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150 East Tenth Street
Harper Hall 152
Claremont, CA 91711 -5909

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a. REPORT	b. ABSTRACT	c. THIS PAGE			Hrushikesh Mhaskar
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RPPR Final Report
as of 02-Oct-2018

Agency Code:

Proposal Number: 66723CS **Agreement Number: W911NF-15-1-0385**

INVESTIGATOR(S):

Name: Ph.D. Hrushikesh Mhaskar
Email: hmhaska@gmail.com
Phone Number: 6269147022
Principal: Y

Organization: **Claremont Graduate University**
Address: 150 East Tenth Street, Claremont, CA 917115909
Country: USA
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Phone: (626) 914-7022
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STEM Degrees: 0 **STEM Participants:** 0

Major Goals: The main objective of our research program is to develop new mathematical theories, along with effective methods, efficient algorithms, and near real-time implementation schemes, for the decomposition of any (blind--source) real-world signal or time series into its primary signal building blocks, called atomic components; thereby introducing innovative powerful signal processing methods for time--frequency analysis of non--stationary and non--linear signals. The mathematical tools developed in our work can also be applied to the study of machine learning problems, and in particular, to gain some valuable insight in the expressive power of deep neural networks.

Accomplishments: See attached pdf file.

Training Opportunities: Nothing to Report

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Results Dissemination: Research Findings
(1) Our research findings resulted in fourteen (14) published papers and one (1) manuscript submitted for publication. They are available for download from https://arxiv.org/find/cs/1/au:+Mhaskar_H/0/1/0/all/0/1 or https://www.researchgate.net/profile/Charles_Chui/publications
(2) Hrushikesh Mhaskar co-edited a two volume collection of papers: Novel methods in harmonic analysis, Vol. 1, 2, Birkh\"auser, 2017.
(3) Newly founded Open-Access Mathematics Journal:
Charles Chui founded the open-access journal, “Mathematics of Computation and Data Science”, as a section of Frontiers of Applied Mathematics and Data Science, which was officially launched in September, 2016, with Charles as its Chief Editor, and Hrushikesh Mhaskar serving on the Board of Associate Editors. Please go to: http://journal.frontiersin.org/journal/applied-mathematics-and-statistics
(4) International Conferences, Workshop, and Colloquia presentations:
Hrushikesh Mhaskar:
(a) Harmonic analysis, Oberwolfach, August, 2015
(b) High dimensional approximation, Bonn, September, 2015
(c) International conference on applied mathematics, Hong Kong, May, 2016 (Invited talk)
(d) University of Central Florida, Orlando, October, 2016 (Invited colloquium)
(e) Banff International Research Station workshop on applied harmonic analysis, massive data sets, machine learning, and signal processing, Oaxaca, October, 2016 (Invited talk)
(f) Indian Mathematical Consortium-Americal Mathematical Society meeting, Varanasi, December, 2016 (Invited talk)
(g) Indian Mathematical Society Annual meeting, Kalyani, December, 2016 (Plenary talk)
(h) Indian Institute of Techology, Chennai, January, 2017 (Invited colloquium)
(i) Indian Institute of Techology, Mumbai, January, 2017 (Invited colloquium)
(j) International workshop on optimal point configurations and orthogonal
polynomials, Castro Urdiales, Spain, April, 2017 (Invited talk)
(k) International conference on computational harmonic analysis, Fudan University, Shanghai, China, May, 2017 (Invited talk)
(l) Harmonic analysis, Oberwolfach, March 2018
(m) San Diego State University, April, 2018 (invited seminar)
(n) University of California, Santa Barbara, April, 2018 (invited colloquium)
(o) Massachussetts Institute of Technology, Cambridge, May 2018 (invited seminar)
(p) International conference on applied mathematics, Hong Kong, May 2018 (invited talk)
(q) Frontiers of high dimensional computation, MATRIX, Sedgwick, Australia, June 2018 (invited talk)

Charles Chui:

(a) Helmholtz Zentrum Munich, Germany, August, 2016 (Invited Colloquium Speaker)
(b) Hasenwinkel, Germany, September, 2016 (Plenary Speaker, International workshop of Approximation Methods and Data Analysis)
(c) University of Maryland, February, 2017 (Norbert Wiener Center Distinguished Lecturer, FFT 2017)
(d) Hong Kong, March, 2017 (Plenary Speaker, International Conference on Mathematics of Data Science)
(e) Stanford University, California, June 2017 (Invited participant, Math+Stats
+X Conference to honor David Donoho’s 60 th Birthday)
(f) Analysis Symposium in honor of Professor Jaap Korevaar's 95-th Birthday, University of Amsterdam, Netherlands, January 24, 2018 (Main speaker)

Honors and Awards: 1. International workshop on approximation methods and data analysis in honor of Hrushikesh Mhaskar on the occasion of his 60th birthday (Hasenwinkel, Germany, September, 2016)
2. Hrushikesh Mhaskar was invited as Scholar-in-Residence at Indian Institute of Techology, Gandhinagar, India in December, 2016.
3. Charles Chui was selected the Norbert Wiener Distinguished Lecturer for the year 2017.

Protocol Activity Status:

Technology Transfer: Nothing to Report

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ARTICLES:

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Applied and Computational Harmonic Analysis
Publication Identifier Type: DOI Publication Identifier: 10.1016/j.acha.2016.06.007 47 48 1 1063-5:
Volume: 44 Issue: 3 First Page #: 611
Date Submitted: 9/28/18 12:00AM Date Published: 5/1/18 7:36AM
Publication Location:
Article Title: A unified framework for harmonic analysis of functions on directed graphs and changing data
Authors: Hrushikesh N. Mhaskar
Keywords: Kernel construction, directed graphs, changing data problems, extension problems, wavelet--like representation, Tauberian theorem
Abstract: We present a general framework for studying harmonic analysis of functions in the settings of various emerging problems in the theory of diffusion geometry. We study the question of constructing kernels for embedding a directed graph into a manifold, and argue that our construction is essentially the only way to do so using discretizations of kernels. We consider the question of how functions defined on one space evolves to another space. We introduce a new definition of distance between points on two spaces, construct localized kernels based on the two spaces and certain interaction parameters, and study the evolution of smoothness of a function on one space to its lifting to the other space via the landmarks. We develop novel mathematical tools that enable us to study these seemingly different problems in a unified manner.
Distribution Statement: 1-Approved for public release; distribution is unlimited.
Acknowledged Federal Support: **Y**

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Applied and Computational Harmonic Analysis
Publication Identifier Type: DOI Publication Identifier: doi.org/10.1016/j.acha.2016.12.005
Volume: 44 Issue: 1 First Page #: 165
Date Submitted: 9/28/18 12:00AM Date Published: 1/1/18 8:00AM
Publication Location:
Article Title: Representation of functions on big data associated with directed graphs
Authors: C. K. Chui, H. N. Mhaskar, and X.Zhuang
Keywords: Function approximation, Directed graphs, clustering, local harmonic analysis
Abstract: This paper is an extension of the previous work of Chui, Filbir, and Mhaskar (Appl. Comput. Harm. Anal. 38 (3) 2015:489-509), not only from numeric data to include non--numeric data as in that paper, but also from undirected graphs to directed graphs (called digraphs, for simplicity). Besides theoretical development, this paper introduces effective mathematical tools in terms of certain data--dependent orthogonal systems for function representation and analysis directly on the digraphs. In addition, this paper also includes algorithmic development and discussion of various experimental results on such data--sets as CORA, Proposition, and Wiki--votes.
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Volume: 45 Issue: First Page #: 436
Date Submitted: 9/28/18 12:00AM Date Published:
Publication Location:
Article Title: A Fourier-invariant method for locating point-masses and computing their attributes
Authors: C. K. Chui, H. N. Mhaskar
Keywords: point mass separation, Hermite moments, localized kernels, polynomial frames
Abstract: Motivated by the interest of observing the growth of cancer cells among normal living cells and exploring how galaxies and stars are truly formed, the objective of this paper is to introduce a rigorous and effective method for counting point-masses, determining their spatial locations, and computing their attributes. Based on computation of Hermite moments that are Fourier-invariant, our approach facilitates the processing of both spatial and Fourier data in any dimension.
Distribution Statement: 1-Approved for public release; distribution is unlimited.
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Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Applied and Computational Harmonic Analysis
Publication Identifier Type: DOI Publication Identifier: doi.org/10.1016/j.acha.2017.12.007
Volume: Issue: First Page #:
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Publication Location:
Article Title: A unified method for super-resolution recovery and real exponential-sum separation
Authors: C. K. Chui, H. N. Mhaskar
Keywords: multivariate super-resolution, separation of exponential sums, backward solution of heat equation, Gaussian kernels
Abstract: In this paper, motivated by diffraction of traveling light waves, a simple mathematical model is proposed, both for the multivariate super-resolution problem and the problem of blind-source separation of real-valued exponential sums. This model facilitates the development of a unified theory and a unified solution of both problems in this paper. Our consideration of the super-resolution problem is aimed at applications to fluorescence microscopy and observational astronomy, and the motivation for our consideration of the second problem is the current need of extracting multivariate exponential features in magnetic resonance spectroscopy (MRS) for the neurologist and radiologist as well as for providing a mathematical tool for isotope separation in Nuclear Chemistry. The unified method introduced in this paper can be easily realized by processing only finitely many data, sampled at locations that are not necessarily prescribed in advance, with computational scheme consisting only of mat
Distribution Statement: 1-Approved for public release; distribution is unlimited.
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Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Front. Appl. Math. Stat.
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Volume: Issue: First Page #:
Date Submitted: 9/28/18 12:00AM Date Published: 5/29/18 7:00AM
Publication Location:
Article Title: Deep nets for local manifold learning
Authors: C. K. Chui, H. N. Mhaskar
Keywords: Deep nets, manifold learning, neural networks
Abstract: The problem of extending a function f defined on a training data $\mathcal{C}$ on an unknown manifold $\mathbb{X}$ to the entire manifold and a tubular neighborhood of this manifold is considered in this paper. A deep learning algorithm is developed for finding a local coordinate system for the manifold without eigen-decomposition, which reduces the problem to the classical problem of function approximation on a low dimensional cube. Deep nets (or multilayered neural networks) are proposed to accomplish this approximation scheme by using the training data. Our methods are universal, assuring optimal (a priori) error bounds on the output in terms of the number of derivatives of the target function, without assuming this prior smoothness. The accuracy of approximation adjusts itself automatically depending only upon the local smoothness of the target function.
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Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Journal of Computational Physics
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Volume: 365 Issue: First Page #: 149
Date Submitted: 9/28/18 12:00AM Date Published: 3/21/18 2:00PM
Publication Location:
Article Title: Minimum Sobolev norm interpolation of scattered derivative data
Authors: S. Chandrasekaran, C. H. Gorman, H. N. Mhaskar
Keywords: minimum Sobolev norm, Birkhoff interpolation, data-defined manifolds, diffusion polynomials.
Abstract: We study the problem of reconstructing a function on a manifold satisfying some mild conditions, given data of the values and some derivatives of the function at arbitrary points on the manifold. While the problem of finding a polynomial of two variables given the values of the polynomial and some of its derivatives at exactly the same number of points as the dimension of the polynomial space is sometimes impossible, we show that such a problem always has a solution in a very general situation if the degree of the polynomials is sufficiently large. We give estimates on how large the degree should be, and give explicit constructions for such a polynomial even in a far more general case. As the number of sampling points at which the data is available increases, our polynomials converge to the target function on the set where the sampling points are dense. Numerical examples in single and double precision show that this method is stable, efficient, and of high-order.
Distribution Statement: 1-Approved for public release; distribution is unlimited.
Acknowledged Federal Support: **Y**

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 4-Under Review
Journal: inverse problems
Publication Identifier Type: Publication Identifier:
Volume: Issue: First Page #:
Date Submitted: 9/28/18 12:00AM Date Published:
Publication Location:
Article Title: Stable soft extrapolation of entire functions
Authors: D. Batenkov, L. Demanet, H. N. Mhaskar
Keywords: super-resolution, band-limited functions, weighted approximation, soft extrapolation
Abstract: In this paper we consider soft extrapolation of entire functions of finite order and type (containing the class of bandlimited functions as a special case), multiplied by a super-exponentially decaying window (such as a Gaussian). We consider a weighted least-squares polynomial approximation with judiciously chosen number of terms and a number of samples which scales linearly with the degree of approximation. It is shown that this simple procedure provides stable recovery with an extrapolation factor which scales logarithmically with the perturbation level and is inversely proportional to the characteristic lengthscale of the function. The algorithm is asymptotically minimax, in the sense that there is essentially no better algorithm yielding meaningfully lower error over the same smoothness class. Our results show that the amount of achievable super-resolution is inversely proportional to the object size, and therefore can be significant for small objects.
Distribution Statement: 1-Approved for public release; distribution is unlimited.
Acknowledged Federal Support: **Y**

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: frontiers in applied mathematics and statistics
Publication Identifier Type: DOI Publication Identifier: doi.org/10.3389/fams.2018.00014
Volume: Issue: First Page #:
Date Submitted: 9/28/18 12:00AM Date Published: 5/17/18 2:00PM
Publication Location:
Article Title: Construction of Neural Networks for Realization of Localized Deep Learning
Authors: C. K. Chui, S.-B. Lin, D.-X. Zhou
Keywords: deep neural networks, local manifold learning
Abstract: The subject of deep learning has recently attracted users of machine learning from various disciplines, including: medical diagnosis and bioinformatics, financial market analysis and online advertisement, speech and handwriting recognition, computer vision and natural language processing, time series forecasting, and search engines. However, theoretical development of deep learning is still at its infancy. The objective of this paper is to introduce a deep neural network (also called deep-net) approach to localized manifold learning, with each hidden layer endowed with a specific learning task. For the purpose of illustrations, we only focus on deep-nets with three hidden layers, with the first layer for dimensionality reduction, the second layer for bias reduction, and the third layer for variance reduction. A feedback component is also designed to deal with outliers. The main theoretical result in this paper is the order of approximation of the regression function.
Distribution Statement: 1-Approved for public release; distribution is unlimited.
Acknowledged Federal Support: **Y**

CONFERENCE PAPERS:

Publication Type: Conference Paper or Presentation **Publication Status:** 1-Published
Conference Name: Thirty-First AAAI Conference on Artificial Intelligence
Date Received: 23-Aug-2017 Conference Date: 05-Feb-2017 Date Published: 14-Feb-2017
Conference Location: San Fransisco, CA
Paper Title: When and why are deep networks better than shallow ones?
Authors: H.N. Mhaskar, Q. Liao, and T. A. Poggio
Acknowledged Federal Support: **Y**

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Accomplishments

Title: Non–stationary signal analysis with applications to blind-source processing and imaging
Contract number: W911NF1510385
Reporting period: July 15, 2015–July 14, 2018.
Lead Agency: Army Research Office
Scientific division/directorate: Research Area 5, Computing Science
Technical area/program officer: Area 5.1, Dr. J. Michael Coyle

The research during this project has resulted in fourteen (14) published papers [10, 18, 19, 11, 21, 22, 13, 20, 17, 8, 9, 2, 7, 5], and one submitted manuscript [1].

Introduction

The starting point of our investigation is the problem of separating atomic components of a non-stationary signal

$$F(t) = \sum_{k=1}^{K} f_k(t), \qquad f_k(t) = A_k(t)\exp(i\phi_k(t)). \tag{1}$$

We developed novel mathematical tools for solving this problem. This research suggested new application areas for these tools, notably including a mathematical understanding of deep learning and a similar analysis of non-numeric signals represented as functions on a directed graph. We have already discussed many of these works in detail in the previously submitted annual reports. In this report, we will describe in detail the new results obtained during the last year of the project, and summarize the accomplishments over the entire project.

Scientific barriers and our approach

Prior to the start of our project, there were two main techniques for separating components of a blind-source non-stationary signal: empirical mode decomposition (EMD) and synchro-squeezing transform (SST). EMD is a purely ad hoc method, and has many disadvantages in terms of accurately separating components with frequencies close to each other. SST is founded on solid mathematics, but requires an a priori knowledge of the number of components. Also, an elaborate procedure is required to recuperate the components after the frequencies are detected. We proved that a non-stationary signal admits an approximation of the form

$$F(t+u) = \sum_{k=1}^{K} f_k(t)\exp(iu\phi_k'(t)), \qquad t, u \in \mathbb{R}. \tag{2}$$

For each fixed t, this is a stationary signal in u. This allows us to determine the number K of components and find the components $f_k(t)$ using certain localized kernels developed during the PI's previously funded ARO grants. Combining these with EMD as a preprocessing tool, we are able to determine accurately both the number of components and the components themselves of a non-stationary signal even when the frequencies are very close by [10].

The signal $F(t+u)$ as a function of u is clearly the Fourier transform of a discretely supported measure: $\sum_{k=1}^{K} f_k(t)\delta_{\phi_k'(t)}$. Simple as it is, this observation is a seminal one.

In many applications in biology, astronomy, etc., the observations are not in the form (2), but may be of the form

$$\frac{1}{(4\pi\sigma^2)^{q/2}}\int_{\mathbb{R}^q} \exp\left(-\frac{|x-y|^2}{4\sigma^2}\right) d\mu(y), \tag{3}$$

for some measure μ. The problem of recuperating μ from the information of the form (3) is a highly ill-posed inverse problem, which is typically solved using regularization techniques. This problem is equivalent to the problem of recuperating the measure given its integrals against Hermite functions. We obtained localized kernels based on Hermite polynomials using theoretical results in [19, 18] to solve this problem in the case when μ is a discrete measure [8, 9]. During the last year of the grant, we considered the harder problem when μ is a compactly supported measure, not necessarily discrete.

The second important direction suggested by the seminal observation above is to consider non-numeric signals, represented as functions on graphs, as well as numerical signals supported on changing manifolds (such as video signals, where the feature space is different from frame to frame). The first question to consider is what should play the role of the Fourier transform in these cases. We addressed this question by developing harmonic analysis on changing manifolds ([19]) and directed graphs ([11]).

The mathematical tools developed during this research enabled us to enter into a new emerging area: the analysis of deep networks. The main problem in this area is to explain why deep networks do a much better job of classification than shallow networks. This question was analyzed in a series of papers [21, 22, 13, 7, 5]. Another question is to design deep networks specific to a given application rather than the ad hoc approach used in practice. In [20], we have illustrated one such idea with applications to the diabetic blood sugar prediction which are better than most state of the art results in academic literature.

Accomplishments of the last year

The paper [1] considers the problem of recuperating a compactly supported measure μ from its transform of the form (3), without assuming that μ is a discrete measure. By taking Fourier transform and rescaling judiciously, we assume the observations in the form

$$g(\omega) = \hat{F}(\omega) + \epsilon(\omega) = \exp(-\omega^2/2)f(\omega) + \epsilon(\omega), \tag{4}$$

where $f(\omega)$ is the Fourier transform of μ and ϵ is a perturbation. Since the problem of recuperation of μ is equivalent to that of recuperation of f, we consider the problem of finding f on the whole real line, given its values on a finite interval of the form $[-\Omega, \Omega]$. It is already observed in [12] that this problem does not have a solution when μ is not a discrete measure. Therefore, the scientific barrier is to obtain the minimal value of Ω and the minimal number of observations of g so as to allow a reasonable approximation to f on the whole real line (and even the complex domain), and estimate the accuracy of this approximation on different parts of the complex domain. From an approximation theory point of view (without the perturbation ϵ), this problem was studied in [16, 14, 15]. This previous work gave a complete characterization of the support of μ in terms of approximation properties of f. The novelty of [1] is twofold. We prove the correct bounds on the error in recuperation of f in different parts of the complex domain, and show that these bounds cannot be improved upon by demonstrating a "dark object". Secondly, we prove that a straightforward solution of a suitable least square problem leads to the recuperation.

The paper [17] is an invited paper in celebration of the 80th birthday of Professor Ian Sloan. The scientific barrier is as follows. In the our work on manifold learning, supported by previous ARO grants to the PI, a crucial role is played by certain quadrature formulas. However, these formulas cannot be obtained without the knowledge of the eigenfunctions of the Laplace-Beltrami operator on the unknown manifold. In this paper, we give a construction of approximate quadrature formulas without such a knowledge, and study certain properties of these approximate quadrature formulas.

The paper [2] is a novel application of the mathematical tools developed during this project and the projects of the PI funded previously by ARO. The scientific barrier is as follows. In image completion, it is necessary to extend a function together with its derivatives given the values of the function and the derivatives at finitely many points [4, 6]. Similar problems appear also in numerical solution of partial differential equations on domains with slits [3]. It is well known that it is impossible in general to interpolate both the values of a multivariate function and its derivatives by polynomials if the degree of the interpolatory polynomial is required to be the minimal commensurate with the data. This problem is known as Birkhoff interpolation problem. We demonstrate that if the degree of the interpolatory polynomial is proportional to the minimal separation among the points at which the data is available, then the problem can always be solved and results in convergent interpolatory processes. This is a substantial generalization of a line of research started by the famous mathematician Erdős. We consider interpolation on arbitrary manifolds using what we have called diffusion polynomials.

The objective of [5] is to introduce a deep neural network (also called deep-net) approach to localized manifold learning, with each hidden layer endowed with a specific learning task. For the purpose of illustrations, we only focus on deep-nets with three hidden layers, with the first layer for dimensionality reduction, the second layer for bias reduction, and the third layer for variance reduction. A feedback component is also designed to deal with outliers. The main theoretical result in this paper is the order $\mathcal{O}(m^{-2s/(2s+d)})$ of approximation of the regression function with regularity s, in terms of the number m of sample points, where the (unknown) manifold dimension d replaces the dimension D of the sampling (Euclidean) space for shallow nets.

Conclusions

1. We gave efficient algorithms to separate the components of a non-stationary signal even in the presence of very close-by frequencies.
2. Our algorithms are simple, and are demonstrated to have a superior performance to that of both EMD and SST techniques. In particular, unlike these approaches, we are able to determine the number of components present in the signal.
3. We extended this theory to the case when the signal is a convolution of a Gaussian with a compactly supported measure, contaminated with some noise.
4. We developed harmonic analysis on a directed graph, a first of its kind accomplishment, that enables us to formulate and solve the separation problem in non-numeric settings, where the signal is given on a directed graph.
5. We developed a theoretical understanding for out-of-sample extension problems in the setting of manifold learning, demonstrating how deep networks can be constructed in this setting for this purpose.
6. We obtained fundamental results to explain when and why deep networks give a superior performance to shallow networks.

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