nice).

The last stage of the project focused on the simulation of non-orthogonal, oblique, projectile impacts, of various angles, for various impact velocities and hypervelocities [1]. The oblique impact revealed curved penetration paths which can make wall penetration impossible. This work became computationally quite demanding though well feasible on the Northwestern supercomputer cluster.

Numerous comparisons with published test data on the projectile penetration tests have been made. For each test series on one and the same concrete, the material parameters were calibrated on one test and then, using the same parameters, all the other tests have been predicted. All the predictions of exit velocities and penetration depths of projectiles, as well as entry and exit craters, were quite accurate. The particle sizes predicted by the model were compare to published test results, especially those of Denis Grady from Sandia Lab, and a good match was obtained.

There are many possible ramifications. The theory can certainly be applied to impacts in various terrorist incidents, e.g., impact of an aircraft jet engine into a nuclear containment. It can be extended to impacts of meteorites and collisions in space, although in the case of high velocities some generalization are needed for the case of nanoscale particles. A simple ramification is to the fragmentation of metallic hollow spherical shell caused by an explosion in the interior, in which case the fragmentation is driven by the kinetic energy of the expansive volumetric strain rate filed in the forming finite-size fragments. This ramification agrees with the previous theory of D. Grady. Rock blasting would be another possible ramification.

Peridynamics: Is that a viable competing approach, a better alternative to the present model?

It is not, by far (see [8]). The most important ramification has been a critical examination of the so-called "peridynamics", which has recently been widely promoted as a better approach to dynamic fracturing of concrete and rocks and has received enormous government funding, In [8] It was shown that this theory is deeply flawed. The following conclusions were made:

(1) The state-based peridynamics is strongly dispersive, significantly more so than the bond-based peridynamics. It has multiple wavelengths, one of them longer than the horizon, for which the phase velocity vanishes. This means that the state-based version acts as an excessively strong filter for high-frequency waves.

(2) The wave dispersion behaviors and the nonlocality in the bond-based and state-based peridynamics are coupled. The material characteristic lengths caused by (i) the heterogeneity of material and (ii) the softening damage and plasticity cannot be distinguished. They are not independently controllable, whereas in the modern nonlocal-strain averaging theories, they are. The fact that the latter depends on material strength and fracture energy, and the former does not, is not captured by peridynamics.

(3) Compared to the traditional numerical methods, the convergence rate for both the bond-based and state-based versions is suboptimal. The computation time for each time increment is longer than it is in the finite difference method that shares with peridynamics the same regular grid of nodes.

(4) The node-skipping (or particle-skipping) force interactions, which are a salient distinguishing feature of peridynamics, are a fiction, except on the atomic scale. In reality, the force gets transmitted to the second and farther neighboring particles (or nodes) by displacements and rotations of the intermediate particles, rather than by some particle permeating potential

(although, in fiber composites, a sort of long-distance interaction exists, engendered by fibers, some new form of peridynamics would be needed to capture the differences between various types of fibrous microstructures).

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(5) Despite using intermediate nodes within the horizon, the morphology of microstructure is not captured by the existing forms of peridynamics.

(6) Although, in bond-based peridynamics, the dimension of the micromodulus as a fundamental characteristic of material stiffness (e.g., N/m⁶ in 3D) can be justified mathematically, it is strange from the physical viewpoint.

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- [1] Luo, Wen, Chau, Viet-Tuan, & Bazant, Z.P. (2018). "Effect of high-rate dynamic comminution on penetration of projectiles of various velocities and impact angles into concrete", IJF in review; arxiv.org/submit/2429248
- [2] Kirane, K., Su, Y, Bazant, Z.P. (2015). "Strain-rate dependent microplane model for high-rate comminution of concrete under impact based on kinetic energy release theory." Proc. Royal Soc. A 471 (Oct.), paper 20150535, pp. 1--8; DOI: 10.1098/rspa.2015.0535.
- [3] Su, Y., Bazant, Z.P., (2015). Int. J. of Fracture 193 (1), 77-85.
- [4] Bazant, Z.P., and Su, Yewang (2015). ASME J. of Applied Mechanics 82 (March), pp. 031007-1--031007-5.
- [5] Bazant, Z.P., & Caner, F.C. (2014). "Impact comminution of solids due to local kinetic energy of high shear strain rate: I. Continuum theory and turbulence analogy." J. of the Mechanics and Physics of Solids 64, 223--235.
- [6] Caner, F.C. & Bazant, Z.P. (2014). Dito - II. Microplane model and verification." J. Mech. Phys. Solids 64, 236--248.
- [7] Bazant, Z.P., & Caner, F.C. (2013). "Comminution of solids caused by kinetic energy of high shear strain rate, with implications for impact, shock and shale fracturing." Proc., National Academy of Sciences 110 (48), 19291--19294.
- [8] Bazant, Z.P., Luo, Wen. (2016). "Wave dispersion and basic concepts of peridynamics compared to classical nonlocal models." J. Appl. Mech. 83, 111004-1---111004-16 (doi: 10.1115/1.40343119).

Training Opportunities: 4 doctoral students and 4 postdocs were trained in modern fracture and damage mechanics of quasibrittle materials

Results Dissemination: The results were published in 8 refereed journal articles in leading refereed journals, as follows:

- [1] Luo, Wen, Chau, Viet-Tuan, and Bazant, Z.P. (2018). "Effect of high-rate dynamic comminution on penetration of projectiles of various velocities and impact angles into concrete", submitted to Intern. Jour. of Fracture; arxiv.org/submit/2429248
- [2] Kirane, K., Su, Yewang, Bazant, Z.P. (2015). "Strain-rate dependent microplane model for high-rate comminution of concrete under impact based on kinetic energy release theory." Proc. Royal Soc. A 471 (Oct.), paper 20150535, pp. 1--8; DOI: 10.1098/rspa.2015.0535.
- [3] Su, Yewang, Balv zant, Z.P., Zhao, Youxuan, Salvato. M., and Kirane, K. (2015). "Viscous energy dissipation of kinetic energy of particles comminuted by high-rate shearing in projectile penetration, with potential ramification to gas shale." {em Int. J. of Fracture} 193 (1), 77-85.
- [4] Bazant, Z.P., and Su, Yewang (2015). "Impact comminution of solids due to progressive crack growth driven by kinetic energy of high-rate shear." ASME J. of Applied Mechanics 82 (March), pp. 031007-1--031007-5.
- [5] Bazant, Z.P., and Caner, F.C. (2014). "Impact comminution of solids due to local kinetic energy of high shear strain rate: I. Continuum theory and turbulence analogy." J. of the Mechanics and Physics of Solids 64, 223--235.
- [6] Caner, F.C. and Bazant, Z.P. (2014). "Impact comminution of solids due to local kinetic energy of high shear strain rate: II. Microplane model and verification." J. of the Mechanics and Physics of Solids 64, 236--248.
- [7] Bazant, Z.P., and Caner, F.C. (2013). "Comminution of solids caused by kinetic energy of high shear strain rate, with implications for impact, shock and shale fracturing." Proc., National Academy of Sciences 110 (48), 19291--19294.
- [8] Bazant, Z.P., Luo, Wen, Chau, Viet T., and Bessa, M.A. (2016). "Wave dispersion and basic concepts of peridynamics compared to classical nonlocal models." J. of Applied Mechanics ASME 83 (Nov.) 111004-1---111004-16 (doi: 10.1115/1.4034319).

Also in 6 conference presentations, with abstracts in conference proceeding.

Also in 7 invited guest seminars at various universities.

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Honors and Awards: -- 2015 ASCE established the Zdenek P. Bazant Medal for Damage Prevention}, established in 2015 by the Am. Soc. of Civil Engrs. as an overall society medal administered Eng. Mech. Institute in consultation with USNC-TAM;

<http://www.asce.org/templates/award-detail.aspx?id=11613>

-- 2015 elected Foreign Member of Royal Society of London (ForMemRS); citation Apr. 30: "Zdenek Bazant discovered the scaling law for the energetic size effect in quasi-brittle structural failure, bridging ductility and brittleness. He verified the law experimentally for many important materials, showed its use for measuring fracture characteristics, and conceived the crack-band and non-local models now widely used in industry and commercial programmes to analyse quasi-brittle structural failure."

-- 2105, Elected Honorary Member, RILEM (Int. Union of Res. in Mat. & Str.), Paris.

2015 State Prize: 2016 Austrian Cross of Honor for Science and Art I. Class, state honor conferred by President of Austria in Imperial Palace (Hofburg), Vienna.

-- 2017 elected Foreign Member, Academy of Athens (national academy of Greece).

-- 2017 elected Foreign Member. National Academy of Engineering of India.

-- 2018 received Alfred M. Freudenthal Medal, ASCE. (given once every two years). Cited for "Developing a comprehensive theory of probabilistic mechanics of strength, lifetime, and size effect of quasi-brittle structures."

-- Citation H-index: 122 (on Google, incl. self. cit.), 67,000 citations in total

Protocol Activity Status:

Technology Transfer: Numerous presentations at conferences, incl. plenary lectures, and guest seminars.

Collaboration with H. Viswanatha and E. Rougier at Los Alamos National Laboratory

Lecture and discussions at Aberdeen Proving Grounds

PARTICIPANTS:

Participant Type: Graduate Student (research assistant)

Participant: Kedar Shrikant Kirane

Person Months Worked: 15.00

Funding Support:

Project Contribution:

International Collaboration:

International Travel:

National Academy Member: N

Other Collaborators:

Participant Type: Graduate Student (research assistant)

Participant: Viet Tuan Chau

Person Months Worked: 15.00

Funding Support:

Project Contribution:

International Collaboration:

International Travel:

National Academy Member: N

Other Collaborators:

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)

Participant: Yewang Su PhD

Person Months Worked: 4.00

Funding Support:

Project Contribution:

International Collaboration:

International Travel:

National Academy Member: N

Other Collaborators:

Participant Type: Graduate Student (research assistant)

Participant: Wen Luo

Person Months Worked: 6.00

Funding Support:

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Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)

Participant: Ferhun C Caner PhD

Person Months Worked: 5.00

Funding Support:

Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Graduate Student (research assistant)

Participant: Youxuan Zhao

Person Months Worked: 1.00

Funding Support:

Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)

Participant: Marco Salviato PhD

Person Months Worked: 2.00

Funding Support:

Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)

Participant: Miguel Bessa A. PdD

Person Months Worked: 1.00

Funding Support:

Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

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Effect of High-Rate Dynamic Comminution on Penetration of Projectiles of Various Velocities and Impact Angles into Concrete

Wen Luo*, Viet T. Chau[†] and Zdeněk P. Bažant[‡]

September 22, 2018

Abstract: The dynamic ‘overstress’, i.e., the apparent increase of strength of concrete at very high strain rates (10 s^{-1} to 10^6 s^{-1}) experienced in projectile impact and penetration, has recently been explained by a new theory with partial analogy to turbulence. The increase is attributed to comminution of concrete driven by the release of local kinetic energy of the shear strain rate field of forming fragments which, in projectile impact problems, exceeds the strain energy of the fragments by orders of magnitude. This theory gives the particle size distribution and the additional local kinetic energy density, ΔK , proportional to the deviatoric strain rate square. To match test results, ΔK must be dissipated during finite element simulations of impact. In previous simulations, ΔK was, at first, dissipated by an artificial equivalent viscosity (not empirical but predicted by the theory). Later it was found that dissipation by upscaling material tensile strength is equally effective. This theoretically justified upscaling is adopted here since it is more realistic when microplane constitutive model M7 for fracturing damage in concrete is used. All artificial damping is eliminated from the computer program. While previous simulations with the comminution theory were limited to orthogonal impacts, and only the cases of penetration of slabs of various thickness by projectile of one velocity and penetration depths for different velocities, the present study also analyzes further test data on oblique impacts at various impact angles up to 35° , and on the exit velocities and penetration depths of projectiles of different velocities. For each test series on one and the same concrete, the material parameters are calibrated on one test and then, using the same parameters, all the other tests are predicted. All the predictions of exit velocities and penetration depths of projectiles, as well as entry and exit craters, are quite accurate.

1 Introduction

In the analysis of impact of projectiles into concrete, the apparent material strength and energy dissipation appear to increase sharply as the strain rate increases above $10/\text{s}$ and becomes major for $10^4/\text{s}$. This phenomenon is also called ‘dynamic overstress’. Several existing continuous [16, 18] and discrete [10, 11] models were developed to model the impact problem of concrete and to explain the considerable increase of apparent material strength. It was recently proposed and

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demonstrated [1, 2, 6] that the physical justification be sought in material fracturing driven not only by the release of strain energy, as in classical fracture mechanics, but also, and mainly, by the release of local kinetic energy of shear strain-rate field within forming particles, comminuting concrete into small fragments. At the rate of $10^4 s^{-1}$, the kinetic energy of the forming particle exceeds the maximum possible strain energy in the particle by two orders of magnitude. Thus, the comminution process can dissipate much of the kinetic energy of projectile impact.

This phenomenon of comminution is not covered by the conventional damage constitutive laws based on standard quasi-static triaxial testing, which predict much larger fragments. The theory proposed in [2, 6], which bears some analogy to turbulence, was shown to give a strain-rate-dependent expression for the additional kinetic energy density ΔK that is dissipated during projectile penetration by the fracture energy of fine material fragments.

In the initial finite element modeling [1, 2, 6], ΔK was dissipated by an additional shear viscosity η obtained from the calculated kinetic energy release. This formulation, which is natural for rate-dependent energy dissipation, led to a model with partial analogy to the theory of turbulence [2]. In the initial, simplified, approach, η was considered to depend only on the strain rate. In a subsequent more rigorous approach [17, 4], η was considered to also depend on the strain-rate derivative. Later it was realized that what matters in finite element modeling is the magnitude of energy dissipation and not the precise way of energy dissipation in the computer program. In subsequent computer programming, the energy was dissipated by increasing the strength limit as a function of the strain rate, which was more convenient for the microplane model as a constitutive damage model for concrete [14]. All these approaches led to a good match of the test data on the exit velocities of projectiles from concrete walls of different thicknesses, and the latter also matched well the measured penetration depths of projectiles of different velocities.

In the current study, the effect of ΔK is introduced similarly to [14]—by scaling up the material strength in the microplane model M7. The strain-dependent strength limits (called stress-strain boundaries) of the microplane damage constitutive model are raised based on the shear strain rate so as to dissipate the additional kinetic density ΔK in the forming particles. It is also assumed that the particles form instantly. This method leads to strength increase depending on both the strain rate and its derivative.

The present goal is to verify the capability of the M7 material model with scaled boundaries to predict: 1) the projectile exit velocity; 2) depth of penetration and 3) the crater shape. To this end, simulations of orthogonal and oblique projectile impact on concrete walls are conducted. The results are then compared with the test data found in the literature. First, orthogonal fixed-velocity impacts into targets of various thicknesses are analyzed. Then, orthogonal impacts into a concrete slab of fixed thickness, sufficiently thin for perforation, are computationally simulated for various striking velocities; the same for the oblique impacts into thin and thick concrete slabs. In the thin slab case, the exit (or residual) velocities are compared, while in the thick slab case, too thick for perforation, the depths of penetration (DOP) are. As will be seen, all these simulations yield good agreement with the experimental data from the literature.

In the analysis of impact of projectiles into concrete, the apparent material strength and energy dissipation appear to increase sharply as the strain rate increases above $10/s$ and becomes major for $10^4/s$. To explain and model this phenomenon, called the 'dynamic overstress', it was recently proposed and demonstrated [1, 2, 6] that the physical justification must be sought in material fracturing driven not only by the release of strain energy, as in classical fracture mechanics, but also, and predominantly, by the release of local kinetic energy of shear strain-rate field within forming particles, comminuting concrete into small fragments. At the rate of $10^4 s^{-1}$, the kinetic energy of the forming particle exceeds the maximum possible strain energy

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2 Review of Microplane Model M7 and Scaling of Its Boundaries

The microplane model M7 [5] is the latest version in a series of progressively improved microplane models labelled M0, M1, M2, $\dots$, M6, developed first for concrete (and then extended to other quasibrittle materials). The basic idea of the microplane model is to express the constitutive law not in terms of tensors, but in terms of the vectors of stress and strain acting on a generic plane of any orientation in the material microstructure, called the microplane. The use of vectors is analogous to the Taylor models used for plasticity of polycrystalline metals,

but there are major conceptual differences [7].

To avoid model instability in postpeak softening, a kinematic constraint is used instead of a static one. That is, the strain vector on each microplane is obtained by projecting the macroscopic strain tensor onto the microplane. So we have,

$$\varepsilon_N = \varepsilon_{ij}N_{ij}, \quad \varepsilon_M = \varepsilon_{ij}M_{ij} \quad \text{and} \quad \varepsilon_L = \varepsilon_{ij}L_{ij} \quad (1)$$

where ε_N , ε_M and ε_L are the magnitudes of the three strain vectors corresponding to each microplane, and $N_{ij} = n_i n_j$, $M_{ij} = (n_i m_j + m_i n_j)/2$ and $L_{ij} = (n_i l_j + l_i n_j)/2$ are the projection tensors; n , m and l are the three mutually orthogonal normal and tangential unit vectors characterizing that microplane, and the subscripts i ($1 = 1, 2, 3$) denote cartesian components. Instead of defining the constitutive law in terms of the stress and strain tensors, it is defined on each microplane, which allows calculating the microplane stress increments from the given microplane strain increments. Finally, to ensure internal equilibrium, a variational principle (principle of virtual work) is used to relate the stresses on the microplanes (σ_N , σ_M and σ_L) to the macro-continuum stress tensor σ_{ij} , which is expressed as

$$\frac{2\pi}{3}\sigma_{ij}\delta\varepsilon_{ij} = \int_{\Omega}(\sigma_N\varepsilon_N + \sigma_M\delta\varepsilon_M + \sigma_L\delta\varepsilon_L)d\Omega \quad (2)$$

In the microplane model M7, the microstresses are subjected to strain-dependent boundaries (or strength limits) of four types: 1) the tensile normal boundary captures the progressive tensile fracturing; 2) the compressive volumetric boundary captures phenomenon such as pore collapse under extreme pressures; 3) the compressive deviatoric boundary captures softening in compression; and 4) the shear boundary captures friction. During each loading step, if any microplane stress exceeds one of the boundaries, it is returned to the boundary with displacement being fixed.

Besides, the failure envelope of concrete is known to depend largely on the confining pressure, which is believed to help prevent crack from developing. This effect is automatically captured by the formulation of the microplane model [6] and the corresponding failure envelope can be calibrated by tuning the 5 basic model parameters ($k_1 \sim k_5$) according to test results.

To capture the material comminution effect, microplane stress boundaries $\Delta\sigma_D$, $\Delta\sigma_M$ and $\Delta\sigma_L$ are adjusted in each load step as a function of the deviatoric strain rate $\dot{\varepsilon}_D$. For simplicity, $\Delta\sigma_D$, $\Delta\sigma_M$ and $\Delta\sigma_L$ are assumed to be the same and are denoted as Δf . In the spirit of turbulence analogy [1, 14], for a given deviatoric strain rate $\dot{\varepsilon}_D$, the drop in kinetic energy per unit volume is obtained as:

$$-\Delta K = c_k \rho h^2 \dot{\varepsilon}_D^2 \quad (3)$$

where h = average particle size, $c_k = I_p/(2hV_p)$, ρ = mass density, V_p and I_p are the average volume and polar moment of inertia of each comminuted prism about its axis, respectively. Combining the expression of ΔK with the equation of the principle of virtual work, one can get the increment of microplane stress boundaries [14]:

$$\Delta f = \Delta\sigma_D = \Delta\sigma_M = \Delta\sigma_L = A_1 \dot{\varepsilon}_D^{-4/3} \ddot{\varepsilon}_D, \quad (4)$$

where A_1 is a constant to be calibrated. This expression shows that, to capture the energy dissipation due to comminution, it is necessary to make the boundary a function of both the first and second time derivatives of the effective deviatoric strain.

Apart from the additional parameter A_1 for the effect of impact comminution and elastic modulus E controls the vertical scaling of stress-strain curves, microplane model M7 has 5

major free parameters to be calibrated based on quasi-static tests: $k_1 \cdots k_5$. Based on [5], their meanings are: k_1 : implements a radial scaling of stress-strain curves (which, in combination with vertical scaling by E , can also give horizontal scaling); k_2 : controls the height of horizontal asymptote of the frictional boundary; k_3 : controls the shape of the volumetric boundary; k_4 : controls the shape of the volumetric boundary; and k_5 : controls the volumetric-deviatoric coupling at low pressures. Since the mechanical properties of materials found in the literature are usually limited to Young’s modulus, Poisson’s ratio and compressive strength, we mainly rely on k_1 and E in calibration to fit the compressive strength while the rest of the parameters are set to default values. It turns out that this calibration suffices for acceptable predictions in impact problems.

3 Model Calibration and Verification

All the numerical models are implemented via finite element method in Abaqus 6.13. The microplane model (M7) with scaled boundary is written as a VUMAT user subroutine. In order to capture the interaction between the projectile and the target during impact, both surface and internal nodes of the target are included in the potential region of contact with the projectile. To reduce the computation cost, the numerical model simulates only half of the geometry with a symmetric boundary condition imposed on its central plane. For orthogonal perforation, it would be even more cost-effective to model only a quarter of the geometry, we didn’t do so because we would like to be consistent with the cases of oblique impact, which requires modeling at least one half of the geometry.

3.1 Orthogonal perforation with various target thicknesses

The M7 model with scaled boundaries was first evaluated using the published tests [8, 12] of orthogonal projectile perforation, performed at the Geotechnical and Structure Laboratory of the US Army Engineer Research and Development Center (ERDC) in Vicksburg. These tests used circular slabs of four thicknesses, 127, 216, 254 and 280mm, made of concrete WES-5000, whose standard compression strength was 48 MPa. The slabs were cast in steel culvert pipes of diameter 1.52 m. The projectiles, which were hollow and made of steel, had an ogival-nose (caliber radius head 3.0, length/diameter ratio 7.0 and diameter 50.8mm) and mass 2.3 kg. The projectiles impacted the concrete slabs with the velocity of 310 m/s at the angle of 90°. The perforation tests were carried out two or three times for each thickness of the slab. First, the M7 model was calibrated to fit the test data for concrete type WES-5000, used in these tests. This concrete had Young’s modulus $E = 25$ GPa and Poisson’s ratio $\nu = 0.18$. The optimum values of the M7 parameters, which achieved very good fits of quasi-static uni-, bi- and tri-axial tests, were $k_1 = 11 \times 10^{-5}$, $k_2 = 110$, $k_3 = 30$, $k_4 = 100$ and $k_5 = 1$.

The mesh used to discretize the slabs was uniform, and consisted of cubic elements of average size 7.5mm. The projectile was treated as rigid body since no excessive damage, melt or distortion was observed after the test. To prevent spurious mesh sensitivity, the modelling was performed in the sense of the crack band model, in which the finite-element size (or mesh size) should be equal to the material characteristic length, which characterizes the size of the representative volume element (RVE) of the material and is used as the localization limiter. The element size was considered as 7.5mm, which is about one to two times the maximum aggregate size. Please note that if, in quasi-static problems, the element size is changed, the crack band model requires adjusting the post-peak softening of the damage constitutive law.

This guarantees that energy dissipated in the crack band, which would always localize into one element width, would not change. This method has also been adopted by commercial softwares such as ATENA and OOFEM. However, in projectile impact problems, the deformation is generally so fast that there is not enough time for the cracking damage to localize, thus the post-peak of the damage law need not, and should not, be adjusted.

Fig. 1 shows the evolution of logarithmic strain contour during the impact of projectile into 127mm plain concrete target. Since predicted the crater shapes for various thicknesses are similar, the results for this particular case only are shown. As seen, the crater shape and size are reasonable and realistic.

The relation between the exit (or residual) velocity and target thickness is shown Fig. 1.a. The test data are marked in the figure with large circular points. The case of target thickness 254 mm was used for calibration, for which parameter A_1 is 1×10^{-6} resulted. Then the exit velocities for various target thicknesses (127mm, 160mm, 216mm and 270mm) were predicted using the calibrated parameter. As can be seen, the predicted values (thick curve) show good agreement to the test data (large circles). Fig. 2.b shows the evolution of projectile velocities during impact. After the projectile nose enters the target, its velocity decreases at a constant rate during the tunneling stage. The deceleration of the projectile is given by the slope of the common tangent of the velocity curves and equals $-2.36 \times 10^4 g$, where g is the acceleration of gravity.

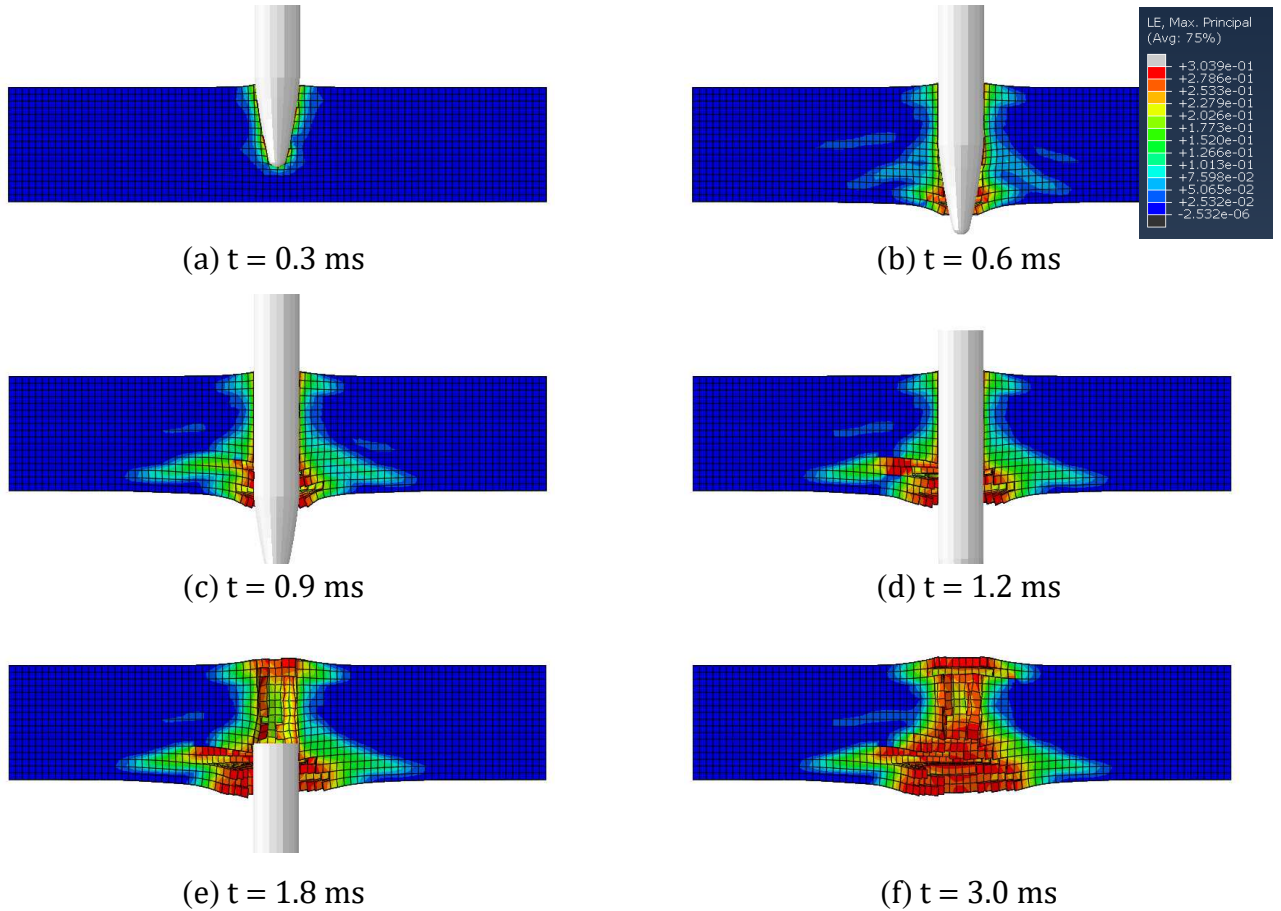


Figure 1: Strain contour evolution during orthogonal projectile impact into 127mm plain concrete target ($v_0 = 310$ m/s).

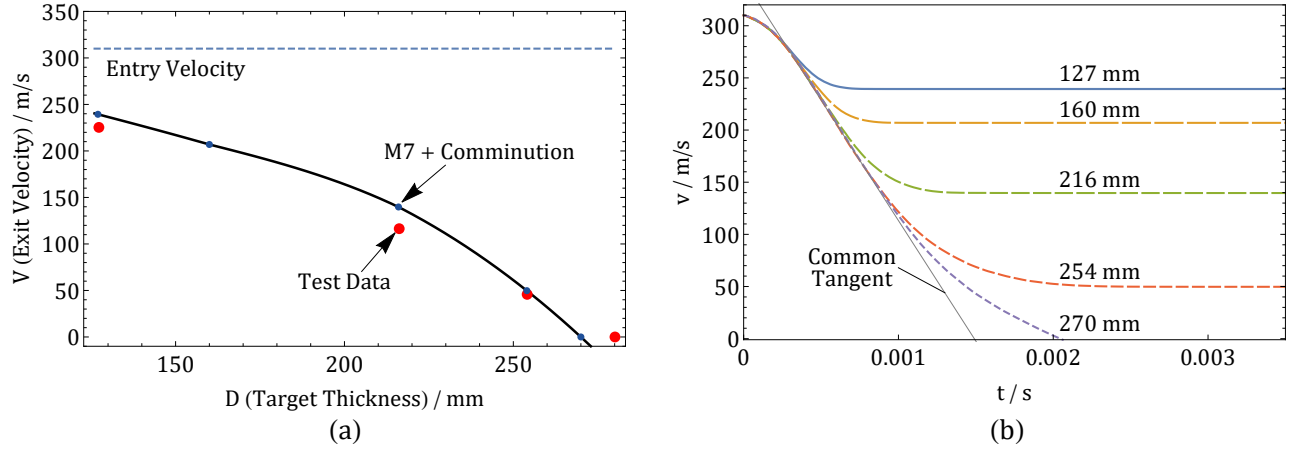


Figure 2: a) Comparison of simulated exit velocities with test data for various target thicknesses
b) Evolution of simulated vertical velocity during impact into targets of various thicknesses ($v_0 = 310$ m/s).

3.2 Orthogonal impacts with various striking velocities

The microplane model M7 with scaled boundaries previously used for constant striking velocities is now used to predict the exit velocity for various striking velocities as well. To this end, the simulations are compared with the impact experiments reported in [13]. In these experiments [13], projectiles of 0.50 kg mass, with ogival nose, were launched by a 30-mm smooth-bore powder gun at striking velocities V_s between 300 and 1100 m/s. The target was located 3.66 m from the end of the gun barrel. The residual velocities were measured by X-ray photography. They showed that the projectiles were not fractured by the targets, and the recovered projectiles showed only minor nose erosion. It is therefore reasonable for computations to treat the projectiles as rigid bodies.

Both normal strength concrete (NSC) and high strength concrete (HSC) targets were tested. Their compression strengths and densities were: $f_c = 48$ MPa and $\rho = 2440$ kg/m³ for NSC target, $f_c = 140$ MPa and $\rho = 2520$ kg/m³ for HSC target.

The fixture allowed the target to be positioned with respect to the prescribed impact location. By changing the location of impact point, one could control whether the projectile would hit the reinforcement. Test data [13] shows that the effect of striking the reinforcement at $V_s = 750$ m/s had a negligible effect. So it suffices to model the target using plain concrete instead.

The microplane model M7 with scaled boundaries was first calibrated using the material properties and one exit velocity in the test data. For the NSC target, the parameters were: $k_1 = 7.5 \times 10^{-5}$, $k_2 = 110$, $k_3 = 30$, $k_4 = 100$, $k_5 = 1$ and $A_1 = 1 \times 10^{-6}$. For the HSC target, $k_1 = 1.3 \times 10^{-4}$, $k_2 = 110$, $k_3 = 30$, $k_4 = 100$, $k_5 = 1$ and $A_1 = 2.9 \times 10^{-6}$. Then the residual (exit) velocities V_r were predicted by simulations with various striking velocities V_s . Comparisons of the test data and model predictions are shown in Fig. 3.a-b. As can be seen, the present results agree with the experiments very well.

3.3 Oblique perforation of plain concrete targets

The microplane model M7 has then be used to extend the analysis to oblique impacts and perforations. It is again found that the predictions of the residual velocities are consistent with

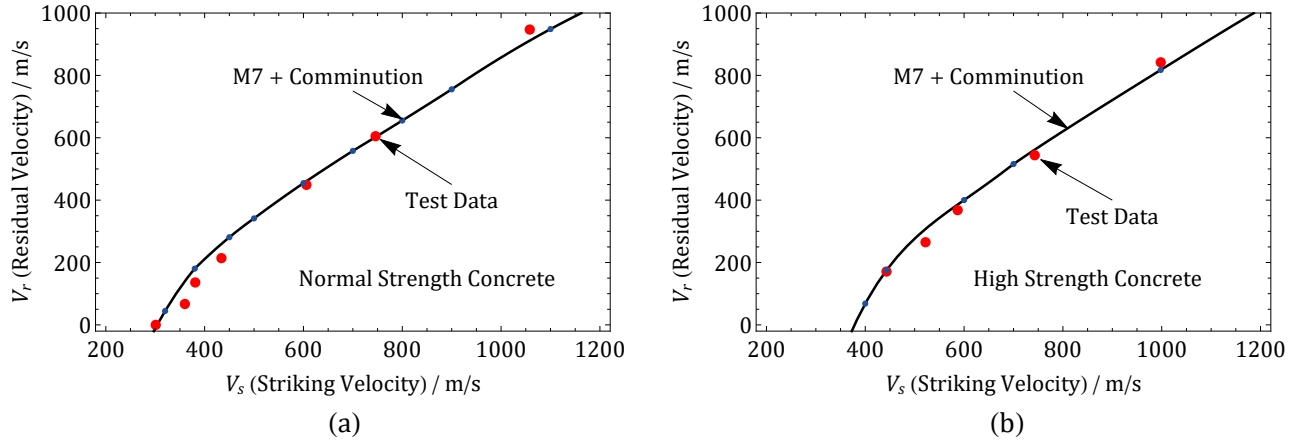


Figure 3: Comparison of numerically predicted residual velocities with test data: a) normal strength concrete (NSC) targets; b) high strength concrete (HSC) targets.

experiments. The test report [9] the projectile velocity along its axis, and the impact angle β ($0 < \beta < 90^\circ$) is defined as the acute angle between the projectile velocity and the outward normal of the impacted surface. Obviously, the orthogonal impact studied previously is the special case where $\beta = 0$.

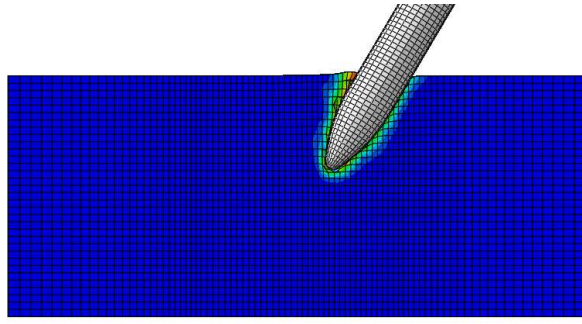
The sharp ogive-nose projectiles of the same geometry (269.4mm long and 50mm in diameter) were used in all the tests. The thickness of the target was $H=250$ mm, the obliqueness was $\beta = 30^\circ$ and other relevant parameters were: $f_c = 40$ MPa, $\rho = 2336$ kg/m³. Besides, three different striking velocities were used: 315, 407 and 610 m/s. In computations, linear hexahedron elements of size approximately 7.5 mm, with reduced integration, were used. Fig. 4 shows the typical strain contour evolution during the process of projectile penetration, with the initial striking velocity being 600 m/s. One can see that the bottom crater is much larger than the top one, which is realistic.

First, the model is calibrated using the case of striking velocity 600m/s. For the normal strength concrete slab the parameters are: $k_1 = 7.5 \times 10^{-5}$, $k_2 = 110$, $k_3 = 30$, $k_4 = 100$, $k_5 = 1$ and $A_1 = 4.5 \times 10^{-6}$. Then the relations between the residual and striking velocities are obtained by interpolation of several predictions (see Fig. 5). It is seen that the predictions (thick lines) match the test data (large points) quite well.

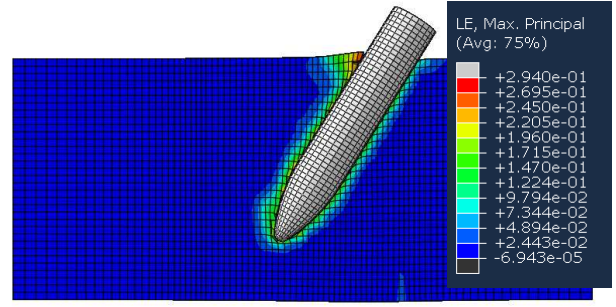
3.4 Oblique impact into thick concrete slabs with different angles

Next our model is used to match the tests of the depth of penetration of projectile into thick plain concrete, reported in [15]. The velocity direction and angle β are defined the same way as before; $0 < \beta < 90^\circ$). The density of concrete target is 2280 kg/m³, the compressive strength is 30 MPa, and the thickness is 800 mm. The projectile has a density of 7800 kg/m³, length of 180 mm and diameter of 20 mm. The projectile material is an alloy of 35 CrMnSiA and is modeled as a rigid body. The calibrated parameters are: $k_1 = 7.5 \times 10^{-5}$, $k_2 = 110$, $k_3 = 30$, $k_4 = 100$, $k_5 = 1$ and $A_1 = 1.2 \times 10^{-6}$.

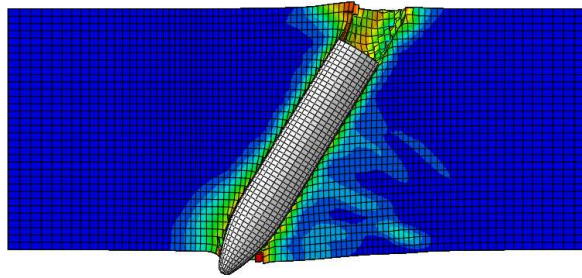
The comparisons of the depth of penetration (DOP) from simulations and experiments are shown in Table. 1. From the table, one can see that when the oblique angle β is smaller than 35° , the predictions agree very well with the experiment and the relative error is always less than 3.6%. When $\beta > 35^\circ$, however, the predicted DOPs are much larger than those measured. This increase of relative error arises mainly from the rigid-body assumption for projectile. Different



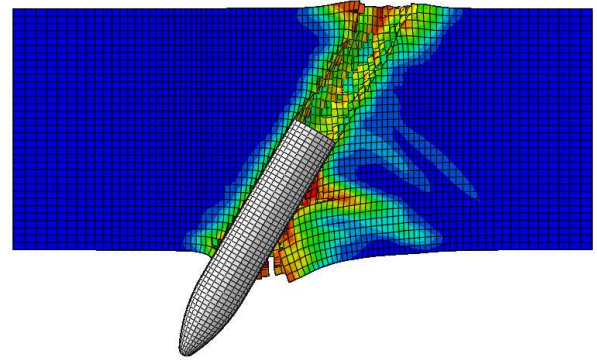
(a) $t = 0.2$ ms



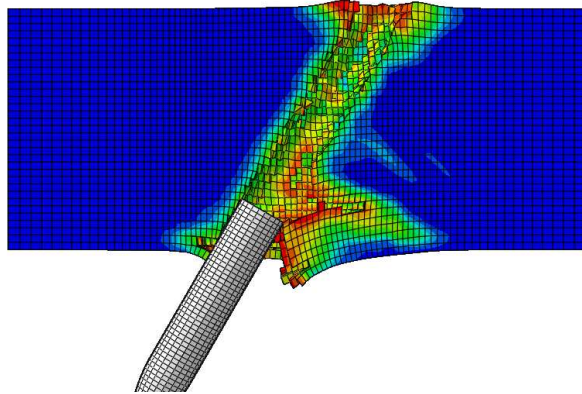
(b) $t = 0.4$ ms



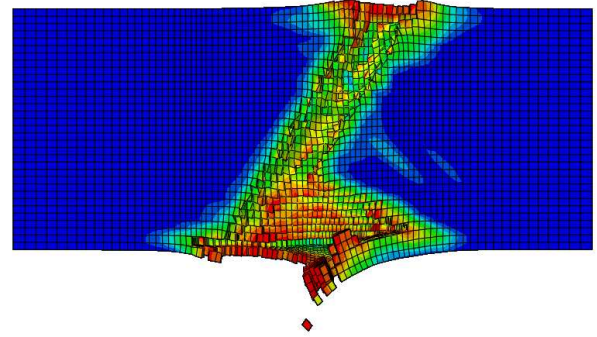
(c) $t = 0.6$ ms



(d) $t = 0.8$ ms



(e) $t = 1.0$ ms



(f) $t = 1.9$ ms

Figure 4: Strain contour evolution during oblique projectile impact into 250mm plain concrete target ($v_0 = 600$ m/s).

from normal perforation tests, steel projectiles could endure significant inelastic deflection when oblique angle is large, with nose bending upward. As a result, the trajectory of projectile is no longer a straight line and it curves up with the nose, leading to a smaller DOP. To model this deviation, it would obviously be necessary to include a realistic constitutive model for the projectile (which is beyond the scope of this study).

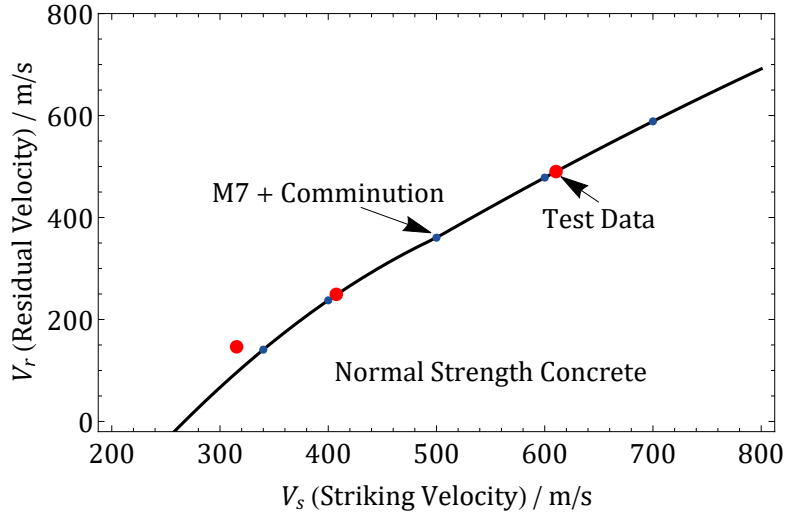


Figure 5: Comparison of numerically predicted residual velocities of projectile with test data ($\beta = 30^\circ$).

Table 1:

Impact Velocity (m/s)	Oblique Angle ($^\circ$)	Tested DOP (cm)	Simulated DOP (cm)
545	28	22.5	23.1
528	30	21.9	22.3
525	30	23	23.8
545	33	21.6	22.4
552	33	20	20.7
537	35	25.2	26
548	35	23	23.8
541	41	15.1	18.9
545	41	16	19.2

4 Conclusions

1. The microplane model M7 with stress-strain boundaries scaled according the deviatoric strain rate provides an effective method for simulating and predicting the projectile impact into concrete slabs. It captures the phenomenon of dynamic overstress by taking account of the local kinetic energy dissipation. It is shown that the model predicts accurately the exit velocities of the projectiles, crater shapes and penetration depths, for both orthogonal and oblique impacts.
2. The local kinetic energy dissipation is calculated by a previously proposed theory in which concrete comminution is driven by the kinetic energy of the strain-rate field of forming particles, which supplies the fracture energy of the fragments about to form. In this theory, bearing partial analogy to the theory of turbulence, the stress boundaries (or strength limits) scale with the $-4/3$ power of the effective deviatoric strain rate and the time derivative of that rate.
3. The rigid-body assumption for the projectiles is valid only when the impact angle deviates

from the normal by less than 35° . For larger angles the path becomes curved and the projectiles would have to be treated as deformable.

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Research



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Strain-rate-dependent microplane model for high-rate comminution of concrete under impact based on kinetic energy release theory

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The apparent increase of strength of concrete at very high strain rates experienced in projectile impact (10 s^{-1} to 10^6 s^{-1}), called 'dynamic overstress', has recently been explained by the theory of release of local kinetic energy of shear strain rate in finite size particles about to form. This theory gives the particle size and the additional kinetic energy density that must be dissipated in finite-element codes. In previous research, it was dissipated by additional viscosity, in a model partly analogous to turbulence theory. Here it is dissipated by scaling up the material strength. Microplane model M7 is used and its stress-strain boundaries are scaled up by factors proportional to the $-4/3$ rd power of the effective deviatoric strain rate and its time derivative. The crack band model with a random tetrahedral mesh is used and all the artificial damping is eliminated. The scaled M7 model is seen to predict the crater shapes and exit velocities of projectiles penetrating concrete walls of different thicknesses as closely as the previous models. The choice of the finite strain threshold for element deletion criterion, which can have a big effect, is also studied. It is proposed to use the highest threshold above which a further increase has a negligible effect.

1. Introduction

Dynamic fracture has been the subject of numerous studies [1–21]. Many aspects are now well understood, especially the dynamic propagation and branching of individual cracks [2,5,17,19,22,23]. The dynamic comminution (i.e. fragmentation, pulverization and crushing) is of interest for many practical purposes, including impact of rocks, concrete, metals, composites and ceramics [3–5,17,21], as well as various industrial grain reduction processes, explosion effects on structures, rock blasting and fracturing of gas or oil shale by chemical explosions or by electro-hydraulic pulsed arc in a horizontal borehole [24,25]. This paper is focused on only one aspect—the projectile impact onto concrete walls.

Predictability of impact effects on concrete has been hampered by the problem of the so-called ‘dynamic overstress’. As the strain rate increases above 1 s^{-1} , the material strength needed to fit data on projectile penetration appears to increase above what is predicted by the standard rate effects, which include the activation energy controlled bond breakage at crack tips and viscoelasticity of the material between the cracks. At strain rates above 10^4 s^{-1} , this increase becomes enormous (e.g. [26]). Some investigators [7] dealt with this problem by simply adjusting the strain-dependent strength limits (or boundaries) of microplane model M4 so as to fit the exit velocities of a projectile penetrating a wall. Lacking physical justification, such an approach cannot be expected to be predictive in different situations.

In [27,28], it was proposed that the physical justification of ‘dynamic overstress’ must be sought in the fracturing of concrete which is driven not by the release of strain energy, as in classical fracture mechanics, but by the release of kinetic energy of shear strain rate field within forming particles as the concrete is getting comminuted into small fragments. At the rate of 10^4 s^{-1} , the kinetic energy of the forming particle exceeds the maximum possible strain energy in the particle by two orders of magnitude. Thus the comminution process can dissipate a lot of energy [11–15], is not covered by the damage constitutive laws based on standard triaxial testing which leads to much larger fragments. This new theory, which bears some analogy to turbulence, was shown to give a strain-rate-dependent expression for the additional density $\Delta\mathcal{K}$ of kinetic energy that drives the comminution and must be dissipated in the finite-element code. There are various methods to dissipate it, and apparently it does not matter which one is adopted.

- (1) One method to dissipate $\Delta\mathcal{K}$, which is more natural for rate-dependent stress, is to introduce additional shear viscosity η calculated from the calculated kinetic energy release. This led to a model with partial analogy to turbulence theory [27,29]. As a simplified approach, η was considered to depend only on the strain rate [27,28]. In a rigorous approach, η was considered to also depend on the strain-rate derivative [30]. Both approaches led to a good match of test data on the exit velocities of projectiles from concrete walls of different thicknesses, and the latter also matched well the measured penetration depths of projectiles of different velocities.
- (2) As another method, rather than treating the interface fracture between forming particles as instantaneous, it was considered to grow at a finite (though very high) speed, according to a power law of kinetic energy release rate, similar to the Charles–Evans law for static subcritical crack growth. In this way, the same data could again be matched well [31].
- (3) A third method, which can match the data on projectile exit velocity and penetration depths equally well, is studied here. Similar to [7], the strain-dependent strength limits (called boundaries) of the microplane damage constitutive model M7 are raised as a function of the shear strain rate, but not by a factor found [7] by matching the projectile penetration data, which would not be predictive. Rather, they are raised so as to dissipate the additional kinetic density $\Delta\mathcal{K}$ of the shear strain rate field in the forming particles. It is also assumed that the particles form instantly, as in method 1. This method leads to strength increase depending on both the strain rate and its derivative. The post-comminution behaviour, including subsequent further comminution and clustering into

bigger particle groups to release the kinetic energy that is being dissipated by inter-group friction, is also discussed and modelled. The present formulation makes it possible to eliminate the artificial dampings of all types, which are normally embedded in commercial finite-element codes but are not predictive as they are not justified physically.

In an attempt to handle the ‘dynamic overstress’, an increase of the microplane stress–strain boundaries (or strength limits) was already introduced by Adley *et al.* [7,8] (using model M4). They scaled the boundaries as a function of the strain rate $\dot{\epsilon}$ by a rate amplification factor χ given as,

$$\chi_i = \dot{\epsilon}_{\text{norm}}^{c_i}, \quad \text{where} \quad \dot{\epsilon}_{\text{norm}} = \max\left(\frac{\dot{\epsilon}}{\dot{\epsilon}_{\text{quasistatic}}}, 1.0\right), \quad (1.1)$$

where c_i are empirical exponents, different for each boundary. They obtained these exponents by fitting the missile penetration data. They achieved a good fit of the observed exit velocities of projectiles penetrating concrete walls of different thicknesses, as well as the crater shapes. However, the connection to the data from quasi-static material testing of concrete, which is the only available source of nonlinear triaxial, damage and fracture properties, was thus lost. The modified microplane model used in these impact simulations could not fit the material test data. Therefore, the model cannot be predictive for other situations.

2. Brief overview of the kinetic energy theory of comminution

The physical source of ‘dynamic overstress’ during impact was traced to the dissipation of the local kinetic energy of shear strain rate within finite size comminuting particles [27]. Let ϵ_{Dij} denote the deviatoric strain tensor and the superior dot the derivatives with respect to time t . Then, the density of kinetic energy of effective deviatoric strain rate $\dot{\epsilon}_D = \sqrt{\dot{\epsilon}_{Dij}\dot{\epsilon}_{Dij}/2}$ is dissipated by creating interface fractures resulting in many particles.

Consider an idealized dynamic fracture process in which the solid is comminuted to identical prismatic particles of length h and hexagonal cross-section of side $h/2$, at a deviatoric strain rate $\dot{\epsilon}_D$ (figure 1). Analysis of the kinematics and comparison of the kinetic energy of particles before (figure 1b) and after (figure 1c) the interface fracture showed [27] that, for a motion in the plane of maximum shear strain, the local kinetic energy of the particles that are about to form, per unit volume of material, $\Delta\mathcal{K}$, is additive to, and separable from, the global kinetic energy.

The global kinetic energy corresponds to the motion of the centres of the particles whose formation is imminent. For a given $\dot{\epsilon}_D$, the drop in kinetic energy per unit volume is found to be [27]:

$$-\Delta\mathcal{K} = c_k \rho h^2 \dot{\epsilon}_D^2, \quad (2.1)$$

where $c_k = I_p/(2hV_p)$, ρ = mass density, $V_p = 3\sqrt{3}h^3/8$ and $I_p = 5\sqrt{3}h^4/128$ = volume and polar moment of inertia of each hexagonal prism about its axis, respectively.

In reality, the size of comminuted particles is never uniform but varies randomly within a certain range, $s \in (h, H)$ where h, H = minimum and maximum sizes, and s = variable particle size. Schuhmann’s empirical power law [32] is adopted to describe the cumulative distribution. It gives the volume fraction of particles of sizes between h to s :

$$F(s) = \frac{s^k - h^k}{H^k - h^k}, \quad s \in (h, H), F(s) \in (0, 1), \quad (2.2)$$

where $(k \approx 0.5)$ is the empirical constant [27].

Then, introducing the simplifying hypothesis that the kinetic energy drop for variable particle size can be obtained by summing the energy losses for all infinitesimal intervals $(s, s + ds)$ calculated for each interval as if the particle size and shape were uniform. Integration furnishes

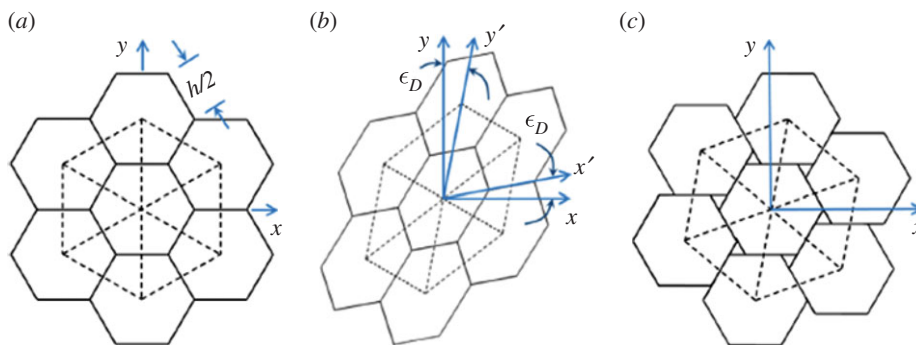


Figure 1. Schematic of material comminution into prismatic hexagonal particles: (a) undeformed regime; (b) sheared regime and (c) comminuted regime. (Online version in colour.)

the kinetic energy drop for particles of all sizes per unit volume [27]:

$$-\Delta\mathcal{K} = \int_{s=h}^H c_k \rho s^2 \dot{\epsilon}_D^2 dF(s) = C_k \rho h^2 \dot{\epsilon}_D^2, \quad (2.3)$$

where

$$C_k = \frac{k}{k+2} \frac{(H/h)^{k+2} - 1}{(H/h)^k - 1} c_k. \quad (2.4)$$

The interface area of uniform particles of size s per unit volume of material is c_s/s , where c_s is the dimensionless constant. For particle size distributed according to equation (2.2), the particle interface area per unit volume of material is obtained as [27]:

$$S = \int_{s=h}^H \frac{c_s}{s} dF(s) = \frac{C_s}{h}, \quad (2.5)$$

where

$$C_s = \frac{c_s k}{k-1} \frac{(H/h)^{k-1} - 1}{(H/h)^k - 1} \quad (2.6)$$

is a dimensionless constant [27]. Equation (2.5) adds, or integrates, the energy dissipations for different particle sizes and is calculated under the assumption that for each size there is a regular array of equal-size particles.

Then, considering the dynamic fracture criterion [27,30] to be $-\Delta\mathcal{K} = S\Gamma$, an overall energy balance is imposed (where Γ is the interface fracture energy of the comminuting particles). Substitution of equations 2.3 and 2.5 gives the minimum particle size as

$$h = C_r \left(\frac{\Gamma}{\rho \dot{\epsilon}_D^2} \right)^{1/3}, \quad (2.7)$$

where

$$C_r = \left(\frac{c_s}{c_k} \frac{k+2}{k-1} \frac{(H/h)^{k-1} - 1}{(H/h)^{k+2} - 1} \right)^{1/3}, \quad (2.8)$$

where C_r is a dimensionless constant. As seen in equation (2.7), the particle size is proportional to the $\frac{2}{3}$ power of the effective deviatoric strain rate. A higher effective deviatoric strain rate gives a greater kinetic energy release, which leads to smaller particles, and thus to a greater interface area and more dissipation.

In [27], equation 2.8 was used as an explicit formula in which the value of H/h was estimated. It should be noted that this estimate might have some errors. An alternate approach could be using a variable H/h , where H is constant but h is not. But this would require a reformulation

of the theory and could not be used in arriving at equation (2.7). Such a theory might be more complicated without a significant improvement in predictions. Hence, we assume a constant H/h .

Then, substitution of equation (2.7) into equation (2.3) gives the kinetic energy drop in terms of only the effective deviatoric strain rate and the material parameters:

$$-\Delta\mathcal{K} = A\dot{\epsilon}_D^{2/3}, \quad (2.9)$$

where $A = (C_0 \Gamma^2 \rho)^{1/3}$ and $C_0 = C_k^3 C_r^2$ is a dimensionless constant. Note that assuming a hexagonal shape of particles is not a strict requirement of the formulation. In reality the shapes of the particles are sure to be variable, but the basic functional form of the equations is not expected to change.

Furthermore, note that in an actual impact event, $\dot{\epsilon}_D$ need not be constant during comminution of a given material volume. So it is useful to obtain the drop in kinetic energy in one time increment of the comminution process. Accordingly, as proposed in [30] the increment of drop in kinetic energy is given by

$$d(-\Delta\mathcal{K}) = \frac{2A}{3} \dot{\epsilon}_D^{-1/3} d\dot{\epsilon}_D. \quad (2.10)$$

Thus, in a given strain increment, the drop in kinetic energy depends not only on the strain rate but also its increment. Any scaling of the boundaries of a constitutive model to account for comminution, must be consistent with this dependence. Such a formulation is proposed in the next section, for the microplane model M7.

3. Microplane model M7 with scaled boundaries

The microplane model M7 is the latest version in a series of progressively improved microplane models labelled M0, M1, M2, ..., M6, developed first for concrete (and then extended to other materials). The microplane model, supplemented by some localization limiter with material characteristic length, has been proven to give rather realistic predictions of the constitutive and damage behaviour of quasi-brittle materials over a broad range of loading scenarios, including uniaxial, biaxial and triaxial loadings with post-peak softening, compression-tension load cycles, opening and mixed mode fractures, tension-shear failure and axial compression followed by torsion [33,34].

The basic idea of the microplane model is to express the constitutive law not in terms of tensors, but in terms of the vectors of stress and strain acting on a generic plane of any orientation in the material microstructure, called the microplane. The use of vectors is analogous to the Taylor models used for plasticity of polycrystalline metals, but with important conceptual differences. Firstly, to avoid model instability in post-peak softening, a kinematic constraint is used instead of a static one [33]. Thus, the strain (rather than stress) vector on each microplane is the projection of the macroscopic strain tensor. So we have,

$$\epsilon_N = \epsilon_{ij} N_{ij}, \quad \epsilon_M = \epsilon_{ij} M_{ij} \quad \text{and} \quad \epsilon_L = \epsilon_{ij} L_{ij}, \quad (3.1)$$

where ϵ_N , ϵ_M and ϵ_L are the magnitudes of the three strain vectors corresponding to each microplane, and $N_{ij} = n_i n_j$, $M_{ij} = (n_i m_j + m_i n_j)/2$ and $L_{ij} = (n_i l_j + l_i n_j)/2$, n , m , and l being the three mutually orthogonal normal and tangential unit vectors characterizing that microplane, and the subscripts i and $j = 1, 2, 3$. Secondly, a variational principle (principle of virtual work) relates the stresses on the microplanes (σ_N , σ_M and σ_L) to the macro-continuum stress tensor σ_{ij} , to ensure equilibrium; it is expressed as

$$\frac{2\pi}{3} \sigma_{ij} \delta \epsilon_{ij} = \int_{\Omega} (\sigma_N \delta \epsilon_N + \sigma_M \delta \epsilon_M + \sigma_L \delta \epsilon_L) d\Omega. \quad (3.2)$$

This expression means that, within a unit sphere, the virtual works of macro-stresses and micro-stresses must be equal (for details, see [33,35]). In the microplane model M7, the micro-stresses are subjected to strain-dependent boundaries (or strength limits) of four types, *viz.*:

- (1) the tensile normal boundary—to capture progressive tensile fracturing;
- (2) the compressive volumetric boundary—to capture phenomenon such as pore collapse under extreme pressures;
- (3) the compressive deviatoric boundary—to capture softening in compression; and
- (4) the shear boundary—to capture friction.

The M7 also includes the quasi-static rate effects [36], which consist of a rate-dependent crack opening and growth controlled by the activation energy of bond breakage, and of viscoelasticity of the material between the cracks. However, the quasi-static rate effects in M7 have been calibrated for strain rates of about 1 s^{-1} , and would have to be recalibrated.

As comminution is induced by local shear strains, we assume that in a given strain increment $d\epsilon_{ij}$, the additional energy to be dissipated must equal the additional distortional strain energy given by $\Delta\sigma_{Dij}\epsilon_{Dij}$, where $\Delta\sigma_{Dij}$ is the additional deviatoric stress that results due to scaled boundaries. To express it in terms of the microplane stresses, we first define the volumetric stress on the microplane level, σ_V , as

$$\frac{2\pi}{3} \frac{\sigma_{kk}}{3} \delta\epsilon_{mmm} = \int_{\Omega} \sigma_V \delta\epsilon_V d\Omega. \quad (3.3)$$

Substituting $\delta\epsilon_V = \delta\epsilon_{mmm}/3$ and $\int_{\Omega} = 2\pi d\Omega$, the volumetric stress on the microplane level can be expressed as $\sigma_V = \sigma_{kk}/3$. Also, the microplane normal stresses $\sigma_N = \sigma_D + \sigma_V$, where σ_D is the deviatoric stress on the microplane level. Likewise, for microplane strains, we have $\epsilon_N = \epsilon_D + \epsilon_V$. Note that $\epsilon_V = \epsilon_{kk}/3$ is the same for all microplanes. Then, subtracting equation (3.3) from (3.2), we get

$$\frac{2\pi}{3} \left(\sigma_{ij} - \frac{\sigma_{kk}}{3} \right) \delta\epsilon_{ij} = \int_{\Omega} (\sigma_D + \sigma_V)(\delta\epsilon_D + \delta\epsilon_V) + \sigma_M \delta\epsilon_M + \sigma_L \delta\epsilon_L - \sigma_V \delta\epsilon_V d\Omega. \quad (3.4)$$

Now noting that $\int_{\Omega} \sigma_D \delta\epsilon_V = \int_{\Omega} \sigma_V \delta\epsilon_D = 0$, we get

$$\frac{2\pi}{3} \sigma_{Dij} \delta\epsilon_{ij} = \int_{\Omega} (\sigma_D \delta\epsilon_D + \sigma_M \delta\epsilon_M + \sigma_L \delta\epsilon_L) d\Omega. \quad (3.5)$$

Then, noting that $\delta\epsilon_D = \delta\epsilon_N - \delta\epsilon_V = \delta\epsilon_{ij} N_{ij} - \delta\epsilon_{ij} \delta_{ij}/3$, $\delta\epsilon_M = \delta\epsilon_{ij} M_{ij}$ and $\delta\epsilon_L = \delta\epsilon_{ij} L_{ij}$, we get an expression for the macroscale deviatoric stress tensor as,

$$\frac{2\pi}{3} \sigma_{Dij} = \int_{\Omega} \sigma_D \left(N_{ij} - \frac{\delta_{ij}}{3} \right) + \sigma_M M_{ij} + \sigma_L L_{ij} d\Omega. \quad (3.6)$$

This expression implies that a change in the macro-scale deviatoric stress can be achieved by scaling only the deviatoric and frictional boundaries of M7. Let the change in the microplane deviatoric and shear stresses be $\Delta\sigma_D$, $\Delta\sigma_M$ and $\Delta\sigma_L$ for a change in deviatoric stress of $\Delta\sigma_{Dij}$. So we have,

$$\frac{2\pi}{3} \Delta\sigma_{Dij} = \int_{\Omega} \Delta\sigma_D \left(N_{ij} - \frac{\delta_{ij}}{3} \right) + \Delta\sigma_M M_{ij} + \Delta\sigma_L L_{ij} d\Omega. \quad (3.7)$$

Now, in the interest of simplicity, we assume that both the deviatoric and frictional boundaries are scaled by the same amount, on each microplane. So, $\Delta\sigma_D = \Delta\sigma_M = \Delta\sigma_L = \Delta f$, and

$$\frac{2\pi}{3} \Delta\sigma_{Dij} = \int_{\Omega} \Delta f \left(N_{ij} - \frac{\delta_{ij}}{3} \right) + \Delta f M_{ij} + \Delta f L_{ij} d\Omega. \quad (3.8)$$

This may be simplified as

$$\Delta\sigma_{Dij} = \Delta f C_{ij}, \quad (3.9)$$

where,

$$C_{ij} = \frac{3}{2\pi} \int_{\Omega} \left(N_{ij} - \frac{\delta_{ij}}{3} \right) + M_{ij} + L_{ij} d\Omega, \quad \text{which is constant.} \quad (3.10)$$

Δf may be taken out of the integral as we assume it to be the same for each microplane. Now, the next task is to relate the quantity Δf to the energy dissipated due to comminution, ΔK . To ensure

the aforementioned energy balance, the work of the additional deviatoric stress must dissipate an energy equal to the drop of kinetic energy of strain rate field caused by comminution in each increment. So,

$$d(-\Delta K) = \Delta \sigma_{Dij} d\epsilon_{Dij}, \quad (3.11)$$

where $i, j = 1, 2, 3$ and d denotes a small increment. So we have

$$d(-\Delta K) = \Delta f C_{ij} d\epsilon_{Dij}. \quad (3.12)$$

Now, using equation 2.10 and taking the derivative with respect to time on both sides, we get

$$\frac{2A}{3} \dot{\epsilon}_D^{-1/3} \ddot{\epsilon}_D = \Delta f C_{ij} \dot{\epsilon}_{Dij}. \quad (3.13)$$

Multiplying both sides by $\dot{\epsilon}_{Dij}$, we get

$$\frac{2A}{3} \dot{\epsilon}_D^{-1/3} \ddot{\epsilon}_D \dot{\epsilon}_{Dij} = 2\Delta f C_{ij} \dot{\epsilon}_D^2, \quad (3.14)$$

as $\dot{\epsilon}_D = \sqrt{\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}/2}$. Thus we have,

$$\Delta f C_{ij} = \frac{A}{3} \dot{\epsilon}_D^{-7/3} \ddot{\epsilon}_D \dot{\epsilon}_{Dij}. \quad (3.15)$$

Note that the right-hand side of this expression is consistent with [30]. To obtain the scalar value Δf , we now calculate the effective values of both sides, by taking square root of the inner product with itself. So we have

$$\Delta f (C_{ij}C_{ij})^{1/2} = \frac{\sqrt{2}A}{3} \dot{\epsilon}_D^{-7/3} \ddot{\epsilon}_D \dot{\epsilon}_D. \quad (3.16)$$

Thus, the scalar Δf is expressed as

$$\Delta f = A_1 \dot{\epsilon}_D^{-4/3} \ddot{\epsilon}_D. \quad (3.17)$$

Here A_1 is a constant to be calibrated, and is given by $A_1 = \sqrt{2}A/3\bar{C}$ and $\bar{C} = \sqrt{C_{ij}C_{ij}}$. Therefore, the deviatoric and frictional boundaries of M7 are scaled as $F = F_{qs}(1 + A_1 \dot{\epsilon}_D^{-4/3} \ddot{\epsilon}_D)$, where $F_{qs} = F_0(1 + h(\dot{\epsilon}))$, the boundary that is already scaled to account for the quasi-static rate effects [36], and F_0 is the original unscaled M7 boundary. This expression shows that to capture the energy dissipation due to comminution, it is necessary to make the boundary a function of both the first and second time derivatives of the effective deviatoric strain.

4. Concrete slab perforation by projectile impact

The M7 model with scaled boundaries was evaluated using the tests of projectile perforation, performed at the Geotechnical and Structure Laboratory of the US. Army Engineer Research and Development centre (ERDC), Vicksburg [8,37]. These tests used circular slabs of four thicknesses, 127, 216, 254 and 280 mm, made of concrete WES-5000, whose standard compression strength was 48 MPa. The slabs were cast in steel culvert pipes of diameter 1.52 m, sufficient to approximate the response for a semi-infinite radius (despite that non-reflecting finite elements producing no backward waves were used at the boundary). The projectiles, which were hollow and made of steel, had an ogival-nose (caliber radius head 3.0, length/diameter ratio 7.0 and diameter 50.8 mm) and weight of 2.3 kg. The projectiles impacted the concrete slabs with the velocity of 310 m s^{-1} at the angle of 90° . The perforation tests were carried out two or three times for each thickness of the slab. First, the M7 model was calibrated to fit the test data for concrete WES-5000, used in these tests. This concrete had Young's modulus $E = 25 \text{ GPa}$, and Poisson's ratio $\mu = 0.18$. The optimum values of M7 parameters, which achieved very good fits of quasi-static uni-, bi- and tri-axial tests, were $k_1 = 11 \times 10^{-5}$, $k_2 = 110$, $k_3 = 30$, $k_4 = 100$ and $k_5 = 10^{-4}$ [33,34].

The mesh used to discretize the slabs was random but statistically uniform, and consisted of tetrahedral elements of average size 7.5 mm. The projectile was considered rigid as no obvious

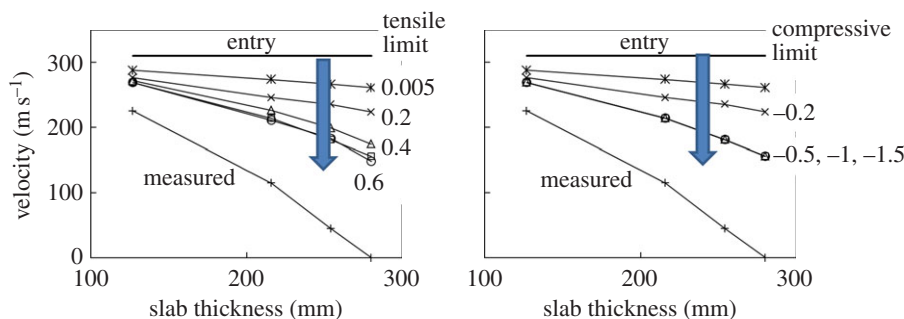


Figure 2. Predicted exit velocities using basic M7 model with no rate effects, for various element deletion thresholds. (Online version in colour.)

damage, melt or deformation was observed after the test. To prevent spurious mesh sensitivity, the modelling was performed in the sense of the crack band model, in which the finite-element size (or mesh size) should be equal to the material characteristic length, which characterizes the size of the representative volume element (RVE) of the material and is used as the localization limiter. The element size was considered as 7.5 mm, which is about one to two times the maximum aggregate size. Note that if, in quasi-static problems, the element size is changed, the crack band model requires adjusting the post-peak softening of the damage constitutive law so that the energy dissipated in the crack band (localized into one element width) would not change [38] (which is what is done in commercial software such as ATENA or OOFEM [39]). But, in projectile impact problems, the deformation is generally so fast that there is not enough time for the cracking damage to localize, and so the post-peak of the damage law need not, and should not, be adjusted.

Also note that, while the apparent strength and fracture energy depend on the strain rate, the crack band width itself does not. It is a material property that can be measured, e.g. as the minimum possible spacing of parallel quasi-static macro-cracks. It depends on material heterogeneity, and is usually equal to 1 to 3 maximum inhomogeneity sizes. In dynamic fracture, where the localization is suppressed by high deformation rates, the finite-element solver based on the crack band model automatically simulates (in a diffuse way, of course) the formation of multiple cracks and crack branching. The additional energy dissipation at very high strain rates is accounted for by material law adjustment, one type of which is presented here.

(a) Element deletion criterion

For these analyses, it was necessary to remove excessively distorted elements to avoid termination of analysis due to a negative Jacobian. This was done using an element deletion criterion based on the maximum principal strain. Thresholds of the criterion were set independently for tension and compression, to a value such that a further increase of the threshold did not make a difference to the predicted exit velocities. Initially, considering the unscaled M7 model, a sensitivity study was conducted using various tensile and compressive threshold values. The results are shown in figure 2*a,b*. It can be seen that, beyond 40% tensile strain and 100% compressive strain, the exit velocities do not change much. Accordingly, these values were set as the thresholds for the element deletion criterion. For the sake of comparison, the results using the basic M7 with the element deletion criterion from [28] (tensile threshold = 0.5% and no compressive threshold) and [30] (tensile threshold = 100% and compressive threshold = -20%) are also shown. It is seen that those deletion thresholds led to an overestimation of the exit velocity. This is rectified here by using sufficiently higher thresholds.

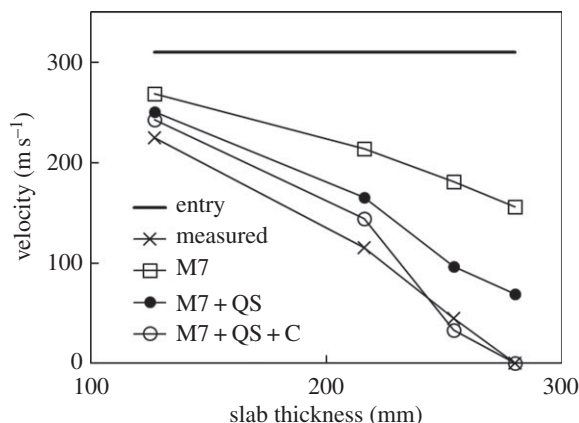


Figure 3. Comparison of the exit velocities predicted by M7 with no rate effects, with quasi-static rate effects and with comminution effects and the experimental data.

(b) Quasi-static rate effects

Next, using the aforementioned element deletion criterion, the quasi-static rate effects were employed, and the projectile exit velocities were predicted (with the same parameters as mentioned in [28]). These effects mainly refer to the rate dependence of cohesive crack opening and follow from the activation energy theory of bond ruptures. These effects are accounted for by scaling all the boundaries of M7 as $F_{qs} = F_0(1 + C_{R2}\ln(2\dot{\epsilon}/C_{R1}))$ [36], where $\dot{\epsilon} = \sqrt{\frac{1}{2}\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}}$. The parameter values C_{R1} and C_{R2} were determined in [28] to be 4×10^{-6} and 0.022, respectively. The exit velocities were again predicted by including these effects, as shown in figure 3. It is seen that these effects caused the predicted velocities to change significantly. In previous studies [28], use of a non-conservative element deletion criterion possibly underestimated the magnitude of these effects, which is again rectified here. However, it is seen that the quasi-static rate effects still do not suffice to correctly predict the exit velocities. So now we add the effects of comminution by scaling the deviatoric and friction boundaries of M7, as described in §3.

(c) Comminution effects

Parameter A_1 was calibrated by fitting one data point in figure 3, which led to the value $A_1 = 3 \times 10^{-7}$. The remaining points were then predicted. Figure 3 shows the predicted exit velocities for all the four slabs tested. As can be seen, the predictions of exit velocities are reasonably good, for all the data points. For the smaller slabs, the exit velocity is slightly overestimated while for the larger slab it is slightly underestimated. The fit could further be improved by relaxing the assumption that both the deviatoric and friction boundaries are scaled by the same factor. But this has not been explored as the errors are small and the data are too limited for calibrating two factors. To further assess the predictions, the predicted crater shapes for the slabs are compared with the measured ones, as shown in figure 4. It can be seen that the shapes can be matched very well, especially those at the exit side. This serves to show that the proposed scaling of M7 boundaries accounts for the phenomenon of comminution quite well.

(d) Finite strain effects

The basic form of microplane model M7 applies only to small strains. In comminution, however, very large strains might occur. To deal with them, a finite strain generalization of the microplane model M4, a predecessor of M7, was proposed in [40]. The same generalization is possible for model M7.

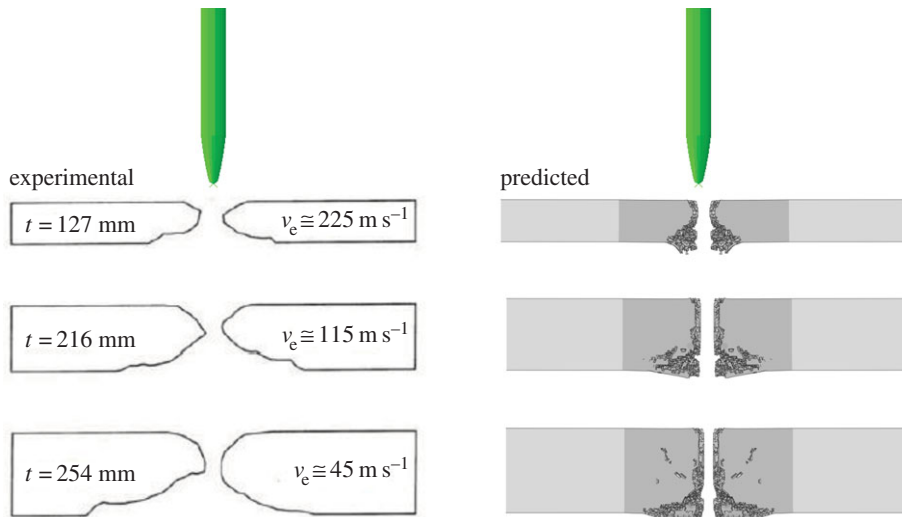


Figure 4. Comparison of the crater shapes predicted by M7 with comminution effects and the experiment data. (Online version in colour.)

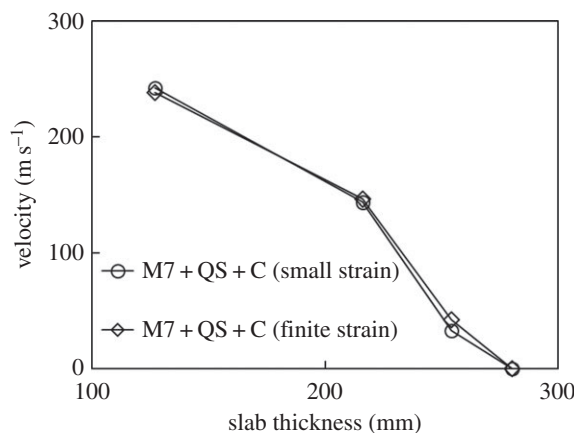


Figure 5. Comparison of the exit velocities predicted by the small and finite strain versions of M7, both include comminution effects.

According to [40], the finite strain generalization of the microplane model is formulated in terms of the stretches and shear angles calculated from the Green–Lagrange strain measure. Only for this strain measure, the stretch in normal direction to one microplane and the shear angles on that microplane can be calculated from the strain components on that same microplane exactly. For other strain measures, they cannot as the stretch and shear angles on one microplane depend also on the strain components on other microplanes. For the Green–Lagrange strain measure, the volumetric strain ϵ_V is obtained as, $\epsilon_V = \epsilon_0 + \epsilon_0^2/2$, where $\epsilon_0 = (J - 1)/3$ and $J = \det(F)$, F being the deformation gradient. For large strains, the volumetric–deviatoric split of the strain tensor is in general multiplicative. However, for concrete, the ϵ_V magnitude can hardly exceed 3%, which is small enough to allow additive split, as used in M7 (and earlier in M4 [40]).

The best choice of the stress tensor for this case was shown to be the back-rotated Cauchy stress tensor $S = R^T \sigma R$, where R is the rotation tensor and σ is the Cauchy stress tensor. The Cauchy stress tensors enables the microplane stress vectors to have clear physical meaning, which simplifies the control of hydrostatic pressure sensitivity, internal friction and dilatancy of material. The projectile impact analyses were rerun with this generalization, and included both quasi-static

rate effects, and the kinetic energy rate effect. The finite strain generalization did change the results but not significantly, as seen in figure 5. The reason could be that even though the total strains are high, the rotations are not.

5. Conclusion

- (1) The microplane model M7 with boundaries modified according to the kinetic energy theory of comminution is an effective approach to simulate projectile impact effects on concrete slabs and offers possibilities of further refinement. It can accurately predict projectile exit velocities, crater shapes and penetration depths.
- (2) It is necessary to scale up the deviatoric and friction boundaries of M7, to achieve a rate-dependent increase of the macroscopic deviatoric stress. Greater versatility could be obtained by scaling up the deviatoric and friction boundaries by different factors. But there are not enough test data to calibrate two independent factors.
- (3) The scaling of the boundaries (or strength limits) must be proportional to both: (i) the $-4/3$ power of the effective deviatoric strain rate and (ii) the time derivative of that rate.
- (4) The crack band model with a random tetrahedral mesh leads to accurate predictions of the crater shapes.
- (5) An element deletion threshold is necessary to run impact analyses. The choice of the threshold has a significant effect on the results. Chosen is a high enough threshold for maximum principal strain such that a further increase would not change the results appreciably.
- (6) The finite strain generalization of the microplane model did not have a significant effect on the predicted exit velocities, perhaps because the material rotations were not too high.
- (7) What is most important is that the finite-element code dissipate the correct energy required by the kinetic theory of comminution. But how exactly this energy is dissipated does not seem important. Dissipation modes in terms of additional viscosity, rate dependence of interface fracture, and scaling of strength limits give similar results. While the viscosity approach is more natural for rate-dependent stress, the present approach has the advantage of greater versatility for further refinements, such as the possibility of different scalings of different microplane boundaries, which might be needed for various extensions such as hypervelocity impact, impact on rock, or explosive comminution of rock, e.g. shale.

Data accessibility. All the test data used were taken from the open literature.

Authors' contributions. The first author, K.K. developed the formulation of boundary scaling in detail, implemented it as an Abaqus subroutine, conducted the finite-element analyses test data fitting and wrote the first draft of the paper. The second author Y.S. contributed in the development of the formulation via technical discussions, provided the finite-element meshes and also reviewed and improved the draft. The corresponding author Z.P.B. conceived the basic idea of kinetic energy release from finite-size particles, directed the work and finalized the text. All the authors approved the publication.

Conflict of interests. We declare we have no conflicting interests.

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Viscous energy dissipation of kinetic energy of particles comminuted by high-rate shearing in projectile penetration, with potential ramification to gas shale

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Abstract While dynamic comminution is of interest to many processes and situations, this work is focused on the projectile impact onto concrete walls, in which predictions have been hampered by the problem of the so-called ‘dynamic overstress’. Recently, in analogy with turbulence, Bažant and Caner modeled the overstress as an additional viscous stress generated by apparent viscosity that accounts for the energy dissipation due to kinetic comminution of concrete into small particles at very high shear strain rate. Their viscosity estimation, however, was approximate since it did not satisfy the energy balance exactly. Here their model is extended and refined by ensuring that the drop of local kinetic energy of high shear strain rate of form-

ing particles must be exactly equal to the energy dissipated by interface fracture of these particles. The basic hypothesis is that the interface fracture occurs instantly, as soon as the energy balance is satisfied. Like in the preceding work, this additional apparent viscosity is a power function of the rate of the deviatoric strain invariant. But here the power exponent is different, equal to $-7/3$, and the apparent viscosity is found to be proportional also to the time derivative of the rate of that invariant, i.e., to the second derivative of the shear strain. It is assumed that the interface fracture that comminutes the material into small particles occurs instantly, as soon as the local kinetic energy of shear strain rate in the forming particles becomes equal to the energy required to form interface fractures. The post-comminution behavior, including subsequent further comminution and clustering into bigger particle groups to release the kinetic energy that is being dissipated by inter-group friction, is discussed and modeled. The present formulation makes it possible to eliminate the artificial damping of all types, which is normally embedded in commercial finite element codes but is not predictive since it is not justified physically. With the aforementioned improvements, and after implementation into the new microplane model M7 for fracturing damage in concrete (which includes the quasi-static rate effects), the finite element predictions give superior agreement with the measured exit velocities of steel projectiles penetrating concrete walls of different thicknesses and with the measured depths of penetration into concrete blocks by projectiles of different veloci-

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ties. Finally it is pointed out that the theory presented may also be used to predict proximate fragmentation and permeability enhancement of gas shale by powerful electric pulsed-arc explosions in the borehole.

Keywords Comminution · Fragmentation · High strain rate · Impact · Dynamic fracture · Fracture mechanics · Concrete · Gas shale

1 Introduction

Dynamic comminution (i.e., fragmentation, crushing and pulverization), due to high shear strain rate occurs in many situations and processes, ranging from projectile or meteorite impact on rocks, concrete, composites, metals and ceramics (Deshpande and Evans 2008; Doyoyo 2002; Ferri et al. 2010; Mescall and Weiss 1984; Wei et al. 2009), to fracturing of gas or oil shale by chemical explosions or electro-hydraulic pulsed arc in the horizontal borehole (Hemmert et al. 2010; Maurel et al. 2010). There exists a vast literature on the subject (Adley et al. 2012; Cadoni et al. 2001; Camacho and Ortiz 1996; Deshpande and Evans 2008; Doyoyo 2002; Ferri et al. 2010; Forquin et al. 2008; Frank et al. 2012; Gailly and Espinosa 2002; Gatuingt and Pijaudier-Cabot 2002; Grady 1982; Grady 1998; Grady and Kipp 1979; Grady and Kipp 1995; Grady 1990; Kožar and Ožbolt 2010; Mescall and Weiss 1984; Mott 1947; Ožbolt et al. 2011; Shih et al. 1998; Wei et al. 2009) and many aspects are well understood, including the micro-mechanics of dynamic propagation and branching of individual cracks (Freund 1990).

The material comminution represents an additional process of energy dissipation. The multiscale transition needed to translate the aforementioned models for individual cracks into global energy dissipation of vast number of cracks and develop a general constitutive law usable in large finite element programs for impact seems hardly feasible. The most important constitutive phenomenon, which has eluded understanding for decades, is the dynamic overstress. It consists of an apparent increase of strength that begins at the shear strain rate of about 1/s and becomes enormous at shear strain rates $> 10^4$ /s. Some authors were able to simulate the dynamic overstress under impact by either introducing into the finite element model various kinds of suitably strong artificial damping, or by arbitrarily increasing the strength or yield limits of the material so

as to fit some impact test data (Adley et al. 2012). However, such ad hoc modifications deprive the constitutive law of its experimental foundation in the fitting of laboratory tests of uni-, bi- and tri-axial behavior under diverse loading paths. Consequently, such modifications cease to be predictive in new situations, although they might generate realistically looking videos. A sound way to model the dynamic overstress must be a physically justified refinement, rather than ad hoc modification, of the damage constitutive model.

Such refinement has recently been attempted with considerable success in the work of Bažant and Caner (2013), Bažant and Caner (2014) and Caner and Bažant (2014). The underlying constitutive model was the microplane model M7 for concrete, which includes only the quasi-static rate effects. M7 was combined with a new theory in which the particle-producing interface fracture of concrete was considered to be driven by the release of the kinetic energy of the local shear strain field, rather than by the release of strain energy as in the classical fracture mechanics. This was justified by the fact that the former energy release can be several magnitudes larger than the latter. The energy dissipated by comminution into particles was modeled by additional viscosity based on the energy dissipated by the comminution (while all the types of artificial damping usually embedded in commercial dynamic finite element programs were removed). The model agreed reasonably well with published test data (Cargile 1999) on the exit velocities of projectiles penetrating concrete walls of different thicknesses. Nevertheless the model contained one crude oversimplification, which is removed in the present study.

The oversimplification in the previous work (Bažant and Caner 2013; Bažant and Caner 2014; Caner and Bažant 2014) was due to calculating the additional shear viscosity so as to yield a dissipation rate proportional to the deviatoric stress invariant of the shear stress required to drive interface fracture. What prevented this simplification from being unrealistic was that this invariant had the correct physical dimension, the energy per unit volume. But the value of the proportionality constant had to be left to data fitting, and the relationship of that invariant to energy dissipation and viscosity was only approximate, even though satisfactory fits of the exit velocities of penetrating projectiles were obtained. Here it is shown how to calculate the additional viscosity in a fully consistent manner, to obtain more accurate results.

The previous work also introduced the hypothesis that the interface fracture that produces the comminuted particles occurs instantaneously. This point is debatable. On one hand, the fracture doubtless begins at the same instant at many forming particles, rather than propagating from one particle to the next. This supports the hypothesis of instantaneity. On the other hand, the fracture certainly propagates along each particle interface, and the propagation is surely rate-dependent (this alternative is explored in a parallel paper [Bažant and Su 2015](#)). The present work develops the former alternative, i.e., the instantaneous fracture formation at all particle interfaces while the balance of fracturing and viscous energy dissipation is strictly maintained.

2 Drop of kinetic energy density caused by comminution: a review

Let ϵ_{Dij} denote the deviatoric strain tensor and the superior dot, $\dot{\cdot}$, the derivatives with respect to time t . Consider an idealized dynamic fracture process in which the solid is comminuted to identical prismatic particles of length h and hexagonal cross section of side $h/2$, as shown in Fig. 1. The kinetic energy of effective deviatoric strain rate $\dot{\epsilon}_D = \sqrt{\dot{\epsilon}_{Dij}\dot{\epsilon}_{Dij}/2}$ becomes, at a certain moment, high enough to create sudden fracture at particle interfaces.

Analyzing the kinematics and comparing the kinetic energy of particles before (Fig. 1b) and after (Fig. 1c) the comminution, [Bažant and Caner \(2014\)](#) showed that, for a motion in the plane of maximum shear strain, the local kinetic energy of the particles that are about to form, per unit volume of material, $\Delta\mathcal{K}$, is additive to, and separable from, the kinetic energy of the global motion. The global kinetic energy corresponds to the movement of the centers of the particles whose formation is imminent. Thus the drop of kinetic energy per unit volume is found to be

$$-\Delta\mathcal{K} = c_k \rho h^2 \dot{\epsilon}_D^2 \quad (1)$$

where $c_k = I_p/(2hV_p)$, ρ = mass density, $V_p = 3\sqrt{3}h^3/8$ and $I_p = 5\sqrt{3}h^4/128$ = volume and polar moment of inertia of each hexagonal prism about its axis, respectively.

In reality, the size of comminuted particles is never uniform but varies randomly within a certain range, $s \in (h, H)$ where h, H = minimum and maximum sizes, and s = variable particle size. Schuhmann's empirical power law ([Charles 1957](#); [Ouchterlony 2005](#); [Schuhmann \(1940\)](#)) is adopted to describe the cumulative distribution (Fig. 2). It gives the volume fraction of particles of sizes between h to s :

$$F(s) = \frac{s^k - h^k}{H^k - h^k}, \quad s \in (h, H), F(s) \in (0, 1) \quad (2)$$

where $(k \approx 0.5)$ = empirical constant ([Bažant and Caner 2014](#)).

We now introduce the simplifying hypothesis that the kinetic energy drop for variable particle size can be obtained by summing the energy losses for all infinitesimal intervals $(s, s + ds)$ calculated for each interval as if the particle size and shape were uniform. Integration furnishes the kinetic energy drop for particles of all sizes per unit volume ([Bažant and Caner 2014](#)):

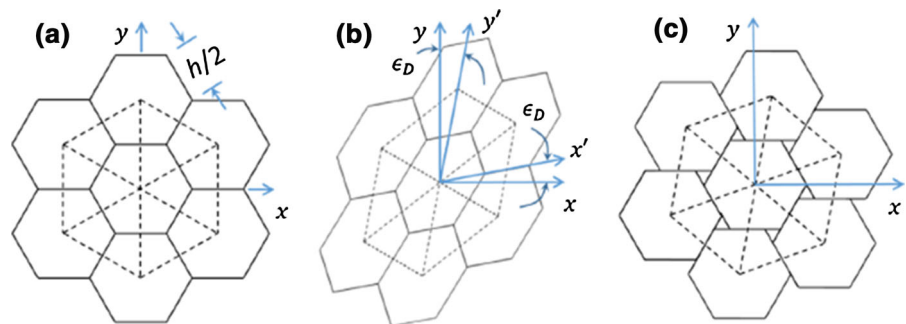
$$-\Delta\mathcal{K} = \int_{s=h}^H c_k \rho s^2 \dot{\epsilon}_D^2 dF(s) = C_k \rho h^2 \dot{\epsilon}_D^2 \quad (3)$$

$$\text{where } C_k = \frac{k}{k+2} \frac{(H/h)^{k+2} - 1}{(H/h)^k - 1} c_k \quad (4)$$

C_k = dimensionless constant. For $H/h = 10$ and $k = 0.5$, $C_k = 1.519$.

The interface area of uniform particle of size s per unit volume of material is c_s/s where c_s = dimensionless constant. For particle size distributed according to Eq. (2), the particle interface area per unit volume of material is obtained as ([Bažant and Caner 2014](#)):

Fig. 1 Schematic illustration of material comminution into prismatic hexagonal particles: **a** undeformed regime; **b** sheared regime; **c** comminuted regime



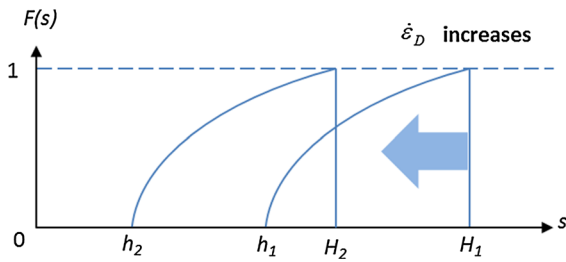


Fig. 2 Illustration of the distribution of particle size subject to increase of the effective deviatoric strain rate

$$S = \int_{s=h}^H \frac{c_s}{s} dF(s) = \frac{C_s}{h} \quad (5)$$

where $C_s = \frac{c_s k}{k-1} \frac{(H/h)^{k-1}-1}{(H/h)^k-1}$ is a dimensionless constant (Bažant and Caner 2014). For $H/h = 10$ and $k = 0.5$, $C_s = 1.0465$.

Equation (5) adds, or integrates, the energy dissipations for different particle sizes are calculated under the assumption that for each size there is a regular array of equal-size particles. This is certainly a simplification, and it must be admitted that its error could be significant for a very broad range. In reality a large forming particle will be in contact with a small one. However, kinetically driven interface fracture creating adjacent particles of different size is hardly amenable to an analytical solution needed for the constitutive law. The error due to this simplification could be explored numerically, but is beyond the scope of this study.

3 The criterion of dynamic fracture

Two forms of dynamic fracture criterion were considered in (Bažant and Caner 2014; Bažant and Caner (2013)):

$$\text{Either } d(\Delta\mathcal{K} + S\Gamma)/dS = 0 \quad (6)$$

$$\text{or } -\Delta\mathcal{K} = S\Gamma \quad (7)$$

where Γ = interface fracture energy of the comminuting particles. The former criterion, Eq. (6), is analogous to classical fracture mechanics in which the strain energy is being dissipated gradually at one place, the tip of a propagating crack. This criterion (Eq. 6) implies that the fracture will propagate once it is triggered. The latter one (Eq. 7) imposes an overall energy balance condition which would not guarantee incremental energy balance at the beginning of propagation. Here, however, it is hard to imagine propagation. In dynamic comminution, the fracture forms and

the kinetic energy is getting dissipated simultaneously and suddenly at many places, at all the interfaces corresponding to the same point of the smearing macro-continuum. So, the latter criterion, Eq. (7), appears more realistic. Nevertheless, either criterion leads to equations of the same form and the difference would matter only in a fully predictive model that is not calibrated by macroscale tests.

Substitution of Eqs. (3) and (5) into Eq. (7) gives the minimum particle size as

$$h = \left(\frac{C_a \Gamma}{\rho \dot{\epsilon}_D^2} \right)^{1/3} \quad (8)$$

where

$$C_a = \frac{C_s}{C_k} = \frac{c_s}{c_k} \frac{k+2}{k-1} \frac{(H/h)^{k-1}-1}{(H/h)^{k+2}-1} \quad (9)$$

which is a dimensionless constant; for $H/h = 10$ and $k = 0.5$, $C_a = 0.6891$. Note that here C_a is slightly different from its value in the first version of the theory (Bažant and Caner 2014), in which the former dynamic fracture criterion was adopted.

As seen in Eq. (8), the particle size is proportional to the $-2/3$ power of the effective deviatoric strain rate. A higher effective deviatoric strain rate gives a greater kinetic energy release, which leads to smaller particles, and thus to a greater interface area and more dissipation. Furthermore, substitution of Eq. (8) into Eq. (3) gives the kinetic energy drop in terms of only the effective deviatoric strain rate and the material parameters:

$$-\Delta\mathcal{K} = (C_0 \Gamma^2 \rho)^{1/3} \dot{\epsilon}_D^{2/3} \quad (10)$$

where $C_0 = C_k^3 C_a^2$ is a dimensionless constant. For $H/h = 10$ and $k = 0.5$, $C_0 = 1.6632$.

In Bažant and Caner (2013) and Bažant and Caner (2014), it was proposed that the type of behavior is decided by the dimensionless number

$$B_a = \frac{-\Delta\mathcal{K}}{\tau_0^2 / (2G_s)} \quad (11)$$

which represents the ratio of the kinetic energy density of shear to the maximum possible strain energy density, the latter being $\tau_0^2 / (2G_s)$ where τ_0 = shear strength and G_s = elastic shear modulus of the solid. Kinetic comminution must happen when $B_a \gg 1$ and cannot happen when $B_a \ll 1$. For concrete, the transition occurs at $\dot{\epsilon}_D \approx 1/s$. In Bažant and Caner (2013) and Bažant and Caner (2014), Eq. (11) could be transformed to the form: $B_a = G / (C_g \tau^2) (\Gamma^2 \rho \dot{\epsilon}_D^2)^{1/3}$. Here this expression cannot be derived because the oversimplifying

hypothesis from the previous works is not adopted here. However, for physical reasons, this expression is probably applicable at least as an approximation.

4 Viscosity dissipating comminution energy in computations

The material comminution represents an additional mechanism of the dissipation of energy. The precise way in which the comminution energy is dissipated in numerical simulations appears to make little difference for the macroscopic structural effects of impact. In simulations, the energy either can be dissipated as the work of a body force or can be incorporated into the constitutive law as an additional stress. The two ways are equivalent, in principle. An additional viscous stress is preferred here since it is easier to program and allows incorporating different types of comminution.

Since the comminution is induced by shear, the additional stress resisting the motion should be applied only to the shear stress components. The work of additional shear stresses must dissipate an energy equal to the drop of kinetic energy of strain rate field caused by comminution; i.e.

$$d(-\Delta\mathcal{K}) = s_{ij}^A d\epsilon_{Dij} \quad (12)$$

where $i, j = 1, 2, 3$, $d(\dots)$ denotes a small increment during time increment $d\Delta t$. This increment is easy to understand and program if the additional stress is defined as a viscous stress;

$$s_{ij}^A = \eta_D \dot{\epsilon}_{Dij} \quad (13)$$

Then, Eqs. (10–13) give the viscosity as

$$\eta_D = \frac{1}{2\dot{\epsilon}_D^2} \frac{d(-\Delta\mathcal{K})}{dt} = \frac{(C_0\Gamma^2\rho)^{1/3}\ddot{\epsilon}_D}{3\dot{\epsilon}_D^{7/3}} \quad (14)$$

It is found that, in contrast to the previous simplified formulation (Bažant and Caner 2014), the viscosity is not only related to the effective deviatoric strain rate, but is also proportional to its derivative, i.e. the 2nd time derivative of strain. The comminution occurs only if the effective deviatoric strain rate changes, and no additional comminution occurs during the time intervals in which the rate is constant, which is a significant difference from the previous formulation (Bažant and Caner 2013; Bažant and Caner 2014).

To verify the theory, consider the following simple shear variation:

$$\epsilon_{12} = \epsilon_{21} = \frac{t^2}{(\Delta t)^2} \epsilon_{max} \quad (15)$$

while all other strain components remain zero. Here ϵ_{max} is the shear strain at $t = \Delta t$. According to Eq. (10), the kinetic energy drop during $t \in [0, \Delta t]$ is obtained as:

$$-\Delta\mathcal{K} = (C_0\Gamma^2\rho)^{1/3} \left(\frac{2\epsilon_{max}}{\Delta t} \right)^{2/3} \quad (16)$$

On the other hand, the additional stress for the simple shear case is

$$s_{12}^A = s_{21}^A = \frac{(C_0\Gamma^2\rho)^{1/3}}{3} \left(\frac{2t^4\epsilon_{max}}{(\Delta t)^2} \right)^{-1/3} \quad (17)$$

The work of the additional stress on the deviatoric strain can be calculated by integration over $t \in [0, \Delta t]$;

$$D = 2 \int_{\epsilon_{12}=0}^{\epsilon_{max}} s_{ij}^A d\epsilon_{12} = (C_0\Gamma^2\rho)^{1/3} \left(\frac{2\epsilon_{max}}{\Delta t} \right)^{2/3} \quad (18)$$

As is seen, the work of the additional stress calculated according to the present formulation exactly equals the kinetic energy drop induced by comminution, i.e. $D = -\Delta\mathcal{K}$. So the new version of the kinetic comminution theory works correctly.

5 Post-comminution behavior

The comminution process (Fig. 1) is a transient process. It occurs virtually instantaneously. However, what happens if the effective deviatoric strain rate increases or decreases in the post-comminution regime, i.e., after the comminution occurred? Since, according to experiments, the particle size s is not uniform but distributed between certain limits, from h to H , it is natural to expect that the previously formed particles in practice instead of uniform size, the bigger size particles can be comminuted further into smaller particles when the strain rate increases, while the relative size distribution still holds true, as shown in Fig. 2. Thus, during a strain rate increase, further interfaces are created to dissipate more kinetic energy. The additional stress and viscosity that dissipate this energy can be obtained from Eqs. (12–14).

The effective deviatoric strain rate as a function of time can in practice evolve in a complicated manner, as shown by the response in Fig. 3 separated into stages I, II, III and IV. The comminution occurs at the beginning of stage I and Eqs. (12–14) apply in that stage. In stage II, the effective deviatoric strain rate decreases, and so the additional stress and viscosity are zero, because the strain rate is not big enough to create smaller particles.

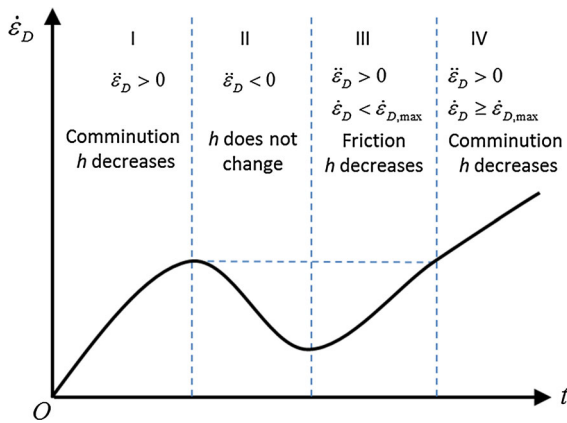


Fig. 3 Various situations of the strain rate in practice

In stage III, though, it is reasonable to assume that the small comminuted particles cluster together into bigger particle groups and release kinetic energy due to the increases of effective deviatoric strain rate. The kinetic energy released, which can be calculated by Eqs. (12–14) as well, may be dissipated by the friction between particles. Further comminution must occur in stage IV as the effective deviatoric strain rate exceeds its previous maximum, and the particles are comminuted into still smaller particles. Therefore, the viscosity and additional stress in Eqs. (12–14) should be applied only when $\dot{\epsilon}_D > 0$ and $\ddot{\epsilon}_D > 0$. This behavior can be implemented into any existing constitutive law for the effect of comminution.

6 Application of microplane model M7 and quasi-static rate effects

Microplane model M7 (Caner and Bažant 2012) was chosen since it is at present the only continuum damage model that can fit all of the 22 existing types of laboratory tests uni-, bi-, and tri-axial behavior under various loading and unloading histories with post-peak softening. Since the fracture process zone size in concrete is typically about 0.3 m, damage localization and nonlocal concepts are not very important in representing the laboratory tests. M7 was calibrated to fit the test data for concrete WES-5000 used in the tests of projectile impact. This concrete had compression strength 48 MPa, Young's modulus $E = 25$ GPa, and Poisson's ratio $\nu = 0.18$. The optimum values of M7 parameters, achieving very good fits of quasi-static uni-, bi-

and tri-axial tests, were $k_1 = 11 \times 10^{-5}$, $k_2 = 110$, $k_3 = 30$, $k_4 = 100$ and $k_5 = 10^{-4}$ (Caner and Bažant 2012; Caner and Bažant (2012)).

The M7 includes the quasi-static rate effects, which consist of a rate-dependent crack opening and growth controlled by the activation energy of bond breakage, and of viscoelasticity of the material between the cracks. The quasi-static rate effects suffice to capture the rate-dependent behavior up to the strain rate of about 1/s, but not beyond. In theory, they should extend into very high strain rates, even in excess of 10^4 /s, though alone they are insufficient to fit test data.

In the zones farther away from the penetrating projectile, the strain rate is not extremely high. In these zones, only the quasi-static rate effects play a role, while the kinetic comminution effect discussed here does not. To capture the quasi-static rate effect due to cohesive crack opening controlled by the activation energy theory of bond ruptures, the M7 stress-strain boundaries on the microplane level (i.e., the strain-dependent strength limits for microplane stress and strain vectors) are scaled vertically in proportion to the function $F(\epsilon) [1 + C_{r2} \ln(2\dot{\epsilon}/C_{r1})]$. Here, $F(\epsilon)$ is the boundary at negligible strain rate, $\dot{\epsilon}$ is an invariant of the strain tensor, $C_{r1} = 4 \times 10^{-6}$ and $C_{r2} = 0.022$. The effect of viscoelasticity between the cracks is taken into account by replacing the elastic constants with a creep operator, for which, in the case of impact modeling, a single spring-dashpot Maxwell rheologic model suffices (Bažant et al. 2000; Caner and Bažant 2000; Bažant and Caner 2014; Caner and Bažant (2014)). These quasi-static rate effects can capture the apparent strength increase in unconfined compression tests at rates only up to about 1/s (which is nevertheless sufficient for the effects of earthquake or remote explosions).

7 Projectiles penetrating and perforating concrete walls

Although the new microplane model M7 (Caner and Bažant 2012; Caner and Bažant (2012)) for concrete can provide excellent fits of the experiment data from diverse types of uniaxial, biaxial and triaxial tests, its application to projectile impact severely overestimated the exit velocities of projectiles perforating concrete walls and the depths of penetration of concrete blocks. The previous enhancement of M7 with

simplified calculation at additional viscosity led to major improvement (Bažant and Caner 2013; Caner and Bažant (2014)), but the agreement with data was imperfect. Here, we demonstrate that the present theory of kinetic comminution leads to virtually perfect agreement.

The tests of projectile perforation, performed at the Geotechnical and Structure Laboratory of the US Army Engineer Research and Development center (ERDC), Vicksburg (Cargile 1999; Frank et al. (2012)), used circular slabs of four thicknesses, 127, 216, 254 and 280 mm, made of the concrete WES-5000, whose standard compression strength was 48 MPa. The slabs were cast in steel culvert pipes of diameter 1.52 m, sufficient to approximate the response for a semi-infinite radius (in spite of that, non-reflecting finite elements producing no backward wave were used at the boundary). The projectiles, which were hollow and made of steel, had an ogive-nose (caliber radius head 3.0, length/diameter ratio 7.0, and diameter 50.8 mm) and weight of 2.3 kg. They were considered as rigid since no obvious damage, melt or deformation was observed after the test. The projectiles impacted the concrete slabs with the velocity of 310 m/s at the angle of 90°. The heat generation was neglected based on simple estimates. The perforation tests were carried out two or three times for each thickness of the slab.

Figure 4 shows that the exit velocities of the projectiles were grossly underestimated, provided that all the types of artificial damping unsupported by mechanics and experiments, which are typically embedded in the commercial finite element codes, were eliminated. The additional viscosity formulated here provides a major improvement, even better than the simplified additional viscosity in Bažant and Caner (2013) and Bažant and Caner (2014). The kinetic comminution viscosity was based on $H/h = 10$, $k = 0.5$. There is only one unknown parameter for the present kinetic comminution model, namely $\alpha = C_0 \Gamma^2 \rho$. It can be calibrated by fitting only one data point in Fig. 4, which led to the value $\alpha = 11034 \text{ N}^2 \text{ Kg/m}^5$ ($C_0 = 1.66$, $\Gamma = 1.7 \text{ N/m}$, $\rho = 2300 \text{ Kg/m}^3$). The remaining points are then predicted. As can be seen, the predictions of exit velocities are excellent, for all the data points. The distributed stress with comminution is usually higher than without, since the additional viscous stress is added to implement the effect of comminution.

For the penetration test (Cargile 1999), the CSPC concrete block was big enough to be treated as semi-

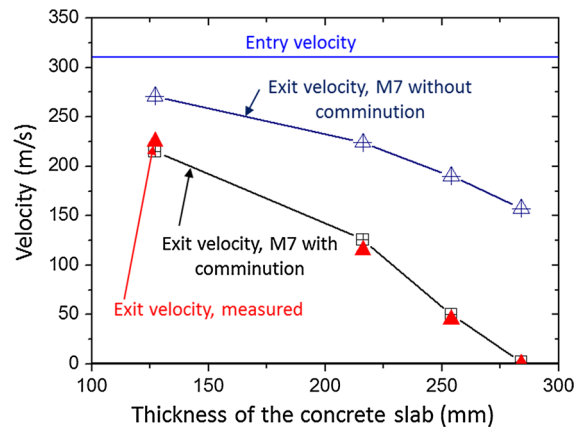


Fig. 4 Comparison of the exit velocity predicted by M7 with and without implementation of comminution effect and the experiment data for projectile perforation onto the concrete wall with thicknesses of 127, 216, 254 and 280 mm

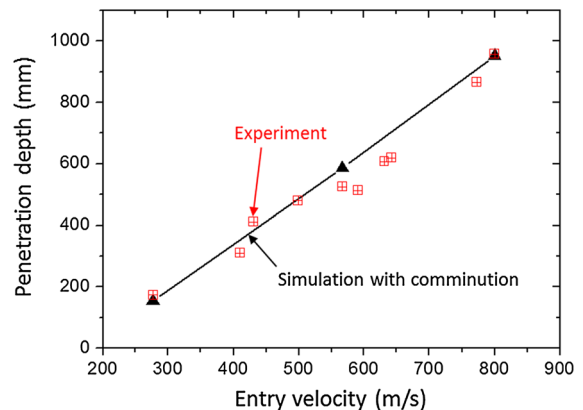


Fig. 5 Comparison of the penetration depth predicted by M7 with implementation of comminution effect and the experiment data. The entry velocity of the projectile ranges from 277 to 800 m/s

infinite. The projectiles, made of steel, were hollow, had an ogive nose, length 242.4 mm, diameter 25.9 mm and weight 0.9 kg, and are here considered as rigid since they were found undeformed after the test. The entry velocities ranged from 277 to 800 m/s. The penetration depths measured are shown in Fig. 5. The blocks consisted of the CSPC concrete, which was rather similar to the WES-5000 concrete. Therefore, the parameters for WES-5000 are applied here. Again, only one free parameter, $\alpha = C_0 \Gamma^2 \rho$, was available to fit the data, and the value $\alpha = 61090 \text{ N}^2 \text{ Kg/m}^5$, along with $C_0 = 1.66$, $\Gamma = 4.0 \text{ N/m}$, $\rho = 2300 \text{ Kg/m}^3$, provided the optimum fit. As shown in Fig. 5, three typical cases

with entry velocities of 277, 567 and 800 m/s were simulated. The thicknesses of the concrete blocks were 760, 1070 and 1830 mm, respectively, according to the experiment. The simulations agree with the test data very well.

For comparison, the predictions without the consideration of the comminution were also done. The penetration depths were 531 and 1606 mm for the first and the third cases, respectively. However, for the second case, the exit velocity calculated without comminution was 153.7 m/s, which is far from the measured value.

Besides the exit velocity and penetration depth, the shape of the entry and exit craters is another result that should be compared with experiments. It depends on various factors, such as the criterion for element deletion, which must be used to avoid excessive element distortion. Here, the elements with principal strain below -20% or above 100% are deleted. The craters obtained by the present simulations do not fit the experiments well; they are much too narrow. This problem will be studied further.

8 Conclusions and final comments

1. The recent idea (Bažant and Caner 2014) to model the dynamic overstress in projectile penetration by means of comminution driven by local kinetic of shear strain rate is here extended and refined by deriving from the energy balance a more accurate expression for the additional apparent viscosity that generates the overstress.
2. The basic hypothesis is that the interface fracture that comminutes the material into small particles occurs instantly, as soon as energy equivalent of the local kinetic energy of forming particles becomes equal to the fracture energy.
3. In contrast to the previous simplified formulation, the additional apparent viscosity, which supplements the quasi-static rate effects, involves a different exponent of the effective deviatoric strain rate and depends not only on the effective deviatoric strain rate but also on its time derivative, i.e., on the second derivative of the effective deviatoric strain.
4. Because of the second derivative, no further kinetic comminution is under way if the effective deviatoric strain rate remains constant.
5. The post-comminution behavior, including subsequent further comminution and clustering into bigger particle groups to release the kinetic energy that dissipated by friction between the groups, is discussed and incorporated into the present improved theory.
6. When the present model is implemented in a commercial finite element code, the artificial damping of all types, normally embedded in these codes, must be eliminated since it is non-physical, non-predictive.
7. In combination with microplane model M7 which also includes the quasi-static rate effects, excellent predictions are obtained for the exit velocities of projectiles penetrating walls of different thicknesses and for the depths or penetration of projectiles of different velocities into concrete blocks.
8. As a final comment, a rational theory of high rate shear comminution might have an important use in the technology of hydraulic fracturing of gas or oil shale. Oil companies are currently investigating powerful explosions in the horizontal wellbore, which would have to be done by electro-hydraulic pulsed arc since enough chemical explosive cannot be packed into the narrow wellbore. Although, due to rapid front attenuation in 3D shock front expansion, only a part of the frac stage could be comminuted, the fragmentation near the wellbore could stimulate the flow through the cracks, mitigate proppant screening and enhance permeability.

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Impact Comminution of Solids Due to Progressive Crack Growth Driven by Kinetic Energy of High-Rate Shear

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A new theory, inspired by analogy with turbulence, was recently proposed to model the apparent dynamic overstress due to the energy that is dissipated by material comminution during penetration of missiles into concrete walls. The high-rate interface fracture comminuting the material to small particles was considered to be driven by the release of kinetic energy of high-rate shear of the forming particles, and the corresponding energy dissipation rate was characterized in the damage constitutive law by additional viscosity. However, in spite of greatly improved predictions for missile impact and penetration, the calculation of viscosity involved two simplifications—one crude simplification in the calculation of viscosity from the shear strain rate, and another debatable simplification in treating the comminution as an instantaneous event, as in the classical rate-independent fracture mechanics. Presented is a refined theory in which both simplifications are avoided without making the theory significantly more complicated. The interface fracture is considered to be progressive and advance according to Evans' power law extended to the fast growth of interface crack area. The growth rate of interface cracks naturally leads to an additional viscosity, which allows close matching of the published test data. In combination with the microplane damage constitutive model M7 for concrete, the refined theory gives a close match of the exit velocities of missiles penetrating concrete walls of different thicknesses and of the penetration depths of missiles of different velocities into a massive block. [DOI: 10.1115/1.4029636]

Keywords: missile impact, penetration, fragmentation of concrete, dynamic fracture, crack growth rate, viscous damping, particle size distribution, turbulence, finite element analysis, dynamic overstress

1 Introduction

The understanding and prediction of dynamic fracture and comminution of materials is important for missile of meteorite impact, ground shock, explosions, mining, fracturing of gas or oil shale, and various industrial processes. Vast literature exists [e.g., 1–24] but it need not be reviewed here in detail since a review was provided in a recent paper [25].

One major difficulty at high-rate fracture has, for many years, been the phenomenon of apparent “dynamic overstress.” This means that the material strength or energy dissipation at strain rates $\gg 1/s$ apparently increases by order of magnitude above the value predicted by statically measured strength and the quasi-static rate effects due to bond ruptures at crack tips and viscoelasticity of material between the cracks. Many attempts for numerical simulation of dynamic overstress under missile impact on concrete or rock have been made. But despite the use of the most realistic damage constitutive model, such as the microplane model, the deceleration of penetrating missile was severely underestimated. The data could be fitted only with a large ad hoc, physically unexplained, increase of either the damping or the strength limits of the material [2], by which, of course, predictability in new situations and the experimental basis in uni-, bi-, and tri-axial material tests with postpeak damage have been forfeited.

A caveat must here be mentioned. Commercial codes, such as ABAQUS, automatically insert artificial damping both in the solver and the finite elements. This generates a bulk viscosity pressure that is linear or quadratic in the volumetric strain rate. The user is often unaware of it and some such measures are not even clear from the manual. It is important to check for such artificial damping. Properly, one should eliminate all such artificial damping from the commercial program and use only physically justified damping or viscosity. Introduction of sufficiently strong artificial damping can largely or totally obviate the problem of dynamic overstress. This is probably the reason why some authors have recently claimed that the missile impact data can be reproduced numerically with no additional viscosity, but apparently they have not checked whether different choices of damping might be needed to fit very different situations. Similar comments apply to the ad hoc artificial choice of elevated strength limits in the microplane model [2].

The reason for insufficiency of the quasi-static damage constitutive models obviously is that they account only for the degree of fragmentation seen in static laboratory tests by which they are calibrated. In these static tests, the size of the smallest fragments is roughly 1/3 of the maximum aggregate size in concrete. But under impact, much smaller particles are generated. This is what must lead to a great increase of energy dissipation or apparent viscosity.

Although the micromechanics of dynamic comminution, which includes the propagation and branching of cracks, their interference with waves, and the interaction of a few cracks, has been well understood for some time [1], a work attempting to develop a

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constitutive model that could be used in large finite element codes for predicting the impact and penetration of missiles into concrete or rock appeared only very recently [25–27].

In that recent work, a new idea was introduced into the dynamic analysis of high velocity impacts or powerful explosions—the dynamic comminution into fine particles under compression is caused not by the release of the strain energy, as in classical fracture mechanics, but by the release of the kinetic energy of high shear strain rate of particles that are about to form. Depending on the particle size, this kinetic energy can exceed the maximum possible value of the stored strain energy by orders of magnitude, and its analysis leads to a model partly analogous to turbulence.

In a preceding work [25,27], the latest version, M7, of the microplane model for concrete was enhanced by material viscosity calculated under two simplifying hypotheses:

- (1) One, rather crude, simplification was that the additional shear viscosity was calculated so as to yield a dissipation rate proportional to the stress invariant of shear stress required to drive interface fracture. This simplification was admissible because that invariant had the dimension of energy per unit volume. But the value of the proportionality constant had to be left to data fitting, and the relationship of that invariant to energy dissipation and viscosity was only approximate, although very good fits of penetration data were obtained.
- (2) The second hypothesis was that the interface fracture producing the comminuted particles occurs instantaneously.

In reality, the interface fracture must, of course, occur progressively. This is modeled here by a high-rate extension of Evans's power law for subcritical crack growth (although it remains debatable whether this approach properly reflects the simultaneity of crack growth at many interfaces). The apparent viscosity is then calculated from energy balance.

2 Local Kinetic Energy Density Due to Material Comminution

To simplify analysis, the comminution of a solid is assumed to produce identical hexagonal prisms with side $h/2$ and length h to be determined later (Fig. 1). The solid undergoes shear strain ϵ_{Dij} at the rate $\dot{\epsilon}_{Dij}$, as depicted by transition from Figs. 1(a) and 1(b) (the superior dot denotes derivative with respect to time t). At a certain moment, the effective deviatoric strain rate $\dot{\epsilon}_D = \sqrt{\dot{\epsilon}_{Dij}\dot{\epsilon}_{Dij}/2}$ becomes high enough for the local kinetic energy to suffice for creating interface shear fractures and slips that separate the particles. In this process, the kinetic energy of shear strain field gets converted into the interface fracture energy. This energy can be calculated as the difference between the kinetic energies per unit volume of the original solid before (Fig. 1(b)) and after (Fig. 1(c)) the comminution. This yields [25] the kinetic energy drop per unit volume of solid

$$-\Delta\mathcal{K} = c_k \rho h^2 \dot{\epsilon}_D^2 \quad (1)$$

where ρ is the mass density and $c_k = I_p/(2hV_p)$. Also, $I_p = 5\sqrt{3}h^4/128$ and $V_p = 3\sqrt{3}h^3/8$ are the polar moment of inertia and volume of each hexagonal prism, respectively. Usually $H/h = 10$ –100 and $H = 1/10$ of the size of fragments that dominate in static crushing of the solid. An important point in derivation of Eq. (1) is that the kinetic energy of the strain rate field is separable from, and additive to, the kinetic energy of the global motion in the shear plane, defined by the velocities of particle centers [25].

In practice, the particle size is always randomly distributed over a certain range, $s \in [h, H]$, where h and H now denote the minimum and maximum particle sizes. An empirical distribution, the Schuhmann power law [28], well justified in mining literature [28–30], is considered. Accordingly, the volume fraction of particles of size $h < s$ is

$$F(s) = \frac{s^k - h^k}{H^k - h^k} \quad [s \in (h, H), F(s) \in (0, 1)] \quad (2)$$

where k is an empirical constant; $k \approx 0.5$. Assuming that energy contributions of particles of various sizes s can be superposed, the kinetic energy drop for particles of all sizes per unit volume of the original solid is thus obtained as

$$-\Delta\mathcal{K} = \int_{s=h}^H c_k \rho s^2 \dot{\epsilon}_D^2 dF(s) = C_k \rho h^2 \dot{\epsilon}_D^2 \quad (3)$$

where $C_k = c_k k(r^{k+2} - 1)/[(k+2)(r^k - 1)]$ = dimensionless constant and $r = H/h$ [25]. For $H/h = 10$ and $k = 0.5$, $C_k = 1.5186$.

The interface area per unit volume of particles of size s can be calculated as c_s/s , where $c_s = 1 + 4/\sqrt{3}$ = dimensionless constant. For the particles distributed according to the Schuhmann law, the particle interface area per unit volume is

$$S = \int_{s=h}^H \frac{c_s}{s} dF(s) = \frac{C_s}{h} \quad (4)$$

Here, $C_s = c_s k(r^{k-1} - 1)/[(k-1)(r^k - 1)]$ = dimensionless constant [25]. Here, it is assumed that $H/h = 10$ and then $k = 0.5$, $C_s = 1.0465$.

The type of behavior is decided by the dimensionless number

$$B_a = \frac{-\Delta\mathcal{K}}{\tau_0^2/(2G_s)} \quad (5)$$

which represents the ratio of the kinetic energy density of shear to the maximum possible strain energy density, the latter being $\tau_0^2/(2G_s)$ where τ_0 = shear strength and G_s = elastic shear modulus of the solid. Kinetic comminution must happen when $B_a \gg 1$

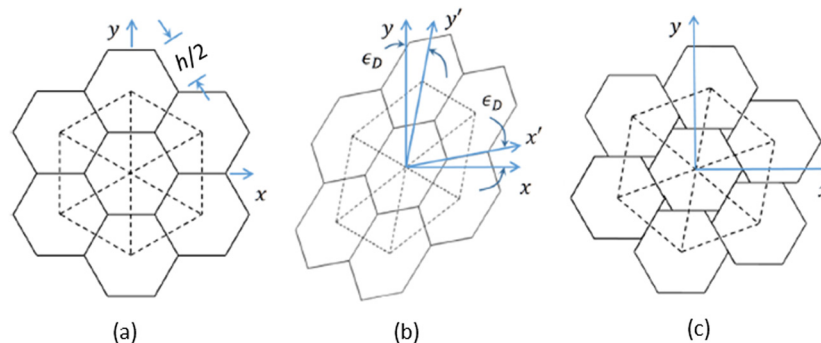


Fig. 1 Schematic illustration of material comminution into prismatic hexagonal particles: (a) undeformed state, (b) shearing regime, and (c) comminuted regime

and cannot happen when $B_a \ll 1$. For concrete, the transition occurs at $\dot{\epsilon}_D \approx 1/s$. The expression for B_a in terms of h , Γ , and ρ given in Refs. [25,26] is not exactly true here although the general definition in Eq. (5) is the same.

3 Continuous Evolution of Comminution According to Evans' Law for Static Crack Growth

Two theoretically possible forms of dynamic fracture criterion were considered in Ref. [25]. One, which was similar to the classical fracture mechanics, assumed that fracture occurs as soon as it is triggered by incremental energy balance, i.e., $-\Delta K/dS = \Gamma = \text{interface fracture energy of the material}$. The other form assumed a total energy balance, $-\Delta K = \Gamma S$, which seems more reasonable since, rather than propagating, the fracture occurs simultaneously at numerous interfaces (in a parallel study [31], the latter form is used with success to develop a more rigorous theory for viscous energy dissipation accounting more accurately for the work of comminution).

However, in both forms of fracture criterion a complete interface fracture of all particles was assumed to occur instantaneously. In reality, the interface fracture must evolve at a certain finite rate, even though that rate may be extremely high. It might, of course, be so high, compared to the strain rate, that the assumption of instantaneous fracture might be closer to the truth. A debatable point though this is, here we pursue the hypothesis that a gradual rather than instantaneous comminution fracture is more realistic.

Aside from the extended Evans' law of crack growth [32,33], which is adopted here for fast growth of interface fracture surface, some other kinetic equations of a similar form have been also proposed for dynamic crack growth. Particularly, Paliwal and Ramesh [34], based on the work by Freund [1] and Deng [35], introduced for fast crack growth rates the following general equation: $\dot{l} = c_{\max}[(K_I - K_{IC})/(K_I - K_{IC}/2)]^2$. They applied it to the modeling of the damage evolution of brittle materials containing dilute wing cracks [34].

Even though Evans' has never been used for fast dynamic growth, it is nevertheless adopted here because it has a solid physical foundation. It was derived from Kramer's rule of transition rate theory [36] Eqs. (6)–(12) and [33] Eqs. (29)–(34) coupled with the activation energy concept. This derivation showed that Evans' law must not be limited to very slow rates but must also apply to very fast rates. According to this law,

$$\dot{S} = A_{TK} \left(\frac{K}{K_c} \right)^n = A_T \left(\frac{\mathcal{G}}{\Gamma} \right)^{n/2} \quad (6)$$

$$\text{where } A_T = A e^{-Q_0/kT} \quad (7)$$

Here, K and K_c are stress intensity factor and its critical value (fracture toughness), $\mathcal{G}$ = energy release rate (with respect to S rather than time), n = empirical constant (typically about 10), A = material constant, Q_0 = activation energy of fracture growth, k = Boltzmann constant, and T = absolute temperature. As argued before, the total form of energy balance at interface crack formation appears to be more realistic than the incremental form, i.e., $S\dot{\mathcal{G}} = -\Delta K$. Thus, the rate of the interface fracture area becomes

$$\dot{S} = A_T \left(\frac{-\Delta K}{\Gamma S} \right)^{n/2} \quad (8)$$

Substitution of Eqs. (3) and (4) into Eq. (8) gives the governing equation for the rate of interface area

$$\dot{S} = \frac{C_T}{S^{3n/2}} \left(\frac{\rho \dot{\epsilon}_D^2}{\Gamma} \right)^{n/2} \quad (9)$$

Here, $C_T = A_T (C_k C_s^2)^{n/2}$ is a constant. For numerical analysis, this differential equation needs to be converted to an incremental

difference equation. To this end, the equation is integrated by separation of variables under the assumption that the strain rate, $\dot{\epsilon}_D$, is constant during time step $\Delta t = t_{\text{new}} - t_{\text{old}}$. The integration yields

$$S_{\text{new}} = \left[C_T \left(\frac{\rho \dot{\epsilon}_D^2}{\Gamma} \right)^{n/2} m \Delta t + S_{\text{old}}^m \right]^{1/m}, \quad m = 1 + \frac{3n}{2} \quad (10)$$

Here, S_{old} and S_{new} are the interface areas at the beginning and end of the time increment, respectively.

As proposed in Ref. [25], in computations the kinetic energy drop can be dissipated by the work of an additional stress, which is more convenient to implement than dissipation by an additional body force. Energy conservation requires that the work of additional stress be equal to the kinetic energy drop that is converted to the energy of new interface; i.e.,

$$\Gamma \Delta S = s_{ij}^A \Delta \epsilon_{Dij} \quad (11)$$

Here, $i, j = 1, 2, 3$; s_{ij}^A is the deviatoric stress tensor doing work on $\dot{\epsilon}_{Dij}$ during material comminution. The additional stress could also be represented as a viscous stress

$$s_{ij}^A = \eta_D \dot{\epsilon}_{Dij} \quad (12)$$

where η_D = viscosity. Equations (9), (11), and (12), together with the replacement of S by S_{new} , gives

$$\eta_D = C_T \frac{\rho^{n/2}}{2\Gamma^{n/2-1}} \frac{\dot{\epsilon}_D^{n-2}}{S_{\text{new}}^{3n/2}} \quad (13)$$

in which the interface area S_{new} is given by Eq. (10). The use of S_{new} instead of S_{old} represents a backward difference approximation, which requires iterations with repeated use of Eq. (10). The backward difference approximation is more stable than a forward or central difference approximation, in which S_{new} in Eq. (13) would be replaced by S_{old} or by $(S_{\text{old}} + S_{\text{new}})/2$.

If the particle size is of interest, it may be calculated for each location and each time step from Eq. (4), i.e., $h = C_s/S$. In the previous version of the theory [25] inspired by analogy with turbulence, a version in which the interface fracture was considered to occur instantaneously (as in classical fracture mechanics), it was found that

$$h \propto \dot{\epsilon}_D^{-2/3}, \quad \Delta K \propto \dot{\epsilon}_D^2/3, \quad \eta_D \propto \dot{\epsilon}_D^{-1/3} \quad (14)$$

where $\propto$ is the proportionality sign.

These simple relations have analogies in the theory of turbulence, the eddies being considered as analogous to the comminuted particles. The analogies can be detected in the eddy size, in the kinetic energy dissipation by viscous friction between the eddies, and in the effective viscosity. However, in the present theory with finite rate of interface fracture growth, these simple relations are not valid. Nevertheless, from the physical viewpoint, they should be approached asymptotically as S increases, because the fracture propagation criterion in linear elastic fracture mechanics is the limit case of Evans' law for $A_T \rightarrow \infty$.

4 Verification by Tests of Impact With Perforation or Partial Penetration of Concrete Walls

Circular slabs, made of concrete WES-5000 with mass density 2300 kg/m³, were impacted by projectiles at the Geotechnical and Structural Laboratory of the U.S. Army Engineer Research and Development Center (ERDC), Vicksburg [9,37]. All the slabs were cast in steel culvert pipes with the diameter of 1.52 m. Their thicknesses were 127, 216, 254, and 280 mm. The impacting projectiles had a diameter of 50.8 mm, weighed 2.3 kg, had a

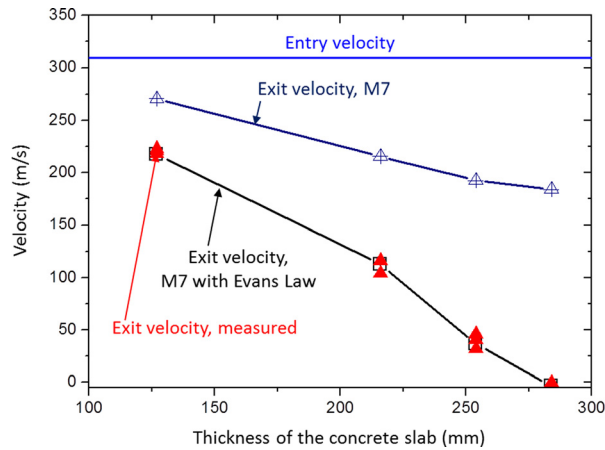


Fig. 2 Comparison of the exit velocity predicted by two methods (M7 with and without implementation of comminution effect) with the experimental data for missile perforation of concrete walls (the thicknesses of impacted walls range 127–280 mm)

hollow steel body with an ogival nose (caliber radius head = 3.0, length/diameter ratio = 7.0), and were considered as rigid since they were found undeformed after the test. They hit the concrete slab at the center, orthogonally (the impact angle of 90 deg), with the entry velocity of 310 m/s. For each slab thickness, the perforation was repeated two or three times. No damage was discerned on the projectiles after the tests.

As the damage constitutive law, microplane model M7 [38,39] was used. It was calibrated to reproduce the Young's modulus of $E = 25$ GPa, Poisson ratio $\nu = 0.18$, and the mean compressive strength of $\bar{f}_t = 48$ MPa. The M7 scaling parameters [39] were $k_1 = 11 \times 10^{-5}$, $k_2 = 110$, $k_3 = 30$, $k_4 = 100$, and $k_5 = 10^{-4}$. With these parameters, model M7 fits well all the basic types of static laboratory tests of uni-, bi-, and triaxial behavior with postpeak softening and size effect, as published in the literature, and the tests of WES-5000 concrete for various special loading paths [38,39].

Without the additional viscosity, the use of the calibrated microplane model M7 led to severe overestimation of the measured exit velocities (see the dashed curve in Fig. 2), as previously reported by Caner and Bazant [27]. However, with the present comminution theory, the measured velocities and penetration depths were matched closely. The optimum fits shown correspond to the interface fracture energy of $\Gamma = 100$ N/m, Evans' law exponent of $n = 3$, and prefactor $A_T = 2 \times 10^9$ m²/s (it is interesting that this optimum value of n is much lower than the n -value for static crack growth, which is 10–20).

Energy absorbing elements were used in finite element computations at the slab perimeter, to avoid spurious rebound waves. The projectile was in the simulations and thus considered to be rigid. This was partly justified by the fact that no visually discernible damage to the projectile was found in the experiments. So the deformation of the projectile must have been elastic, but this is very small.

For $A_T = 2 \times 10^9$ m²/s, the four predicted exit velocities agree with all the experimental data closely (Fig. 2). They also match the data better than the comminution simulations with simplified viscosity presented in Ref. [27]; see Fig. 2.

The tests of penetration depths used a slightly different concrete, the CSPC concrete [37], but it was so similar to the WES-5000 concrete that the same mechanical parameters, besides the adjustment of A_T , could be used. The concrete wall was big and thick enough to be considered as semi-infinite. The projectile had an ogive nose, weighed 0.9 kg, and length of 242.4 mm and diameter of 25.9 mm. The entry velocities ranged from 277 m/s to 800 m/s.

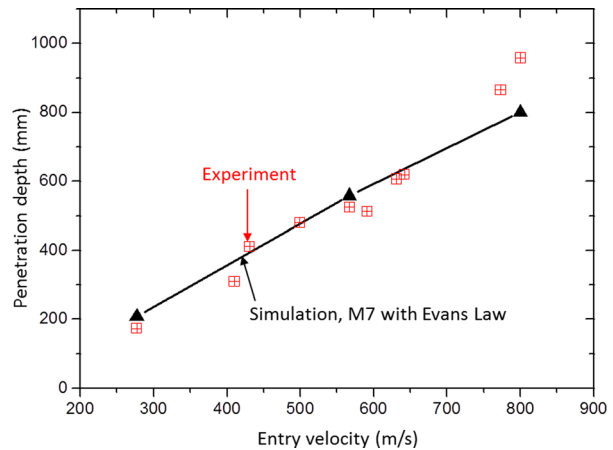


Fig. 3 Comparison of the penetration depth predicted by two methods (M7 with and without implementation of comminution effect) and the experimental data (the entry velocities of the missiles range from 277 m/s to 800 m/s)

As shown in Fig. 3, the penetration depth increases with the increase of the entry velocity. For $A_T = 5 \times 10^8$ m²/s, Fig. 3 documents that the present comminution model can give excellent agreement with the observed penetration depths.

Finally it may be pointed out that, despite the strain softening in the constitutive model, there is, here, no need for nonlocal damage modeling with a material characteristic length. The reason is that the strain rate was so high that the inertia must have prevented the damage localization instability from taking place at the same rate and also so high that nonlocal damage interactions could not occur within time periods shorter than the time a sound wave would need to travel the material characteristic length.

5 Conclusions

- (1) Extending the recently proposed idea [40] to model the dynamic overstress or enhanced resistance to missile penetration, explosion or shock wave transmission due to comminution of compressed material caused by the release of kinetic energy of high-rate shearing, this paper shows that the finiteness of the rate of interface crack growth can be taken into account using Evans' power law for subcritical crack growth.
- (2) Based on Evans' law, the growth rate of interface crack area is considered as a function of the shear strain rate, current interface area, material density, and interface fracture energy.
- (3) What is important for computations is to dissipate the energy of fracturing at the correct rate. This is easily achieved by incorporating into the damage constitutive model an additional shear viscosity, which is found to be a function of the current shear strain rate, total interface area between comminuted particles, fracture energy, and mass density.
- (4) At the same time, any artificial damping automatically introduced in commercial finite element programs must be eliminated, both on the structural and element levels.
- (5) This additional rate effect comes on top of the quasi-static rate effects due to the formation of macrocracks and to the viscoelasticity of material between the macrocracks, which are normally embedded in a quasi-static constitutive law such as microplane model M7. The additional rate effect asymptotically vanishes as the strain rate decreases (in practice, below about 1/s).
- (6) The additional viscosity can be derived from energy balance, by equating the work of fracturing to the macro-dissipation of energy by viscosity.

- (7) As a difference from previous theory, the additional viscosity is proportional not simply to the $-1/3$ power the strain rate, but to both the $n - 2$ power of that rate and also to the $-3 n/2$ power of the interface area, which itself depends on the strain rate history ($n = \text{Evans' law exponent}$).
- (8) Although the problem is highly nonlinear, it is possible to formulate an iterative explicit finite-element algorithm in which the interface area increments are calculated from the current total interface area and the average strain rate in the time step. The algorithm converges well and, with a supercomputer, delivers realistic results, as demonstrated for systems with millions of nodal displacements.
- (9) Computer simulations in which the microplane constitutive damage model M7 is enriched with the present additional viscosity lead to excellent fits of published test data on the projectile penetration of concrete walls of different thicknesses and on the depth of penetration of projectiles of different velocities into a large concrete block.
- (10) The use of Arrhenius law for the Evans law prefactor should allow generalizations to variable temperature.
- (11) In conjunction with the use of a realistic damage constitutive model such as microplane model M7, the high-rate simulations remain anchored in all the experimentally evidenced uni-, bi-, and triaxial damage properties postpeak softening (which, for M7, are characterized by 22 different types of quasi-static tests of concrete [39]).

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Impact comminution of solids due to local kinetic energy of high shear strain rate: I. Continuum theory and turbulence analogy

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ABSTRACT

The modeling of high velocity impact into brittle or quasibrittle solids is hampered by the unavailability of a constitutive model capturing the effects of material comminution into very fine particles. The present objective is to develop such a model, usable in finite element programs. The comminution at very high strain rates can dissipate a large portion of the kinetic energy of an impacting missile. The spatial derivative of the energy dissipated by comminution gives a force resisting the penetration, which is superposed on the nodal forces obtained from the static constitutive model in a finite element program. The present theory is inspired partly by Grady's model for expansive comminution due to explosion inside a hollow sphere, and partly by analogy with turbulence. In high velocity turbulent flow, the energy dissipation rate gets enhanced by the formation of micro-vortices (eddies) which dissipate energy by viscous shear stress. Similarly, here it is assumed that the energy dissipation at fast deformation of a confined solid gets enhanced by the release of kinetic energy of the motion associated with a high-rate shear strain of forming particles. For simplicity, the shape of these particles in the plane of maximum shear rate is considered to be regular hexagons. The particle sizes are assumed to be distributed according to the Schuhmann power law. The condition that the rate of release of the local kinetic energy must be equal to the interface fracture energy yields a relation between the particle size, the shear strain rate, the fracture energy and the mass density. As one experimental justification, the present theory agrees with Grady's empirical observation that, in impact events, the average particle size is proportional to the $(-2/3)$ power of the shear strain rate. The main characteristic of the comminution process is a dimensionless number B_0 (Eq. (37)) representing the ratio of the local kinetic energy of shear strain rate to the maximum possible strain energy that can be stored in the same volume of material. It is shown that the kinetic energy release is proportional to the $(2/3)$ -power of the shear strain rate, and that the dynamic comminution creates an apparent material viscosity inversely proportional to the $(1/3)$ -power of that rate. After comminution, the interface fracture energy takes the role of interface friction, and it is pointed out that if the friction depends on the slip rate the aforementioned exponents would change. The effect of dynamic comminution can simply be taken into account by introducing the apparent viscosity into the material constitutive model, which is what is implemented in the paper that follows.

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1. Introduction

In spite of the recent advances in the constitutive modeling of concrete, the finite element models for impact of missiles onto the walls of concrete structures often severely overestimate the depth of penetration and, in the case of perforation, the exit velocity. By contrast with impact on thin walls (< 10 cm) or high rate tensile fracture, the penetration of thick walls causes comminution of a significant portion of concrete into fine particles ($0.01 \text{ mm}^{-1} \text{ mm}$). The underestimation of the exit velocities and penetration depths is severe even when the finite element code uses a highly realistic constitutive model such as the new microplane model M7 (Caner and Bažant, 2013; Caner et al., 2013), which is an improvement of model M4 (Bažant et al., 2000) and provides very good fits of virtually the complete range of the experimental data from diverse types of uniaxial, biaxial and triaxial tests, including the tests of vertex effect, tensile and shear fracturing and the compression-shear behavior under very high confinement. The additional resistance to penetration due to the viscoelastic rate effect and to the rate-dependence of crack growth does not suffice by far for obtaining correct predictions of impact.

The macroscopic constitutive equation with softening damage, calibrated by standard laboratory tests at low strain rates, cannot describe material comminution into sub-mesoscale particles. It can capture only the energy dissipation by mesoscale fragmentation, i.e., the creation of fragments of the same order of magnitude as the dominant mesoscale material inhomogeneities, such as the largest aggregate pieces in concrete. This is the only kind of comminution that occurs in the standard laboratory tests of damage behavior. This limitation applies even if the constitutive equation is enhanced by the material rate effects, which include viscoelasticity and the rate dependence of mesoscale crack growth.

The purpose of the present paper is to show how a constitutive model can take into account the energy dissipated by material comminution at very high strain rates (see Fig. 1 adapted from Reinhardt and Weerheijm, 1991; Kipp et al., 1980). The spatial derivative of the energy dissipated by comminution represents a compressive force resisting the penetration, which has the effect of greatly increasing the finite element nodal forces obtained with a standard macroscopic constitutive model.

The literature on the analysis of impact is vast and great progress has been achieved in many directions (Mott, 1947; Grady and Kipp, 1979; Grady, 1982; Mescall and Weiss, 1984; Grady, 1985, 1990; Grady and Kipp, 1995; Camacho and Ortiz, 1996; Grady, 1998; Shih et al., 1998; Cadoni et al., 2001; Gailly and Espinosa, 2002; Gatuingt and Pijaudier-Cabot, 2002; Doyoyo, 2002; Forquin et al., 2008; Deshpande and Evans, 2008; Wei et al., 2009; Grady, 2010; Ferri et al., 2010; Kožar and Ožbolt, 2010; Adley et al., 2012; Ožbolt et al., 2011; Frank et al., 2012). As it appears, however, there exists no constitutive law that would capture the effect of material comminution and could be used to formulate an initial-boundary value problem underlying a finite element formulation. One important advance has been the development of the so-called “Mescall” models (Mescall and Weiss, 1984; Doyoyo, 2002; Deshpande and Evans, 2008; Wei et al., 2009; Kožar and Ožbolt, 2010; Ferri et al., 2010). But although they describe the nucleation and branching of dynamically propagating individual cracks they do not provide a constitutive model for a finite element code, describing the behavior of a material comminuting into a vast number of tiny fragments. The present work aims to provide for the Mescall zone an alternative model that is usable in finite element programs.

In the community of computational missile impact simulation, the recent work of Adley et al. (2012) and Frank et al. (2012) is worth noting. They adapted to high rate deformation a dynamic finite element code with microplane model M4 in which the strain-dependent strength limits (called stress-strain boundaries) were scaled up sharply so as to fit the test data on missile penetration. However, after this kind of purely empirical adjustment, the constitutive model used no longer fits the multitude of the standard uni-, bi-, and tri-axial laboratory tests of concrete by which the microplane model was calibrated. It also no longer fits (Ožbolt et al., 2011) high-rate tests of fracture specimens loaded dynamically in opening mode I, in which the high-rate deformation occurs almost exclusively in tension. These tests are in perfect agreement with the unadjusted microplane model M4 incorporating solely the quasi-static strain rate effects, which provide only a mild and smooth strength increase as the strain rate increases. The point to note is that, in contrast to missile impact, the notched specimens of Ožbolt and Reinhardt exhibit no material comminution. Adley et al.'s empirical adjustment thus loses

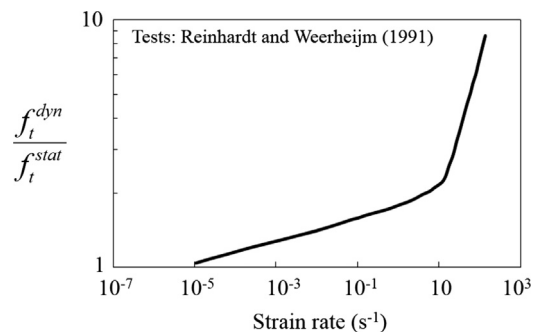


Fig. 1. Typical diagram of concrete tensile strength as a function of strain rate, roughly following the test data of Reinhardt and Weerheijm (1991).

predictive capability except for situations very similar to those for which the microplane strength limits have been adjusted.

The theory proposed here is partly inspired by analogy with turbulence. In high velocity turbulent flow, the energy dissipation rate is greatly enhanced by the formation of micro-vortices (eddies) which dissipate energy by viscous shear stress. By analogy, it is assumed that the energy dissipation at fast deformation of a confined solid gets greatly enhanced by the release of kinetic energy of high shear strain rate of forming particles. Another inspiration for the present model is Grady's model for explosion in a hollow sphere (Grady, 1982), in which the kinetic energy of volumetric strain is considered as the driving force of comminution. Grady calculated and experimentally justified that, in rapid volumetric expansion, the average particle size is

$$\bar{s} = \left(\frac{48\Gamma}{\rho \dot{\epsilon}_V^2} \right)^{1/3} \quad (1)$$

where Γ is the interface fracture energy, ρ is the mass density and $\dot{\epsilon}_V$ is the expansive volumetric strain rate. In later works (Grady, 1990; Grady and Kipp, 1995; Grady, 1998, 2010) he demonstrated by experiments that this equation can be extended to comminution under impact, but did not justify it theoretically. Here the extension of Grady's equation to impact is theoretically justified by noting that comminution under impact must be driven by the release of the local kinetic energy of shear strain rate. This basic idea is then used to develop a coherent constitutive model.

The presently proposed formulation is a continuum model that smears the sub-scale cracking. A number of continuum models that homogenize the sub-scale cracking of concrete, sea ice and other brittle heterogeneous materials have been proposed in the past (e.g. Bažant and Lin, 1988; Jirásek and Bažant, 1995). Gao and Klein (1998) and Klein and Gao (1998) consider a random network of cohesive bonds on the microscale and obtain the continuum model via the Cauchy–Born rule, i.e., by equating the strain energy of the continuum to the potential energy of the micro-scale bonds. However, all these models consider the fracturing to be caused by the release of strain energy. The present model is based on the fact that, at high rate deformation, it is the release of local kinetic energy that matters. This idea was outlined in a brief paper (Bažant and Caner, 2013) motivated by the recently proposed concept of shock fracturing of gas shale, and is here developed in detail.

This paper presents the theory. The following paper (Caner and Bažant, this issue) presents the numerical validation by large-scale finite element simulations of missile impact and by the simulation of Hopkinson bar tests of strength. These simulations use the microplane model M7 with comminution enhancement.

2. Local kinetic energy density of comminuting micro-particles at high shear rate

Consider first a simple idealized comminution process in which the material is comminuted to identical particles (Fig. 2a–d). The particles must fill the space completely. At first, we consider the particles to be identical. The only possible repetitive regular subdivisions of the material in the plane of maximum shearing are the squares, isosceles triangles, and regular hexagons. Among them, the hexagonal prisms are the likeliest as they have the least interface area per unit volume. More complex space-filling geometries, such as truncated octahedron, could be considered. But it turns out that the particle shape is not terribly important. It has no effect on the form of the resulting equations and causes only relatively small changes in their coefficients.

For graphical two-dimensional visualization, it is more instructive to begin discussion with a subdivision into squares (or cubes), as depicted in Fig. 2a for the undeformed state. During deformation, orthogonal coordinate axes x and y , along with the lines of the square subdivision (yet to occur), rotate by angle ω . Simultaneously, these lines get skewed relative to axes x and y by angles ϵ_D representing pure shear (Fig. 2b).

At a certain moment, the strain rate becomes high enough for the kinetic energy of deforming material to suffice for creating the fractures that rapidly comminute the material into separate identical square particles shown in Fig. 2c. As that happens, the particles regain their original undeformed shape, i.e., become squares again, while the centers of the comminuted particles still conform to the same macroscopic displacement field, which means that the dashed lines

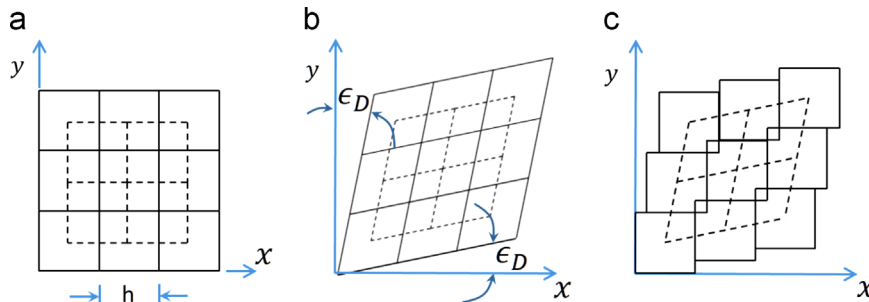


Fig. 2. Comminution of material into prismatic square particles, showing the velocities in terms of displacements during time Δt (note that displacements are supposed to be infinitesimal, in which case the overlaps at the square corners are second-order small and thus negligible).

connecting these centers are identical in Fig. 2b and c (we ignore the necessary crushing of the particle corners since their volume is second-order small).

As the particles return to their near-original shape, they release their kinetic energy $\Delta\mathcal{K}$ while the opposite faces of neighboring particles slip against each other, as marked by double arrows in Fig. 2c.

The global kinetic energy, which excludes the kinetic energy of shear strain rate of the particles, is in two dimensions, defined as

$$\bar{K} = \sum_i h^3 \frac{\rho}{2} (\dot{u}_0^2 + \dot{v}_0^2)_i \quad (2)$$

(where $\dot{u}_0, \dot{v}_0$ are the velocity components of the centers $i = 1, 2, 3, \dots$ of the particles or of the eddies, and h is the side of the squares) remains unchanged as the particles separate.

For the subdivision of a plane into regular hexagons (Fig. 3a), we consider the particles in three dimensions to be hexagonal prisms of a length equal to the corner-to-corner diameter h of the hexagon. The volume of one particle and the interface area S of all the particles per unit volume of material are, respectively,

$$V_p = c_v h^3, \quad c_v = \frac{3\sqrt{3}}{8} \doteq 0.6495 \quad (3)$$

$$S = \frac{c_s}{h}, \quad c_s = 1 + \frac{4}{\sqrt{3}} \doteq 3.3094 \quad (4)$$

where c_s is the dimensionless constant. Note that since each two neighboring particles share the same interface, $S = \frac{1}{2} S_p / V_p$, where S_p is the surface area of one hexagonal prism.

Let the Cartesian coordinates x, y, z be placed so that plane (x, y) be the plane of maximum shear strain rate among all possible orientations. The field of displacements is considered to consist of pure shear strain ϵ_D and rotation ω . The maximum shear strain rate, denoted as $\dot{\epsilon}_D$, is chosen to represent pure shear, in which case $\dot{\epsilon}_D$ also represents the effective deviatoric strain rate, i.e.

$$\dot{\epsilon}_D = \sqrt{\dot{\epsilon}_{Dij} \dot{\epsilon}_{Dij} / 2} \quad (5)$$

(repetition of tensorial subscripts implies summation). Here ϵ_{Dij} is the deviatoric strain tensor in Cartesian coordinates, the superior dots denote the time derivatives, and the shear angle rate is $\dot{\gamma} = 2\dot{\epsilon}_D$.

At first the hexagonal cells, not yet separated and of a size not yet known (Fig. 3b), undergo shear strain ϵ_D in conformity with the material as a whole. However, a quasibrittle material such as concrete cannot undergo a large strain and so, at a certain moment, the material will fracture into the hexagonal prisms. This sudden fracturing releases the strain in the hexagonal particles and allows them to regain their original shape. As they do, they must rotate against each other by angle ϵ_D (Fig. 3c).

It is considered that, during this whole process, the particle centers move so as to follow the field of macroscopic pure shear strain ϵ_D and simultaneous material rotation rate $\dot{\omega}$ (Fig. 3c) in plane (x, y) . The effect of hydrostatic pressure will be included later. Explosive volume expansion, which leads to a different type of comminution, is not considered.

Before comminution, the displacement velocities $\dot{u}$ and $\dot{v}$ in the directions of current (Eulerian) coordinates x and y whose origin is placed into the particle center are, in the plane of maximum shear rate,

$$\dot{u} = \dot{u}_0 - \dot{\omega}y + \dot{\epsilon}_D y, \quad \dot{v} = \dot{v}_0 + \dot{\omega}x + \dot{\epsilon}_D x \quad (6)$$

After fracturing and separation of the particles these velocities change to

$$\dot{u}^+ = \dot{u}_0 - \dot{\omega}y, \quad \dot{v}^+ = \dot{v}_0 + \dot{\omega}x \quad (7)$$

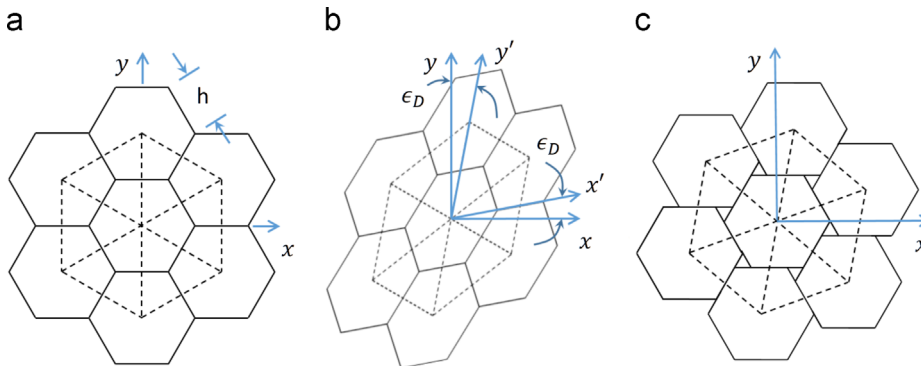


Fig. 3. Comminution of material into prismatic hexagonal particles, showing the velocities in terms of displacements during time Δt (note that displacements are supposed to be infinitesimal, in which case the gaps at the hexagon corners are second-order small and thus negligible).

The drop in kinetic energy of each cell is

$$-V_p \Delta \mathcal{K} = h \int_A \frac{\rho}{2} (\dot{u}^2 + \dot{v}^2 - (\dot{u}^+)^2 - (\dot{v}^+)^2) dA \quad (8)$$

$$-V_p \Delta \mathcal{K} = \frac{h\rho}{2} \dot{\epsilon}_D^2 \int_A (x^2 + y^2) dA + h\rho \int_A [\dot{v}_0 \dot{\epsilon}_D x + \dot{u}_0 \dot{\epsilon}_D y + \dot{\epsilon}_D \dot{\omega} (x^2 - y^2)] dA \quad (9)$$

$$-V_p \Delta \mathcal{K} = \frac{h\rho}{2} \dot{\epsilon}_D^2 \int_A r^2 dA = \frac{h\rho}{2} I_p \dot{\epsilon}_D^2 \quad (10)$$

or, per unit volume of material,

$$\Delta \mathcal{K} = -c_k \rho h^2 \dot{\epsilon}_D^2 \quad (11)$$

where

$$I_p = \frac{\sqrt{3}}{32} h^4 \doteq 0.05413 h^4, \quad c_k = \frac{I_p}{2hV_p} \doteq 0.04167 \quad (12)$$

here $\Delta \mathcal{K}$ is the drop of kinetic energy of the particle per unit volume, c_k is the dimensionless coefficient of kinetic energy, ρ is the mass density, A is the area of the hexagon, r is the radial coordinate, $I_p = I_x + I_y$ is the centroidal polar moment of inertia of the hexagon (and I_x, I_y is the moments of inertia about axes x and y , the orientation of which does not matter).

Note that, if the global motion is characterized by the velocities of particle centroids, the second integral in Eq. (9) vanishes. Actually this is exactly true only when $I_x = I_y$, which occurs for cubes, but only approximately for hexagons. Also note that the material rotation velocity $\dot{\omega}$ has no effect on $\mathcal{K}$. Indeed, it ought to be so, since rigid body rotations can cause no fracturing. Thus it is proven that the global and local kinetic energy densities are, in two dimensions, separable.

3. Separability of local and global kinetic energy in three dimensions

In three dimensions, the axial vector of local rotation rate, $\dot{\omega}$, does not have to be orthogonal to plane (x, y) of maximum shear rate. It may be decomposed as

$$\dot{\omega} = \dot{\omega}_N + \dot{\omega}_P \quad (13)$$

where $\dot{\omega}_N, \dot{\omega}_P$ are the axial vectors of the components normal to plane (x, y) of maximum shear and parallel to that plane. Instead of Eq. (8), the density difference between the total kinetic energy and the kinetic energy of corresponding to velocity vector $\mathbf{v}_0$ at particle center becomes

$$-V_p \Delta \mathcal{K} = \frac{\rho}{2} \int_{V_p} [(\mathbf{v}_0 + \dot{\omega} \times \mathbf{x} + \dot{\epsilon} \cdot \mathbf{x})^2 - (\mathbf{v}_0 + \dot{\omega} \times \mathbf{x})^2] dV \quad (14)$$

$$-V_p \Delta \mathcal{K} = \frac{\rho}{2} \int_{V_p} (\dot{\epsilon} \cdot \mathbf{x})^2 dV + \rho \int_{V_p} (\dot{\epsilon} \cdot \mathbf{x}) \cdot (\dot{\omega}_N \times \mathbf{x}) dV + \rho \int_{V_p} (\dot{\epsilon} \cdot \mathbf{x}) \cdot (\dot{\omega}_P \times \mathbf{x}) dV \quad (15)$$

where the integral of $\mathbf{v}_0 \cdot (\dot{\epsilon} \cdot \mathbf{x})$ has vanished by virtue of symmetry; $\dot{\epsilon}$ is the strain rate tensor, $\dot{\epsilon} \cdot \mathbf{x}$ is the strain rate vector on plane (x, y) ; and $\mathbf{x}$ is the vector of coordinates x, y centered at particle center. The second integral in Eq. (15) corresponds to rotation within plane (x, y) of maximum shear rate $\dot{\epsilon}_D$, and so the first two integrals are equivalent to the previous two-dimensional analysis in Eq. (10).

The third integral in Eq. (15) corresponds to rotation in a plane normal to the plane (x, y) of maximum shear rate $\dot{\epsilon}_D$. This integral is in general nonzero. But it seems impossible that a rotation in a plane normal to (x, y) , i.e., a rotation about a vector lying in plane (x, y) , could contribute to comminution in plane (x, y) . It could affect comminution only in a plane normal to (x, y) . Therefore, the kinetic energy corresponding to the last term in Eq. (15) can be ignored, and Eq. (11) should be applicable to all situations.

4. Partial analogy with turbulence

It is interesting that the kinetic energy $\mathcal{K}_{shear}$ of a particle deforming by pure shear at rate $\dot{\epsilon}_D$ happens to be the same as the kinetic energy $\mathcal{K}_{eddy}$ of the same particle rotating as a rigid body at angular rate $\dot{\omega} = \dot{\epsilon}_D$. Indeed, the squares of the velocity magnitudes in the comminuting particle with shear strain rate $\dot{\omega}_D$ and in a turbulence vortex (eddy) treated approximately as a small rigid domain of area A with angular flow velocity $\dot{\omega}$ can be used, respectively, to calculate the local kinetic energies:

$$\mathcal{K}_{shear} = \int_A \frac{\rho}{2} |\mathbf{v}|^2 dA = \int_A \frac{\rho}{2} [(\dot{\epsilon}_D y)^2 + (\dot{\epsilon}_D x)^2] dA \quad (16)$$

$$\mathcal{K}_{eddy} = \int_A \frac{\rho}{2} |\mathbf{v}|^2 dA = \int_A \frac{\rho}{2} [(\dot{\omega} y)^2 + (\dot{\omega} x)^2] dA \quad (17)$$

Now note that when $\dot{\omega} = \dot{\epsilon}_D$,

$$\mathcal{K}_{\text{shear}} = \mathcal{K}_{\text{eddy}} \quad (18)$$

even though the velocity vectors at the corresponding points do not have the same directions.

This observation reveals a partial analogy with turbulence (Tennekes and Lumley, 1972), which is what inspired the present theory. In both comminution and turbulence, the micro-level kinetic energy (Eq. (16) or (17)) augments the kinetic energy of the macro-level part of the turbulent flow of a fluid, or the macrolevel kinetic energy of the assembly of the comminuting particles, which in both cases is equal to

$$\sum_i \frac{\rho}{2} (\dot{u}_0^2 + \dot{v}_0^2)_i \quad (19)$$

where $\dot{u}_0, \dot{v}_0$ denote the velocity components of the centers $i = 1, 2, 3, \dots$ of the particles or of the eddies. The micro-level kinetic energy gets dissipated by fluid viscosity in the eddies of turbulent flow, or the by the energy of interface fracture of the comminuting particles. In both cases, minimization of the total energy of motion requires a micro-level energy dissipation mechanism consisting of eddy formation or comminution.

When the local motion of a shear strain field continues after the break of the interface, interface slips (like those from (b) to (c) in Fig. 3) must get repeated. But in contrast to the rotational motion in a turbulent eddy, the interface slips cannot continue indefinitely because the shear angle cannot be infinite.

5. Approximate generalization to randomly distributed particle sizes

Limiting consideration to micro-particles of one size would be an oversimplification. It is well-known that, in all sorts of dynamic comminution, the particle sizes vary randomly. The frequency distribution of particle sizes may approximately be described by Schuhmann's power law (Schuhmann, 1940; Charles, 1957; Ouchterlony, 2005; Cunningham, 1987). This law was found to give realistic results for, e.g., the particle sizes and the resisting force due to comminution of concrete slabs during the collapse of the World Trade Center towers (Bažant et al., 2008). The Schuhmann law is described by a power-law cumulative distribution function of particle size s :

$$F(s) = \frac{s^k - h^k}{H^k - h^k} \quad [s \in (h, H), F(s) \in (0, 1)] \quad (20)$$

where k is a empirical constant (typically $k \approx 0.5$), $h = s_{\min}$ is the minimum micro-particle size, and $H = s_{\max}$ is the maximum micro-particle size (usually $H/h = 10$ – 100). Other particle size distributions have also been used (Mott, 1947; Grady, 1990) and the analysis that follows can be readily adapted to any one of them.

Within the size interval $(s, s + ds)$, the number of micro-particles per unit volume is $dF(s)/s^3$. The average particle size is $\bar{s} = \int_h^H s dF(s)$ or

$$\bar{s} = \bar{c}h \quad (21)$$

$$\text{where } \bar{c} = \frac{k}{k+1} \frac{(H/h)^{k+1} - 1}{(H/h)^k - 1} \quad (22)$$

Since, for size s , the particle interface area per unit volume is c_s/s , the combined interface area of all the micro-particles or random sizes per unit volume is

$$S = \int_{s=h}^H \frac{c_s}{s} dF(s) = \frac{C_s}{h} \quad (23)$$

$$\text{where } C_s = \frac{c_s k}{k-1} \frac{(H/h)^{k-1} - 1}{(H/h)^k - 1} \quad (24)$$

C_s is a dimensionless constant. It may be checked (with the aid of L'Hospital rule) that the limit of Eq. (23) for $H/h \rightarrow 1$ is c_s , as it must.

Note that, in this calculation, we did not address the fact that particles of unequal sizes cannot all be hexagonal prisms. This means that their closest packing may leave some empty interparticle spaces and would thus require a certain volume dilation, and also that particles of diverse shapes will be created. For simplicity, we do not analyze this dilation and shape diversity.

To calculate the kinetic energy loss, we introduce the simplifying hypothesis that it is possible to superpose the contributions calculated for various uniform particle sizes as if the particles of each size were distributed regularly. The loss of kinetic energy of the particles of all the sizes per unit volume (dimension J/m^3 or N/m^2) may then be obtained, according to Eq. (11), as follows:

$$\Delta \mathcal{K} = - \int_{s=h}^H c_k \rho s^2 \dot{\epsilon}_D^2 dF(s) \quad (25)$$

The integration yields

$$\Delta\mathcal{K} = -C_k \rho h^2 \dot{\epsilon}_D^2 \quad (26)$$

$$\text{where } C_k = \frac{k}{k+2} \frac{(H/h)^{k+2} - 1}{(H/h)^k - 1} c_k \quad (27)$$

C_s is a dimensionless constant. For the limit case of particles of one size h , one may check that $\lim_{H/h \rightarrow 1} C_k = c_k$, as required. To illustrate the effect of the choice of particle shape, assuming $H/h = 100$, one obtains for hexagonal and cubical particles respectively $C_s = 0.331, 0.300$ and $C_k = 92.6, 138.9$. These differences are not large and the form of the equations remains the same.

6. Release of local kinetic energy of strain rate as the driving force of dynamic fracturing

The total energy of the comminuted particles per unit volume is

$$\mathcal{F} = \mathcal{K} + S\Gamma \quad (28)$$

where Γ is the interface fracture energy of the comminuting particles (dimension J/m^2 or N/m). To formulate the criterion of dynamic fracturing, one must choose between two theoretically possible forms:

$$\text{either } \frac{\partial \mathcal{F}}{\partial S} = 0 \quad (29)$$

$$\text{or } -\Delta\mathcal{K} = S\Gamma \quad (30)$$

In the former case, by analogy to classical fracture mechanics, it is assumed that once the fracture is triggered in a part of surface S , it will propagate. In the latter case, it is assumed that the entire surface forms simultaneously. The latter case would apply if the continuum strain rate were so high that the duration of comminution would require the fracture to propagate faster than the maximum possible crack propagation rate, which is the Raleigh wave speed. The results of analysis for both cases are very similar and differ only in dimensionless empirical coefficients. In what follows we will consider that the former case applies. Thus, according to Eq. (29), it is required that

$$-\frac{\partial(\Delta\mathcal{K})}{\partial S} = -\frac{\partial(\Delta\mathcal{K})/\partial h}{dS/dh} = \Gamma \quad (31)$$

This equation is similar to that used in Grady's (1982) analysis of tensile comminution due to high volumetric strain rate in an explosion within the center of a hollow sphere (see also Freund, 1990, Eq. (8.7.3)). It is seen to be analogous to the energy release criterion of fracture mechanics. In fact, it could have been derived directly from the condition of criticality: When the comminution gets under way, the rate of release of kinetic energy (with respect to S) at the given kinematic constraint (i.e., at given $\dot{\epsilon}_D$) must be equal to the surface energy Γ .

Note that Eq. (31) may also be considered as a special case of Eqs. (5.3.2) and (5.3.20) in Freund (1990) when the strain energy density is neglected. But these are valid only for moderately high strain rates in which the kinetic energy density does not exceed the strain energy density in the material by an order of magnitude or more.

Eq. (30) imposes an overall energy balance condition which does not ensure incremental energy balance and thus does not guarantee the interface fracture to begin. On the other hand, like in classical fracture mechanics, Eq. (31) means that as soon as energy balance is satisfied incrementally, the fracture can occur, and must occur if the kinetic energy density is increased further.

Substitution of Eqs. (26) and (23) into (31) and differentiation furnish the minimum particle size:

$$h = s_{\min} = \left(\frac{C_a \Gamma}{\rho \dot{\epsilon}_D^2} \right)^{1/3} \quad (32)$$

$$\text{where } C_a = \frac{c_s}{c_k} \frac{(k+2)}{(k-1)} \frac{(k-1)(H/h)^k - k(H/h)^{k-1} + 1}{k(H/h)^{k+2} - (k+2)(H/h)^k + 2} \quad (33)$$

If $H/h = 100$, then $C_a = 0.032$ for hexagonal prisms and $C_a = 0.019$ for cubical prisms. Thus we see again that the precise particle shape does not make a large difference. According to Eq. (21), the average particle size is then obtained as $\bar{s} = \bar{c}h$.

The maximum particle size H is approximately known. It must be one order of magnitude smaller than d_a , i.e.,

$$H \approx 0.1 d_a \quad (34)$$

because fragmentation into particles of the same order of magnitude as d_a is covered by the constitutive law based on static material tests with post-peak softening. Therefore, H must be about 2–4 mm for normal concrete, and 0.05–2 mm for high strength concrete. Thus, to calculate h , one must substitute Eq. (32) into (33) and then solve the resulting nonlinear algebraic equation for h iteratively, e.g., by the Newton method. The ratio H/h then follows.

Eq. (32) agrees with Grady's formula (1), which was justified empirically by observations of particle sizes in impact tests. This provides one verification of the present theory.

Substitution of Eq. (32) into Eq. (26) further yields

$$-\Delta\mathcal{K} = (C_0 \Gamma^2 \rho)^{1/3} \dot{\epsilon}_D^{2/3} \quad \text{where } C_0 = C_k^3 C_a^2 \quad (35)$$

where $C_0 = 822$ for hexagonal prisms and $C_0 = 1013$ for cubical particles, assuming $H/h = 100$. Again the difference that results from different particle shapes is not large.

7. Dimensionless number separating fractures driven by kinetic and strain energies

What is the critical strain rate beyond which the fracturing driven by the release of kinetic energy of strain rate, $-\Delta\mathcal{K}$, dominates over the fracturing of classical type driven by the release of strain energy $\mathcal{U}$? To answer it, we must also include $\mathcal{U}$ in the energy balance. As is clear from Fig. 3, the domains of the particles to form are initially deformed by locally uniform shear strain $\gamma = 2\epsilon_D$ and after comminution fracture they become undeformed. So the strain energy release due to comminution fracture is $\mathcal{U} = \tau^2/2G$ where G is the elastic shear modulus and τ is the shear stress. According to Eq. (35), we have

$$-\frac{\Delta\mathcal{K}}{\mathcal{U}} = B_a \quad (36)$$

$$B_a = \frac{G}{C_g \tau^2} (\Gamma^2 \rho \dot{\epsilon}_D^2)^{1/3} \quad (37)$$

$$\text{where } C_g = C_0^{-1/3}/2 = 1/(2C_k C_a^{2/3}) = 0.2656 \text{ for } H/h = 10 \quad (38)$$

where C_g is a dimensionless geometry factor depending on the particle shape and the type of frequency distribution of particle sizes. For $H/h = 10$ and for the hexagonal particles distributed according the Schuhmann law, $C_g = 0.2656$. For other shapes, distributions and ratios H/h , different values will be obtained but they will generally be of the order of 1. B_a is a dimensionless number, an indicator of the onset of comminution, characterizing the importance of kinetic shear fracturing in the comminution process. The comminution is

$$\begin{aligned} &\text{kinetic energy driven if } B_a \gg 1 \\ &\text{in transition if } B_a \approx 1 \\ &\text{absent or static if } B_a \ll 1 \end{aligned} \quad (39)$$

Since τ at comminution cannot be larger than the shear strength or yield stress τ_0 , the use of $\tau = \tau_0$ in the condition $B_a \gg 1$ gives a sufficient condition for comminution to be driven by the kinetic energy alone.

In the flow of fluids, the dimensionless Reynolds number gives the threshold beyond which the local kinetic energy due to eddies begins to control the resistance to flow. Likewise, B_a is a dimensionless number that defines the threshold beyond which the local kinetic energy due to shear strain rate begins to control the resistance to deformation. This is another feature of the turbulence analogy.

Expressing $\dot{\epsilon}_D$ from Eq. (32) and substituting it into (37), one gets an alternative expression for the dimensionless indicator:

$$B_a = \frac{C_a^{1/3}}{C_g} \frac{G\Gamma}{\tau_0^2 h} \quad (40)$$

8. Some numerical estimates and discussion

Let us now estimate the dimensionless number B_a for shear comminution of concrete. While, at the macroscale, the fracture energy of concrete is about 100 J/m^2 , for sub-millimeter particles it is probably much smaller, perhaps $\Gamma = 10 \text{ J/m}^2 = 10 \text{ kg/s}^2$, because the fracture process zone is much narrower. Further we may consider $\tau_0 = 3 \text{ MPa} = 3 \times 10^6 \text{ kg s}^{-2} \text{ m}^{-1}$, $G = 10.59 \text{ GPa} = 10.59 \times 10^9 \text{ kg s}^{-2} \text{ m}^{-1}$, and $\rho = 2300 \text{ kg/m}^3$. Assuming $H/h = 10$ and $H = 9.43 \text{ mm}$ (or $\dot{\epsilon}_D = 100 \text{ s}^{-1}$), we have $B_a = 5.85$. Considering the strain rates $\dot{\epsilon}_D = 10^4/\text{s}$ and $10^6/\text{s}$, we get from Eq. (36):

$$-\frac{\Delta\mathcal{K}}{\mathcal{U}} = 126 \quad \text{and} \quad 2715 \quad (41)$$

respectively. So, at these rates, the deformation is almost totally dominated by the kinetic energy of strain rate. In a similar way, one finds that

$$-\Delta\mathcal{K}/\partial\mathcal{U} = 1 \quad \text{when } \dot{\epsilon} = 7.075/\text{s}. \quad (42)$$

Since the values of Γ and τ_0 at sub-millimeter scale are highly uncertain, the foregoing estimates must, of course, be regarded as very crude. It is better to calibrate the computational model by fitting impact test data.

Let us now estimate the particle size. We assume the same values of Γ and ρ as before and (for $\dot{\epsilon}_D = 10^4 \text{ s}^{-1}$, $H/h = 10$ and $k = 0.5$) we calculate $C_a = 1.929$. From Eq. (32) with (33), we get the minimum and mean particle sizes:

$$h = 0.4377 \text{ mm for } \dot{\epsilon}_D = 10^4/\text{s}, \quad h = 0.0943 \text{ mm for } \dot{\epsilon}_D = 10^5/\text{s} \quad (43)$$

$$\bar{s} = 2.066 \text{ mm for } \dot{\epsilon}_D = 10^4/\text{s}, \quad \bar{s} = 0.4452 \text{ mm for } \dot{\epsilon}_D = 10^5/\text{s} \quad (44)$$

Under high confining hydrostatic pressure p , say $p = 10\tau_0$, the effective fracture energy might increase significantly. This would increase the foregoing estimates of threshold and particle size. But under high enough pressure the material would behave plastically and would not comminute.

Another point to note is that the release of strain energy $\mathcal{U}$ does not lead to dependence of fracturing on the strain rate. Rather, it leads to a dependence on the magnitude of applied stress or strain, which is already accounted for by the softening part of the constitutive relation.

9. Implementation of kinetic rate effect

Energy conservation requires that $\mathcal{K} = \mathcal{D} =$ energy dissipated per unit volume (dimension J/m^3). As the same time, since the stress (dimension N/m^2) is the energy per unit volume, $\Delta\mathcal{K} = \sigma^A =$ additional stress due to comminution. There are two possible approaches to enforce this energy dissipation in a finite element program:

(1) Either as a distributed body force

$$\mathbf{f} = \partial\mathcal{K}/\partial\mathbf{x} \quad (\Delta\mathcal{K} = \mathcal{D} = \sigma^A) \quad (45)$$

which gets translated into equivalent nodal forces ($\mathbf{x}$ = global coordinate vector formed from Cartesian coordinates x_i , $i = 1, 2, 3$);

(2) Or as an additional stress, $\sigma^A = \mathcal{D}$, to be implemented in the constitutive equation. We favor this approach as it seems simpler for programming, and also is more versatile as it allows generalization to different types of comminution.

Since one may write $\sigma^A = \mathcal{D} = \eta\dot{\epsilon}$, the additional (apparent) viscous stress σ^A may be implemented as kinetic (or apparent) viscosity $\eta = \mathcal{D}/\dot{\epsilon} = -\Delta\mathcal{K}/\dot{\epsilon}$. But this equation would be acceptable only in a uniaxial model. In a triaxial constitutive model, the additional viscous stress should be applied only to the stress components that represent shear, i.e., as additional deviatoric stress components s_{ij}^A , which must be added to the stresses obtained from the standard quasi-static constitutive model.

To ensure tensorial invariance, powers of the tensorial components are inadmissible. Only a tensorial invariant can be raised to a power. So we use again the effective deviatoric strain rate given by Eq. (5). Since the energy density has the same dimension as the stress, it is convenient to introduce equivalent viscosity η_D such that the viscous stress–strain relation

$$s_{ij}^A = \eta_D \dot{\epsilon}_{Dij} \quad (46)$$

where

$$\eta_D = (C_0 \Gamma^2 \rho)^{1/3} \dot{\epsilon}_D^{-1/3} \quad (47)$$

would give the same energy dissipation density as Eq. (35) for any possible deviatoric strain rate tensor $\dot{\epsilon}_{Dij}$, in the variational sense. Now, because the energy density is the same as the stress, s_{12}^A must be equal to $\Delta\mathcal{K}$ when all other tensorial components vanish. Since $\Delta\mathcal{K}$ is a tensorial invariant, it is thus necessary that

$$-\Delta\mathcal{K} = \sqrt{s_{ij}^A s_{ij}^A / 2} \quad (48)$$

In view of Eq. (46), we may then write

$$\sqrt{s_{ij}^A s_{ij}^A / 2} = \eta_D \sqrt{\dot{\epsilon}_{Dij} \dot{\epsilon}_{Dij} / 2} = \eta_D \dot{\epsilon}_D \quad (49)$$

where $\dot{\epsilon}_D = \sqrt{\dot{\epsilon}_{Dij} \dot{\epsilon}_{Dij} / 2}$. This is a tensorial invariant which simplifies to pure shear rate $\dot{\epsilon}_D$ when $\dot{\epsilon}_{D12}$ is the only nonzero component.

This analysis shows that the energy sink due to the comminution process may be modeled by equivalent apparent viscosity η_D . This is the simplest way to implement the comminution effect in finite element programs. It can be combined with any constitutive law.

The enhancement of the dissipative viscous resistance to shearing is a similar feature as the enhancement of viscous resistance caused by eddies in turbulent flow. This is another aspect of analogy with turbulence.

A stress increase proportional to $\dot{\epsilon}^{2/3}$ gives an enormous rate effect on the apparent material strength, but only at very high strain rates. To illustrate this, consider that the strength or yield limit is scaled up by a rate-dependent factor, r , and note that, according to the test data in the literature, the rate effect beyond that explained by viscoelasticity and static crack growth rate is detectable only for rates $\dot{\epsilon} > 1/\text{s}$. Knowing this fact suffices to estimate the rate effect magnitude. Assuming

that the error of the data is not below 1%, we have $1^{2/3}r = 0.01$, from which $r = 0.01$. So, e.g., at the strain rate of $10^5/s$ one has $(10^5)^{2/3}r = 21.5$. This is the ratio by which the stress must be increased, by means of apparent viscosity.

10. Generalization to kinetic energy of volume expansion and its apparent viscosity

In some situations, the rate of volumetric strain $\varepsilon_V = \varepsilon_{kk}/3$ can be so high that the volumetric kinetic energy is significantly contributing to the comminution (such a case was shown by Grady (1982, 1985) in his analysis of an explosion within a hollow sphere). In that case, Eqs. (6) need to be generalized as

$$\dot{u} = \dot{u}_0 - \dot{\omega}y + \dot{\varepsilon}_D y + \dot{\varepsilon}_{Ex} x, \quad \dot{v} = \dot{v}_0 + \dot{\omega}x + \dot{\varepsilon}_D x + \dot{\varepsilon}_{Ex} y \quad (50)$$

in which

$$\dot{\varepsilon}_{Ex} = \left\langle \frac{d}{dt} \left\langle \frac{\varepsilon_{kk}}{3} \right\rangle \right\rangle = \begin{cases} \dot{\varepsilon}_V & \text{if } \dot{\varepsilon}_V > 0 \text{ and } \varepsilon_V > 0 \\ 0 & \text{otherwise} \end{cases} \quad (51)$$

and t is the time, $\dot{\varepsilon}_{Ex}$ represents the expansive strain rate which is such that no comminution occurs if either the volumetric strain rate is compressive or if the volumetric strain is compressive (i.e., negative). Noting that Eq. (7) remains unchanged, one finds that Eqs. (26) and (35) for the drop of kinetic energy per unit volume of material must now be generalized as follows:

$$-\Delta\mathcal{K} = C_k \rho h^2 (\dot{\varepsilon}_D^2 + \dot{\varepsilon}_{Ex}^2) \quad (52)$$

$$-\Delta\mathcal{K} = (C_0 \Gamma^2 \rho)^{1/3} (\dot{\varepsilon}_D^2 + \dot{\varepsilon}_{Ex}^2)^{1/3} \quad (53)$$

This means that, in the remaining equations, $\dot{\varepsilon}_D^2$ must now be replaced by $\dot{\varepsilon}_D^2 + \dot{\varepsilon}_{Ex}^2$. Hence, Eq. (32) must be generalized as

$$h = s_{min} = \left(\frac{C_a \Gamma}{\rho (\dot{\varepsilon}_D^2 + \dot{\varepsilon}_{Ex}^2)} \right)^{1/3} \quad (54)$$

Eq. (46) must now be combined with the additional apparent tensile volumetric viscous stress, as follows:

$$s_{ij}^A = \eta_D \dot{\varepsilon}_{ij}, \quad \sigma_V^A = \eta_V \dot{\varepsilon}_{Ex} \quad (55)$$

in which

$$\eta_V = (C_0 \Gamma^2 \rho)^{1/3} \dot{\varepsilon}_{Ex}^{-1/3} \quad (56)$$

here η_V is the volumetric kinetic (or apparent) viscosity.

Numerical simulations nevertheless show that the generalization for $\dot{\varepsilon}_{Ex}$ has virtually no effect in impact problems.

11. Static-to-kinetic transition of rate effect and apparent viscosity

Eq. (47) cannot be applied to very small loading rates, for two reasons: (1) numerical, because the viscosity would become infinite when $\dot{\varepsilon}_D \rightarrow 0$ and would thus cause the computer program to diverge if $\dot{\varepsilon}_D$ happens to be small enough and (2) physical, because the comminution by local kinetic energy does not exist according to Eq. (39) when $B_a \ll 1$. Thus, there must be a transition from finite to zero viscosity, centered at $B_a = 1$.

To deduce this transition, we must replace function $\dot{\varepsilon}_D^{-1/3}$ by a function that is asymptotically finite or zero when $\dot{\varepsilon}_D \rightarrow 0$ but remains asymptotically unchanged for high $\dot{\varepsilon}_D$. To this end, we express $\dot{\varepsilon}_D$ from Eq. (37) and substitute it into Eq. (47) to express the viscosity in terms of B_a :

$$\eta_D = \frac{\Gamma}{\tau_0} \left(\frac{C_0^{2/3} G \rho}{C_g B_a} \right)^{1/2} \quad (57)$$

This equation may now be replaced by the following:

$$\eta_D = \frac{\Gamma}{\tau_0} \left(\frac{C_0^{2/3} G \rho}{C_g} \frac{B_a^{n-1}}{1+B_a^n} \right)^{1/2} \quad (58)$$

where n is an empirical constant controlling how sharp the transition is; $n \geq 1$ (calculations used $n=2$).

The foregoing equation is the final expression for the kinetic viscosity, with corrected low-rate asymptotics. It is identical to Eqs. (47) and (57) when $B_a \gg 1$ and gives a zero or negligible finite viscosity when $B_a \ll 1$, as required by (39).

12. Possible variability of Γ , work of friction and effect of pressure

When a high shear strain rate continues after the initial comminution by interface fracture, Γ plays the role of frictional work, i.e., the work of frictional shear stress τ on unit displacement. If $\dot{\varepsilon}_D$ after initial comminution decreases while still

remaining in the comminution range according to Eq. (37), the energy balance requires the frictional slip to be concentrated into interfaces spaced farther apart than particle size. Thus groups of particles moving as virtually rigid blocks must form and Γ then represents the work of friction between these blocks. On the other hand, if the shear strain rate after the initial comminution increases further, the particles already comminuted must be getting comminuted further (or fractured) to smaller sizes.

Since friction decreases with increasing slip rate, the frictional work may be decreasing with increasing strain rate. This effect could be approximately captured by setting

$$\Gamma \propto \dot{\epsilon}_D^{-q} \quad (q \geq 0) \quad (59)$$

However, to verify such an effect and determine the value of q , further more detailed test data will be necessary. In data fitting thus far, the possible effect of $\dot{\epsilon}$ on Γ has been neglected.

Note that exponent $-1/3$ in Eq. (47) has been derived for constant Γ . However, if Γ as a characteristic of post-fracture frictional work decreases with the slip rate or shear strain rate according to the power law in Eq. (59), it would reduce the exponent in Eq. (47) below $-1/3$. However, the particle size given by Eq. (32) would not be affected since it depends on the fracture energy rather than the subsequent frictional slip.

When the high-rate shearing occurs under high hydrostatic pressure p , the effective value of Γ in the sense of fracture energy as well as frictional work may increase as a function of p .

It may be noted that a different fracture energy and particle size effects have been observed in the case of dynamic erosion of a solid surface by impinging hard particles (Tilly and Sage, 1970). However, this is a fundamentally different problem to which the present theory does not apply.

13. Conclusions

1. The local kinetic energy of motion associated with high shear strain rates ($> 10^2/\text{s}$) is sufficient to provide the surface fracture energy necessary for comminution of materials such as concrete into fine particle.
2. At very high strain rates, kinetic energy of strain rate is orders of magnitude higher than the strain energy. Hence, the classical fracture mechanics does not apply.
3. The dissipated energy depends on the particle size distribution, which is here assumed to follow the Schuhmann power law.
4. The minimum or average particle size follows from the condition that the rate of release of kinetic energy of shear strain rate must be equal to the rate of energy dissipation due to growing area of interparticle fractures. The minimum particle size predicted for missile impact at 310 m/s is of the order of 0.15 mm.
5. The main characteristic of the comminution effect is a dimensionless number B_a (Eq. (37)) representing the ratio of local kinetic energy of shear strain rate to the maximum possible strain energy that can be stored in the same volume of material.
6. The present theory indicates that the density of kinetic energy available for comminution is proportional to the $(2/3)$ power of the shear strain rate, the particle size or crack spacing is proportional to the $(-2/3)$ power of that rate, and the energy dissipation by comminution is equivalent to a shear viscosity decreasing as the $(-1/3)$ power of that rate. For a strain rate increase from $1/\text{s}$ to $10^5/\text{s}$, the result is a roughly 20-fold increase of apparent material strength due to comminution.
7. After comminution, the role of interface fracture energy changes to the work of interface frictional slip per unit relative displacement. If the interface friction depends on the slip rate, the exponent of strain rate giving the effective viscosity can change.
8. As one experimental justification, the present theory agrees with Grady's empirical observation that, in impact events, the average particle size is inversely proportional to the $(2/3)$ power of the shear strain rate (Eq. (1)).
9. The theory can be extended in a similar way by including particle comminution due to kinetic energy of volumetric strain rate, although, for missile impact, the volume expansion is not important.
10. The theory leads to a rate-dependent modification of constitutive equation calibrated only for static strain rate effects.

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Appendix A. The alternative of increasing strength or yield limit

Since the energy per unit volume (dimension J/m^3 , with $\text{J}=\text{N m}$) has the same dimension as the stress (dimension N/m^2), an increase in the strain rate may alternatively be considered to cause an increase in the strength or yield limit in the constitutive law. In the microplane model, the compressive deviatoric boundary curve for deviatoric stress σ_D , and the normal boundary for tensile normal stress σ_N , respectively, may thus be scaled up by the kinetic factors:

$$r_D = (C_D I^2 \rho)^{1/3} \langle \dot{\varepsilon}_D \rangle^{2/3}, \quad r_V = (C_V I^2 \rho)^{1/3} \langle \dot{\varepsilon}_N \rangle^{2/3} \quad (\text{A.1})$$

where ε_D is now the microplane deviatoric strain component, ε_N the microplane normal strain component, and C_V, C_D microplane stiffness constants.

Modeling of the enhanced resistance to high-velocity missile penetration by raising of the boundaries on the microplanes was attempted by [Adley et al. \(2012\)](#) and [Frank et al. \(2012\)](#). They raised the boundaries purely empirically, in a way that allowed them to fit their missile penetration data.

However, in the microplane model there are many boundaries and each has a different and complicated shape controlling the triaxial behavior. Thus different boundaries and different parts of each boundary would have to be raised by different ratios, which would reshape the boundaries. Unfortunately, it appears to be next to impossible to do that without loosing the foundation of the microplane model in the static triaxial test data. This is a serious objection to this kind of empirical approach.

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Corrigendum

Corrigendum to Impact comminution of solids due to local kinetic energy of high shear strain rate: I. Continuum theory and turbulence analogy [J. Mech. Phys. Solids 64 (2014) 223–235]



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The authors regret that the following figure was not included. This figure should have been included on p. 6 as Fig. 4, with the remaining figure numbers sequentially updated.

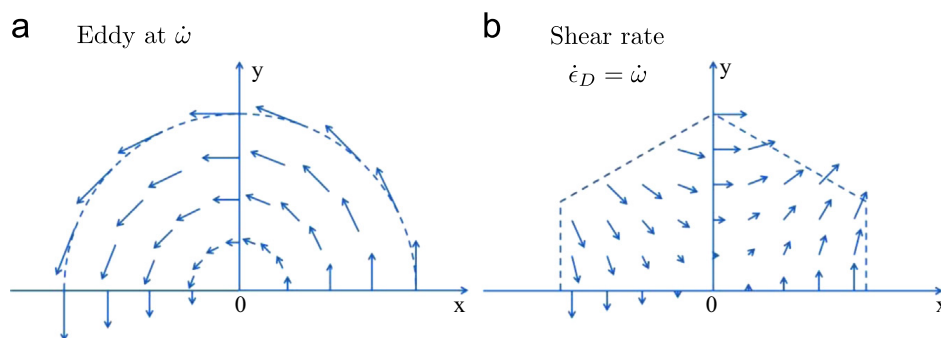


Fig. 4. Velocity vector fields of (a) idealized eddy rotating as a rigid body at angular velocity ω , and (b) pure shear. For deviatoric strain rate $\dot{\epsilon}_D = \omega$, both fields have the same kinetic energy for the same area.

Furthermore, the authors regret a numerical error in Eq. (2) which affected values of several parameters, as follows:

Eq. (12) should read $I_p = 5\sqrt{3}/128 h^4 \doteq 0.06766 h^4$, $c_k = I_p/2hV_p \doteq 0.05208$. The correction in I_p requires the following corrections of the numerical values of various dimensionless constants throughout the paper: In the paragraph following Eq. (27), $C_k = 115.7, 185.2$. In Eqs. (30) and (31), replace ΔK by $|\Delta K|$. In the paragraph that follows Eq. (33), $C_a = 0.026$ and $C_a = 0.015$. In the paragraph of Eq.(35), $C_0 = 1028$ and $C_0 = 1351$. In Eq. (38), C_g should be 0.2466. In the first paragraph of Section 8, $h = 8.75$ mm. In Eq. (41), $-\Delta K/U = 136$ and in Eq. (42), $\dot{\epsilon} = 6.325/\text{s}$. In Eq. (43), change 0.4377 mm to 0.4063 mm and 0.0943 mm to 0.0875 mm. In the paragraph of Eq. (43), $C_a = 1.543$. In Eq. (44), change 2.066 mm to 1.918 mm and 0.4452 mm to 0.4133 mm. In Fig. 3a, change distance label h to $h/2$. No change is needed in the data fits of part 2 because the parameters were scaled to fit thickest slab test.

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Impact comminution of solids due to local kinetic energy of high shear strain rate: II—Microplane model and verification

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ABSTRACT

The new theory presented in the preceding paper, which models the dynamic comminution of concrete due to very high shear strain rate, is now compared to recent test data on the penetration of projectiles through concrete walls of different thicknesses, ranging from 127 to 254 mm. These data are analyzed by an explicit finite element code using the new microplane constitutive model M7 for concrete, which was previously shown to provide the most realistic description of the quasi-static uni-, bi- and tri-axial test data with complex loading path and unloading. Model M7 incorporates the quasi-static strain rate effects due viscoelasticity and to the rate of cohesive crack debonding based on activation energy of bond ruptures, which are expected to extend to very high rates. Here model M7 is further enhanced by apparent viscosity capturing the energy dissipation due to the strain-rate effect of comminution. The maximum shear strain rates in the computations are of the order of 10^5 s^{-1} . The simulations document that, within the inevitable uncertainties, the measured exit velocities of the projectiles can be matched quite satisfactorily and the observed shapes of the entry and exit craters can be reproduced correctly.

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1. Introduction

The preceding Part 1 (Bažant and Caner, *this issue*) presented a new theory that allows the energy dissipation due to dynamic comminution of material under impact to be modeled through the constitutive relation which can be readily implemented in standard finite element programs. The comminution is assumed to be driven by the release of local kinetic energy of high shear strain rate caused by impact, and the energy dissipation due to comminution and inter-particle slip is modeled by additional apparent viscosity.

In the present part 2, the theory is being applied to the problem of penetration of missiles through concrete walls of various thicknesses. Fitting of previously published impact test data is used to calibrate and validate the new theory.

Because of the complexity of triaxial softening damage in concrete, the microplane model is selected as the constitutive law, to be enriched by the comminution rate effect. The microplane model has already been used with great success for the simulations of various dynamic effects on hardened concrete structures, including the effects of explosions, impact and

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groundshock. In contrast to plastic and other tensorial constitutive models, various microplane models have been shown capable of predicting the correct entry and exit crater shapes of penetrating missiles (Bažant et al., 2000a; Adley et al., 2012).

However, prediction of the exit velocities of perforating projectiles proved to be much more challenging. Recently a new model based on the microplane model M4, called the high-rate brittle microplane model, has been developed at U.S. Army Corps of Engineers (Adley et al., 2012). To fit the exit velocities, the microplane stress–strain boundaries (i.e., strain-dependent yield limits) were scaled up drastically, both vertically and horizontally, as functions of the strain rate (Adley et al., 2012). But the scaling ratios were empirical, with no theory behind them. Their empirical nature unfortunately severed the connection to the numerous types of static uni-, bi-, and tri-axial tests by which M4 was calibrated. Besides, in the sense of crack-band or nonlocal concept, the model in Adley et al. (2012) implied a major increase of fracture energy, while the previous studies showed the fracture energy to be only weakly dependent on the strain rate. This study will show that an extended microplane model, labeled M7R, which incorporates the theory of material comminution expounded in Part I, can fit the exit velocities satisfactorily without losing the capability to fit all the static tests.

2. Review of microplane model and its version M7

The internal friction in materials is usually modeled as a relation between the first stress invariant and the second deviator invariant, without any reference to the direction of slip. This is not realistic. The plastic or frictional slip occurs in fact only on planes of a certain specific orientation. Likewise, almost all of the other inelastic deformations in concrete microstructure, such as tensile cracking and compressive axial splitting, occur on planes of well defined orientations. These orientations can be captured by the idea of Taylor (1938) who proposed to formulate the constitutive law in terms of the vectors of stress and strain acting on planes of various orientations in the material. This idea was first developed for plasticity of polycrystalline metals under the name of slip theory of plasticity (Batdorf and Budianski, 1949) and has by now led to powerful plasticity models called the Taylor models (Butler and McDowell, 1998; Rice, 1971). In these models, the stress vectors on the slip planes are the projections of the continuum stress tensor, which is called the static constraint, and the plastic slips on all the slip planes are simply superposed.

In the early 1980s, it was realized (Bažant, 1984; Bažant and Oh, 1985) that Taylor models are unstable in the case of strain-softening damage due to diffuse microcracking. To ensure model stability, the static constraint had to be changed to the kinematic constraint, in which the strain vector is the projection of the strain tensor onto a generic plane in the microstructure, called the microplane. Also the elastic deformation had to be included on the microplane level. The static equivalence of the stress tensor with stress vectors on microplanes of all possible spatial orientations was enforced by the principle of virtual work, which gives the continuum stress tensor

$$\sigma_{ij} = \frac{3}{2\pi} \int_{\Omega} (\sigma_N N_{ij} + \sigma_L L_{ij} + \sigma_M M_{ij}) \, d\Omega \quad (1)$$

This is an integral over a unit hemisphere Ω , in which σ_{ij} = stress tensor, $\sigma_N, \sigma_L, \sigma_M$ = normal and two shear components of the microplane stress vector, and N_{ij}, L_{ij}, M_{ij} = geometrical coefficients defined later. This integral is evaluated approximately according to an optimum Gaussian integration formula as a weighted sum over all the discrete microplanes whose normals are chosen to correspond to the integration points of this formula.

Compared to the classical tensorial constitutive models based on tensorial invariants, the microplane concept has a number of advantages: (1) It is conceptually simpler, since the strain vectors on the microplanes can be intuitively related to crack opening and slip. (2) The orientation of cracking and slip can be captured. (3) The so-called vertex effect is automatic, while being beyond the capability of the tensorial constitutive models. (4) Apparent deviations from normality in the sense of tensorial plasticity models are represented, thanks to the fact that the microplane model is equivalent to a large set of simultaneously active yield surfaces intersecting at the same point of the stress space, for each of which the normality rule can be satisfied (this is particularly important for dynamic loading, with highly nonproportional loading paths). (5) The kinematic constraint of the microplanes of different orientation automatically simulates the cross effects such as the shear dilatancy and pressure sensitivity. (6) Combinations of loading and unloading on different microplanes provide a complex path dependence and automatically reproduce the Bauschinger effect and the hysteresis under cyclic loading. (7) The dependence of the current yield or strength limits on the strain components (rather than on scalar hardening–softening parameters) is easy to take into account. (8) In cyclic loading, fatigue is automatically simulated by accumulation of residual stresses on the microplanes after each load cycle. (9) Finally, though not important for concrete, anisotropy can be easily captured.

Early on, the microplane model was computationally too demanding for full structures. But thanks to the rise of computer power, the microplane model is now being used in systems with tens of millions finite elements. Many progressively improved versions of the microplane model, now labeled M1, M2, ..., M7, have been developed for concrete, and so have other variants for fiber-reinforced concrete, fiber–polymer composites, sandstone, shale, clay, rigid foam, shape memory alloy and some soft tissues. Model M4 (Bažant et al., 2000b; Caner and Bažant, 2000) was used with success in large-scale simulation of various explosions and groundshock, and with partial success in missile impact and penetration of concrete walls. The new model M7 (Caner and Bažant, 2013a,b), used and refined in this work, brings about further significant improvements. It eliminates unrealistic lateral strains in postpeak tensile softening, gives damage dependent unloading and works even for cyclic loading with softening.

Let us now briefly review the basics of microplane formulation. According to the kinematic constraint, the normal component of the strain vector on a generic microplane of orientation defined by its unit normal n_i is

$$\epsilon_N = n_i n_j \epsilon_{ij} = N_{ij} \epsilon_{ij} \quad (2)$$

where subscripts $i, j = 1, 2, 3$ refer to Cartesian coordinates x_i (Fig. 1). The in-plane shear strain vector on each microplane is represented by its two in-plane orthogonal components in the directions of unit in-plane coordinate vectors $\vec{m}$ and $\vec{l}$ which are generated randomly on each microplane. These components are (Bažant et al., 2000b; Bažant and Oh, 1985)

$$\epsilon_M = \frac{1}{2} (n_i m_j + n_j m_i) \epsilon_{ij}; \quad \epsilon_{ij} = M_{ij} \epsilon_{ij}; \quad \epsilon_L = \frac{1}{2} (n_i l_j + n_j l_i) \epsilon_{ij} = L_{ij} \epsilon_{ij} \quad (3)$$

To be able to model the pressure-sensitive compressive behavior of concrete, it is necessary to split the microplane normal strains and stresses into their volumetric and deviatoric parts

$$\epsilon_N = \epsilon_V + \epsilon_D; \quad \sigma_N = \sigma_V + \sigma_D \quad (4)$$

where $\epsilon_V = \epsilon_{kk}/3$ and $\epsilon_D = (N_{ij} - \delta_{ij}/3) \epsilon_{ij}$; σ_V and σ_D are computed from ϵ_V and ϵ_D by the microplane constitutive law. In contrast to previous microplane models, this split is in M7 introduced only for inelastic compressive behavior, and not for the elastic part of response nor inelastic tensile behavior.

The inelastic softening behavior in microplane model best characterized by means of separate strength limits, or stress-strain boundaries, on the microplane stress components. Within the boundaries, the behavior is considered to be elastic, and it should be noted that in spite of that the macroscopic response represents the pre-peak nonlinearity because different microplanes enter the strain softening behavior at different moments of loading. The inelastic behavior, defined on the microplanes in terms of σ_N in tension and in terms of σ_V and σ_D in compression, can simply be specified as functions of the corresponding ϵ_N , ϵ_V and ϵ_D , while the shear stress components σ_M and σ_L are limited by boundaries depending on σ_N and ϵ_V .

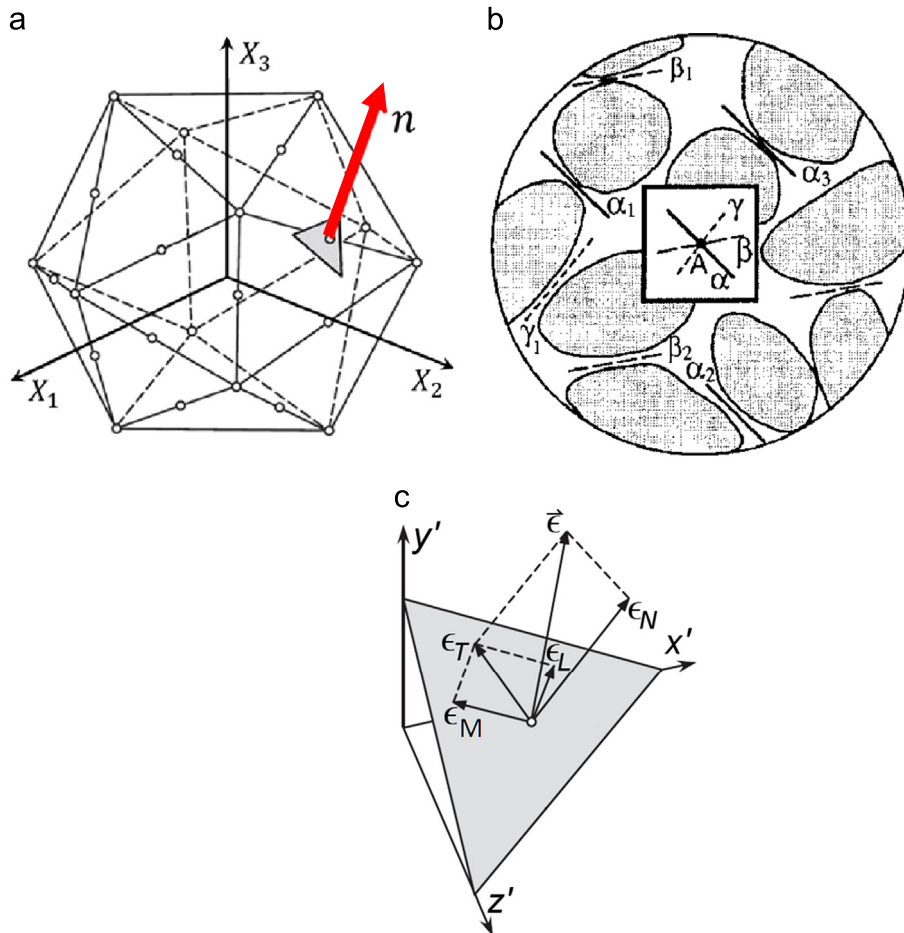


Fig. 1. (a) One Gauss point for each microplane and a typical normal at a microplane, (b) the concrete mesostructure with failure planes between aggregates modeled as microplanes, (c) the kinematic constraint giving rise to the microplane normal and shear strain components.

Accordingly, the tensile strength limit for normal microplane stresses, which governs tensile fracturing, can be defined as (see Fig. 2a and Eq. (23) in [Caner and Bažant, 2013a](#))

$$\sigma_N^+ = \mathcal{F}_N(\epsilon_N, \sigma_V) \quad (5)$$

The normal boundary, which governs the compressive failure, is given by

$$\sigma_N^- = \sigma_V + \sigma_D = \mathcal{F}_V(\epsilon_V, \epsilon_I, \epsilon_{III}) + \mathcal{F}_D(\epsilon_D, \epsilon_V) \quad (6)$$

see Eqs. (13)–(14) and (16)–(18) in [Caner and Bažant \(2013a\)](#), and Fig. 2b and c, respectively; ϵ_I and ϵ_{III} are the maximum and minimum principal strains.

The microplane frictional yield surface is expressed as (see Fig. 2d and Eqs. (26)–(32) in [Caner and Bažant, 2013a](#))

$$\sigma_\tau = \mathcal{F}_T(\epsilon_\tau, \epsilon_V, \sigma_N), \quad \epsilon_\tau = \sqrt{\epsilon_L^2 + \epsilon_M^2}, \quad \sigma_\tau = \sqrt{\sigma_L^2 + \sigma_M^2} \quad (7)$$

Based on extensive calibrations by test data for many concretes, the stress–strain boundaries are in M7 characterized by five free parameters k_i ($i=1, 2, \dots, 5$), which can be easily adjusted according to basic data from uniaxial and triaxial compression tests and uniaxial tensile tests. Besides, the definitions of the boundaries include about twenty fixed parameters, which can be considered the same for all concretes. They have been identified by computer optimization of the fits of numerous data of different types. For details, see [Caner and Bažant \(2013a,b\)](#).

Based on the given elastic modulus E of concrete and given Poisson ratio ν , the kinematic constraint without volumetric–deviatoric split leads to the following values of the microplane normal and shear stiffness constants (Eq. (20a), (20b) in [Bažant and Prat, 1988](#))

$$E_N = \frac{E}{1-2\nu}, \quad E_T = E_N \frac{1-4\nu}{1+\nu} \quad (8)$$

(also $E_N = K/3$ where K =bulk modulus). Obviously since both E_N and E_T must be non-negative, only Poisson's ratios in the range $\nu \in [-1, 0.25]$ can be reproduced. This range of ν is sufficient for concrete, for which $\nu \approx 0.18$. But it would not suffice

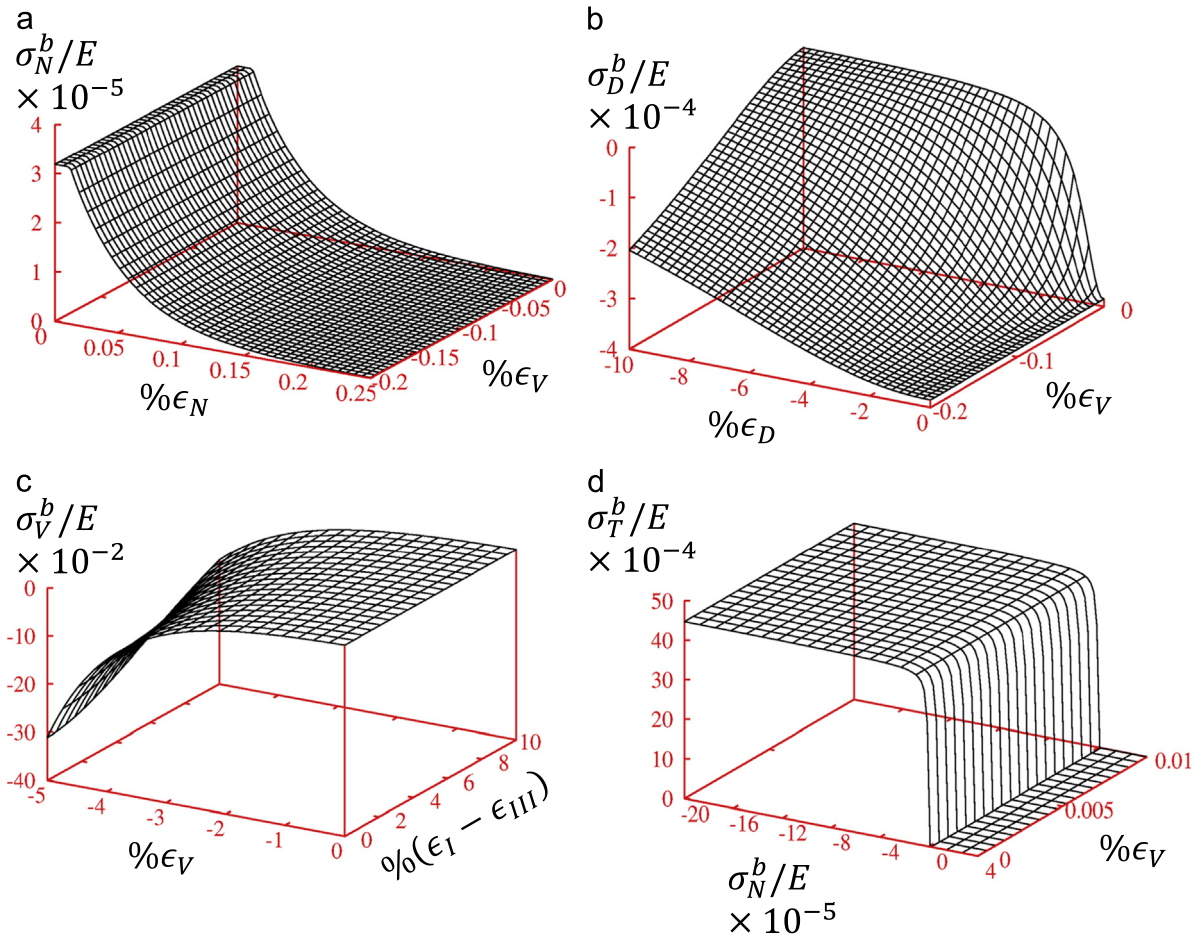


Fig. 2. (a) Microplane normal boundary, (b) microplane deviatoric boundary, (c) microplane volumetric boundary and (d) microplane frictional boundary.

for a general material, for which $\nu \in [-1, 0.5]$. In that case, to get the full range of ν , the microplane model M7 must be coupled in series with an isotropic shear-deformable elastic element of an infinite bulk modulus $K' = \infty$ and a finite shear modulus $G' > 0$ (Bažant and Oh, 1985) (see Fig. 2 in Caner and Bažant, 2013a; Caner et al., 2013).

The thermodynamic requirement of non-negative density of energy dissipation is discussed in detail in Caner and Bažant (2013a).

3. Calibration of model M7 by WES-5000 test data

Numerous earth penetration simulations in the literature have used the Eulerian coordinates, which are the coordinates of the current state and do not allow keeping the memory of the initial virgin state of the material. This approach is suitable for plastic materials. However, for brittle or quasibrittle materials, and particularly for the microplane model of concrete, memory of the initial virgin state is essential. Therefore, the updated Lagrangian approach, in which the material points are characterized by their coordinates in the initial state, must be used. Calculations show that the predicted geometry of the exit crater, as well as the deceleration of projectile prior to exit, depend strongly on the softening damage behavior of the material, whose characterization depends on the initial virgin state.

Given the complexity of concrete behavior, the constitutive law has been verified against a large set of experimental data covering virtually all the distinct types of experiments that characterize concrete, whose number is about 20 (Caner and Bažant, 2013b). A complicating feature of verification is that the test data used to verify Model M7 were obtained by numerous researchers in different laboratories, using different concretes. For calibration of model M7 for WES-5000 concrete, not only classical laboratory test data such as uniaxial and triaxial compression data, but also the test data obtained under unconventional loading paths (Cargile, 1999) have been employed, as shown in Fig. 3.

The same free model parameters, namely $E = 25$ GPa, $k_1 = 11 \times 10^{-5}$, $k_2 = 110$, $k_3 = 30$, $k_4 = 100$ and $k_5 = 1 \times 10^{-4}$, have been used for all the present data fits. Although in two cases the simulations show significant differences from the test data, the fit of a great majority of the simulations is good. Model M7 has also been verified by comparisons with the tests of compression–tension load cycles, the tests of mixed-mode crack propagation (see Figs. 4, 5, 6 and 7 in Caner and Bažant, 2013b), and the tests of explosions on plain concrete slabs.

The fits are performed by taking into account the meaning of the coefficients in terms of various particular aspects of the mechanical behavior of concrete. This kind of fitting procedure is more tedious than the automatic procedures such as the Levenberg–Marquardt optimization algorithm or the genetic algorithm but the resulting optimum values of the parameter preserve the robustness of the model in simulating general stress states. Complex automatic fitting procedures often yield parameter values that correspond to very close fits but almost always such parameter values render the model unstable in simulations of multiaxial states of stress. This is because it is not entirely possible to convey the meaning of each model parameter to the automatic fitting procedure.

4. Data from missile penetration tests

Slabs made of normal strength concrete, called the WES-5000 concrete, were perforated by projectiles at the Geotechnical and Structures Laboratory of the U.S. Army Engineer Research and Development Center (ERDC), Vicksburg (Frank et al., 2012). The slabs were circular, of thicknesses of 127, 216 and 254 mm, and were cast in steel culvert pipes of diameter of 1.52 m, two or three slabs for each thickness. The exit velocities were averaged. The projectiles had an ogive nose, weighed 2.3 kg and hit the slab with the entry velocity of 310 m/s and at an impact angle of 90° (with deviations of $< 1^\circ$). The measured exit velocities of the projectiles were 225 m/s, 115 m/s and 45 m/s for the three slab thicknesses, respectively (Frank et al., 2012; Cargile, 1999). The geometries of the entry and exit craters were determined after the tests. Slabs of three different thicknesses were penetrated. As seen in finite element simulations, the strain rates in the thinnest slabs remained very high throughout the entire duration of the penetration. In the thickest slabs, the strain rates were very high initially but became only moderately high later on.

To obtain material data, standard cylindrical specimens of the same WES-5000 concrete were tested under both conventional and unconventional triaxial load paths (Caner and Bažant, 2000, 2013a). Frank et al. (2012). Various material models were calibrated with these quasistatic test data, and subsequently used to calculate the exit velocities as well as the entry and exit crater shapes in all three slabs. Despite using a variety of models, it turned out to be very challenging to match the test results (Frank et al., 2012).

5. Quasi-static rate effect in Model M7

As is clear from the preceding Part I (Bažant and Caner, this issue), there are two kinds of deformation rate effects—quasistatic and comminutive. The quasistatic rate effects are expected on theoretical grounds to extend to extreme rates and so they must be part of the material model even for such rates. Besides, even in projectile impact and perforation, the zones farther away from the missile are not deforming at extreme rates and exhibit only the quasistatic rate effects which play a role in wave attenuation and diffraction, and the same is true when the projectile forms the exit crater at greatly reduced velocity.

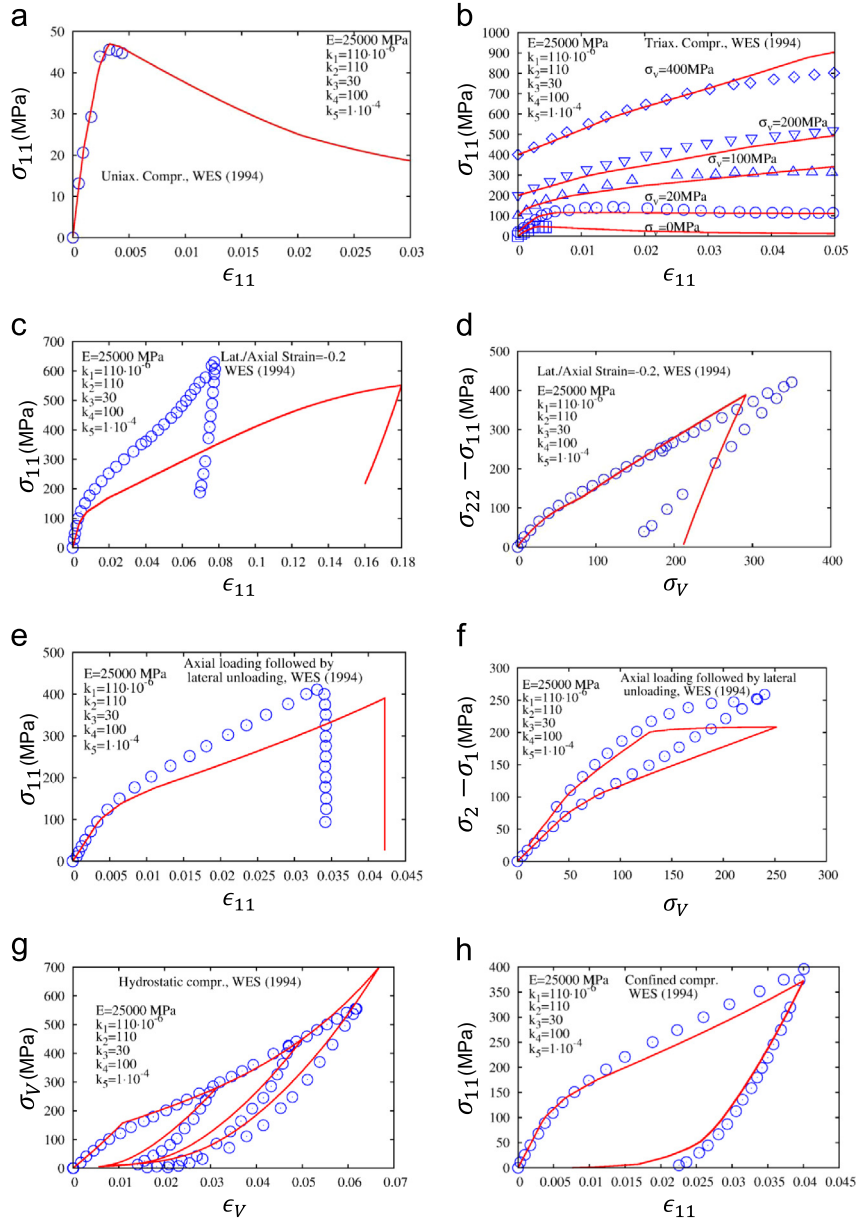


Fig. 3. Benchmark set of test data obtained with one and the same concrete, the WES-5000 concrete, used to evaluate the model and optimize material parameters.

Another way to generate rate effects in fracture modeling is through dynamic crack propagation and crack branching (e.g. Doyoyo, 2002). This approach has been pursued for metals and ceramics (see comments on the “Mescall” models in Part I). But it has not been elaborated for concrete and has not been developed in the form of a macro-continuum constitutive equation for finite element programs. In a general sense, though, the present comminution model and the dynamic crack branching are similar in that both simulate creation of small particles. The present model can in fact be regarded as an alternative to creation of a Mescall zone.

5.1. Effect of cohesive crack separation rate

One quasistatic rate effect, already introduced into microplane model M4 (Bažant et al., 2000a), is due to the rate dependence of cohesive crack opening w . It must follow the activation energy theory of bond ruptures. In this formulation, the rate dependence of the opening w of a cohesive crack (also called ‘fictitious’ crack) is described as (see Fig. 1a in Bažant

and Li, 1997 and Fig. 1 in Li and Bažant, 1997)

$$\dot{w} = k_0 \sinh\left(\frac{T_0}{T} \frac{\sigma - f^0(w)}{k_r N_b(w)}\right) e^{-(Q/k)(1/T - 1/T_0)} \quad (9)$$

where $\dot{w} = dw/dt$, t =time; σ =cohesive (crack-bridging) stress; k_0, k_r, T^0, Q, k =constants (k =Boltzmann constant; Q =activation energy of interatomic bond ruptures, T =absolute temperature; T_0 =given reference temperature, and $N_b(w)$ =number of surviving bonds spanning the cohesive crack, per unit area. Function $f^0(w)$ describes the softening law of the cohesive crack at an opening rate that is at the lower limit of the range of rates for which Eq. (9) is to be applied or has been calibrated (normally the rate of loading in static material tests in the laboratory). The adiabatic heating of concrete in an impact event has probably a negligible effect on $\dot{w}$, and in that case one may set $T \approx T_0$. Eq. (9) can be imagined to correspond to the rheologic model in which a rate-independent cohesive crack element is coupled in parallel with a nonlinear damper at each point of the cohesive crack (see Fig. 1b in Li and Bažant, 1997; Bažant et al., 2000a). The sinh-function in Eq. (9) ensues from the activation energy theory (rate-process theory, e.g. Krausz and Krausz, 1988) for bond ruptures.

The ratio $\sigma/N_b(w)$ represents the transmitted stress per bond, which is what matters for the activation energy theory. Obviously, N_b must decrease with increasing crack opening w and must drop to 0 when w becomes so large that $\sigma = f(w) = 0$. It has been assumed that

$$N_b(w) = C_b f^0(w) \quad (10)$$

where C_b is some proportionality constant.

The macroscopic strain softening may be imagined to be the result of the openings of many parallel cohesive cracks. Denoting by s_{cr} their average spacing, one may write

$$\dot{\epsilon} = \frac{\dot{w}}{s_{cr}} + \frac{\dot{\sigma}}{E} \approx \frac{\dot{w}}{s_{cr}} \quad (11)$$

where ϵ =average macroscopic normal strain in the direction perpendicular to the parallel cracks, E =Young's modulus of uncracked material; $\dot{\sigma}/E$ represents the elastic strain rate, which is normally negligible by comparison if the crack is opening. It is by virtue of this observation that the approximation in Eq. (11) is justified.

Substituting $\dot{w} = s_{cr} \dot{\epsilon}$ and Eq. (10) into Eq. (9) and solving the equation for σ , one obtains $\sigma = F(\epsilon)$ where

$$F(\epsilon) = \mathcal{F}(\epsilon) \left[1 + C_2 \operatorname{asinh}\left(\frac{\dot{\epsilon}}{C_1}\right) \right] \quad (12)$$

with

$$C_1 = C_0 e^{-(Q/R)(1/T - 1/T_0)}, \quad C_2 = C_r T/T_0, \quad \dot{\epsilon} \geq 0 \quad (13)$$

Here $C_0 = k_0/s_{cr}$, $C_r = k_r C_b$ and $f^0(w) = \mathcal{F}(\epsilon)$; C_0 and C_r are constants to be determined from tests. When the temperature is about the same as the temperature in the laboratory tests used for calibration, which is about 25 °C ($T_0 = 298$ K), then $C_1 = C_0$ and $C_2 = C_r$, and C_1 and C_2 become temperature independent, i.e., constants.

Since $\operatorname{asinh} x = \ln(x + \sqrt{x^2 + 1})$, the asymptotic approximation for $x^2 \gg 1$ is $\operatorname{asinh} x \approx \ln(2x)$. In Eq. (12), this occurs if the loading rate is so large that $\dot{\epsilon}^2 \gg C_1$. Then Eq. (12) takes the form

$$F(\epsilon) \approx \mathcal{F}(\epsilon) \left[1 + C_2 \ln\left(\frac{2\dot{\epsilon}}{C_1}\right) \right] \quad (14)$$

The logarithmic function is normally used for the rate effect on the yield limit in metal plasticity (e.g., in wavecode EPIC).

Function $F(\epsilon)$ may be interpreted as the stress–strain boundary on the microplane corresponding to strain rate $\dot{\epsilon}$, and $\mathcal{F}(\epsilon)$ has the meaning of the static stress–strain boundary that corresponds to a vanishing strain rate (a boundary that approximately applies to the small loading rates of static material tests). The transformation from the static boundary $\mathcal{F}(\epsilon)$ to the rate-dependent (dynamic) boundary represents, according to Eq. (12), a vertical scaling of the boundary curve (see Fig. 1c in Bažant et al., 2000a).

It is logical to assume that the normal and deviatoric boundaries adhere to Eq. (12). Thus, the following relations can be written on the microplane level:

$$F_N = \mathcal{F}_N [1 + C_{R2} R(\dot{\xi})] \quad (15)$$

$$F_D = \mathcal{F}_D [1 + C_{R2} R(\dot{\xi})] \quad (16)$$

$$F_\tau = \mathcal{F}_\tau [1 + C_{R2} R(\dot{\xi})] \quad (17)$$

where, however, $\operatorname{asinh}(\dot{\epsilon}/C_1)$ is replaced by function $R(\dot{\xi})$ of global strain rate measure $\dot{\xi}$, which satisfies the tensorial invariance restrictions. The following simple definition is used:

$$R(\dot{\xi}) = \operatorname{asinh}(\dot{\xi}/C_{R1}) \approx \ln(2\dot{\xi}/C_{R1}) \quad \text{with } \dot{\xi} = \sqrt{\frac{1}{2} \dot{\epsilon}_{ij} \dot{\epsilon}_{ij}} \quad (18)$$

where the logarithmic approximation of R is admissible only if $\dot{\xi}^2 \gg c_{R1}$; c_{R1} , c_{R2} are material rate constants, analogous to C_1 , which have been calibrated by test data. They can be considered as fixed parameters, which need not be adjusted by the user and are applicable to all concretes; $\dot{\epsilon}_{ij}$ are strain rate components with subscripts i, j ($= 1, 2, 3$) referring to Cartesian components, and repetition of subscripts implies summation; and $\dot{\xi}$ is a non-negative invariant of $\dot{\epsilon}_{ij}$ (note that the square root of the second invariant of the strain rate tensor would be inappropriate because the cracking strain involves volume change).

5.2. Viscoelastic rate effect

The viscoelasticity (or creep) of concrete is properly taken into account at the microplane level. The creep is fully characterized by the compliance function $J(t, t')$, representing the strain at time t caused by a unit uniaxial stress applied at time t' , and by the creep Poisson ratio ν , which can be approximately taken as constant ($\nu = 0.18$). The constitutive law, based on the principle of superposition, consists of a matrix integral equation over the stress tensor history. For the analysis of impact and penetration, the aging of concrete may be neglected. In that case $J(t, t') \approx J(t - t')$ = function of only the time lag $t - t'$. Computationally it is more effective to approximate the integral-type constitutive law with a Kelvin chain or Maxwell chain. This leads to a system of first-order matrix differential equations in time for the partial strains or stresses of the chain, whose values from the preceding time step need to be stored.

Since long-time creep is not of interest here, the entire Maxwell chain is not needed. Therefore (in similarity to the approach of [Özbolt and Bažant, 1992](#)), the compliance function may be approximated for the duration of loading (e.g., the impact event) by a single spring-dashpot Maxwell rheologic model (see Fig. 1d in [Bažant et al., 2000a](#)). For uniaxial stress, this model is characterized by the stress strain relation:

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \quad (19)$$

where E = elastic modulus and η = viscosity.

Assuming that, during each time step Δt , the strain rate $\dot{\epsilon}$ is constant, the solution of Eq. (19) within the time step beginning at time t_i is

$$\sigma(t) = \eta \dot{\epsilon} + (\sigma_i - \eta \dot{\epsilon}) e^{-E(t-t_i)/\eta} \quad (20)$$

Setting $t = t_i + \Delta t$ and $\dot{\epsilon} \approx \Delta \epsilon / \Delta t$, this may be rearranged to the following quasi-elastic incremental stress-strain relation

$$\Delta \sigma = E'' \Delta \epsilon - \Delta \sigma'' \quad (21)$$

with the notations

$$E'' = \frac{1 - e^{-\zeta}}{\zeta} E, \quad \Delta \sigma'' = (1 - e^{-\zeta}) \sigma, \quad \zeta = \Delta t / \tau_1 \quad (22)$$

where σ is the value at the beginning of the time step Δt , and $\tau_1 = \eta/E$ = relaxation time of the Maxwell model. Eq. (22) define the exponential algorithm for linear viscoelasticity ([Bažant, Chairman, 1988](#)). Unlike the central difference algorithm, this algorithm is unconditionally stable no matter how large the ratio ζ is. Note that, in computer calculations, an overflow may occur in Eq. (22) when ζ is very small. To prevent it, the first terms of the Taylor series expansions of E'' and $\Delta \sigma''$ must be used for very small ζ .

5.3. Discussion of quasi-static rate effects

Numerous test data on the apparent unconfined compressive strength under uniaxial compression of various concretes show that, with increasing strain rate, the apparent strength increases sharply (see Fig. 1 in [Cotsovos and Pavlović, 2008](#)). These data can be predicted well up to a strain rate of about 1 s^{-1} by model M7 with quasi-static rate effect, and also by other models that incorporate the activation energy based rate effect. But at higher strain rates, a proper comparison of model predictions with experimental results would require the strain rate distributions across the specimen. However, it is next to impossible to record these distributions in experiments at very high rates of loading. The numerical model predictions of what is left of the specimen body can also be compared to the remains of the specimens tested. For example, the so-called lattice discrete particle models (e.g. [Cusatis et al., 2003, 2011a,b](#)) can predict the mesoscale fragmentation behavior of the specimens during high strain rate tests, though not yet the comminution on a finer scale. Continuum models such as model M7 can be used to simulate approximately the effects of both the mesoscale fragmentation and finer scale comminution, for example, by deleting the failed elements and lumping the deleted element masses to rigid particles at the nodes. The geometric shape that consists of the surviving elements at the end of the simulation can be compared to the surviving specimen parts from the experiments.

The confined compression behavior at high strain rates of concrete is of interest for perforation of concrete slabs by projectiles as well as explosions. [Forquin et al. \(2008\)](#) reported Hopkinson pressure bar tests of concrete “MB50” cast in a steel tube. In these tests, shock waves are generated to create the desired strain rate in the concrete core of the specimens. To calibrate model M7, only the Young modulus has been adjusted to $E = 70 \text{ GPa}$, while the other model parameters have been considered with their reference values reported in [Caner and Bažant \(2013b\)](#). The values of quasistatic strain-rate

parameters were determined as $C_{R1} = 4 \times 10^{-6}$ and $C_{R2} = 0.022$. Fig. 4a depicts the hoop strain test data measured by strain gauge G3 and their predictions by model M7. The predictions by model M7 are completely satisfactory.

Next, using model M7 calibrated for the “MB50” concrete used in these tests, the axial stress–strain response of that concrete to high strain rates is extracted, as shown in Fig. 4b. The responses for the strain rates 141 s^{-1} and 221 s^{-1} are visually indistinguishable, meaning the continuous curve in that figure is probably the maximum response that can be obtained for such concretes cast in steel tubes using the activation energy based rate effect theory. The dashed curve in this figure corresponds to quasi-static laboratory strain rates. These results indicate that, when the inertial effects are minimized, a relatively small strain rate effect exists in the response of concrete under confinement even when the strain rates are significantly larger than the quasi-static rates.

6. Commminution rate effect in model M7 and simulations of missile penetration

The sharp increase of the strength amplification factor for strain rates above 10 s^{-1} (e.g. see Fig. 1 in Cotsovos and Pavlović, 2008 or Fig. 1 in Bažant and Caner, this issue) documents that good prediction of the exit velocities of projectiles is hardly possible using only quasi-static strain rate effects. This is documented by Fig. 3 showing the calculation results for microplane model M7 calibrated for the WES-5000 concrete. To resolve the strain rates and keep the discretization error low, small enough tetrahedral elements (of size 7.5 mm) have been employed in the finite element analyses. Both the compression and tension performance of these elements employing the M7 constitutive model have been verified.

A series of finite element calculations of projectile perforation have been performed using model M7 with only the quasi-static strain rate effects. The results are shown as the dashed curve in Fig. 5. The calculated exit velocities far exceed the experimental values. Thus, the quasi-static rate effects are clearly insufficient to explain the observed exit velocities of these projectiles.

It should be mentioned that these velocities could roughly be obtained by calculations with an enhanced friction coefficient. However, assuming a higher friction coefficient for high slip rates is questionable since other experimental

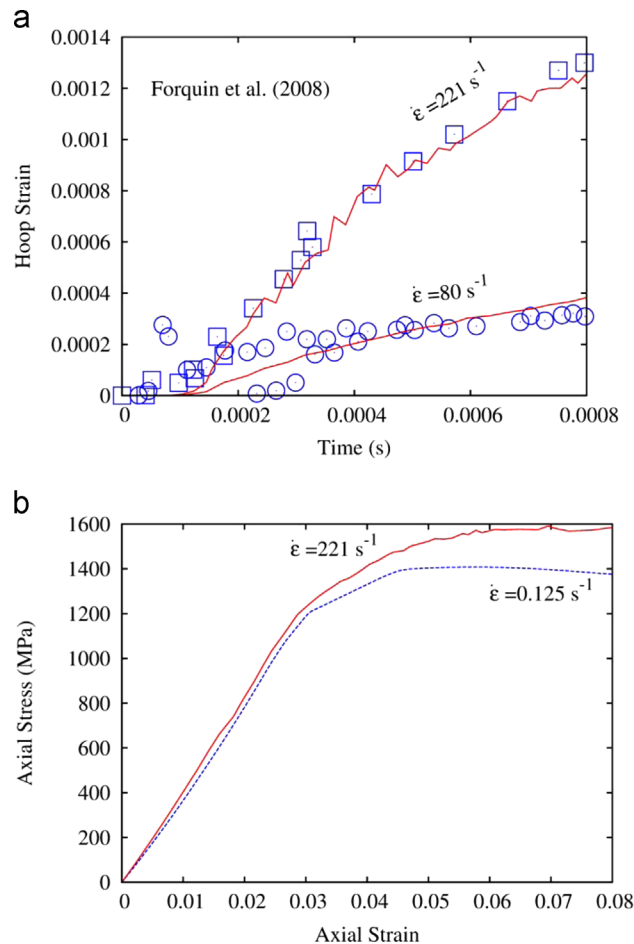


Fig. 4. (a) Comparison of hoop strain test data at strain rates of 80 s^{-1} and 221 s^{-1} of MB50 concrete cast in steel tubes (symbols) and predictions by the model M7 (lines); (b) axial stress–strain behavior of MB50 concrete at quasistatic strain rates (dashed line) and at a strain rate of 221 s^{-1} (continuous line).

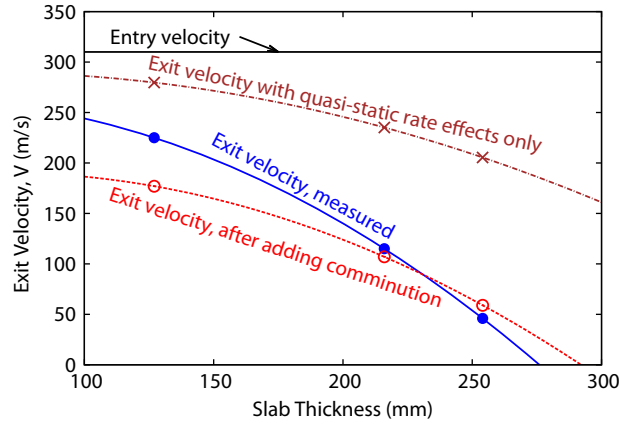


Fig. 5. Comparison of measured exit velocities and predictions by model M7 using only quasi-static strain rate effect and both quasi-static strain rate effect and effect of comminution of concrete.

evidence shows the friction coefficient to decrease with an increasing slip rate. Also, a blunter nose profile different from the one used in the tests could help to reduce the penetration velocity.

It was also examined whether the effect of the frictional boundary, which can greatly increase with the confining pressure, could suffice to explain the drop of exit velocities for thicker walls. However, the calculated confining pressure was not high enough. Actually, in absence of large enough pressure on the missile sides, the dynamic sliding leads to a reduction of friction. Thus, although the rate effect is here applied also at the microplane frictional boundary, it is ineffective in reducing the missile speed and cannot explain the observed exit velocities.

In Frank et al. (2012), the entry and exit crater geometries, as well as the exit velocities of the projectiles, were matched by imposing on microplane model M4 a major empirical expansion of the static microplane stress–strain boundaries (or strain-dependent yield limits) determined so as to fit the penetration data. The expansion was both vertical and horizontal. Although the vertical expansion of these microplane yield limits might be explained by the comminution effect (coupled with the quasi-static rate effects), a horizontal expansion of these boundaries can hardly be justified theoretically. In fact, a horizontal expansion of the boundaries implies an increase of ductility, but the existing experimental evidence indicates that, as the strain rate increases, concrete becomes more brittle. Still more importantly, the fracture energy is known not to change much with the strain rate (Jansen et al., 1995; Yon et al., 1992).

Another significant parameter of the perforation of concrete slabs is the shape of the projectile tip. Intuitively, a blunt projectile would be less effective than a sharp one in perforating slabs. The projectile tip has been specified in Frank et al. (2012) as an ogive nose with a caliber radius head of 3.0 and a length to diameter ratio of 7. The diameter of the projectile was 50.8 mm. For the definition of the caliber radius head, the formula given in Simms and Berger (2010) can be used. More details of the dimensions of the projectile are given in Cargile (1999).

The impossibility of predicting the correct exit velocities of the projectiles with the same quasi-static rate-dependent microplane constitutive law and the same element sizes revealed that there must be a different type of rate effect that dissipates far more energy than the quasi-static rate effect. In the preceding companion paper (Bažant and Caner, this issue), it is postulated that this rate effect arises from the comminution of material into very fine particles at extremely high strain rates. This comminution rate effect has been implemented as a nonlinear viscous element (or dashpot) which is coupled in parallel with the microplane model and generates the shear viscosity

$$\eta_D = \frac{\Gamma}{\tau_0} \left(\frac{C_0^{2/3} G_p}{C_g} \frac{B_a}{1 + B_a^2} \right)^{1/2} \quad (23)$$

leading to the comminution induced shear stress tensor given by

$$\tau_{ij} = \eta_D \dot{\epsilon}_{ij} \quad (24)$$

where $\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij} - \delta_{ij} \dot{\epsilon}_{kk}$ and η_D = shear viscosity; B_a is the dimensionless number characterizing the ratio of kinetic energy of shear strain rate to the maximum possible strain energy, defined in Part I along with C_0 and G_g .

Fig. 5 shows, by dashed curves with circles, the exit velocities of projectiles predicted by three different versions of model M7:

- (1) basic M7 for quasi-static multiaxial behavior of concrete;
- (2) M7 enhanced by the quasi-static strain rate effects (which include the crack opening rate and viscoelasticity); and
- (3) M7 with the comminution rate effect and the quasi-static rate effects.

The exit velocity predictions which include the effect of comminution are clearly superior.

Model M7 including all of these rate effects has been coded into VUMAT subroutine of ABAQUS. The finite element analysis of the perforation of concrete walls was performed in the sense of crack band model. In these perforation simulations, the slabs are discretized uniformly using 4-node constant strain tetrahedral elements of size approximately 7.5 mm. The effect of switching to higher-order elements has been found negligible, in both crater formation and exit velocities, except for an increase in the computational cost of each simulation.

The boundary effects, such as wave reflection at the boundary of the mesh, have been removed in some of the simulations by attaching a single layer of infinite (or wave transmitting) finite elements at all exterior boundaries. But the removal of these boundary effects has resulted in negligible changes in both crater formation and exit velocities. By contrast, in explosion simulations, the boundary effects are known to play a crucial role in the degree of damage inflicted upon a concrete slab.

The crater geometries in three dimensions and a half of the projectile are shown in Fig. 6a–c for slab the thicknesses of 127 mm, 216 mm and 254 mm respectively. The thick black line represents the measured profile of the craters (see Fig. 3 in Frank et al., 2012). In the simulations, the elements whose maximum principal strain reached 0.005 or more have been eroded so that the computer simulation could proceed (otherwise the maximum time increments allowed by numerical stability would become too short and the computer simulation would abort). This threshold of element erosion corresponds

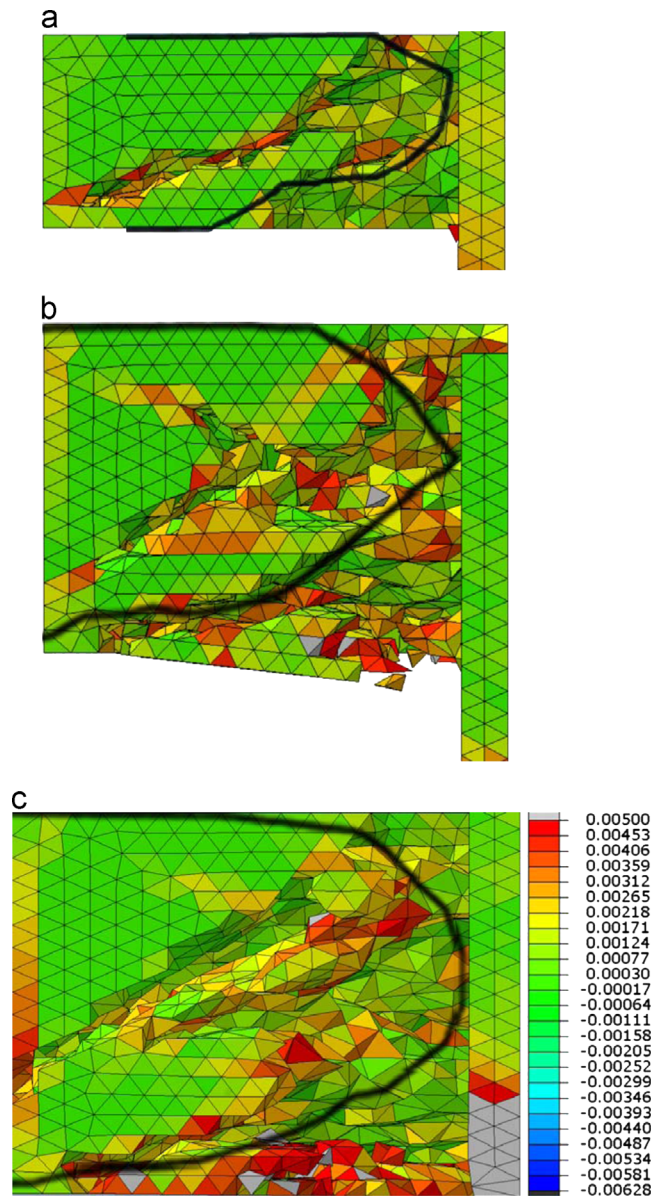


Fig. 6. Comparison of measured crater shapes (thick black curve) and predictions by model M7 using both quasi-static strain rate effect and effect of comminution of concrete in slabs of thickness (a) 127 mm, (b) 216 mm, (c) 254 mm and the legend for max. principal strains.

to material failure in the direction of the maximum principal strain under quasi-static loading conditions. Failure is assumed even though some of these eroded elements might still be carrying some compressive stress perpendicular to the maximum tensile stress direction.

7. Uncertainties and errors due to possible effects of moisture content, age, curing, etc.

The overestimation of the exit velocity of the thin slabs might not be an error of the present theory. It might be caused by various extraneous factors. For example, since the thinner slabs dry faster, the thin slab must have had a lower specific moisture content than the thicker one if the curing type and durations were about the same. Hopkinson bar tests reported in [Cadoni et al. \(2001\)](#) showed that a lower moisture content of concrete can lead to a significant decrease of strength under high-rate loading. This increase might explain why the measured exit velocity for the thin (and thus drier slab) is lower than the present prediction shown in [Fig. 5](#). However, the data are so limited that no meaningful calculations are possible at this time. Especially, it is not clear whether the slab were tested right after moist curing, or whether they were exposed to drying before the test, and whether the period of drying exposure was the same for each slab. The environmental humidity and the permeability of concrete are also not known.

There are further factors that could have skewed the result. From [Cargile \(1999\)](#) it is not clear whether the specimens had the same age at the time of the tests and whether they experienced the same environmental and curing history. For example, if the thinner slab was appreciably older than the thicker ones, the chemical process of hydration could have caused the strength and fracture energy of the thin slabs to be significantly higher than it was in the thicker slabs. Furthermore, according to [Cargile \(1999\)](#), the slabs were clearly not cast from the same batch of concrete, and thus random differences between different batches of the same concrete might have caused strength differences as high as 8%.

So it is at present impossible to conclude whether the overestimation of the exit velocity of the thin plate, seen in [Fig. 5](#), is an error of the present theory, or an error of the data themselves. Further experiments are needed.

8. Conclusions

1. If only the quasi-static rate effects due to crack opening rate and to viscoelasticity are taken into account, the observed exit velocities in the perforation of concrete slabs by projectiles cannot be matched by computer simulations.
2. Missile penetration can produce strain rates on the order of 10^5 s^{-1} . Such strain rates must be causing comminution of concrete resulting in dissipation of enormous energy.
3. Comminution of concrete at such extremely high strain rates can explain the additional dissipation needed to match the exit velocities of perforating projectiles.
4. The simulation results agree with the theoretical prediction that the comminution is equivalent to additional shear stresses of concrete proportional to the $2/3$ -power of deviatoric strain rate, which is in turn equivalent to a deviatoric (shear) viscosity proportional to the $-1/3$ -power of the deviatoric strain rate.
5. Including in the calculations a comminution due to rate of volume expansion has almost no effect on the exit velocities of the projectiles. Hence, the comminution must be explained in terms of the shear strain rate.
6. Including a strain-rate dependent friction between the projectile and the surrounding concrete cannot help to explain the additional dissipation needed to match the exit velocities.
7. The underestimation of the exit velocity of the thinnest slab might be explicable by taking into account the Hopkinson bar tests indicating that a drier concrete has a lower strength, and thus less resistance to missile penetration. However, the data are too scant for meaningful calculations. The underestimation could easily be explained by possible defects in the design and control of the experiments and by missing data on water content, age, curing, etc.
8. The microplane model M7 coupled with both the comminution rate effect and the quasi-static strain rate effects can correctly predict the geometries of the both the entry and exit craters and gives overall a satisfactory match of the observed exit velocities of the projectiles.

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Comminution of solids caused by kinetic energy of high shear strain rate, with implications for impact, shock, and shale fracturing

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Although there exists a vast literature on the dynamic comminution or fragmentation of rocks, concrete, metals, and ceramics, none of the known models suffices for macroscopic dynamic finite element analysis. This paper outlines the basic idea of the macroscopic model. Unlike static fracture, in which the driving force is the release of strain energy, here the essential idea is that the driving force of comminution under high-rate compression is the release of the local kinetic energy of shear strain rate. The density of this energy at strain rates $>1,000/\text{s}$ is found to exceed the maximum possible strain energy density by orders of magnitude, making the strain energy irrelevant. It is shown that particle size is proportional to the $-2/3$ power of the shear strain rate and the $2/3$ power of the interface fracture energy or interface shear stress, and that the comminution process is macroscopically equivalent to an apparent shear viscosity that is proportional (at constant interface stress) to the $-1/3$ power of this rate. A dimensionless indicator of the comminution intensity is formulated. The theory was inspired by noting that the local kinetic energy of shear strain rate plays a role analogous to the local kinetic energy of eddies in turbulent flow.

fracture mechanics | dynamic fracture | shale gas | dimensional analysis

The previous studies of high-rate dynamic fracture of rocks, concretes, ceramics, composites, and metals have dealt mainly with the nucleation, propagation, and branching of dynamically propagating cracks, their interference with elastic or shock waves, and the mechanism of development of the zones of densely distributed fractures, called the Mescall zones (1–7). However, a comminution model in the form of a macroscopic constitutive equation that could be used in large dynamic finite element programs for global response of structures is apparently still unavailable.

Seeking such a constitutive model, we begin with the analysis of a simple idealized process in which the solid is comminuted to identical particles (Fig. 1). Among simple space-filling regular subdivisions in the plane of maximum shear, regular hexagons are the most likely because they give the smallest surface-to-volume ratio (Fig. 1A) and thus require the minimum energy to form. In the direction normal to the hexagons, we assume the particles to be prismatic.

Consider that, at a certain moment, the strain rate (shown in Fig. 1B as a displacement regarded as infinitesimal) becomes high enough for the kinetic energy of shear strain rate to suffice for creating the fractures and interface slips that separate the particles of as yet unknown size. As that happens, the particles release their local kinetic energy, slip against each other, and regain their original undeformed shape, while the particle centers conform to the same macroscopic velocity field (Fig. 1C).

For the sake of simplicity and clarity, we will first outline a 2D analysis of comminution in the plane of maximum shear strain rate denoted as $\dot{\epsilon}_D$, and leave the 3D generalization for later discussion. Before comminution or slip, the displacement velocities in the directions of current (Eulerian) coordinates x and y , whose origin is attached to the particle centroid, are $\dot{u} = \dot{u}_0 - \dot{\omega}y + \dot{\epsilon}_D y$ and $\dot{v} = \dot{v}_0 + \dot{\omega}x + \dot{\epsilon}_D x$, in which the superior dots denote time derivatives. After comminution or slip, these velocities become $\dot{u}^+ = \dot{u}_0 - \dot{\omega}y$

and $\dot{v}^+ = \dot{v}_0 + \dot{\omega}x$. Assuming particle symmetry with respect to axes x and y , we have $\int_A x dA = \int_A y dA = \int_A xy dA = 0$, and for the drop of kinetic energy of the hexagonal prisms per unit volume we obtain

$$\Delta \mathcal{K} = -\frac{h}{V_p} \int_A \frac{\rho}{2} (\dot{u}^2 + \dot{v}^2 - (\dot{u}^+)^2 - (\dot{v}^+)^2) dx dy = -c_k \rho h^2 \dot{\epsilon}_D^2. \quad [1]$$

Here A = particle area, ρ = mass density, $c_k = I_p / (2hV_p)$, $V_p = (3\sqrt{3}/8)h^3$ = particle volume, and $I_p = (\sqrt{3}/32)h^4$ = polar moment of inertia of each hexagonal prism of side h and length h . Note that the macroscopic velocities of material rotation ω and displacements u_0 and v_0 have no effect on $\mathcal{K}$, which means that the local kinetic energy density is separable from the kinetic energy density of the macroscopic 2D motion defined by the velocities of particle centroids.

It is interesting that the kinetic energy $\mathcal{K}_{\text{shear}}$ of a particle deforming by pure shear at rate $\dot{\epsilon}_D$ happens to be the same as the kinetic energy $\mathcal{K}_{\text{eddy}}$ of an eddy rotating as a rigid body of the same size at angular rate $\dot{\omega} = \dot{\epsilon}_D$ (Fig. 2). Thus, we see a partial analogy with turbulence (8), which is what inspired the present theory. In both comminution and turbulence, the microlevel kinetic energy augments the kinetic energy of the macrolevel motion. The microlevel kinetic energy is dissipated by fluid viscosity in the eddies of turbulent flow, or by the energy, Γ , of interface fracture or subsequent frictional slip. However, unlike an eddy, the local shear strain motion cannot continue indefinitely.

Significance

Fragmentation, crushing, and pulverization of solids, referred to as comminution, has long been of keen interest for mining, tunneling, explosions, meteorite impact, missile impact, ground shock, terrorist attacks, and various industrial processes. Recently, interest surged in comminution of gas or oil shale as a way to enhance the permeability of shale mass by orders of magnitude. Particularly intriguing is a proposed, environmentally friendlier, alternative to hydraulic fracturing in which the fracturing would be achieved by shock waves from explosions or electrohydraulic pulsed arc in a horizontal borehole. The discharge of contaminated water would, in this case, be negligible. In all these problems the energy dissipation density is a key parameter to predict. Here a theory to do so is outlined.

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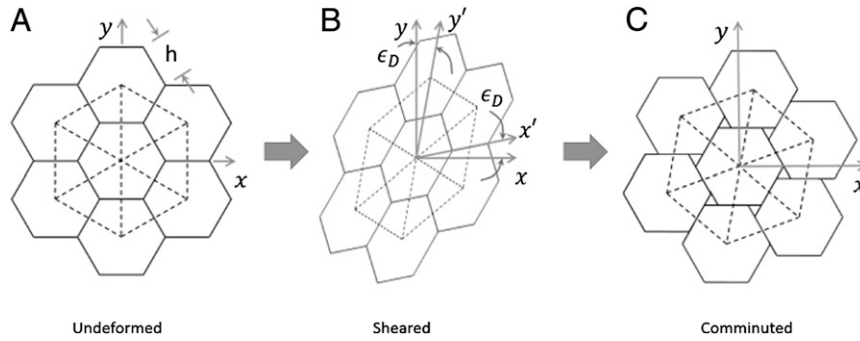


Fig. 1. Example of comminution of material into prismatic hexagonal particles; the velocities are shown as infinitesimal displacements (in which case the gaps at the hexagon corners are second-order small and thus negligible). (A) Undeformed material, (B) sheared material, and (C) comminuted material.

In reality, particle sizes h vary randomly, according to a certain distribution, for which the following cumulative distribution [Schuhmann's power law (9–12)] is adopted:

$$F(s) = \frac{s^k - h^k}{H^k - h^k} [s \in (h, H), F(s) \in (0, 1)]. \quad [2]$$

Here k = empirical constant ($k \approx 0.5$), s = variable particle size, and h and H = minimum and maximum particle sizes (usually $H/h = 10$ to 100). Because the macroscopic quasi-static constitutive law with nonlocalized strain softening includes the energy dissipation corresponding to material crushing into particles of the size d_a of the largest material inhomogeneities, such as the maximum aggregate size in concrete, H should be considered to be one order of magnitude smaller (i.e., about $0.1d_a$). The average particle size is $\bar{s} = \int_h^H s dF(s) = \bar{c}h$, the combined interface area per unit volume is $\bar{S} = C_s/h$, and the loss of kinetic energy of the shear strain rate of the particles of all sizes per unit volume is $\Delta\mathcal{K} = -\int_h^H c_k \rho s^2 \dot{\epsilon}_D^2 dF(s) = -C_k \rho h^2 \dot{\epsilon}_D^2$, where $\bar{c}$, C_s , and C_k are dimensionless constants. For $H/h = 100$ and for hexagonal or cubical particles, $C_s = 0.331$ or 0.300 , and $C_k = 92.60$ or 138.9 , respectively (the values for cubical particles are included to show that the particle shape does not make much difference).

Assuming that all of a kinetic energy decrement $\Delta\mathcal{K}$ is dissipated by an interface fracture energy increment $\Gamma\Delta S$, the energy balance condition is $\delta(\Delta\mathcal{K}) = \Gamma\delta S$; Γ = shear fracture energy or, after break, the work of friction per unit interface area (which is considered as constant because its dependence on the confining pressure and slip velocity is not known. So, the interface fracture (or frictional slip) can occur when

$$-\frac{\partial(\Delta\mathcal{K})}{\partial S} = -\frac{\partial(\Delta\mathcal{K})/\partial h}{dS/dh} = \Gamma. \quad [3]$$

After substitutions into Eq. 3 one gets

$$h = \left(\frac{C_a \Gamma}{\rho \dot{\epsilon}_D^2} \right)^{1/3}, \quad [4]$$

where C_a is a dimensionless constant. If $H/h = 100$, then $C_a = 0.032$ for hexagonal prisms and 0.019 for cubical particles. Again, the precise particle shape does not make a major difference.

It should be mentioned that Eqs. 4 and 3 are similar to Grady's (13) equation, derived and experimentally verified for tensile comminution caused by a high volumetric strain rate driven by explosion within a hollow sphere. Later, Grady (14) verified empirically (although not theoretically) that $h \propto \dot{\epsilon}_D^{-2/3}$ holds also for the impact of missiles, in which the role of volume expansion is

negligible. This fact serves as one experimental verification of the present theory (Eq. 4).

Note that, more generally, and specifically for a single dynamically propagating crack in two dimensions, the dynamic energy release rate involving both the elastic strain energy $\mathcal{U}$ and the kinetic energy with inertial effects was rigorously formulated in equations 5.3.2 and 5.3.20 of Freund's book (7). Eq. 3 may be regarded as a simplification of these equations to the case of negligible $\mathcal{U}$ and at the same time a generalization and adaptation to a smeared continuum representation of many diffuse cracks characterized by their combined surface S .

Substitution of Eq. 4 into Eq. 1 further yields

$$\Delta\mathcal{K} = -(C_0 \Gamma^2 \rho)^{1/3} \dot{\epsilon}_D^{2/3}, \quad [5]$$

where C_0 is a certain dimensionless constant; $C_0 = 822$ for hexagonal prisms and $1,013$ for cubical particles. This expression suggests how to implement the energy sink owing to comminution in macroscopic structural analysis. Note that $\Delta\mathcal{K}$ has the dimension of stress and can be interpreted as such.

To obtain a 3D generalization, it is convenient to introduce equivalent viscosity η_D such that the viscous stress strain relation $s_{ij}^A = \eta_D \dot{\epsilon}_{Dij}$ would give the same energy dissipation density as Eq. 5 for any deviatoric strain rate tensor $\dot{\epsilon}_{Dij}$, in the variational sense; here s_{ij}^A is the additional deviatoric stress caused by the comminution. Now we note that the energy density is the same as the stress, that s_{12}^A must be equal to $\Delta\mathcal{K}$ when all other tensorial components vanish, and that $\Delta\mathcal{K}$ must be a tensorial invariant. To satisfy these three conditions, it is necessary that $-\Delta\mathcal{K} = \sqrt{s_{ij}^A s_{ij}^A}/2$.

Because $s_{ij}^A = \eta_D \dot{\epsilon}_{Dij}$ in terms of viscosity, we may write $\sqrt{s_{ij}^A s_{ij}^A}/2 = \eta_D \sqrt{\dot{\epsilon}_{Dij} \dot{\epsilon}_{Dij}}/2 = \eta_D \dot{\epsilon}_D$, where now $\dot{\epsilon}_D = \sqrt{\dot{\epsilon}_{Dij} \dot{\epsilon}_{Dij}}/2$, which simplifies to pure shear $\dot{\epsilon}_D$ when $\dot{\epsilon}_{D12}$ is the only nonzero component. Therefore, the energy sink owing to the comminution process may be modeled by the equivalent viscosity

$$\eta_D = (C_0 \Gamma^2 \rho)^{1/3} \dot{\epsilon}_D^{-1/3}. \quad [6]$$

Unlike Eq. 1, this 3D generalization is only approximate because, in three dimensions, the local kinetic energy density of strain rate is not separable from the total kinetic energy density. There is an energy cross-term involving both $\dot{\epsilon}_{Dij}$ and general rotation tensor $\dot{\omega}_{ij}$, which is here neglected. However, calculations indicate that this cross-term is not important.

Viscosity η_D can easily be implemented in the constitutive relation in a finite element program. It may be noted that the enhancement of dissipative viscous resistance to shearing is again a feature analogous to the enhancement of viscous resistance caused by eddies in turbulent flow.

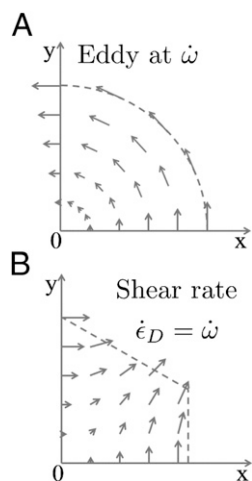


Fig. 2. Fields (in the first quadrant) of local velocity vectors giving the same kinetic energy density for (A) rigid body rotation and (B) pure shear deformation of the same magnitude.

In view of the partial analogy with turbulence (Fig. 2), can one formulate a dimensionless indicator analogous to the Reynolds number, which would decide when the comminution dominates the energy dissipation? One can. The strain energy density stored in the material may be expressed as $\mathcal{U} = \tau^2/2G$, where G is the elastic shear modulus and τ is the shear stress. When $\Delta\mathcal{K} \gg \mathcal{U}$, then obviously the comminution cannot be caused by the release of strain energy and the release of kinetic energy is the only possible energy source for the comminution. Therefore, we may define the dimensionless number $B_a = -\Delta\mathcal{K}/\mathcal{U}$ or

$$B_a = \frac{G}{C_o \tau^2} (\Gamma^2 \rho \dot{\epsilon}_D^2)^{1/3}, \quad [7]$$

which has the property that the comminution is

kinetic energy driven	if $B_a \gg 1$	
in transition	if $B_a \approx 1$	[8]
static or absent	if $B_a \ll 1$.	

Substituting $\tau = \tau_0 =$ maximum shear stress that can be resisted by the material (i.e., the yield strength), one has the sufficient condition for the comminution to be driven by the kinetic energy release. The equivalent viscosity may also be uniquely expressed in terms of B_n .

Note that exponent $-1/3$ in Eq. 6 is only an approximation for constant Γ . Although no relevant data exist at present, Γ as a characteristic of postfracture frictional work is likely to decrease with the slip rate; if it decreases as a power law it would reduce the exponent in Eq. 6 below $-1/3$. Further note that if, after comminution, $\dot{\epsilon}_D$ further increases, particles already comminuted are getting comminuted to smaller sizes. However, if $\dot{\epsilon}_D$ decreases while remaining in the comminution range according to Eq. 7, energy balance requires the frictional slip to be concentrated into interfaces between groups of particles moving as virtually rigid bodies (h then represents the size of these groups).

Finite element simulations indicate that, in practical applications such as impact, the rate of expansive volumetric strain rate, ϵ_{Ex} , plays no significant role. For explosions in a shale mass, however, the rates of shear strain and of volumetric expansion may both be important. One can show that, in that case, the foregoing theory can be easily generalized; for example, $\bar{\epsilon}_D^{-1/3}$ in Eq. 6 needs to be replaced by $(\dot{\epsilon}_D + \dot{\epsilon}_V)^{-1/3}$.

Some Results and Discussion

Although a practically most intriguing application of dynamic comminution modeling may be the fracturing of gas or oil shale by electro-hydraulic pulsed arc (15, 16) or by chemical explosion in the pipe of a horizontal borehole, no data on this recently discussed alternative technology exist in the public domain. Nevertheless, a confirmation of the present comminution model can be obtained by fitting the data on the measured exit velocity, v_e , of projectiles penetrating concrete walls of different thicknesses.

One such set of data is shown in Fig. 3 (17), which shows v_e as a function of the thickness D of the wall. The entry velocity of the missile is 310 m/s. The v_e values are computed with an explicit dynamic finite element program using the microplane model, first under the assumption that the only rate effects are the quasistatically calibrated rate effects, which consist of viscoelasticity of intact concrete between the cracks and of the rate of bond breakage at the fracture front controlled by activation energy (18). As seen, this simple assumption leads to a gross overestimation of the exit velocities. However, when the presently formulated equivalent viscosity owing to kinetic comminution is included, the data for the two thicker walls are fitted perfectly (in ref. 17, the strength and strain limits of the microplane constitutive laws were elevated horizontally by more than an order of magnitude and vertically by more than three times, with no physical justification, so as to fit these data).

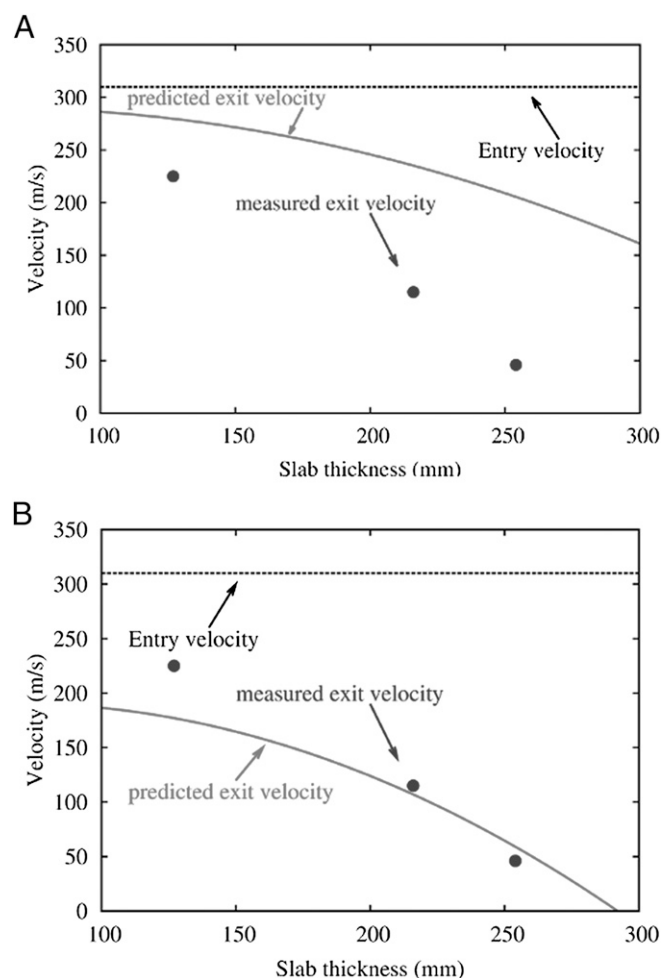


Fig. 3. Comparison of measured exit velocities and finite element predictions using microplane constitutive model M7 (A) with quasi-static strain rate effects only and (B) with both quasi-static strain rate effects and effect of comminution of concrete (the experimental data points are taken from ref. 17).

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Wave Dispersion and Basic Concepts of Peridynamics Compared to Classical Nonlocal Damage Models

The spectral approach is used to examine the wave dispersion in linearized bond-based and state-based peridynamics in one and two dimensions, and comparisons with the classical nonlocal models for damage are made. Similar to the classical nonlocal models, the peridynamic dispersion of elastic waves occurs for high frequencies. It is shown to be stronger in the state-based than in the bond-based version, with multiple wavelengths giving a vanishing phase velocity, one of them longer than the horizon. In the bond-based and state-based, the nonlocality of elastic and inelastic behaviors is coupled, i.e., the dispersion of elastic and inelastic waves cannot be independently controlled. In consequence, the difference between: (1) the nonlocality due to material characteristic length for softening damage, which ensures stability of softening damage and serves as the localization limiter, and (2) the nonlocality due to material heterogeneity cannot be distinguished. This coupling of both kinds of dispersion is unrealistic and similar to the original 1984 nonlocal model for damage which was in 1987 abandoned and improved to be nondispersive or mildly dispersive for elasticity but strongly dispersive for damage. With the same regular grid of nodes, the convergence rates for both the bond-based and state-based versions are found to be slower than for the finite difference methods. It is shown that there exists a limit case of peridynamics, with a micromodulus in the form of a Delta function spiking at the horizon. This limit case is equivalent to the unstabilized imbricate continuum and exhibits zero-energy periodic modes of instability. Finally, it is emphasized that the node-skipping force interactions, a salient feature of peridynamics, are physically unjustified (except on the atomic scale) because in reality the forces get transmitted to the second and farther neighboring particles (or nodes) through the displacements and rotations of the intermediate particles, rather than by some potential permeating particles as on the atomic scale. [DOI: 10.1115/1.4034319]

1 Introduction

Peridynamics is a recently promulgated radically new nonlocal theory of solid mechanics with an intriguing new name.² It eschews Cauchy's concept of stress and strain and is based on central force interactions between material points at various finite distances³ [e.g., Refs. 2–18]. The novel idea of peridynamics [13] is to perform the nonlocal spatial (weighted) averaging over displacement *differences*, whereas in the classical nonlocal damage model [19–23], the averaging is performed over *inelastic* strains or over a scalar *softening* parameter that depends on the displacement *derivatives*. The introduction of nonlocality into the continuum model is essential for preventing spurious localization of softening damage [19]. On the other hand, an artificial or excessive nonlocality introduced into elasticity may raise challenging questions when compared to the nonlocal damage model, especially with respect to wave dispersion, material property characterization, and numerical discretization.

This study is focused on the wave dispersion properties. Silling [13], and Weckner and Abeyaratne [24], obtained a general expression for the dispersion relation of the bond-based peridynamic continuum and evaluated the expression by using different

types of micromoduli. Zimmermann [25] compared the dispersion behavior of discrete bond-based peridynamic networks based on intermolecular forces with classical gradient and integral type nonlocal continua. For integral-type nonlocal elasticity and for various gradient formulations, the dispersion relation was studied, e.g., by Jirásek [26]. However, a comprehensive investigation of the various versions of peridynamics is still lacking.

This work analyzes the wave propagation behavior obtained from the continuous and discrete descriptions for both bond- and state-based peridynamics. The effects originating from the discretization of continuum peridynamics are isolated from the effects of the governing laws of the different peridynamic formulations. A comparison with the classical nonlocal continua is also provided. The main objectives of this study are fourfold:

- (1) Analytical examination of the wave dispersion properties of linear peridynamics;
- (2) Comparison of peridynamic continuum to the earlier continuous nonlocal models for quasibrittle materials [19–23,27–33] (although comparisons to the extensive work on lattice particle models for quasibrittle materials [e.g., Refs. 34–37] are beyond the present scope);
- (3) Comparison of important metrics such as critical time step, numerical error as well as rate of convergence of the discrete peridynamic formulations with classical numerical methods, such as finite difference schemes and the finite element method.
- (4) Discussion of some problematic or improvable physical aspects of the current forms of peridynamics.

The wave dispersion analysis will pursue an approach similar to that used earlier for spurious wave reflection in nonuniform

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²In Greek, “peri” means “about” and “dyna” means “force”. Although “dynamics” is normally understood as the opposite of “statics”, “peridynamics” is applied to statics, too.

³In this regard, it bears partial similarity to the 1977 “network model” of Burt and Dougill [1] for concrete, which was, however, used only to simulate the strain-softening constitutive relation

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finite element meshes [38,39] and for the original nonlocal models of distributed damage [40] (see also p. 926 in Ref. [27]).

2 Overview of Peridynamic Continuum Models

2.1 Bond-Based Peridynamics. The continuum limit of a discrete (bond-based) peridynamic network [13] is a special kind of a nonlocal continuum defined at every point X by the following equation of motion [9,14]:

$$\rho(X)\ddot{u}(X,t) = F(X,t) + b(X,t) \quad (1)$$

$$\text{where } F(X,t) = \int_V f[u(X_S,t) - u(X,t), X_S - X, t] dV \quad (2)$$

where X, X_S = Lagrangian coordinate vectors of paired continuum points, u, u_S = displacement vectors of these points, t = time, ρ = mass density, V = nonlocal averaging domain bounded by a horizon radius, δ (or briefly “horizon”), b = distributed body force, $F(X,t)$ = elastic resisting force, and f = force density transmitted by the nodal pair to point of coordinate vector X in the undeformed state, called the “pairwise response function,” whose dimension is N/m^6 , N/m^5 , or N/m^4 , respectively, for three-dimensional (3D), two-dimensional (2D), and one-dimensional (1D) continuum. Force f is calculated as [9,14] (Fig. 1)

$$f(u_S - u, X_S - X, t) = \frac{C(X_S - X)s(u_S - u, X_S - X, t)}{n(u_S - u, X_S - X, t)} \quad (3)$$

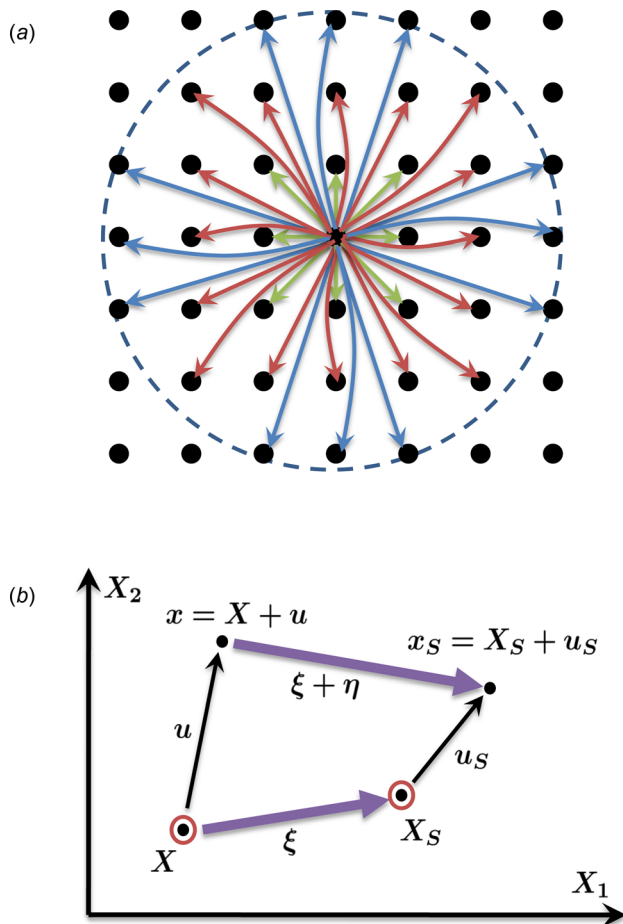


Fig. 1 (a) Schematic illustration of how particles in a horizon interact through bond forces. **(b)** Nodal pair vectors in original and deformed position and displacement vectors of paired nodes.

where $u_S = u(X_S, t)$ and $u = u(X, t)$;

$$n(u_S - u, X_S - X, t) = \frac{u_S + X_S - u - X}{||u_S + X_S - u - X||} \quad (4)$$

$$s(u_S - u, X_S - X) = \frac{||u_S + X_S - u - X||}{||X_S - X||} - 1 \quad (5)$$

Here $C(X_S - X)$ is a specified material property called micromodulus which has the same dimension as f (although these dimensions have been perfectly justified mathematically, it is strange from the physical viewpoint that the material stiffness needs to be fundamentally characterized by quantities of typical dimensions such as N/m^6 in 3D).

2.2 State-Based Peridynamics. The state-based version of peridynamics, developed by Silling et al. [15], brings about a significant improvement in the material modeling capability. In this version, an effective strain tensor is constructed from dyadic products of all the pairwise force vectors emanating from a node. The use of tensors circumvents the Poisson ratio restrictions of bond-based peridynamics (1/4 in 3D, 1/3 in 2D) and allows using the standard tensorial constitutive relations (as well as the microplane models). The average of the displacement differences over all the pairs is used to approximate the displacement derivative, which was shown in Ref. [18] to coincide, for uniform grids, with the so-called Lanczos derivative (a particular case of the reproducing kernel approximation [41]). Connection to other mesh-free methods was elucidated by Bessa et al. [18].

In state-based peridynamics, with correspondence to classical constitutive laws, the conservation of linear momentum equation is formulated as [15]

$$\rho(X)\ddot{u}(X,t) = \int_V \alpha(X_S - X) [\sigma(X_S, t) \cdot K^{-1}(X_S) + \sigma(X, t) \cdot K^{-1}(X)] \cdot X_S dV + b(X, t) \quad (6)$$

where $\alpha(X_S - X)$ is the weight function, typically used in the peridynamics literature as the window (or separator) function, and $K(X)$ is the so-called shape tensor

$$K(X) = \int_V \alpha(X_S - X)(X_S - X) \otimes (X_S - X) dV \quad (7)$$

and where the stress at each point of the peridynamic continuum is obtained from the constitutive law for that material, $\sigma(X, t) \equiv \sigma(F_{\text{def}}(X, t))$, applied to the nonlocal deformation gradient tensor, which is

$$F_{\text{def}}(X, t) = \left(\int_V \alpha(X_S - X)(X_S - X) \otimes (X_S - X) dV \right) \cdot K^{-1}(X) dV \quad (8)$$

This tensor is substituted into the aforementioned constitutive law giving σ , which is then substituted into Eq. (6). Here the lower case x and x_S are used to indicate the current positions of the points whose reference positions are X and X_S , respectively.

3 Wave Propagation in Peridynamic Continuum

Wave propagation problems are often used as a probe to examine the behavior of various mechanical models. If the models are nonlocal or inelastic, they introduce dispersion. Comparing the dispersion relations of such models then becomes a useful way to distinguish the effect of the different parameters and formulations.

In this section, we present 1D dispersion relations for wave propagation in peridynamic continua, as well as a comparison with classical nonlocal models. Multidimensional analyses are

also presented when appropriate but, since the results are similar to the 1D analyses, the detailed derivations are relegated to Appendices A and B. Note that the dispersion relations presented in this section are valid only for continua. To solve the continuum governing equation, one must discretize it both in time and space. After discretization, additional dispersion and nonlocal effects are introduced. Their consequences will be discussed in Sec. 4.

Since we are focusing on linear elastic continua, a wave of any shape can be either expanded into a Fourier series, if periodic, or represented by its Fourier Transform, if aperiodic. So it suffices to analyze one harmonic wave component

$$u(X, t) = a e^{i\kappa(X-vt)} \quad (9)$$

where X may be replaced by x when small deformations are considered; i = imaginary unit ($i^2 = -1$), v = phase velocity [e.g., Ref. 42], and κ = wave number (or spatial frequency) = number of wavelengths within length 2π ($\kappa v = \omega$ = circular frequency which, in classical elasticity, is proportional to κ as v is constant). Note that the expression in Eq. (9) is a complex form of the harmonic solution. To recover the real solution, the fundamental complex solutions must appear in pairs with their complex conjugate $a e^{-i\kappa(X-vt)}$. In other words, we admit that both κ and ω could be negative numbers. In signal processing, ω is also called *phasor*, and its absolute value $|\omega|$ is the angular frequency that can be measured in experiments.

3.1 Waves in Bond-Based 1D Peridynamic Continuum.

Consider now the special case of Eq. (1) representing an infinite 1D elastic nonlocal peridynamic continuum. If the deformations are small, the equation of motion at any point X may be written as

$$\rho(X)\ddot{u}(X, t) = \int_{-\infty}^{\infty} \alpha(X_S - X) f(u_S - u, X_S - X, t) dX_S + b(X, t) \quad (10)$$

where

$$f(u_S - u, X_S - X) = C(X_S - X) \left(\frac{|u_S + X_S - u - X|}{|X_S - X|} - 1 \right) \text{sgn}(u_S + X_S - u - X) \quad (11)$$

For convenience, we use again the previously introduced separator (or window) function, which is in 1D defined as

$$\alpha(X_S - X) = \begin{cases} 1 & \text{for } |X_S - X| < \delta \\ 0 & \text{for } |X_S - X| \geq \delta \end{cases} \quad (12)$$

This function separates from the rest of the body the nonlocal domain within the horizon, which, in 1D, constitutes the interval $(-\delta, \delta)$.

Equation (10) can be simplified by introducing a convenient change of variables $\xi = X_S - X$, and by splitting the integration to eliminate the absolute values in the integrand, bearing in mind that for $X_S > X$ we must have $X_S + u_S > X + u$, and for $X_S < X$ we must have $X_S + u_S < X + u$ (which means that the material point initially located at X cannot pass through the material point that is initially located at X_S).

After some algebra, Eq. (10) can be written as

$$\rho(X)\ddot{u}(X, t) = \int_{-\infty}^{\infty} \alpha(\xi) C(\xi) \frac{u(X + \xi, t) - u(X, t)}{|\xi|} d\xi + b(X, t) \quad (13)$$

The macroscopic elastic (Young's) modulus in standard elasticity can be related to the micromodulus $C(\xi)$ in Eq. (13) by means of energy equivalence

$$E = \frac{1}{2} \int_{-\infty}^{\infty} C(\xi) |\xi| d\xi \quad (14)$$

Based on dimensional analysis, $E = k_e \bar{C} \delta^2$ where $\bar{C}$ is the average of $C(\xi)$ and k_e is some dimensionless constant.

Upon substitution of Eq. (9) into Eq. (13), one finds that time t and amplitude a can be eliminated, and one gets

$$-\rho \kappa^2 v^2 = \int_{-\infty}^{\infty} \frac{\alpha(\xi) C(\xi)}{|\xi|} (e^{i\kappa\xi} - 1) d\xi = \int_{-\delta}^{\delta} C(\xi) \frac{\cos \kappa\xi - 1}{|\xi|} d\xi + i \int_{-\delta}^{\delta} C(\xi) \frac{\sin \kappa\xi}{|\xi|} d\xi \quad (15)$$

Since $C(\xi) \sin \kappa\xi / |\xi|$ is an odd function, and the domain is symmetric, the last integral vanishes. The square of the phase velocity can then be expressed as

$$v^2 = \frac{2}{\rho \kappa^2} \int_0^{\delta} C(\xi) \frac{1 - \cos \kappa\xi}{\xi} d\xi \quad (16)$$

It can be shown that the elastic limit can always be recovered if the micromodulus satisfies (14). Starting from Eq. (16) and approximating $\cos \kappa\xi$ by $1 - (\kappa\xi)^2/2$

$$v^2 = \frac{2}{\rho \kappa^2} \int_0^{\delta} C(\xi) \frac{1 - \cos \kappa\xi}{\xi} d\xi \approx \frac{2}{\rho \kappa^2} \int_0^{\delta} C(\xi) \frac{\kappa^2 \xi^2}{2\xi} d\xi = \frac{1}{\rho} \int_0^{\delta} C(\xi) \xi d\xi \quad (17)$$

we see that the last integral corresponds to the elastic modulus E , given by Eq. (14). So the elastic limit for a shrinking horizon, $\delta \rightarrow 0$, is verified for an arbitrary micromodulus $C(\xi)$ that satisfies Eq. (14)

$$\lim_{\delta \rightarrow 0} v = \sqrt{\frac{E}{\rho}} \quad (18)$$

Particular solutions for Eq. (16) can be found for specific micromodulus $C(\xi)$. For example, considering a uniform micromodulus $C(\xi) = \bar{C}$, we can evaluate Eq. (16) and obtain

$$\left| \frac{v}{v_0} \right| = \frac{2}{\kappa \delta} \sqrt{\gamma - \text{Ci}(\kappa \delta) + \ln(\kappa \delta)}, \quad v_0^2 = \delta^2 \frac{\bar{C}}{2\rho} \quad (19)$$

where $\gamma \approx 0.577216$ = Euler gamma constant, and Ci is the "cosine integral" function, defined by

$$\text{Ci}(X) = - \int_X^{\infty} \frac{\cos t}{t} dt \quad (20)$$

For 2D bond-based peridynamic continuum with a constant micromodulus $C(\xi, \eta) = \bar{C}$, similar dispersion relations for in-plane dilatational plane waves can be derived. Since the horizon in 2D becomes a disk, the integral over the horizon is evaluated numerically, and the result is given together with the 1D dispersion curve in Fig. 2. The derivation of this result is given in the Appendix A.

Figure 2 presents the phase velocity of 1D bond-based peridynamics (solid line) given by Eq. (16) as a function of the number of wavelengths L per horizon δ defined as $\nu_L = \kappa \delta / 2\pi$ since κ is the number of wavelengths per 2π . Note that Eq. (16) always has two solutions; we denote $v^+(\kappa \delta)$ and $v^-(\kappa \delta)$ as the solutions that converge to $\pm v_0$ as $\kappa \delta \rightarrow 0$ respectively. Since $v^-(\kappa \delta) = -v^+(\kappa \delta)$ and they are symmetric to the x -axis, only $v^+(\kappa \delta)$ is plotted.

3.2 Waves in State-Based 1D Peridynamic Continuum.

Considering now the special case of Eq. (6) for an infinite 1D elastic nonlocal state-based peridynamic continuum, the conservation of linear momentum becomes

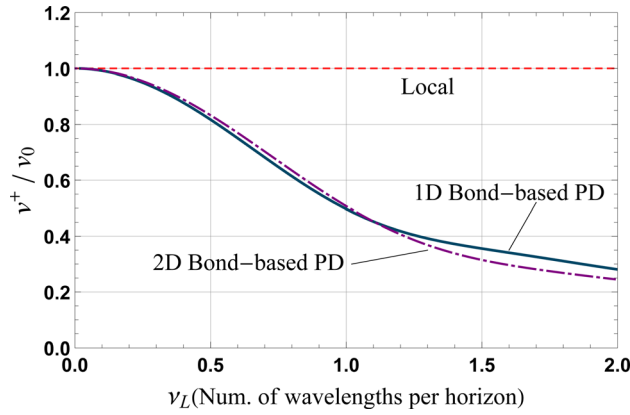


Fig. 2 Comparison of phase velocity for wave propagation in bond-based peridynamic (PD) continuum: (1) 1D bond-based peridynamic continuum with uniform micromodulus; and (2) 2D bond-based peridynamic continuum with uniform micromodulus. The dashed line corresponds to local model (classical elasticity).

$$\rho(X)\ddot{u}(X,t) = \int_{-\infty}^{\infty} \alpha(\xi) [\sigma(X+\xi,t)K^{-1}(X+\xi) + \sigma(X,t)K^{-1}(X)] \xi d\xi + b(X,t) \quad (21)$$

where $\xi = X_S - X$ was introduced as before. The shape tensor gives the same scalar value at every point

$$K(X) = K(X+\xi) = \int_{-\infty}^{\infty} \alpha(\xi)\xi^2 d\xi = \frac{2\delta^3}{3} \quad (22)$$

where the separator function $\alpha(\xi)$ is defined as before, in Eq. (12).

The nonlocal deformation gradient at two different points X and X_S is defined by

$$F(X,t) = \frac{\partial x}{\partial X} = \frac{\int_{-\infty}^{\infty} \alpha(\xi)(x_S - x)\xi d\xi}{\int_{-\infty}^{\infty} \alpha(\xi)\xi^2 d\xi} = \frac{3}{2\delta^3} \int_{-\delta}^{\delta} (\xi + u_S - u)\xi d\xi \quad (23)$$

$$F(X_S,t) = \frac{3}{2\delta^3} \int_{-\delta}^{\delta} (\eta + u_R - u_S)\eta d\eta \quad (24)$$

where $\eta = X_R - X_S$.

Evaluating the above results for the harmonic wave considered previously, Eq. (9), we can obtain the phase velocity as a function of the wave number, as done previously for the bond-based case. Recall that the small strain measure, $\varepsilon = 1/2(\mathbf{F} + \mathbf{F}^T) - \mathbf{I}$, can be used to compute the stress in Eq. (21) because we are dealing with a linear elastic problem. Hence, according to Eqs. (23) and (24), the small strain at the two different points is obtained as

$$\varepsilon(X,t) = \frac{\int_{-\infty}^{\infty} \alpha(\xi)(u_S - u)\xi d\xi}{\int_{-\infty}^{\infty} \alpha(\xi)\xi^2 d\xi} = \frac{3}{\delta^3} i e^{i\kappa(X-vt+\xi)} \frac{-\delta\kappa \cos(\delta\kappa) + \sin(\delta\kappa)}{\kappa^2} \quad (25)$$

$$\varepsilon(X_S,t) = \frac{\int_{-\infty}^{\infty} \alpha(\eta)(u_R - u_S)\eta d\eta}{\int_{-\infty}^{\infty} \alpha(\eta)\eta^2 d\eta} \quad (26)$$

$$= \frac{3}{\delta^3} i e^{i\kappa(X-vt+\xi)} \frac{-\delta\kappa \cos(\delta\kappa) + \sin(\delta\kappa)}{\kappa^2} \quad (27)$$

from which the stress at the two respective points is simply obtained by Hooke's law $\sigma = E\varepsilon$, leading to the following simplification of Eq. (21) (neglecting the body force):

$$\rho(X)\ddot{u}(X,t) = \frac{3}{2\delta^3} \frac{3E}{\delta^3} i e^{i\kappa(X-vt+\xi)} \frac{-\delta\kappa \cos(\delta\kappa) + \sin(\delta\kappa)}{\kappa^2} \int_{-\delta}^{\delta} \xi (e^{i\kappa\xi} + 1) d\xi \quad (28)$$

This equation can further be simplified to determine the phase velocity using the harmonic wave given by Eq. (9)

$$\frac{v^+}{v_0} = \frac{3[\sin(\delta\kappa) - \delta\kappa \cos(\delta\kappa)]}{\delta^3 \kappa^3} = \frac{3[\sin(2\pi\nu_L) - (2\pi\nu_L)\cos(2\pi\nu_L)]}{(2\pi\nu_L)^3} \quad (29)$$

where ν_L is the number of wavelengths L per horizon δ , defined in Sec. 3.1 as $\nu_L = \kappa\delta/2\pi$. This result is shown in Fig. 3. Again, the result of 2D analysis similar to this case is derived

$$\frac{v^+}{v_0} = 8 \frac{J_2(\kappa\delta)}{(\kappa\delta)^2} = 8 \frac{J_2(2\pi\nu_L)}{(2\pi\nu_L)^2} \quad (30)$$

where $J_2(x)$ is the Bessel function of the first kind. In addition, the comparison with the dispersion relation of bond-based peridynamics is shown in Sec. 3.3.

3.3 Comparison of Bond- and State-Based Peridynamic Continua. Sections 3.1 and 3.2 derived different dispersion relations for bond- and state-based peridynamic continua (see Eqs. (19) and (29)).

Figure 4 shows these phase velocities as a function of the number ν_L of wavelengths L per horizon δ . According to this plot, both peridynamic continua are dispersive, as can be seen by the deviations from the elastic wave speed v_0 . The figure shows that the phase velocities are real for all wave numbers κ , but decrease with ν_L and tend to 0 as $\nu_L \rightarrow \infty$ (at constant ρ , δ , and $\bar{C}$ or E). As already mentioned, the limit of vanishing horizon, $\delta \rightarrow 0$, corresponds to the standard elastic continuum for both formulations of peridynamics. High values of $\nu_L = \kappa\delta/2\pi$ correspond to short

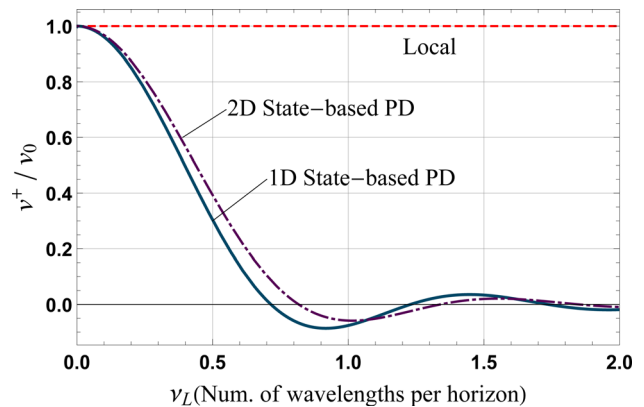


Fig. 3 Comparison of phase velocity for wave propagation in state-based peridynamic (PD) continua: (1) 1D state-based peridynamic continuum with uniform micromodulus; and (2) 2D state-based peridynamic continuum with uniform micromodulus. The dashed line corresponds to the local model (classical elasticity).

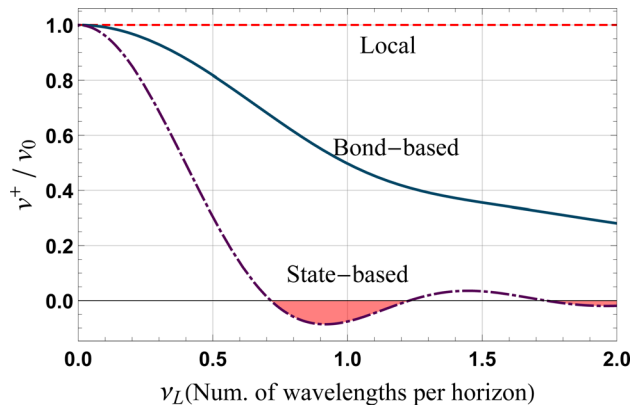


Fig. 4 Comparison of phase velocity for 1D wave propagation: (1) bond-based peridynamic continuum, and (2) state-based peridynamic continuum (both with uniform micromodulus). The dashed line corresponds to classical elasticity.

wavelength waves and small values to long ones. The maximum value of ν_L in the plot is 2, considering that wavelengths shorter than twice of the horizon are not captured for the most common case of $\delta = 3h$, where h is the nodal spacing.

Observing Fig. 4, it is clear that state-based peridynamics is much more dispersive than the bond-based formulation because the phase velocity in state-based peridynamic continuum drops much faster and can be negative for some ν_L (see the shadow in Figs. 4 and 5). Furthermore, in state-based peridynamics, the phase velocity vanishes for a series of wave numbers, representing zero-energy deformation modes.

Peridynamics is intended to tackle mainly the problems of fracture, where the stress, strain, and displacement fields can have strong discontinuities. The dispersion relation of state-based peridynamics indicates that its behavior may be undesirable when analyzing wave propagation in real solids, especially when the stress waves have shock fronts close to Heaviside step function, in which case the components of high wave number have relatively large amplitudes when compared to well-behaved functions. Thus the strong dispersion may lead to a large error.

Figure 5 shows the dispersion diagram obtained for both formulations by evaluating the normalized circular frequency $\omega^+ \nu_L$ as a function of ν_L .

3.4 Comparison With Classical Nonlocal Models. Particularly interesting is a comparison of peridynamics to the classical nonlocal models for softening damage. The first nonlocal models

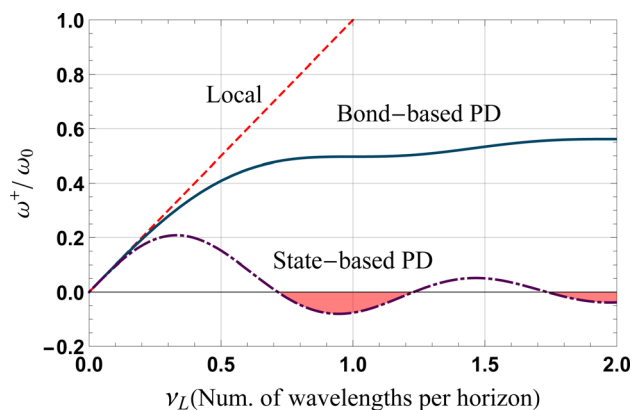


Fig. 5 Comparison of dispersion diagram for 1D wave propagation: (1) bond-based peridynamic (PD) continuum with uniform micromodulus; and (2) state-based peridynamic (PD) continuum. The dashed line corresponds to classical elasticity.

formulated in the 1960s [43–48] (cf. [33]) were intended for continuum smoothing of heterogeneous microstructure in elasticity and, later, in hardening plasticity. In the first version of a nonlocal model with softening damage, proposed in 1984 in Refs. [19] and [20], the nonlocal concept was exploited as a localization limiter for finite element computations. We begin by this version and then discuss its 1987 revision.

3.4.1 Original Nonlocal Model for Softening Damage. The incremental small-amplitude linearization of the original 1984 nonlocal model for softening damage [19,33] uses the constitutive law

$$\sigma(x) = E\bar{\epsilon}(x), \quad \bar{\epsilon}(x) = \int_{-\infty}^{\infty} \alpha(s)\epsilon(x+s)ds \quad (31)$$

where $\bar{\epsilon}$ is nonlocal strain, $\alpha(s)$ is nonlocal weight function, σ and ϵ are understood as small increments from the initial damaged state, and E is the tangential modulus; also, we replace X with x .

The limit of the original nonlocal model for vanishing length l of the nonlocal averaging region is obtained by setting $\alpha(s) = l\delta(s)$ where $\delta(s)$ = Dirac delta function. This limit recovers local elasticity

$$\bar{\epsilon}(x) = \frac{1}{l} \int_{-l/2}^{l/2} \alpha(s)\epsilon(x+s)ds = \int_{-l/2}^{l/2} \epsilon(x+s)\delta(s)ds = \epsilon(x) \quad (32)$$

Substituting Eq. (31) and the harmonic wave component $u = ae^{ik(x-vt)}$ into the equation of motion $\rho\ddot{u} = \partial\sigma/\partial x$, one obtains, after rearrangements

$$v^2 = v_0^2\hat{\alpha}(\kappa), \quad v_0 = \sqrt{E/\rho} \quad (33)$$

$$\text{where } \hat{\alpha}(\kappa) = \int_{-\infty}^{\infty} \alpha(s)e^{iks}ds \quad (34)$$

Here, v_0 is the phase velocity of local elastic continuum and $\hat{\alpha}(\kappa)$ is the Fourier transform of nonlocal weight function $\alpha(s)$. The effective nonlocal characteristic length l is defined so that the areas under $\alpha(r)$ and under the rectangle of width l and height 1 be equal (note that in the 2D or 3D generalizations, the equivalent rectangle is replaced by a uniform weight over a circle or sphere).

It was shown [27,40] that to allow propagation of harmonic waves of any frequency and prevent zero-energy modes of instability, $\hat{\alpha}(\kappa)$ must be positive for all κ (cf. [40] or [27, Eq.13.10.14]). This excluded in 1984 the use of rectangular and triangular distributions of $\alpha(s)$, although superimposing a Dirac delta function at $s = 0$ worked, and was modeled by element imbrication with an elastic overlay [40]). We consider here another acceptable and frequently used choice for $\alpha(s)$ [40], namely, the Gaussian distribution

$$\alpha(s) = \frac{1}{\sqrt{2\pi}s_d} e^{-s^2/2s_d^2} \quad (35)$$

where s_d is the standard deviation. If $\alpha(s)$ is Gaussian, then $\hat{\alpha}(\kappa)$ is also Gaussian, i.e.,

$$\hat{\alpha}(s) = e^{-\kappa^2 s_d^2/2} \quad (36)$$

In nonlocal averaging models, the characteristic size l of the averaging domain is usually considered to be the size of a domain with uniform weight that gives the same area or volume under the weight function $\alpha(s)$. For the Gaussian distribution, this gives $l = s_d$.

The relation between the horizon, δ , and the characteristic size, l , of the averaging domain depends on the criterion chosen. As a

reasonable criterion, we may require the classical 1984 nonlocal averaging model to have the same dispersion properties as the bond-based peridynamics. This is achieved, in 1D, approximately for

$$\delta = 2l \quad (37)$$

The phase velocity given by Eq. (33) for $\delta = 2l$ is plotted in Fig. 6. The figure demonstrates that the original 1984 nonlocal averaging model and the bond-based peridynamics (which is equivalent in 1D) have similar dispersion relations.

The state-based peridynamics is much more dispersive, as already discussed.

Calculations show that the elastic behavior of the 1D bond-based peridynamics and of the 1D original 1984 nonlocal model can be made identical by a proper ratio of s_d and δ .

Both peridynamic formulations and the original 1984 nonlocal model have a common feature that should be regarded as unrealistic: They do not distinguish, and cannot independently control, the wave dispersion due to elastic heterogeneity and to softening damage or distributed fracturing. This prompted in 1987 the rejection of the original model, as explained next.

3.4.2 Nonlocal Softening Damage With Local Elastic Strain.

For some simple but convenient weight functions, particularly the rectangular one, the Fourier transform in Eq. (33) is nonpositive, which allows zero-energy periodic modes of instability (see Refs. [49] and [27, Eq.13.10.14]). The imbricate approximation, used to overcome this problem [20], led to programming complexities (and did not allow the damage stress to soften to zero). For this reason, the original nonlocal damage model was abandoned and, in 1987, a new nonlocal model was introduced [21–23]. In that model, elasticity can be local, unless nonlocality is introduced separately, while the damage is independently nonlocal, which suffices to prevent strain localization into a zone of zero volume.

The 1D linearized small-amplitude incremental (or tangential) constitutive equation for a domain of initially uniform inelastic state may be most simply written as

$$\begin{aligned} \text{for } \epsilon \leq \epsilon_1 \text{ or } d\epsilon < 0 : \sigma(X) &= E\epsilon(X) \\ \text{otherwise : } \sigma(X) &= E\epsilon(X) - (E - D) \int_{-\infty}^{\infty} \alpha(s) [\epsilon(X+s) - \epsilon(X)] ds \end{aligned} \quad (38)$$

where σ and ϵ here represent small increments from the initially uniform inelastic state (positive increments, since unloading

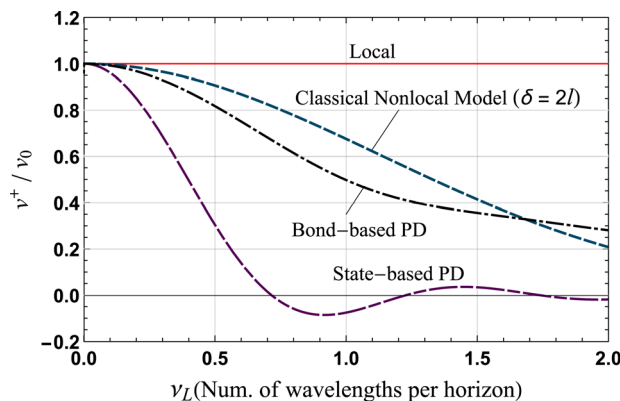


Fig. 6 Comparison of phase velocity for 1D wave propagation for bond-based and state-based peridynamic (PD) continuum with uniform micromodulus and original nonlocal model for softening damage considering different nonlocal characteristic length l . The dashed line corresponds to a local model (classical elasticity).

follows a different tangent modulus). The strains, ϵ , defined by displacement derivatives (gradients), are assumed to be positive since negative strains (unloading) follow a different modulus E . The stress–strain law is, for sake of simplicity, assumed to be bilinear, with ϵ_1 = elastic limit and E = initial elastic modulus, D = hardening modulus if it is non-negative, and softening modulus if it is negative (for simplicity, we neglect the rate dependence of E , and particularly the fact that a sudden big rate increase can revert the effective E from negative to positive [50]).

If the material is in the elastic range ($\epsilon \leq \epsilon_1$), and if no material heterogeneity is considered, the phase velocity v is independent of the wave number, κ , as in a classical elastic continuum (i.e., the plot of v versus κ is a horizontal line, as in Fig. 6). If the material is in the inelastic range, regions with different moduli E and D inevitably develop. Then the wave dispersion occurs due to the switching of incremental stiffness between E and D . An exception is a purely unloading wave, which is the same as a wave in the elastic regime because the incremental modulus for unloading is E at all points.

3.4.3 Weakly Nonlocal Models. These are gradient-type models whose limiting elastic case is in 1D most simply epitomized by

$$\sigma(X) = E\bar{\epsilon}(X); \quad \bar{\epsilon}(X) = (1 + l^2 \nabla^2) \epsilon(X) \quad (39)$$

where $\bar{\epsilon}$ is the weakly nonlocal strain and, in one dimension, $\nabla^2 = d^2/dX^2$. Substituting this, and $\epsilon = du/dX$ and $u = a e^{ik(X-vt)}$, into the foregoing equation of motion, $E d\sigma/dX = \rho d^2u/dt^2$, we find that amplitude a and the exponentials cancel out and we get

$$v^2 = v_0^2 (1 - l^2 \kappa^2) \quad (40)$$

$$\text{or } \frac{v}{v_0} = \sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{l^2}{v_0^2} \omega^2}} \quad (41)$$

Note that the wave dispersions due to elastic heterogeneity and to inelastic behavior cannot be independently controlled. Also note, for $\omega < v_0/2l$, there are two different phase velocities, both real and nonzero, whereas for $\omega \geq v_0/2l$, the velocity v becomes complex. These are abnormal properties, which are one reason why this weakly nonlocal model fell in disfavor.

This explains why the only nonlocal models for softening damage that are implemented in commercial codes are the ones based on Eq. (38) and on the crack band model [51], which also exhibits local elastic strain (used, e.g., in commercial codes ATENA, DIANA, and SBETA, and in the open-source code OOFEM).

In addition, note that the crack band model [e.g., Ref. 32], which is in practice a widely used alternative to damage nonlocality, has the advantage of unambiguous boundary conditions. It avoids the problem of nonlocal averaging domain that protrudes through the boundary or interface.

4 Wave Propagation in Discrete Peridynamic Networks

4.1 Effect of Discretization on Nonlocality, Dispersion, and Aliasing. Numerical dispersion is a common phenomenon and major cause of error in numerical methods, such as the finite element method (FEM) and finite difference method (FD). In computational mechanics, these classical methods are used to solve dynamic problems of the form:

$$\rho \ddot{u} = \nabla \sigma + b \quad (42)$$

where $\sigma = E : (\nabla u)$ and E = tangential moduli tensor. Note that, for uniform E , this governing equation is nondispersive, and only

after discretization it becomes dispersive. The discretization often generates *heterogeneity* and inherent *nonlocality* because the mass is lumped at discrete points and these points are forced to interact with other nodes that lie at a *finite* distance. Thus the discrete system is intrinsically dispersive. Intuitively, the influence induced by discretization becomes negligibly small as the mesh is refined. So, we see that the discretization inevitably leads to numerical dispersion.

In nonlocal models, the nonlocality is imposed deliberately, making the continuum dispersive in its nature, as observed in Section 3. However, in order to solve the governing equations numerically, they must be transformed into a discrete form, which inevitably introduces additional nonlocality and dispersion.

Once the continuous equation is discretized into a finite number of (N) nodes or particles in space. The system has only a finite number (N) of degrees of freedom, the spatial operator on $\mathbf{u}$ in the continuum equation of motion, Eq. (42), becomes an $N \times N$ matrix $\mathbf{L}$, and the displacement field, $\mathbf{u}(\mathbf{x})$, becomes an $N \times 1$ column vector $\mathbf{u}$. Since this operator must be self-adjoint, its discrete form $\mathbf{L}$ must also be symmetric regardless of the boundary conditions. Thus, its eigenvalues are all real-valued and the multiplicity of each eigenvalue must add up to N . In other words, there are at most N distinct eigenvalues (wave numbers) that the discrete system can represent.

Thus, even though we could have a continuous phase velocity curve from the Fourier analysis, the real phase velocities represent only a finite number of points sampled from the curve. In addition, the minimum wavelength that the system can represent, or the resolution of the system, is $2h$ (at least one trough and one crest in the wavelength), and so the maximum absolute value of wave number $|\kappa_{\max}|$ is π/h , which is fully determined by the particle size h (or nodal spacing). This means that the N wave numbers of the system must fall in the range $-\pi < \kappa h < \pi$.

The higher wave number components that fall outside this range will be represented as lower wave number components inside the band, which is called *aliasing*. For instance, if a signal $u(x_j) = e^{i\kappa_0 x_j} = e^{i\kappa_0(jh)}$ satisfies $-\pi < \kappa_0 h < \pi$, then other signals whose wave numbers satisfy the condition $\kappa h = \kappa_0 h + 2m\pi$, $m = \pm 1, \pm 2, \dots$ will end up with wave numbers that are exactly the same as $u(x_j)$ after sampling.

Since a signal can be represented by the system only if its wave number falls in the unique region $-\pi < \kappa h < \pi$, it is, in signal processing, called the *reconstruction zone* or, in solid state physics, the *Brillouin zone*. Therefore, in order to show the dispersion relation and phase velocity of various discrete models, one should always plot the curves that fall in the reconstruction zone.

4.2 Discretized Bond-Based Peridynamics. In practice, peridynamics is always used in the discrete form. As the node network is refined, the discrete system converges to the nonlocal peridynamic continuum. Consider the horizon that spans, in 1D, the interval $(-\delta, \delta)$ to be subdivided by $n = k + l + 1$ nodes of coordinates X_m ($m = -k, -k + 1, \dots, 0, 1, 2, \dots, l$), which, in general, need not be equally spaced. Let the distance between two adjacent points be given by $h_s = |X_{s+1} - X_s|$ and let the points be ordered such that $X_{m+1} > X_m$. Then, the spatial discretization of Eq. (13) within the horizon of point X_0 for a uniform micromodulus and a zero body force becomes

$$\rho \ddot{u}(X, t) = \sum_{s=-k, s \neq 0}^l \frac{2E}{\delta^2} \frac{u(\xi_s + X) - u(X, t)}{|\xi_s|} h_s \quad (43)$$

where $\xi_m = X_m - X_0$ and $\alpha(\xi_m) = 1$ within the horizon. Note that, for a uniform spatial discretization, we have $\xi_m = mh$.

Considering the same plane wave as before

$$u(\xi_s + X, t) = a e^{i\kappa(\xi_s + X - vt)} \quad (44)$$

we find the phase velocity to be

$$v^2 = \frac{2E}{\rho(\kappa\delta)^2} \sum_{s=-k, s \neq 0}^l \frac{1 - e^{i\kappa\xi_s}}{|\xi_s|} h_s \quad (45)$$

Denoting $v_0 = \sqrt{E/\rho}$, we can rewrite this equation in terms of a dimensionless phase velocity and a dimensionless wave number $\kappa\delta$. It suffices to consider only the absolute value of the normalized velocity

$$\left| \frac{v}{v_0} \right| = \frac{\sqrt{2}}{\kappa\delta} \sqrt{\sum_{s=-k, s \neq 0}^l \frac{1 - e^{i(\kappa\delta)\xi_s/\delta}}{|\xi_s|} h_s} \quad (46)$$

This equation was derived for arbitrarily spaced 1D grids of points within the horizon δ . However, as explained by Hirt [52], Fourier analysis is applicable to linear equations with constant coefficients. Therefore, Eq. (46) should be evaluated for the case of uniform grids only

$$\left| \frac{v}{v_0} \right| = \frac{\sqrt{2}}{\kappa\delta} \sqrt{\sum_{s=1}^{(n-1)/2} \frac{2 - e^{i(\kappa\delta)sh/\delta} - e^{-i(\kappa\delta)sh/\delta}}{s}} \quad (47)$$

$$= \frac{2}{\kappa\delta} \sqrt{\sum_{s=1}^m \frac{1 - \cos(\kappa h s)}{s}} \quad (48)$$

where $h_s = h$ for all s and $k = l = (n-1)/2 = m$ due to symmetry. The imaginary part of the wave velocity was canceled out, also due to symmetry.

Figure 7 shows the dimensionless phase velocities obtained for the 1D peridynamic model with uniform micromodulus considering the total number of points within the horizon, $n = 5$. Different from the continuous case, the independent variable χ is defined as $\kappa h/2\pi$, which represents the number of wavelengths per particle.

This choice is implied by the fact that the particle (or nodal) spacing, h , decides the resolution of the system, instead of the horizon δ . From the figure, it can be seen that the phase velocity is more than 20% higher than the classical elastic wave propagation speed for large wavelengths, and only converges to the same value if a large number of points n within the horizon is considered (this is the so-called δ -convergence, in which the curve tends to be the continuous one). This means that a numerical simulation of wave propagation using bond-based peridynamics is not only dispersive (forming a wave behind the wave front) but also leads to the

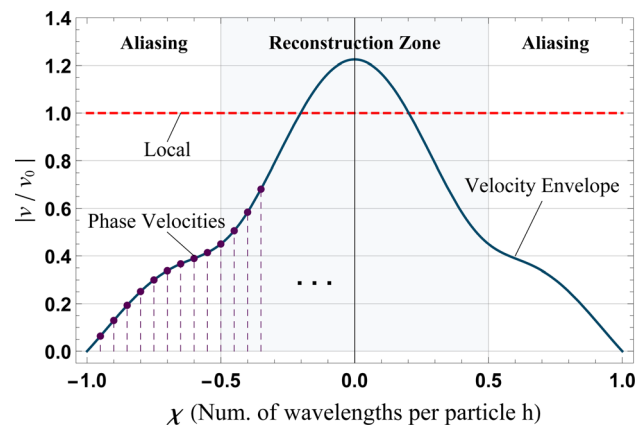


Fig. 7 Phase velocity for 1D bond-based peridynamic model with a uniform micromodulus, considering a uniform grid with 5 points within the horizon. The thick curve represents the phase velocity envelope of the discretized system, and the points on the curve represent the discrete phase velocities of the system. The dashed line represents the phase velocities of linear elastic model.

formation of a wave that travels roughly 20% faster than the elastic wave speed in classical continuum, which might be undesirable.

In addition, from Eq. (48), for $\delta = (n-1)h/2 = mh$ we have

$$\left| \frac{v}{v_0} \right| = \frac{2}{m} \sqrt{\sum_{s=1}^m \frac{1 - \cos(\kappa h s)}{s(\kappa h)^2}} \quad (49)$$

From Eq. (49), one can see that $|v/v_0|$ is 0 for $\kappa h = 2\pi$. Similar cutoffs for wavelengths or frequencies occur, of course, for all discrete methods, including the finite element methods [e.g., Ref. 38,39] and finite difference methods. The dispersion diagram (i.e., w/w_0 versus χ) is also shown in Fig. 8. Note that w_0 is the natural frequency of the particles and $w_0 = v_0/h = \sqrt{(EA/h)/(\rho Ah)} = \sqrt{k_0/m_0}$, where k_0 and m_0 are the spring constant and particle mass, respectively.

4.3 Discretized State-Based Peridynamics. State-based peridynamics was fundamentally analyzed for uniform grids in Ref. [18]. For these grids with spacing $h_m = h$, for all m , the discretization of Eq. (6) considering the window function in Eq. (12) and a number n of points within the horizon δ leads to the following shape tensor [18]:

$$K(X) = \sum_{m=1}^L 2m^2 h^2 = \frac{L(L+1)(2L+1)}{3} h^2 \quad L = \frac{n-1}{2} \quad (50)$$

which is the same at every point. This can be used for discretizing the nonlocal deformation gradient given by Eq. (8)

$$\begin{aligned} F(X, t) &= \frac{\partial x}{\partial X} = \frac{\sum_{m=1}^L [(x_{m^+} - x_{m^-})mh]}{K(X)} \\ &= \frac{3}{L(L+1)(2L+1)h} \sum_{m=1}^L m(x_{m^+} - x_{m^-}) \end{aligned} \quad (51)$$

where the notation x_{m^+} indicates the current position of the m th point to the right of point X , and x_{m^-} the m th point to the left (both points being at the same reference distance from X). The foregoing expression can be rewritten as a function of the displacement u of the points, instead of their current position x .

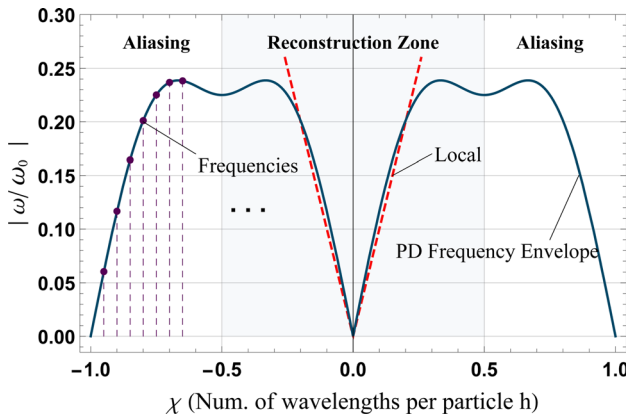


Fig. 8 Dispersion diagram for 1D bond-based peridynamic (PD) model with uniform micromodulus considering a uniform grid with 5 points within the horizon. Note that the y -coordinate stands for the magnitude of normalized frequency, $|w/w_0|$, where $w = \chi v$. Note that the region $-0.5 < \chi < 0.5$ in the plot is the reconstruction zone, and the region outside shows the aliasing region.

Considering that $x_{m^+} = u(X + \xi_{m^+}, t) + X_{m^+}$, one may, after some algebra, write Eq. (51) as

$$F(X, t) = 1 + \frac{3}{L(L+1)(2L+1)h} \sum_{m=1}^L m[u(X + \xi_{m^+}, t) - u(X + \xi_{m^-}, t)] \quad (52)$$

The discretization of the conservation of linear momentum equation (6), with the body force neglected, follows then from the foregoing results (recall that K is the same at every point for uniform grids):

$$\rho(X) \ddot{u}(X, t) = \frac{3}{L(L+1)(2L+1)h} \sum_{m=1}^L m[\sigma(X + \xi_{m^+}, t) + \sigma(X + \xi_{m^-}, t)] \quad (53)$$

For a linear elastic material with Young's modulus E , the stress at each point of the above equation is given by

$$\begin{aligned} \sigma(X + \xi_{m^+}, t) &= E \frac{\partial x_{m^+}}{\partial X} = \frac{3E}{L(L+1)(2L+1)h} \\ &\quad \sum_{r=1}^L r[u(X + \xi_{m^+} + \xi_{r^+}, t) - u(X + \xi_{m^+} + \xi_{r^-}, t)] \end{aligned} \quad (54)$$

$$\begin{aligned} \sigma(X + \xi_{m^-}, t) &= E \frac{\partial x_{m^-}}{\partial X} = \frac{3E}{L(L+1)(2L+1)h} \\ &\quad \sum_{r=1}^L r[u(X + \xi_{m^-} + \xi_{r^+}, t) - u(X + \xi_{m^-} + \xi_{r^-}, t)] \end{aligned} \quad (55)$$

Considering the displacement in the plane wave in Eq. (9), and calculating the stress at each point from Eqs. (54) and (55), Eq. (53) for the conservation of linear momentum leads to the following phase velocity:

$$v^2 = \left(\frac{6}{L(L+1)(2L+1)h} \right)^2 \frac{E}{\rho} \sum_{m=1}^L \left\{ \frac{m \sin(\kappa mh)}{\kappa^2} \sum_{r=1}^L r \sin(\kappa rh) \right\} \quad (56)$$

which, upon considering that $v_0 = \sqrt{E/\rho}$, can be rewritten as a dimensionless phase velocity that is a function of the dimensionless wave number $\kappa\delta$

$$\left| \frac{v}{v_0} \right| = \frac{6}{L(L+1)(2L+1)h} \sqrt{\sum_{m=1}^L \left\{ \frac{m \sin[(\kappa\delta)mh/\delta]}{(\kappa\delta)^2} \sum_{r=1}^L r \sin[(\kappa\delta)rh/\delta] \right\}} \quad (57)$$

Figure 9 shows the phase velocity when the number of grid points within the horizon is the same as in the bond-based case ($n = 5$). Comparing Fig. 9 with Fig. 7, one can see that, even though the velocity envelope shows nonzero phase velocity after $\chi = \kappa h/2\pi = 0.5$ or $\lambda = 2\pi/\kappa = 2h$, the actual velocity drops, for the discretized state-based formulation, to zero at that point.

Thus, the dispersion error due to numerical discretization is much larger than in the bond-based case. Figure 10 presents the circular frequency for the same grid.

Furthermore, from Eq. (56), one can see that $v = 0$ at $\chi = 0.5$ is true for any m . So, we must conclude that the state-based

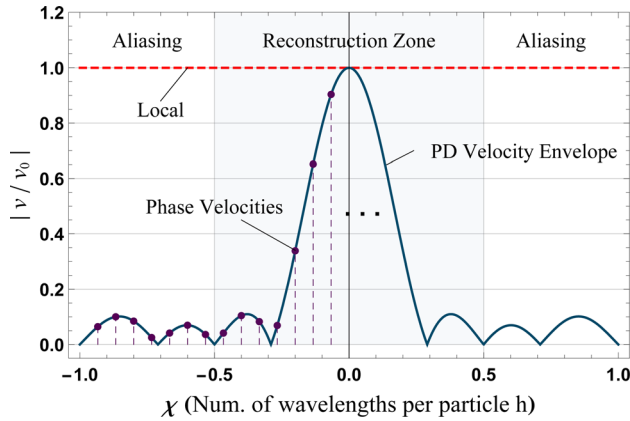


Fig. 9 Phase velocity for 1D state-based peridynamic (PD) model considering a uniform grid with 5 points within the horizon. The dashed line represents the continuum solution, while the dotted line represents the classical elasticity solution. The domain $-0.5 < \chi < 0.5$ is the reconstruction zone.

peridynamics, unlike bond-based, acts as a very effective filter for the high-frequency stress waves. This is in agreement with the fact that the state-based peridynamics approximates derivatives via the so-called Lanczos differentiators, which are used in digital filters [53], as pointed out by Bessa et al. [18].

4.4 Convergence of Peridynamics

4.4.1 Spatial Numerical Error of Peridynamics. As suggested by Silling and Lehoucq [54], the peridynamic continuum can be discretized into the following form:

$$\frac{\rho_j}{\Delta t^2} (u_j^{n+1} - 2u_j^n + u_j^{n-1}) = \sum_{s \in \mathcal{H}} f_{sj} V_j + b_j \quad (58)$$

where the second derivative of $\ddot{u}$ is discretized by the second-order central difference scheme.

Again, consider the 1D wave propagation problem, where $b_i = 0$ and $2\delta = (2m+1)h$. For the bond-based peridynamics, Eq. (58) becomes

$$\begin{aligned} \frac{\rho}{\Delta t^2} (u_j^{n+1} - 2u_j^n + u_j^{n-1}) &= \sum_{s=-m, s \neq 0}^m C_{js} \frac{u_{j+s}^n - u_j^n}{|s|h} h \\ &= \sum_{s=1}^m C_{js} \frac{u_{j+s}^n - 2u_j^n + u_{j-s}^n}{s} \end{aligned} \quad (59)$$

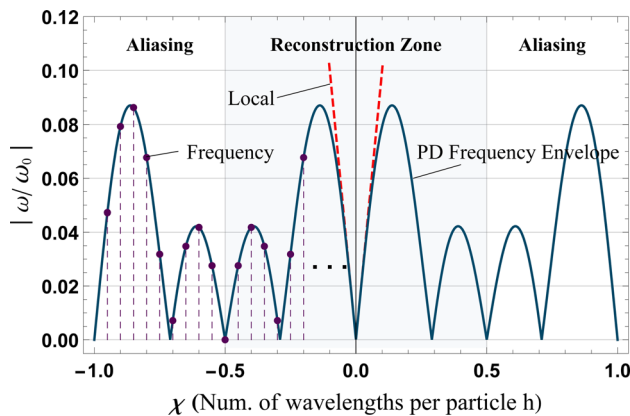


Fig. 10 Dispersion diagram for 1D state-based peridynamic (PD) model considering a uniform grid with 5 points within the horizon. The dashed line represents the continuum solution. Note that the shaded region $-0.5 < \chi < 0.5$ in the plot is the reconstruction zone.

For $m = 1$ and $C_{js} = 2E/\delta^2 = 4E/3h^2$ (the value calculated in the continuous case for constant micromodulus in 1D), Eq. (59) obviously becomes

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = \left(\frac{4E}{3\rho} \right) \frac{u_{j+s}^n - 2u_j^n + u_{j-s}^n}{h^2} \quad (60)$$

This is very similar to the second-order central difference discretization (with a 3-point stencil) of the two-way wave equation $\ddot{u} = c_0^2 u''$, except that the phase speed is replaced by $2c_0/\sqrt{3}$.

Similarly, for the cases where $m > 1$, the right-hand side of Eq. (59) is just a linear combination of the discrete second derivative (Laplacian) operator. After expanding the expression, one realizes the similarities to the central difference scheme with a $(2m+1)$ -point stencil. In this regard, note that the horizon in peridynamics serves as the counterpart of the stencil in finite differences.

It is interesting to note that a finite difference (FD) scheme with a $(2m+1)$ -point stencil always has $(2m)^{\text{th}}$ -order accuracy (i.e., the error $e \sim O(h^{2m})$), while the bond-based peridynamics for a general micromodulus is always only second-order accurate because of the influence of the micromodulus. The micromodulus can be chosen as either uniform or as any other positive, symmetric, and compactly supported function as long as it does not lead to instability. Choosing a different micromodulus affects the coefficients C_{js} for u_{j+s}^n and u_{j-s}^n , leading to values that are in general not the same as in the finite difference scheme.

From a purely numerical standpoint, these new coefficients enhance dispersion but they also affect the convergence rate. This means that, for approximation of the local behavior, peridynamics is less efficient than the finite difference method.

Nevertheless, the fixed combination of coefficients that are defined in the finite difference schemes appears to be the only ones that can guarantee $(2m)^{\text{th}}$ -order accuracy. So, if these coefficients are different, the scheme is limited to second order accuracy due to the $(1, -2, 1)$ coefficient combination on the right-hand side of Eq. (59). For the state-based peridynamics, the same conclusion can be reached because of the arbitrariness of the aforementioned micromodulus. A comparison is given below for the bond-based peridynamics model (5 points within the horizon) and the central difference scheme with a 5-point stencil. The fourth-order central difference scheme reads

$$\frac{\partial^2}{\partial t^2} u(x_j, t_n) = c_0^2 \frac{-u_{j+2}^n + 16u_{j+1}^n - 30u_j^n + 16u_{j-1}^n - u_{j-2}^n}{12h^2} + O(h^4) \quad (61)$$

For the bond-based peridynamics with a 5-point horizon, and constant micromodulus (second-order)

$$\frac{\partial^2}{\partial t^2} u(x_j, t_n) = c_0^2 \frac{4u_{j+2}^n + 8u_{j+1}^n - 24u_j^n + 8u_{j-1}^n + 4u_{j-2}^n}{25h^2} + O(h^2) \quad (62)$$

Comparing Eq. (61) and Eq. (62), one can see that, to have the fourth-order accuracy, u_{j+2}^n and u_{j-2}^n must have coefficient -1 , i.e., the micromodulus $C(x - \xi)$ would have to be *negative* for some $|\xi| < \delta$ so as to cancel out part of the nonlocality. However, this is not physically justifiable and could possibly lead to instability. The most commonly used micromoduli are either constant (uniform) or cone-shaped functions, which are all positive within the horizon. So, the peridynamic formulations (both bond-based and state-based), cannot be more than second-order accurate in space.

In applications, usually there are 5–11 nodes in the horizon ($m = 2 \sim 5$). The foregoing analysis shows that, in the elastic range, the bond-based peridynamics can be regarded as a modified finite difference scheme that runs in each time increment at the computational cost of a $(2m)^{\text{th}}$ -order scheme, but has only second-order accuracy.

This explains why peridynamics is slower than traditional computational methods such as the finite difference method or the finite element method.

4.4.2 Temporal Error of Peridynamics and Its Coupling With Spatial Error. The most common way to discretize $\ddot{u}$ is, upon second-order central differencing

$$\ddot{u}(x_j, t_n) = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} + O(\Delta t^2) \quad (63)$$

The theory of discrete-time dynamical systems shows that this scheme is nondissipative in time and all the numerical errors come from dispersion. Combining the spatial error from Sec. 4.4.1, the error of peridynamic schemes can be expressed as

$$e = A\Delta x^2 + B\Delta t^2 \quad (64)$$

where A and B are constants. Thus, we can fairly say that the error (not efficiency) of peridynamic schemes is similar to finite difference schemes that are second order in space and time (2-2 schemes).

4.4.3 Numerical Experiments. To illustrate the dispersion of peridynamics, a numerical experiment in MATLAB has been conducted. Suppose there is an elastic bar with one end ($x = 0$) free and the other one ($x = 1$) fixed. Young's modulus and cross-sectional area are 200 GPa and 10^{-4} m^2 , respectively. The initial displacement field is chosen as an aperiodic rectangular wave (see Fig. (11(a))) and the initial velocity is zero. The horizon δ is chosen as $2.5 h$, where h is the particle length created by discretization.

One can see that the horizon will change when we refine the mesh, and so this corresponds to the so-called m -convergence (if we leave aside the point that the horizon in an internal material length which should not be adjusted when the mesh is changed).

Finally, the space and time are discretized from the bond-based peridynamics by Eqs. (62) and for (63), respectively. The total number of nodes is 1000, and $\Delta t/\Delta t_c$ the values of 0.5, 0.99975, and 1.005 are used.

After $t = 0$, one would expect that the initial square wave would split into two parts that look almost the same but propagate in opposite directions. One moving to the left, which would hit the free end and reflect (with no change in sign) to the right, and the other moving to the right, which would hit the fixed end and reflect to the left (with a change in sign). To compare the results for various time increments Δt , the simulation is stopped at $t = 1 \times 10^{-4} \text{ s}$ (0.1 ms). At this moment, the left-moving wave has already been reflected to the right, following the right propagating one, and the right-moving wave has not hit the fixed end yet. The results are shown in Figs. 11(b)–11(d).

From Figs. 11(b) and 11(c), Gibb's effect is observed at the wave front and the numerical error is manifested by dispersion (long bumpy tails of the main waves). Note that for common 2-2 FD schemes, the spatial and temporal errors cancel each other when they run very close to the critical time step ($\Delta t \simeq \Delta t_c$). This phenomenon is not observed in peridynamics (see Fig. 11(c))—the larger the time step, the larger the dispersion is. When $\Delta t \simeq \Delta t_c$ (Fig. (11(c))), the dispersion error is much higher than $\Delta t = 0.5\Delta t_c$. As the time step is increased further ($\Delta t > \Delta t_c$), the solution becomes unstable, as expected.

Having chosen the initial displacement to be a rectangular distribution, we can highlight the wave dispersion when strong discontinuities appear (cracks). In the 1D case, this displacement field is not realistic. But it is likely to show up in 2D or 3D when a crack is generated, e.g., when the dynamic crack branching is simulated, which is the case often presented in the peridynamic literature and lectures. It is obvious that a strong discontinuity leads to a much larger energy (or amplitude) in the high frequency components of the displacement field, which will lead to larger dispersion. For example, when one assumes a Gaussian distribution with the standard deviation of 0.05 m as the initial

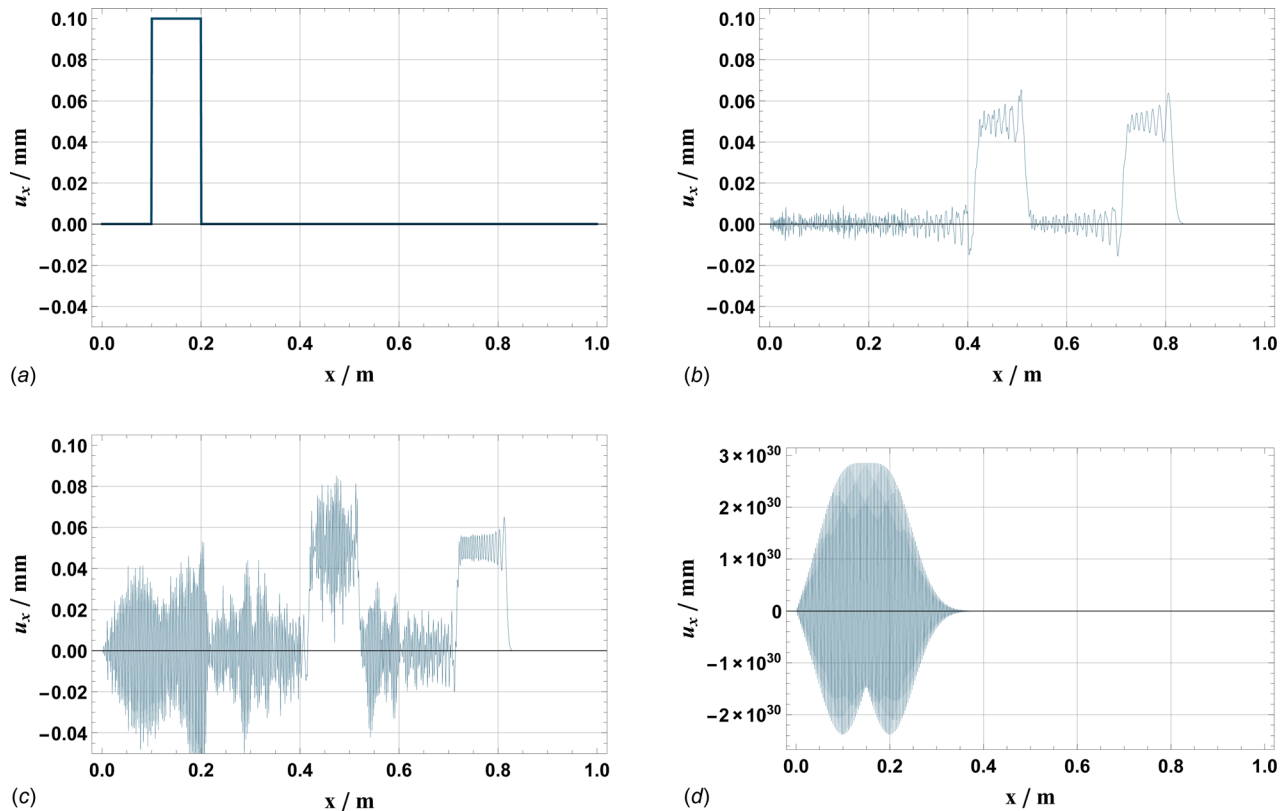


Fig. 11 Wave propagation in an elastic bar

displacement field, one needs only 200 particles to get a good approximation of the elastic wave. But when one assumes a rectangular wave as the initial field, one needs about 1000 particles.

5 Discussion of Some Problematic or Improvable Aspects of Peridynamics

5.1 Fictitious Node-Skipping Force Interactions. A fundamental aspect that is problematic is the physical representation of the interactions between the nodes of a peridynamic network. These interactions emulate the transmission of forces between the atoms of a solid, in the manner of computer codes for molecular dynamics (MD).

To explain the problem, consider Fig. 12(a) which schematically shows the transmission of force from atom 0 to its first, second, and third neighbors, labeled 1, 2, 3. The force is transmitted to the third neighbor (atom 3) by an interatomic potential (such as Morse or Lenard-Jones) which permeates through the first and second neighbors (atoms 1 and 2). But peridynamics is intended and used to simulate solids on a higher than atomic scale. For a heterogeneous solid consisting of particles or grains, peridynamics implicitly considers a direct force transmission to occur between all paired particles, skipping the particles lying in between, as sketched in Fig. 12(b). The particle skipping forces are immediately obvious for the original bond-based peridynamics.

For the state-based peridynamics, the particle skipping forces are not immediately obvious. The reason is that the pair interactions from all the interparticle bond pairs of various orientations emanating from the central node (i.e., from 0 to 1, 2, 3 in Fig. 12(a)) are buried in dyadic products of all the nodal pair vectors.

For clarity, consider, e.g., the second (unnumbered) equation on page 112 of Ref. [17] defining the shape tensor $\mathbf{K}$, which is in

our notation reproduced in Eq. (6). In that equation, many of the vectors $(\mathbf{X}_S - \mathbf{X})$ in the dyadic product skip the intermediate particles, yet the tensor $\mathbf{K}$ is subsequently used in Eq. (8) to calculate the nodal force $\mathbf{F}$. Consequently, force $\mathbf{F}$ depends on the particle-skipping intermediate nodes indirectly. So, in terms of the force transmission at distance, the state-based and bond-based peridynamics are identical. And if the particle-skipping interactions were omitted, one would obtain a very different form of state-based peridynamics.

The classical nonlocal models (as well as the crack band model [e.g., Refs. 27,33,51,55]) also imply interactions over finite distances. But these interactions consist of the stress dependence on the strain average over the whole nonlocal domain, which is defined as equivalent to the representative volume element (RVE) of the material. In view of the randomness of particulate or fibrous microstructure, and of the fact that nonlocality of damage is caused mainly by microcrack interactions [28,30], such averaging is a physically reasonable simplifying hypothesis made in the spirit of the classical homogenization theory. It is also simpler than peridynamics because, in standard usage, the strain averaging needs only the relative displacements of the boundary points of the nonlocal domain and does not need the displacements of the points inside the nonlocal domain, which are in reality highly random.

5.2 Problems With Peridynamic and Nonlocal Boundary Conditions. For both peridynamics and the classical nonlocal averaging models, the formulation of boundary conditions is fraught with ambiguity. For points whose horizon, or nonlocal averaging domain, protrudes outside the boundary, there are several options to deal with the protruding part, and none seems better physically justified than another.

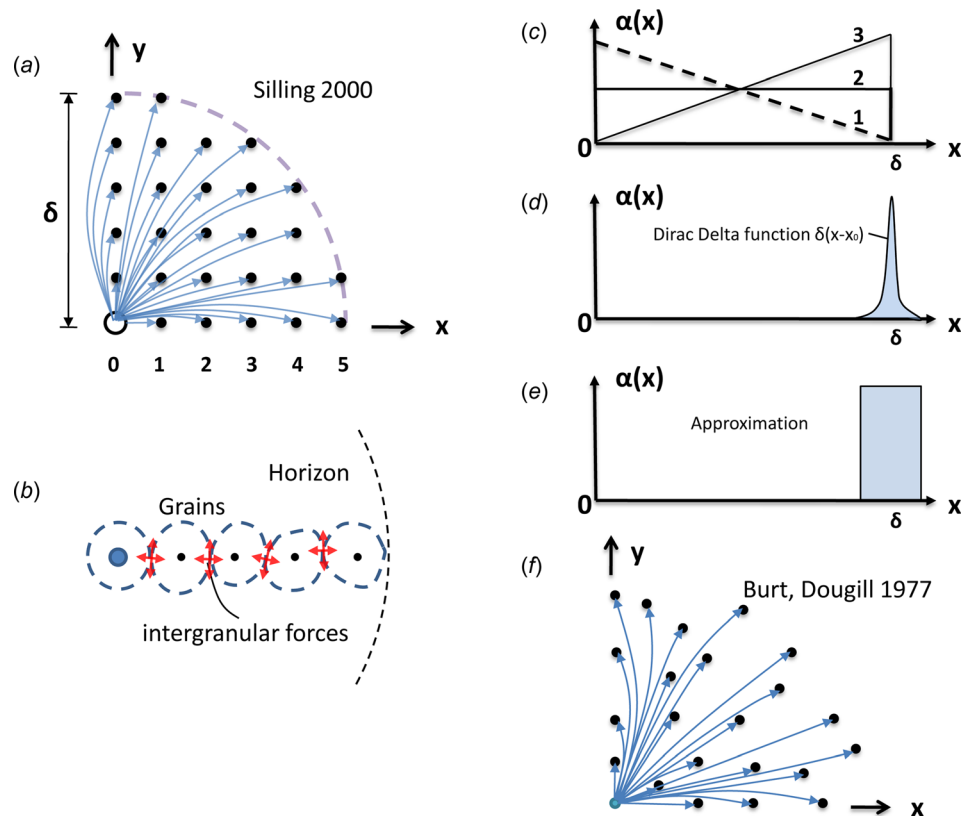


Fig. 12 (a) Direct long-range interparticle interactions in a peridynamic continuum, (b) indirect interactions between particles through intergranular forces, (c) three common cases of micromodulus distribution, (d) micromodulus with a Dirac delta function at $|\xi| = \delta$, (e) approximation of the Dirac delta function at $|\xi| = \delta$, (f) Burt and Dougill's constitutive model for progressive failure of heterogeneous media

This is a problem for all nonlocal models except the crack band model [51] and the recent nonlocal model with a damage boundary layer [56].

5.3 Capturing Morphology of Heterogeneous Microstructure and Comparison to Lattice Discrete Particle Model (LDPM). The classical nonlocal models do not capture the effect of the morphology of a heterogeneous microstructure, such as the particle sizes, shapes, or configurations. This is not surprising since the nonlocal strain is defined by averaging over the whole domain, ignoring the interior points.

However, the peridynamic computations, unlike the classical nonlocal ones, use an array of interior discrete nodes located inside the horizon domain. This suggests that the morphology of microstructure should be taken into account by peridynamics in some way. But it is not clear how.

By contrast, the lattice discrete particle model (LDPM) [e.g., Refs. 34–36] takes naturally the morphology inside the RVE directly into account.

5.4 Disregard of Interparticle Shears and Particle Rotations. Referring to Fig. 12(b), the internal force gets transmitted from particle 0 to particle 3 indirectly, through the contact or interface zones of neighboring particles or grains (or fibers). Based on unsatisfactory experience with the early lattice models for concrete (which assumed central normal forces only, i.e., no rotations), it was concluded in the early 1990s that one must take into account also the shear forces, which depend on the relative rotations of the adjacent particles (Fig. 12(c)). The rotations and interparticle shears are the hallmark of the lattice discrete particle model (LDPM) [36].

The existing versions of peridynamics, however, ignore the shears generated by rotations of neighboring particles. This is obvious for the bond-based version, but not so obvious for the state-based version. True, the stress and strain tensors in the latter versions do include shear strains and stresses. However, this is true in the sense of continuum homogenization only, since these tensors are constructed solely from the dyadic product of central force vectors that skip particles and ignore the rotations and shears.

It may also be noted that, in one generalization of peridynamics called “nonordinary” materials, rotational degrees of freedom of particles were included (see, e.g., Ref. [5], and also refer to Fig. 5, Eq. (132), and page 153 in Ref. [15]). The nonordinary version involved rotation-resisting interparticle springs and noncentral nodal force vectors (at an angle with the pair vectors). However, this version again implied direct interaction of the rotations of *nonneighboring particles*, somehow permeating through the intermediate particles. Referring to Fig. 12(b), the relative rotation of the first versus the third (or fourth) particle is not what controls shear. Only the first versus the second, or the second versus the third, etc., does.

Like the state-based peridynamics, the classical nonlocal damage models, too, do not take these rotations and shears into account. But these classical models are simpler, having, in the simple form used in practice, no need for the kinematics of interior nodes within the nonlocal averaging domain.

By contrast, the LDPM [34–36] does take these rotations and shears explicitly into account. For concrete, the LDPM has been shown to give superior predictions of the inelastic behavior and failure. The LDPM, together with the microplane model, are the only damage constitutive models for the RVE that can fit the full range of test data for concrete. This range includes over 20 different types of tests under uni-, bi-, and triaxial loading and shear loading, proportional and nonproportional, tensile and compressive, with unloading and load cycles (by contrast, the fits of merely one to three of them are typically presented to, supposedly, justify numerous concrete models in the computational literature). For concrete, numerous damage tests exist since the specimens are roughly of the same size as the RVE.

5.5 Lack of Independent Controllability of Wave Dispersions Caused by Elastic Heterogeneity and by Softening Damage or Plasticity. In peridynamics, these two sources of wave dispersion are coupled. This feature is unrealistic. It is shared with the original 1984 integral-type nonlocal model for softening damage [19,33] and served as the main motivation to formulate in 1987 a new nonlocal model [21], in which elasticity and damage are decoupled.

The crucial point to note is that there are two different physical reasons for introducing nonlocality into continuum mechanics [21]:

- (1) the need to capture elastic wave dispersion due to heterogeneity of the material microstructure (particles, grains, fibers, etc.) (see the original 1960s nonlocal models [e.g., Refs. 43–48]); and
- (2) the need to impose a localization limiter preventing spurious localization of softening damage into a line or surface of zero volume, and thus to enforce a finite size of fracture process zone (FPZ) [27,57]. The FPZ size is governed mainly by Irwin’s [58] material characteristic length

$$l_0 = \frac{EG_f}{\sigma_0^2} \quad (65)$$

where E = Young’s modulus, σ_0 = material tensile strength (a parameter in the cohesive crack model or crack band model), and G_f = material fracture energy ($G_f = K_{Ic}^2/E$ where K_{Ic} = fracture toughness).

These two sources of wave dispersion are independent and should be independently controllable. In peridynamics, they are not.

Both sources of wave dispersion are related to material heterogeneity, but in fundamentally different ways. The elastic wave dispersion depends only on the elastic stiffness of the material phases, such as the matrix, particles, fibers, etc., whereas the dispersion due to damage or fracturing depends on the FPZ size which is governed also, and mainly, by the fracture energy G_f and the cohesive strength σ_0 of the material (which, in turn, depend on the strengths, fracture energies, and interfacial strengths of the phases). For quasibrittle heterogeneous materials such as concrete, the elastic behavior can usually be considered as local and nondispersive (except for extremely short wavelengths), while the wave dispersion is limited to fracturing damage.

The nonlocal models for plasticity and damage can be justified by physical as well as computational arguments, the latter to avoid spurious mesh sensitivity. These models include not only the tensorial and microplane versions of the original nonlocal continuum with local elastic strain [21–23] but also the Peerlings and de Borst [59] continuum with nonlocal inelastic strain governed by the Helmholtz equation. They also include the crack band model [51,60], which is now widely used in the civil and aerospace industry to control mesh sensitivity, and in large wave codes (such as EPIC or Pronto at ERDC, Vicksburg, MS, or MARS at ES3, San Diego, CA) to predict the effects of impact, explosions, and ground shock [61,62].

When the elastic dispersion of waves with short wavelength needs to be captured by a continuum model, the nonlocal model with local elastic strain can be enriched by a second material characteristic length for the elastic part, optimized only according to the sizes of inhomogeneities and their elastic properties. This is an optional independent feature of the nonlocal damage model.

5.6 How Dispersion in Bond-Based Peridynamics Differs From State-Based Formulation. Bond-based peridynamics and state-based peridynamics were demonstrated to have different dispersive behaviors even in one dimension. The bond-based formulation shows moderate dispersion diagrams that are similar to

other classical nonlocal models, as highlighted previously. However, this formulation has a few important limitations [13,14]: direct force interaction skipping intermediate nodes, disregard of interparticle shear and particle rotations, restrictions on the Poisson ratio, and the need for specific constitutive laws, which so far lack proper experimental validation.

These limitations of bond-based peridynamics arise from the initial parallelism established between bond-based peridynamics and molecular dynamics, where the transmission of force from the central atom to its first, second, and third neighbors is effected by an interatomic potential (such as Morse or Lennard-Jones) which permeates through the first and second neighbors. At higher than atomic scale, such force transmission is nonphysical as it ignores the force transmission through the intermediate particles (see Fig. 12(a)). In reality, the force gets transmitted from particle 0 to particle 3 indirectly, through particle contacts (see Fig. 12(b)). Extensive experience has shown that one must take into account not only the normal forces in the contacts but also the shear forces which depend on the relative rotations of the adjacent particles.

5.7 Numerical Error and Convergence. The rate of convergence is here shown to be always of the second order in space and time, for all the versions of peridynamics, regardless of the number of particles within the horizon. The same convergence rate (of the second-order) is obtained by using a horizon (or stencil) with significantly fewer nodes.

If we disregard the fixed value of the horizon as a material property, a much higher convergence rate could be achieved by allowing the micromodulus to be negative in a part of the horizon such that the coefficients of nodal displacements within the horizon would be exactly the same as in the corresponding higher-order finite difference scheme, similar to what was done in Ref. [18]. However, this kind of improvement of convergence rate is not accessible in peridynamics, not only because of the fixed finite value of the horizon but also because a negative micromodulus would lack physical meaning and could lead to material instability. So, the accuracy improvement of finite differences achieved by an increasing stencil size cannot be matched in peridynamics.

Moreover, for the classic finite difference and finite element methods, the spatial and temporal errors cancel each other when the system time step is very close to the critical time step. This is not true for peridynamics because the coefficients used by both bond-based and state-based formulations differ from the finite difference method.

To sum up, both the continuum state-based theory and its discretized version are strongly dispersive and have multiple wavelengths for which the phase velocity vanishes or assumes a small value compared to the elastic wave propagation speed. This observation is supported by a previous investigation [18] showing that the state-based peridynamics approximates derivatives by Lanczos finite differences, often used in digital signal filtering due to their strong dispersion properties.

5.8 Existence of a Limit Case Where Peridynamics is Identical to Classical Nonlocal Strain Averaging. Consider the special case of a micromodulus having a Dirac-delta spike δ_D at the horizon, another spike at the center, and zero values everywhere else (Fig. 12(d)); i.e.,

$$\alpha(\xi)C(\xi) = C_0\mu(\xi), \quad \mu(\xi) = \delta_D(|\xi| - \delta) + p\delta_D(|\xi|) \quad (66)$$

where $p = \text{constant}$ (note that $\alpha(\xi)$ must be included here, to avoid integrating a product of Dirac and Heaviside functions). Eq. (66) means that only interactions of the central point with the points on the horizon remain, being equivalent to spatial averaging and those with the points inside get wiped out. Then the peridynamic 1D equation of motion, Eq. (13), reduces to the equation:

$$\begin{aligned} \rho(X)\ddot{u}(X,t) - b(X,t) &= C_0 \int_{-\infty}^{\infty} [\delta_D(|\xi| - \delta) + p\delta_D(\xi)] \frac{u(X+\xi,t) - u(X,t)}{|\xi|} d\xi \\ &= C_0 \int_{-\infty}^0 [\delta_D(-\xi - \delta) + p\delta_D(\xi)] \frac{u(X+\xi,t) - u(X,t)}{-\xi} d\xi \end{aligned} \quad (67)$$

$$\begin{aligned} &+ C_0 \int_0^{\infty} [\delta_D(\xi - \delta) + p\delta_D(\xi)] \frac{u(X+\xi,t) - u(X,t)}{\xi} d\xi \\ &= C_0 \frac{u(X,t) - u(X-\delta,t)}{-\delta} + C_0 \frac{u(X+\delta,t) - u(X,t)}{\delta} \end{aligned} \quad (68)$$

$$\begin{aligned} &+ C_0 p \lim_{s \rightarrow 0} \frac{u(X,t) - u(X-s,t)}{-2s} + C_0 p \lim_{s \rightarrow 0} \frac{u(X+s,t) - u(X,t)}{2s} \\ &= C_0 \delta \frac{u(X+\delta,t) - 2u(X,t) + u(X-\delta,t)}{\delta^2} \end{aligned} \quad (69)$$

$$+ C_0 p \lim_{s \rightarrow 0} s \frac{u(X+s,t) - 2u(X,t) + u(X-s,t)}{2s^2} \quad (70)$$

$$= C_0 \delta \left[\frac{\partial^2 u(X,t)}{\partial X^2} \right] + C_0 p \lim_{s \rightarrow 0} \frac{s}{2} \frac{\partial^2 u(X,t)}{\partial X^2} \quad (71)$$

$$\text{Therefore} \quad \rho(X)\ddot{u}(X,t) = \frac{\partial \bar{\sigma}(X,t)}{\partial X} + b(X,t) \quad (72)$$

$$\text{where} \quad \bar{\sigma}(X,t) = E\bar{\epsilon}(X,t), \quad \bar{\epsilon}(X,t) = \left[\frac{\partial u(X,t)}{\partial X} \right], \quad E = C_0 \delta \quad (73)$$

$$\begin{aligned} \text{and} \quad \left[\frac{\partial u(X,t)}{\partial X} \right] &= \frac{u(X+\delta,t) - u(X-\delta,t)}{2\delta} \\ &= \int_{-\delta}^{\delta} w(s)\epsilon(X+s,t) ds \end{aligned} \quad (74)$$

Here $\epsilon = \text{continuum strain}$; an overbar denotes averaging over the interval $(-\delta, \delta)$, and $w(s) = \text{nonlocal weight function}$ which, for the sake of simplicity, is here taken as uniform, equal to 1.

Equation (72) is the equation of motion of 1D nonlocal continuum, which reduces to the standard local 1D equation of motion in the limit of $\delta \rightarrow 0$. An interesting point is that the Dirac delta function δ_D function at the center, which was introduced with a view to the stabilization of imbricate continuum alluded to later, has no effect on the result (here it must also be admitted that the splitting of the Dirac delta function into two separate limits in Eq. (69) is unconventional; it could be avoided but the result would be the same, and the derivation less compact).

Now we see that the special limit case of peridynamics with a micromodulus in the form of the Dirac delta spike at the horizon is identical to the simple special case of the original 1984 nonlocal continuum with a uniform weight function [27,33,51,55], which was in 1984 called the imbricate continuum [40] and was later abandoned. As it turned out, this nonlocal continuum model allows periodic zero-energy modes of instability, due to the fact that the Fourier transform of the uniform, or rectangular, weight function $w(s)$ is not everywhere positive [27,40]. So, the present limit case of peridynamics, too, exhibits zero-energy modes of instability.

The imbricate continuum was stabilized by adding to the uniform weight function $w(s)$ a Dirac delta function $p\delta_D(s)$ with $p > 0.17847$ [40] at the center, which was equivalent to an overlay with a local elastic continuum. But, despite this stabilization, the imbricate nonlocal model was in 1987 abandoned and replaced by Eq. (38). It is interesting that the present limit case of peridynamics cannot be stabilized in this way because the limit in Eq. (69) vanishes.

6 Conclusions

- (1) The state-based peridynamics is strongly dispersive, significantly more so than the bond-based peridynamics. It has multiple wavelengths, one of them longer than the horizon, for which the phase velocity vanishes. This means that the state-based version acts as a particularly strong filter for high-frequency waves.
- (2) The wave dispersion behaviors and the nonlocality in the bond-based and state-based peridynamics are coupled. The material characteristic lengths caused by (i) the heterogeneity of material and (ii) the softening damage and plasticity cannot be distinguished. They are not independently controllable, whereas in the modern nonlocal-strain averaging theories, they are. The fact that the latter depends on material strength and fracture energy, and the former does not, is not captured by peridynamics.
- (3) Compared to the traditional numerical methods, the convergence rate for both the bond-based and state-based versions is suboptimal (although this is a price to pay for nonlocality in general). The computation time for each time increment is longer than it is in the finite difference method that shares with peridynamics the same regular grid of nodes.
- (4) The node-skipping force interactions, which are a salient distinguishing feature of peridynamics, are a fiction, except on the atomic scale. In reality, the force gets transmitted to the second and farther neighboring particles (or nodes) by displacements and rotations of the intermediate particles, rather than by some particle permeating potential (although, in fiber composites, a sort of long-distance interaction exists, engendered by fibers, some new form of peridynamics would be needed to capture the differences between various types of fibrous microstructures).
- (5) Despite using intermediate nodes within the horizon, the morphology of microstructure is not captured by the existing forms of peridynamics.
- (6) Although not used before, there exists a limit case of peridynamics that is equivalent to unstabilized imbricate nonlocal continuum and exhibits zero-energy periodic modes of instability. It is the case of a micromodulus in the form of a Delta function spiking at the horizon.
- (7) Although, in bond-based peridynamics, the dimension of the micromodulus as a fundamental characteristic of material stiffness (e.g., N/m⁶ in 3D) is perfectly justified mathematically, it is strange from the physical viewpoint. A better way of defining the elastic (and inelastic) material properties might be desired. The state-based thermodynamics has no such problem since the stress in Eq. (6) delivered by the tensorial constitutive law has the regular physical dimension of N/m².

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Appendix A: Waves in 2D Bond-Based Peridynamics

The general equations of motion, Eqs. (1)–(3), for bond-based peridynamics, and for small deformations in plane (x, y) , can be rewritten as

$$\rho \ddot{u}(x, y, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha(r, s) f(x, y, r, s) \, ds \, dr \quad (A1)$$

$$f(x, y, r, s) = C \frac{u(x+s, y+r) - u(x, y)}{\sqrt{r^2 + s^2}} \quad (A2)$$

$$\alpha(r, s) = \begin{cases} 1 & \text{for } \sqrt{r^2 + s^2} < \delta \\ 0 & \text{for } \sqrt{r^2 + s^2} \geq \delta \end{cases} \quad (A3)$$

Let us now seek a solution in terms of one spectral component of a 2D plane wave with wave number κ in the x direction

$$u(x, y, t) = A e^{i\kappa(x-vt)} \quad (A4)$$

where A is the amplitude vector, coaxial with u , and v is the velocity magnitude (a scalar). Substituting this into Eq. (A1), one can eliminate both time t and the vectors, which gives

$$\kappa^2 v^2 = \frac{C}{\rho} \int_{-\delta}^{\delta} \int_{-\sqrt{\delta^2-r^2}}^{\sqrt{\delta^2-r^2}} \frac{1 - e^{i\kappa r}}{\sqrt{r^2 + s^2}} \, ds \, dr \quad (A5)$$

Since the integrand is symmetric with respect to the horizon circle, we have

$$\kappa^2 v^2 = \frac{4C}{\rho} \int_0^{\delta} \int_0^{\sqrt{\delta^2-r^2}} \frac{1 - e^{i\kappa r}}{\sqrt{r^2 + s^2}} \, ds \, dr \quad (A6)$$

With the notation $\bar{C} = \delta^3 C/E$ and $v_0 = \sqrt{E/\rho}$, we obtain

$$\left(\frac{v}{v_0}\right)^2 = \bar{C} I(\kappa \delta) \quad (A7)$$

where

$$I(\kappa \delta) = \frac{4}{\kappa^2 \delta^3} \int_0^{\delta} \int_0^{\sqrt{\delta^2-r^2}} \frac{1 - \cos(\kappa r)}{\sqrt{r^2 + s^2}} \, ds \, dr \quad (A8)$$

Since v^+/v_0 must approach 1 as $\kappa \delta$ tends to 0, $\bar{C}$ must be equal to $1/I(0)$. So we have

$$\left(\frac{v}{v_0}\right)^2 = \frac{I(\kappa \delta)}{I(0)} \quad (A9)$$

Now we may integrate $I(\kappa \delta)$ numerically and evaluate Eq. (A9). This gives a similar dependence of v on κ as the 1D; see the plot in Fig. 2. This similarity means that the properties of elastic peridynamics can be qualitatively studied in one dimension.

Appendix B: Waves in 2D State-Based Peridynamics

Consider now a longitudinal harmonic wave of $u(x, t)$ propagating in the x -direction in a 2D state-based peridynamic continuum. The conservation of momentum can be expressed as

$$\rho(X) \ddot{u}(X, t) = \int_{\Omega} \alpha(X_s - X) [\sigma(X_s, t) K^{-1}(X_s) + \sigma(X, t) K^{-1}(X)] X_s \, d\Omega(X_s) \quad (B1)$$

where $\sigma = C : \varepsilon$, and $\varepsilon = (F + F^T)/2 - I$. The deformation gradient tensor F is defined as

$$F(X, t) = K^{-1}(X) \int_{\Omega} \alpha(X_s - X) [(X_s - X) \otimes (x_s - x)] \, d\Omega(X_s) \quad (B2)$$

where x_s and x are the points corresponding to X_s and X in the deformed configuration

$$K(X_s) = K(X) = \int_{\Omega} \alpha(X_s - X) [(X_s - X) \otimes (X_s - X)] \, d\Omega(X_s) \quad (B3)$$

In 2D, the position difference in deformed and undeformed configurations and displacement difference can be written explicitly as

$$\mathbf{X}_s - \mathbf{X} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mathbf{u}_s - \mathbf{u} = \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}, \quad \mathbf{x}_s - \mathbf{x} = \begin{pmatrix} \xi + \delta u \\ \eta + \delta v \end{pmatrix} \quad (\text{B4})$$

Thus, the shape tensors $\mathbf{K}(\mathbf{X}_s)$ and $\mathbf{K}(\mathbf{X})$ are

$$\mathbf{K}(\mathbf{X}_s) = \mathbf{K}(\mathbf{X}) = \int_{-\delta}^{\delta} \int_{-\sqrt{\delta^2 - \xi^2}}^{\sqrt{\delta^2 - \xi^2}} \begin{pmatrix} \xi^2 & \xi\eta \\ \xi\eta & \eta^2 \end{pmatrix} d\xi d\eta = \frac{\pi\delta^4}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{B5})$$

Then the deformation gradient tensor $\mathbf{F}$ becomes

$$\mathbf{F}(\mathbf{X}) = \mathbf{K}^{-1} \int_{\Omega} \alpha(\xi, \eta) \begin{pmatrix} \xi^2 + \xi\delta u & \xi\eta + \eta\delta u \\ \xi\eta + \xi\delta v & \eta^2 + \eta\delta v \end{pmatrix} d\xi d\eta \quad (\text{B6})$$

Substituting the expressions for $\mathbf{K}$ and $\mathbf{F}$ into Eq. (B2), we have

$$\boldsymbol{\varepsilon}(\mathbf{X}, t) = \mathbf{K}^{-1} \int_{\Omega} \alpha(\xi, \eta) \begin{pmatrix} \xi\delta u & (\eta\delta u + \xi\delta v)/2 \\ (\eta\delta u + \xi\delta v)/2 & \eta\delta v \end{pmatrix} d\xi d\eta \quad (\text{B7})$$

Similarly, $\mathbf{F}(\mathbf{X}_s)$ and $\boldsymbol{\varepsilon}(\mathbf{X}_s, t)$ can be derived by fixing $\mathbf{X}$ and $\mathbf{X}_s$ and introducing another pair of dummy variables r and s

$$\mathbf{F}(\mathbf{X}_s, t) = \mathbf{K}^{-1} \int_{\Omega} \alpha(r, s) \begin{pmatrix} \xi^2 + \xi\delta u' & \xi\eta + \eta\delta u' \\ \xi\eta + \xi\delta v' & \eta^2 + \eta\delta v' \end{pmatrix} dr ds \quad (\text{B8})$$

$$\boldsymbol{\varepsilon}(\mathbf{X}_s, t) = \mathbf{K}^{-1} \int_{\Omega} \alpha(r, s) \begin{pmatrix} \xi\delta u' & (\eta\delta u' + \xi\delta v')/2 \\ (\eta\delta u' + \xi\delta v')/2 & \eta\delta v' \end{pmatrix} dr ds \quad (\text{B9})$$

where $\delta u' = u(X + \xi + r, Y + \eta + s) - u(X + \xi, Y + \eta)$ and $\delta v' = v(X + \xi + r, Y + \eta + s) - v(X + \xi, Y + \eta)$ are the displacement differences at point $(X + \xi, Y + \eta)$. Substituting now $\delta v = \delta v' = 0$, $\delta u = e^{i\kappa(X-vt)}(e^{i\kappa\xi} - 1)$ and $\delta u' = e^{i\kappa(X+\xi-vt)}(e^{i\kappa r} - 1)$ into Eqs. (B7) and (B9), we have

$$\boldsymbol{\varepsilon}(\mathbf{X}, t) = \frac{4}{\pi\delta^4} e^{i\kappa(X-vt)} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\varepsilon}(\mathbf{X}_s, t) = \frac{4}{\pi\delta^4} e^{i\kappa(X+\xi-vt)} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad (\text{B10})$$

$$\text{where } A = \int_{-\delta}^{\delta} \int_{-\sqrt{\delta^2 - \xi^2}}^{\sqrt{\delta^2 - \xi^2}} (e^{i\kappa\xi} - 1) \xi d\eta d\xi \\ = 4i \int_0^{\delta} \xi \sqrt{\delta^2 - \xi^2} \sin(\kappa\xi) d\xi \quad (\text{B11})$$

Assuming a linear elastic constitutive law, $\sigma_{ij} = \delta_{ij}(\lambda + G)\varepsilon_{kk} + 2G\varepsilon_{ij}$, the stresses can be determined from the foregoing strains

$$\boldsymbol{\sigma}(\mathbf{X}, t) = \frac{4(\lambda + 2G)}{\pi\delta^4} e^{i\kappa(X-vt)} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \\ \boldsymbol{\sigma}(\mathbf{X}_s, t) = \frac{4(\lambda + 2G)}{\pi\delta^4} e^{i\kappa(X+\xi-vt)} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad (\text{B12})$$

Substitution into Eq. (B1) gives

$$(-i\kappa v)^2 \rho \begin{pmatrix} e^{i\kappa(X-vt)} \\ 0 \end{pmatrix} = A \frac{32(\lambda + 2G)}{\pi^2\delta^8} e^{i\kappa(X-vt)} \\ \int_0^{\delta} 2\sqrt{\delta^2 - \xi^2} \begin{pmatrix} \xi(1 + e^{i\kappa\xi}) \\ 0 \end{pmatrix} d\xi \quad (\text{B13})$$

Since the second row in the vectors is zero, we calculate only the first equation. Next, we cancel out $e^{i\kappa(X-vt)}$ on both sides and drop the antisymmetric part of the integral

$$-\left(\frac{v}{v_0}\right)^2 = \frac{16A^2}{\kappa^2\pi^2\delta^8} \quad (\text{B14})$$

where $v_0 = \sqrt{(\lambda + 2G)/\rho}$ and A is given in Eq. (B14). Then, evaluating A gives

$$A = 2i \frac{\pi\delta^2 J_2(\delta\kappa)}{\kappa}, \quad (\text{B15})$$

where $J_2(x)$ is the Bessel function of the first kind. Substitution into Eq. (B14) now furnishes

$$\left(\frac{v}{v_0}\right)^2 = \left[8 \frac{J_2(\delta\kappa)}{(\kappa\delta)^2}\right]^2 = \left[8 \frac{J_2(2\pi\nu_L)}{(2\pi\nu_L)^2}\right]^2 \quad (\text{B16})$$

The curve of v^+/v_0 is plotted in Fig. 3.

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