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Invariant Vector Representation of a Qutrit in a Cascade Model

by Vinod K Mishra

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14. ABSTRACT The qutrit state density matrix is of order 3 and depends on eight parameters in the most general case. Visualization of this 8-D state space is practically impossible with commonly used 8-D vectors. Recently, a 3-D vector representation of the qutrit state space (also called invariant vector representation [IVR]) was proposed. In this report, we present the time–evolution of the IVR vectors for the qutrit cascade or Ξ -model to emphasize the advantages of the IVR.					
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1. Introduction

The qutrit (three internal states) comes next in complexity after qubit (two internal states) as a resource for quantum information processes like quantum sensing.¹ The qubit density matrix is of order 2 and it depends on three parameters for the most general mixed states, but only on two parameters for pure states. It can be easily visualized using Bloch sphere representation in which the pure states are represented by points on the Bloch sphere. On the other hand, the qutrit density matrix is of order 3 and it depends on eight parameters in the most general case. Visualization of the 8-D qutrit state space is practically impossible using the 8-D vectors commonly used in the Gell-Mann representation.² Recently, another 3-D representation of the qutrit state space was proposed based on density matrix invariants (called invariant vector representation [IVR]).³ These vectors also reside on the surface of a sphere and help one visualize the dynamics of a qutrit state. In this report, we apply IVR to the cascade configuration of a qutrit (also known as the Ξ -model^{2,4}) and show its utility in understanding the dynamics of the model.

2. Invariant Vector Representation (IVR) of a Qutrit: Basic Results

The density matrix ρ based on the spin-1 representation of a qutrit is given as^{5,6}

$$\rho = \begin{bmatrix} \omega_1 & \frac{1}{2}(q_3 + ia_3) & \frac{1}{2}(q_2 - ia_2) \\ \frac{1}{2}(q_3 - ia_3) & \omega_2 & -\frac{1}{2}(q_1 + ia_1) \\ \frac{1}{2}(q_2 + ia_2) & -\frac{1}{2}(q_1 - ia_1) & \omega_3 \end{bmatrix}. \quad (1)$$

The parameters of ρ are related to the expectation values of expressions involving spin-1 components and their combinations.

$$\omega_i = \langle S_i^2 \rangle = \text{Tr}(\rho S_i^2), \quad (2a)$$

$$a_i = \langle S_i \rangle = \text{Tr}(\rho S_i), \quad (2b)$$

$$q_k = \langle S_i S_j + S_j S_i \rangle = \text{Tr}\{\rho(S_i S_j + S_j S_i)\}, k \neq i, j. \quad (2c)$$

The invariant 3-D vectors are given by the following relations.⁵

First invariant vector:

$$\vec{w} = \{\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}\}. \quad (3a)$$

Based on the trace relation,

$$Tr(\rho_{\Xi}) = \omega_1 + \omega_2 + \omega_3 = 1, \sum_{i=1}^3 w_i^2 = 1. \quad (3b)$$

Second invariant vector:

$$\vec{u} = \left\{ \sqrt{\omega_1^2 + (q_1^2 + a_1^2)/2}, \sqrt{\omega_2^2 + (q_2^2 + a_2^2)/2}, \sqrt{\omega_3^2 + (q_3^2 + a_3^2)/2} \right\}. \quad (4a)$$

Based on the trace relation,

$$Tr(\rho_{\Xi}^2) = \sum_{i=1}^3 u_i^2 \leq 1. \quad (4b)$$

Third invariant vector:

$$\vec{v} = \left\{ \sqrt{X + 3(q_1^2 + a_1^2)/2}, \sqrt{X + 3(q_2^2 + a_2^2)/2}, \sqrt{X + 3(q_3^2 + a_3^2)/2} \right\}. \quad (5a)$$

Here,

$$X = \frac{1}{3} - 2\omega_1\omega_2\omega_3 - \frac{1}{2}(a_2a_3q_1 + a_3a_1q_2 + a_1a_2q_3 - q_1q_2q_3). \quad (5b)$$

Based on the trace relation,

$$3Tr(\rho_{\Xi}^2) - 2Tr(\rho_{\Xi}^3) = \sum_{i=1}^3 v_i^2 \leq 1. \quad (5c)$$

The vectors $\vec{u}$ and $\vec{v}$ represents the second and third density matrix invariants of a qutrit. The bounds on the vector-norms are, in general, $\sum_{i=1}^3 u_i^2 \leq 1$ and $\sum_{i=1}^3 v_i^2 \leq 1$, with equality signs holding for a pure state.

For a pure state, the nine density matrix parameters are related via the following five relations.

$$\text{1 relation} \quad Tr(\rho_{\Xi}) = \omega_1 + \omega_2 + 3 = 1, \quad (6a)$$

3 relations

$$\frac{1}{4} \begin{pmatrix} q_1^2 + a_1^2 \\ q_2^2 + a_2^2 \\ q_3^2 + a_3^2 \end{pmatrix} = \begin{pmatrix} \omega_2\omega_3 \\ \omega_3\omega_1 \\ \omega_1\omega_2 \end{pmatrix}, \quad (6b)$$

1 relation

$$a_2a_3q_1 + a_3a_1q_2 + a_1a_2q_3 - q_1q_2q_3 = 8\omega_1\omega_2\omega_3. \quad (6c)$$

So, finally, we have only four independent parameters, which is the correct number of independent degrees of freedom for a pure qutrit state.

3. Time-Dependent Eigenvectors of the Qutrit Cascade or Ξ -Model

The isolated qutrit states with their energies are taken to be $|0\rangle (E_0)$, $|1\rangle (E_1)$, and $|2\rangle (E_2)$ with $E_2 > E_1 > E_0$ average energy $\bar{E} = (E_0 + E_1 + E_2)/3$. Define $\varepsilon_1 = (-2E_0 + E_1 + E_2)/3$, and $\varepsilon_2 = (-E_0 - E_1 + 2E_2)/3$, then the starting energies are related with them as

$$E_0 = \bar{E} - \varepsilon_1, \quad (7a)$$

$$E_1 = \bar{E} - (\varepsilon_2 - \varepsilon_1) = \bar{E} - \varepsilon, \quad (7b)$$

$$E_2 = \bar{E} + \varepsilon_2. \quad (7c)$$

Here $\varepsilon = \varepsilon_2 - \varepsilon_1 = (E_0 + E_2 - 2E_1)/3 > 0$ has been assumed for later analysis ($\varepsilon < 0$ case is similar). After taking the zero of energy at $\bar{E}$, the Hamiltonian of the qutrit in an external field is given by

$$H = \begin{bmatrix} -\varepsilon_1 & (g_1 - ig_2)\phi & (g_3 - ig_4)\phi \\ (g_1 + ig_2)\phi & -\varepsilon & (g_5 - ig_6)\phi \\ (g_3 + ig_4)\phi & (g_5 + ig_6)\phi & \varepsilon_2 \end{bmatrix}. \quad (8)$$

We specialize to the equidistant level Ξ -model in which, (i) $\varepsilon_2 = \varepsilon_1$, (ii) couplings are equal, in other words, $g_1 = g_3 = g_5$, $g_2 = g_4 = g_6$, and (iii) there is no coupling between first and third levels. Then the Hamiltonian becomes

$$H_{\Xi} = \begin{bmatrix} -\varepsilon_1 & \phi(g_1 - ig_2) & 0 \\ \phi(g_1 + ig_2) & 0 & \phi(g_1 - ig_2) \\ 0 & \phi(g_1 + ig_2) & \varepsilon_1 \end{bmatrix}. \quad (9)$$

Let

$$g_1 + ig_2 = Ge^{i\delta}, \quad (10a)$$

$$\omega = \sqrt{\varepsilon_1^2 + 2\phi^2 G^2}, \quad (10b)$$

$$\varepsilon_1 = \omega \cos\theta, \quad (10c)$$

$$\phi G\sqrt{2} = \omega \sin\theta. \quad (10d)$$

Then we get

$$H_{\Xi} = \omega \begin{bmatrix} -\cos\theta & \frac{1}{\sqrt{2}}\sin\theta e^{-i\delta} & 0 \\ \frac{1}{\sqrt{2}}\sin\theta e^{i\delta} & 0 & \frac{1}{\sqrt{2}}\sin\theta e^{-i\delta} \\ 0 & \frac{1}{\sqrt{2}}\sin\theta e^{i\delta} & \cos\theta \end{bmatrix}. \quad (11)$$

The stationary eigenvectors for eigenvalues $(-\omega, 0, \omega)$ of H_{Ξ} are found respectively as

$$|0\rangle = \begin{pmatrix} e^{-i\delta} \cos^2 \frac{\theta}{2} \\ -\frac{1}{\sqrt{2}} \sin\theta \\ e^{i\delta} \sin^2 \frac{\theta}{2} \end{pmatrix}, |1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\delta} \sin\theta \\ \cos\theta \\ -\frac{1}{\sqrt{2}} e^{i\delta} \sin\theta \end{pmatrix}, |2\rangle = \begin{pmatrix} e^{-i\delta} \sin^2 \frac{\theta}{2} \\ \frac{1}{\sqrt{2}} \sin\theta \\ e^{i\delta} \cos^2 \frac{\theta}{2} \end{pmatrix}. \quad (12)$$

The time-dependent Schrödinger's equation for the Ξ -model is given as

$$i \frac{\partial}{\partial t} \begin{pmatrix} |\Psi_0(t)\rangle \\ |\Psi_1(t)\rangle \\ |\Psi_2(t)\rangle \end{pmatrix} = H_{\Xi} \begin{pmatrix} |\Psi_0(t)\rangle \\ |\Psi_1(t)\rangle \\ |\Psi_2(t)\rangle \end{pmatrix}. \quad (13)$$

As before, the interacting field ϕ is time-independent and the initial conditions are: (i) $|\psi_0(t=0)\rangle = 1$, and (ii) $|\psi_1(t=0)\rangle = 0 = |\psi_2(t=0)\rangle$. The solutions for the time-dependent eigenvectors are

$$\begin{pmatrix} |\Psi_0(t)\rangle \\ |\Psi_1(t)\rangle \\ |\Psi_2(t)\rangle \end{pmatrix} = \cos^2 \frac{\theta}{2} e^{i(\omega t + \delta)} |0\rangle + \frac{1}{\sqrt{2}} \sin\theta e^{i\delta} |1\rangle + \sin^2 \frac{\theta}{2} e^{-i(\omega t - \delta)} |2\rangle. \quad (14)$$

They can be rewritten as

$$\begin{pmatrix} |\Psi_0(t)\rangle \\ |\Psi_1(t)\rangle \\ |\Psi_2(t)\rangle \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{1}{2} \sin^2 \theta\right) \cos\omega t + \frac{1}{2} \sin^2 \theta + i \sin\omega t \cos\theta \\ \frac{1}{\sqrt{2}} \sin\theta [\cos\theta (1 - \cos\omega t) - i \sin\omega t] e^{i\delta} \\ -\frac{1}{2} \sin^2 \theta (1 - \cos\omega t) e^{2i\delta} \end{pmatrix}. \quad (15)$$

4. IVR of a Qutrit: Ξ -Model

The density matrix of the qutrit cascade or Ξ -model is calculated as

$$\rho_{\Xi} = \begin{pmatrix} |\Psi_0(t)\rangle \\ |\Psi_1(t)\rangle \\ |\Psi_2(t)\rangle \end{pmatrix} \begin{pmatrix} \langle \Psi_0(t) | & \langle \Psi_1(t) | & \langle \Psi_2(t) | \end{pmatrix}. \quad (16)$$

We express the resulting density matrix as the spin-1 representation given earlier. Then for the qutrit Ξ -model, the following are the expressions for the density matrix parameters.

$$\omega_1 = \frac{1}{4}(3 + \cos 2\theta)\cos^2 \omega t, \quad (17a)$$

$$\omega_2 = \frac{1}{4}(3 + \cos 2\theta)\sin^2 \omega t, \quad (17b)$$

$$\omega_3 = \frac{1}{4}(1 - \cos 2\theta), \quad (17c)$$

$$\begin{pmatrix} q_1 \\ a_1 \end{pmatrix} = \frac{\sin \theta}{2\sqrt{2}}(1 - \cos 2\theta)(1 - \cos \omega t) \left[\cos \theta (1 - \cos \omega t) \begin{pmatrix} \cos \delta \\ -\sin \delta \end{pmatrix} - \sin \omega t \begin{pmatrix} \sin \delta \\ \cos \delta \end{pmatrix} \right], \quad (17d)$$

$$\begin{pmatrix} q_2 \\ a_2 \end{pmatrix} = \frac{1 - \cos 2\theta}{8}(1 - \cos \omega t) \left[-\{(1 - \cos 2\theta) + (3 + \cos 2\theta)\cos \omega t\} \begin{pmatrix} \cos 2\delta \\ \sin 2\delta \end{pmatrix} + 4\cos \theta \sin \omega t \begin{pmatrix} -\sin 2\delta \\ \cos 2\delta \end{pmatrix} \right], \quad (17e)$$

$$\begin{aligned} \begin{pmatrix} q_3 \\ a_3 \end{pmatrix} &= \frac{\sin \theta}{4\sqrt{2}} \cos \theta [-(5 + 3\cos 2\theta) + 4(1 + \cos 2\theta)\cos \omega t \\ &+ (1 - \cos 2\theta)\cos 2\omega t] \begin{pmatrix} \cos \delta \\ -\sin \delta \end{pmatrix} \\ &+ [2(3 + \cos 2\theta)\sin \omega t + (1 - \cos 2\theta)\sin 2\omega t] \begin{pmatrix} \sin \delta \\ \cos \delta \end{pmatrix}. \end{aligned} \quad (17f)$$

Due to the structure of the Ξ -model, there are only two independent parameters (ω, θ) . Using the earlier expressions, the IVR vectors for Ξ -model are found to be

1) First-order invariant vector ($\vec{w} = \{\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}\}$):

The angles in spherical representation are:

Colatitude angle or the angle between the IVR vector and z-axis

$$\psi_1 = \cos^{-1}(\sqrt{\omega_3}) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\sin \theta\right). \quad (18a)$$

It is time-independent.

Azimuthal angle or the angle between projection of IVR on XY-plane and the x-axis

$$\chi_1 = \tan^{-1} \sqrt{\frac{\omega_2}{\omega_1}} = \omega t \text{ (modulo } 2\pi). \quad (18b)$$

It has linear time-dependence.

2) Second-order invariant vector ($\vec{u} = \{\sqrt{\omega_1^2 + 2\omega_2\omega_3}, \sqrt{\omega_2^2 + 2\omega_3\omega_1}, \sqrt{\omega_3^2 + 2\omega_1\omega_2}\}$):

The IVR angles for $\vec{u}(\psi_2, \chi_2)$ are:

Colatitude angle or the angle between the IVR vector and Z-axis:

$$\psi_2 = \cos^{-1} \sqrt{\omega_3^2 + 2\omega_1\omega_2} , \quad (19a)$$

Azimuthal angle or the angle between projection of IVR on XY-plane and the X-axis

$$\chi_2 = \tan^{-1} \sqrt{\frac{\omega_2^2 + 2\omega_3\omega_1}{\omega_1^2 + 2\omega_2\omega_3}} . \quad (19b)$$

We calculate and plot $(\psi_2, \omega t)$ and $(\chi_2, \omega t)$ for values of $\theta = 3.0$ radians and 5.0 radians as shown in Figs. 1–4.

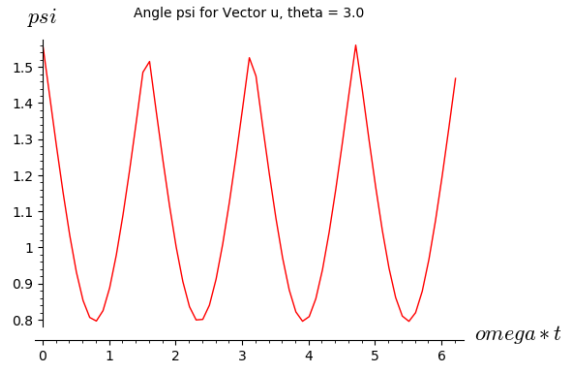


Fig. 1 Colatitude angle of the second-order invariant vector u as a function of ωt for $\theta = 3$ radians

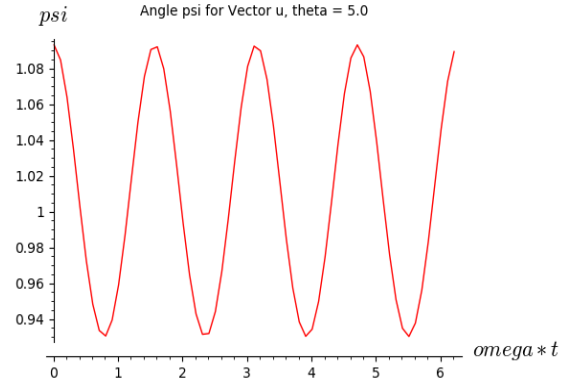


Fig. 2 Colatitude angle of the second-order invariant vector u as a function of ωt for $\theta = 5$ radians

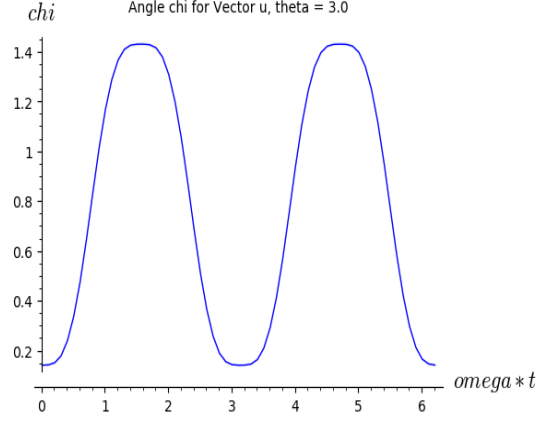


Fig. 3 Azimuthal angle of the second-order invariant vector $\mathbf{u}$ as a function of ωt for $\theta = 3$ radians

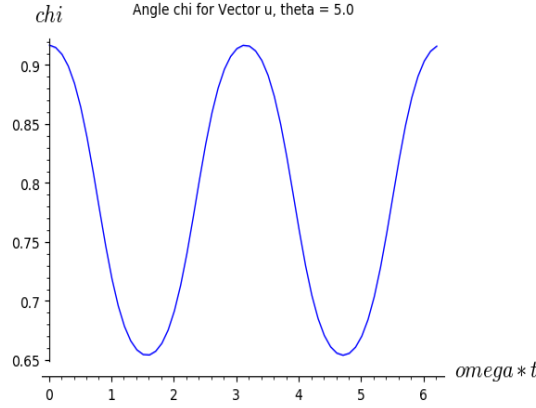


Fig. 4 Azimuthal angle of the second-order invariant vector $\mathbf{u}$ as a function of ωt for $\theta = 5$ radians

These angles show sinusoidal-like variations of the spherical angles of IVR vectors for two widely varying values of undressed energy given by $\varepsilon_1 = \omega \cos \theta$. It is conjectured that similar behavior will also be present in the $\vec{u}(\psi_2, \chi_2)$ of qutrit Λ and V models.

3) Third-order invariant vector: $\vec{v} = \left\{ \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}} \right\}$.

The third-order vector turns out to be a constant one, for the Ξ -model and angles are

$$\psi_3 = \cos^{-1} \sqrt{\frac{1}{3}} \cong 55^\circ, \quad (20a)$$

$$\chi_3 = \tan^{-1}(1) = 45^\circ. \quad (20b)$$

It is conjectured that the third invariant vector is always constant and has the angles given previously for pure qutrit state. The spherical angles associated with the other two vectors, $\vec{w}(\psi_1, \chi_1)$ and $\vec{u}(\psi_2, \chi_2)$, capture the time-dependent dynamic essence of this model.

5. Conclusion and Next Steps

The 3-D IVR vectors representing qutrit states captured the essential dynamics of the cascade or Ξ -model. The model qutrit state is pure, and so has fewer degrees of freedom. Out of three vectors, only one was found to display complex behavior. Out of the remaining two vectors, one is a constant and another has a linearly time-dependent azimuthal angle.

On the other hand, the IVR is capable of displaying the full static or dynamic behavior of a mixed qutrit state with all 8 degrees of freedom as well. In that situation, the qutrit dynamics are expressed by the behavior of three parameters each of $\vec{u}$ and $\vec{v}$ and two parameters of $\vec{w}$ as it has unit length by definition. This is a significant advance compared with the traditional approach based on Gell-Mann special unitary group of order 3 [SU(3)] matrices. The IVR will be applied in the future to study the relative performance of different qutrit models for quantum sensing.

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