

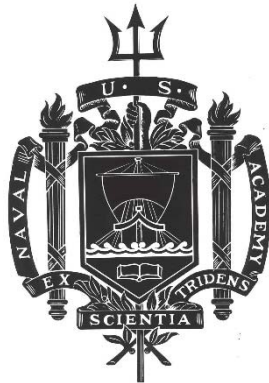
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Quantum Singularities in Black Hole Spacetime Systems with Timelike Classical Singularities

by

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UNITED STATES NAVAL ACADEMY
ANNAPOLIS, MARYLAND

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**QUANTUM SINGULARITIES IN BLACK HOLE SPACETIME SYSTEMS WITH
TIMELIKE CLASSICAL SINGULARITIES**

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Abstract

General relativity is currently the best theory that describes gravitation, one of the four basic forces of nature. A thorough understanding of general relativity is essential for navigation systems reliant upon GPS to operate with accurate and precise measurements over time. Classical general relativity predicts the existence of irremovable singularities, points in space where a mathematical description of the spacetime “breaks down” due to geodesic incompleteness. These singularities are found in a host of relativistic spacetimes, including those of observable astrophysical objects such as black hole systems. In our study, timelike curvature singularities associated with a group of super-extremal spacetimes are analyzed with a quantum wave packet in place of geodesic incompleteness. In this case, these singularities may be “removed” or “healed” without imposing boundary conditions. The super-extremal spacetimes explored include overcharged and overspinning black hole systems with naked singularities and different assumptions such as penetration by a cosmic string, a higher dimensional universe background, or in expanding or contracting cosmologies.

The technique to determine quantum singularities focuses on analysis of the spatial segment of the minimally coupled, relativistic Klein Gordon wave operator for a massive scalar particle and whether it is essentially self-adjoint. Both, Weyl’s limit-point, limit-circle criterion and deficiency indices are used to determine self-adjointness. By using self-adjointness properties, the spacetime can be characterized as quantum mechanically singular or non-singular. This method can help in determining the behavior of quantum particles, including the Higgs and other bosons found in spontaneous symmetry breaking models, near a black hole singularity. Our results indicate the quantum wave operator is not essentially self-adjoint for the spherically-symmetric spacetimes and spacetimes with cosmic strings. Hence, they contain quantum singularities. Finally, for a wave packet in the vicinity of a rotating black hole’s naked singularity, the wave operator is not essentially-self adjoint indicating quantum singularities exist and the ring singularity is not “healed”.

Keywords: quantum mechanics, singularity, black hole, general relativity, essential self-adjointness, functional analysis

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1 Introduction

General relativity is currently the best theory that describes gravitation, one of the four basic forces of nature. It is therefore important to understand both the mathematical structure and the predictions of general relativity. Since basic theorems of general relativity predict the existence of singularities, it is important to study the nature of these singularities. Therefore, I have chosen to study solutions to Einstein's equations that possess both classical and quantum singularities (technical definitions of classical and quantum singularities will be discussed in the sections "Classical Singularities" and "Quantum Singularities" to follow). Once the existence of classical singularities is determined, there exists the potential to "heal" the classical singularities. In theory, this is done by treating the system quantum mechanically meaning and probing the classical singularity with a test scalar wave packet. The classical singularity is resolved if the resultant spacetime is determined to be quantum mechanically non-singular. A spacetime is quantum mechanically non-singular if the quantum particle is not "affected" by the singularity. For example, a singularity does not affect a quantum particle when an infinite potential exists that repels the particle. Technically speaking, a spacetime is quantum mechanically non-singular if the wave packet is uniquely determined by three factors: the initial wave packet, the manifold, and the metric, without having to put any boundary conditions on the singularity [1]. The more direct way to see if a spacetime is quantum mechanically non-singular is to determine if the spatial segment of the wave operator is essentially self-adjoint on $C_0^\infty(\Sigma)$ in $L^2(\Sigma)$ where Σ is a spatial slice. A static spacetime is quantum mechanically-singular if the spatial segment of the wave operator is not essentially self-adjoint and vice versa. Thus, the process for resolving classical singularities and classifying quantum singularities is made significantly more efficient by analyzing a given operator's self-adjointness properties.

There are two primary methods for determining the essential self-adjointness of this operator: the use of Weyl's limit-point, limit-circle (LP-LC) theorems and the calculation of von Neumann deficiency indices. The first method, using LP-LC theorems, allows the user to make conclusions based on the nature of a one-dimensional Schrödinger equation and its associated "effective potential". The second method makes conclusions based on the dimensions of deficiency spaces which are populated by the eigenfunctions of complex eigenvalues for the spatial portion of the Klein-Gordon (KG) operator. Together, these two methods arrive at the same result in determining whether the spacetime in question is quantum mechanically singular or non-singular.

In general, classifying black hole systems with timelike singularities as quantum mechanically singular or non-singular is mechanical to a degree. The method involves examining the line element for a given spacetime, determining the classical singularities via geodesic incompleteness, applying a quantum test wave packet in the form a minimally coupled, massive KG equation, determining an effective potential, and finally using Weyl's LP-LC approach to see if the operator is essentially self-adjoint or not. This entire process has already been completed for some static and conformally static spacetimes [1]. More in depth explanations of these different spacetimes, such as static versus stationary and spherically symmetric versus axially symmetric, can be found in Appendix F. The challenge arises in applying this approach to more complicated black hole systems, such as the Reissner-Nordström (RN) black hole penetrated by a cosmic string, higher dimensional RN black holes, RN anti de Sitter (AdS) or de Sitter (dS) systems, and a Kerr black hole.

To begin with this entire investigation though, it is necessary to first understand the theoretical physics concepts used to describe the universe in which we will be operating - special relativity and general relativity. From here, we can then move on to a description of the specific black hole systems which will be the subject of our analysis and are direct predictions of general relativity. We then have opened the door to discuss the nature of classical singularities and how the characteristics of these singularities may change by application of a quantum particle via a wave equation. A brief overview of the key points of functional analysis needed to understand Weyl's LP-LC technique will then be presented. Finally, the analysis and characterization of these black hole systems will be conducted and we will determine whether the classical singularity was "healed" or whether quantum singularities exist.

2 Background

2.1 Special Relativity

The theory of special relativity developed by Einstein in 1905 can be encompassed in two simple postulates which upturned Newtonian mechanics and led to a massive shift in scientific paradigms [2]. Specifically, the two postulates are as follows:

- 1) *Principle of Special Relativity: All inertial observers are equivalent.*
- 2) *Constancy of Velocity of Light: The velocity of light in a vacuum is the same in all inertial systems.*

For clarification, an inertial system is a system moving at constant velocity (not accelerating or rotating). Therefore, the first postulate states that the laws of physics should be the same for every observer moving at a constant velocity or no velocity, while the second postulate states that these observers should always measure the same speed of light in a vacuum, regardless of their velocity. The shift in paradigm comes with the second postulate, which states that the speed of light is an invariant quantity, meaning the nature of time between the two observers can now change (and specifically, be Lorentz transformed, a property discussed in Appendix D). Hence, this means that the time elapsed for a given event and the length measured for a given object can change for one observer compared to another observer who is moving. In other words, one coordinate system measures time ticking slower or faster than a coordinate system moving at a different speed as well as the moving object's length to be shorter or longer than its length measured at rest. The mathematical relationships for time and length measured by the observer moving with an observed event and another observer measuring from a relative stationary frame of reference, are termed Lorentz transformations and can be found in Appendix D.

Important conclusions which can be made from analysis of Lorentz transformations include phenomena known as length contraction and time dilation. During length contraction, the moving object, or object that appears to be moving, is measured to have a smaller length measured in the “stationary” reference frame compared to the length measured in the moving frame (the seemingly “known” length). A visual of this is provided in Figure 1, which demonstrates how an object moving faster has its length contracted more and more due to this phenomenon. During time dilation, the clock which is moving (or appears to be moving) in the frame of reference ticks slower by a constant factor called γ . This can be visualized in Figure 2, where a light beam from a laser inside a rocket travels up to the ceiling of the rocket and then returns to the floor. The observer inside the rocket measures the event of the light traveling to the ceiling and back down occurring in a shorter period of time compared to the observer standing stationary on land outside the rocket, but observing the same event. This phenomenon becomes extremely important when considering satellites tens of thousands of miles above the surface of Earth moving significantly different than an observer on the surface. These systems include nearly any type of communications or global positioning system (GPS) based satellite.

With that being said, the larger factor in the significant time dilation which occurs between communication satellites in space and users on the surface is actually derived from gravitational fields, although special relativity also must also be taken into account. As will be depicted in the next section, the lower gravitational field in space compared to the surface results in time dilation between the clocks on the satellite and the clocks synchronized with Earth, creating an error in systems reliant upon these clocks. In fact, the errors due to time dilation and clock synchronization are on the order of around 11 kilometers per day, which becomes a significant problem over the course of even a couple hours if relativity is not taken into account.

Additionally, when Einstein's theory of special relativity is analyzed with respect to Newtonian dynamics (this includes properties such as momentum, force, and energy) the conclusion is made that the energy of a resting object can be directly equated to mass. In other words, energy is mass and mass is energy (for a resting object, if it is moving, then it has this rest energy plus some additional kinetic energy). This famous

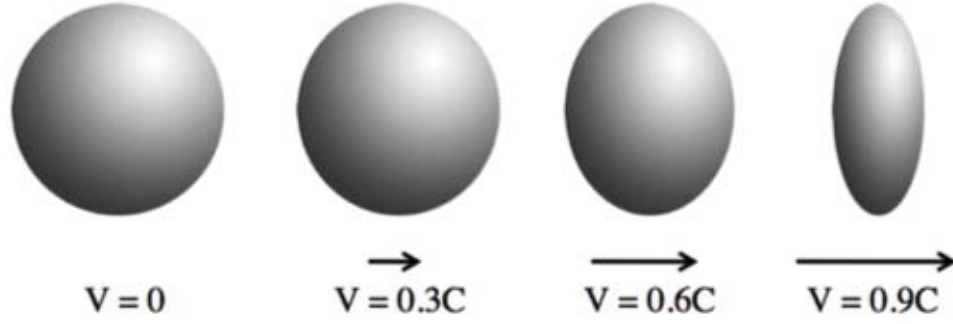


Figure 1: Length contraction of a moving object. Here the observer moving at the same speed as the sphere measures a longer length than a stationary observer measuring the sphere as it travels past them [3].

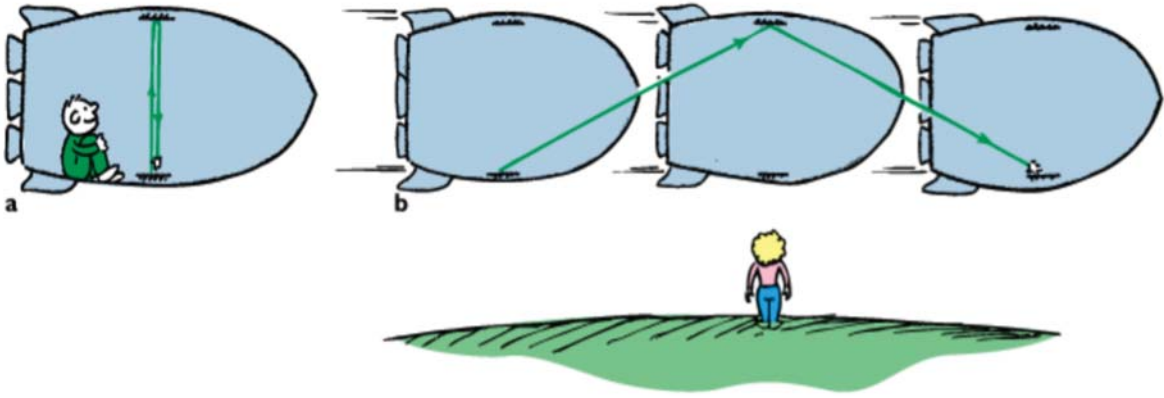


Figure 2: Time dilation between two inertial observers. Here the observer in the rocket measures a shorter time for the same event as the stationary observer [4].

relationship, $\Delta E = \Delta mc^2$, has resulted in the ability to construct some of the modern marvels of our species and has had a profound effect on the evolution of our civilization. For instance, this discovery in theoretical physics contributed to feats such as nuclear fission reactors which now can power cities as well as our Navy's entire submarine and aircraft carrier fleet and the atomic and fusion weaponry which allowed the United States to end the second World War. While these tangible, technological constructs are all direct results of an abstract theoretical development in the fundamental understanding of the physics which drives our universe, they are only a few examples of the impact of special relativity. In fact, in order to develop a model which can be used to predict larger scale phenomena, gravitation and acceleration need to be included - this became the development of Einstein's general theory of relativity.

2.2 General Relativity

The theory established in 1915 by Albert Einstein is the basis behind our ability to gain insight and successfully predict with great accuracy celestial phenomena, including many phenomena which have been experimentally verified. These phenomena include the existence of gravitational waves, gravitational lensing, gravitational frequency shifts, and the aforementioned gravitational time dilation [5]. Furthermore, Einstein utilized the following five concepts to develop the field equations which define his general theory of relativity:

- 1) Mach's Principle

- 2) Principle of Equivalence
- 3) Principle of Covariance
- 4) Principle of Minimal Gravitational Coupling
- 5) Principle of Correspondence

A definition and in depth description of each of these concepts is available in Appendix E. However, only the most important and relevant of these will be discussed here, which are the principle of equivalence and Mach's principle. The strong form of the principle of equivalence states that the motion of a gravitational test particle in a gravitational field is independent of its own mass and composition. If we work in reverse and invoke one of the famous results of special relativity, $E = mc^2$ which says mass is energy and energy is mass, then we derive what is known as the weak form of the principle of equivalence. The weak form states that the gravitational field is coupled to everything since matter is energy and energy is matter. Overall then, if we take the strong form and weak form of the principle together, then we can arrive at the most well known definition for the principle of equivalence.

Principle of Equivalence: There are no local experiments which can distinguish non-rotating free fall in a gravitational field from uniform motion in space in the absence of a gravitational field

In other words, an observer linearly accelerated relative to another inertial frame is identical, at least locally, to an observer at rest in a gravitational field. For instance, in Figure 3 the observer stationary on Earth is in a gravitational field and thus experiences the same laws of physics as the observer in the rocket in deep space (no gravitational field) who is accelerating upwards at 9.81 m/s^2 . Hence, gravitational forces are simply the result of acceleration (movement), not a mystical force which Newton had once thought.

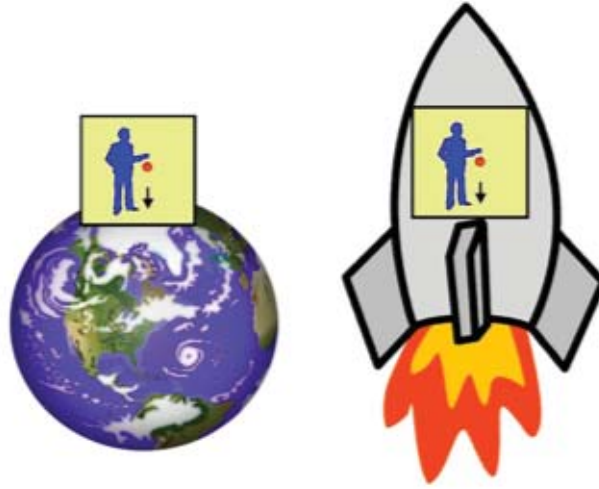


Figure 3: The principle of equivalence. Both observers experience the same laws of physics.

Second is Mach's Principle, which is an idea based on inertial frames and the fact that the large-scale distribution of matter in the universe determines local (small-scale) phenomena. It additionally provides a new definition for inertial frames. Explicitly, it states the following:

Mach's Principle: Inertial frames are those which are not accelerating relative to distant fixed stars.

The first implication of his principle is that matter, specifically the matter in his “fixed stars” (which encompasses most of the matter in the universe), determines the geometry of spacetime. In other words, matter - otherwise known as mass - warps spacetime around it, which results in the illusion of a fictitious force Newton termed gravity. What is actually occurring when gravity is at play, is that an object with significant mass, such as the Sun, warps the spacetime around it and as a result on this warping less massive objects, such as Earth, “fall” towards the more massive object. A relatable metaphor would be a trampoline - the warping of spacetime is analogous to a heavy object causing a depression in the center of the trampoline such that smaller objects on the edge of the trampoline are “forced” to fall towards the heavier object in the center. This can be seen visually in Figure 4 which has the Sun in the center with the Earth orbiting around it. Another implication of Mach’s Principle includes the fact that without matter, then there is no geometry - it is impossible to speak of geometry in the absence of the item which causes it (matter). Finally, Mach’s principle implies that a body in an otherwise empty universe should possess no inertial properties (it should have no movement). This is because there is nothing with the body to interact with, thus there is no reference frame to give that body motion relative to.

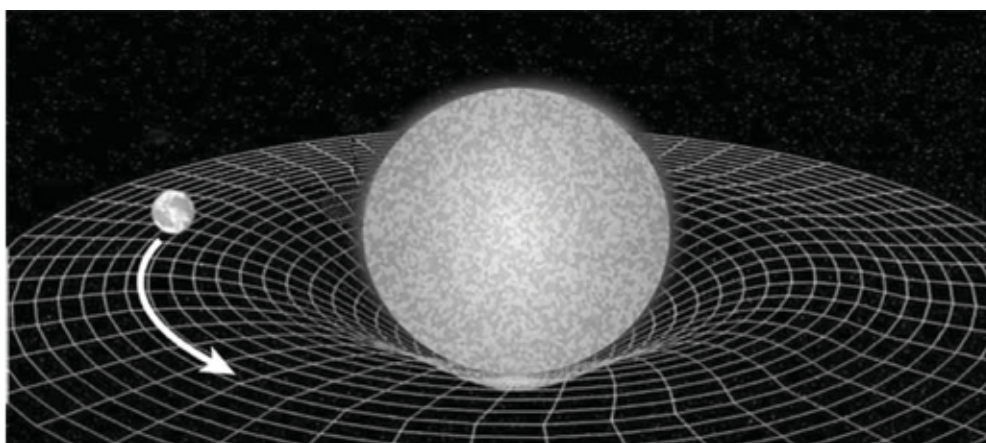


Figure 4: Mach’s Principle. The Sun warps spacetime, causing the Earth to feel a gravitational force and fall into orbit [6].

A few of the important consequences of general relativity include the prediction of celestial objects such as black holes and celestial phenomena such as gravitational waves, gravitational lensing, and the successful prediction of the precession of planetary orbits around the Sun. Black holes, to be explained in more depth in the subsequent section, are celestial objects of incredible density. In fact, these objects are so dense that general relativity is necessary in order to describe them because Newtonian mechanics breaks down.

The theory of general relativity also predicts that light itself - a massless wave packet - is affected by gravity in a phenomena termed gravitational lensing. Gravitational lensing is the bending of light as it passes through an incredibly strong gravitational field. In fact, it was this prediction that allows observers to indirectly infer the presence of black holes in a region of space because the bending of light due to the black hole’s extraordinarily strong gravitational field is observable. A photograph of this is shown below in Figure 6(a), which was an actual observation made from the Hubble Space Telescope’s Camera 3, followed by a theoretical model in Figure 6(b). The rings in Figure 6(a) are known as Einstein Rings, and are composed of light from stars and galaxies directly behind a massive object, in this case the yellow-red galaxy at the center of the image and not a black hole. Additionally, there is very strong evidence for black holes as a result of how they affect their companion star when part of a binary system as well as the strong x-ray emissions they produce from the matter which is being accelerated near the event horizon in a region known as the accretion disk. Gravitational waves are a separate phenomena which can occur when two incredibly dense objects, such as black holes, merge. In fact, gravitational waves have recently been observed by the Laser Interferometer Gravitational Observatory (LIGO) which observed the merger of two black holes nearly 1.3 billion light years from Earth. A theoretical diagram of the effect of a set of gravitational waves on the fabric of spacetime itself (see Appendix F for more information on the fabric of spacetime) can be seen in Figure

6(c). One of the first and most important confirmations of Einstein’s relativity came when his explanation of why Mercury’s perihelion shift, or the change in the closest point Mercury’s orbit reaches relative to the Sun, was observed to be slightly larger than previously calculated was proven correct [2]. However, today we are near certain of the existence of these objects, especially due to the recent release of the first image of a black hole in human history for data taken in 2017 by the Event Horizon Telescope (EHT) group of the supermassive black hole at the center of the galaxy Messier 87. This image can be found in Figure 5 [7].

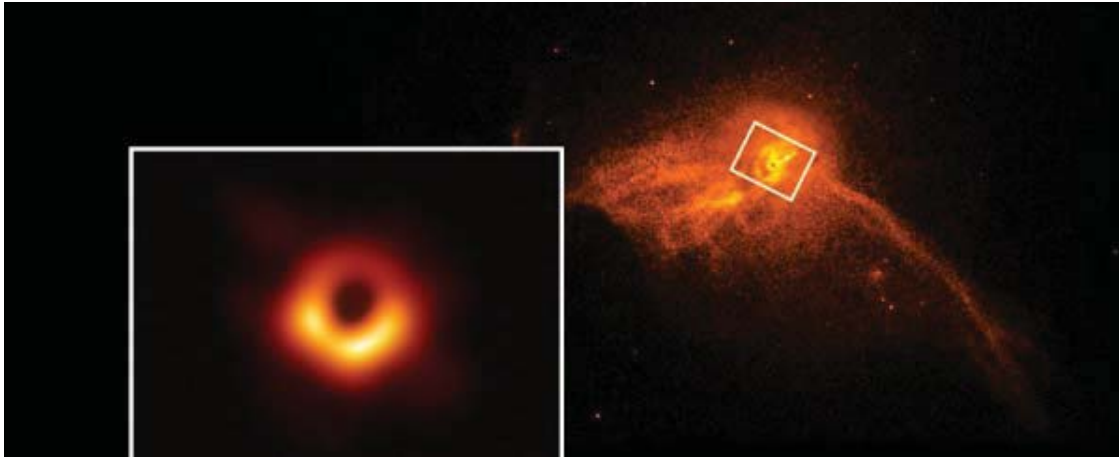


Figure 5: The first image of a black hole. The “image” (which is actually a modeled intensity map based on analysis from an interferometric array) was constructed by the Event Horizon Telescope (EHT) group and shows the supermassive blackhole at the center of Messier 87 [7].

2.3 Black Holes

As stated above, one of the major predictions of Einstein’s relativistic theories was the presence of celestial objects known as black holes. A black hole is a celestial object of finite mass, but all of the finite mass is concentrated on a single point at the center of the black hole, making the density infinite at that point. Thus, since all of the mass of the black hole is on a single point, the spacetime for a black hole is “dragged” to infinity. This concept of how a black hole warps spacetime can be found in Figure 7.

The center of a black hole is one example of a classical singularity - a point where basic mathematical principles, such as never dividing by zero, are violated. However, what makes the black hole special is the fact that there is a certain distance from the center - a certain boundary - that once a particle or observer crosses they can no longer transmit information back past this boundary. In essence, this is due to the fact that at this radius, the velocity necessary to escape the gravitational field of the black hole is equal to the speed of light which is the upper limit on the possible speed for any particle or observer in the universe. Hence, if the escape speed becomes greater than the speed of light past this radius, then nothing, not even light and thus information, can be transmitted back out of the black hole. The technical name for this radius is the Schwarzschild radius and the name for the region with that radius is referenced as the event horizon. The event horizon is in fact another example of a classical singularity (it is technically a removable classical singularity, or a coordinate singularity, which is described in more in depth in the “Classical Singularities” section). However, predicting what occurs for observers near the event horizon can be done by theorists and without much difficulty because it only signals a breakdown of coordinates. The difficulty comes in predicting what happens to the particle or the observer once they enter regions extremely close to the center of the black hole.

The center of a black hole is also a classical singularity, but different from the event horizon, a notion discussed in the “Classical Singularities” section to follow. The key idea is that if the distance to this singularity is extremely small, the theories of quantum mechanics must be invoked. These theories can

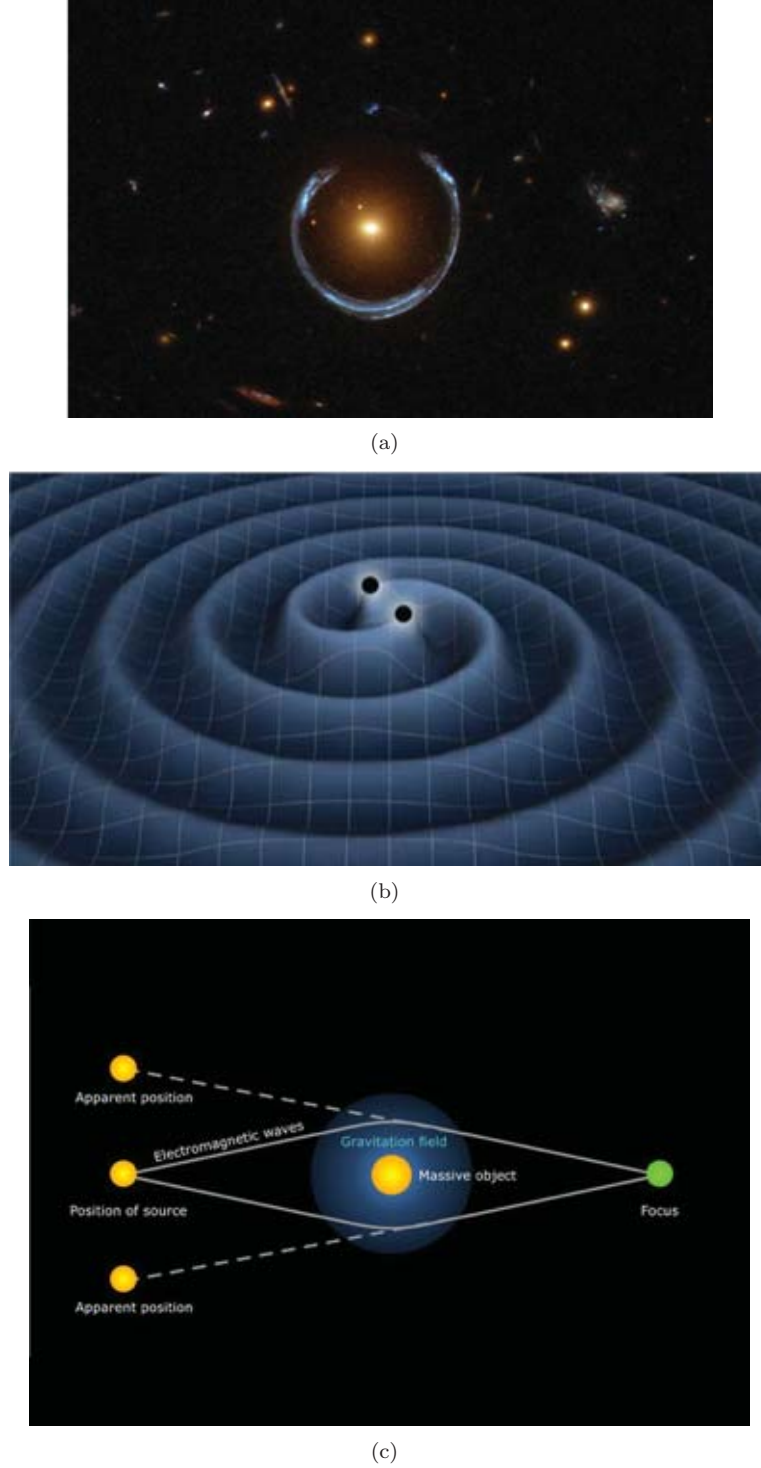


Figure 6: Gravitational waves and lensing effects. In (a) gravitational lensing photographed from the Hubble telescope is shown [8]. The ring of light is termed an Einstein ring. In (c) a 2-D model of gravitational lensing shows how light is bent by the massive object in the center [9]. In (b) a simulation of gravitational waves from the merger of two black holes, such as those which were observed by LIGO in 2015, are shown [10].

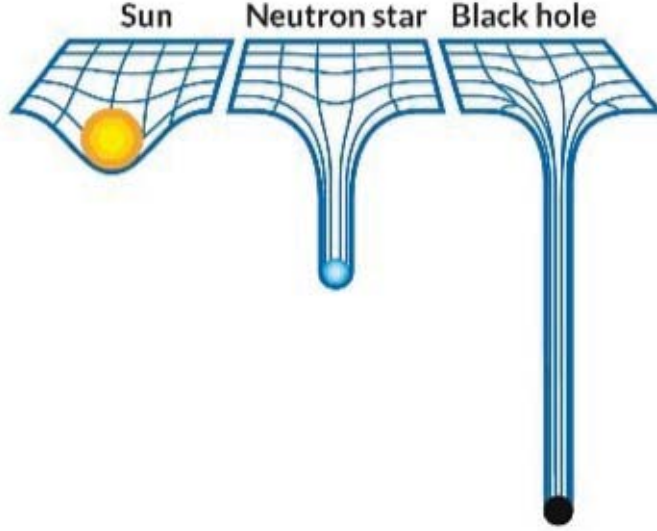
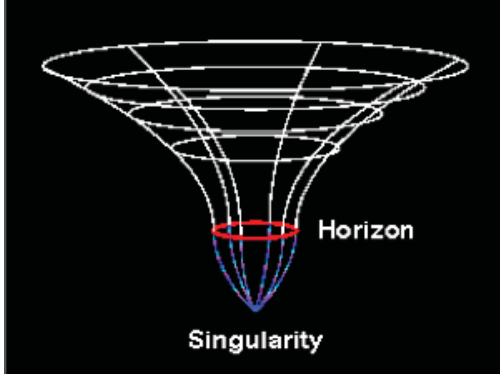


Figure 7: The effect a black hole has on spacetime, which is to drag it down to infinity [11].

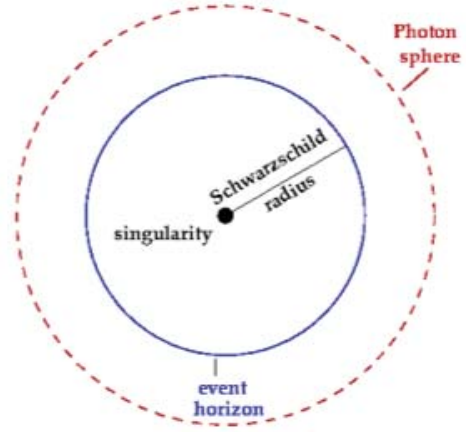
predict with incredible accuracy and precision what occurs on the extraordinary small scale of the universe or when there are extremely large energies, while relativity explains what happens on the large scale of the universe. Thus, black holes are of such immense scientific interest because when scientists attempt to predict what occurs for a particle or observer as they enter distances very near the singularity of the black hole, they must attempt to reconcile relativity and quantum mechanics - something no one has been able to accomplish with significant success or experimental confirmation through today.

Moreover, when discussing black holes and the specifics of their mathematical description it becomes necessary to classify them and to take into account different phenomena which may be at play. For example, the most simplistic black hole is one which is not rotating, not charged, and exists within a surrounding universe which has a flat geometry of spacetime (see Appendix F for more details on the geometry of spacetime around the black hole system). This flavor of black hole is termed a Schwarzschild black hole, and is a well studied model of a celestial system. The structure of a Schwarzschild black hole can be found in Figures 8(a)-(c). Another type of black hole is the Reissner-Nordström (RN) black hole which is identical to the Schwarzschild black hole in all respects other than the fact that it has electric charge. The structure of a RN black hole can be found in Figures 8(d)-(e). Likewise, a black hole without charge, but that is rotating, is termed a Kerr black hole. The structure of a Kerr black hole can be found in Figures 8(f)-(g). If different geometries of spacetime are included, for example a spherical geometry or hyperbolic geometry surrounding the black hole system, then the space surrounding the black hole is termed de Sitter (dS) or anti de Sitter (AdS) space respectively. Thus, a system such as the RN black hole in AdS space might be a possible black hole system. For the purposes of this investigation, the primary systems of interest are variations of the RN black hole system such as with a cosmic string, in higher dimensions, and in AdS or dS space; however, a negative mass Schwarzschild black hole and Kerr black hole will be addressed as well. It is important to note that all of the systems analyzed in through this project involve super-extremal cases of the “standard” black hole systems described above. A super-extremal case means that the black hole system is either overcharged for the RN systems (technically meaning that the charge of the black hole ϵ , or more precisely ϵ^2 , is much greater than the mass of the black hole M) or overspinning for the Kerr system (similarly, this means that the angular momentum of the black hole a , or more precisely a^2 , is much greater than the mass of the black hole M or M^2). These systems are particularly notable because in super-extremal cases, the classical singularity at the center of the black hole becomes a “naked” singularity. A naked singularity indicates a black hole system with a central singularity without an event horizon in between the singularity and distances extremely far away from the singularity (near infinity). It should be noted that theories by Roger Penrose (see [12]) indicate the idea of cosmic censorship – that in physically

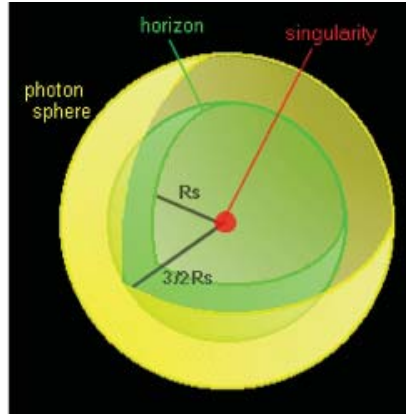
acceptable spacetimes, singularities should be concealed or shrouded behind an event horizon. A more in depth discussion of super-extremal cases and their structure can be found in Appendix F as well.



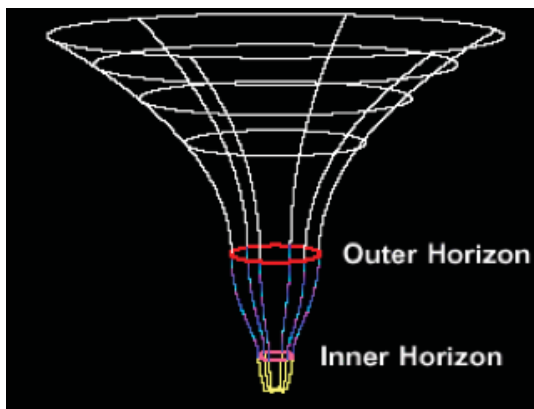
(a)



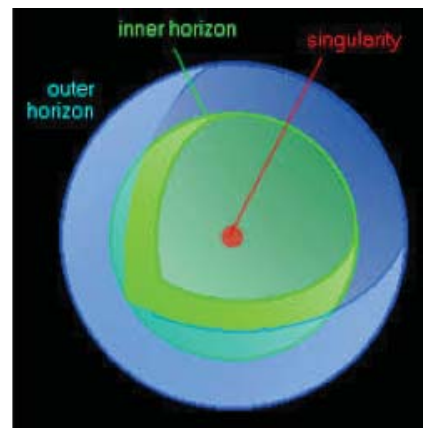
(b)



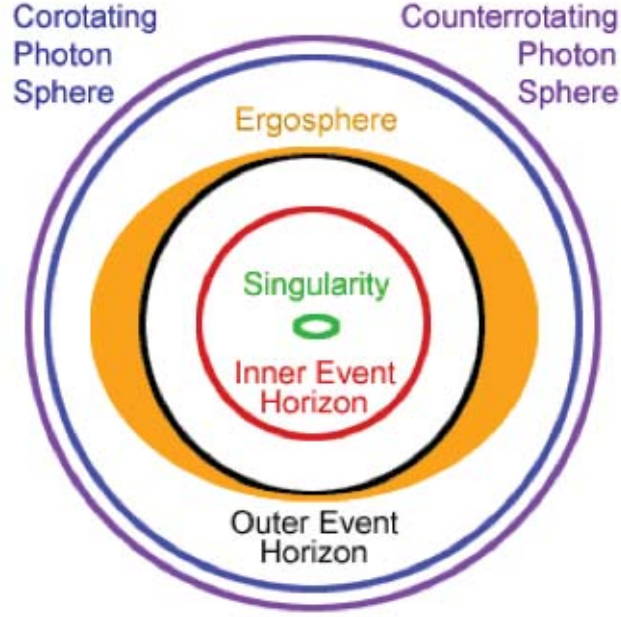
(c)



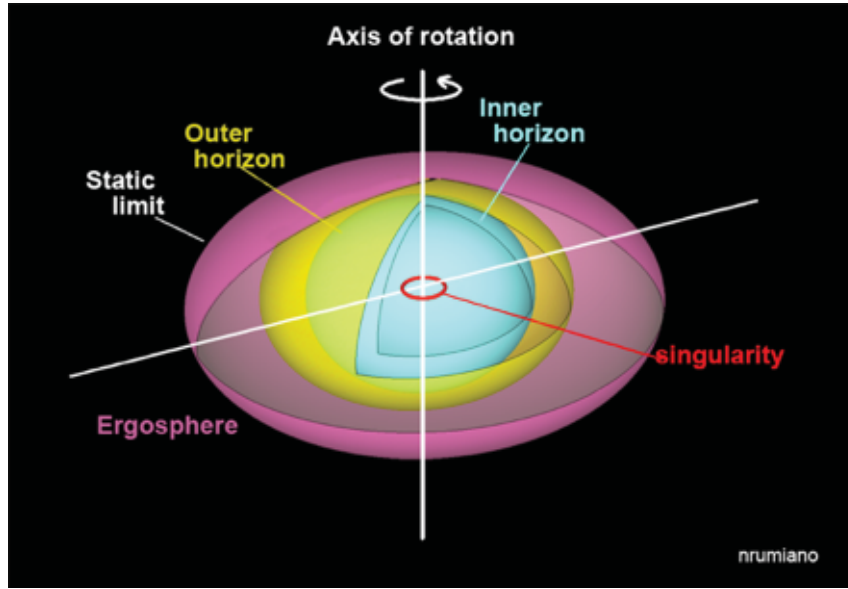
(d)



(e)



(f)



(g)

Figure 8: Models for Schwarzschild, RN, and Kerr black holes. A 2-D model of the spacetime warping caused by a Schwarzschild black hole is shown in (a) [13] followed by a 2-D drawing of its structure in (b) [14], and a 3-D drawing of the structure in (c) [15]. A 2-D model of the spacetime warping caused by a RN black hole is shown in (d) [16] followed by a 3-D drawing of the structure in (e) [17]. A 2-D model of the spacetime warping caused by a Kerr black hole is shown in (f) [18] followed by a 3-D drawing of its structure in (g) [19].

2.4 Cosmic Strings

Cosmic strings are theoretical “topological defects”, formed in the early universe during what are known as phase transitions. As matter began to cool in the seconds immediately following the big bang, it underwent

changes in phase analogous to steam condensing. As the process continued, the four forces of nature - electromagnetism, weak nuclear force, strong nuclear force, and gravity - began to separate and become distinct as well. The phase transition of the universe's early matter did not occur uniformly. Rather, as with boiling water, "bubbles" of the new matter became to form. These bubbles would expand and eventually collide with each other. It is here at the boundaries of the phase transitions that cosmic strings would have been formed [20].

These cosmic strings would be strands of mass-energy forming a lattice work of imperfections across the universe [21]. This can be visualized in Figure 9. Because these strings would be remnants of the very early universe - when it was less than 10-35 seconds old - grand unification would still hold within them. There would be no separation of the strong from electroweak forces, nor distinction of matter (although gravity would have been separated out). It is possible that even if these structures had existed at some time in the past, they have all collapsed by now. However, if they still exist, they would be incredibly long, stretching across the visible universe. They also would be incredible slim, with a thickness of about 10-28 cm. However, these strings are incredibly dense as well; a ten kilometer string would weigh as much as our planet [21]! Thus, they have very interesting gravitational effects. The tremendous tension in the strings created by the expanding universe would create extremely rapid oscillations and vibrations in them as well [22].

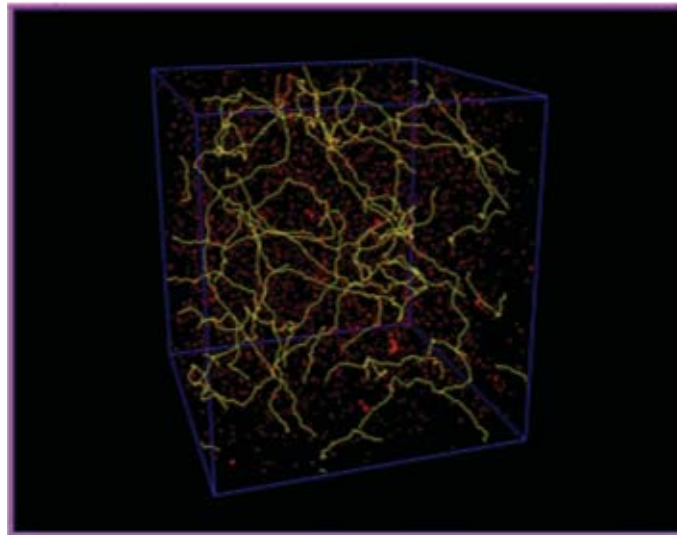


Figure 9: A model of cosmic strings forming a lattice throughout space [23]

Although cosmic strings have never been detected, there is much work being put into this endeavor. In 2000, T. Damour and A. Vilenkin showed cosmic strings were able to produce high levels of gravitational waves. The gravitational wave detectors LIGO and VIRGO may eventually prove helpful toward detecting a string [24]. Much earlier in the 1980's, a different method of detection was explored. The extreme density of a cosmic string would warp spacetime around it enough to bend light. If a source such as a distant galaxy lay on the other side of a cosmic string from us, its light would take two separate paths. We would see two separate images of the exact same galaxy in close proximity (see Figure 10 for a conceptual picture on how a cosmic string warps spacetime by removing a wedge and how the resultant geometry bends parallel light rays). Such a phenomenon was thought to have been observed in 2004 by scientists in Russia [25]. However, after closer observation by the Hubble Space Telescope, unimpeded by Earth's atmosphere, the images were determined to be two separate galaxies [26]. The detection of a cosmic string will have to wait for another day.

While cosmic strings were originally thought to have been the seeds of the galaxies, this theory has been ruled out in the past several years. However, interest in cosmic strings has remained high lately due to their possible relation to M-theory (the modern version of superstring theory) [27]. An interesting feature of M-theory is the possible existence of higher dimensions (as many as 26 in some cases). The integration of

a cosmic string into each variation of the black hole systems depicted in the previous section is covered in subsequent analysis as a result. The mathematical implications of adding a cosmic string into the celestial system can be found in Appendix A.

2.5 Classical Singularities

Classical singularities are commonplace in general relativistic spacetimes as indicated by the theorems of Hawking and Penrose (see, e.g., [28]). In their most basic form, classical singularities are points which have been removed from a manifold - a smooth surface. Thus, the singularity creates a boundary within a manifold. The aforementioned theorems indicate that classical singularities are defined by geodesic incompleteness, meaning at least in the timelike and null cases, that classical particle paths come to an abrupt end at the boundary of the singularity (see Appendix B for more information on geodesics). An abrupt end is represented mathematically through discontinuities in the geodesic equation where an element diverges. However, it is important to recognize that there exist non-removable singularities and removable singularities. Removable singularities can be eliminated from a geodesic equation through a coordinate transformation. An example includes a Schwarzschild black hole spacetime which has a removable singularity at $r = 2M$, the event horizon (where M is the mass of the black hole in units of length and r is the radial distance from the center of the black hole). This singularity can be resolved by transforming to Eddington-Finkelstein coordinates where $\bar{t} = t + 2M \ln(r - 2M)$ (see Appendix F for more information and clarification of the symbols used in Eddington-Finkelstein coordinates). On the other hand, non-removable singularities cannot be resolved in the same way. An example of a non-removable singularity includes the center of a Schwarzschild black hole when $r = 0$, which cannot be removed by a coordinate shift.

Furthermore, irremovable classical singularities can be categorized according to their strength [28, 27]. Specifically, there exist three categories: quasiregular, scalar curvature, and non-scalar curvature singularities. The first of these, quasiregular singularities, are of the lowest strength and physically can be thought of as representing holes in the fabric of spacetime, making them topological. Mathematically, quasiregular singularities occur when the Riemann tensor components and their derivatives are bounded. One example of a quasiregular singularity is a conical singularity. At a conical singularity all physical quantities represented by invariants in the tensor formalism will reach finite values in the limit of the point itself. Since spacetime is smooth around the point, but not at the point itself, and the physical quantities as well as the metric reach finite values, the point can be visualized as a cone with a singularity at the tip. An instance where quasiregular singularities appear would be in the Schwarzschild black hole with a cosmic string. The cosmic string creates a conical spacetime. As a visualization, you can imagine the cosmic string as removing a wedge from a plane, resulting in the formation of a cone from the plane curling in on itself in an attempt to remain smooth. Thus, the cosmic string parameter, which is a function of the string's density μ , determines the angle of the wedge to be removed from the spacetime. A visual of this can be found in Figure 10.

Stronger singularities include the types of curvature singularities, scalar and non-scalar. The differentiation between scalar and non-scalar comes from calculation of different scalar invariants, such as the Ricci scalar and the Kretschmann scalar (see Appendix C for more information). Scalar curvature singularities occur if any scalar constructed from the metric, the totally anti-symmetric tensor, or Riemann tensor are not bounded. Non-scalar curvature singularities occur when all of the scalars constructed from these three tensors are bounded. In essence, if any of the scalar quantities within these diverge, then the singularity is designated as a scalar curvature singularity. Scalar curvature singularities are the most common, such as at the center of black holes and in cosmologies which consider the beginning or end of the universe to be characterized by a 'bang' (often termed the Big Bang for the theories regarding the origins of our universe). An example of a scalar curvature singularity from our commonly used Schwarzschild black hole can be seen from review of the Kretschmann scalar $R_K = 48mM^2r^{-6}$ indicating a non-removable scalar curvature singularity occurs at $r = 0$ because the scalar diverges to infinity at this point. Non-scalar curvature singularity examples include singularities found in whimper cosmologies which are cosmologies defined by an infinitely large value for the scaling factor in comoving coordinates (whimper cosmologies result in the universe beginning or ending with zero density unlike the 'bang' cosmologies which indicate a scaling factor approaching zero

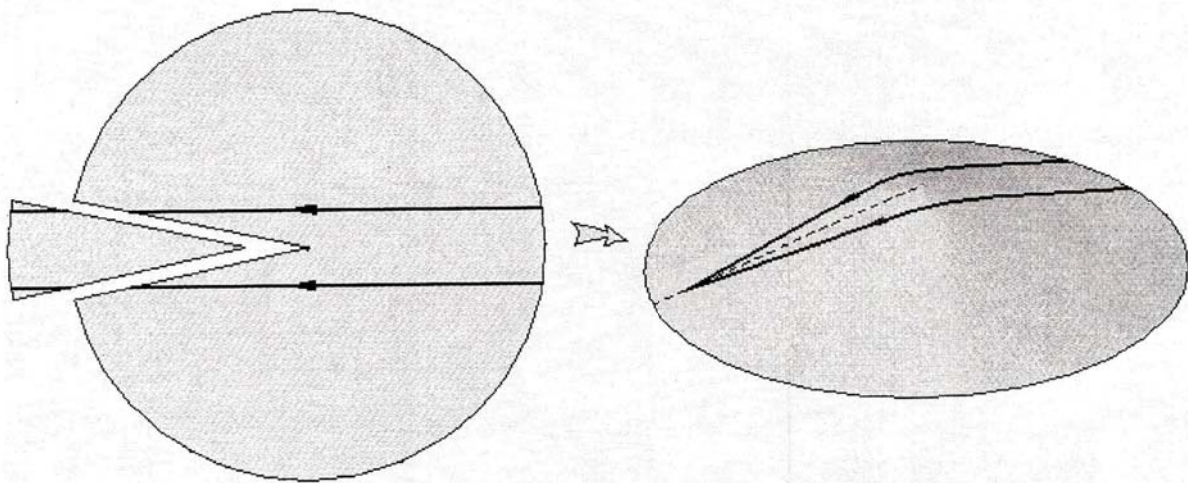


Figure 10: The canonical spacetime caused by a cosmic string [23]

with a density approaching infinity).

2.6 Quantum Mechanics and Quantum Particles

As stated earlier, when analyzing what happens for particles and observers near the central singularity of a black hole or naked singularities, the scale becomes small enough where quantum mechanics must be invoked. Quantum mechanics is yet another abstract theoretical physics theory which has led to serious, tangible leaps forward in technology. For example, the discovery of quantum mechanics led Americans John Bardeen, William Shockley, and Walter Brattain to invent the transistor, revolutionizing the modern world and catapulting our civilization into the computerized era. Another technological result of quantum mechanics includes the development of magnetic resonance imaging (MRI) which has been a key tool in identifying medical problems and allowing for proper treatments in millions of people over the past several decades. To clarify though, this project does not completely invoke quantum mechanics. In the course of this project, the classical particle which was originally considered is swapped out for a quantum particle, but the background of the universe in which the quantum particle is being considered (the black hole or naked singularity) is still classical. This is an important distinction, because it allows the researcher to analyze the situation using classical assumptions about the universe depicted by theories of general and special relativity. In order to understand what differences arise when analyzing a quantum particle versus a classical particle, we must first start with what a classical particle is.

A classical particle is an object with finite physical properties - mass, position, speed, momentum, energy, etc. Furthermore, a classical particle can be thought of in nearly every situation as simply a rock with some set of predetermined set of measurable quantities. On the other hand, a quantum particle is an object where its physical properties are described by a probability distribution. Thus, instead of having a finite, measurable value for the momentum or the speed, the object only has a probability of having a specific measurable value which you cannot predetermine until you actually physically measure the object's properties. A quantum particle is also called a wave packet because in quantum mechanics the particle is assumed to behave both like a particle and wave simultaneously. Moreover, the probability distribution is described by a mathematical function called the wavefunction which is technically a "probability density". The set of figures below (Figures 11(a) and 11(b)) depict classical particles versus the probability driven quantum particles. One prominent example is the location of electrons in the hydrogen atom depicted in Figure 11(b) where the denser (more solidly colored) regions represent high probability and the more

sparsely dotted areas represent a lower probability of finding an electron. In this picture nodes depict regions where destructive interference occurs - a phenomena resultant from the wave nature of the electron. A three dimensional version of this is found in Figure 11(c) where taller spikes represent higher probabilities of finding electrons. A quantum particle's wavefunction is determined by solving the quantum particle's Schrödinger equation (it is a solution to the Schrödinger equation). The Schrödinger equation is one type of equation which admits wavelike solutions. Other types of equations admit wavelike solutions, such as equation 1, which is a second order partial differential equation describing the evolution of water waves, light waves, and sound waves. In short, a wave equation describes how waves move. For example, a length of string can behave like a wave when moved in a certain way which can be depicted using the following one dimensional scalar wave equation.

$$\frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \quad (1)$$

where y is the vertical displacement of the string, x is the position along the string, t is the elapsed time, and v is the velocity of the wave. Thus, quantum particles are described by wavefunctions which are determined directly from Schrödinger equations. Moreover, the test scalar wave packets in this investigation will be analyzed utilizing a relativistic scalar wave equation, also known as the relativistic Klein-Gordon (KG) wave equation, which implements a KG operator. We will then perform a substitution mathematically to take this differential equation to a Schrödinger equation, allowing us to determine the behavior of our quantum particle. The KG wave equation for a relativistic scalar particle of mass m is written in natural units ($G = c = 1$) as,

$$(\square - m^2) \Phi = 0 \quad (2)$$

$$(-g)^{1/2} ((-g)^{1/2} g^{\mu\nu} \Phi_{,\nu})_{,\mu} = m^2 \Phi \quad (3)$$

where g represents the determinant of the spacetime metric, Φ represents the wave function, and $\square$ represents the d'Alembert operator ($\square = \frac{\partial^2}{\partial t^2} + \nabla^2$ where $\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$). The notation used here with the μ and ν indices is known as Einstein summation notation, and more information on how this notation can be translated into a more understandable form can be found in Appendix C.

More importantly, as noted earlier we can perform a mathematical substitution after expanding the KG equation and write the resultant equation in a certain notation known as the Schrödinger form. When written in this method, the equation takes the general form,

$$\hat{H}\Psi(r) = E\Psi(r)$$

where $\hat{H}$ is a mathematical operator known as the Hamiltonian operator, $\Psi(r)$ is the wavefunction, and E is an energy eigenvalue of the Hamiltonian. The important idea here is that by studying specific properties of the Hamiltonian operator when the equation is written in this form (those properties being the self-adjointness properties which can be found in Appendix G) we can determine whether or not our spacetime is quantum mechanically singular or non singular. Thus, the final explanation needed then is what quantum mechanically singular or non singular truly means.

2.7 Quantum Singularities

As stated earlier, there are times when analyzing a classical singularity by applying a quantum particle resolves or "heals" the singularity. However, caution must be exercised here with the terminology. In this

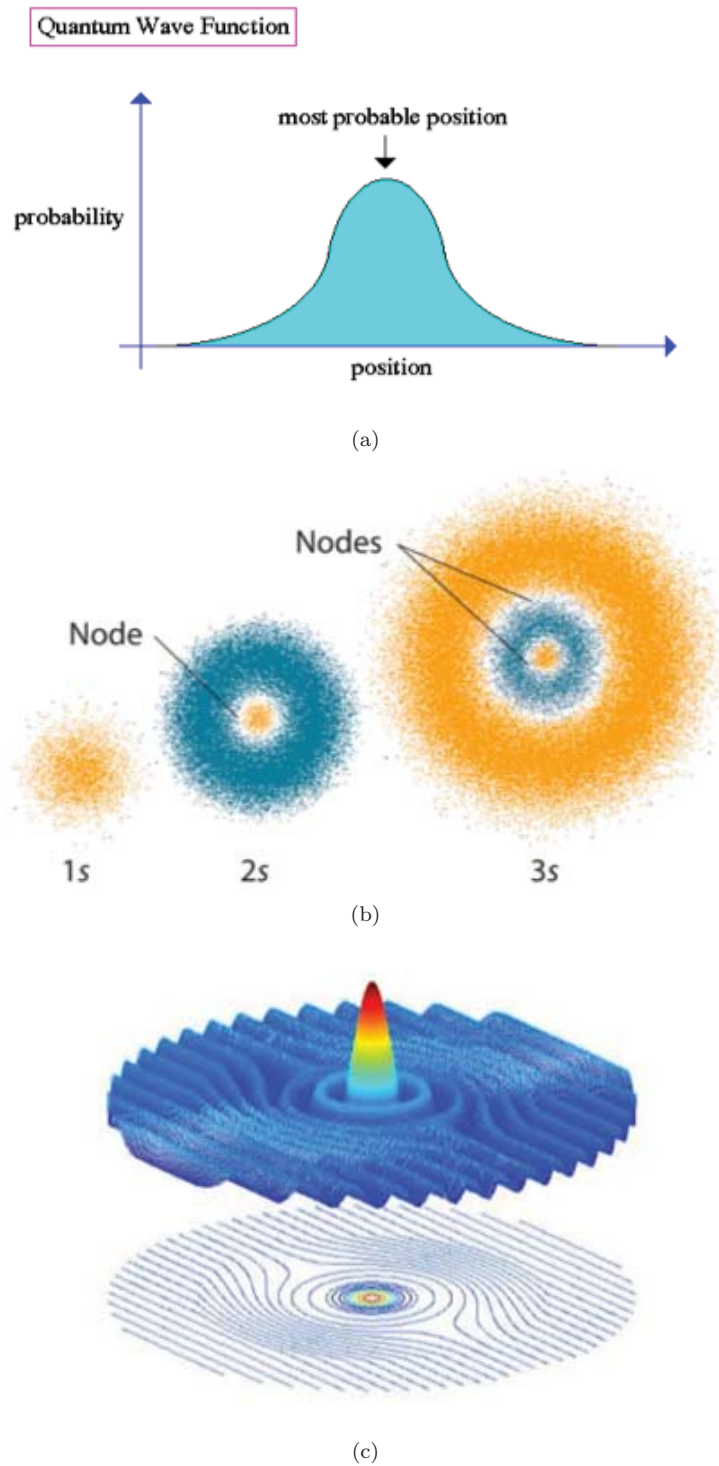


Figure 11: Wavefunctions and probabilities for quantum particles. A wavefunction for a 1-D quantum particle is shown in (a) [29]. In (b) the picture represents the probability of finding electrons in a hydrogen atoms [30]. More densely populated areas indicate a higher probability. A wavefunction for a 3-D quantum particle is shown in (c) [31].

report, the singularities being discussed and analyzed are still in a classical background spacetime – there is no quantization of gravity itself happening here. With that in mind, we say the spacetime is quantum mechanically non-singular, meaning it contains no quantum singularities and is “healed”, if the wave packet is not “affected” by the singularity. For example, an instance where a quantum particle is not affected by the singularity would be in the case of an infinite repulsive potential barrier which drives the wavefunction, and thus the probability of finding the particle there, to zero.

Formally, a static spacetime, which includes negative mass Schwarzschild black holes, RN black holes, RN AdS-dS space, and black holes with cosmic strings, is quantum mechanically non-singular if the evolution of a test scalar wave packet, representing a quantum particle, is uniquely determined by the initial wave packet, manifold and metric, without having to put any boundary conditions on the singularity [1]. Conversely, a static spacetime is quantum singular if the spatial segment of the wave operator is not essentially self-adjoint [32] on $C_0^\infty(\Sigma)$ in $L^2(\Sigma)$ where Σ is a spatial slice. If the operator is a matrix, to be self-adjoint means that the operator’s conjugate transpose must be equal to the operator itself ($A = A^*$ where A^* is the adjoint) and the domain of the operator is equal to the domain of the adjoint ($Dom(A) = Dom(A^*)$). This definition can be generalized to infinite dimensional space as well.

The term “essentially self-adjoint” is derived from functional analysis [32], and means that the first condition above is met and the second can be met by expanding the domain of the operator or the adjoint so that it is true (see Appendix G for more information on the concepts mentioned in this paragraph based on functional analysis, essential self-adjointness, and Weyl’s LP-LC theorems which are tests based on such ideas). Specifically, if A is not essentially self-adjoint the future time development of the wave function is ambiguous, thus resulting in a quantum singularity as defined by Horowitz and Marolf [1]. In fact, a theorem of Weyl [32] directly relates the essential self-adjointness of the Hamiltonian operator to the behavior of the potential in an effective one-dimensional Schrödinger equation derived from the radial equation (again, see Appendix G for more information on Weyl’s theorems). In turn, this relation determines the behavior of the scalar-wave test packet. Hence, in order to use Weyl’s LP-LC technique to derive quantum singularities, the wave equation must be put into a one-dimensional radial equation of the well known Schrödinger form $Hu(x) = Eu(x)$ where E is a constant and where any singularity is assumed to be at $x = 0$. Hence, if we re-examine this equation and explicitly define the Hamiltonian operator to be $H = \frac{-d^2}{dx^2} + V(x)$ and $\Psi(x)$ to be the time independent wave function both on $x \in (0, \infty)$, then we must simply analyze the self-adjointness properties of the Hamiltonian operator and we can determine whether or not the spacetime is quantum singular or non-singular.

3 Limit Point, Limit Circle Analysis

3.1 Spherically Symmetric Black Hole Spacetimes

The starting point for analysis of any black hole system is the line element (or, equivalently, the metric). More information on line elements, metrics, and geodesics can be found in Appendices A and B. The line element for a generic spherically symmetric black hole spacetime in 3+1 dimensions (meaning three spatial and one temporal dimension) and in natural units where $G = c = 1$ has the form,

$$ds^2 = -f^2 dt^2 + \frac{dr^2}{f^2} + s^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (4)$$

where $f \equiv f(r)$ and $s \equiv s(r)$. The metric $g_{\mu\nu}$ for the line element in equation 4 in matrix form is,

$$\begin{pmatrix} \frac{1}{f^2} & 0 & 0 & 0 \\ 0 & s^2 \sin^2 \theta & 0 & 0 \\ 0 & 0 & s^2 & 0 \\ 0 & 0 & 0 & -f^2 \end{pmatrix}$$

and the inverse metric $g^{\mu\nu}$ in matrix form is,

$$\begin{pmatrix} f^2 & 0 & 0 & 0 \\ 0 & \frac{\csc^2 \theta}{s^2} & 0 & 0 \\ 0 & 0 & \frac{1}{s^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{f^2} \end{pmatrix}$$

In general, this is the form taken by any black hole spacetime which has spherical symmetry. This is extremely important, because if we are able to do the LP-LC analysis for the general case of any spherically symmetric black hole spacetime, then all is left to do is input the particular f^2 and s^2 to determine the results for a specific system such as a RN or a higher dimensional RN AdS-dS case. Returning to LP-LC analysis, we use the metric in equation (4) to write a minimally coupled, massive scalar KG wave equation. Then, we separate variables into temporal, angular, and radial equations in the usual way as done by Konkowski and Helliwell [33]. The KG wave equation with the inclusion of gravitational interaction takes the form

$$(-g)^{-1/2}((-g)^{1/2}g^{\mu\nu}\Phi_{,\nu})_{,\mu} = m^2\Phi$$

where g is the determinant of the metric tensor, $g^{\mu\nu}$ is the inverse metric tensor, m represents the mass of the particle, and Φ represents the wavefunction. Rewriting this equation and decrypting the Einstein summation notation results in,

$$\sum_{i=1}^4 \sum_{j=1}^4 \frac{\frac{\partial}{\partial x_i} \left(\sqrt{-g} g^{ij} \frac{\partial \Phi(r, \Theta, \phi, t)}{\partial x_j} \right)}{\sqrt{-g}} = m^2 \Phi \quad (5)$$

Once expanded, this results in a second order differential equation of the form,

$$2f(r)f'(r)\Phi_r + f(r)^2 \left(\frac{2s'(r)\Phi_r}{s(r)} + \Phi_{rr} \right) - \frac{\Phi_{tt}}{f(r)^2} + \frac{\Phi_{\theta\theta} + \cot(\Theta)\Phi_\theta + \csc^2(\Theta)\Phi_{\phi\phi}}{s(r)^2} = m^2\Phi \quad (6)$$

where $\Phi_r \equiv \frac{\partial \Phi}{\partial r}$ and similarly for other variables. This differential equation can be separated by assuming a solution of the form $\Phi \sim R(r)T(t)Y(\theta, \phi)$. When variables are separated into temporal, radial, and angular equations, the resultant differential equations are

$$2ff'\frac{R'}{R} + f^2 \left(\frac{2s'R'}{sR} + \frac{R''}{R} \right) + \frac{q^2}{f^2} - \frac{\ell(\ell+1)}{s^2} - m^2 = 0 \quad (7)$$

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \ell(\ell+1) \right] Y = 0$$

$$T'' + q^2 T = 0$$

By inspection, we observe that the angular and temporal equations have known solutions. The angular equation for $Y(\theta, \phi)$ matches that of standard spherical harmonics on the 2-sphere and the temporal equation yields solutions which are exponential. Thus, the solutions are,

$$Y_{\ell}^{\tilde{m}}(\theta, \phi) \equiv \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-\tilde{m})!}{(\ell+\tilde{m})!}} P_{\ell}^{\tilde{m}}(\cos \theta) e^{i\tilde{m}\phi}$$

$$T \sim e^{iqt}$$

where, in general, $P_{\ell}^m(x)$ are associated Legendre polynomials (see [34] for a definition of these functions). Next is the radial equation, equation 7. Our goal is to manipulate this equation in order to attain a one dimensional Schrödinger equation from which we can gather an effective potential. To do so, we make the substitution $R(r) = \frac{u(x)}{g(r)}$ and solve the resultant equation for $g(r)$ so first derivatives in $u(x)$ disappear. In order to make this substitution, additional information is needed as to how x is related to r , and to be exact, how $\frac{dx}{dr}$ is related to $f(r)$ and $s(r)$. This information is housed in the normalization constraint in equation 7,

$$\int dr d\theta d\phi R^* R \sqrt{\frac{g_3}{-g_{tt}}} = 4\pi \int^r \frac{s^2}{f^2} R^* R dr \quad (8)$$

$$= 4\pi \int^r \frac{u^*(x)u(x)dr}{f^2} \quad (9)$$

$$= 4\pi \int^x u^*(x)u(x)dx \quad (10)$$

where g_3 represents the spatial determinant of the metric and g_{tt} is the temporal coefficient. The result of equation 9 being equal to equation 10 is that the following must be true,

$$\frac{dx}{dr} = \frac{1}{f^2} \rightarrow x(r) = \int^r \frac{dr}{f^2} \quad (11)$$

Under these constraints, the substitution of $R(r) = \frac{u(x)}{g(r)}$ yields $g(r) = s(r)$ in order for the coefficient of first derivative terms to go to zero. From here, we apply the substitution $R(r) = \frac{u(x)}{s(r)}$ and put it in the form $Hu(x) = Eu(x)$ where E is a constant and $H = \frac{-d^2}{dx^2} + V(x)$. In this Schrödinger equation, x is proportional to r and the classical singularity remains at $x = 0$. This equation becomes,

$$\left(-\frac{d^2}{dx^2} + \frac{f^2}{s} \frac{d}{dr} \left(f^2 \frac{ds}{dr} \right) + \frac{\ell(\ell+1)}{s^2} f^2 + m^2 f^4 \right) u = q^2 u$$

Therefore, for generic metrics of the form in equation 4 (namely that of metrics which have spherical symmetry), the potential will always take the following form [33]:

$$V(r) = \frac{f^2}{s} \frac{d}{dr} \left(f^2 \frac{ds}{dr} \right) + \frac{\ell(\ell+1)}{s^2} f^2 + m^2 f^4 \quad (12)$$

3.1.1 Negative Mass Schwarzschild Black Hole

In particular for a Schwarzschild black hole system we have,

$$\begin{aligned} f^2(r) &= 1 - \frac{2M}{r} \\ s^2(r) &= r^2 \end{aligned} \quad (13)$$

For our case, we will consider a negative mass Schwarzschild black hole, so that f^2 becomes,

$$f^2(r) = 1 + \frac{2M}{r} \quad (14)$$

where M is now positive. We consider the negative mass Schwarzschild black hole for the same reason described in the ‘‘Black Holes’’ section - to remove the event horizon located at $r = 2M$ and make our singularity a naked singularity. If this is done, then we can simplify our equations near the curvature singularity at zero and then near infinity for LP-LC analysis and deficiency index analysis. Because there is no event horizon or other singularity between the two regions, the results from our analysis about whether the system is quantum singular or non-singular are still valid. Since most of the difficult work has been worked out for a generic spherically symmetric black hole system, we begin with equation 11 and using it to determine $x(r)$ and then $r(x)$ for the negative mass Schwarzschild black hole,

$$x(r) = \int^r \frac{dr}{(1 + \frac{2M}{r})}$$

In the assumption that we are very near the classical curvature singularity located at $r = 0$, the $1/r^2$ term will dominate (in other words, the term with the largest power in the denominator), so we can approximate this integral by eliminating the other terms so that,

$$x_0(r) \approx \int^r \frac{dr}{\frac{2M}{r}} = \frac{r^2}{4M}$$

Simultaneously, equation 12 can now be used to approximate $V(r)$ near $r = 0$. By substituting f^2 and s^2 and again only considering the term with the largest power in the denominator, this is found to be,

$$\begin{aligned} V(r) &= \frac{(1 + \frac{2M}{r})}{r} \frac{d}{dr} \left(1 + \frac{2M}{r} \right) + \frac{\ell(\ell+1)(1 + \frac{2M}{r})}{r^2} + m^2 \left(1 + \frac{2M}{r} \right) \\ V(r) &= \left(\frac{1}{r} + \frac{2M}{r^2} \right) \left(\frac{-2M}{r^2} \right) + \ell(\ell+1) \left(\frac{1}{r^2} + \frac{2M}{r^3} \right) + m^2 \left(1 + \frac{2M}{r} \right) \\ V(r) &= \left(\frac{-2M}{r^3} + \frac{-4M^2}{r^4} \right) + \ell(\ell+1) \left(\frac{1}{r^2} + \frac{2M}{r^3} \right) + m^2 \left(1 + \frac{2M}{r} \right) \end{aligned} \quad (15)$$

where, $\lim_{r \rightarrow 0} V(r) = \frac{-4M^2}{r^4}$. Substituting to obtain $V_0(x)$ we find,

$$V_0(x) \approx \frac{-1}{4x^2}$$

With a relationship for the effective potential near zero $V_0(x)$, the LP-LC theorems of Reed and Simon [32] can now be applied to determine the essential self-adjointness of the spatial portion of the KG operator, consequently determining if the spacetime is quantum singular or non-singular at the same time (remember that we are not quantizing gravity and still are working on a classical background spacetime without back reaction). Near $x = 0$, we will use the definition of LP-LC in order to characterize the potential. To do so, we must determine the square integrability of both solutions to the one dimensional Schrödinger equation formed by the new potential,

$$-\psi'' + V(x)\psi = \lambda\psi \quad (16)$$

Since the definition applies for any possible eigenvalue (i.e. any possible value of λ where $\lambda = q^2$) we select the easiest equation to analyze - that of $\lambda = 0$. Inserting $V_0(x)$ and zero for λ , we obtain,

$$-\psi''(x) - \frac{1}{4x^2}\psi(x) = 0 \quad (17)$$

This differential equation is an Euler equation [34], and therefore to solve it we assume a polynomial form to the solution. The solution and its first two derivatives become,

$$\psi = x^n \quad (18)$$

$$\psi' = nx^{n-1} \quad (19)$$

$$\psi'' = n(n-1)x^{n-2} \quad (20)$$

By inserting these equations into equation 17 and solving for n we find $n \rightarrow \frac{1}{2}$. The first solution therefore becomes $\psi_1 = x^{1/2}$. Armed with one solution, we assume a second solution of the form $\psi = vx^{1/2}$ where v is a function of x . The second solution and its first two derivatives become,

$$\psi = vx^{1/2}$$

$$\psi' = v'x^{1/2} + \frac{v}{2}x^{-1/2}$$

$$\psi'' = v''x^{1/2} + v'x^{-1/2} - \frac{v}{4}x^{-3/2}$$

Again, by inserting these into 17 and solving for v we find $v \rightarrow \ln x$. Therefore, the second solution becomes $\psi_2 = x^{1/2} \ln x$. We now check the two solutions to determine if they are square integrable. The first solution behaves as follows,

$$\int_0^{const.} |x^{1/2}|^2 dx = \int_0^{const.} x dx = \frac{1}{2}x^2 \Big|_0^{const.} < \infty$$

therefore making it square integrable. The second solution behaves as follows,

$$\begin{aligned}
\int_0^{const.} |x^{1/2} \ln x|^2 dx &= \int_0^{const.} x(\ln x)^2 dx \\
&= \frac{1}{4} x^2 (2(\ln x)^2 - 2 \ln x + 1) \Big|_0^{const.} \\
&= I
\end{aligned}$$

The value of I is indeterminate in its current form at $x = 0$. Therefore, we must determine the limit of I as x goes to 0 using L'Hospital's rule,

$$\begin{aligned}
\lim_{x \rightarrow 0} I &= \frac{(\ln x)^2}{2x^{-2}} - \frac{\ln x}{2x^{-2}} \\
&\stackrel{L'H}{=} \frac{(2 \ln x) \frac{1}{x}}{-4x^{-3}} + \frac{\frac{1}{x}}{4x^{-3}} \\
&\stackrel{L'H}{=} \frac{\ln x}{-2x^{-2}} \\
&\stackrel{L'H}{=} \frac{\frac{1}{x}}{4x^{-3}} \\
&= \frac{x^2}{4} \\
&= 0 < \infty
\end{aligned}$$

Because both solutions are square integrable, the potential is in the limit-circle case near $x = 0$ (and thus $r = 0$, where the curvature singularity is located). In order to determine what is happening to the particle near infinity, we must re-examine equation 15 in the limit as x approaches infinity instead of zero,

$$V(r) = \left(\frac{-2M}{r^3} + \frac{-4M^2}{r^4} \right) + \ell(\ell+1) \left(\frac{1}{r^2} + \frac{2M}{r^3} \right) + m^2 \left(1 + \frac{2M}{r} \right)$$

where,

$$\lim_{r \rightarrow \infty} V(r) \approx m^2 \quad (21)$$

Although previously we had to make a final substitution to change from r to x , an inspection of equation 11 as r goes to infinity yields the conclusion that $x_\infty(r) = r$ so the form of equation 21 does not change. With this new effective potential, we will use theorems associated with LP-LC instead of the definition itself. Specifically, we will use Theorem X.8 of Reed and Simon [32]. The theorem says we must find an $M(x)$ which meets the following three criteria in order for the potential $V(x)$ to be limit point near ∞ ,

- i $V(x) \geq -M(x)$
- ii $\int_1^\infty (M(x))^{-1/2} dx = \infty$
- iii $\frac{M'(x)}{(M(x))^{3/2}} < \infty$

By selecting $M(x) = k$ we meet these three criteria,

- i $V_\infty(x) \geq -k$
- ii $\int_1^\infty (k)^{-1/2} dx = \left. \frac{x}{\sqrt{k}} \right|_1^\infty = \infty$
- iii $\frac{k'}{k^{3/2}} = 0 < \infty$

As a result, the potential near infinity is in the limit-point case [32]. In all, we have proved that a negative mass Schwarzschild black hole system is limit-circle near the curvature singularity at $r = 0$ and limit-point near distances infinitely far away from the center of the system. Due to the system not being in the limit-point case at both zero and infinity, the spatial portion of the KG operator is not essentially self-adjoint; therefore, the system is quantum singular.

3.1.2 A Reissner Nordström Black Hole

The line element for RN spacetimes has the same form found in equation 4 because this system is spherically symmetric. However, f^2 and s^2 now become,

$$f^2 = 1 - \frac{2M}{r} + \frac{\varepsilon^2}{r^2} \quad (22)$$

$$s^2 = r^2 \quad (23)$$

where ε represents the charge of the black hole system. Recall that in this investigation, all of the variations of our RN black hole system are super-extremal cases, so that it is overcharged. Mathematically speaking, this indicates $\varepsilon^2 \gg M^2$ such that there are no event horizons (see Appendix F for more information including Penrose diagrams). The next step is to use equation 11 to determine $x(r)$ and then $r(x)$ very near the classical singularity located at $r = 0$ as before,

$$x_0(r) \approx \int^r \frac{dr}{\frac{\varepsilon^2}{r^2}} = \frac{r^3}{3\varepsilon^2}$$

Using equation 12 to approximate $V_0(r)$ near $r = 0$ (i.e. taking only the term with the largest power of r in the denominator) we find,

$$V_0(r) \approx \frac{-2\varepsilon^4}{r^6}$$

And likewise,

$$V_0(x) \approx \frac{-2}{9x^2} \quad (24)$$

Again, near $x = 0$, we will use the definition of LP-LC in order to characterize the potential by solving equation 16 with the potential described by equation 24 again with $\lambda = 0$,

$$-\psi''(x) - \frac{2}{9x^2}\psi(x) = 0 \quad (25)$$

Again, this differential equation is an Euler equation [34], and therefore to solve it we assume a polynomial form to the solution using equations 50, 51, and 52. By inserting these equations into equation 25 and solving for n we find $n \rightarrow \frac{1}{2} \pm \frac{1}{6} = (\frac{2}{3}, \frac{1}{3})$. The two solutions therefore become $\psi_1 = x^{2/3}$ and $\psi_2 = x^{1/3}$. We now check the two solutions to determine if they are square integrable. The solutions behave as follows,

$$\begin{aligned} \int_0^{const.} |x^{2/3}|^2 dx &= \int_0^{const.} x^{4/3} dx = \frac{3}{7} x^{7/3} \Big|_0^{const.} < \infty \\ \int_0^{const.} |x^{1/3}|^2 dx &= \int_0^{const.} x^{2/3} dx = \frac{3}{5} x^{5/3} \Big|_0^{const.} < \infty \end{aligned}$$

therefore making both square integrable. Because both solutions are square integrable, the potential is in the limit-circle case near $x = 0$. We now re-examine equation 12 in the limit as x approaches infinity instead of zero. However, in doing so we find $V_\infty(x)$ matches that of 21. Since it takes exactly the same form, this potential is in the limit-point case as well near infinity. Similarly, because the RN system is not in the limit-point case at both zero and infinity, the system is quantum singular.

3.1.2.1 Anti-de Sitter and de Sitter Spacetime

Now we will consider the RN case in AdS or dS space, meaning we are assuming a geometry of spacetime. Mathematically, this is done by the inclusion of Λ into the line element for the RN black hole system (see Appendix E for more information on the meaning of Λ including various sign differences). However, it is very important to note that when this is done, even in the overcharged case of the RN black hole system, we introduce a removable classical singularity into our spacetime (an effective event horizon) termed the “cosmological horizon”. This cosmological horizon is nested between our two regions of interest for the LP-LC theorems and may impact the conclusions we draw for the system as a whole when we use these theorems. As a result, it should be laid out explicitly in the beginning that we are considering the effects of this cosmological horizon to be negligible, thereby keeping our curvature singularity as a naked singularity. For any further black hole systems which include an AdS or dS surrounding spacetime we will assume that this approximation is true and that our singularity is naked.

Furthermore, because the only difference between AdS and dS is the sign of Λ , the system is often termed a RN AdS-dS black hole system. This is the convention we will use in this investigation as well. The line element for RN AdS-dS spacetimes includes,

$$f^2 = \left(1 - \frac{2M}{r} + \frac{\varepsilon^2}{r^2} - \frac{\Lambda r^2}{3} \right) \quad (26)$$

$$s^2 = r^2 \quad (27)$$

where Λ is the cosmological constant. In this case, $x(r)$, $V(r)$, and $V(x)$ become,

$$x_0(r) \approx \int^r \frac{dr}{\frac{\varepsilon^2}{r^2}} = \frac{r^3}{3\varepsilon^2} \quad (28)$$

$$V_0(r) \approx \frac{-2\varepsilon^4}{r^6}$$

$$V_0(x) \approx \frac{-2}{9x^2}$$

Astonishingly, the potential near zero for the RN AdS-dS exactly matches the RN system without inclusion of the cosmological constant (the potential in equation 24). Due to the potential having the exact same form, we can directly conclude that near zero the potential is in the limit-circle case as before. However, near infinity the potential changes form. Using equation 15 with an f^2 and s^2 described by equations 26 and 27 and taking the limit as r goes to infinity we find,

$$\lim_{r \rightarrow \infty} V(r) \approx r^2 \Lambda \frac{(2\Lambda - 3m)}{9}$$

Using equation 28, we arrive at $V_\infty(x)$,

$$V_\infty(x) \approx \frac{\Lambda(2\Lambda - 3m)\varepsilon^{4/3}}{9^{4/3}} x^{2/3}$$

To determine whether this potential is limit-point or limit-circle, we will again use Theorem X.8 of Reed and Simon [32], this time selecting $M(x) = kx^{4/3}$ to meet the three criteria,

- i $V_\infty(x) \geq -kx^{2/3}$
- ii $\int_1^\infty (kx^{2/3})^{-1/2} dx = \left. \frac{3x^{2/3}}{2\sqrt{k}} \right|_1^\infty = \infty$
- iii $\frac{(kx^{2/3})'}{(kx^{2/3})^{3/2}} = \frac{2k}{3k^{3/2}x^{4/3}} = \frac{2}{3\sqrt{k}x^{4/3}} < \infty$

As a result, the potential near infinity is in the limit-point case. This proves that a RN AdS-dS black hole system is limit-circle near the curvature singularity at $r = 0$ and limit-point near distances infinitely far away from the center of the system, making the system quantum singular.

3.2 Higher Dimensional Spacetimes

Next we will consider a spacetime which has an additional spatial dimension - an angular dimension α . This spacetime is termed a 4+1 dimensional spacetime. Interest in higher dimensional spacetimes has remained high lately for the same reasons as that of cosmic strings - the possible relation to M-theory (the modern version of superstring theory) [27]. An interesting feature of M-theory is the possible existence of higher dimensions (as many as 26 in some cases). The line element for a generic, spherically symmetric 5D spacetime is,

$$ds^2 = -f^2 dt^2 + \frac{dr^2}{f^2} + s^2(d\theta^2 + \sin^2 \theta(d\phi^2 + \sin^2 \phi d\alpha^2)) \quad (29)$$

The metric $g_{\mu\nu}$ for the line element in equation 29 in matrix form is,

$$\begin{pmatrix} \frac{1}{f^2} & 0 & 0 & 0 & 0 \\ 0 & s^2 \sin^2 \theta & 0 & 0 & 0 \\ 0 & 0 & s^2 \sin^2 \theta \sin^2 \phi & 0 & 0 \\ 0 & 0 & 0 & s^2 & 0 \\ 0 & 0 & 0 & 0 & -f^2 \end{pmatrix}$$

and the inverse metric $g^{\mu\nu}$ in matrix form is,

$$\begin{pmatrix} f^2 & 0 & 0 & 0 & 0 \\ 0 & \frac{\csc^2 \theta}{s^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\csc^2 \theta \csc^2 \phi}{s^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s^2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{f^2} \end{pmatrix}$$

Due to the addition of the angular coordinate α , we must derive a new generic form of the potential $V(r)$ and relationship for our one dimensional coordinate $x(r)$. Starting with the minimally coupled, massive KG equation (found in equation 5) and the above matrices to determine g and g^{ij} , we arrive at the following differential equation

$$-\frac{\Phi_{tt}}{f^2} + \frac{1}{s^2 \sin^2 \theta} \left(\frac{\Phi_{\alpha\alpha} + \Phi_{\phi\phi} + \cot \phi \Phi_{\alpha}}{\sin^2 \phi} + \frac{2}{\tan \theta} \Phi_{\theta} + \Phi_{\theta\theta} \right) + 2f \frac{df}{dr} \Phi_r + f^2 \left(\frac{3}{s} \frac{ds}{dr} \Phi_r + \Phi_{rr} \right) = m^2 \Phi \quad (30)$$

This separates to $\Phi \sim R(r)T(t)Y^{k\ell\tilde{m}}(\theta, \phi, \alpha)$ where $Y^{k\ell\tilde{m}}(\theta, \phi, \alpha)$ are the spherical harmonics for the 3-sphere [35]. The radial equation then becomes,

$$R'' + \left(\frac{2f'}{f} + \frac{3s'}{s} \right) R' + \left(\frac{q^2}{f^4} - \frac{\ell(\ell+1)}{f^2 s^2} - \frac{m^2}{f^2} \right) R = 0 \quad (31)$$

The goal is exactly the same as before - to make a substitution for $R(r)$ changing this radial equation to the one dimensional Schrödinger equation $Hu(x) = Eu(x)$. The substitution $R(r) = \frac{u(x)}{g(r)}$ is made and the resultant equation is solved for $g(r)$ so first derivatives in $u(x)$ disappear. Under these constraints, we find,

$$\frac{dg}{dr} = \frac{3g}{3s} \frac{ds}{dr} \quad (32)$$

$$g(r) = s^{3/2} \quad (33)$$

Simultaneously, we find the relationship between x and r from the normalization condition for 4 + 1 dimensional spacetimes (the same normalization equation which led to equations 9 and 10 with the addition of integration over the α coordinate). Integrating over $0 \leq \theta \leq \pi, 0 \leq \phi \leq \pi, 0 \leq \alpha \leq 2\pi$ and using $\sqrt{\frac{g_4}{-g_{00}}} = \frac{s^3 \sin^2 \theta \sin \phi}{f^2}$

$$\int dr d\theta d\phi d\alpha R^* R \sqrt{\frac{g_4}{-g_{tt}}} = 2\pi^2 \int^r \frac{s^3}{f^2} R^* R dr \quad (34)$$

where g_4 is the spatial determinant of the metric and g_{tt} is still the temporal coefficient. Changing from $R(r)$ to $u(x)$ within the integral, equation 34 becomes,

$$2\pi^2 \int^r \frac{u^*(x)u(x)dr}{f^2} = 2\pi^2 \int^x u^*(x)u(x)dx$$

where,

$$\frac{dx}{dr} = \frac{1}{f^2}, \quad x(r) = \int^r \frac{dr}{f^2}$$

Substituting back into equation 31 for $R(r)$ we find,

$$V(r) = \frac{3f^2}{2} \left(\frac{1}{s} \frac{d}{dr} (f^2 s') - \frac{1}{2} \frac{d}{dr} \left(\frac{1}{s} \right) \right) + \frac{\ell(\ell+1)f^2}{s^2} + m^2 f^2 \quad (35)$$

3.2.1 RN AdS - dS Spacetimes

Since we have worked out the effective potential for generic higher dimensional systems, we can now specify the system of interest. For the RN AdS-dS cases, the f^2 and s^2 become [33],

$$f^2 = \left(1 - \frac{2M}{r^2} + \frac{\varepsilon^2}{r^4} - \frac{\Lambda r^2}{6} \right) \quad (36)$$

$$s^2 = r^2 \quad (37)$$

Using equations 36 and 37 for the 5D RN AdS and dS spacetimes, we insert these into equation 35 and take the limit as r approaches 0. In this limit we again only consider the term with the largest power of r in the denominator. Thus, we arrive at an approximate potential of,

$$V_0(x) \approx \frac{-6\varepsilon^4}{r^{10}}$$

Exchanging r for $x(r)$ we arrive at the effective potential,

$$x_0(r) \approx \int^r \frac{dr}{\frac{q^2}{r^4}} = \frac{r^5}{5\varepsilon^2} \quad (38)$$

$$V_0(x) \approx \frac{-6}{25x^2}$$

Applying this potential to equation 16 we find,

$$-\psi''(x) - \frac{6}{25x^2}\psi(x) = 0 \quad (39)$$

This results in another Euler equation, and therefore we again use equations 50, 51, and 52. By inserting these equations into equation 39 and solving for n we find $n \rightarrow \frac{1}{2} \pm \frac{1}{10} = (\frac{2}{5}, \frac{3}{5})$. The two solutions therefore become $\psi_1 = x^{2/5}$ and $\psi_2 = x^{3/5}$. We now check the two solutions to determine if they are square integrable. The solutions behave as follows

$$\int_0^{const.} |x^{2/5}|^2 dx = \int_0^{const.} x^{4/5} dx = \frac{5}{9} x^{9/5} \Big|_0^{const.} < \infty$$

$$\int_0^{const.} |x^{3/5}|^2 dx = \int_0^{const.} x^{6/5} dx = \frac{5}{11} x^{11/5} \Big|_0^{const.} < \infty$$

therefore making both square integrable. Because both solutions are square integrable, the potential is in the limit-circle case near $x = 0$. We now re-examine equation 12 in the limit as x approaches infinity instead of zero. Using equation 15 with an f^2 and s^2 described by equations 36 and 37 and taking the limit as r goes to infinity we find,

$$\lim_{r \rightarrow \infty} V(r) \approx r^2 \frac{\Lambda(\Lambda - 3m)}{18}$$

Using equation 38, we arrive at $V_\infty(x)$

$$V_\infty(x) \approx \frac{5^{2/5} \Lambda(\Lambda - 3m) \varepsilon^{4/5}}{18} x^{2/5}$$

A plot of the effective potentials near infinity for varying values of the cosmological constant Λ for the RN AdS - dS and higher dimensional RN AdS - dS cases can be found in Figure 12(a) and Figure 12(b) respectively.

To determine whether this potential is limit-point or limit-circle, we will again use Theorem X.8 of Reed and Simon [32], this time selecting $M(x) = kx^{2/5}$ to meet the three criteria,

- i $V_\infty(x) \geq -kx^{2/5}$
- ii $\int_1^\infty (kx^{2/5})^{-1/2} dx = \left. \frac{5x^{4/5}}{4\sqrt{k}} \right|_1^\infty = \infty$
- iii $\frac{(kx^{2/5})'}{(kx^{2/5})^{3/2}} = \frac{(\frac{2k}{5x^{3/5}})}{(k^{3/2}x^{3/5})} = \frac{2}{5\sqrt{k}x^{6/5}} < \infty$

As a result, the potential near infinity is in the limit-point case. This proves that a higher dimensional RN AdS-dS black hole system is limit-circle near the curvature singularity at $r = 0$ and limit-point near distances infinitely far away from the center of the system, making the system quantum singular.

3.3 Axially Symmetric Black Hole Spacetimes

3.3.1 Cosmic Strings

Now we will consider all of the previously analyzed black hole systems, but with the inclusion of a cosmic string. With the inclusion of a cosmic string penetrating through the center of the black hole, the line element no longer has the property of being spherically symmetric (see Appendix A for definitions of spherical symmetry and axial symmetry); however, the line element is now axially symmetric. As a result, we cannot use the generic potentials derived in the previous section and must derive new generic potentials for systems including a cosmic string. The line element becomes,

$$ds^2 = -f^2 dt^2 + \frac{dr^2}{f^2} + s^2 (d\theta^2 + \tilde{\alpha}^2 \sin^2 \theta d\phi^2)$$

In this metric the parameter $\tilde{\alpha}$ is representative of the azimuthal deficit which occurs due to the nature of the conical singularity associated with the cosmic string. To analyze this metric utilizing the same LP-LC method used in the prior cases, we must restart from the minimally coupled, massive KG equation (found

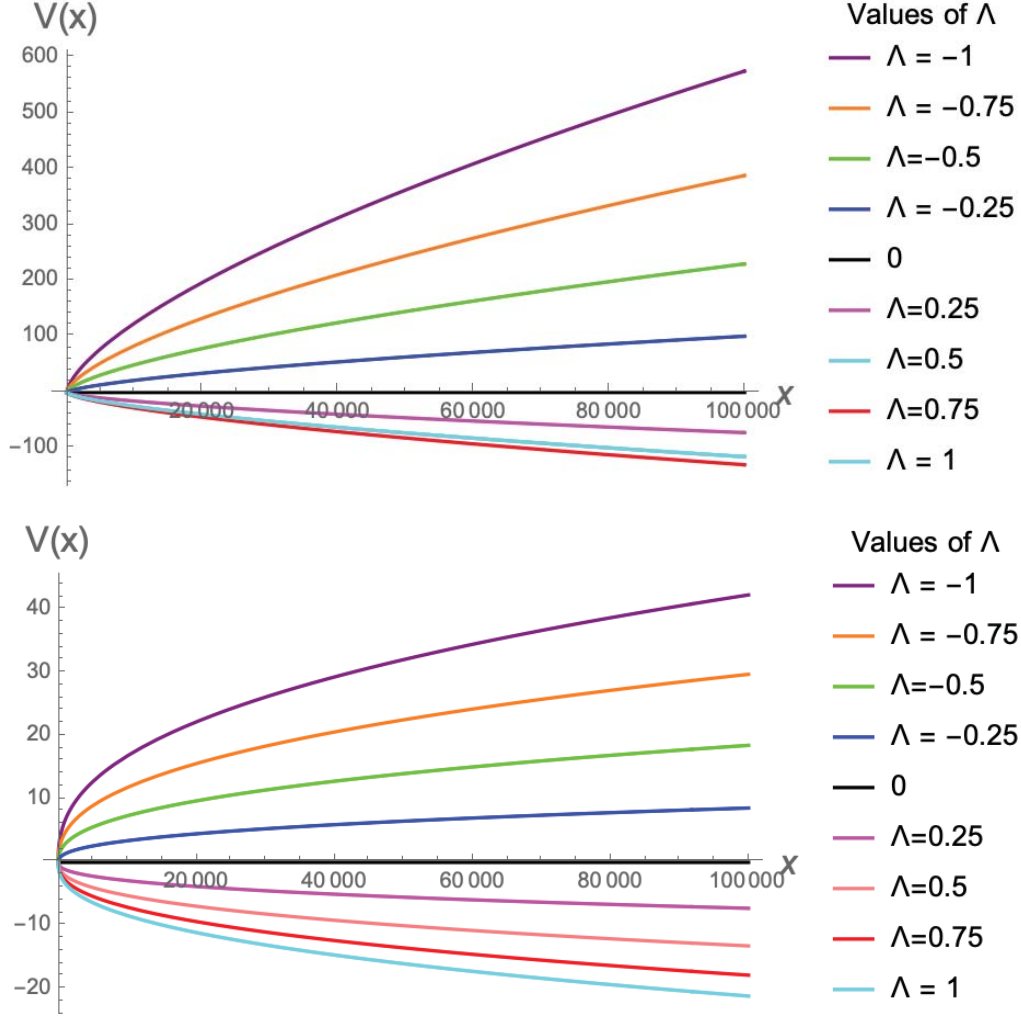


Figure 12: The effective potential near infinity for varying values of Λ for the RN AdS - dS and higher dimensional RN AdS - dS black hole systems. In these plots, the mass of the scalar particle and charge of the black hole are assumed to be one in arbitrary units (i.e. $m = 1$ and $\varepsilon = 1$).

in equation 5), solve via separation of variables, etc. The separation of variables results in the following equations,

$$2f(r)f'(r)\frac{R'(r)}{R(r)} + f(r)^2 \left(\frac{2s'(r)R'(r)}{s(r)R(r)} + \frac{R''(r)}{R(r)} \right) + \frac{q^2}{f(r)^2} - \frac{\lambda(\lambda+1)}{s(r)^2} - m^2 = 0$$

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\tilde{\alpha}^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \tilde{\lambda}(\tilde{\lambda}+1) \right] Y(\theta, \phi) = 0$$

$$T''(t) + q^2 T(t) = 0$$

These equations are identical to the radial, angular, and temporal equations solved for previously in the spherically symmetric cases, except with the separation constant ℓ changed to a new separation constant $\tilde{\lambda}$. The new separation constant, $\tilde{\lambda}$, is connected to the original separation constants $\tilde{m}$ and ℓ , as well as the

cosmic string parameter $\tilde{\alpha}$, through the following relationship (see [36] for more information),

$$\tilde{\lambda} = \ell - |\tilde{m}| + \frac{|\tilde{m}|}{\tilde{\alpha}}$$

The spherical harmonics are different as a result and are visually compared to the spherical harmonics on the 2-sphere without a cosmic string in Figure 13 for the first five modal solutions. The spherical harmonics, labeled $Y_{\tilde{\lambda}}^{\tilde{m}}(\theta, \phi)$, now take the form,

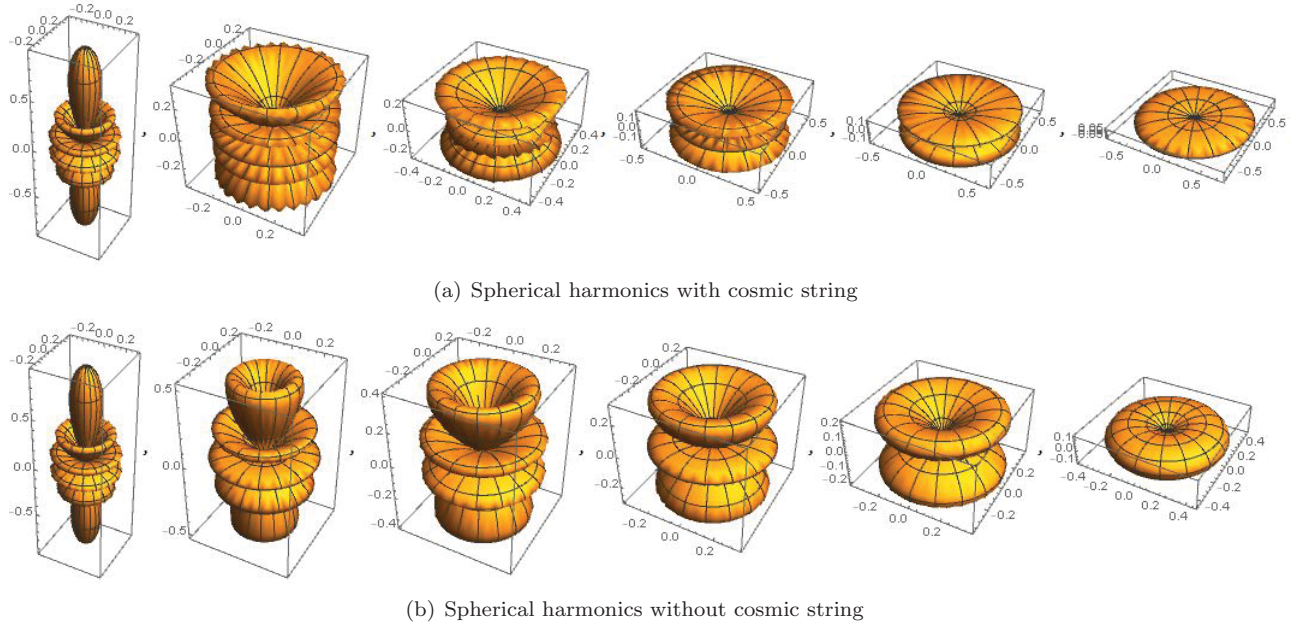


Figure 13: The differences in the spherical harmonics are shown for the addition of a cosmic string. Notice how with the cosmic string included, the spherical harmonics are flattened out due to the conical singularity.

$$\sqrt{\frac{(2\tilde{\lambda} + 1) \Gamma(\tilde{\lambda} + \frac{|\tilde{m}|}{\tilde{\alpha}} + 1)}{4\pi \Gamma(\tilde{\lambda} - \frac{|\tilde{m}|}{\tilde{\alpha}} + 1)}} e^{i\tilde{m}\phi} P_{\tilde{\lambda}}^{-\frac{|\tilde{m}|}{\tilde{\alpha}}}(\cos \theta) \quad (40)$$

Because $\tilde{\lambda}$ appears in both the radial and angular equations, the radial solution should be altered also. However, the only difference is that the ℓ in our original radial equation has been substituted for the new $\tilde{\lambda}$, so the radial solution will be identical to before with the inclusion of $\tilde{\lambda}$. In other words, f^2 and s^2 are still general. As a result, the final effective potentials near zero and infinity solved for previously can be reused by simply substituting $\tilde{\lambda}$ for ℓ . Armed with this information, we can then look at individual black hole systems and see if the effective potentials near zero and infinity depend on ℓ and if this changes conclusions when ℓ is swapped out for $\tilde{\lambda}$.

3.3.1.1 RN Black Hole Spacetime

In particular, near $x = 0$ equation 24 does not change at all and as x approaches infinity equation 21 yields that the ℓ will be substituted out for $\tilde{\lambda}$, but takes the same form. The former indicates that the RN black hole system penetrated by a cosmic string is limit-circle near zero. The latter, because $\tilde{\lambda} > 0$, indicates

that this black hole system is in the limit-point case as x goes to infinity. This leads to the conclusion that the system is quantum singular.

3.3.1.2 RN AdS-dS Black Hole Spacetime

Because the effective potentials for the RN AdS-dS black hole system do not include ℓ at all, the LP-LC analysis remains unchanged. In light of this fact, the result is that the RN AdS-dS black hole system penetrated by a cosmic string is quantum mechanically singular.

3.3.1.3 Higher Dimensional RN AdS-dS Spacetime

A black hole system which requires further explanation is the higher dimensional RN AdS-dS system penetrated by a cosmic string. The line element for this system is,

$$ds^2 = -f^2 dt^2 + \frac{dr^2}{f^2} + s^2 (d\theta^2 + \sin^2 \theta (d\phi^2 + \alpha^2 \sin^2 \phi d\chi^2))$$

where α now represents the cosmic string parameter (formerly denoted $\tilde{\alpha}$) and χ now represents our additional spatial dimension. Starting with the minimally coupled, massive KG equation (found in equation 5) and this metric, the angular, radial, and temporal equations become the following,

$$\begin{aligned} R'' + \left(\frac{2f'}{f} + \frac{3s'}{s} \right) R' + \left(\frac{q^2}{f^4} - \frac{k(k+2)}{f^2 s^2} - m^2 f^2 \right) R &= 0 \\ \left[\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial}{\partial \phi} \right) + \frac{1}{\alpha^2 \sin^2 \phi} \frac{\partial^2}{\partial \chi^2} + 2 \cot \theta \frac{d}{d\theta} + \frac{d^2}{d\theta^2} + k(k+2) \right] Y(\theta, \phi, \chi) &= 0 \\ T''(t) + q^2 T(t) &= 0 \end{aligned}$$

The angular equation results in the spherical harmonics for the 3-sphere, as before, with a minor alteration. The spherical harmonics for the 3-sphere are written as $Y^{k\ell\tilde{m}} = H^{k\ell}(\theta) Y_{\ell}^{\tilde{m}}(\phi, \chi)$ where $Y_{\ell}^{\tilde{m}}(\phi, \chi)$ are the spherical harmonics for the 2-sphere (this is assuming the notation used in [35]). The equations above dictate that the old $Y_{\ell}^{\tilde{m}}(\phi, \chi)$ are shifted to the new $Y_{\lambda}^{\tilde{m}}(\phi, \chi)$ which are the spherical harmonics with cosmic string parameter λ for the 2-sphere stated in equation 40. The $H^{k\ell}$ are likewise shifted to $H^{k\lambda}$, substituting out all values of ℓ for those of λ without any additional changes.

Similarly, the radial equation is changed only by the substitution of $\ell(\ell+1)$ to $k(k+2)$. However, in reviewing the effective potentials both near the curvature singularity at zero and as x approaches infinity, it can be seen that ℓ does not appear in these potentials. Due to this fact, k and λ will be absent from the LP-LC analysis entirely. Thus, despite the angular and radial solutions changing form due to the penetration of the cosmic string, the result does not change – the higher dimensional RN AdS-dS black hole system with a cosmic string included is still quantum mechanically singular.

3.3.2 Kerr Black Holes

The last black hole system to be analyzed in this investigation is arguably the most complex – the rotating black hole. A rotating black hole, termed a Kerr black hole, presents issues because it is not spherically

symmetric and contains terms in the line element mixing temporal and spatial dimensions. It, like the cosmic string spacetime, is axially symmetric but less amenable to simple analysis because it is stationary and not static (see Appendix A for a definition of stationary and static metrics). In particular, the line element for a rotating black hole in Boyer-Lindquist coordinates and in natural units where $G = c = 1$ is [2],

$$ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} (adt - (r^2 + a^2)d\phi)^2 \quad (41)$$

where,

$$\Delta = r^2 + a^2 - 2Mr \quad (42)$$

$$\rho = r^2 + a^2 \cos^2 \theta \quad (43)$$

In this line element, $a = \frac{J}{M}$ where J is the angular momentum of the rotating black hole and M is the mass of the black hole (note that in natural units M and a have units of length). It is important to note that we are considering only the super-extremal case of this black hole system in which it is overspinning, meaning that $a^2 \gg M^2$. This black hole system allows for a naked singularity and no event horizons between zero and infinity, as before. The metric $g_{\mu\nu}$ for the line element in equation 41 in matrix form is,

$$\begin{pmatrix} \frac{\rho^2}{\Delta} & 0 & 0 & 0 \\ 0 & \frac{\Delta \sin^2 \theta}{\rho^2} & 0 & -\frac{2amr \sin^2 \theta}{\rho^2} \\ 0 & 0 & \rho^2 & 0 \\ 0 & -\frac{2amr \sin^2 \theta}{\rho^2} & 0 & \frac{2mr}{\rho^2} - 1 \end{pmatrix}$$

and the inverse metric $g^{\mu\nu}$ in matrix form is,

$$\begin{pmatrix} \frac{\Delta}{\rho^2} & 0 & 0 & 0 \\ 0 & \frac{(\rho^2 - 2mr)}{\Delta \rho^2 \sin^2 \theta} & 0 & -\frac{2amr}{\Delta \rho^2} \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ 0 & -\frac{2amr}{\Delta \rho^2} & 0 & -\frac{\Sigma}{\rho^2} \end{pmatrix}$$

where $\Sigma = \frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta$. In both of these matrix representations, $g_{3\nu}$ and $g_{\mu 3}$ (i.e. any term in the 4th row or 4th column) represent coefficients associated with any temporal differentials in the line element. The quantum test packet will be described using the massive, minimally coupled KG equation written as,

$$(-g)^{-1/2} ((-g)^{1/2} g^{\mu\nu} \Phi_{,\nu})_{,\mu} = m^2 \Phi$$

In this analysis the assumed solution is called the reduced wave equation and takes the form $\Phi(t, r, \theta, \phi) = e^{i(k\phi)} f(t, r, \theta)$. This model of solution is popular for exploring new symmetries to the solutions of the wave equation. Using the metric in equation 41 with this version of the KG operator and the reduced wave solution, the resultant differential equation takes the following form,

$$\frac{\partial^2 f}{\partial t^2} + \frac{4aMrki}{\Delta \Sigma} \frac{\partial f}{\partial t} = \frac{1}{\Sigma} \left[\frac{\partial}{\partial r} \left(\Delta \frac{\partial}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \left(k^2 \left(\frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) + m^2 \rho^2 \right) \right] f$$

We are interested in encountering the classical ring singularity at $r = 0$. By reviewing the line element in equation 41 with the notion of geodesic incompleteness, this is possible on any path where $r^2 + a^2 \cos^2 \theta = 0$. One path which allows this to happen is one within the equatorial plane such that $\theta = \frac{\pi}{2}$ and then performing analysis very, very close to $r = 0$. Alternatively, the path with $r = 0$ could have been chosen with θ allowed to vary anywhere between $\theta + \epsilon$ and $\theta - \epsilon$ where $\epsilon \ll 1$. For the remainder of this analysis we will focus on the case where $\theta = \frac{\pi}{2}$ and r is approaching zero (although an infinite number of alternative paths are possible). With this restriction, we input $\theta = \frac{\pi}{2}$ noting that $\frac{\partial^2 f}{\partial \theta^2}$ and $\frac{\partial f}{\partial \theta}$ are both equal to zero because θ is a constant. Because we are considering only the equatorial plane where $\theta = \frac{\pi}{2}$ in tandem with the idea that k is related to the orbital quantum number of the quantum wave packet, any solution in the equatorial plane only admits the *s-wave* mode with $k = 0$. Inputting this value for k , the wave equation takes the following form [37],

$$\frac{\partial^2 f}{\partial t^2} = \frac{\Delta}{r^2 \Xi} \left[\frac{\partial}{\partial r} \left(\Delta \frac{\partial}{\partial r} \right) - m^2 r^2 \right] f \quad (44)$$

where $\Xi = r^2 + a^2 + \frac{2Ma^2}{r}$. This can be solved via separation of variables. By inspection we can see our reduced wave solution now becomes $\psi = e^{ik\phi} e^{iqt} R(r)$. Inputting the $T(t) = e^{iqt}$ solution for the temporal equation, the resultant radial equation is,

$$R'' + \frac{(2r - 2M)}{\Delta} R' + \frac{r^2 \Xi}{\Delta^2} (q^2 - m^2 r^2) R = 0 \quad (45)$$

We make the substitution $R(r) = \frac{u(x)}{g(r)}$ where $g(r)$ is solved such that first derivative terms in $u(x)$ sum to zero. To make this substitution the relationship between x and r must be known as well, and it can be found through the normalization integral for $3 + 1$ dimensional spacetime and using the line element from equation 41,

$$\int_{\Sigma_t} \sqrt{-g^{tt} g_3} R^* R d^3 \Sigma_t = \int dr d\phi d\theta R^* R \sin \theta \quad (46)$$

Because we are interested in the equatorial plane, we set $\theta = \frac{\pi}{2}$ and take the normalization over $2 + 1$ dimensional spacetime which yields,

$$\int dr d\phi R^* R \frac{r^2 \Xi}{\Delta} \sim \int \frac{r^2 \Xi dr u^* u}{\Delta g^2(r)} = \int dx u^* u$$

From this relationship we obtain an equation for $\frac{dx}{dr}$,

$$\frac{dx}{dr} = \frac{r^2 \Xi}{\Delta g^2(r)}$$

When we solve for $g(r)$ we find $g(r) \rightarrow (r^2 \Xi)^{1/4}$. When we make the full substitution back into our radial equation for $R(r)$, the resultant one-dimensional Schrödinger equation we find,

$$u'' + \left(q^2 + \frac{8(M - r)(2r^3 + a^2 r + Ma^2) + \Delta^2(a^2 - 9r^2)}{\Xi^2 r^4} + \frac{5\Delta^2(a^2 + 3r^2)^2}{\Xi^3 r^4} + \frac{5\Delta^2}{\Xi r^4} - m^2 r^2 \right) u = 0$$

	Black Hole Type				
Applicable Region	Negative Mass Schwarzschild	RN	RN AdS-dS	Higher dimensional RN AdS-dS	Kerr
Near $r = 0$	$\frac{-1}{4x^2}$	$\frac{-2}{9x^2}$	$\frac{-2}{9x^2}$	$\frac{-6}{25x^2}$	$\frac{-20}{9x^2}$
As $r \rightarrow \infty$	m^2	m^2	$\frac{5^{2/5}\Lambda(\Lambda-3m^2)\varepsilon^{4/5}}{18}x^{2/5}$	$\frac{\Lambda(2\Lambda-3m^2)\varepsilon^{4/3}}{9^{4/3}}x^{2/3}$	m^2x^2

Table 1: Effective potentials

This one dimensional Schrödinger equation takes the desired form of $\frac{d^2u}{dx^2} + (E - V)u = 0$, with V still in terms of the original radial coordinate r . Specifically, $V(r)$ is,

$$V(r) = - \left[\frac{8(M-r)(2r^3 + a^2r + Ma^2) + \Delta^2(a^2 - 9r^2)}{\Xi^2 r^4} + \frac{5\Delta^2(a^2 + 3r^2)^2}{\Xi^3 r^4} + \frac{5\Delta^2}{\Xi r^4} - m^2 r^2 \right] \quad (47)$$

From this point, it is easier to look near the ring singularity located at $r = 0$, and as r goes to infinity as well, and approximate the potential prior to making the substitution for x . As r goes to zero, $\Delta \rightarrow a^2$, $\Xi \rightarrow \frac{2Ma^2}{r}$, and the largest powers of r in the denominator will dominate the equation since those terms will tend towards infinity the fastest. The potential $V_0(r)$ thus becomes,

$$V_0(r) \approx -\frac{5a^2}{2Mr^3}$$

Now the relationship between x and r must be determined. Again approximating near zero, this relationship becomes,

$$x_0(r) = \int^r \frac{r^2 \Xi dr}{\Delta(g(r))^2} = \int^r \frac{r \sqrt{\Xi} dr}{\Delta} \approx \int^r \frac{\sqrt{r} 2M dr}{a} = \frac{2\sqrt{2M} r^{3/2}}{3a} \quad (48)$$

Using this to obtain $V_0(x)$,

$$V_0(x) \approx -\frac{20}{9x^2}$$

A plot of the effective potentials near zero for the negative mass Schwarzschild, RN AdS-dS, higher dimensional RN AdS-dS, and Kerr cases can be found in Figure 14. Likewise, a table showing the effective potentials near zero and near infinity can be found in Table 1.

With a relationship for the effective potential near zero $V_0(x)$, the LP-LC theorems of Reed and Simon [32] can now be applied to determine the essential self-adjointness of the spatial portion of the KG operator, consequently determining the quantum nature of the spacetime at the same time. Near $x = 0$, we will use the definition of LP-LC in order to characterize the potential. To do so, we must determine the square integrability of both solutions to the one dimensional Schrödinger equation formed by the new potential,

$$-\psi'' + V(x)\psi = \lambda\psi$$

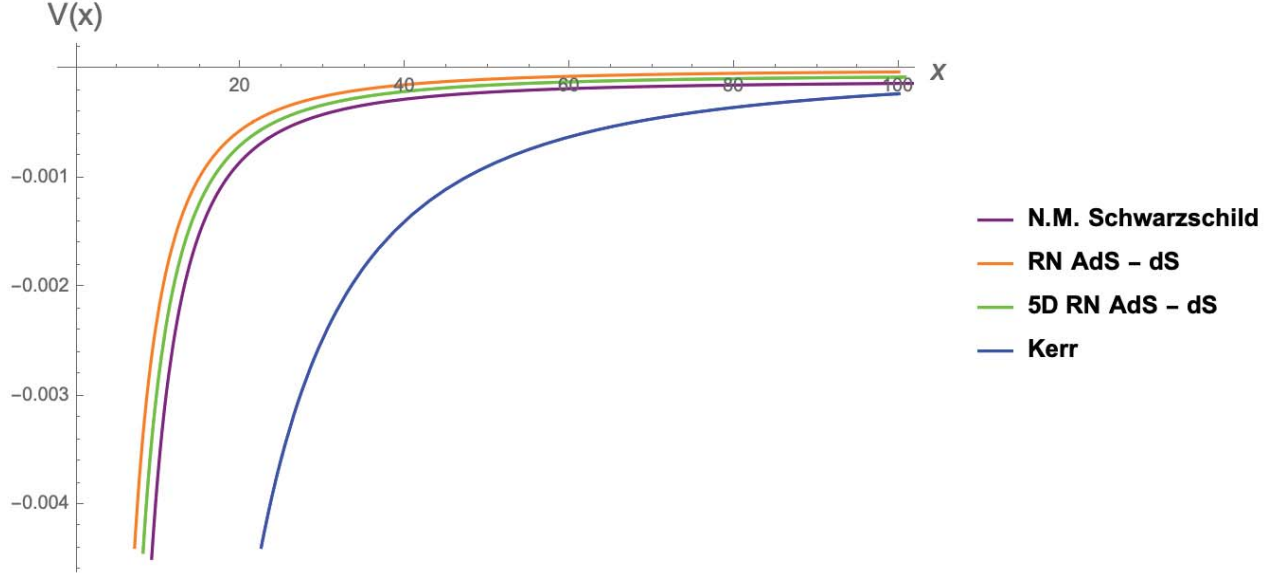


Figure 14: The effective potential near zero for the negative mass Schwarzschild, RN AdS-dS, and higher dimensional RN AdS-dS cases.

Since the definition applies for any possible eigenvalue (i.e. any possible value of λ) we select the easiest equation to analyze - that of $\lambda = 0$. Inserting $V_0(x)$ and zero for λ , we obtain,

$$-\psi''(x) - \frac{20}{9x^2}\psi(x) = 0 \quad (49)$$

This differential equation is an Euler equation [34], and therefore to solve it we assume a polynomial form to the solution. The solution and its first two derivatives become,

$$\psi = x^\gamma \quad (50)$$

$$\psi' = \gamma x^{\gamma-1} \quad (51)$$

$$\psi'' = \gamma(\gamma-1)x^{\gamma-2} \quad (52)$$

By inserting these equations into equation 49 and solving for γ we find $\gamma_{\pm} \rightarrow \frac{1}{2} \pm \frac{i\sqrt{76}}{6}$. The solutions are $\psi_+ = x^{\gamma_+}$ and $\psi_- = x^{\gamma_-}$. We now check the two solutions to determine if they are square integrable. The solutions behave as follows,

$$\begin{aligned} \int_0^{const.} x^{\gamma_+*} x^{\gamma_+} dx &= \int_0^{const.} x^{1/4} dx = \frac{4}{5} x^{5/4} \Big|_0^{const.} < \infty \\ \int_0^{const.} x^{\gamma_-*} x^{\gamma_-} dx &= \int_0^{const.} x^{1/4} dx = \frac{4}{5} x^{5/4} \Big|_0^{const.} < \infty \end{aligned}$$

therefore making both square integrable. Because both solutions are square integrable, the potential is in the limit-circle case near $x = 0$. Now the effective potential can be analyzed near infinity. As r goes to infinity, $\Delta \rightarrow r^2$ and $\Xi \rightarrow r^2$, and the term with the smallest power of r in the denominator will dominate

because it will tend towards zero the slowest. If equation 47 is taken in the limit as r approaches infinity, it becomes the following,

$$V_\infty(r) \approx m^2 r^2$$

The relationship between x and r must be determined again. Approximating equation 48 near infinity, this relationship becomes,

$$x_\infty(r) = \int^r \frac{r\sqrt{\Xi}dr}{\Delta} \approx \int^r dr = r$$

Therefore $V_\infty(x)$ is simply,

$$V_\infty(x) \approx m^2 x^2$$

With this effective potential, we will use theorems associated with LP-LC instead of the definition itself. Specifically, we will use Theorem X.8 of Reed and Simon [32]. The theorem says we must find an $M(x)$ which meets the following three criteria to have $V(x)$ in the limit point case near infinity,

- i $V(x) \geq -M(x)$
- ii $\int_1^\infty \left(\frac{1}{M(x)}\right)^{-1/2} dx = \infty$
- iii $\frac{M'(x)}{(M(x))^{3/2}}$ is bounded near ∞

By selecting $M(x) = kx^2$ we meet these three criteria

- i $V_\infty(x) \geq -kx^2$
- ii $\int_1^\infty (kx^2)^{-1/2} dx = k \ln x|_1^\infty = \infty$
- iii $\frac{(kx^2)'}{(kx^2)^{3/2}} = \frac{(2kx)}{(kx^3)} = -\frac{2}{x^2} < \infty$

As a result, the potential near infinity is in the limit-point case [32]. In all, we have proved that a Kerr black hole system is limit-circle near the ring singularity at $r = 0$ and limit-point near distances infinitely far away from the center of the system. Due to the system not being in the limit-point case at both zero and infinity, the spatial portion of the KG operator is not essentially self-adjoint; therefore, the system is quantum singular.

4 Deficiency Indices

The second method to characterize the quantum singular nature of these black hole systems is to use von Neumann deficiency indices. In essence, this method solves for the eigenfunctions associated with

complex eigenvalues of the spatial portion of the KG operator. As stated earlier, wave operators governing the behavior of quantum particles should always have real eigenvalues (which are related to measurable properties of the system); therefore, it makes sense that a wave operator with eigenfunctions populating a non-zero dimensional subspace with complex eigenvalues would be quantum singular. Technically speaking, this means solving,

$$(\hat{A} \pm i) \Phi = 0 \quad (53)$$

By determining the dimensions of the spaces spanned by the functions $\Phi_{\pm}$, we can determine whether our KG operator has self-adjoint extensions, unique self-adjoint extensions (or equivalently, essential self-adjointness), or no self-adjoint extensions (see [38] on more information about self-adjoint extensions). In other words, the deficiency index is how many of the two solutions are square integrable (i.e. in our Hilbert space $L^2(0, \infty)$), and by using the deficiency index for the positive and negative values of i we can make conclusions about self adjointness. These deficiency indices are often expressed as $\langle n_+, n_- \rangle$ where n_+ corresponds to the deficiency index for the $+i$ eigenvalue and n_- corresponds to the $-i$ eigenvalue. Specifically, the possibilities are: 1) the indices match (i.e. the dimensions of the spaces are equal), in which case self adjoint extensions exist 2) they are both zero, in which case the operator has a unique self-adjoint extension (it is essentially self adjoint) or 3) they do not match and the operator is neither essentially self adjoint nor equipped with self adjoint extensions.

4.1 Spherically Symmetric Black Hole Spacetime

For a generic spherically symmetric black hole system, we begin with equation 53 where,

$$\hat{A} = 2f(r)^3 f'(r) \frac{d}{dr} + f(r)^4 \left(\frac{2s'(r) \frac{d}{dr}}{s(r)} + \frac{d^2}{dr^2} \right) + \frac{f(r)^2 \left(\frac{d^2}{d\theta^2} + \cot(\theta) \frac{d}{d\theta} + \csc^2(\theta) \frac{d^2}{d\phi^2} \right)}{s(r)^2} - m^2 f(r)^2$$

Applying $\hat{A}$ to Φ we arrive at the same relationship as equation 6. Inserting that into 53, the resultant differential equation is

$$2ff'\Phi_r + f^2 \left(\frac{2s'\Phi_r}{s} + \Phi_{rr} \right) + \frac{\Phi_{\theta\theta} + \cot(\Theta)\Phi_{\theta} + \csc^2(\Theta)\Phi_{\phi\phi}}{s^2} - m^2 \pm \frac{i\Phi}{f^2} = 0$$

In this equation we can separate variables exactly the same as done for equation 6. The angular equation again results in the standard spherical harmonics for the 2-sphere $Y_{\ell}^{\tilde{m}}(\theta, \phi)$ and the temporal equation again yields a complex exponential. The resultant radial equation is,

$$R'' + 2 \left(\frac{f'}{f} + \frac{s'}{s} \right) R' - \left(\frac{\ell(\ell+1)}{f^2 s^2} + \frac{m^2}{f^2} \pm \frac{i}{f^4} \right) R = 0 \quad (54)$$

Unlike the LP-LC analysis where we were able to determine a generic form of the potential for any f^2 and s^2 , we must analyze each black hole system individually because we are looking for the direct solutions $R_{\pm}(r)$ to equation 54.

4.1.1 Negative Mass Schwarzschild Black Hole

When you insert equation 14 and 13 for $f(r)$ and $s(r)$ into equation 54 and take the limit as r approaches zero, you arrive at the following radial equation,

$$R'' + \frac{1}{r}R' - \left(\frac{\ell(\ell+1)}{2Mr}\right)R = 0$$

This radial equation matches the generic differential equation with Bessel function solutions with $a = 0$, $c = 1/2$, and $b = i\sqrt{\frac{2\ell(\ell+1)}{M}}$ (see [34] for details on the notation used here). The two solutions $R_1(r)$ and $R_2(r)$ written as a linear combination are then,

$$R(r) = C_1 I_0 \left(\sqrt{\frac{2\ell(\ell+1)r}{M}} \right) + C_2 K_0 \left(\sqrt{\frac{2\ell(\ell+1)r}{M}} \right)$$

A plot of the normalization argument for both solutions including various modes can be found in Figure 15. In order to analyze the solutions for square integrability near zero, we will use expansions for $I_0(x)$ and $K_0(x)$ as $x \rightarrow 0$. The solutions using these expansions become, [39],

$$R(r) \approx C_1 \left(1 + \frac{\ell(\ell+1)r}{2M} + \mathcal{O}(4) \right) + C_2 \left(\ln \sqrt{\frac{2\ell(\ell+1)r}{M}} \right)$$

Where we will ignore terms of order 4 or higher ($\mathcal{O}(4)$). To determine square integrability of the solutions, we must use the normalization requirement for $R(r)$ from the right hand side of equation 8.

$$4\pi \int_0^{const.} \frac{r^2 R^* R dr}{1 + \frac{2M}{r}} \approx 4\pi \int_0^C \frac{r^2}{\frac{2M}{r}} \left(C_1 + \frac{C_1 \ell(\ell+1)r}{2M} + C_2 \ln \sqrt{\frac{2\ell(\ell+1)r}{M}} \right)^2 dr$$

The expansion of the integrand involves calculating six different integrals. For simplicity, the six are evaluated separately below,

$$\begin{aligned} 1) \quad & \frac{C_1^2}{2M} \int_0^C r^3 dr = \frac{C_1^2 r^4}{8M} \Big|_0^C < \infty \\ 2) \quad & \frac{C_1^2 \ell(\ell+1)}{2M^2} \int_0^C r^4 dr = \frac{C_1^2 \ell(\ell+1) r^5}{10M^2} \Big|_0^C < \infty \\ 3) \quad & \frac{C_1^2 \ell^2(\ell+1)^2}{8M^3} \int_0^C r^5 dr = \frac{C_1^2 \ell^2(\ell+1)^2 r^6}{48M^3} \Big|_0^C < \infty \\ 4) \quad & \frac{C_1 C_2}{2M} \int_0^C r^3 \ln \sqrt{\frac{2\ell(\ell+1)r}{M}} dr = \left(\frac{-r^4}{32} + \frac{1}{4} r^4 \ln \sqrt{\frac{2\ell(\ell+1)r}{M}} \right) \Big|_0^C \\ & \approx \lim_{r \rightarrow 0} \left(\frac{-r^4}{32} + \frac{1}{4} r^4 \ln \sqrt{\frac{2\ell(\ell+1)r}{M}} \right) < \infty \end{aligned}$$

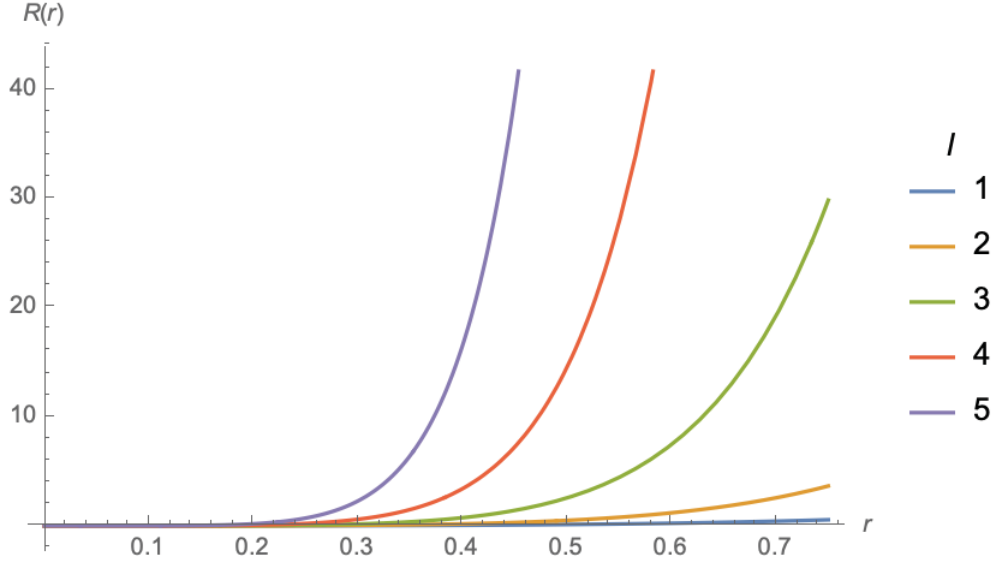
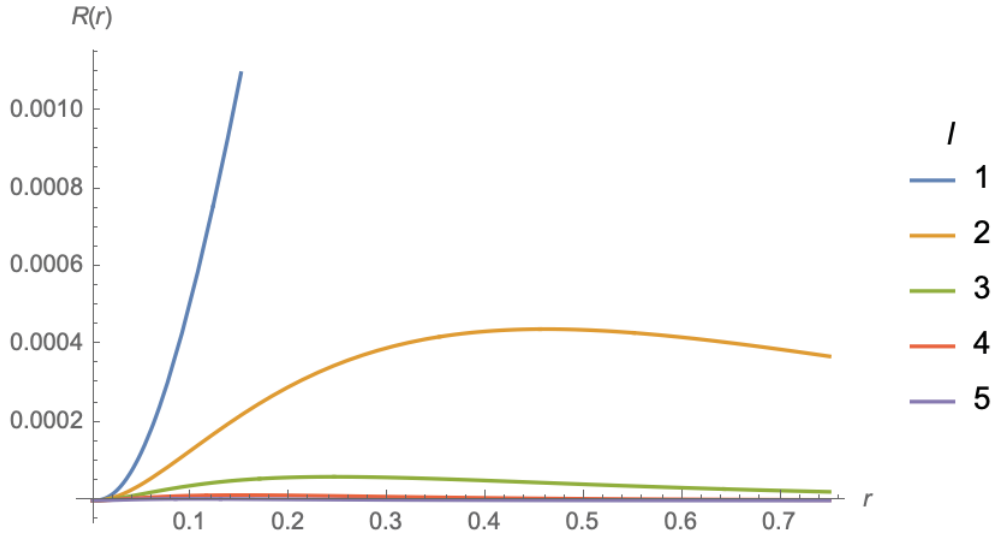
(a) Normalization argument for $R_1(r)$ (b) Normalization argument for $R_2(r)$

Figure 15: The normalization arguments for the two radial solutions for various modes as $r \rightarrow 0$ for the Negative Mass Schwarzschild case. In this plot, M is set to 1. Notice how both solutions are square integrable as r approaches zero.

$$\begin{aligned}
 5) \quad & \frac{C_1 C_2 \ell(\ell+1)}{2M^2} \int_0^C r^4 \ln \sqrt{\frac{2\ell(\ell+1)r}{M}} dr = \left(\frac{-r^5}{50} + \frac{1}{5} r^5 \ln \sqrt{\frac{2\ell(\ell+1)r}{M}} \right) \Big|_0^C \\
 & \approx \lim_{r \rightarrow 0} \left(\frac{-r^5}{50} + \frac{1}{5} r^5 \ln \sqrt{\frac{2\ell(\ell+1)r}{M}} \right) < \infty \\
 6) \quad & \frac{C_2^2}{2M} \int_0^C r^3 \left(\ln \sqrt{\frac{2\ell(\ell+1)r}{M}} \right)^2 dr = \left(\frac{-r^4}{128} - \frac{1}{16} r^4 \ln \sqrt{\frac{2\ell(\ell+1)r}{M}} + \frac{1}{4} r^4 \ln^2 \sqrt{\frac{2\ell(\ell+1)r}{M}} \right) \Big|_0^C
 \end{aligned}$$

$$\approx \lim_{r \rightarrow 0} \left(\frac{-r^4}{128} - \frac{1}{16} r^4 \ln \sqrt{\frac{2\ell(\ell+1)r}{M}} + \frac{1}{4} r^4 \ln^2 \sqrt{\frac{2\ell(\ell+1)r}{M}} \right) < \infty$$

Thus, at least near $r = 0$, we have found both of the solutions in $R(r)$ to be square integrable for both the $+i$ and $-i$ case eigenvalues, meaning they belong to our Hilbert Space. The follow on step is to obtain solutions near infinity. The radial equation near infinity takes the form,

$$R'' + \frac{2}{r}R' + (m^2 \pm i)R = 0 \quad (55)$$

The solutions to this equation can be written as a complex exponential sum,

$$R(r) = C_3 \frac{e^{-\kappa r}}{r} + C_4 \frac{e^{\kappa r}}{r}$$

where $\kappa = \sqrt{\pm i - m^2}$. A plot of the real and imaginary parts of this solution can be found in Figure 16(a) and Figure 16(b). In this case, the magnitude squared $|R|^2$ becomes,

$$R^*R = \frac{1}{r^2} (|C_3|^2 e^{-2\zeta r} + C_3^* C_4 e^{\pm 2\eta r} + C_3 C_4^* e^{\mp 2\eta r} + |C_4|^2 e^{2\zeta r}) \quad (56)$$

where $\zeta = Re[\kappa]$ and $\eta = Im[\kappa]$. Testing this solution with equation 8 near infinity we find,

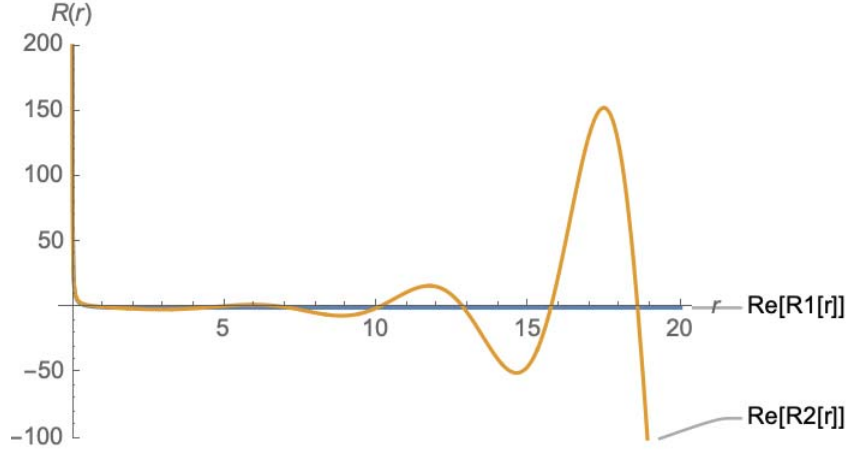
$$4\pi \int_C^\infty \frac{r^2 R^* R dr}{1 + \frac{2M}{r}} \approx \int_C^\infty (|C_3|^2 e^{-2\zeta r} + C_3^* C_4 e^{\pm 2\eta ir} + C_3 C_4^* e^{\mp 2\eta ir} + |C_4|^2 e^{2\zeta r}) dr$$

The two terms in the center of this integral can be combined into a single cosine term by writing C_3 and C_4 as $\tilde{r}_1 e^{i\theta_1}$ and $\tilde{r}_2 e^{i\theta_2}$ respectively, and then using the relationship between cosine and complex exponentials. However, this integral is finite only if $C_4 = 0$, thus eliminating the second solution. If this condition is met, then for both the $+i$ and $-i$ case there is one square integrable solution near infinity, and therefore in our Hilbert space.

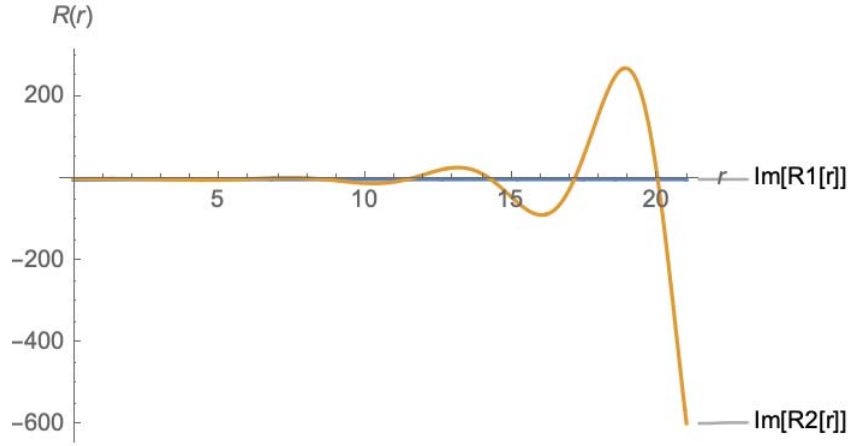
To summarize, as $r \rightarrow 0$ we calculated both solutions to be square integrable and as $r \rightarrow \infty$ we found only one solution to be square integrable. Recall that the dimension of our deficiency spaces, $n_\pm = \dim(\mathcal{N}_\pm)$, or the deficiency index, is determined by counting the number of linearly independent solutions which belong to our Hilbert Space, in this case $L^2(0, \infty)$. While we have two linearly independent, square integrable solutions near zero, the two solutions must converge to our lone solution in the asymptotic region near infinity (see [40] for a thorough explanation). As a result, $R(r)$ has one solution which belongs to our Hilbert Space and spans the entire domain of interest for both the $+i$ and $-i$ case. Hence, the deficiency indices are $\langle 1, 1 \rangle$. Because the deficiency indices are $\langle 1, 1 \rangle$, we arrive at the conclusion that the spatial portion of the KG operator is not essentially self-adjoint, but is equipped with an infinite number of self-adjoint extensions parameterized by $U(1)$, i.e. a phase $e^{i\theta}$ (see [38] for specific details). This agrees with the conclusions derived from Weyl's LP-LC theorems, and together they indicate that the spacetime is quantum mechanically singular because the evolution of the wavepacket is ambiguous.

4.1.2 Reissner Nordström Black Hole

When you insert equation 22 and 23 for $f(r)$ and $s(r)$ into equation 54 and take the limit as r approaches zero, you arrive at the following radial equation,



(a) Real part of each radial solution near infinity



(b) Imaginary part of each radial solution near infinity

Figure 16: The radial solutions as $r \rightarrow \infty$ for the Negative Mass Schwarzschild case. In (a) and (b) the first solution is denoted $R1[r]$ and the second solution is denoted $R2[r]$. This is also the solution near infinity for the RN and Kerr cases. In this plot, $m = 1$. Notice as r tends towards infinity, both the real and imaginary parts of the second solution, $R2[r] = C_4 \frac{e^{\kappa r}}{r}$, become more and more unstable resulting in the solution not being square integrable in the asymptotic region.

$$R'' + \frac{2M}{\varepsilon^2} R' - \left(\frac{\ell(\ell+1)}{\varepsilon^2} \right) R = 0 \quad (57)$$

The solutions to this differential equation written as a linear combination are,

$$R(r) = C_1 e^{\omega_1 r} + C_2 e^{\omega_2 r}$$

where,

$$\omega = \frac{-M}{\varepsilon^2} \pm \frac{\sqrt{M^2 + \ell(\ell+1)\varepsilon^2}}{\varepsilon^2}$$

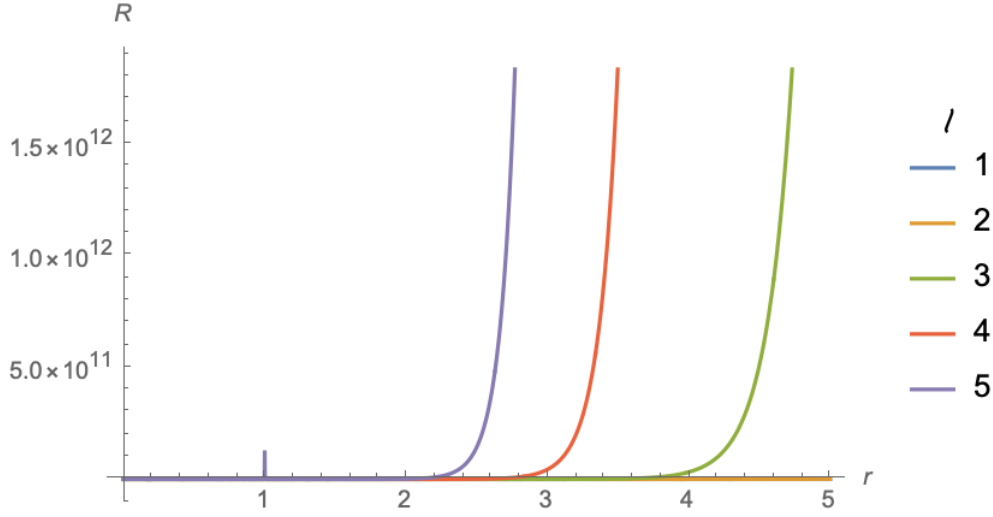
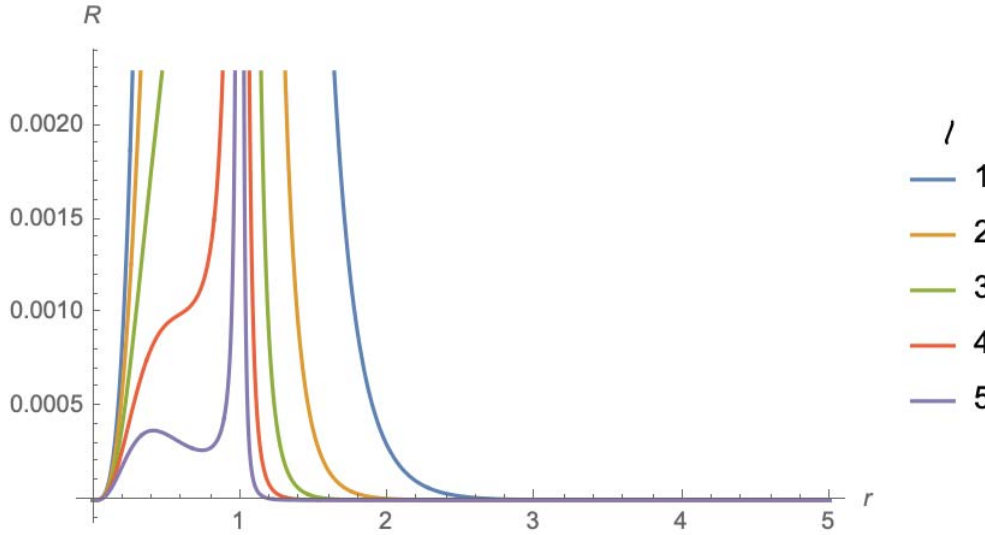
(a) Normalization argument for $R_1(r)$ (b) Normalization argument for $R_2(r)$

Figure 17: The normalization argument for both radial solutions including various modes as $r \rightarrow 0$ for the RN and RN AdS-dS cases. The Kerr black hole system's normalization argument also takes a similar form near zero, but without any modal dependence on ℓ . In this plot, M and ε are set to 1. In both (a) and (b) the plots demonstrate that the solutions are square integrable for very small values of r . The large values for the normalization argument in (b) near $r = 1$ represent the removable singularity which would be present in a normal RN system. This singularity shows itself in this plot because M and ε have both been set to 1 - the same number. Recall though that we are considering the overcharged case of the RN system, meaning for our case $\varepsilon^2 \gg M^2$, so this singularity in the plot can be ignored because the singularity is naked in this case.

A plot of the normalization argument for the two solutions individually can be found in Figure 17(a) and Figure 17(b). To determine square integrability of the solutions, we must use the normalization requirement for $R(r)$ from the right hand side of equation 8,

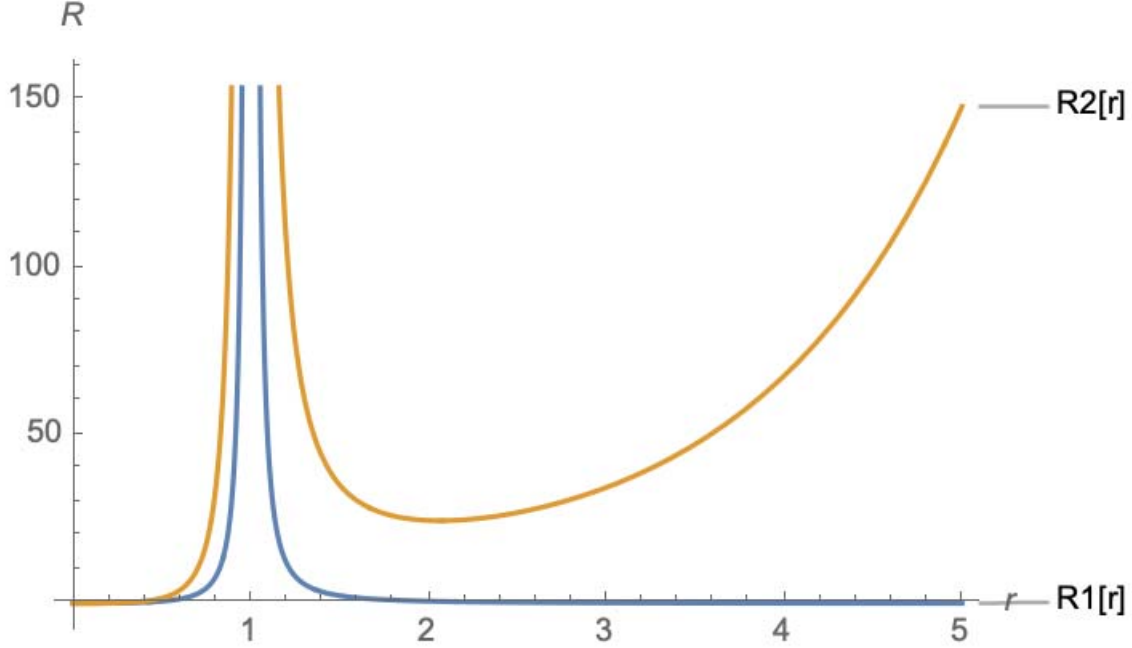


Figure 18: The normalization argument for the radial solutions of the RN case as $r \rightarrow \infty$. This is also the approximate form of the normalization argument for the Kerr black hole radial solutions in the asymptotic region with a and M set to 1. Again, the large values for the normalization argument near $r = 1$ represent the removable singularity which would be present in a normal RN. This singularity shows itself in this plot because M and ε have both been set to 1 - the same number. For the Kerr case, M and a are the variables set to 1. Recall though that we are considering the overcharged case of the RN system, meaning for our case $\varepsilon^2 \gg M^2$, and the overspinning case of the Kerr system, meaning $a^2 \gg M^2$, so this singularity in the plot can be ignored because the singularity is naked for both spacetimes.

$$4\pi \int_0^{const.} \frac{r^2 R^* R dr}{1 + \frac{2M}{r} + \frac{\varepsilon^2}{r^2}} \approx 4\pi \int_0^C \frac{r^4}{\varepsilon^2} (C_1 e^{\omega_1 r} + C_2 e^{\omega_2 r})^2 dr \quad (58)$$

$$= \int_0^C \frac{r^4}{\varepsilon^2} (C_1^2 e^{2\omega_1 r} + 2C_1 C_2 e^{(\omega_1 + \omega_2)r} + C_2^2 e^{2\omega_2 r}) dr \quad (59)$$

By inspection, at least near zero, we have found both of the solutions in $R(r)$ are square integrable for both the $+i$ and $-i$ case eigenvalues. The follow on step is to obtain solutions near infinity. The radial equation near infinity takes the form,

$$R'' + \frac{2}{r}R' + (m^2 \pm i)R = 0$$

This radial equation is identical to equation 63, yielding the same solutions. A plot of the normalization argument for the RN black hole case as r approaches infinity can be found in Figure 18 (as we will determine in a subsequent section, the normalization arguments for the Kerr black hole system in the asymptotic region take a similar form). In the asymptotic region near infinity, the normalization argument will be identical as well. Because of this, the conclusion for the square integrability of the radial solutions near infinity will be the same as the negative mass Schwarzschild black hole system. Specifically, this means that the radial solutions are square integrable if $C_4 = 0$, again eliminating the second solution near infinity. Because the deficiency

indices are $\langle 1, 1 \rangle$, we arrive at the identical conclusion that the spatial portion of the KG operator is not essentially self-adjoint, but is equipped with an infinite number of self-adjoint extensions parameterized by $U(1)$, i.e. a phase $e^{i\theta}$. This agrees with the conclusions derived from Weyl's LP-LC theorems, and together they indicate that the spacetime of the RN black hole system is quantum mechanically singular because the evolution of the wavepacket is ambiguous.

4.1.2.1 Anti-de Sitter and de Sitter Spacetime

When you insert equation 26 and 27 for $f(r)$ and $s(r)$ into equation 54 and take the limit as r approaches zero, you arrive at the following radial equation,

$$R'' + \frac{2M}{\varepsilon^2} R' - \left(\frac{\ell(\ell+1)}{\varepsilon^2} \right) R = 0$$

This radial equation is identical to equation 57 and yields the same solutions. Similarly, the normalization integral reduces to the same as the RN case as well near zero. As a result, the square integrability of the radial solutions near zero are determined to be the same as the RN black hole system (specifically, both solutions are square integrable). The follow on step is to obtain solutions near infinity. The radial equation near infinity takes the form,

$$R'' + \frac{4}{r} R' = 0$$

This radial equation yields the following solutions written as a linear combination,

$$R(r) = \frac{C_3}{r^3} + C_4$$

The normalization integral near infinity reduces to the following for these solutions,

$$4\pi \int_{const.}^{\infty} \frac{r^2 R^* R dr}{1 - \frac{2M}{r} + \frac{\varepsilon^2}{r^2} - \frac{\Lambda r^2}{3}} \approx 4\pi \int_{const.}^{\infty} \frac{-3}{\Lambda} \left(\frac{C_3^2}{r^6} + C_4^2 + \frac{C_3 C_4}{r^3} \right) dr$$

In this case, the integral is finite only if $C_4 = 0$, eliminating the second solution. For both the $+i$ and $-i$ case, one of the solutions near infinity is square integrable and therefore in our Hilbert space ($L^2(0, \infty)$), indicating the deficiency indices for the entire space are $\langle 1, 1 \rangle$. Thus, we arrive at the conclusion that the spatial portion of the KG operator is not essentially self-adjoint, but is equipped with an infinite number of self-adjoint extensions parameterized by $U(1)$. This agrees with the conclusions derived from Weyl's LP-LC theorems, and together they indicate that the spacetime of the RN AdS-dS black hole system is quantum mechanically singular because the evolution of the wavepacket is ambiguous.

4.2 Higher Dimensional Spacetimes

For the generic higher dimensional spacetime, the same process is used, but the operator $\hat{A}$ is different. The operator is the spatial portion of the KG operator for this system, which using equation 30 becomes,

$$\hat{A} = 2f(r)^3 f'(r) \frac{d}{dr} + f(r)^4 \left(\frac{3s'(r)}{s(r)} \frac{d}{dr} + \frac{d^2}{dr^2} \right) + \frac{f(r)^2}{s(r)^2 \sin^2 \theta} \left(\frac{d^2}{d\phi^2} + \cot(\phi) \frac{d}{d\phi} + \csc^2(\phi) \frac{d^2}{d\chi^2} \right) + \frac{2f(r)^2 \cot \theta}{s(r)^2} \frac{d}{d\theta} + \frac{1}{s(r)^2} \frac{d^2}{d\theta^2} - m^2 f(r)^2$$

With this new operator, we can begin specific deficiency index analysis for individual black hole systems.

4.2.1 RN AdS - dS Black Hole Spacetimes

Applying the new $\hat{A}$ to Φ , inserting the result into equation 53, and solving for the radial equation near zero yields the following relationship,

$$R'' - \frac{1}{r} R' + \frac{\ell(\ell+1)r^2}{\varepsilon^2} R = 0 \quad (60)$$

The solutions to this radial equation written as a linear combination take the form,

$$R(r) = C_1 \cos(\omega r^2) + C_2 \sin(\omega r^2)$$

where, $\omega = \frac{\sqrt{\ell(\ell+1)}}{2\varepsilon}$. A plot of the normalization argument for one of these solutions can be found in Figure 19,

To test square integrability, the normalization requirement for higher dimensional spacetime must be used, specifically equation 34. The integral in question near zero becomes,

$$2\pi^2 \int^r \frac{s^3}{f^2} R^* R dr \approx \int_0^C \frac{r^7}{\varepsilon^2} (C_1^2 \cos^2(\omega r^2) + C_2^2 \sin^2(\omega r^2) + 2C_1 C_2 \cos(\omega r^2) \sin(\omega r^2)) dr$$

When evaluated near zero, the integral above is finite. As a result, both of the solutions are square integrable and thus in our Hilbert Space. Analyzing the system near infinity, the radial equation takes the form,

$$R'' + \frac{5}{r} R' = 0 \quad (61)$$

This equation has the solutions,

$$R(r) = \frac{C_3}{r^4} + C_4$$

A plot of the normalization arguments for these solutions can be found in Figure 20 alongside the normalization argument of the RN AdS-dS radial solutions near infinity. Testing square integrability again using equation 34, this time near infinity,

$$2\pi^2 \int^r \frac{s^3}{f^2} R^* R dr \approx \int_C^\infty \left(\frac{-r}{\Lambda} \right) \left(\frac{C_3^2}{r^8} + C_4^2 + \frac{C_3 C_4}{r^4} \right) dr$$

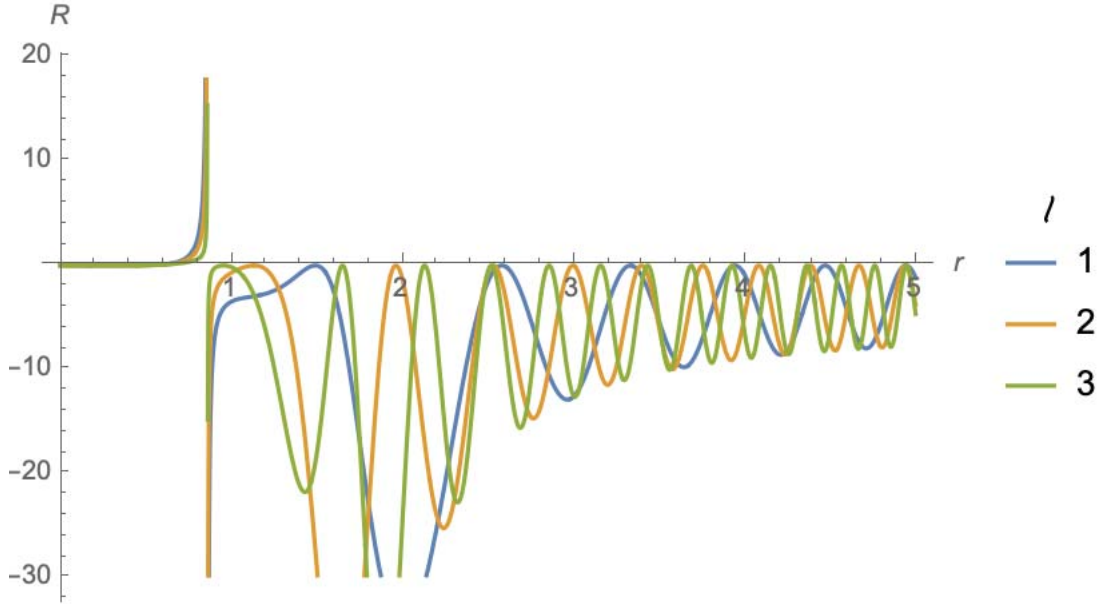


Figure 19: The normalization argument for one of the radial solutions including various modes near zero for the higher dimensional RN AdS-dS case. The second radial solution is not shown since it is a sine wave, making it a shifted version of the plot shown above. In this plot, $\varepsilon = 1$. Notice how both solutions will be square integrable for very small values of r . Also note that the large values for the normalization argument near $r = 1$ represent the removable singularity which would be present in a normal higher dimensional RN AdS-dS system. This singularity shows itself in this plot because M and ε have both been set to 1 - the same number. Recall though that we are considering the overcharged case of the higher dimensional RN AdS-dS system, meaning for our case $\varepsilon^2 \gg M^2$, so this singularity in the plot can be ignored because the singularity is naked in this case.

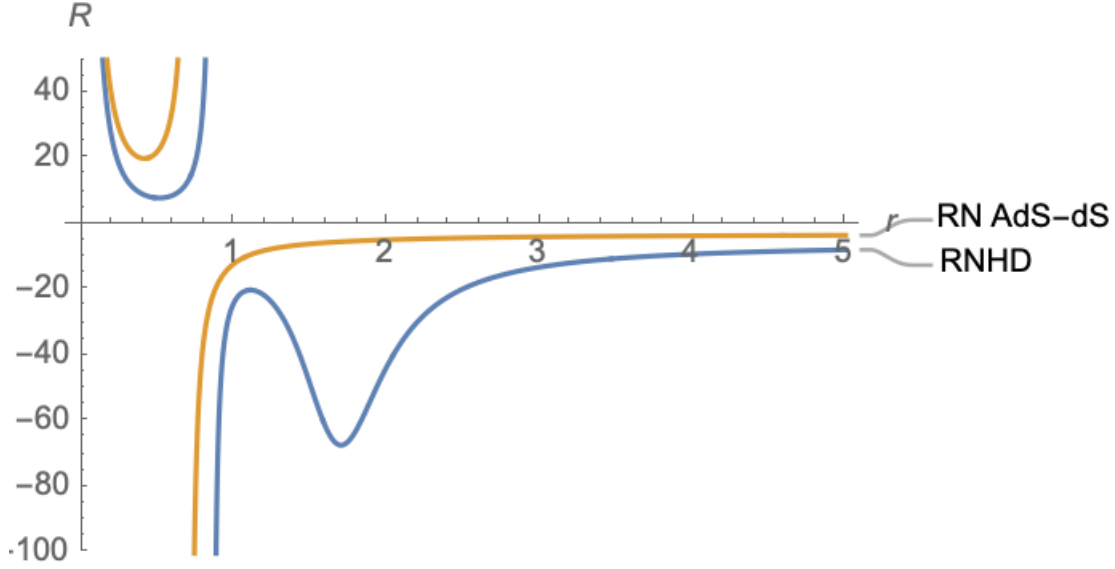


Figure 20: The normalization argument for the radial solutions of the higher dimensional RN AdS-dS and 3+1 dimensional RN AdS-dS cases as $r \rightarrow \infty$. There are several important aspects of this plot. First, note that the large values for the normalization argument near $r = 1$ represent the removable singularity which would be present in a normal higher dimensional RN AdS-dS or 3+1 dimensional RN AdS-dS system. This singularity shows itself in this plot because M and ε have both been set to 1 - the same number. Recall though that we are considering the overcharged case of the higher dimensional RN AdS-dS and 3+1 dimensional RN AdS-dS system, meaning for our case $\varepsilon^2 \gg M^2$, so this singularity in the plot can be ignored because the singularity is naked in this case. Second, notice that although both systems seem to be square integrable in the asymptotic region (as r becomes extremely large), the integral is in fact divergent, similar to how the integral of $\frac{1}{x}$ is not square integrable as x goes to infinity.

This integral is only finite near infinity if C_4 is set to zero, thus throwing out the second solution. However, if this is the case then for both the $+i$ and $-i$ case, there is one square integrable solution and thus our deficiency indices are again $(1, 1)$. The operator is not essentially self-adjoint and the spacetime is categorized as quantum mechanically singular.

4.3 Axially Symmetric Black Hole Spacetimes

4.3.1 Cosmic Strings

Again we will use equation 53, but this time the operator becomes,

$$\hat{A} = 2f(r)^3 f'(r) \frac{d}{dr} + f(r)^4 \left(\frac{2s'(r) \frac{d}{dr}}{s(r)} + \frac{d^2}{dr^2} \right) + \frac{f(r)^2 \left(\alpha^2 \left(\frac{d^2}{d\theta^2} + \cot(\theta) \frac{d}{d\theta} \right) + \csc^2(\theta) \frac{d^2}{d\phi^2} \right)}{\alpha^2 s(r)^2} - m^2 f(r)^2$$

and α representing the cosmic string parameter. From inspection, we see the only equation affected by the α parameter is the angular equation. This results in the spherical harmonics for the cosmic string detailed in the previous section (see Figure 13). In the radial equation given by equation 54 the only change is to

again swap out the integer ℓ for $\tilde{\lambda}$. We can now specify the individual black hole system by inserting f^2 and s^2 .

4.3.1.1 RN and RN AdS-dS Black Hole Spacetimes

In the deficiency index analysis for the RN and RN AdS-dS cases penetrated by a cosmic string, the change from ℓ to $\tilde{\lambda}$ does not affect the final calculations of square integrability because the normalization argument is unchanged (or changed by only the constant α which has no influence on square integrability). Thus, for these two cases the deficiency indices are confirmed to be $\langle 1, 1 \rangle$ in correspondence with the LP-LC conclusion that the operator in question is not essentially self-adjoint and the spacetime is quantum singular.

4.3.1.2 Higher Dimensional RN AdS - dS Black Hole Spacetimes

For the higher dimensional spacetimes penetrated by a cosmic string, we can utilize a similar argument. Using equation 53 where,

$$\hat{A} = 2f(r)^3 f'(r) \frac{d}{dr} + f(r)^4 \left(\frac{3s'(r)}{s(r)} \frac{d}{dr} + \frac{d^2}{dr^2} \right) + \frac{f(r)^2}{\alpha^2 s(r)^2 \sin^2 \theta} \left(\alpha \left(\frac{d^2}{d\phi^2} + \cot(\phi) \frac{d}{d\phi} \right) + \csc^2(\phi) \frac{d^2}{d\chi^2} \right) + \frac{2f(r)^2 \cot \theta}{s(r)^2} \frac{d}{d\theta} + \frac{1}{s(r)^2} \frac{d^2}{d\theta^2} - m^2 f(r)^2$$

Using this operator, the only equation affected by the α parameter is the angular equation. This results in the higher dimensional spherical harmonics for the cosmic string detailed in the previous section. In the radial equations given by equation 60 and equation 61 the only change is to again swap out the integer ℓ for λ . In the deficiency index analysis the change from ℓ to λ does not affect the final calculations of square integrability. Thus, the deficiency indices are confirmed to be $\langle 1, 1 \rangle$ in correspondence with the LP-LC conclusion that the operator in question is not essentially self-adjoint and the spacetime is quantum singular.

4.3.2 Kerr Black Holes

In essence, let us recall that this method solves for the eigenfunctions associated with complex eigenvalues of the spatial portion of the KG operator. Technically speaking, this means solving,

$$(\hat{A} \pm i) \Psi = 0 \tag{62}$$

where, $\hat{A}$ is the spatial portion of the KG operator. In other words, we determine $\hat{A}$ by separating the temporal and spatial parts of the operator so that the wave equation appears as,

$$\frac{\partial^2 \Phi}{\partial t^2} = -\hat{A} \Phi$$

For the Kerr black hole, we are considering the *s-wave* (i.e. $k = 0$ mode) in the equatorial plane (i.e. $\theta = \frac{\pi}{2}$), so the spatial part of the KG operator can be read off of equation 44 as,

$$\hat{A} = -\frac{\Delta}{r^2\Xi} \left(\frac{\partial}{\partial r} \left(\Delta \frac{\partial}{\partial r} \right) - m^2 r^2 \right)$$

Applying $\hat{A}$ to Ψ you arrive at the same equation as equation 45. Taking that and inserting that into 62, the resultant differential equation is,

$$R'' + \frac{(2r - 2M)}{\Delta} R' - \left(\frac{m^2 r^2}{\Delta} \pm \frac{ir^2\Xi}{\Delta^2} \right) R = 0$$

When you take the limit as r approaches zero for this radial equation, you arrive at the following differential equation,

$$R'' - \frac{2M}{a^2} R' = 0$$

The solution to this differential equation is,

$$R(r) = C_1 + C_2 e^{\omega r}$$

where $\omega = \frac{2M}{a^2}$. To determine square integrability of the solutions, we must use the normalization requirement for $R(r)$ from the left hand side of equation 46.

$$\sim \int_0^{const.} \frac{r^2 \Xi R^* R dr}{\Delta} \approx \int_0^C r (C_1^2 + 2C_1 C_2 e^{\omega r} + C_2^2 e^{2\omega r}) dr = \frac{C_1^2 r^2}{2} + \frac{2C_1 C_2 e^{r\omega} (r\omega - 1)}{\omega^2} + \frac{C_2^2 e^{2\omega r} (2\omega r - 1)}{4\omega^2} \Big|_0^{const.}$$

Thus, at least near $r = 0$, our solution is square integrable for both the $+i$ and $-i$ case eigenvalues. The follow on step is to obtain solutions near infinity. The radial equation near infinity takes the form,

$$R'' + \frac{2}{r} R' + (m^2 \pm i) R = 0 \tag{63}$$

The solutions to this equation can be written as a complex exponential sum,

$$R(r) = C_1 \frac{e^{-\kappa r}}{r} + C_2 \frac{e^{\kappa r}}{r}$$

where $\kappa = \sqrt{\pm i - m^2}$. In this case, $|R|^2$ takes the same form as equation 56. Testing this solution with equation 8 near infinity we find,

$$4\pi \int_C^\infty \frac{r^2 \Xi R^* R dr}{\Delta} \approx \int_C^\infty (|C_3|^2 e^{-2\zeta r} + C_3^* C_4 e^{\pm 2\eta i r} + C_3 C_4^* e^{\mp 2\eta i r} + |C_4|^2 e^{2\zeta r}) dr$$

The two terms in the center of this integral can be combined into a single cosine term by writing C_3 and C_4 as $\tilde{r}_1 e^{i\theta_1}$ and $\tilde{r}_2 e^{i\theta_2}$ respectively, and then using the relationship between cosine and complex exponentials. This integral is fine only if either $C_4 = 0$. If this condition is met, then for both the $+i$ and $-i$ case the solution is square integrable and therefore in our Hilbert space. As a result, the deficiency indices are $\langle 1, 1 \rangle$.

Because the deficiency indices are $\langle 1, 1 \rangle$, we arrive at the conclusion that the spatial portion of the KG operator is not essentially self-adjoint for the Kerr black hole, but is equipped with an infinite number of self-adjoint extensions parameterized by $U(1)$. This agrees with the conclusions derived from Weyl's LP-LC theorems, and together they indicate that the spacetime is quantum mechanically singular because the evolution of the wavepacket is ambiguous.

5 Conclusion

Using LP-LC theorems (see the Appendix in Section G), all systems were found to be limit circle near the scalar curvature singularity at $x = 0$ and limit point as x goes to infinity. Through review of Weyl's Theorem, our Hamiltonian is only essentially self adjoint if $V(x)$ is limit point at both zero and infinity. This indicates that for spherically symmetric black hole systems including negative mass Schwarzschild systems, RN or RN AdS-dS systems, and higher dimensional RN AdS-dS systems, as well as all of these systems with a cosmic string penetrating through the center, the Hamiltonians are not essentially self adjoint, meaning they are quantum mechanically singular [32] when probed by spin-0 massive particles. Similarly, Kerr black hole systems, which are not spherically symmetric or static due to rotation, have also been shown to have a Hamiltonian which is not essentially self adjoint when a minimally coupled, massive KG wave operator was applied. Hence, it is quantum singular as well. However, within the systems already analyzed there exist points of interest which were not inspected as well as black hole systems which were not discussed. Future analysis through LP-LC methods could be used in these regions to determine the behavior of quantum particles. These regions include the horizons of the systems, especially the Cauchy horizons, which recently have been proven to be unstable, a phenomenon known as a kink singularity [41].

Furthermore, our results were verified through analysis of the deficiency spaces for a wave packet described by a minimally coupled, massive scalar KG equation. For all systems named above, the deficiency indices were calculated to be $n_+ = n_- = 1$. Deficiency indices of $\langle 1, 1 \rangle$ conclude the spatial portion of the KG operator is not essentially self-adjoint, but is equipped with a family of self-adjoint extensions parameterized by $U(1)$, where $U(1)$ is a unitary group [42] (more information on self-adjoint extensions can be found in [38] which provides an extensive overview). This agrees with the conclusions derived from Weyl's LP-LC theorems, and together they indicate that the spacetime is quantum mechanically singular because the evolution of the wavepacket is ambiguous.

6 References

- [1] Gary T. Horowitz and Donald Marolf. Quantum probes of spacetime singularities. *Physical Review D*, 52(10):5670, 1995.
- [2] Ray D'Inverno. *Introducing Einstein's Relativity*. Clarendon Press, 1992.
- [3] Available at <https://www.askamathematician.com/2011/01/q-why-does-lorentz-contraction-only-act-in-the-direction-of-motion/>, [Online; accessed 6-March-2019].
- [4] P.G. Hewitt. *Conceptual Physics*. Addison Wesley, 2002.
- [5] Albert Einstein. *Relativity: The Special and General Theory*. Read Books, 2009.
- [6] Available at http://www.uh.edu/~jclarage/astr3131/lectures/4/einstein/Einstein_stanford_Page7.html, [Online; accessed 6-March-2019].
- [7] Ru-Sen Lu, Avery E Broderick, Fabien Baron, John D Monnier, Vincent L Fish, Sheperd S Doeleman, and Victor Pankratius. Imaging the supermassive black hole shadow and jet base of m87 with the event horizon telescope. *The Astrophysical Journal*, 788(2):120, 2014.

- [8] ESA/Hubble and NASA. Astronomy picture of the day, 2011. Available at http://apod.nasa.gov/apod/image/1112/lensshoe_hubble_3235.jpg, [Online; accessed 6-March-2019].
- [9] Redshift live. Available at https://www.redshift-live.com/ext/en/magazine/articles/Astronomy/15450-Bending_the_light-1.html, [Online; accessed 6-March-2019].
- [10] NASA. Available at <https://www.bmcc.cuny.edu/news/news.jsp?id=11433>, [Online; accessed 6-March-2019].
- [11] James Provost. Available at <https://scienceblogs.com/startswithabang/2015/06/30/astroquizzical-how-does-gravity-escape-from-a-black-hole-synopsis>, [Online; accessed 31-March-2019].
- [12] R. Penrose. Gravitational collapse: The role of general relativity. *Riv. Nuovo Cim.*, 1:252–276, 1969. [Gen. Rel. Grav.34,1141(2002)].
- [13] Andrew Hamilton. More about the schwarzschild geometry. <https://jila.colorado.edu/~ajsh/bh/schwp.html>, 2006. [Online; accessed 30-March-2019].
- [14] Some further notes on black holes. Available at <https://galileospendulum.org/2011/09/02/some-further-notes-on-black-holes/>, [Online; accessed 31-March-2019].
- [15] N. Rumiano. Schwarzschild black hole. http://www.daviddarling.info/encyclopedia/S/Schwarzschild_black_hole.html. [Online; accessed 30-March-2019].
- [16] Andrew Hamilton. Charged black holes: The reissner-nordström geometry. <https://jila.colorado.edu/~ajsh/bh/rn.html>, 2001. [Online; accessed 30-March-2019].
- [17] N. Rumiano. Reissner-nordstrom black hole. http://www.daviddarling.info/encyclopedia/R/Reissner-Nordstrom_black_hole.html. [Online; accessed 30-March-2019].
- [18] Martin Silvertant. Are black hole sone-dimensional dots? Available at <https://www.quora.com/Are-black-holes-one-dimensional-dots>, [Online; accessed 31-March-2019].
- [19] N. Rumiano. Kerr black hole. http://www.daviddarling.info/encyclopedia/K/Kerr_black_hole.html. [Online; accessed 30-March-2019].
- [20] A. Vilenkin and E.P.S. Shellard. *Cosmic Strings and Other Topological Defects*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2000.
- [21] Marcia Bartusiak. If you like black holes you’ll love cosmic strings. *Discover*, 9:60, April 1988.
- [22] Deborah Konkowski and Thomas Helliwell. Quantum singularity of quasiregular spacetimes. *General Relativity and Gravitation*, 33(6):1131–1136, 2001.
- [23] Carlos Martins and Andrew Sornborger. Cosmic strings and large scale structure. http://www.damtp.cam.ac.uk/research/gr/public/cs_1ss.html, 1996. [Online; accessed 20-January-2019].
- [24] Thibault Damour and Alexander Vilenkin. Gravitational wave bursts from cosmic strings. *Physical Review Letters*, 85(18):3761, 2000.
- [25] Marcus Chown. Is it cosmic strings we’re seeing? *New Scientist*, 9:60, July 2005.
- [26] Makoto Natsuume. The singularity problem in string theory. *arXiv preprint gr-qc/0108059*, 2001.
- [27] George F.R. Ellis and Bernd G. Schmidt. Singular space-times. *General Relativity and Gravitation*, 8(11):915–953, 1977.
- [28] Stephen W. Hawking and George F.R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1973.

- [29] James Schombert. Quantum wave function. Available at <https://scienceblog.com/54718/mathematicians-can-conjure-matter-waves-inside-an-invisible-hat/>, [Online; accessed 31-March-2019].
- [30] Joshua Halpern, Scott Sinex, and Scott Johnson. 6.6:3d representation of orbitals, 2018. Available at <https://scienceblog.com/54718/mathematicians-can-conjure-matter-waves-inside-an-invisible-hat/>, [Online; accessed 31-March-2019].
- [31] Mathematicians can conjure matter waves inside an invisible hat. Available at <https://scienceblog.com/54718/mathematicians-can-conjure-matter-waves-inside-an-invisible-hat/>, [Online; accessed 31-March-2019].
- [32] Michael Reed and Barry Simon. *Fourier Analysis, Self-adjointness*. Methods of modern mathematical physics. Academic Press, 1975.
- [33] Thomas M. Helliwell and Deborah A. Konkowski. Quantum singularities in spherically symmetric, conformally static spacetimes. *Physical Review D*, 87(10):104041, 2013.
- [34] M.L. Boas. *Mathematical Methods in the Physical Sciences*. Wiley, 2006.
- [35] Lee Lindblom, Nicholas W Taylor, and Fan Zhang. Scalar, vector and tensor harmonics on the three-sphere. *General Relativity and Gravitation*, 49(11):139, 2017.
- [36] Adrian C Ottewill and Peter Taylor. Vacuum polarization on the schwarzschild metric threaded by a cosmic string. *Physical Review D*, 82(10):104013, 2010.
- [37] O Gurtug and M Halilsoy. Quantum particle probe of the kerr naked singularity. *The European Physical Journal Plus*, 132(1):41, 2017.
- [38] Guy Bonneau, Jacques Faraut, and Galliano Valent. Self-adjoint extensions of operators and the teaching of quantum mechanics. *American Journal of physics*, 69(3):322–331, 2001.
- [39] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, ninth dover printing, tenth gpo printing edition, 1964.
- [40] Akihiro Ishibashi and Akio Hosoya. Who’s afraid of naked singularities? probing timelike singularities with finite energy waves. *Physical Review D*, 60(10):104028, 1999.
- [41] Hideki Maeda, Takashi Torii, and Tomohiro Harada. Novel cauchy-horizon instability. *Physical Review D*, 71(6):064015, 2005.
- [42] D M Gitman, I V Tyutin, and B L Voronov. Self-adjoint extensions and spectral analysis in the calogero problem. *Journal of Physics A: Mathematical and Theoretical*, 43(14):145205, 2010.
- [43] Henry Martyn Taylor. *Euclid’s Elements of Geometry*. Number bks. 1-6 in Pitt press mathematical series. The University Press, 1893.
- [44] Wikimedia Commons, 2007. Available at https://en.wikipedia.org/wiki/Light_cone#/media/File:World_line.svg, [Online; accessed 31-March-2019].
- [45] Andrew Hamilton. Penrose diagram of a schwarzschild black hole. <https://jila.colorado.edu/~ajsh/insidebh/penrose.html>. [Online; accessed 31-March-2019].
- [46] Penrose diagrams. Available at <http://physics.oregonstate.edu/coursewikis/GGR/book/ggr/penrose>, [Online; accessed 31-March-2019].

Acronyms

AdS anti de Sitter. 3, 5, 12, 20, 21, 27, 28, 30, 31, 34, 37, 44, 47, 48, 51, 53, 58, 67

dS de Sitter. 3, 5, 12, 20, 21, 27, 28, 30, 31, 34, 37, 44, 47, 48, 51, 53, 58, 67

KG Klein-Gordon. 5, 18, 21, 23, 26, 29, 31, 34, 35, 37, 39, 40, 43, 46, 47, 51, 52, 53

LP-LC limit-point, limit-circle. 5, 20, 21, 23, 25, 26, 27, 31, 34, 37, 39, 40, 43, 46, 47, 51, 52, 53

RN Reissner-Nordström. 3, 5, 12, 13, 20, 21, 26, 27, 28, 30, 31, 33, 34, 37, 43, 44, 46, 47, 48, 51, 53, 58, 67, 68, 69, 70

Appendix A Line Elements

In Euclid’s *Elements*, he described five postulates which defined flat geometry [43]. Thus, to describe the distance between two points in a flat geometry, we will use what is called a Euclidean line element ds . To do so, we will begin with the Pythagorean theorem. In three-dimensions, the distance Δs between two points can be defined using the Pythagorean theorem:

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Although two viewers may measure the x , y , and z position of the two points using different coordinate systems, the distance between the points will be the same for both (assuming that they are measuring using the same units). The distance is “invariant”. The line element is a more generalized version of the distance formula, but it also does not change regardless of the coordinate system used, making it invariant also. The Euclidean line element is defined as follows:

$$ds^2 = dx^2 + dy^2 + dz^2$$

which is identical to the distance formula given from the Pythagorean theorem except with differentials instead of differences. Note that the terms line element and metric are often used interchangeably, but the metric technically refers to the coefficients of the line element - see Appendix C for more information on metrics. Furthermore, as indicated earlier this line element can be rewritten in terms of different coordinates which equivalently define the same space. For example, in spherical coordinates the line element becomes the following:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (64)$$

These line elements are employed in classical Newtonian mechanics which relies on the notion that time is also an invariant coordinate. However, when Albert Einstein introduced his theory of special relativity in 1905, he relied upon the notion that the speed of light is invariant and time is no longer invariant [5]. Since time is no longer invariant (it can change) during a change in coordinate systems, it was introduced as a new coordinate into the line element. The line element is still used to define distance, but now the coordinate system is termed a spacetime. Due to special relativity, as will be discussed in Appendix D, the distance between two points is no longer the same if measured by two different observers when one of

the observers is moving, because of Lorentz contraction. The situation is even more complicated if the two points are separated in time as well as in space. Thus, a different measure must be used to measure the effective "distance" between two events.

In four-dimensional spacetime, the analog to distance is the "spacetime interval". Although time comes in as a fourth dimension, it is treated differently than the spatial dimensions. Minkowski space hence differs in important respects from four-dimensional Euclidean space. The fundamental reason for merging space and time into spacetime is that space and time are separately not invariant, which is to say that, under the proper conditions, different observers will disagree on the length of time between two events (because of time dilation) or the distance between the two events (because of length contraction). But special relativity provides a new invariant, called the "spacetime interval", which combines distances in space and in time. All observers who measure time and distance carefully will find the same spacetime interval between any two events. Suppose an observer measures two events as being separated in time by Δt and a spatial distance Δx . Then the spacetime interval $(\Delta s)^2$ between the two events that are separated by a distance Δx in space and by $\Delta ct = c\Delta t$ in the ct -coordinate is:

$$(\Delta s)^2 = (\Delta ct)^2 - (\Delta x)^2$$

or for three space dimensions,

$$(\Delta s)^2 = (\Delta ct)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \quad (65)$$

where the constant c , the speed of light, converts the units used to measure time (seconds) into units used to measure distance (meters). The equation above is similar to the Pythagorean theorem, except with a minus sign between the Δct^2 and the Δx^2 terms. The reason is that unlike distances in Euclidean geometry, intervals in Minkowski spacetime can be negative or zero. If Δs^2 , or ds^2 for the line element, is positive, the spacetime interval is timelike, meaning that two events are separated by more time than space. If ds^2 is negative, the spacetime interval is spacelike, meaning that two events are separated by more space than time. Spacetime intervals are zero when $x = \pm ct$. In other words, the spacetime interval between two events for an object moving at the speed of light is zero. Such an interval is termed lightlike or null.

A few additional definitions of terms which arise in this investigation include what it means for a line element (or, equivalently, a metric) to be static, stationary, spherically symmetric, or axially symmetric. A static spacetime is invariant for time reversal (i.e. does not change when $t \rightarrow -t$) while a stationary spacetime does not explicitly depend on time. A spherically symmetric metric is invariant under rotation (i.e. there is no "special" choice of θ or ϕ coordinate for the system) while an axially symmetric metric is invariant under any azimuthal rotation (in other words, it does not depend explicitly on ϕ). For spacetime, the simplest version of the line element is called the Minkowski line element, written as,

$$ds^2 = \pm (dt^2 - dx^2 - dy^2 - dz^2) \quad (66)$$

where the sign choice of the ± 1 in front is arbitrary. This is the same as equation 65 with the differences exchanged for differentials. Often times the four-dimensional coordinates of choice are (ct, x, y, z) instead of (t, x, y, z) because this is a more "realistic" way of thinking about spacetime since all the coordinates now have units of length instead of three with units of length and one with a unit of time. In these coordinates, the line element can be equivalently written as,

$$ds^2 = \pm (c^2 dt^2 - dx^2 - dy^2 - dz^2) \quad (67)$$

However, in special and general relativity the value of c is often set to equal 1 (this is called using

relativistic units), so the constant in front of dt^2 disappears. In spherical coordinates the Minkowski line element behaves as expected, written as,

$$ds^2 = \pm (c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2) \quad (68)$$

where,

$$0 \leq r \leq \infty \quad 0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi$$

In this project, we applied Einstein's theory of general relativity to our spacetime, resulting in more complex line elements. These line elements are dependent upon the spacetime within which we are operating, which in turn depends on the type of celestial object we are looking to examine. The first and simplest of these situations will be applying general relativity to a negative mass Schwarzschild black hole, a well studied scenario which has a line element as follows:

$$ds^2 = - \left(1 + \frac{2M}{r} \right) dt^2 + \frac{dr^2}{\left(1 + \frac{2M}{r} \right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where,

$$0 < r \leq \infty \quad 0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi$$

and M is the mass of the black hole. If a cosmic string were to pierce the black hole, then a parameter α would need to be added into the azimuthal term, written formally as,

$$ds^2 = - \left(1 + \frac{2M}{r} \right) dt^2 + \frac{dr^2}{\left(1 + \frac{2M}{r} \right)} + r^2 (d\theta^2 + \alpha^2 \sin^2 \theta d\phi^2)$$

where,

$$0 \leq r \leq \infty \quad 0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi\alpha$$

More specifically, the parameter $\alpha = 1 - 4\mu$ and stems from the development of Grand Unification Theory, where μ is the linear mass density of the cosmic string. It is simple to see with this relationship how removal of the cosmic string by setting $\mu = 0$ allows for $\alpha = 1$ and the Schwarzschild line element to appear again.

More complicated line elements will be examined when more complicated celestial systems are introduced. These systems include the Reissner-Nordström (RN) charged black holes, the Kerr rotating black hole, and possibly the RN situation in de Sitter (dS) space [indicating a spherical cosmological geometry] and anti de Sitter (AdS) [indicating a hyperbolic cosmological geometry]. Additionally, the presence of a penetrating cosmic string will be examined in the basic RN case and the higher dimensional RN case because of the importance of higher dimensional celestial systems to string theory. A cosmic string will also be included for the AdS and dS cases; however, it will not be included in the analysis of the Kerr black hole. As a preview, here is an example of the RN line element for a charged black hole without a cosmic string,

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{\varepsilon^2}{r^2} \right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{\varepsilon^2}{r^2} \right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where ε is the charge of the black hole which, when equal to zero, recovers the Schwarzschild line element.

Appendix B Geodesics

A geodesic is the path that represents the extremal distance between two points. For simple geometries, mainly those which are non-relativistic, the extremal distance represents the shortest path between the two points. In the case of Euclidean space (flat space), the geodesic represents a straight line, termed the Euclidean geodesic. In the case of constrained surfaces and non-Euclidean space, the geodesic still represents the shortest path, but is no longer linear. For example, for a particle constrained to a spherical non-relativistic geometry, a two-dimensional spherical shell (one type of non-Euclidean geometry), the geodesic is not a line but a curve termed a great circle. These great circles are called Riemannian geodesics [2]. Considering how line elements define distance for a given spacetime, they play a vital role in determining geodesics for that specific space. For instance, in the line element of relativistic coordinate systems such as from equation 67 or 68, the term containing the time coordinate is negative while other terms containing spatial coordinates are positive. Thus, in relativistic spacetime, the geodesic no longer represents the shortest path between two points, but represents the extremal path between the two points. An extremal path is the path which is invariant under coordinate transformations for the spacetime (it does not change when shifting coordinate systems).

The method of approach to determining geodesics includes applying the Euler-Lagrange equation to the space, which utilizes variational calculus in order to derive the extremal path. The Euler-Lagrange equation, also known as the geodesic equation, is written as,

$$\frac{d}{dx}L - \frac{d}{ds} \left(\frac{d}{dx}L \right) = 0$$

where $\dot{x} = \frac{dx}{ds}$, $L = ds^2$ (the line element), and s is the affine parameter (or the proper time in a Lorentzian metric for a timelike particle). In this equation, x represents each coordinate of the system (i.e. the equation must be applied for each coordinate of the system). Using the same example of the two-dimensional sphere in $\mathbb{R}^3$, we can work through the formalism to arrive at the solution for the geodesic. The Euclidean line element in spherical coordinates defined by equation 64 with a fixed radius $r = R$ which constrains the particle to the surface of the sphere is written as,

$$ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

Dividing by ds^2 provides a more useful form,

$$L = 1 = R^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \quad (69)$$

Applying the Euler-Lagrange equation to both the θ and ϕ coordinates produces,

$$\ddot{\theta} = \sin \theta \cos \theta \dot{\phi}^2 \quad (70)$$

$$\dot{\phi} = \frac{C}{\sin^2 \theta} \quad (71)$$

where C is an arbitrary constant. By solving for $\dot{\theta}$ in equation 69, using the chain rule to derive $\frac{\partial \phi}{\partial \theta}$, and then integrating, we arrive at a solution for $\phi(\theta)$,

$$\phi(\theta) = \cos^{-1} \left(\frac{R^2 C}{\sqrt{1 - R^4 C^2}} \cot \theta \right) + \phi_0$$

Solve for $\cot \theta$ to obtain the geodesic,

$$\cot \theta = A \cos (\phi - \phi_0) \quad (72)$$

$$A = \frac{\sqrt{1 - R^4 C^2}}{R^2 C} \quad (73)$$

Hence, we have calculated the geodesic for a particle constrained to the surface of a sphere, which is in fact a great circle. This can be proven by performing a coordinate transformation to Cartesian coordinates, resulting in the following solution,

$$z = c_1 x + c_2 y \quad (74)$$

$$c_1 = A \cos \phi_0 \quad (75)$$

$$c_2 = A \sin \phi_0 \quad (76)$$

where z can be visualized as a plane centered at the origin $(0, 0, 0)$. Note that all the geodesics described in this Appendix are Euclidean, meaning they are spacelike, while spacetime uses a Lorentzian metric (or line element) which presents the possibility of timelike or null geodesics. See the previous Appendix for an explanation of these differences.

Appendix C Tensors

One important aspect of line elements, geodesics, and relativity in general, is that these concepts can be written in an extremely concise form through the use of tensors. Additionally, the most fundamental advantage of tensors is that when the laws of physics, and in particular general relativity, are written using the tensor formalism, they hold for all possible coordinate systems and transformations between coordinate systems. As a result, in order to define tensors exactly you must define how they act under coordinate transformations. If we were to transform from an n -dimensional coordinate system $x^b = (x^1, \dots, x^n)$ where $b = (1 \dots n)$ to another coordinate system $x'^a = f^a(x^1, \dots, x^n)$ where $a = (1 \dots n)$ then the total differential of the new coordinate system would be written as,

$$dx'^a = \sum_{b=1}^n \frac{\partial x'^a}{\partial x^b} dx^b \quad (77)$$

$$\left[\frac{\partial x'^a}{\partial x^b} \right] = \begin{bmatrix} \frac{\partial x'^1}{\partial x^1} & \cdots & \frac{\partial x'^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x'^n}{\partial x^1} & \cdots & \frac{\partial x'^n}{\partial x^n} \end{bmatrix}$$

where $\left[\frac{\partial x'^a}{\partial x^b}\right]$ is the Jacobian matrix. This can be written more concisely if we imply summation over any repeated indices, known as the Einstein summation notation. Then equation 77 becomes,

$$dx'^a = \frac{\partial x'^a}{\partial x^b} dx^b$$

Furthermore, if we now expand this definition beyond infinitesimals and to sets of physical quantities $X^b = (X^1, \dots, X^n)$ in our old coordinate system, we arrive at the definition of a contravariant tensor of rank 1 which takes the quantities to our new coordinate system,

$$X'^a = \frac{\partial x'^a}{\partial x^b} X^b$$

where $\left[\frac{\partial x'^a}{\partial x^b}\right]$ is evaluated at some point of interest. Similarly, a covariant tensor does an analogous operation in the reverse manner, written formally as,

$$X'_a = \frac{\partial x'^b}{\partial x^a} X_b$$

Moreover, we can also expand these definitions to define contravariant and covariant tensors of higher orders or rank. The rank of a tensor is determined by the number of derivatives taken to transform into the new coordinate system. An example of a rank 2 contravariant tensor is,

$$X'^{ab} = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} X^{cd} \quad (78)$$

An instance where a rank 2 contravariant tensor may be necessary would be if the product of the contravariant vector Y^a and Z^b was a desired operation to be represented in a new coordinate system. In this case, $Y^a Z^b$ would equal X^{ab} which could then be transformed using equation 79. More complicated transformations or desired relationships between the physical quantities will result in mixed tensors with a valence (p, q) where p is the rank of the contravariant tensor and q is the rank of the covariant tensor. An example of a rank 3 mixed tensor of valence $(2, 1)$ is written as,

$$X'^{ab}_c = \frac{\partial x'^a}{\partial x^d} \frac{\partial x'^b}{\partial x^e} \frac{\partial x'^f}{\partial x^c} X^{de}_f \quad (79)$$

Likewise, a tensor of rank 0 is termed a scalar invariant because it transforms as,

$$\phi' = \phi$$

Algebraic operations for tensors of the same valence are as expected for addition, subtraction, and scalar multiplication. Relating back to line elements and geodesics, the line element is thus written extremely concisely for a coordinate system as,

$$ds^2 = g_{ab} dx^a dx^b$$

where g_{ab} is a rank 2 covariant tensor termed the metric. If the metric can be written with all ± 1 terms along the diagonal then it is termed flat. Another aspect of the metric is the square of the length or norm of a contravariant vector, defined as,

$$X^2 = g_{ab}(x)X^a X^b$$

Furthermore, the signature of the metric is defined as the sum of the signs of the diagonal terms, which also holds importance in determining the line element and thus the spacetime in question. Hence, line elements can be completely distinguished by their metrics, which then in turn can determine the structure of the geodesic for a given geometry or spacetime. Other various quantities we can define using this formalism include the inverse of the metric, g^{ab} , as

$$g_{ab}g^{cd} = \delta_a^c = \begin{cases} 0 & a \neq c \\ 1 & a = c \end{cases}$$

where δ_a^c is defined as the Kronecker delta. Additionally, by defining the metric this way the transformation between covariant and contravariant, or vice versa, is also more easily determined

$$g^{ab}X_b = X^a \quad (80)$$

$$g_{ab}X^b = X_a \quad (81)$$

Important tensors utilized in the analysis of different celestial systems, particularly those of the black hole systems analyzed in this project, include the Riemann tensor R_{bcd}^b (also known as the curvature tensor because it gives the curvature at each point of spacetime), Ricci tensor R_{ab} , and Ricci scalar R defined below respectively

$$\nabla_{[c}\nabla_{d]}X^a = \frac{1}{2}R_{bcd}^a X^b \quad (82)$$

$$R_{ab} = g^{cd}R_{dacb} \quad (83)$$

$$R = g^{ab}R_{ab} \quad (84)$$

where the bracket notation is the anti symmetric part defined by $\nabla_{[c}\nabla_{d]} = \frac{1}{2}(\nabla_c\nabla_d - \nabla_d\nabla_c)$. This leads to another important aspect of tensors to realize which includes the various algebraic and calculus based operations and how those operations act when utilizing tensors. For example, two tensors of the same valence can be added and subtracted in the expected fashion - component by component - without trouble. If a tensor is of a different valence than another tensor and the two are to interact algebraically, then one of the tensors can be contracted by setting a raised index and a lowered index of a single tensor equal to one another, and then summed accordingly. For instance, to contract X_{bcd}^a to Y_{cd} you could perform the following contraction.

$$X_{bcd}^a \xrightarrow{\text{contraction on } a \text{ and } b} X_{bcd}^a = Y_{cd}$$

A similar effect can be achieved by multiplying the tensor which needs to be contracted by the Kronecker delta function as well.

However, defining the tensor operations in terms of calculus actions is more difficult because our results do not always remain tensors. For instance, the partial derivative of a tensor is not a tensor, and is thus not

a desired operation. On the same note though, taking derivatives is a necessary piece of physics, and so the concept of the covariant derivative (the effective equivalent of a derivative with tensors) must be introduced. The operation is defined as follows:

$$\nabla_c X^a = \partial_c X^a + \Gamma_{bc}^a X^b$$

where ∇_c is the covariant derivative operator, ∂_c is the ordinary partial derivative operator, and Γ_{bc}^a are termed Christoffel symbols. As a result of the covariant derivative on a tensor X^a , valence (1,0), we finish with a tensor of valence (1,1), but we remain with a tensor. One additional important point to note is that Christoffel symbols transform according to the following laws:

$$\Gamma_c'^{ab} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x'^b}{\partial x^e} \frac{\partial x^f}{\partial x'^c} \Gamma_{de}^f - \frac{\partial x^d}{\partial x'^b} \frac{\partial x^e}{\partial x'^c} \frac{\partial^2 x'^a}{\partial x^d \partial x^e}$$

In terms of relative operations, the last important operation to define includes that of Killing equations and their respective solutions, termed Killing vector fields. The Killing equations are derived by considering the two types of transformations which tensors can undergo - discrete transformations, such as reflections, and continuous ones, such as translations and rotations. Transformations which are invariant are of significant interest, and those transformations which leave important quantities such as the metric unchanged are extremely sought after. The latter of these two, continuous transformations, are the most important in physics models and can be obtained through Killing vector fields. Other important terms include isometry, which means a transformation where a metric is the same function of its argument as the original metric is of its argument (the arguments are different from one another though). Thus, the condition for an infinitesimal isometry results in the Killing equations,

$$L_x g_{ab} = \nabla_b X_a + \nabla_a X_b = 0$$

where L_x is known as the Lie derivative and is defined as follows,

$$L_x T^{ab} = X^c \partial_c T^{ab} - T^{ac} \partial_c X^b - T^{cb} \partial_c X^a \quad (85)$$

$$X^a(x(u)) = \frac{dx^a}{du} \quad (86)$$

Appendix D Equations of Special Relativity

The mathematical relationship between time and space for two different inertial observers mentioned in the body of the report are written as follows:

$$t' = \frac{t - vx}{\sqrt{1 - v^2}} \quad (87)$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} \quad (88)$$

$$y' = y \quad (89)$$

$$z' = z \quad (90)$$

where t is the proper time, v is the velocity of the moving frame, t' is the time measured by the observer seeing the event occur at two separate locations, x' is the proper length, and x is the length measured by a moving observer relative to the object. These transformations can be written more usefully as,

$$t' = \gamma \left(t - \frac{vx}{c^2} \right) \quad (91)$$

$$x' = \gamma (x - vt) \quad (92)$$

$$y' = y \quad (93)$$

$$z' = z \quad (94)$$

where $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Thus, when in relativistic units (setting $c = 1$ for convenience) we recover the prior version of the Lorentz transformations. Notice the important point that Lorentz transformations mix time and space in the direction of motion. This accounts for time dilation and length contraction.

Appendix E Principles of General Relativity

In the body of this report, the idea of Mach's principle and the principle of equivalence were discussed; however, there remain three more core principles fundamental to the theory of general relativity. The next principle to discuss then is the principle of general covariance, which is derived from the principle of general relativity. The principle of general relativity dictates that all observers are equivalent. Essentially, this statement implies that some global inertial observer cannot exist. Additionally, it implies that for a curved metric, then the laws of physics should be equivalent regardless of the coordinate system selected (regardless of the observer). Since properties do not change (are invariant) under coordinate transformations when using tensors, this led directly to the development of the principle of general covariance.

Principle of General Covariance: The equations of physics should have tensorial form.

The principle of minimal gravitational coupling arises from Occam's razor and the fact that in going from special relativity to general relativity we should not add in unnecessary terms. Similar to Mach's Principle, it is not precise, but does help in the formulation of Einstein's field equations.

Principle of Minimal Gravitational Coupling: No terms explicitly containing the curvature tensor should be added in making the transition from the special to the general theory.

Finally, the principle of correspondence simply indicates how general relativity simplifies to former and more specialized theories, such as the theory of special relativity, the Newtonian theory of gravitation, and Newtonian mechanics without gravitation, in limiting cases. A schematic of the principle of correspondence is in Figure 21.

The net result of these five principles was the development of Einstein's field equations. Mathematically, the vacuum field equations of general relativity can be written in terms of the Ricci tensor.

$$R_{\mu\nu} = 0$$

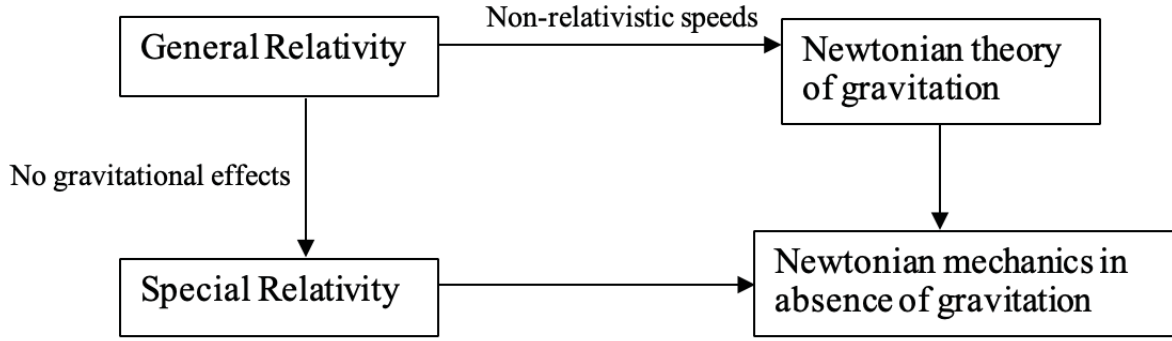


Figure 21: Principle of Correspondence [2]

These can be written equivalently in terms of the Einstein tensor, which is a function of the Ricci tensor, the metric, and the Ricci scalar.

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

On the other hand, the full field equations of general relativity include a term $T_{\mu\nu}$ called the energy-momentum tensor which accounts for the presence of all fields other than gravitation. However, special relativity implies the equivalence of mass and energy; thus, it implies in general relativity that all forms of energy serve as sources of gravitational fields (another paraphrasing of the weak form of the principle of equivalence). As a result, the Einstein tensor is related to the energy-momentum tensor through the coupling constant $\kappa = \frac{8\pi G}{c^4}$ in non-relativistic units, or $\kappa = 8\pi$ in relativistic units. Relativistic units, or natural units, set G and c equal to 1.

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Additionally, the energy-momentum tensor can be construed to account for other selective fields, such as the electro-magnetic field. By defining the Maxwell tensor as,

$$F^{ab} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

The Maxwell energy-momentum tensor can be defined as,

$$T_{ab} = \frac{1}{4\pi} \left(-g^{cd} F_{ac} F_{bd} + \frac{1}{4} g_{ab} F_{cd} F^{cd} \right)$$

Furthermore, another term called the cosmological term $\Lambda g_{\mu\nu}$ with a cosmological constant Λ , which can be thought of as the energy density of the universe, was added by Einstein into the full field equations of general relativity. The addition of this term was to try and develop a static, closed solution to the equations for a universe uniformly filled with matter and in essence prevent gravity from evolving a non-static universe. The full field equations are now written in relativistic units as,

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Due to the physical evidence of a non-static, expanding universe, the cosmological term is often removed by setting $\Lambda = 0$, and it is assumed naturally to be very small on a local scale. The importance of the cosmological term only presents itself on a cosmological scale, or a quantum scale. Consequently, if $\Lambda > 0$ then matter experiences a ‘cosmic repulsion’ which tends to disperse matter to infinity, if $\Lambda < 0$ then matter experiences a ‘cosmic attraction’ which tends to bring matter contract matter towards other matter in addition to gravitation, and $\Lambda = 0$ then we recover the original full field equations.

Appendix F Spacetime

One of the primary features of relativity is the discarding of time as an invariant quantity and integrating it into the space as a new coordinate, thereby developing a four-dimensional space called spacetime. Thus, an event in spacetime now has coordinates (t, x, y, z) , or (ct, x, y, z) if it is preferable to scale the time axis. The simplest spacetime is the Minkowski spacetime, or simply termed flat space, which is defined as a four-dimensional manifold endowed with a flat metric of signature -2 (see equation 66). Specifically, in Minkowski spacetime the line element is the aforementioned Minkowski line element. From the line element and the associated metric, we can define other useful terminology regarding the norm or length of a vector in Minkowski spacetime including timelike, spacelike, and null vectors.

$$\left\{ \begin{array}{ll} \text{timelike} & \text{if } X^2 > 0 \\ \text{spacelike} & \text{if } X^2 < 0 \\ \text{null or lightlike} & \text{if } X^2 = 0 \end{array} \right.$$

If the tangent vector of a curve is everywhere timelike, then the curve is a timelike curve (respectively for spacelike and null). Timelike curves represent tracks on which material particles or observers can travel. Additionally, the nature of null vectors in Minkowski spacetime by this definition satisfy the equation,

$$0 = g_{ab} X^a X^b$$

Due to this, we can develop the notion of the null cone in Minkowski coordinates - a double cone bounded by the set of all null vectors. If we define the timelike vector $T^b = (1, 0, 0, 0)$, then the timelike or null vector X^a is said to be future pointing if $g_{ab} X^a T^b > 0$ and past pointing if $g_{ab} X^a T^b < 0$. The set of all future pointing and past pointing vectors form a future light cone and past light cone, shown pictorially in Figure 22.

Furthermore, using a number of coordinate transformations we can develop a better visual picture of the space through the implementation of Penrose diagrams. First, we rewrite the metric g_{ab} (also known as the

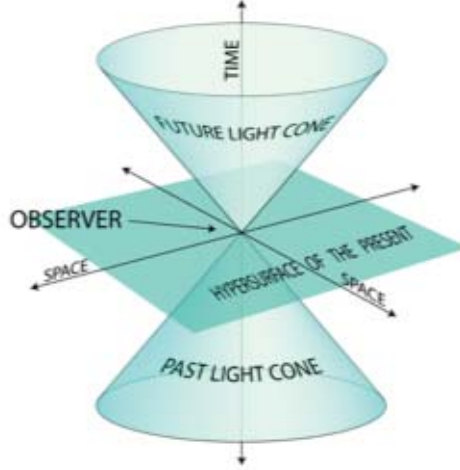


Figure 22: The future and past light cone [44].

physical metric) in terms of an unphysical metric $\bar{g}_{ab}$ in a conformal relation, meaning

$$\bar{g}_{ab} = \Omega^2 g_{ab}$$

Next, we transform coordinates as follows:

$$v = t + r \tag{95}$$

$$w = t - r \tag{96}$$

$$p = \tan^{-1} v \tag{97}$$

$$q = \tan^{-1} w \tag{98}$$

$$t' = p + q \tag{99}$$

$$r' = p - q \tag{100}$$

This results in the conformal factor of $\Omega = \frac{1}{4} \sec^2 p \sec^2 q$ and the unphysical line element,

$$\bar{ds}^2 = dt'^2 - dr'^2 - \sin^2 r' (d\theta^2 + \sin^2 \theta d\phi^2)$$

This process is known as conformal compactification and the region now defined by the unphysical line element and respective domains of $(t' + r')$, $(t' - r')$, and r' is called compactified Minkowski spacetime. This is shown with two dimensions suppressed in Figure 23(a). In this figure t' is represented by T and r' is represented by R . The result of these coordinate transformations is the bringing of infinity into a finite boundary for help in visualizing the timelike curves for particles on different spacetime diagrams. A Penrose diagram is then a diagram which takes any conformally compacted spacetime and draws it on a spacetime diagram. For instance, by taking compactified Minkowski spacetime and drawing a spacetime diagram, we arrive at Figure 23(b). In this diagram, I^+ , I^- , i^+ , i^- , and i^0 represent future null infinity, past null infinity, future timelike infinity, past timelike infinity, and spacelike infinity respectively. On these curves, light's path remains the same as a normal spacetime diagram in Minkowski spacetime because it still travels exclusively along 45° paths, making them easy to analyze.

However, Minkowski spacetime is not the only spacetime available, nor is it the spacetime this project focuses on. Other spacetimes previously mentioned also include Schwarzschild, RN, RN AdS-dS, and Kerr

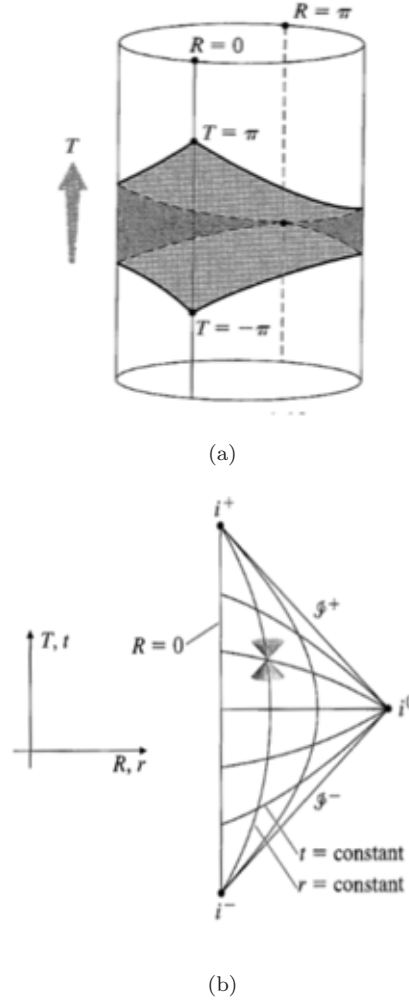
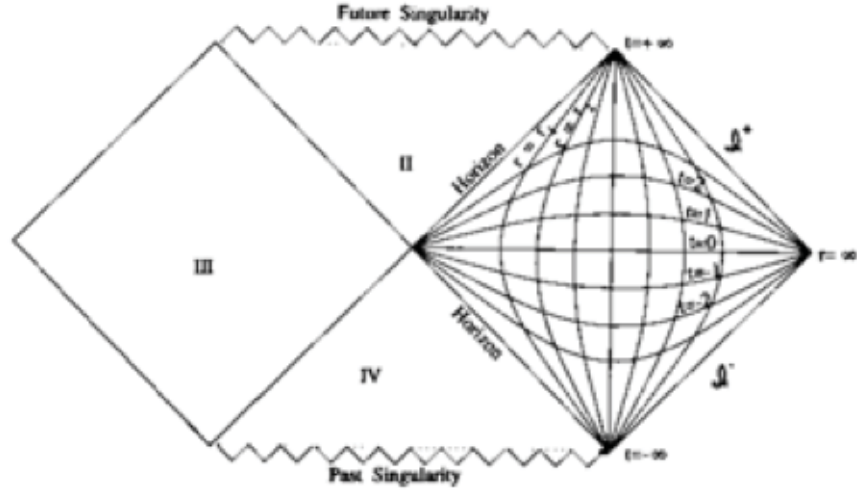


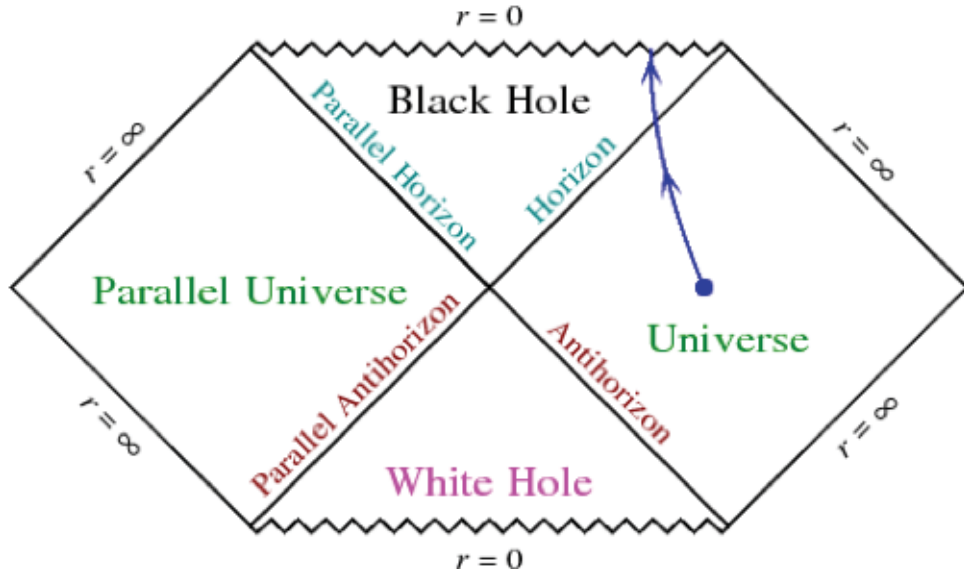
Figure 23: Light cones and Penrose diagrams [2]. The sketch in (a) is a drawing with two dimensions suppressed of a light cone in compactified Minkowski coordinates. In (b) a Penrose diagram for a flat spacetime is shown.

which are all based on the concept of a black hole. Essentially, a Schwarzschild spacetime represents a spherically symmetric (and thus static) solution to the vacuum field equations. The Penrose diagram for a Schwarzschild spacetime can be found in Figure 24(a), while the “maximal extension” is offered in Figure 24(b). In a physically realizable Schwarzschild black hole, the other side of the antihorizon shown in Figure 24(b) lies the star that collapsed, redshifted to blackness. However, the Schwarzschild geometry has a simple mathematical form, and that form can be extended analytically. The mathematical extension, or “maximal extension” which is a purely mathematical construct, consists of a second copy of the Schwarzschild geometry, reversed in time, glued along the antihorizon. The complete analytic extension of the Schwarzschild geometry contains not only a universe and a black hole, but also a parallel universe and a white hole. Again, this is simply a mathematical construction, with no basis in physical evidence or observation. Thus, a more realistic Penrose diagram can be found in Figure 24(c), which dictates a more practical approach to the collapse of a star into the formation of a Schwarzschild black hole,

Analogously, the RN black hole spacetime represents a charged black hole scenario. Thus, the spacetime is developed through the static, asymptotically flat, and spherically symmetric solutions to the full field equations of general relativity with the Maxwell energy-momentum tensor implemented. The spacetime



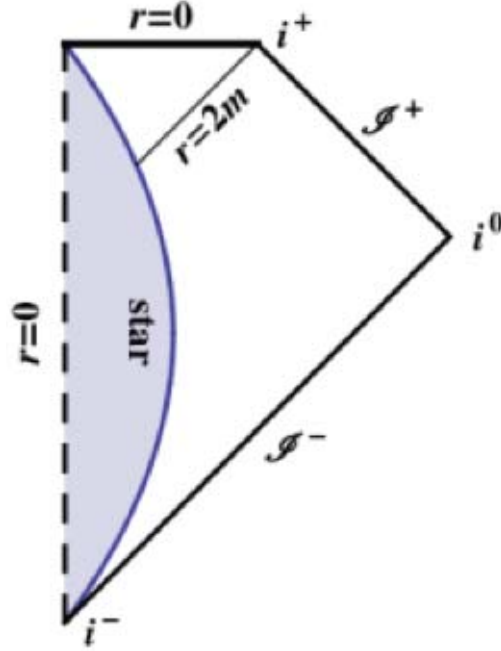
(a)



(b)

diagram of the black hole can be represented best in advanced Eddington-Finkelstein type coordinates $\left[\bar{t} = t + \frac{r_+^2}{r_+ - r_-} \ln(r - r_+) - \frac{r_-^2}{r_+ - r_-} \ln(r - r_-)\right]$ where $r_{\pm} = M \pm (M^2 - \varepsilon^2)$ in Figure 25(a). The Penrose diagram of the maximal analytic extension for the $\varepsilon^2 \ll M^2$ case of such a spacetime is represented in Figure 25(b). Likewise, the Penrose diagram of the $\varepsilon^2 = M^2$ case is represented in Figure 25(c) and of the $\varepsilon^2 \gg M^2$ case (called the overcharged black hole case) in Figure 25(d).

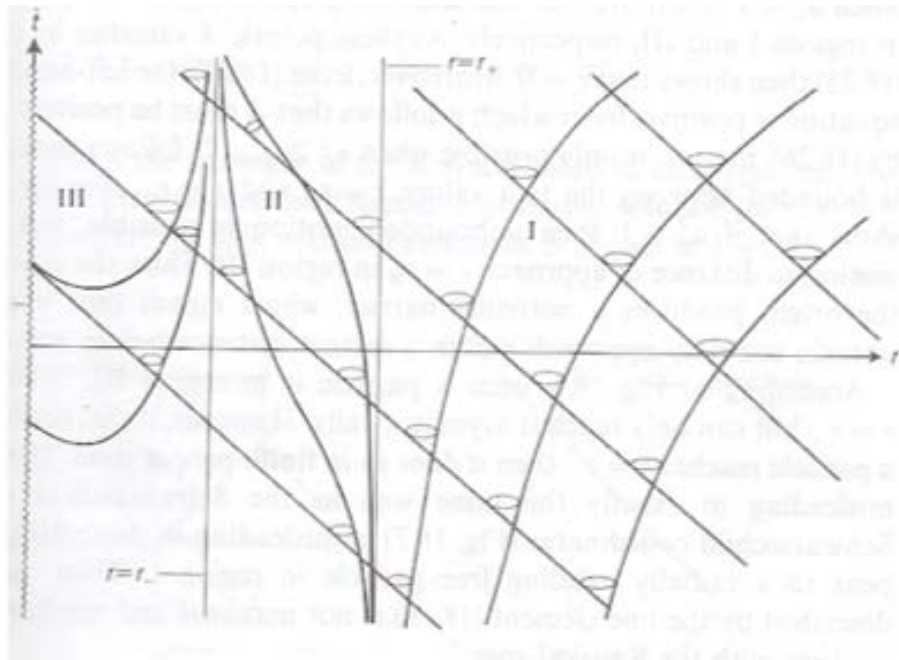
Another possible black hole spacetime to be analyzed was the Kerr black hole spacetime, which represents a rotating black hole. The solution to the vacuum field equations of general relativity, known as the Kerr solution, defines this spacetime and the spacetime diagram can be found in Figure 26(a) in advanced Eddington-Finkelstein type coordinates. A surface diagram of the black hole can be found in Figure 26(b). Likewise, the Penrose diagram of the maximal analytic extension for the $a^2 \ll M^2$ case is represented in the



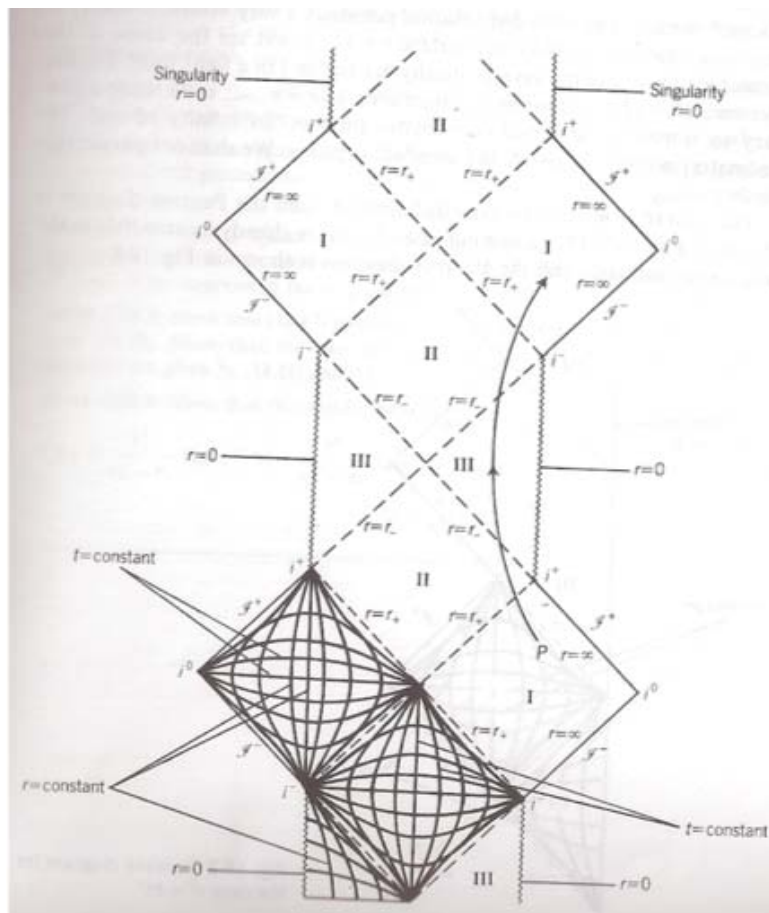
(c)

Figure 24: Penrose diagrams for a Schwarzschild black hole. Both (a) [2] and (b) [45] show a Penrose diagram of a Schwarzschild black hole. However, in (b), the maximal extension of the Schwarzschild black hole system, labels are put in for each region including regions indicating a “parallel universe” and “white hole”. It should be emphasized that the idea of a black hole connecting parallel universes or the notion that white holes are possible are both purely mathematical constructs with no experimental confirmation of their existence – they are included for completeness in the Penrose diagram and can be ignored when considering a physical situation. In (c) a Penrose diagram for a Schwarzschild black hole including the presence of the collapsing core of the star is shown, and this should be considered as the physically realizable situation [46]. Notice how in (a) and (b) the jagged lines are horizontal, indicating a spacelike singularity. If the jagged lines were vertical, as is shown in the images in Figure 25 and Figure 26 and is the case in the negative mass Schwarzschild case, then a timelike singularity would be indicated.

left diagram of Figure 26(c) and of the $a^2 = M^2$ case in the right diagram of Figure 26(c). It is important to note that the Penrose diagram of the overspinning case (the case covered in this report) which involves $a^2 \gg M^2$ looks identical to the $\varepsilon^2 \gg M^2$ RN black hole Penrose diagram. A spatial diagram of the black hole can be found in Figure 26(d). Interestingly enough, the analysis of the Kerr solution suggests the notion that an advanced civilization could in fact live near a rotating black hole and extract energy from the rotation of the black hole. This concept is illustrated in Figure 26(e). Finally, it is important to note that the addition of a cosmic string to any of the scenarios discussed in this section does not alter the Penrose diagrams.



(a)



(b)

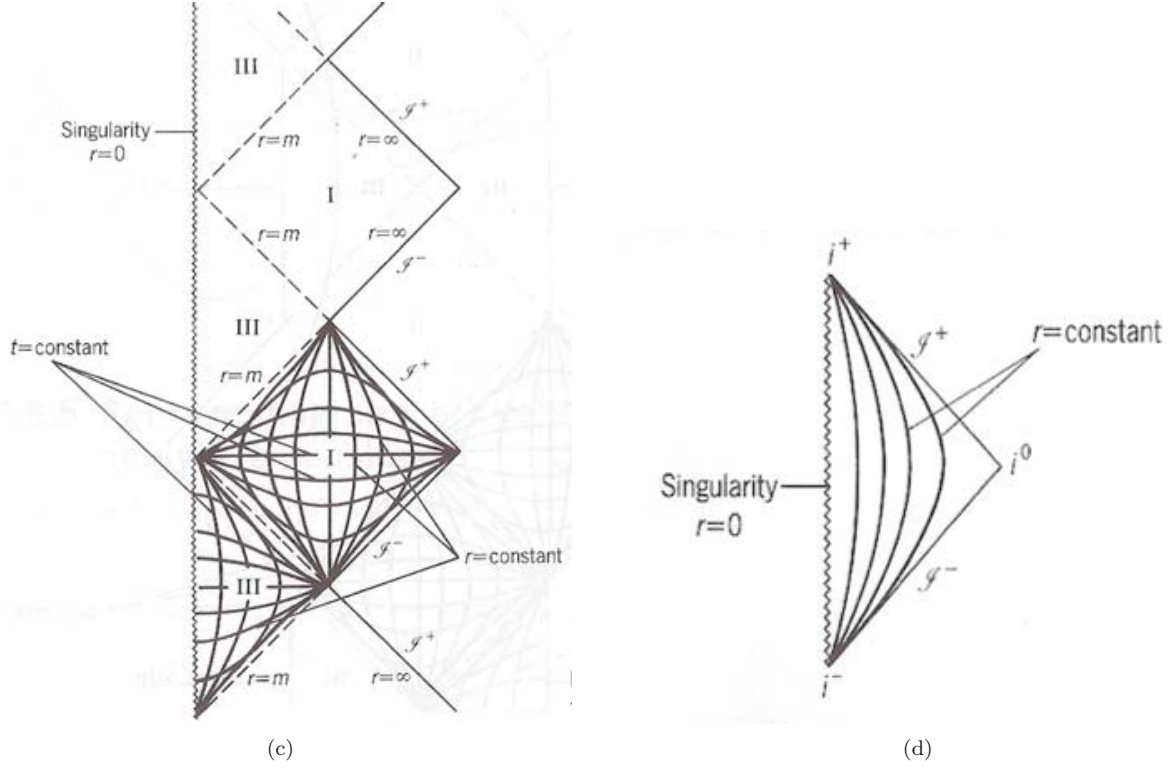
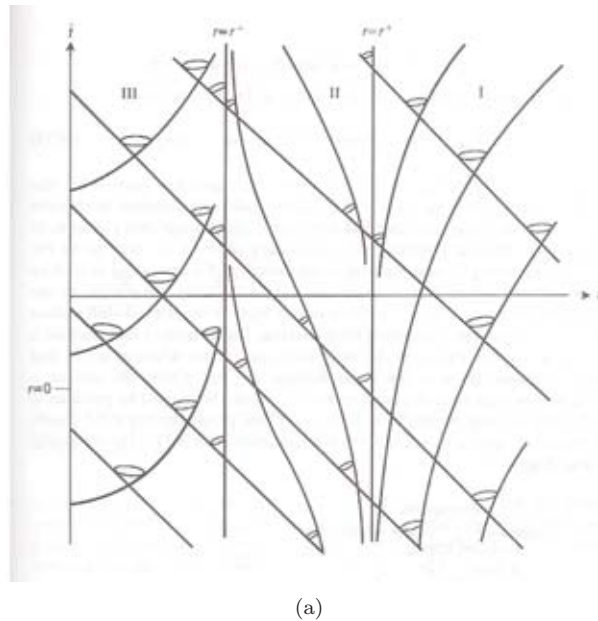
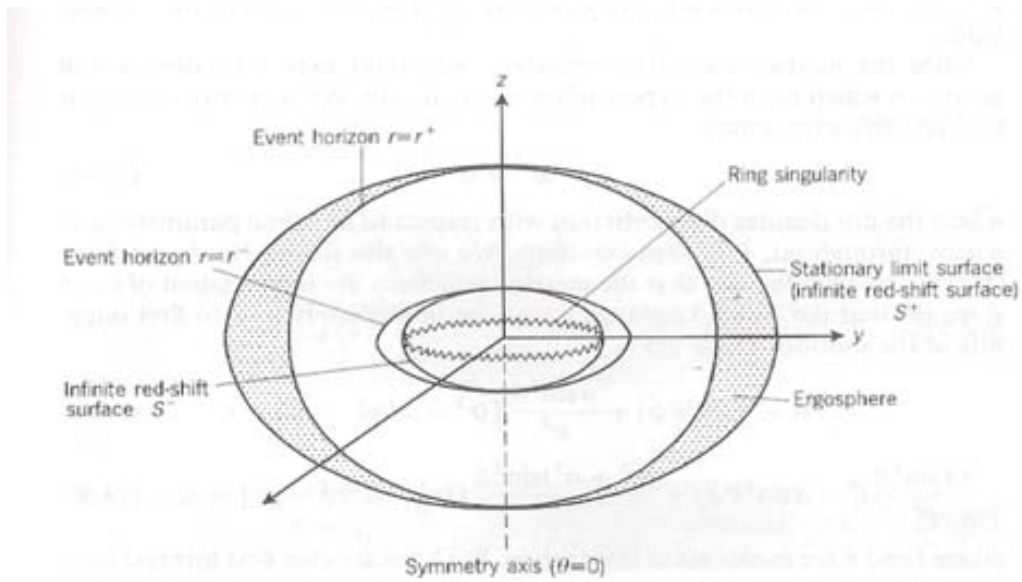
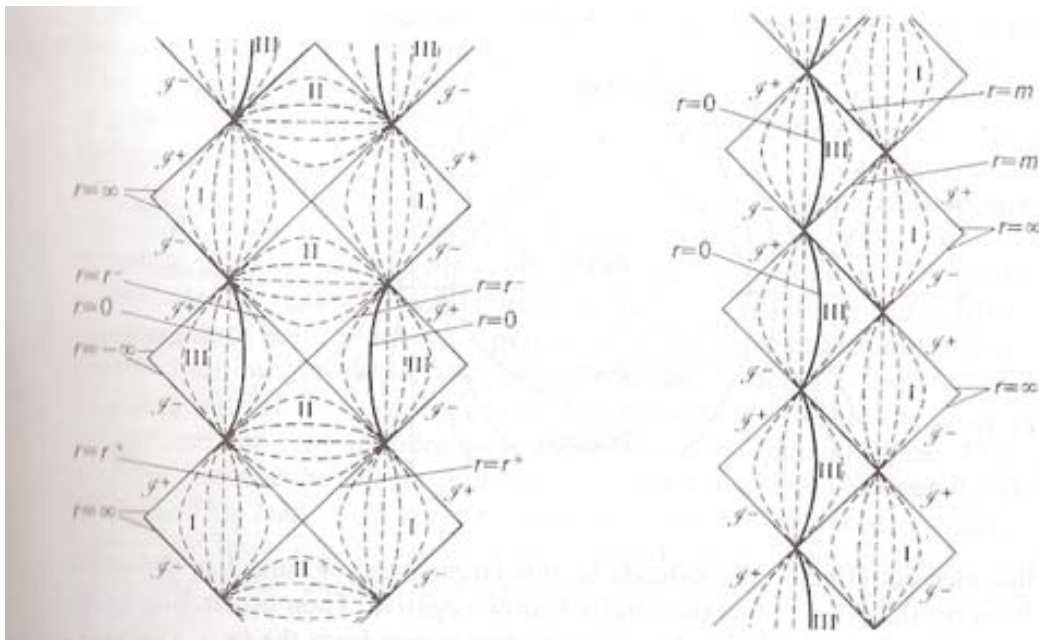


Figure 25: Penrose diagrams for a Reissner-Nordström black hole [2]. The spacetime of the Reissner-Nordström represented in Eddington-Finkelstein coordinates is shown in (a). A Penrose diagram of the maximal analytic extension for the $\varepsilon^2 \ll M^2$ case of such a spacetime is shown in (b). A Penrose diagram of the $\varepsilon^2 = M^2$ case is shown in (c). A Penrose diagram of the $\varepsilon^2 \gg M^2$ case is shown in (d). Notice how the Penrose diagram for the overcharged RN black hole system is not extendable and looks very similar to Figure 23(b) with a timelike singularity at $r = 0$ and no event horizons between zero and infinitely far radial distances.

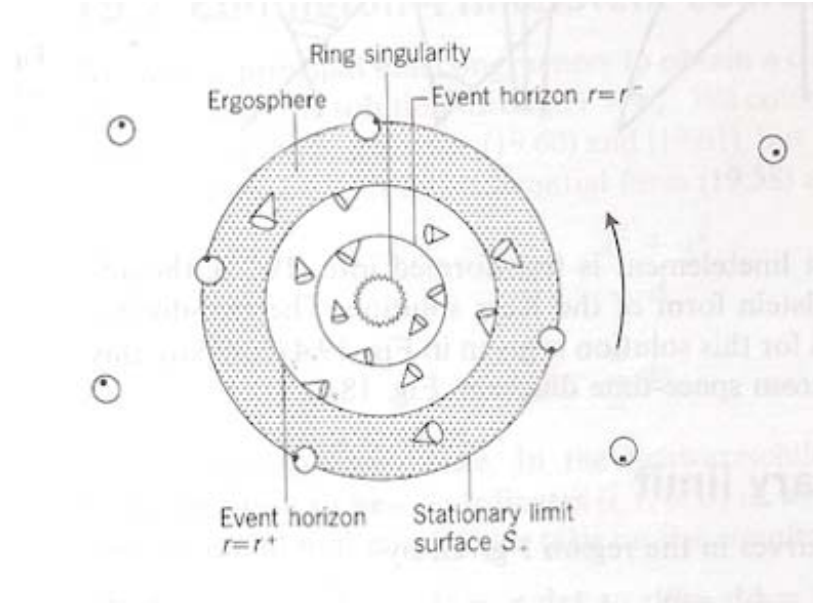




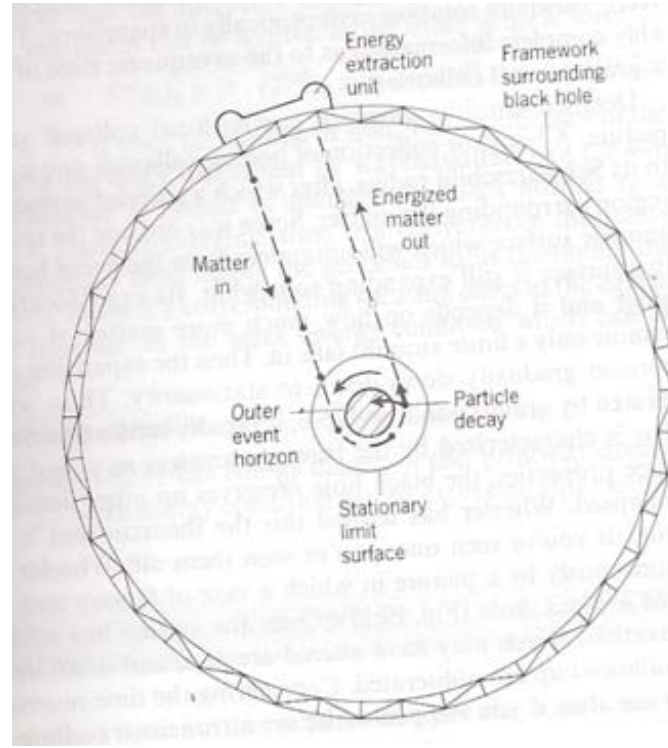
(b)



(c)



(d)



(e)

Figure 26: Spacetime and Penrose diagrams for a Kerr black hole [2]. The spacetime of the Kerr black hole represented in Eddington-Finkelstein coordinates is shown in (a). The surface structure of a Kerr black hole is shown in (b). A Penrose diagram of the maximal analytic extension for the $a^2 \ll M^2$ case is shown on the left in (c) and of the $a^2 = M^2$ case on the right in (c). A spatial diagram of the Kerr black hole is shown in (d). A theoretical visual of the mining of energy from a Kerr black hole is shown in (e). Again, note that the Penrose diagram of the overspinning case (the case covered in this report) which involves $a^2 \gg M^2$ looks identical to the $\varepsilon^2 \gg M^2$ RN black hole Penrose diagram.

Appendix G Limit-Point, Limit Circle Theorems

In order to analyze the test wave packet in quantum mechanics, we will be looking at the wave operator of choice. In order to do this, we must understand the basics of unbounded operators on Hilbert spaces.

Definition 1: A Hilbert space $\mathcal{H}$ is an inner product space which is complete – one in which all Cauchy sequences converge to an element of the space. The Hilbert space we will be using for our work is $L^2(0, \infty)$, the space of square integrable functions (where square integrability is defined in *Definition 4*) equipped with an inner product defined by the following:

$$\langle f | g \rangle = \int_0^\infty f^* g d\mu$$

where μ is the measure. An operator on a Hilbert space is a linear map from one Hilbert space to another. An important side note is that the norm is derived from the inner product. If we have inner product $\langle * | * \rangle$, then $\|x\| = \langle * | * \rangle^{1/2}$.

Definition 2: Consider an operator $T : \mathcal{H} \rightarrow \mathcal{H}$. Its adjoint, T^* is defined as follows: $Dom(T^*) = \{j \in \mathcal{H} : \exists f \in \mathcal{H} \text{ with } \langle T(h) | j \rangle = \langle h | f \rangle \forall h \in Dom(T)\}$ and $T(j) = f$. In layman's terms, an operator is self-adjoint if $(T = T^*)$ and $Dom(T) = Dom(T^*)$.

Definition 3: A symmetric operator T is called essentially self-adjoint if its closure $\bar{T}$ is self-adjoint. In essence, this means that the first condition of *Definition 2* is met and the second can be met by expanding the domain of the operator or the adjoint so that it is true.

Definition 4: A function f is square integrable on the domain D if $\int |f|^2 dx < \infty$ over D .

Definition 5 (see Theorem X.7 in [32]): The potential function $V(x)$ is quantum mechanically complete if $\hat{H} = -\frac{d^2}{dx^2} + V(x)$ is essentially self-adjoint on $C_0^\infty(0, \infty)$. $V(x)$ is said to be complete at ∞ (and also at 0) if at least one solution of $\psi(x)'' = V(x)\psi(x)$ is not in L^2 near ∞ (respectively near 0).

Definition 6 (see Theorem X.6 in [32]): $V(x)$ is in the limit circle case at ∞ (respectively at 0) if for some, and therefore all λ , all solutions of,

$$-\phi''(x) + V(x)\phi(x) = \lambda\phi(x)$$

are square integrable at ∞ (respectively 0). If $V(x)$ is not in the limit circle case at ∞ , then it is said to be in the limit point case.

Theorem 1 (Weyl's Theorem) (also see Theorem X.7 in [32]): Let $V(x)$ be a real-valued function on $(0, \infty)$. Then $\hat{H} = -\frac{d^2}{dx^2} + V(x)$ is essentially self-adjoint on $C_0^\infty(0, \infty)$ if and only if $V(x)$ is in the limit point case at both 0 and ∞ .

Theorem 2 (see Theorem X.8 [32]): Let $V(x)$ be a continuous real-valued function on $(0, \infty)$ and suppose that there exists a positive differentiable function on $M(x)$ so that,

- i $V(x) \geq -M(x)$
- ii $\int_1^\infty (M(x))^{-1/2} dx = \infty$
- iii $\frac{M'(x)}{(M(x))^{3/2}} < \infty$

Then $V(x)$ is in the limit point case (complete) at ∞ .