ARL-TR-8697 • MAY 2019

The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma: 3. Multicomponent Plasma

by Michael Grinfeld and Pavel Grinfeld

NOTICES

Disclaimers

ARL-TR-8697 ● MAY 2019

The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma: 3. The Multicomponent Plasma

by Michael Grinfeld
Weapons and Materials Research Directorate, CCDC Army Research Laboratory

Pavel Grinfeld
Drexel University

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED (From - To)
May 2019	Technical Report	2 January–15 April 2019

4. TITLE AND SUBTITLE
The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma:
3. The Multicomponent Plasma

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Michael Grinfeld and Pavel Grinfeld

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
US Army Combat Capabilities Development Command
Army Research Laboratory
ATTN: FCDD-RLW-PC
Aberdeen Proving Ground, MD 21005

8. PERFORMING ORGANIZATION REPORT NUMBER
ARL-TR-8697

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT
This report is the third part of the study published under the common umbrella *The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma*. In Part 1, we formulated a novel approach to thermodynamics of one-component heterogeneous systems completely or partially filled with a liquid substance in plasma state. The approach is based on the use of the Gibbs variational principles, and it enables efforts to address a variety of problems relating to equilibrium and stability of such systems. In this third part, the results of the Part 1 are generalized for the multicomponent plasma systems.

15. SUBJECT TERMS
plasma, thermodynamics, Gibbs variational principles, plasma stability, equations of state

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
					Michael Grinfeld
a. REPORT	b. ABSTRACT	c. THIS PAGE	UU	15	19b. TELEPHONE NUMBER (Include area code)
Unclassified	Unclassified	Unclassified			(410) 278-7030

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

Contents

1. Introduction

In Part 1 of this series of reports,[1,2] we formulated a novel approach to thermodynamics of heterogeneous systems completely or partially filled with a liquid substance in a plasma state. The approach is based on the use of the Gibbs variational principles, and it enables efforts to address a variety of problems relating to equilibrium and stability of such systems.

The general motivation for this series of reports is discussed in Grinfeld and Grinfeld.[1] The only issue not discussed so far is the reason for separate discussions of the one- and multicomponent cases. The separate analysis of the one-component case is motivated by the methodological reason: the 1-D case is simpler and results in less-cumbersome and more-transparent relationships, as was discussed in Grinfeld and Grinfeld.[1]

The necessity for the multicomponent analysis is mostly physical and absolutely straightforward: there is a strong repulsion between the electric charges of the same sign. This repulsion can be compensated for by the fact that a strong attraction shows up between the charges of opposite signs. This is why practically all macroscopic and nanoscale bodies (atoms and molecules) are almost electrically neutral. Therefore, for the phenomenological macroscopic analysis, we have to deal with, at least, two-component charged liquids: the positively charged component and the negatively charged component. This is the bare minimum; quite often it is necessary to introduce many more components.

For the sake of brevity and transparency, though, we will dwell on the two-component charged liquids, which will be called the electronic and ionic components.

2. The System under Study

Assume that our charged liquid has two components with the mass densities of charges $\rho^e(z)$ and $\rho^i(z)$ with the fixed total amounts of each of them, as follows:

$$\int_\Omega d\Omega \rho^e(z) = M^e, \quad \int_\Omega d\Omega \rho^i(z) = M^i, \tag{1}$$

where M^e and M^i are the total masses of the electronic and ionic liquids. Let σ^e and σ^i be the charge per unit mass of the electronic and ionic liquids, respectively. We assume that the mixture is characterized by the cooperative internal energy density per unit volume,

$$U = U\left(\rho^e, \rho^i, \mathrm{H}\right), \tag{2}$$

and the entropy density H per unit volume.

We postulate the following relationship for the electrostatic energy E_{elect}:

$$E_{eles} \equiv \frac{1}{2}\int_{\Omega} d\Omega \int_{\Omega} d\Omega^* \frac{\left(\sigma^e \rho^e + \sigma^i \rho^i\right)\left(\sigma^e \rho^{e*} + \sigma^i \rho^{i*}\right)}{\left|\vec{z} - \vec{z}^*\right|}. \tag{3}$$

Here and in the following, the notation ρ^{e*} means that the function ρ is treated as function of the independent spatial variable z^*, whereas the notation ρ is treated as function of the independent variable z.

Per the Gibbs[3] methodology, when analyzing equilibrium and stability in the closed thermodynamic systems, we have to minimize the functional Φ,

$$\Phi \equiv \frac{1}{2}\int_{\Omega} d\Omega \int_{\Omega} d\Omega^* \frac{\left(\sigma^e \rho^e + \sigma^i \rho^i\right)\left(\sigma^e \rho^{e*} + \sigma^i \rho^{i*}\right)}{\left|\vec{z} - \vec{z}^*\right|} + \int_{\Omega} d\Omega U\left(\rho^e, \rho^i, \mathrm{H}\right), \tag{4}$$

under the mass constraints (Eq. 1) and the entropy,

$$\int_{\Omega} d\Omega \mathrm{H}(z) = S, \tag{5}$$

where S is the total entropy of the system under consideration.

To address this minimum problem with isoperimetric constraint we have to consider the unconstraint minimum for the functional $\tilde{\Phi}$,

$$\begin{aligned}\tilde{\Phi} = &\int_{\Omega} d\Omega \left[U\left(\rho^e, \rho^i, \mathrm{H}\right) - \Lambda^e \rho_e - \Lambda^i \rho^i - T\mathrm{H}\right] + \\ &\frac{1}{2}\int_{\Omega} d\Omega \int_{\Omega} d\Omega^* \frac{\left(\sigma^e \rho^e + \sigma^i \rho^i\right)\left(\sigma^e \rho^{e*} + \sigma^i \rho^{i*}\right)}{\left|\vec{z} - \vec{z}^*\right|},\end{aligned} \tag{6}$$

where Λ^e, Λ^i, and T are the indefinite Lagrange multipliers, associated with the constraints of Eqs. 1 and 5.

3. First Variation and Equilibrium Conditions

Following the methodology of Grinfeld and Grinfeld,[1] we arrive at the formula of the first variation of the functional $\tilde{\Phi}$, which reads

$$\delta\tilde{\Phi} = \int_{\Omega} d\Omega \begin{bmatrix} \left(U_{\rho^e} - \Lambda^e + \sigma^e \int_{\Omega} d\Omega^* \frac{\sigma^e \rho^{e*} + \sigma^i \rho^{i*}}{\left|\vec{z} - \vec{z}^*\right|} \right) \delta\rho^e + \\ \left(U_{\rho^i} - \Lambda^i + \sigma^i \int_{\Omega} d\Omega^* \frac{\sigma^e \rho^{e*} + \sigma^i \rho^{i*}}{\left|\vec{z} - \vec{z}^*\right|} \right) \delta\rho^i + \\ \left(U_{\mathrm{H}} - T \right) \delta\mathrm{H} \end{bmatrix}. \tag{7}$$

Separating the independent variations in Eq. 7, we arrive at the following equations of equilibrium:

$$\begin{aligned} U_{\rho^e} + \sigma^e \int_{\Omega} d\Omega^* \frac{\sigma^e \rho^{e*} + \sigma^i \rho^{i*}}{\left|\vec{z} - \vec{z}^*\right|} &= \Lambda^e \\ U_{\rho^i} + \sigma^i \int_{\Omega} d\Omega^* \frac{\sigma^e \rho^{e*} + \sigma^i \rho^{i*}}{\left|\vec{z} - \vec{z}^*\right|} &= \Lambda^i \end{aligned}, \tag{8}$$

and

$$U_{\mathrm{H}} = T\,. \tag{9}$$

Summarizing, in order to analyze the equilibrium distributions of the charged particles and the entropy, we have to solve the two integral equations of Eq. 8 combined with the algebraic Eq. 9 and the constraints of Eqs. 1 and 5.

4. Second Variation and the Stability Conditions

For the second variation in vicinity of equilibrium configuration we get the following formula:

$$\delta^2\tilde{\Phi} = \int_{\Omega} d\Omega \begin{pmatrix} U_{\rho_e\rho_e}\delta\rho_e\delta\rho_e + 2U_{\rho_e\rho_i}\delta\rho_e\delta\rho_i + U_{\rho_i\rho_i}\delta\rho_i\delta\rho_i + \\ 2U_{\rho_e\mathrm{H}}\delta\rho_e\delta\mathrm{H} + 2U_{\rho_i H}\delta\rho_i\delta\mathrm{H} + U_{\mathrm{HH}}\delta\mathrm{H}^2 + \\ \int_{\Omega} d\Omega^* \frac{\sigma_e\sigma_e\delta\rho_e^*\delta\rho_e + 2\sigma_e\sigma_i\delta\rho_i^*\delta\rho_e + \sigma_i\sigma_i\delta\rho_i^*\delta\rho_i}{\left|\vec{z} - \vec{z}^*\right|} \end{pmatrix}. \tag{10}$$

Equation 10 can be rewritten as

$$\delta^2\tilde{\Phi} = \int_\Omega d\Omega \begin{pmatrix} U_{\rho_e\rho_e} a_e^2 + 2U_{\rho_e\rho_i} a_e a_i + U_{\rho_i\rho_i} a_i^2 + \\ 2U_{\rho_e \mathrm{H}} a_e h + 2U_{\rho_i H} a_i h + U_{\mathrm{HH}} h^2 + \\ \int_\Omega d\Omega^* \frac{\sigma_e^2 a_e^* a_e + 2\sigma_e \sigma_i a_i^* a_e + \sigma_i^2 a_i^* a_i}{\left|\vec{z} - \vec{z}^*\right|} \end{pmatrix}, \tag{11}$$

where we used the notations

$$a^e \equiv \delta\rho^e\ ,\ a^i \equiv \delta\rho^i\ ,\ h \equiv \delta\mathrm{H}\,. \tag{12}$$

The necessary conditions of stability of an equilibrium configuration is the following: the second variation $\delta^2\tilde{\Phi}$ must be non-negative for all the variations a^e, a^i, and h, satisfying the (isoperimetric-type) constraints

$$\int_\Omega d\Omega a^e(z) = 0,\ \int_\Omega d\Omega a^i(z) = 0,\ \int_\Omega d\Omega h(z) = 0\,. \tag{13}$$

Let us explore the extrema of the integral quadratic form Eq. 11 under the linear integral constraints Eq. 13 and the integral quadratic constraint:

$$\int_\Omega d\Omega \left(a_e^2 + a_i^2 + \kappa^2 h^2\right) = 1\,, \tag{14}$$

where κ is a positive constant.

Using again the method of indefinite multipliers, we arrive at unconstraint minimization of the integral quadratic form

$$\tilde{\Pi}(a,h) \equiv \int_\Omega d\Omega \begin{bmatrix} U_{\rho_e\rho_e} a_e^2 + 2U_{\rho_e\rho_i} a_e a_i + U_{\rho_i\rho_i} a_i^2 + \\ 2U_{\rho_e \mathrm{H}} a_e h + 2U_{\rho_i H} a_i h + U_{\mathrm{HH}} h^2 + \\ \int_\Omega d\Omega^* \frac{\sigma_e^2 a_e^* a_e + 2\sigma_e \sigma_i a_i^* a_e + \sigma_i^2 a_i^* a_i}{\left|\vec{z} - \vec{z}^*\right|} - \\ \alpha^e a^e - \alpha^i a^i - \beta h - \lambda\left(a^I a^I + \kappa^2 h^2\right) \end{bmatrix}, \tag{15}$$

where α^e, α^i, β, and λ are the Lagrange multipliers associated with the constraints of Eqs. 13 and 15.

Varying the functional $\tilde{\Pi}$, we get

$$
\delta\tilde{\Pi}\left(a,h\right)\equiv\int_{\Omega}d\Omega\left[\begin{array}{c}2U_{\rho_e\rho_e}a_e\delta a_e+2U_{\rho_e\rho_i}\left(\delta a_e a_i+a_e\delta a_i\right)+2U_{\rho_i\rho_i}a_i\delta a_i+\\ 2U_{\rho_e\mathrm{H}}\left(\delta a_e h+a_e\delta h\right)+2U_{\rho_i H}\left(\delta a_e h+a_e\delta h\right)+2U_{\mathrm{HH}}h\delta h+\\ \int_{\Omega}d\Omega^*\frac{\sigma_e^2\delta a_e^* a_e+\sigma_e^2 a_e^*\delta a_e+2\sigma_e\sigma_i\left(\delta a_i^* a_e+a_i^*\delta a_e\right)+\sigma_i^2\left(\delta a_i^* a_i+a_i^*\delta a_i\right)}{\left|\vec{z}-\vec{z}^*\right|}-\\ \alpha^e\delta a^e-\alpha^i\delta a^i-\beta\delta h-\lambda\left(2a^e\delta a^e+2a^i\delta a^i+2\kappa^2 h\delta h\right)\end{array}\right].\quad(16)
$$

Using Eq. 16, we proceed as follows:

$$
\delta\tilde{\Pi}\left(a,h\right)\equiv2\int_{\Omega}d\Omega\left[\begin{array}{l}\left(U_{\rho_e\rho_e}a_e+U_{\rho_e\rho_i}a_i+U_{\rho_e\mathrm{H}}h-\frac{1}{2}\alpha^e-\lambda a^e\right)\delta a_e+\\ \left(U_{\rho_e\rho_i}a_{ei}+U_{\rho_i\rho_i}a_i+U_{\rho_i H}h-\frac{1}{2}\alpha^i-\lambda a^i\right)\delta a_i+\\ \left(U_{\rho_e\mathrm{H}}a_e+U_{\rho_i H}a_i-\frac{1}{2}\beta-\lambda\kappa^2 h\right)\delta h+\\ \frac{1}{2}\int_{\Omega}d\Omega^*\frac{\sigma_e^2\delta a_e^* a_e+\sigma_e^2 a_e^*\delta a_e+2\sigma_e\sigma_i\left(\delta a_i^* a_e+a_i^*\delta a_e\right)+\sigma_i^2\left(\delta a_i^* a_i+a_i^*\delta a_i\right)}{\left|\vec{z}-\vec{z}^*\right|}\end{array}\right].\quad(17)
$$

We then get

$$
\begin{array}{l}\int_{\Omega}d\Omega\int_{\Omega}d\Omega^*\frac{\sigma_e^2\delta a_e^* a_e+\sigma_e^2 a_e^*\delta a_e+2\sigma_e\sigma_i\left(\delta a_i^* a_e+a_i^*\delta a_e\right)+\sigma_i^2\left(\delta a_i^* a_i+a_i^*\delta a_i\right)}{\left|\vec{z}-\vec{z}^*\right|}=\\ \int_{\Omega}d\Omega\int_{\Omega}d\Omega^*\frac{2\sigma_e\left(\sigma_e a_e^*+\sigma_i a_i^*\right)\delta a_e+2\sigma_i\left(\sigma_e a_e^*+\sigma_i a_i^*\right)\delta a_i}{\left|\vec{z}-\vec{z}^*\right|}\end{array},\qquad(18)
$$

as implied by the following chain:

$$
\begin{array}{l}\int_{\Omega}d\Omega\int_{\Omega}d\Omega^*\frac{\sigma_e^2\delta a_e^* a_e+\sigma_e^2 a_e^*\delta a_e+2\sigma_e\sigma_i\left(\delta a_i^* a_e+a_i^*\delta a_e\right)+\sigma_i^2\left(\delta a_i^* a_i+a_i^*\delta a_i\right)}{\left|\vec{z}-\vec{z}^*\right|}=\\ \int_{\Omega}d\Omega\int_{\Omega}d\Omega^*\frac{\sigma_e^2\delta a_e^* a_e+\sigma_e^2 a_e^*\delta a_e+2\sigma_e\sigma_i\delta a_i^* a_e+2\sigma_e\sigma_i a_i^*\delta a_e+\sigma_i^2\delta a_i^* a_i+\sigma_i^2 a_i^*\delta a_i}{\left|\vec{z}-\vec{z}^*\right|}=\\ \int_{\Omega}d\Omega\int_{\Omega}d\Omega^*\frac{\sigma_e^2\delta a_e^* a_e+\sigma_e^2 a_e^*\delta a_e+2\sigma_e\sigma_i\delta a_i^* a_e+2\sigma_e\sigma_i a_i^*\delta a_e+\sigma_i^2\delta a_i^* a_i+\sigma_i^2 a_i^*\delta a_i}{\left|\vec{z}-\vec{z}^*\right|}=\\ \int_{\Omega}d\Omega\int_{\Omega}d\Omega^*\frac{\sigma_e^2 a_e^*\delta a_e+\sigma_e^2 a_e^*\delta a_e+2\sigma_e\sigma_i a_e^*\delta a_i+2\sigma_e\sigma_i a_i^*\delta a_e+\sigma_i^2 a_i^*\delta a_i+\sigma_i^2 a_i^*\delta a_i}{\left|\vec{z}-\vec{z}^*\right|}=\\ \int_{\Omega}d\Omega\int_{\Omega}d\Omega^*\frac{2\sigma_e\left(\sigma_e a_e^*+\sigma_i a_i^*\right)\delta a_e+2\sigma_i\left(\sigma_e a_e^*+\sigma_i a_i^*\right)\delta a_i}{\left|\vec{z}-\vec{z}^*\right|}\end{array}.\qquad(19)
$$

Using the relationship of Eq. 18, we can transform Eq. 17 as follows:

$$\delta\tilde{\Pi}(a,h) \equiv 2\int_{\Omega} d\Omega \begin{bmatrix} \left(U_{\rho_e\rho_e} a_e + U_{\rho_e\rho_i} a_i + U_{\rho_e \mathrm{H}} h - \frac{1}{2}\alpha^e - \lambda a^e \right)\delta a_e + \\ \left(U_{\rho_e\rho_i} a_{ei} + U_{\rho_i\rho_i} a_i + U_{\rho_i H} h - \frac{1}{2}\alpha^i - \lambda a^i \right)\delta a_i + \\ \left(U_{\rho_e \mathrm{H}} a_e + U_{\rho_i H} a_i - \frac{1}{2}\beta - \lambda\kappa^2 h \right)\delta h + \\ \int_{\Omega} d\Omega^* \frac{\sigma_e\left(\sigma_e a_e^* + \sigma_i a_i^*\right)\delta a_e + \sigma_i\left(\sigma_e a_e^* + \sigma_i a_i^*\right)\delta a_i}{\left|\vec{z}-\vec{z}^*\right|} \end{bmatrix} . \quad (20)$$

At last, we can transform Eq. 20 to the form

$$\delta\tilde{\Pi}(a,h) \equiv 2\int_{\Omega} d\Omega \begin{bmatrix} \begin{pmatrix} U_{\rho_e\rho_e} a_e + U_{\rho_e\rho_i} a_i + U_{\rho_e \mathrm{H}} h - \frac{1}{2}\alpha^e - \lambda a^e + \\ \int_{\Omega} d\Omega^* \frac{\sigma_e^2 a_e^* + \sigma_e\sigma_i a_i^*}{\left|\vec{z}-\vec{z}^*\right|} \end{pmatrix}\delta a_e + \\ \begin{pmatrix} U_{\rho_e\rho_i} a_{ei} + U_{\rho_i\rho_i} a_i + U_{\rho_i H} h - \frac{1}{2}\alpha^i - \lambda a^i + \\ \int_{\Omega} d\Omega^* \frac{\sigma_i\sigma_e a_e^* + \sigma_i^2 a_i^* \delta a_i}{\left|\vec{z}-\vec{z}^*\right|} \end{pmatrix}\delta a_i + \\ \left(U_{\rho_e \mathrm{H}} a_e + U_{\rho_i H} a_i - \frac{1}{2}\beta - \lambda\kappa^2 h \right)\delta h \end{bmatrix} . \quad (21)$$

Separating the independent variations in Eq. 21, we arrive at the relationships

$$U_{\rho_e\rho_e} a_e + U_{\rho_e\rho_i} a_i + U_{\rho_e \mathrm{H}} h - \frac{1}{2}\alpha^e - \lambda a^e + \int_{\Omega} d\Omega^* \frac{\sigma_e^2 a_e^* + \sigma_e\sigma_i a_i^*}{\left|\vec{z}-\vec{z}^*\right|} = 0 , \quad (22)$$

$$U_{\rho_e\rho_i} a_{ei} + U_{\rho_i\rho_i} a_i + U_{\rho_i H} h - \frac{1}{2}\alpha^i - \lambda a^i + \int_{\Omega} d\Omega^* \frac{\sigma_i\sigma_e a_e^* + \sigma_i^2 a_i^*}{\left|\vec{z}-\vec{z}^*\right|} = 0 , \quad (23)$$

and

$$U_{\rho_e \mathrm{H}} a_e + U_{\rho_i H} a_i - \frac{1}{2}\beta - \lambda\kappa^2 h = 0 . \quad (24)$$

Consider the functions $a_e(z)$, $a_i(z)$, and $h(z)$ satisfying the constraints of Eqs. 13 and 14 and, in combination with the constants α^e, α^i, β, and λ, satisfying Eqs. 22 and 23. Let us treat the system of Eqs. 13, 22, and 23 as a system of six linear equations with respect to six unknowns: $a_e(z)$, $a_i(z)$, $h(z)$, α^e, α^i, and β. Let us call spectral those values for which the system permits nontrivial (spectral) solutions $a_e(z)$, $a_i(z)$, and $h(z)$. What can be said about the spectral values? To answer this question, let us multiply Eq. 22 on $a_e(z)$, Eq. 23 on $a_i(z)$, and integrate the resulting equations over the volume Ω. We have, then, the following relationships:

$$\begin{array}{l}\int\limits_\Omega d\Omega\left(U_{\rho_e\rho_e}a_e^2+U_{\rho_e\rho_i}a_e a_i+U_{\rho_e\mathrm{H}}ha_e-\frac{1}{2}\alpha^e a_e-\lambda a_e^2\right)+\\ \int\limits_\Omega d\Omega\int\limits_\Omega d\Omega^*\frac{\sigma_e^2 a_e^* a_e+\sigma_e\sigma_i a_e a_i^*}{\left|\vec{z}-\vec{z}^*\right|}=0\end{array}. \quad (25)$$

$$\begin{array}{l}\int\limits_\Omega d\Omega\left(U_{\rho_e\rho_i}a_e a_i+U_{\rho_i\rho_i}a_i^2+U_{\rho_i H}a_i h-\frac{1}{2}\alpha^i a_i-\lambda a_i^2\right)+\\ \int\limits_\Omega d\Omega\int\limits_\Omega d\Omega^*\frac{\sigma_i\sigma_e a_e^* a_i+\sigma_i^2 a_i^* a_i}{\left|\vec{z}-\vec{z}^*\right|}=0\end{array}. \quad (26)$$

Adding the relationships of Eqs. 25 and 26, we get

$$\begin{array}{l}\int\limits_\Omega d\Omega\begin{pmatrix}U_{\rho_e\rho_e}a_e^2+2U_{\rho_e\rho_i}a_e a_i+U_{\rho_i\rho_i}a_i^2+\\ U_{\rho_e\mathrm{H}}ha_e+U_{\rho_i H}a_i h-\frac{1}{2}\left(\alpha^e a_e+\alpha^i a_i\right)-\lambda\left(a_e^2+a_i^2\right)\end{pmatrix}+\\ \int\limits_\Omega d\Omega\int\limits_\Omega d\Omega^*\frac{\sigma_e^2 a_e^* a_e+\sigma_e\sigma_i a_e a_i^*+\sigma_i\sigma_e a_e^* a_i+\sigma_i^2 a_i^* a_i}{\left|\vec{z}-\vec{z}^*\right|}=0\end{array}. \quad (27)$$

Multiplying the relationship Eq. 24 by h and integrating the result over the domain Ω, we get

$$\int\limits_\Omega d\Omega\left(U_{\rho_e\mathrm{H}}a_e h+U_{\rho_i H}a_i h-\frac{1}{2}\beta h-\lambda\kappa^2 h^2\right)=0\,. \quad (28)$$

Adding Eqs. 27 and 28, we get

$$\begin{array}{l}\int\limits_\Omega d\Omega\begin{pmatrix}U_{\rho_e\rho_e}a_e^2+2U_{\rho_e\rho_i}a_e a_i+U_{\rho_i\rho_i}a_i^2+2U_{\rho_e\mathrm{H}}ha_e+2U_{\rho_i H}a_i h-\\ \frac{1}{2}\left(\alpha^e a_e+\alpha^i a_i+\beta h\right)-\lambda\left(a_e^2+a_i^2+\kappa^2 h^2\right)\end{pmatrix}+\\ \int\limits_\Omega d\Omega\int\limits_\Omega d\Omega^*\frac{\sigma_e^2 a_e^* a_e+\sigma_e\sigma_i a_e a_i^*+\sigma_i\sigma_e a_e^* a_i+\sigma_i^2 a_i^* a_i}{\left|\vec{z}-\vec{z}^*\right|}=0\end{array}. \quad (29)$$

Now, using Eq. 13, we can rewrite Eq. 29 as follows:

$$\int_\Omega d\Omega \left(U_{\rho_e\rho_e} a_e^2 + 2U_{\rho_e\rho_i} a_e a_i + U_{\rho_i\rho_i} a_i^2 + 2U_{\rho_e \mathrm{H}} h a_e + 2U_{\rho_i H} a_i h \right) +$$
$$\int_\Omega d\Omega \int_\Omega d\Omega^* \frac{\sigma_e^2 a_e^* a_e + \sigma_e\sigma_i a_e a_i^* + \sigma_i\sigma_e a_e^* a_i + \sigma_i^2 a_i^* a_i}{\left|\vec{z}-\vec{z}^*\right|} = \lambda \int_\Omega d\Omega \left(a_e^2 + a_i^2 + \kappa^2 h^2 \right) \cdot \quad (30)$$

If the normalization condition of Eq. 14 is satisfied, we get the following result:

$$\delta^2 \Phi_{spectral} = \int_\Omega d\Omega \begin{pmatrix} U_{\rho_e\rho_e} a_e^2 + 2U_{\rho_e\rho_i} a_e a_i + U_{\rho_i\rho_i} a_i^2 + \\ 2U_{\rho_e \mathrm{H}} h a_e + 2U_{\rho_i H} a_i h \end{pmatrix} +$$
$$\int_\Omega d\Omega \int_\Omega d\Omega^* \frac{\sigma_e^2 a_e^* a_e + \sigma_e\sigma_i a_e a_i^* + \sigma_i\sigma_e a_e^* a_i + \sigma_i^2 a_i^* a_i}{\left|\vec{z}-\vec{z}^*\right|} = \lambda \quad . \quad (31)$$

The relationship of Eq. 31 implies the following necessary conditions of stability: All the spectral values of the system of Eqs. 13, 21, and 23 must be non-negative.

5. Conclusion

We analyzed the problems of equilibrium and stability of two-component liquid plasma. Our analysis is based on the Gibbs variational approach, which does not require any dynamics equations of plasma; it is the source of the strength and weaknesses of this approach.

We calculated explicitly the first (Eq. 7) and second variations (Eq. 10) of the total energy. The first variation permits establishing the equations of equilibrium (Eqs. 8 and 9). The second variation permits to explore stability of equilibrium configurations of plasma. We reduced the investigation of sign-definiteness of the second variation to the verification of the spectrum of eigenvalues of the system of linear integral equations.

6. References

1. Grinfeld M, Grinfeld P. The Gibbs variational method in thermodynamics of equilibrium plasma: 1. General conditions of equilibrium and stability for one-component charged gas. Aberdeen Proving Ground (MD): Army Research Laboratory (US); 2018 Apr. Report No.: ARL-TR-8348.

2. Grinfeld M, Grinfeld P. The Gibbs variational method in thermodynamics of equilibrium plasma: 2. The equation of state for plasma. Aberdeen Proving Ground (MD): Army Research Laboratory (US); 2018 July. Report No.: ARL-TR-8419.

3. Gibbs JW. On the equilibrium of heterogeneous substances. Transactions of the Connecticut Academy of Arts and Sciences. Vol. 3. New Haven (CT): Tuttle, Morehouse, and Taylor; 1874-1878. p. 108–248 and 343–524.

1 (PDF)	DEFENSE TECHNICAL INFORMATION CTR DTIC OCA
2 (PDF)	DIR ARL IMAL HRA RECORDS MGMT FCDD RLD CL TECH LIB
1 (PDF)	GOVT PRINTG OFC A MALHOTRA
3 (PDF)	SANDIA NATL LAB J NIEDERHAUS A ROBINSON C SIEFERT
35 (PDF)	CCDC ARL FCDD RLD M TSCHOPP FCDD RLW T BJERKE S SCHOENFELD FCDD RLW LH B SCHUSTER FCDD RLW M J BEATTY B LOVE FCDD RLW MB G GAZONAS D HOPKINS B POWERS T SANO FCDD RLW MG J ANDZELM FCDD RLW PA S BILYK W UHLIG P BERNING M COPPINGER K MAHAN C ADAMS FCDD RLW PB C HOPPEL T WEERASOORIYA S SATAPATHY FCDD RLW PC R BECKER D CASEM J CLAYTON M GREENFIELD R LEAVY J LLOYD M FERMEN-COKER S SEGLETES A TONGE C WILLIAMS A SOKOLOW FCDD RLW PD R DONEY C RANDOW J RUNYEON G VUNNI