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Majorana Star and Invariant Vector Representations of a Single Qutrit

by Vinod K Mishra

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14. ABSTRACT The Qutrit state density matrix is of order 3 and depends on eight parameters in the most general case. Visualization of this 8-D state space is practically impossible using 8-D vectors commonly used. Recently, a 3-D vector representation of the Qutrit state space (also called invariant vector representation [IVR]) was proposed. In this report, we present the general relationship between the Majorana star representation and IVR for the qutrit and illustrate the differences using the qutrit cascade or Ξ-model.						
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1. Introduction

Quantum sensing[1] uses quantum properties of particles and states to improve the sensitivity and accuracy of measurements beyond classical limits. Currently, qubits are used as the central resource for most of this task, as well as other tasks in quantum computing and communication. Advantages of higher dimensional objects like qutrit[2,3,4] are still under investigation. One of the obstacles faced is the difficulty in visualization of qutrit states.

The qubit density matrix is of order 2 and it depends on three parameters for the most general mixed states, but only on two parameters for pure states. It can be easily visualized using Bloch sphere representation in which the pure states are represented by points on the Bloch sphere and mixed states by the inner points. On the other hand, the qutrit density matrix is of order 3 and it depends on eight parameters in the most general case. Visualization of the 8-D qutrit state space is practically impossible with the 8-D vectors commonly used in the representation based on the Gell–Mann matrices. An alternative method for representing qutrit is the Majorana star representation (MSR) based on multi-qubit spin states.[5,6] In MSR, a qutrit is represented by two unit-length vectors on a Bloch sphere. Yet another 3-D vector representation of the qutrit state space based on density matrix invariants has been proposed.[7] It is based on spin-1 representation matrices[8,9,10] and density matrix invariants. These vectors also reside on the surface of a sphere, but it is not a Bloch sphere. In this report, we compare and contrast the two representations for pure states and find the general transformation between them

2. Qutrit MSR and its Density Matrix

The qutrit is represented by two unit vectors in MSR. Let us choose the two points $P_1(\theta_1, \phi_1)$ and $P_2(\theta_2, \phi_2)$ or equivalently $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ on the Bloch sphere centered at point O. Then the Cartesian and spherical components of these Majorana stars for qutrit are given as

$$\overrightarrow{OP_1} = (x_1, y_1, z_1) = (sin\theta_1 cos\phi_1, sin\theta_1 sin\phi_1, cos\theta_1) \ . \tag{1a}$$

and $$\overrightarrow{OP_2} = (x_2, y_2, z_2) = (sin\theta_2 cos\phi_2, sin\theta_2 sin\phi_2, cos\theta_2) \ . \tag{1b}$$

Let the general qutrit state be given as

$$\begin{pmatrix} C_1 \\ C_0 \\ C_{-1} \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{2}$$

Then the qutrit state coefficients are related by Majorana's star equation specialized to a qutrit.

$$\frac{C_1}{\sqrt{2}}\varsigma^2 - C_0\varsigma + \frac{C_{-1}}{\sqrt{2}} = 0. \tag{3}$$

The solutions of this quadratic equations are

$$\begin{pmatrix} \varsigma_1 \\ \varsigma_2 \end{pmatrix} = \frac{1}{C_1\sqrt{2}}\left[C_0 \begin{pmatrix} + \\ - \end{pmatrix} \sqrt{C_0^2 - 2C_1C_{-1}}\right]. \tag{4}$$

These solutions are related to the starting Majorana star vectors through the following steps.

Step 1: The inverse stereographic projections of $P_1(\theta_1, \phi_1)$ and $P_2(\theta_2, \phi_2)$ give the two points that lie on the equatorial plane of the Bloch sphere. Their Cartesian components are related to those of the stars.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} x_1/(1+z_1) \\ x_2/(1+z_2) \end{pmatrix}. \tag{5a}$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} y_1/(1+z_1) \\ y_2/(1+z_2) \end{pmatrix}. \tag{5b}$$

Step 2: The coordinates are combined to give complex quantities, which are solutions to the Majorana star equation.

$$\begin{pmatrix} \varsigma_1 \\ \varsigma_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \end{pmatrix} = \begin{pmatrix} \exp(i\phi_1)\tan\left(\frac{\theta_1}{2}\right) \\ \exp(i\phi_2)\tan\left(\frac{\theta_2}{2}\right) \end{pmatrix}. \tag{6}$$

Then the pure qutrit state is given by

$$\begin{pmatrix} C_1 \\ C_0 \\ C_{-1} \end{pmatrix} = \frac{1}{N}\begin{pmatrix} 1 \\ (\varsigma_1 + \varsigma_2)/\sqrt{2} \\ \varsigma_1\varsigma_2 \end{pmatrix}, \tag{7a}$$

with normalization given by

$$N = \left(\frac{(1+cos\theta_1)(1+cos\theta_2)}{3+4cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)+sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\cos(\phi_1-\phi_2)}\right)^{1/2}. \tag{7b}$$

The MSR qutrit density matrix then becomes

$$\rho_{MSR} = \begin{pmatrix} C_1 \\ C_0 \\ C_{-1} \end{pmatrix}\begin{pmatrix} \overline{C_1} & \overline{C_0} & \overline{C_{-1}} \end{pmatrix} =$$

$$\frac{1}{N^2}\begin{pmatrix} 1 & (\overline{\varsigma_1} + \overline{\varsigma_2})/\sqrt{2} & \overline{\varsigma_1}\overline{\varsigma_2} \\ (\varsigma_1 + \varsigma_2)/\sqrt{2} & |\varsigma_1 + \varsigma_2|^2/2 & \overline{\varsigma_1}\overline{\varsigma_2}(\varsigma_1 + \varsigma_2)/\sqrt{2} \\ \varsigma_1\varsigma_2 & \varsigma_1\varsigma_2(\overline{\varsigma_1} + \overline{\varsigma_2})/\sqrt{2} & |\varsigma_1\varsigma_2|^2 \end{pmatrix}. \tag{8}$$

3. Invariant Vector Representation (IVR) Density Matrix of a Qutrit

The density matrix ρ based on the spin-1 representation of a Qutrit is given as

$$\rho_{IVR} = \begin{bmatrix} \omega_1 & \frac{1}{2}(q_3 + ia_3) & \frac{1}{2}(q_2 - ia_2) \\ \frac{1}{2}(q_3 - ia_3) & \omega_2 & -\frac{1}{2}(q_1 + ia_1) \\ \frac{1}{2}(q_2 + ia_2) & -\frac{1}{2}(q_1 - ia_1) & \omega_3 \end{bmatrix} . \tag{9}$$

The parameters of the density matrix in the invariant vector representation (IVR) or ρ_{IVR} are related to the expectation values of expressions involving spin-1 components and their combinations.

$$\omega_i =< S_i^2 >= Tr(\rho S_i^2) . \tag{10a}$$

$$a_i =< S_i >= Tr(\rho S_i) . \tag{10b}$$

$$q_k =< S_i S_j + S_j S_i >= Tr\{\rho(S_i S_j + S_j S_i)\}, k \neq i, j . \tag{10c}$$

The qutrit in IVR is represented by three vectors and they are given in Table 1.

Table 1 IVR vectors for qutrit

IVR vectors	Cartesian components	Basis trace relations	Comments
$\vec{w}$	$\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}$	$\sum_{i=1}^{3} w_i^2 = Tr(\rho_\Xi) = \omega_1 + \omega_2 + \omega_3 = 1$	Follows from unit trace relation of density matrix
$\vec{u}$	$\sqrt{\omega_1^2 + (q_1^2 + a_1^2)/2}$, $\sqrt{\omega_2^2 + (q_2^2 + a_2^2)/2}$, $\sqrt{\omega_3^2 + (q_3^2 + a_3^2)/2}$	$Tr(\rho_\Xi^2) = \sum_{i=1}^{3} u_i^2 \leq 1$	Trace idempotence relation for pure states
$\vec{v}$	$\sqrt{X + 3(q_1^2 + a_1^2)/2}$, $\sqrt{X + 3(q_2^2 + a_2^2)/2}$, $\sqrt{X + 3(q_3^2 + a_3^2)/2}$	$3Tr(\rho_\Xi^2) - 2Tr(\rho_\Xi^3) = \sum_{i=1}^{3} v_i^2 \leq 1$	Here $X = \frac{1}{3} - 2\omega_1\omega_2\omega_3 - \frac{1}{2}(a_2 a_3 q_1 + a_3 a_1 q_2 + a_1 a_2 q_3 - q_1 q_2 q_3)$

The vectors $\vec{u}$ and $\vec{v}$ represent the second and third density matrix invariants of a qutrit. The bounds on the vector norms are, in general, $\sum_{i=1}^{3} u_i^2 \leq 1$ and $\sum_{i=1}^{3} v_i^2 \leq 1$, with equality signs holding for a pure state. For a pure state, the eight density matrix parameters are related via the following four relations:

1) $q_1^2 + a_1^2 = 4\omega_2\omega_3$, 2) $q_2^2 + a_2^2 = 4\omega_3\omega_1$, 3) $q_3^2 + a_3^2 = 4\omega_1\omega_2$, and 4) $a_2a_3q_1 + a_3a_1q_2 + a_1a_2q_3 - q_1q_2q_3 = 8\omega_1\omega_2\omega_3$. So finally we have only four independent parameters, which is the correct number of independent degrees of freedom for a pure qutrit state.

4. Relation between the MSR and IVR of a Qutrit

We equate the two density matrices representing the same qutrit entity.

$$\rho_{IVR} = \rho_{MSR}. \tag{11}$$

Then the density matrix elements of the MSR and IVR are related by

$$\omega_1 = \frac{1}{N^2}, \tag{12a}$$

$$\omega_2 = \frac{1}{2N^2}[(\alpha_1 + \alpha_2)^2 + (\beta_1 + \beta_2)^2], \tag{12b}$$

$$\omega_3 = \frac{1}{N^2}(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2) \tag{12c}$$

$$\begin{pmatrix} q_1 \\ a_1 \end{pmatrix} = -\frac{\sqrt{2}}{N^2}\begin{bmatrix} \alpha_1(\alpha_2^2 + \beta_2^2) + \alpha_2(\alpha_1^2 + \beta_1^2) \\ \beta_1(\alpha_2^2 + \beta_2^2) + \beta_2(\alpha_1^2 + \beta_1^2) \end{bmatrix}, \tag{13}$$

$$\begin{pmatrix} q_2 \\ a_2 \end{pmatrix} = \frac{2}{N^2}\begin{pmatrix} \alpha_1\alpha_2 - \beta_1\beta_2 \\ \alpha_1\beta_2 + \alpha_2\beta_1 \end{pmatrix}, \tag{14}$$

and

$$\begin{pmatrix} q_3 \\ -a_3 \end{pmatrix} = \frac{\sqrt{2}}{N^2}\begin{pmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \end{pmatrix}. \tag{15}$$

The invariant vectors have the following expressions.

1) $$\vec{w} = \{\sqrt{\omega_1}, \quad \sqrt{\omega_2}, \quad \sqrt{\omega_3}\} = (sin\psi_1 cos\chi_1, sin\psi_1 sin\chi_1, cos\psi_1)\ . \tag{16}$$

We use the pure qutrit relations given earlier to get the following forms for the other two vectors.

2) $$\vec{u} = \{\sqrt{\omega_1^2 + 2\omega_2\omega_3}, \quad \sqrt{\omega_2^2 + 2\omega_3\omega_1}, \quad \sqrt{\omega_3^2 + 2\omega_1\omega_2}\} = (sin\psi_2 cos\chi_2, sin\psi_2 sin\chi_2, cos\psi_2)\ . \tag{17}$$

3) $$\vec{v} = \left\{\sqrt{\frac{1}{3}}, \quad \sqrt{\frac{1}{3}}, \quad \sqrt{\frac{1}{3}}\right\} = (sin\psi_3 cos\chi_3, sin\psi_3 sin\chi_3, cos\psi_3)\ . \tag{18}$$

The vector $\vec{v}$ reduces to a constant vector for a pure state. This form of third order invariant vector is conjectured to be an indicator of a pure qutrit state.

5. MSR and IVR Vectors of the Qutrit Cascade Model

One can study the relationship between the $(\theta_1, \phi_1, \theta_2, \phi_2)$ and $(\psi_1, \chi_1, \psi_i, \chi_2)$ sets in many ways to map the MSR vectors into IVR ones and vice versa. The following plots given show the relationship between the angles for a particular case in which IVR angles are calculated for the following MSR case:

1) The MSR $\overrightarrow{OP_1}$ vector is fixed (θ_1=1 rad, ϕ_1=1 rad).

2) The azimuth angle of the MSR $\overrightarrow{OP_2}$ vector is fixed (ϕ_2=4 rad).

Figures 1–4 show the behavior of the colatitude angles (ψ_1, ψ_2) and azimuth angles (χ_1, χ_2) of the IVR vectors $\vec{w}$ and $\vec{u}$, respectively.

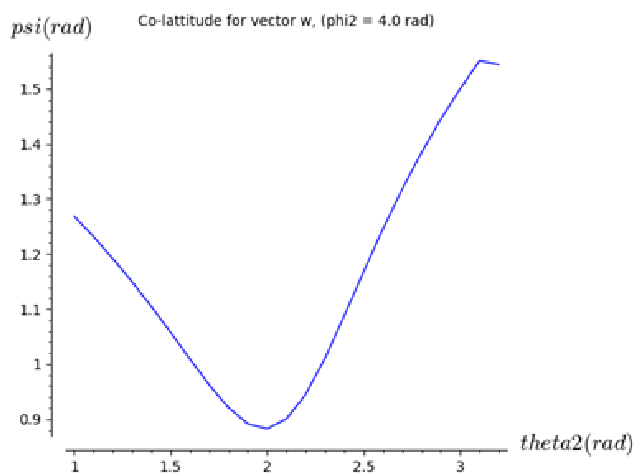

Fig. 1 **Colatitude angle of the first-order invariant vector $\vec{w}$ as a function of MSR colatitude angle θ_2 of $\overrightarrow{OP_2}$ ($\phi_2$is fixed)**

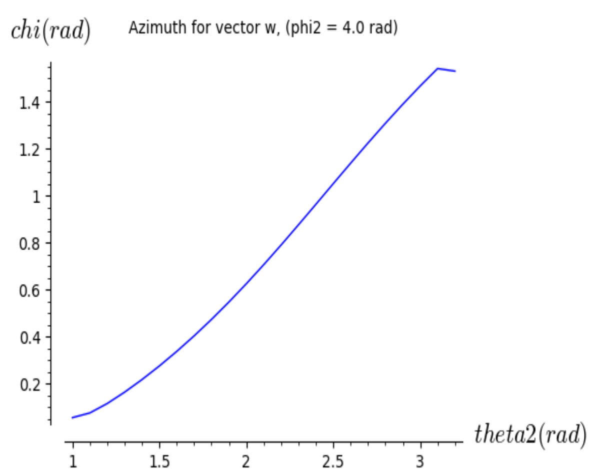

Fig. 2 **Azimuth angle of the first-order invariant vector $\vec{w}$ as a function of MSR colatitude angle θ_2 of $\overrightarrow{OP_2}$ ($\phi_2$is fixed)**

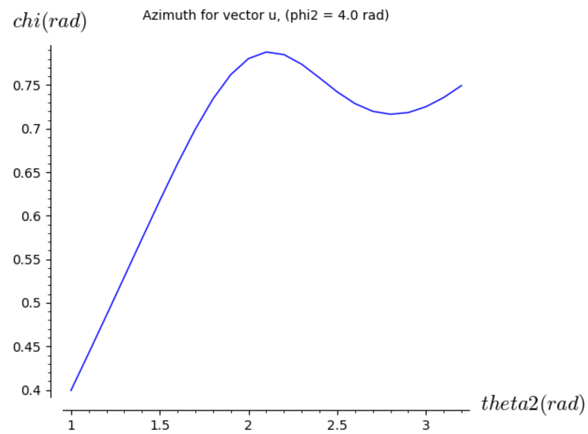

Fig. 3 Colatitude angle of the second-order invariant vector $\vec{u}$ as a function of MSR colatitude angle θ_2 of $\overrightarrow{OP_2}$ ($\phi_2$is fixed)

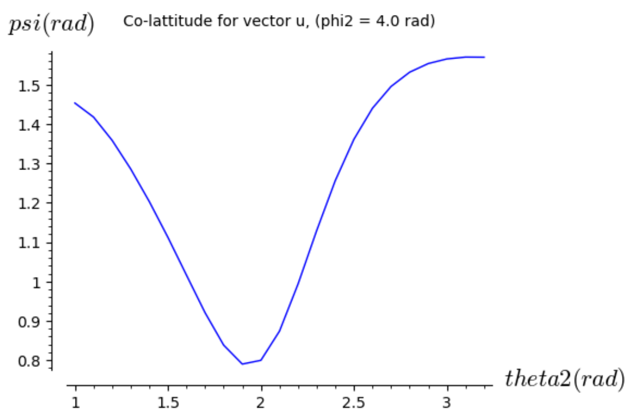

Fig. 4 Azimuth angle of the second-order invariant vector $\vec{u}$ as a function of MSR colatitude angle θ_2 of $\overrightarrow{OP_2}$ ($\phi_2$is fixed)

It is seen that the two vector sets are related nonlinearly, which was expected given the complicated relations between them in the two representations. In practice, it is easier to find the IVR from the given MSR and inverse relations are much harder to calculate.

6. Conclusion and Next Steps

One can see that given the angular variables of Majorana stars for a qutrit, it is very straightforward to calculate the corresponding IVR angles and vice versa. They are related in a complicated manner as seen by the plots given earlier.

The IVR representation is more versatile as it can also represent mixed qutrit states that cannot be represented by Majorana stars. Recently, MSR was extended to mixed states so IVR may be connected to this extended MSR.

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