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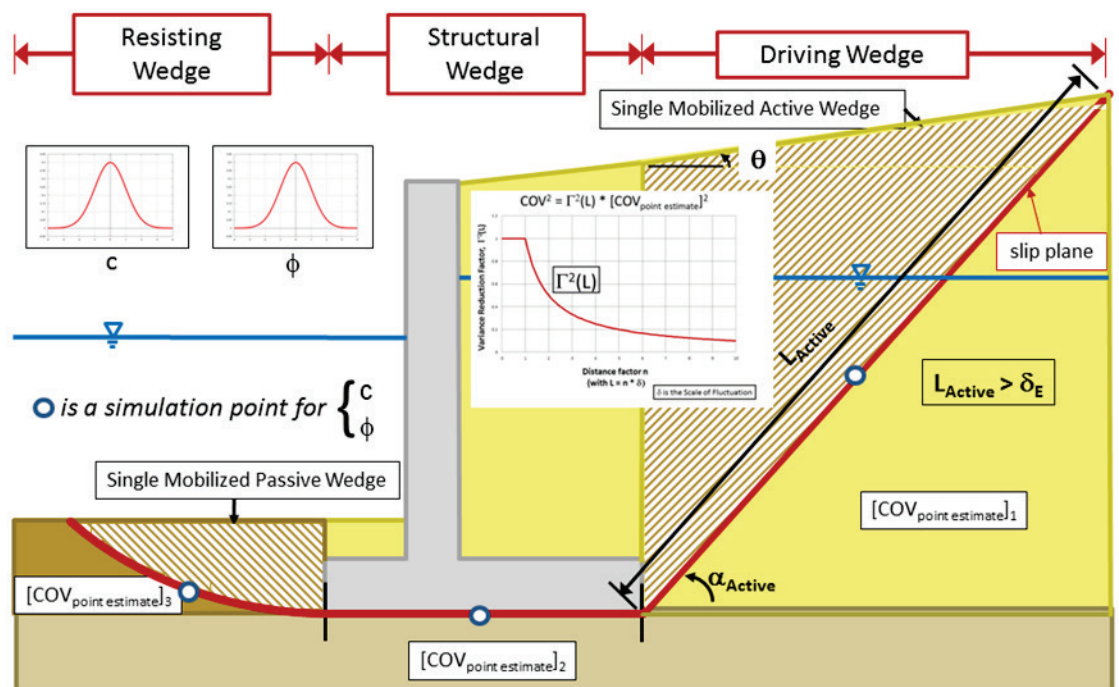
Navigation Systems

Load Factors for Earth Retaining, Reinforced Concrete Hydraulic Structures Based on a Reliability Index (β) Derived from the Probability of Unsatisfactory Performance (PUP)

Phase 1 Study

Robert M. Ebeling and Barry C. White

June 2019



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Phase 1 Study

Robert M. Ebeling and Barry C. White

Information Technology Laboratory

U.S. Army Engineer Research and Development Center

3909 Halls Ferry Road

Vicksburg, MS 39180-6199

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Abstract

This technical report documents the first of a two-phase research and development (R&D) study in support of the development of a combined Load and Resistance Factor Design (LRFD) methodology that accommodates geotechnical as well as structural design limit states for design of the U.S. Army Corps of Engineers (USACE) reinforced concrete, hydraulic navigation structures. To this end, this R&D effort extends reliability procedures that have been developed for other non-USACE structural systems to encompass USACE hydraulic structures. Many of these reinforced concrete, hydraulic structures are founded on and/or retain earth or are buttressed by an earthen feature. Consequently, the design of many of these hydraulic structures involves significant soil structure interaction. Development of the required reliability procedures has been lagging in the geotechnical topic area as compared to those for structural limit state considerations and have therefore been the focus of this first-phase R&D effort. Design of these hydraulic structures involves consideration of up to six geotechnical and structural limit states and a seventh performance-related limit state. Reliability calculations given multiple limit states are facilitated by new numerical procedures with enough precision for a complete LRFD analysis of these reinforced concrete, hydraulic structures.

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Preface

This study and report was authorized by Headquarters, U.S. Army Corps of Engineers (HQUSACE), under Work Unit 675GGJ, “Navigation LRFD,” of the Navigations Systems Research Program. Mr. Charles E. Wiggins, USACE Coastal and Hydraulics Laboratory (CHL), was the Program Manager for the Navigation Systems Research Program during the time in which the majority of this research was conducted. From October 2017 through April 2018, Dr. Duncan B. Bryant, CHL, was acting Program Manager; from January through June of 2017, Dr. Patricia K. DiJoseph, CHL, was Acting Program Manager. At the time of preparation of this report, Mr. Kareem S. El-Naggar was the HQUSACE Navigation Business Line Manager.

The research was performed by the Computational Science and Engineering Division, U.S. Army Engineer Research and Development Center – Information Technology Laboratory (ERDC-ITL). At the time of publication, Dr. Jerrell R. Ballard was, Chief. Mr. W. Jeff Lillycrop (ERDC-CHL) (retired) was the Navigation Technical Director until December 2018. Mr. Charles E. Wiggins has been Navigation Technical Director since January 2019 and during the time of publication of this report. The Deputy Director of ERDC-ITL was Ms. Patti S. Duett, and the Director was Dr. David A. Horner.

COL Ivan P. Beckman was the Commander of ERDC, and Dr. David W. Pittman was the Director.

1 Introduction

1.1 Background

The U.S. Army Corps of Engineers (USACE) has been relying on an allowable stress design (ASD) methodology for design of the various structural features comprising its hydraulic navigation structures. In ASD, also referred to as working stress design, factors of safety (FoS) are implicit through the use of allowable stresses during the structural design process. Allowable stresses are specified for use in the various design limit states for the USACE three categories of design load cases of Usual, Unusual, and Extreme.¹ In the USACE design guidance for geotechnical-dominated limit states and engineering issues, the use of FoS against Ultimate Limit States (ULS) is more straightforward than in the structural design guidance. The appropriate value for the FoS is explicitly stated in this portion of the USACE guidance.

The USACE is currently transitioning to a structural design methodology based on Load and Resistance Factor Design (LRFD). The advantage for LRFD is that it formally takes into consideration the variability in the loads and the resistances separately. LRFD has become an accepted mode of implementation of probability-based limit state design in structural steel and reinforced concrete design. Three of the most recently developed and prominent examples of the LRFD-based building and bridge design criteria are

- American Society of Civil Engineers (2010), ASCE/SEI 7-10 Minimum Design Loads for Buildings and Other Structures
- Canadian Standards Association (2014), Canadian Highway Bridge Design Code, CAN/CSA-S6-14
- American Association of State Highway and Transportation Officials (2010) AASHTO LRFD Bridge Design Specifications.

Development of these three LRFD-based design criteria has been led by the structural engineering community within the civil engineering profession. In recent years, researchers have been proposing means for

¹ One definition of a limit state is “a structural condition beyond which it no longer fulfills the specified design criteria.”

extending LRFD into geotechnical ULS designs. But to date, there are no LRFD-based geotechnical design criteria.

Many of the USACE hydraulic structures are founded on and/or retain earth or are buttressed by an earthen feature. Consequently, the design of many of the USACE hydraulic structures involve significant soil structure interaction (SSI). Generally speaking, SSI is a specialized topic associated with foundation features for the structure and with a structure's ability to retain earth and/or to be buttressed by an earthen mass. Currently, no LRFD methodology exists for SSI analysis of the USACE-type of hydraulic structures.

1.2 Objective

This technical report discusses research in support of the development of a combined LRFD methodology that accommodates structural as well as geotechnical design limit states (especially the ULSs) as used in the design of the USACE hydraulic navigation structures. To this end, this research and development (R&D) effort intends to extend reliability procedures that have been developed for other non-USACE structural systems to encompass USACE hydraulic structures. Additionally, procedural methods for accurate estimation of reliability are assessed for application to the to-be-developed combined LRFD methodology.

1.3 Approach

The authors of this technical report have gathered applicable reliability methodologies and supporting data from the civil engineering LRFD technical community. These methodologies and data come from many structural and geotechnical fields and are interpreted for ultimate use in a combined LRFD application to the USACE hydraulic structures. Numerical procedures to facilitate accurate reliability calculations are also devised to accommodate reliability analysis of the USACE hydraulic structures multiple limit states.

1.4 Report contents

- Chapter 2 discusses reliability analysis as it relates to LRFD of the USACE hydraulic structures. An introduction to considerations for reliability analysis with multiple limit states is included.

- Chapter 3 discusses considerations for reliability analysis for geotechnical states. This includes an introduction into the important concepts of the scale of fluctuation and variance reduction due to spatial averaging.
- Chapter 4 discusses a structural-system's Probability of Unsatisfactory Performance (PUP_{system}) for a hydraulic structure. The three stages to load factor development are introduced. Additionally, three reliability methodologies devised to compute a systems PUP value are summarized.
- Chapter 5 summarizes three engineering procedures that may be used to compute the resultant earth pressure forces imposed on the structural wedge, which contains the hydraulic structure within the three reliability models of Chapter 4. Each of these three engineering computational approaches are used to compute the forces imposed on the structural wedge by the driving wedge and the resisting wedge.
- Chapter 6 summarizes the moment and shear capacity relationships for use in these two structural limit states for reinforced concrete structures such as those used in retaining walls (e.g., T-Walls).
- Chapter 7 summarizes the R&D findings discussed in this report, conclusions to be made from the research conducted to date, and the required next research and development tasks.
- Appendix A discusses individual and multiple Limit States and the computation of the PUP using numerical procedures.
- Observations regarding the Sleep and Duncan (2014) procedure for geotechnical engineering reliability calculations are contained in Appendix B.
- Appendix C discusses the determination of geotechnical soil property distributions/statistics for sparse data.
- Values for Coefficients of Variation (COV) for a variety of different soil types are collected and tabulated in Appendix D, along with typical vertical and horizontal scales of fluctuation.
- Methodologies for dealing with anisotropic scales of fluctuation in sliding stability analyses are discussed in Appendix E.
- Appendix F gives the derivation for an equivalent unit weight of soil for use in a single wedge solution scheme formulation for modeling the driving and the resisting wedges for the potential slip plane.
- Appendix G describes the line of seepage (a.k.a., line of creep) procedure to construct a one-dimensional (1-D) seepage path about a structural wedge that is affected by differing head pressures on either side of the structural wedge and the soil material region hydraulic

conductivity. The results from this rapidly computed, simplified method of analysis are used for determining seepage gradient values along the line of seepage around the perimeter of the structural wedge. These gradients are used in the computation of an equivalent unit weight of a partially saturated or fully saturated soil with seepage for a single wedge analysis of a homogeneous retained soil.

- Appendix H describes a numerical method used for calibrating partial load and resistance safety factors given a target β value. The loads and resistances are modeled with distributions with at most two unknown means. Typically, the resistance variable accounts for one unknown mean with the other unknown mean corresponding to one of the load terms. Load ratios set the relationships of the mean values of the defined loads to the load with the unknown mean value. With these relationships, the ratio of the unknown mean value to the unknown resistance value allows for the definition of the load and resistance factors for the given target β . A supplemental procedure is derived for the calculation of the partial load safety factor given a specified partial resistance safety factor and the procedurally defined partial safety factor ratio. This supplemental procedure is valid for the condition where a single load and resistance variable are used. A second numerical procedure to calculate β given variable load and resistance with distributions, which may have partial load and resistance safety factors applied is also presented. This second numerical procedure can be used as a verification step for the results from the first numerical procedure. An example for each of these procedures is included. The examples are taken from a 2016 USACE LRFD¹ study. Important differences between the results from the Appendix H numerical procedure and the 2016 unpublished LRFD study are explained.

¹ Kent D. Hokens. 2016 (Oct). Unpublished. *USACE Load Factor Development for Design of Hydraulic Concrete Structures*. St. Paul District.

2 Reliability Analysis as It Relates to Load and Resistance Factor Design (LRFD) of the USACE Hydraulic Structures

2.1 LRFD

The equations governing homogeneous soil regions are introduced in this section. The first subsection introduces a general form of the LRFD equation while the second describes the interrelationship between nominal and mean loads and resistance and introduces a bias factor.

2.1.1 A general form of LRFD

The general form of a single ULS (Fenton et al. 2016) in the context of an LRFD design requires that the resistance (i.e., the capacity) be greater than or equal to the load:

$$\phi * R_n \geq \sum_{i=1}^j \eta_i * \gamma_i * L_{ni} \quad (2.1)$$

where:

- ϕ = is the resistance factor to account for variability in the resistance
- R_n = is the nominal resistance (Fenton, et al. (2016) refer to this as the characteristic resistance, $\hat{R}$)¹
- η_i = is a load modifier factor to account for load combinations, the effects of ductility, redundancy, and operational importance (used in some types of LRFD formulations)
- γ_i = is a load factor that accounts for variability in each load i , among the j loads, to which it is assigned
- L_{ni} = is the nominal load i of j different loads (Fenton et al. [2016] refer to this as the characteristic load, $\hat{L}_{ni}$).

¹ Fenton et al. (2016) prefers to use the term *characteristic* rather than *nominal* because it suggests a value that characterizes a design parameter that is uncertain (e.g., a random resistance or load). They believe that the word nominal (or design) does not convey the underlying randomness of the design parameter.

Safety is assured if the factored resistance is at least equal to the factored load. The Equation 2.1 design factors are intended to cover uncertainties or to compensate for the lack of complete information.

Observe in Equation 2.1 that separate factors of ϕ and γ are defined for resistance and loading, respectively, and an equation of this form is unique to each ULS. These partial factors ϕ and γ are individually related to the variability of the resistance and load quantities, respectively, that they are factoring and are used to scale the nominal (or characteristic) design values to more conservative values so that the probability of a particular ULS occurring is acceptably small and within a magnitude acceptable to the USACE. The *PUP* value for a ULS may be equivalently expressed in terms of a reliability index (β) value. Determining the acceptable *PUP* and β values is one of the research topics discussed in this report. Section 2.9 reviews a limited amount of published values for β and/or *PUP* that are associated with select designs or calculated for constructed, non-USACE project systems. The necessary design factor procedure is to be developed for the USACE to obtain designs that achieve a prescribed level of reliability. Looking forward, the level of a hydraulic structural systems reliability will be indicated by values for *PUP* and β in this report.

The authors of this ERDC technical report recognize that multiple ULSs are considered during the design of the USACE hydraulic structures. When considered alone, each ULS has a unique *PUP* and β value. The important ULSs will be identified in Section 2.2. A structural system *PUP*_{system} for the hydraulic structure can be computed from the conjoined ULSs. The results from research by the authors of this report into this topic are discussed in Chapter 4 and Appendix A.

The load and resistance factors (ϕ and γ) are determined using statistics, reliability concepts, and a pre-selected value for probability of failure or, equivalently, β . These relationships are summarized in Section 2.4 for a single ULS. Generally, a factor of unity (1) or less is assigned to ϕ that is applied to the resistances of the material (including soil resistance), and a factor of unity or greater is assigned to γ that is applied to the nominal loads L_{ni} , loads $i = 1$ through j . An example taken from the structural discipline of LRFD design for L_i with i equal to 1 and to 2 would be dead and live, respectively, nominal loads.

When considering a single, displacement-based, Serviceability Limit State (SLS) in foundation design for routine, non-USACE structures, Baecher and Christian (2003) recognize that the LRFD methodology requires that the tolerable (nominal) displacement, δ_{tn} , be greater than or equal to the estimated (nominal) displacement, δ_{ni} :

$$\phi * \delta_{tn} \geq \sum_{i=1}^j \eta_i * \gamma_i * \delta_{ni} \quad (2.2)$$

where:

- ϕ = is the resistance factor to account for variability in the tolerable displacement (i.e., a *resistance* factor)
- δ_{tn} = is the tolerable displacement (viewed as a nominal resistance)
- η_i = is a load induced, displacement modifier factor to account for load combinations, the effects of ductility, redundancy, and/or operational importance (used in some types of LRFD formulations)
- γ_i = is a factor that accounts for variability in each displacement induced by nominal load i , among the j loads, to which it is assigned (i.e., a *load factor*)
- δ_{ni} = is the estimated displacement resulting from load i of j of the different nominal loads.

An example of what Baecher and Christian (2003) refer to as a routine design would be the foundation design for a low-rise building system. Safety is assured if the factored tolerable displacement is at least equal to the sum of displacements for all of the nominal loads.

2.1.2 An interrelationship between nominal and mean loads, resistance and bias factors

The calculated nominal resistance R_n can differ from the mean value of measured resistance. This is particularly true for structural systems in contact with the earth that involve SSI-type of engineering computations. A good example of this is in the case of axially loaded pile foundations, as discussed in Federal Highway Administration (FHWA 2001). To account for this discrepancy between measured resistance R_m and nominal resistance R_n , a bias factor is introduced:

$$R_m = \lambda_R * R_n \quad (2.3)$$

where λ_R represents the resistance bias factor. The bias factor may also be viewed as a correction factor. In the FHWA (2001) axially loaded driven pile foundation example, λ_R was determined as the ratio of the axial pile load capacities measured in the field (i.e., R_m) divided by the axial ULS.

$$\lambda_R = \frac{R_m}{R_n} \quad (2.4)$$

The ULS may be computed using either empirical correlations combined with traditional geotechnical field exploration test data (R_n) or computer programs and geotechnical soil strength data.¹

FHWA (2001) provides a computational method to compute λ_R . One value for λ_R was generated for each pile load test location. The Standard Penetration test (SPT) data were recovered for various soil stratum in one soil boring location near to the select pile load test and was converted into a (calculated) axial pile capacity estimate for R_n using Meyerhoff's (1976) procedure, an established geotechnical engineering correlation.² For each pile load test location, an R_m value was measured, and a corresponding R_n calculation was made using the SPT data recovered from the soil boring made close to the specified pile load test (Table 3-2 in FHWA [2001]). Thus, one λ_R value was computed for each pile load test location using the pairs of R_m and R_n data using Equation 2.4. Statistical processing of the collection of λ_R values was then performed generating mean and standard deviations for the bias factor λ_R based for each pile load test location. For the FHWA (2001) example cited, 24 pile load tests were used in the evaluation of λ_R . The mean (μ), standard deviation (σ), and COV of the ratio of the measured to predicted capacities are 1.22, 0.66, and 54%, respectively. Because the mean λ_R value equals 1.22, this implies that the Meyerhoff's method tends to underpredict, on average, the pile resistance. In summary, there is a difference between what is predicted and what is measured, and the value for λ_R is used to quantify this difference.

¹ Pile (ultimate) axial capacity calculations for R_n may be made using the USACE Computer-Aided Structural Engineering (CASE) computer program CAXPILE (Dawkins 1984) combined with soil strength data. Pile load tests conducted at the site result in a different and more accurate ultimate pile capacity value, designated R_m . FHWA's (2001) logic to apply a bias factor λ_R in their Reliability methodology is an ingenious way to devise a "correction factor" to be applied to the predicted pile capacity calculated using site-specific soil strength data and the geotechnical procedure and/or software (e.g., CAXPILE) that computes ultimate pile capacity.

² The SPT is a common in situ testing method used to determine the geotechnical engineering properties of subsurface soils. It is a simple and inexpensive test to estimate (through correlations) the approximate shear strength parameters for soils as well as relative density of cohesionless soils.

When performing an example reliability analysis of a pile foundation, FHWA (2001) expanded on Equation 2.4 to become

$$\bar{R} = \lambda_R * R_n \quad (2.5)$$

where $\bar{R}$ is the mean value of resistance and λ_R is the resistance bias factor. Similarly, the i^{th} mean load $\bar{L}_i$ is given in FHWA (2001) to be

$$\bar{L}_i = \lambda_{Li} * L_{ni} \quad (2.6)$$

Rearranging, Equations 2.5 and 2.6 become

$$R_n = \frac{\bar{R}}{\lambda_R} \quad (2.7)$$

and

$$L_{ni} = \frac{\bar{L}_i}{\lambda_{Li}} \quad (2.8)$$

Introducing Equations 2.6 and 2.7, the generalized LRFD Equation 2.1 is transformed to

$$\phi * \frac{\bar{R}}{\lambda_R} \geq \sum_{i=1}^j \eta_i * \gamma_i * \frac{\bar{L}_i}{\lambda_{Li}} \quad (2.9)$$

Equation 2.9 allows for the introduction of resistance and/or load bias in an LRFD analysis. When no bias is present, the bias factors may be set equal to unity. In this case and by Equations 2.5 and 2.6, Equation 2.9 reverts to Equation 2.1 with $\bar{R} = R_n$ and $\bar{L}_i = L_{ni}$, respectively.

The FHWA (2001) example discussed above, in the context of the introduction of Equation 2.4 into this report, demonstrates how introduction of a bias factor λ_R into the reliability formulation (Equation 2.9) allows for the correction of limit state model(s). Numerous reliability formulations proposed and/or used in the different civil engineering disciplines (e.g., structural, geotechnical, etc.) make use of bias factors when formulating and conducting reliability analyses.

There has been significant research and proposed code developmental activity across the civil engineering community, particularly within the last

25 years, that is resulting in a variety of proposed reliability-based LRFD methodologies for specific structural systems and/or types of structures. Some of the proposed methodologies have been or are being adopted by organizations that provide structure specific design guidance. In general, the LRFD methodologies are not *one size fits all* but are tailored to the system type and to the historical design practice of the agency overseeing the design methodology for the specific structural system type of interest. Fenton et al. (2016) observes that some non-USACE structural system design codes and Civil Engineer researchers developing LRFD-based design procedures specify that the nominal (or characteristic) load is equal to the mean; others suggest using a cautious estimate of the mean while others specify the use of an upper (or lower) fractile (whichever yields the most conservative result). Their second, companion observation is that the nominal (or characteristic) resistance may be computed using mean strength parameters or using fractiles of the strength parameters. In summary, the bias factor captures the difference between the nominal design value and the mean for both resistance and load terms.

2.2 Limit states for hydraulic navigation structures

In general, there are seven categories of ULSs considered in the design of the USACE hydraulic structures. They are the following:

- Flexural hinging of individual structural members
- Shear failure of individual structural members
- Buckling failure of individual structural members subjected to axial compressive loading
- Excessive displacement of the structural system in its entirety
- Global sliding failure of the structural system in its entirety
- Bearing failure of the foundation to the structural system in its entirety
- Overturning or rotational failure of the structural system about its foundation in its entirety.

The first three are evaluated on a structural member-by-member basis. The latter three are commonly associated with geotechnical aspects of the application of reliability theory to a project feature's design. This report focuses on assessing the latter three limit states for the USACE hydraulic structures because reliability-based LRFD design methodologies are not as advanced on geotechnical related issues as they are on structural related issues.

2.3 Reliability analysis

Reliability studies are performed to determine the *PUP* and β values of a structural system for a limit state or limit states affecting system performance. The reliability of the system is defined as the complement of the probability of unsatisfactory performance ($1-PUP$) and is the probability of proper or satisfactory performance during which the structure behaves safely.

$$Reliability = 1 - PUP \quad (2.10)$$

Thus, the level of a hydraulic structural systems' reliability is indicated by its *PUP* value as well as its corresponding β .

One of the major strengths of the reliability approach, according to Christian et al. (1994), is that design engineers can use statistical-based estimates (of means or expected values) and variances (equal to the square of the standard deviation, σ^2) of engineering parameters (e.g., soil strength) and design based on these uncertainties through a reliability analysis. This leads to a more efficient system design than merely assigning conservative values of the parameters. Generally speaking, the structural community has advanced the application of reliability theory in design methodologies for steel, reinforced concrete, and timber more rapidly than has the geotechnical community for soil foundations and retained soils. One explanation for this was made by Fenton et al. (2016) with their observation that the ground is by far the most variable and hence uncertain of engineering materials. Nevertheless, the geotechnical profession has made progress and advanced to the stage whereby geotechnical design codes are appearing (Fenton et al. 2016). One example would be the Canadian Highway Bridge Design Code, published in 2014 by the Canadian Standards Association. Reliability theory serves as a basis for the quantifiable characterization of load and resistance factors in a LRFD analysis of Canadian bridges. In this document, example values for load and resistance factors for geotechnical ULS are provided for use in the design of Canadian bridge foundations. These load and resistance factors are based on target theoretical maximum lifetime (equal to 75 years for bridges) failure probabilities and equivalent values of β . Using the information provided by Fenton et al. (2016), the authors of this ERDC report made a careful assessment of the logic behind this Canadian bridge foundation design procedure and of the details associated with its development. The assessment was made from the viewpoint of its

application to combined LRFD based design of the USACE hydraulic structures. This assessment is summarized in the following paragraphs.

Minimum service life of 100 years: Note that in ER 1110-2-8159 (HQUSACE 1997), USACE policy requires by regulation that engineering design decisions should be made using life-cycle design as a basis for selection of all project elements such as materials, structural systems, mechanical equipment, and scour protection on all projects. These design decisions will be consistent throughout all project phases, including value engineering studies, and will be based on a minimum project service life of 100 years for major infrastructure projects such as locks, dams, and levees.

After a careful review of Dr. Fenton's description of the theory used (Fenton et al. 2016) and the decisions made during the application of reliability and LRFD theories to Canadian bridge foundation systems, the authors of this report conclude that this LRFD-based bridge foundation design procedure cannot be applied to the USACE hydraulic structures without modification. There are four fundamental items that make the design of a USACE hydraulic structure different from the design of a bridge foundation: They are (1) differences in project service life (e.g., minimum service life of 100 years for the USACE navigation structures according to ER 1110-2-8159 [HQUSACE 1997] versus a 75-year life for bridges); (2) approaches for consideration of the cost of the project in the reliability analysis; (3) the consequence of unsatisfactory performance; and (4) the transition of key features of the USACE current design methodology into LRFD (versus that given in 2014 for the Canadian bridge design industry). When decisions are ultimately made regarding these four items, as well as on other details, it is anticipated that acceptable *PUP* values and their associated β values for USACE hydraulic structures will not be the same as those made for the 2014 Canadian Highway Bridge Design Code. The authors of this report anticipate that a difference in *PUP* and β values means that the load and resistance factors in the two LRFD procedures of analysis will also differ.

It is anticipated that the four factors that were mentioned regarding the 2014 Canadian Highway Bridge Design Code, along with other details, are to be addressed and incorporated in the USACE transition to combined LRFD design. Recall that LRFD design methodology is being facilitated through the use of reliability theory. Because the application of LRFD to geotechnical related *PUP* calculations for ULSs is less advanced than *PUP*

calculations for structural ULSs, this ERDC report focuses on details related to developing a path forward in the application of reliability theory to geotechnical ULSs and the subsequent development of load and resistance factors for use in the design of the USACE hydraulic structures.

An SLS will occur more frequently than a ULS. Thus, the SLS *PUP* value is higher than the ULS *PUP* value. Baecher and Christian (2003) observe that because an SLS *PUP* value is higher, compared to that for the ULS, the allowable settlement of a structure rather than the ultimate bearing capacity of the soil generally controls the design of shallow foundations. They cite Duncan et al. (1989) observation that an SLS has a higher probability of occurrence because they occur at lower absolute loads. Baecher (1996) observes that most routine designs are based on specific serviceability limits, and ultimate limits are checked subsequently. However, the design of pile foundations contradict this trend: The design of pile foundations is controlled by the ULS.

A USACE hydraulic structure is not a routine civil engineering structure in the context of the previously discussed, routine foundation design. Three important differences are (1) the service life of a hydraulic structure is approximately twice the life of a routine building (100 years versus 50 years); (2) the consequences of its failure is far more significant and can impact the Nation's commerce; (3) and its financial investment is significantly more than for a routine building system. The design procedures differ between the two structural systems. USACE hydraulic structures are designed using three different load cases of Usual, Unusual, and Extreme (EM 1110-2-2100) (HQUSACE 2005). In contrast, routine structural systems (e.g., for buildings) and bridges are typically designed for two limit states, an SLS and a ULS. Depending on the structural feature, limiting deformation considerations may be one of the limit states evaluated for at least one and possibly all three load cases for USACE hydraulic structures. The excess deformation observations made regarding the SLS discussed in the previous paragraph with regards to the design of routine structural foundations could be relevant to the design of a hydraulic structure for Usual and possible Unusual load cases. In conclusion, deformation limit state(s) is/are also a point of consideration for the authors of this report with regard to the design of USACE hydraulic structures for their three design load cases.

2.4 Performance function, dimensionless variables and the Probability of Unsatisfactory Performance (*PUP*)

Consider the case of a vector $\mathbf{X}$ defining a set of random variables ($X_1, X_2, \dots, X_j$) of loads and resistances and a single limit state function, designated $g(\mathbf{X})$, among one of those listed in Section 2.2. Reliability is based on the performance of a system as the variables in $\mathbf{X}$ change. Therefore, it is important to know when the system achieves, for example, a ULS. ULS is often associated with system failure and its inability to perform its intended function.

In general, a limit state is the boundary between satisfactory and unsatisfactory performance of a structure (Nowak and Collins 2000, 2013). There are different types of limit states. In non-USACE, civil engineering literature, these limit states are broken into ULS as well as SLS, depending on their severity. ULS are related mostly to the system's loss of load-carrying capacity. As an example, consider the foundation of the hydraulic structure, a ULS state might involve exceeding the bearing capacity of the soil foundation, or the permanent rotation of the structure about a point along its foundation. It could also be associated with the siding of the hydraulic structure into the lock chamber. Serviceability-type limit states are generally related to gradual deterioration, personal discomfort, or maintenance cost.

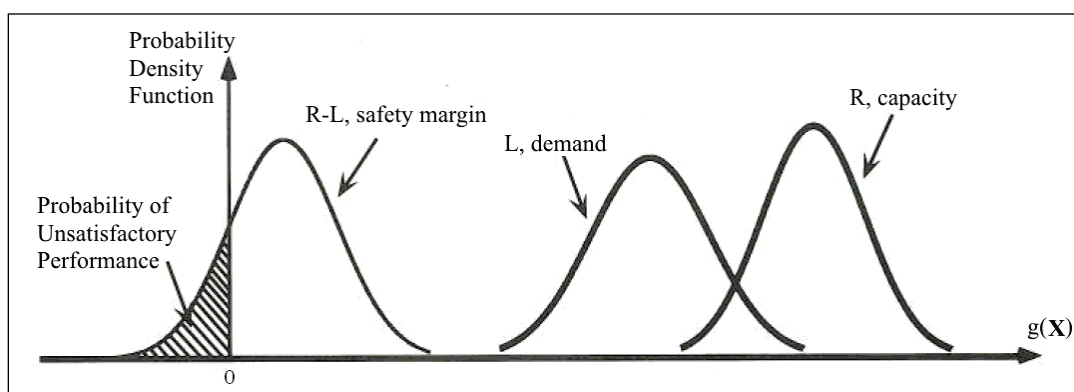
Each limit state boundary is represented mathematically by a performance function that is positive for satisfactory performance of the structure, negative for unsatisfactory, and 0.0 at the transition point of the limit state being investigated. As an example, the performance function can be defined with respect to the capacity of the structure, R , and the demand on the structure, L . With respect to the capacity and demand, the performance function becomes

$$g(\mathbf{X}) = R - L \quad (2.11)$$

with X_1 equal to L and X_2 equal to R . The Equation 2.11 difference is sometimes referred to as the margin of safety. Using this definition of the performance function, satisfactory performance occurs when the capacity (R) exceeds the demand (L), and unsatisfactory performance occurs when the demand (L) meets or exceeds the capacity (R). The limit state occurs when the demand is balanced by the capacity. This can be related to the probabilities of whether the system performance is satisfactory or

unsatisfactory (Figure 2-1). Given that the two variables R and L are each defined by a mean, a variance (equal to the square of the standard deviation) and a distribution (e.g., normal, lognormal), the *PUP* is the portion of the resulting distribution of $(R-L)$ that falls below zero, as depicted and labeled in Figure 2-1. Since it is a ULS, the *PUP* is also referred to as Probability of Failure, P_f , in the civil engineering literature. Note that since this is a probability density function (PDF or ϕ) the area under the $(R-L)$ distribution is equal to 1.0, as are each of the PDF areas under the R and L distributions.

Figure 2-1. Capacity vs. demand related to the *PUP* (after Nowak and Collins 2000, 2013).



The first step in the computation of the *PUP* value (and a corresponding β) for each limit state is to apply the Hasofer-Lind approach (Hasofer and Lind 1974) of reformulating the problem in Gaussian variate space. This space is also known as the normalized, Normal-Gaussian distribution space. Generalizing, the j -dimensional space of the vector $\mathbf{X}$ defining random variables ($X_1, X_2, \dots, X_j$) of loads and resistances is first transformed to a vector space designated $\mathbf{X}'$ by the normalization

$$x'_i = \frac{x_i - \mu_{xi}}{\sigma_{xi}} \quad (2.12)$$

This transformed variable x'_i is described as a transformed basic variable (Baecher and Christian 2003) or as a reduced variable¹. Transformed basic variables have zero means and unit standard deviations (Baecher and Christian 2003). They are dimensionless variables. Rearranging, Equation 2.12 becomes

¹ Ellingwood, B. R. 2015. *Procedures for Developing Reliability-Based Load Criteria for Hydraulic Steel and Concrete Structures*. Final letter report submitted to the URS Group, Inc. U.S. Army Corps of Engineers.

$$x_i = \mu_{xi} + x'_i * \sigma_{xi} \quad (2.13)$$

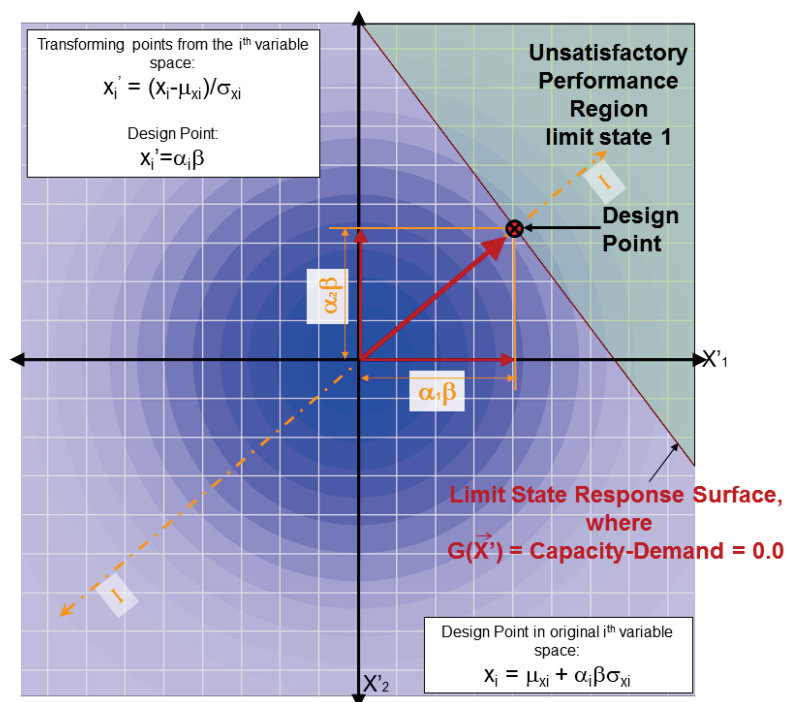
Used in this context, x'_i may be interpreted as the number of standard deviations x_i differs from mean μ_{xi} ; x'_i is expressed as a decimal fraction.

Plan view for a two-variate (i.e., X_1 and X_2) problem, the resulting transformed basic variable space of X'_1 and X'_2 (with subscript $i = 1$ and 2 in Equation 2.12), is shown in Figure 2-2(a). The concentric, blue circles shown in this figure represent the radially symmetric, normalized, Normal-Gaussian PDF distribution. Figure 2-2(b) shows a cross section passing through the origin, designated as I-I in Figure 2-2(a). For this two-dimensional (2-D) transformed basic variable space, the PDF has a peak value of 0.1591 at the origin. The I-I cross section Figure 2-2(b) of the two-variate, normalized, Normal-Gaussian PDF distribution is symmetric about its Figure 2-2(a) center (of $x'_1 = 0$, $x'_2 = 0$) having a peak value of 0.1591. This two-variate, peak PDF value is equal to $(0.3989)^2$: the square of the PDF value at the origin of a single normalized, Normal-Gaussian variate.¹ Figure 2-3 shows a three-dimensional (3-D) rendering of this Figure 2-2.a PDF distribution for a 2-D transformed variate space. Again, note the decreasing magnitude of the PDF values with increased distance in any direction away from the origin in this transformed space in Figure 2-2(b) and Figure 2-3.

¹ Generalizing for j -variate space (where j is the number of variables), the normalized Normal-Gaussian PDF distribution possess a peak PDF value of $(0.3989)^j$ at its origin. The more variables, the lower the peak PDF value will be.

Figure 2-2. The normalized, Normal-Gaussian distribution for two independent variables, X_1 and X_2 .

(a) Plan view of the transformed basic variable space for the variables X'_1 and X'_2 and response surface for a single limit state – overhead view.



(b) The PDF distribution for cross sectional I-I passing through the origin.

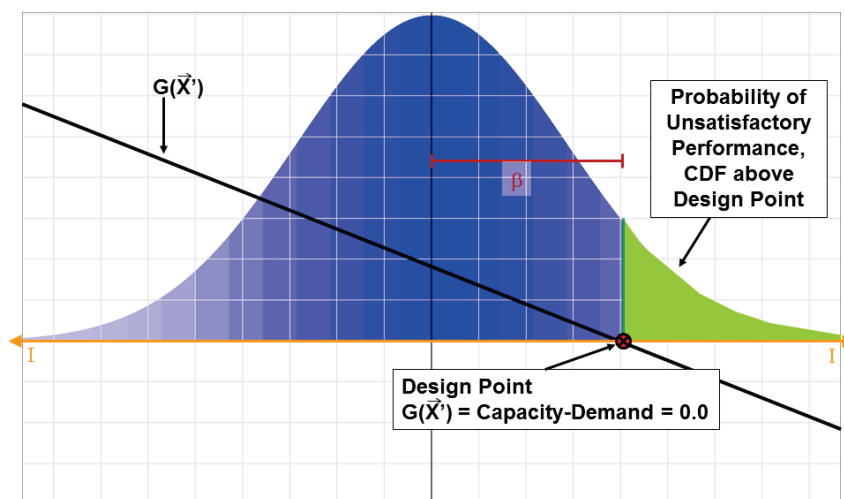
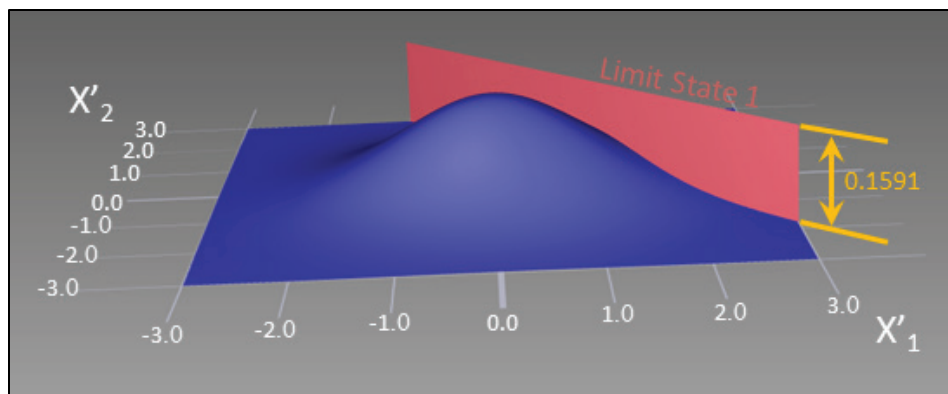


Figure 2-3. 3-D rendering of a normalized, Normal-Gaussian PDF distribution for two independent variables, X_1 and X_2 .



The volume under the entire Figure 2-3 3-D PDF is 1.0. The PDF plot shown in this figure is scaled to a peak value of 0.1591 at $x'_1 = 0$, $x'_2 = 0$. The vertical plane (in red), shows the location of the limit state 1 response surface $g(\mathbf{X}') = 0$. The height of the vertical red plane is terminated in this figure so that it is at this same height of 0.1591 as the peak PDF value. Each X'_i axis ($i = 1, 2$) is labeled using scalar values (e.g., numbers like 2.15 along an axis in this figure). Recall from the definition of x'_i by Equation 2.13 that the value of x'_i expresses the number of standard deviations σ_{xi} that a point x_i is from the mean value μ_{xi} , where x_i is given in basic variable coordinates.

The normalized, Normal-Gaussian distribution space for the example two-variable space of X'_1 and X'_2 is plotted to plus and minus 3.0 standard deviations from the mean in Figure 2-3 and many subsequent normalized, Normal-Gaussian distribution space figures in this report. Accuracy considerations in *PUP* computations by numerical method may require the variable space be extended to as much as plus and minus 5 standard deviation from the mean. This accuracy consideration will be assessed when performing example problems.

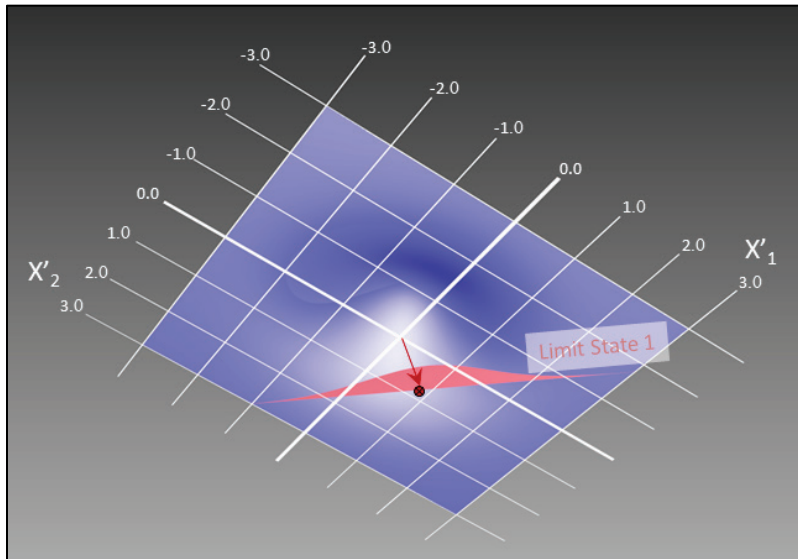
The advantage of analyzing the $g(\mathbf{X}) = 0$ problem in the transformed basic variable space of Figure 2-2(a) is two-fold: (1) Normal-Gaussian distribution relationships may be applied and (2) the variable axes are orthogonal; thus, the transformed variables are independent (i.e., uncorrelated).

A linear limit state response surface is depicted in Figure 2-2(a) where $g(\mathbf{X}')$ of capacity and demand, $(R'-L)$, is equal to zero in the transformed

basic variable space. The closest distance in transformed basic variable space to the response surface $g(\mathbf{X}') = 0$ is designated by a unit vector times a scalar β (designated by the red vector emanating from the origin in Figure 2-2(a)). The scalar β is designated as the reliability index. Its vector projection onto the X'_1 and X'_2 axes are designated as the directional cosine α_1 times β and α_2 times β , respectively, with their lengths labeled in orange in Figure 2-2(a) along each axis.

The Figure 2-2(a) point on the limit state response surface of $g(\mathbf{X}') = 0$ that is closest to the origin of transformed basic variable space to the response surface $g(\mathbf{X}') = 0$ is designated as the design point. This point is shown as a red dot and is labeled as the design point in Figures 2.2(a) and 2.2(b). Observe in Figure 2-2(a) that the unit vector times a scalar β is orthogonal to the limit state response surface at the design point. It is a point (red dot) on the limit state response surface at which the probability density of the joint random variables is greatest, as seen by the Figure 2-4 PDF (red) cross-section view. The red vector emanating from the origin of this transformed basic variable space is the unit vector times the scalar β .

Figure 2-4. Cross section along the position of the limit state response surface through the PDF distribution in a 3-D rendering and the location of the design point — view from underneath.



In transformed basic variable space $\mathbf{X}'_i$ of Figure 2-2.a, the design point is at position

$$x'_i = \alpha_i * \beta_i \quad (2.14)$$

and in the basic random variable space $\mathbf{X}_i$ the design point is located at

$$x_i = \mu_{xi} + \alpha_i * \beta_i * \sigma_{xi} \quad (2.15)$$

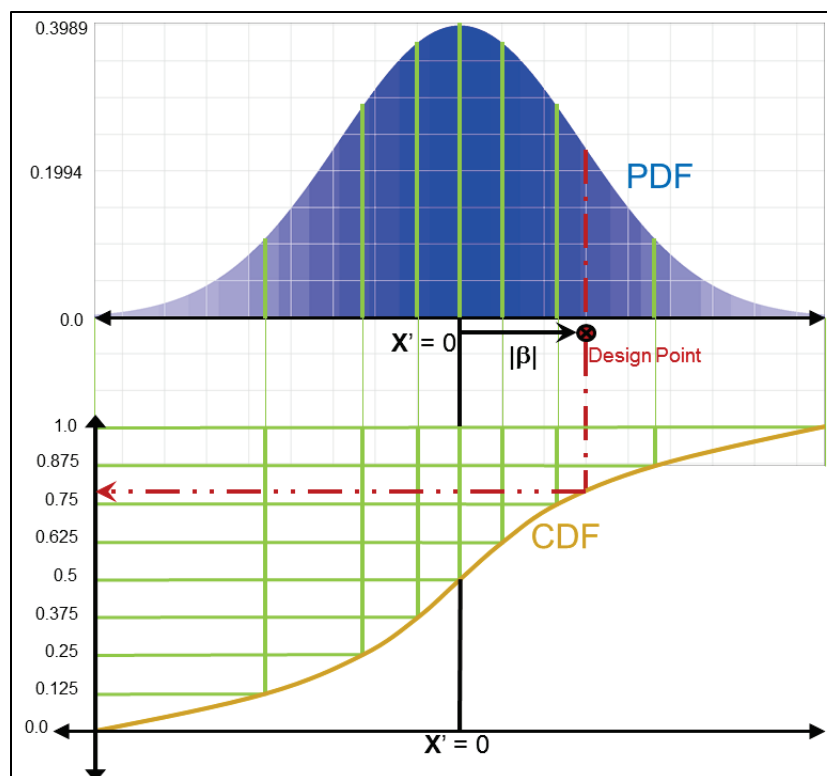
The cross section designated I-I in Figure 2-2(a) and shown in Figure 2-2(b) also identifies the position of the design point.

For a single planar response surface at a distance of magnitude β from the origin (i.e., $x'_1 = 0, x'_2 = 0$) in the transformed space to the design point, the *PUP* value is determined from the Cumulative Distribution Function of β , $CDF(\beta)$, value by

$$PUP = 1 - CDF(\beta) \quad (2.16)$$

with the $CDF(\beta)$ being computed using the standard statistical tabulated data for a normalized, 1-D, Normal-Gaussian distribution variate or by using normal distribution functions that are available in, for example, Excel spreadsheet software. In this evaluation, a PDF with a maximum value of 0.3989 at the origin of the $x' = 0$, normalized, Normal-Gaussian PDF distribution is used (Figure 2-5). The $CDF(\beta)$ is equal to the integral of this PDF from negative infinity up to $X' = 0$ plus β (using the reduced coordinates with a unit standard deviation), as depicted in Figure 1.5. This simplified computational approach avoids the need to perform the sometimes complex numerical integration of the Figure 2-2(a) PDF volume designated as the unsatisfactory performance region limit state 1 and identified as the olive-green region in this figure. In this computation, the PFD distribution has a peak value of 0.3989 at its origin (i.e., $x' = 0$) in transformed space of Figure 2-5. The value of the CDF is computed using the value for β associated with the design point (discussed in preceding paragraphs regarding Figure 2-2).

Figure 2-5. PDF and CDF for a normalized, Normal-Gaussian single variate.



In summary, the Hasofer-Lind transformation simplifies the computational process for the PDF value and for the β value of a single limit state response surface. Its advantage is that all variables are transformed into normalized, Normal-Gaussian variate space, and these transformed variables are independent of one another.

2.5 The interrelationship between LRFD and reliability index for a single Ultimate Limit State (ULS)

The interrelationship between reliability theory and load and resistance factors used in an LRFD methodology is derived by returning to the simple ULS example of Figure 2-1 and Equation 2.11. The single limit state performance function is defined by setting Equation 2.11 equal to zero:

$$g(\mathbf{X}) = (R - L) = 0 \quad (2.17)$$

where R is the capacity (i.e., $X_2 = R$) and L is demand (i.e., $X_1 = L$). Recall that when $(R-L) < 0$, failure occurs. To relate the reliability index to load and resistance factors, Ellingwood (2015) starts by expressing the limit state response surface in reduced transformed basic variable space (also

known as reduced coordinates). The limit state response surface is transformed using the equations

$$r = \frac{(R - \mu_R)}{\sigma_R} \quad (2.18)$$

where r is the normalized variable for resistance in reduced coordinates, R is the basic variable, μ_R is the mean value of R , and σ_R is its standard deviation. Similarly, the reduced coordinates for the load is

$$l = \frac{(L - \mu_L)}{\sigma_L} \quad (2.19)$$

where l is the normalized variable for load in reduced coordinates, L is the basic variable, μ_L is the mean value of L , and σ_L is its standard deviation. Rearranging, Equations 2.18 and 2.19 become

$$R = \mu_R + r * \sigma_R \quad (2.20)$$

$$L = \mu_L + l * \sigma_L \quad (2.21)$$

Introducing Equations 2.20 and 2.21 to the limit state response surface Equation 2.17 yields

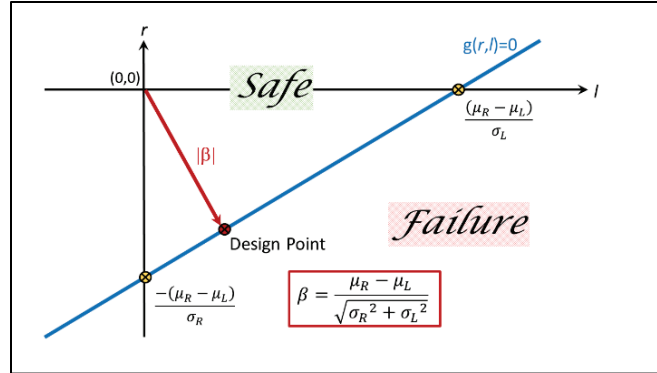
$$g(X) = g(R, L) = g(r, l) = (\mu_R + r * \sigma_R) - (\mu_L + l * \sigma_L) = 0 \quad (2.22)$$

Rearranging,

$$g(r, l) = (\mu_R - \mu_L) + (r * \sigma_R - l * \sigma_L) = 0 \quad (2.23)$$

Figure 2-6 shows the limit state response surface of Equation 2.23, $g(r, l) = 0$ in the 2-D, transformed basic variable space. Observe Equation 2.23 represents a straight line in r, l space.

Figure 2-6. Reliability Index defined as the shortest distance from the origin of the transformed basic variable space to the design point on the limit state response surface function $g(r,l)$ (after Nowak and Collins 2013).



One intercept located along the transformed basic variable axis r is obtained by setting $r = 0$ in equation 2.23 and solving for l . This results in

$$l = \frac{(\mu_R - \mu_L)}{\sigma_L} \quad (2.24)$$

and the other intercept located along the transformed basic variable axis l is obtained by setting $l = 0$ in equation 2.23 and solving for r . This results in

$$r = \frac{-(\mu_R - \mu_L)}{\sigma_R} \quad (2.25)$$

These two axis intercepts of l and r axes are identified in Figure 2-6.

Using geometry of Figure 2-6, Nowak and Collins (2013) calculated the Reliability Index as the shortest distance from the origin of the transformed basic variable space to the $g(r,l)$ limit state response surface (Equation 2.23) to be

$$\beta = \frac{\mu_R - \mu_L}{\sqrt{\sigma_R^2 + \sigma_L^2}} \quad (2.26)$$

Recall that β is a scalar value.

The Reliability Index is related to the PUP (equivalently, the P_f for the ULS) by

$$\beta = -\Phi^{-1}(PUP) \quad (2.27a)$$

as well as by

$$\beta = \Phi^{-1}(1 - PUP) \quad (2.27b)$$

with

$$PUP = \Phi(-\beta) \quad (2.28)$$

The safe region of satisfactory performance occurs when the capacity (R) exceeds the demand (L). It is the region located above the blue $g(r,l) = 0$ line in Figure 2-6. Conversely, unsatisfactory performance occurs when L meets or exceeds R . It is the region located below the blue $g(r,l) = 0$ line in this figure.

Ellingwood¹ gives the design point in transformed basic variable (i.e., reduced coordinate) space as

$$r = -\alpha_R * \beta \quad (2.29)$$

and

$$l = \alpha_L * \beta \quad (2.30)$$

where α_L and α_R are the directional cosines for L and R , respectively. Figure 2-6 supports this observation. Recognizing the (orange) vector decomposition of the red vector shown in Figure 2-2(a), a unit vector times a scalar β value, to the X'_1 and X'_2 axes, the scalar value of β is

$$\beta^2 = (\alpha_R)^2 * \beta^2 + (\alpha_L)^2 * \beta^2 \quad (2.31)$$

Dividing through by β^2 ,

$$(1)^2 = \alpha_R^2 + \alpha_L^2 \quad (2.32)$$

In its most general LRFD form for the basic variable coordinate space of a single limit state response surface is

¹ Ellingwood, B. R. 2015. *Procedures for Developing Reliability-Based Load Criteria for Hydraulic Steel and Concrete Structures*. Final letter report submitted to the URS Group, Inc. U.S. Army Corps of Engineers.

$$R \geq L \quad (2.33)$$

Introducing Equations 2.20, 2.21, 2.29, and 2.30,

$$\mu_R - \alpha_R * \beta * \sigma_R \geq \mu_L + \alpha_L * \beta * \sigma_L \quad (2.34)$$

Recall that the COV is equal to the standard deviation for the variable divided by its mean value. Introducing the COV for variable L and R , Equation 2.34 becomes

$$\mu_R * (1 - \alpha_R * \beta * COV_R) \geq \mu_L * (1 + \alpha_L * \beta * COV_L) \quad (2.35)$$

This equation expresses the requirement for a reliability-based design for a single limit state response surface. The corresponding *PUP* value is calculated by Equation 2.28. The LRFD resistance factor is designated ϕ_R and defined as the term contained within the left-hand side (LHS) parenthesis:

$$\phi_R = (1 - \alpha_R * \beta * COV_R) \quad (2.36)$$

Similarly, the LRFD load factor is designated γ_L and defined as the term contained within the right-hand side (RHS) parenthesis:

$$\gamma_L = (1 + \alpha_L * \beta * COV_L) \quad (2.37)$$

Recall that equations presented thus far are for R and L variables, which are normally distributed. Reviewing Equation 2.35 demonstrates that these resistance and load factors of ϕ_R and γ_L , respectively, are applied to the mean values of R and L .

$$\mu_R * \phi_R \geq \mu_L * \gamma_L \quad (2.38)$$

In the specific application of LRFD to building systems, Ellingwood et al. (1982) observe that the term in parenthesis on the LHS of Equation 2.35 and on the RHS of Equation 2.36 is replaced with

$$\phi_R = e^{-\alpha_R * \beta * COV_R} \quad (2.39)$$

thereby describing an R with a lognormal distribution. Similarly, describing L with a lognormal distribution:

$$\gamma_L = e^{\alpha_L * \beta * COV_L} \quad (2.40)$$

Another feature to be considered in a reliability analysis is to account for the difference between mean resistance μ_R and nominal resistance R_n and between mean load μ_L and nominal load L_{ni} . Nowak and Collins (2013) provide an example as to how they are related for R and L to factored loads (e.g., $\phi_R * R_n$ and $\gamma_{Li} * L_{ni}$, respectively) as shown in Figures 2-7 and 2-8, respectively. They point out that in some recent developments in building design codes, the factored load was taken so that the shaded area in Figure 2-7 was the same for all dead and live building loads components. The factored resistance and factored load plotted in these figures assume the typical case of a ϕ_R less than 1 and γ_{Li} for $i = 1$ to loading j , to be greater than 1 for each load.

Figure 2-7. Relationships among nominal load, mean load, and factored load (after Nowak and Collins 2013).

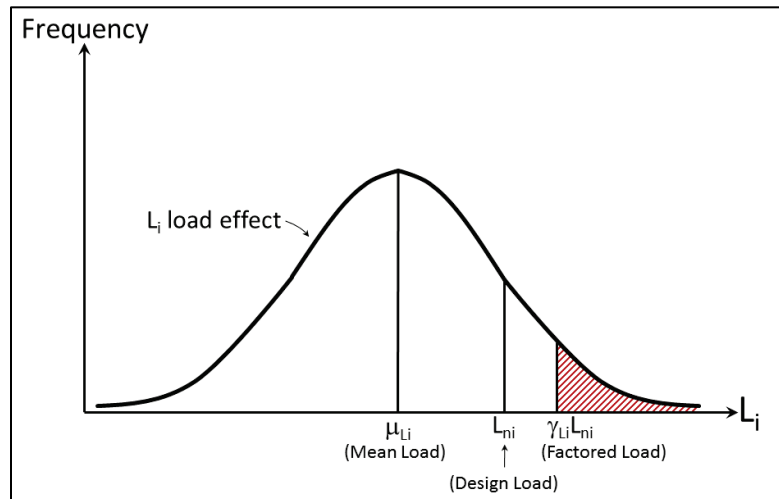
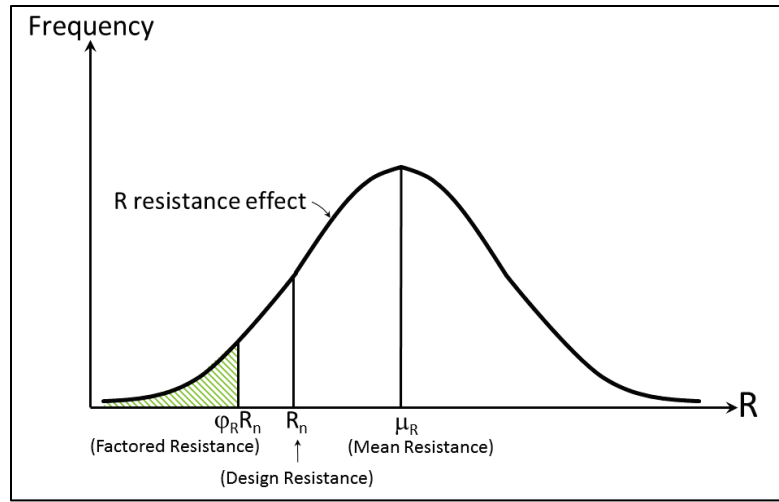


Figure 2-8. Relationships among nominal resistance, mean resistance, and factored resistance (after Nowak and Collins 2013).



To accommodate the situations demonstrated in these two figures, engineers (e.g., Nowak and Collins 2013; Fenton et al. 2016; Ellingwood¹) are introducing a bias factor into the resistance and load factors of ϕ_R and γ_{Li} Equations of 2.36 and 2.37, respectively. The bias factor, λ , is defined as

$$\lambda = \frac{\text{mean load value}}{\text{nominal (design) value}} \quad (2.41)$$

Recall that Fenton et al. (2016) prefers to use the term characteristic (design) value in the denominator rather than nominal (design) value. Thus, for the resistance R , the bias factor is

$$\lambda_R = \frac{\mu_R}{R_n} \quad (2.42)$$

and for the i^{th} load L_i , with $i = 1$ to j loads, each bias factor is

$$\lambda_{Li} = \frac{\mu_{Li}}{L_{ni}} \quad (2.43)$$

Thus, the LRFD resistance factor ϕ_R becomes

$$\phi_R = \lambda_R * (1 - \alpha_R * \beta * COV_R) \quad (2.44)$$

¹ Ellingwood, B. R. 2015. *Procedures for Developing Reliability-Based Load Criteria for Hydraulic Steel and Concrete Structures*. Final letter report submitted to the URS Group, Inc. U.S. Army Corps of Engineers.

and each load factor γ_{Li} becomes

$$\gamma_{Li} = \lambda_{Li} * (1 + \alpha_{Li} * \beta * COV_{Li}) \quad (2.45)$$

for R and L variables which are normally distributed. The generalized, governing LRFD equation for $i = 1$ to j loads is

$$\mu_R * \phi_R \geq \sum_{i=1}^j \mu_{Li} * \gamma_{Li} \quad (2.46)$$

When R possesses a lognormal distribution, Ellingwood¹ states that the resistance factor ϕ_R becomes

$$\phi_R = \lambda_R * e^{-\alpha_R * \beta * COV_R} \quad (2.47)$$

For the two-variable problem being discussed in this section, i is set equal 1 for the single load case.

When L possesses a lognormal distribution, the load factor γ_L is

$$\gamma_L = \lambda_L * e^{\alpha_L * \beta * COV_L} \quad (2.48)$$

for the two-variable problem being discussed in this section with i set equal 1 for the single load case.

In summary, the load and resistance factors of ϕ_R and γ_{Li} (for $i = 1$ to j loads) depend on the values for β , the directional cosines α_R and α_{Li} , and the coefficients of variation COV_R and COV_{Li} , and the bias factors λ_R and λ_{Li} . Ellingwood¹ observes that β was adopted as the reliability metric in first-generation LRFD criteria rather than a PUP value because β values are typically on the order of 2 to 5 while PUP values can be quite small, on the order of 1×10^{-4} or less.

2.6 A numerical procedure to calculate the reliability Index

A simplifying first step applied in the computation of β (and a corresponding PUP value) for each limit state is by using the Hasofer-Lind approach (Hasofer and Lind 1974) of transforming the problem into

¹ Ellingwood, B. R. 2015. *Procedures for Developing Reliability-Based Load Criteria for Hydraulic Steel and Concrete Structures*. Final letter report submitted to the URS Group, Inc. U.S. Army Corps of Engineers.

Gaussian variate space. This Figure 2-2(a) space is also known as the normalized, Normal-Gaussian distribution space. This space is dimensionless and has properties of symmetry and orthogonally, as well as cumulative properties for simple determination of probabilities.

Several techniques are available for calculating β , as well as the directional cosines of Figure 2-2, that define the closest design point among the various limit state response surfaces. Baecher and Christian (2003) discuss the attributes of the different techniques being for reliability analysis as well as the merits and deficiencies of each procedure. Based on experience gained during the development of the USACE CPGA-R software for reliability analysis of pile groups (Ebeling et al. 2013), procedures have been selected for possible use on the engineering problems discussed in this report:

- The First Order Second Moment (FOSM) Method. This method uses the first terms of a Taylor series expansion of the $G(\mathbf{X})$ performance function to estimate its expected value and variance.
- The Advanced Second Moment (ASM) Method. This is a numerical method for finding the minimum distance that a $G(\mathbf{X})$ performance function is from the origin of a defined vector space.

Examples of the application of FOSM procedure as applied in a reliability analysis of a slope are given in Chapter 13 of Duncan et al. (2014) and Chapter 14 of Baecher and Christian (2003). The ASM procedure is discussed in detail in the next subsection.

Alternatively, simulation methods may be used to directly estimate probabilities for exceeding these response surfaces. Simulation techniques are discussed in Ebeling et al. (2013) and Baecher and Christian (2003). The computed *PUP* value can then be converted into a β value. Simulation methods were relied upon in the USACE CPGA-R software for reliability analysis of pile groups (Ebeling et al. 2013). This approach is quite different from the FOSM and ASM methodologies (refer to Ebeling et al. 2013 for its details). It is also identified for possible usage, if deemed necessary.

2.6.1 Advanced Second Moment (ASM) for uncorrelated, normal distributions

The ASM technique, also called Advanced First Order Second Moment (AFOSM), is a method for finding the minimum distance that a capacity-vs.-demand surface, based on the function $G()$, is from the origin of a defined vector space. This differs from the FOSM technique, which approximates a nonlinear capacity-vs.-demand surface with a hyperplanar surface in multidimensional space (Melchers 1999), which forms a linear surface in 2-D space (i.e., considering two independent variables). The FOSM technique uses the linear definition of the capacity-vs.-demand surface to quickly estimate the closest surface point to the origin, without having to perform multiple iterations to find the actual nonlinear capacity-vs.-demand surface's closest point.

For a reliability analysis, the surface function is the limit state boundary function. Recall that this vector space, $\mathbf{X}$, can be transformed into the vector space $\mathbf{X}'$, where the probability of occurrence is based on the Euclidean distance from the origin. In this case, the distance is the β of the design with respect to that point (Ditlevsen 1981). The point (in the normalized vector space, $\mathbf{X}'$, Figure 2-2) on the surface $G(\mathbf{X}) = 0$ (where the design point has been transformed from X -space to X' -space) that is closest to the origin is called the design point. Values of β are shown in Table 2-1 and Figure 2-2, and the β value can be used to determine the probability that this point will be chosen from the variable distributions. In terms of reliability analysis, PUP can be given by $PUP = 1 - \Phi(\beta)$ for an arbitrary axis with a linear limit state, where $\Phi()$ is the cumulative distribution function. Table 2-1 lists values of β and its corresponding PUP values.

Table 2-1. Reliability Index (β) and PUP – Normal distribution.

β	$P_U = 1 - \Phi(\beta)$
0	5.0000000000000E-01
1	1.586552539315E-01
2	2.275013194818E-02
3	1.349898031630E-03
4	3.167124183678E-05
5	2.866515719235E-07

β	$P_U = 1 - \Phi(\beta)$
6	9.865877004245E-10
6.36	1.008768624189E-10
6.71	9.731215833142E-12
7	1.279865102788E-12
8	6.661338147751E-16
9	Out of range for a double-precision variable

The ASM technique works because any hyperplanar surface can be fully described by a directional cosine vector (α) and the distance from the origin (β) for the closest point on the hyperplanar surface (Figure 2-2(a)). The vector describing the resultant of all directional cosine vectors, α , also is identified in Figure 2-2(a). This resultant directional cosine vector (α) is perpendicular to the hyperplanar surface labeled as the ASM Response Surface (i.e., where $G(X) = 0$). This resultant directional cosine vector α is a unit vector, with the magnitude of each component vector along a variable X_i axis, $|\alpha_i|$, being a cosine value for that direction with respect to the unit vector α . The components of the directional cosine vector α_i multiplied by the distance, β , are shown for variables X_1 and X_2 in Figure 2-2(a) and lay along each respective axis. For the x -space, the vector X 's components that define the plane are given by

$$x_i = \mu_{xi} + \alpha_i * \beta_i * \sigma_{xi} \quad (2.15 \text{ bis})$$

At the start of the reliability analysis, the directional cosine α is unknown. However, it can be estimated by determining the vector on the surface, $G(\mathbf{X})$, that is approaching a 0.0 value from the mean value at the fastest rate. The rate at which each component approaches the 0.0 value can be determined by computing the partial derivative of G with respect to each variable ($X_1, X_2, \dots$) at the mean value.

Referring to Figure 2-2(b), which shows the 1-D cross section of a linear $G(\mathbf{X})$ surface, with particular attention paid to the region above the origin at the means of this 2-D normal space, a pair of partial derivatives are made numerically in the positive and negative direction of an individual axis resulting in a total of four partial derivatives for the Figure 2-2(a) two-

variable problem. The direction along each variable axis with the steepest slope approaching $G(\mathbf{X}) = 0.0$ is selected for each variable as the direction of the vector. The magnitudes of each axis, $|\alpha_i|$, are computed according to Equation 2.50.

Recall that the response surface for vector $\mathbf{X}$ is defined by

$$G(\vec{X}) = 0.0 \quad (2.49)$$

This can be performed by setting

$$\alpha_i = \frac{\left(\frac{\partial G(\mathbf{X})}{\partial X_i}\right) \sigma_{X_i}}{\sqrt{\sum_{i=1}^n \left(\frac{\partial G(\mathbf{X})}{\partial X_i}\right)^2 \sigma_{X_i}^2}} \quad (2.50)$$

with the initial trial value for β set to 0.0 so that the initial vector $\mathbf{X}$ is composed of only the mean values. A numerical procedure, along with Equations 2.15, 2.49, and 2.50, is used to calculate the α_i values for each of the i variables.

When the directional cosine (α) has been determined, it is possible to determine β using numerical methods (e.g., the Regula-Falsi method) so that the current design point (vector $\mathbf{X}$) is on the response surface, where Equation 2.15 is satisfied (Figure 2-2(a)).

For nonlinear performance functions, the current design point might not be the point with the minimum distance from the mean values. For this case, partial derivatives for each axis at the current design point (vector $\mathbf{X}$) are computed (Equation 2.50) to determine the direction and magnitudes for a new directional cosine from the origin at the means. The β value is determined from Equation 2.15. This process is performed until the new β and the old β converge within a certain tolerance.

This solution might not reach the absolute minimum distance point to the response surface for particularly complex nonlinear surfaces, rather a local minima. Most engineering problems do not have particularly complex nonlinear response surfaces, so this ASM method will see widespread usage.

2.6.2 Dealing with non-normal distributions for X_i

To handle non-normal distributions, each non-normal distribution variable X must be changed to an equivalent normally distributed random variable X_{Normal} . This is called the Rosenblatt transformation (Rosenblatt 1952). Rackwitz and Fiessler (1976 and 1978) impose two conditions on the equivalent normally distributed random variable that enable the determination of the two parameters of that distribution, μ_{Xi}^N and σ_{Xi}^N , as cited in Singh et al. (2007) and Nowak and Collins (2000, 2013). The CDFs and PDFs of the non-normal and normal distributions should be equivalent at the current design point for the performance function.

These conditions (CDF and PDF) can be expressed, respectively, in

$$\Phi\left(\frac{X_i - \mu_{Xi}^N}{\sigma_{Xi}^N}\right) = F_i(X_i) \quad (2.51)$$

$$\frac{1}{\sigma_{Xi}^N} \phi\left(\frac{X_i - \mu_{Xi}^N}{\sigma_{Xi}^N}\right) = f_i(X_i) \quad (2.52)$$

where F_i is the non-normal CDF, f_i is the non-normal PDF, Φ is the standard normal CDF, and ϕ is the standard normal PDF. From these conditions, it can be shown that the standard deviation and mean, respectively, are

$$\sigma_{Xi}^N = \frac{1}{f_i(X_i)} \phi\left(\frac{X_i - \mu_{Xi}^N}{\sigma_{Xi}^N}\right) = \frac{\phi(\Phi^{-1}[F_i(X_i)])}{f_i(X_i)} \quad (2.53)$$

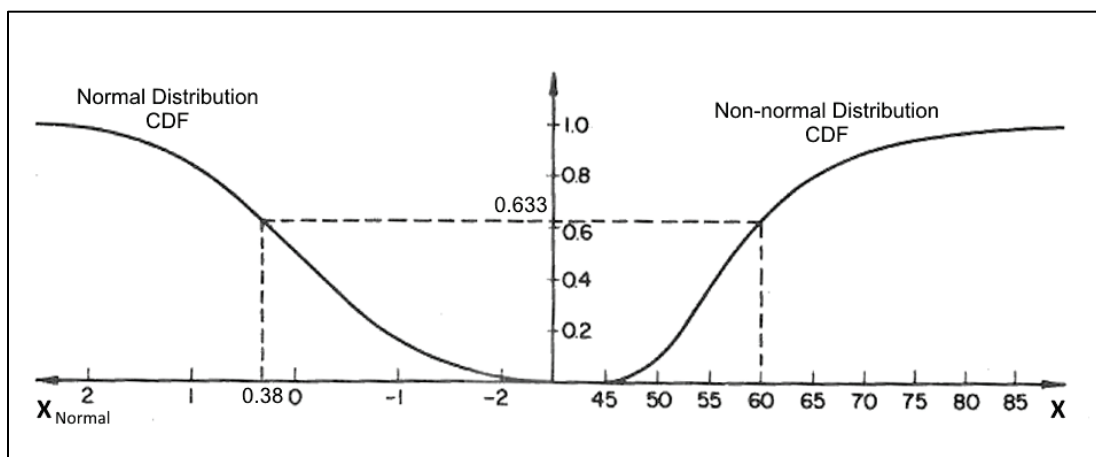
$$\mu_{Xi}^N = X_i - \Phi^{-1}[F_i(X_i)] * \sigma_{Xi}^N \quad (2.54)$$

The PDF of a standard normal distribution is a continuously positive function. Bounded distributions including triangular PDFs are not. While it is possible to get equivalent CDF values between the standard normal distribution and a bounded non-normal distribution (Figure 2-9), it might be impossible to find an equivalent PDF between distributions. In this case, the standard deviation can be calculated and the mean of the standard deviation varied to find the appropriate CDF value. The PDF in this case will be a close estimate at convergence.

Ang and Tang (1984) in their Example B.1 provided an example using annual maximum mile wind speed. This was modeled as a Type II asymptotic distribution of largest values. The CDF (Φ) of this distribution

is displayed on the right side of Figure 2-9 while the CDF (Φ) of a normalized Gaussian distribution (which has a mean $\mu = 0.0$ and standard deviation $\sigma = 1.0$) is shown on the left with positive X_{Normal} axis values growing from right to left. With fixed parameters of $\gamma = 9.0$ and $\beta = 55$ mph in the Type II asymptotic distribution equation (not shown), then for a velocity $v = 60$ mph, the CDF (Φ) of the Type II asymptotic distribution has a value of 0.633. This is an equivalent CDF (Φ) of the normalized Gaussian distribution at a distance of 0.38 standard deviation from the mean origin of the normalized Gaussian distribution.

Figure 2-9. Finding the position of equivalent cumulative probability for a normal distribution as from a non normal distribution (after Ang and Tang 1984).



2.6.3 Dealing with correlated random variables X_1 and X_2

In certain situations, it is unrealistic to believe that variables chosen for a reliability analysis will be independent of one another. In these other instances, such as the case in which the variables are the soil material properties of the angle of internal friction and cohesion for soil layers, variables might be related to some degree. For this discussion, a pair of variables (X_1 and X_2) is correlated with a correlation coefficient (ρ). If the two variables (X_i with $i = 1, 2$) have a standard normal distribution, they can be changed to a pair of non-correlated variables (X^*_i) by solving for two eigenvalues and the corresponding eigenvectors as follows:

$$X_1^* = \frac{1}{2t} \left(\frac{X_1 - \mu_{X_1}}{\sigma_{X_1}} + \frac{X_2 - \mu_{X_2}}{\sigma_{X_2}} \right) \quad (2.55a)$$

$$X_2^* = \frac{1}{2t} \left(\frac{X_1 - \mu_{X_1}}{\sigma_{X_1}} - \frac{X_2 - \mu_{X_2}}{\sigma_{X_2}} \right) \quad (2.55b)$$

where $t = \sqrt{0.5}$. The variances for these two variables are equal to the eigenvalues (λ) as follows:

$$\sigma_{X_1^*}^2 = \lambda_1 = 1 + \rho \quad (2.56a)$$

$$\sigma_{X_2^*}^2 = \lambda_2 = 1 - \rho \quad (2.56b)$$

The directional cosines for these newly uncorrelated values must be determined from the original correlated values using

$$\alpha_{X_1^*} = \frac{\left[\left(\frac{\partial G(\mathbf{0})}{\partial V_1} \right) t \sigma_{X_1} + \left(\frac{\partial G(\mathbf{0})}{\partial V_2} \right) t \sigma_{X_2} \sqrt{1+\rho} \right]}{\left[\left(\frac{\partial G(\mathbf{0})}{\partial X_1} \right)^2 \sigma_{X_1}^2 + \left(\frac{\partial G(\mathbf{0})}{\partial X_2} \right)^2 \sigma_{X_2}^2 + 2\rho \left(\frac{\partial G(\mathbf{0})}{\partial X_1} \right) \left(\frac{\partial G(\mathbf{0})}{\partial X_2} \right) \sigma_{X_1} \sigma_{X_2} \right]^{1/2}} \quad (2.57a)$$

$$\alpha_{X_2^*} = \frac{\left[\left(\frac{\partial G(\mathbf{0})}{\partial X_1} \right) t \sigma_{X_1} - \left(\frac{\partial G(\mathbf{0})}{\partial X_2} \right) t \sigma_{X_2} \sqrt{1-\rho} \right]}{\left[\left(\frac{\partial G(\mathbf{0})}{\partial X_1} \right)^2 \sigma_{X_1}^2 + \left(\frac{\partial G(\mathbf{0})}{\partial X_2} \right)^2 \sigma_{X_2}^2 + 2\rho \left(\frac{\partial G(\mathbf{0})}{\partial X_1} \right) \left(\frac{\partial G(\mathbf{0})}{\partial X_2} \right) \sigma_{X_1} \sigma_{X_2} \right]^{1/2}} \quad (2.57b)$$

and

$$X_1 = \mu_{X_1} - \sigma_{X_1} t \beta (\alpha_{X_1^*} \sqrt{\lambda_1} + \alpha_{X_2^*} \sqrt{\lambda_2}) \quad (2.58a)$$

$$X_2 = \mu_{X_2} - \sigma_{X_2} t \beta (\alpha_{X_1^*} \sqrt{\lambda_1} - \alpha_{X_2^*} \sqrt{\lambda_2}) \quad (2.58b)$$

Ditlevsen (1981) presents a detailed derivation of these equations for transforming correlated random variables into corresponding non-correlated random values.

2.6.4 ASM algorithm

The ASM algorithm presented in this subsection has been modified from the AFOSM method to include ones for working with non-normal distributions and correlated random variables.

The ASM Algorithm

1. Assign the mean value for each random variable as a starting design point value (i.e., $(X_1, X_2, \dots, X_i) = (\mu_1, \mu_2, \dots, \mu_i)$).
2. Compute the standard deviation and mean of the equivalent normal distribution for each non-normal random variable by using Equations

- 2.53 and 2.54 or by adjusting the mean so that an equivalent CDF is determined for the standard deviation of the original data.
3. Compute the partial derivative, $\partial G() / \partial X_i$, of the performance function with respect to each non-correlated random variable evaluated at the design point as needed to satisfy Equation 2.50.
 4. Compute the directional cosine, α_i , for each non-correlated random variable as given in Equation 2.50 at the design point. For correlated pairs of random variables, Equations 2.57a and b should be used.
 5. Compute β by substituting Equation 2.50 for non-correlated random variables and Equations 2.57a and b for correlated random variables into the $G()$ performance function of the vector $\vec{X}$ (Equation 2.15) and satisfying the limit state $G() = 0$ (Equation 2.49) in using a numerical root-finding method.
 6. Compute a new estimate of the design point by substituting the resulting β obtained in Step 5 into Equation 2.15 for non-correlated random variables and Equations 2.58a and b for correlated random variables.
 7. Repeat Steps 2 through 6 until β converges within an acceptable tolerance, δ .

2.6.5 ASM caveats

The ASM method estimates the actual *PUP*. It is accurate only if the limit state, response surface, where $G() = 0$, is linear and approximate when the limit state, response surface is nonlinear. The response surface might be nonlinear if the $G()$ function is nonlinear or if multiple limits states are included in the $G()$ function (as discussed in the next section using Figure 2-10 as an example).

2.7 Reliability analysis with multiple limit states

USACE hydraulic structures are designed considering more than one limit state (e.g., multiple ULs) occurring at the same time. Section 2.2 lists the ULs considered during the design process. Plan view for a two-variate problem in the transformed basic variable space of X'_1 and X'_2 is shown in Figure 2-10, along with two vertical planes, one red and the other green, showing the location of each of their respective limit state response surfaces, with each possessing a $g(\mathbf{X}') = 0$ type relationship. The height of these vertical planes are terminated in this figure so that they are at the same height of 0.1591 ($= 0.3989^2$) as the maximum PDF value at $x'_1 = 0$ and $x'_2 = 0$. The concentric, blue circles shown in this figure represent the

radially symmetric, normalized, Normal-Gaussian PDF distribution. Figure 2-11 shows a 3-D rendering of this Figure 2-10 PDF distribution for this two dimensional transformed Variate space. Note the decreasing magnitude of the PDF values with increased distance in any direction from the origin in Figure 2-11.

Figure 2-10. Plan view of the transformed basic variable space for the variables X'_1 and X'_2 and response surfaces for a two-limit states – overhead view.

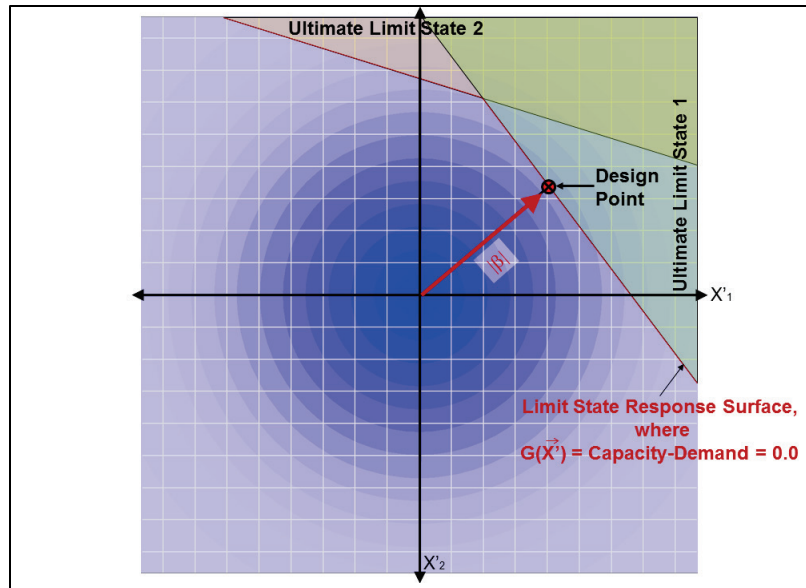
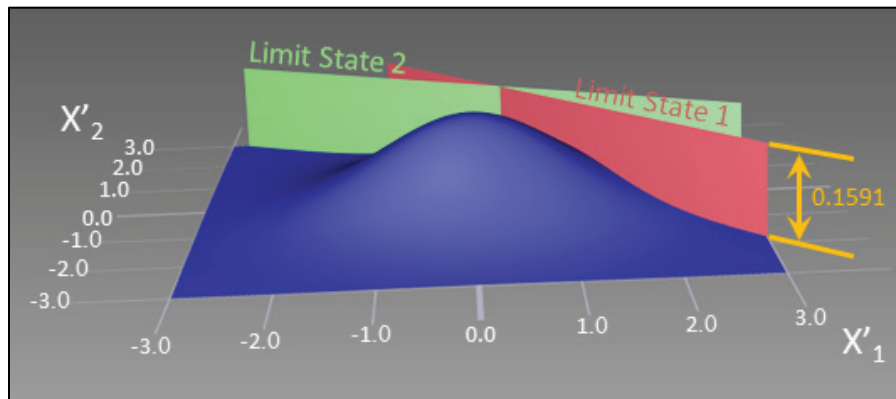


Figure 2-11. 3-D rendering of a normalized, Normal-Gaussian PDF distribution for two independent variables, X'_1 and X'_2 and two limit state response surfaces



The Figure 2-10 region shaded in light brown, green, and olive green designates the failure region (i.e., $g(\mathbf{X}') \leq 0$). Note that regions beyond both of the two limit states in this figure contribute to this *PUP* volume. Conversely, the blue region in this figure designates the safe region:

$g(\mathbf{X}') > 0$. The Reliability Value is again dictated by the fencing of the *PUP* volume provided by the location of these two limit states.

The closest distance in transformed basic variable space to each response surface, $g(\mathbf{X}') = 0$, is calculated. The distance among the two values that is closest to the $(x'_1 = 0, x'_2 = 0)$ origin is then designated as the design point. It is located in Figure 2-10 by a unit vector times a scalar β and designated by the red vector emanating from the origin. In this case, the design point resides on limit state 1 that is located along the base of the Figure 2-11 red plane. Recall β is designated as the reliability index. Observe in Figure 2-10 that the unit vector times a scalar β is orthogonal to the limit state 1 response surface (with $g(\mathbf{X}') = 0$) at the design point. It is also the point on the limit state response surface at which the probability density of the joint random variables is greatest, as seen in Figure 2-4.

The Figure 2-10 designated limit state 2 is shown at its corresponding location in the 2-D PDF distribution in Figure 2-12. Its PDF distribution along the trace of limit state 2 for $g(\mathbf{X}') = 0$ is shown with a viewpoint from below in Figure 2-13, along with the red vector defining its closest distance as measured from the origin. Because the distance of limit state 2 is farther away than for limit state 1 (Figure 2-4), this red dot in Figure 2-13 for limit state 2 is not designated as the design point. Observe that for the green PDF distribution plotted along the location of $g(\mathbf{X}') = 0$ for limit state 2 in Figure 2-13, the largest PDF value occurs at the point designated by the red dot. This dot was the limit state 2 candidate for the design point.

Figure 2-12. 3-D rendering of a normalized, Normal-Gaussian PDF distribution for two independent variables, X_1 and X_2 , and the second limit state response surface.

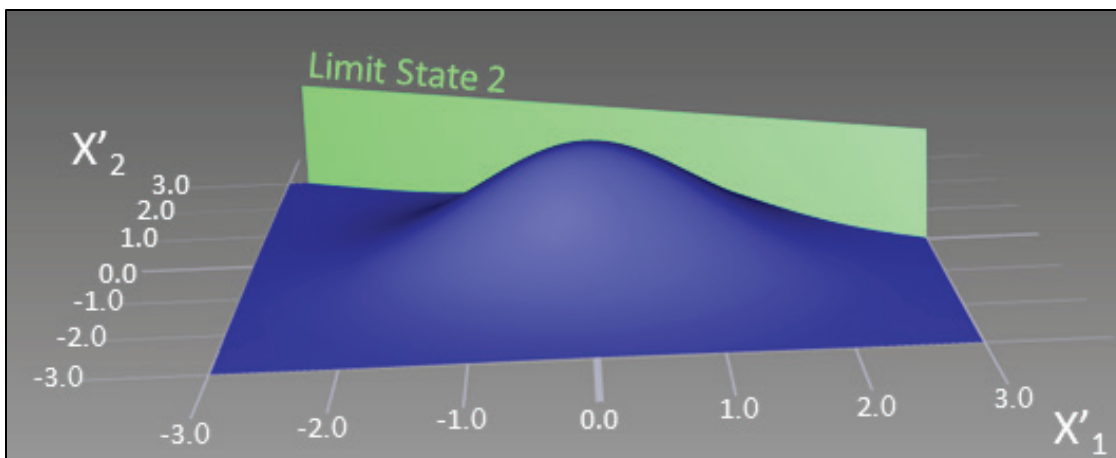


Figure 2-13. Cross section along the position of the limit state response surface through the PDF distribution in a 3-D rendering and the location of the design point for the response surface of limit state 2 – view from underneath.

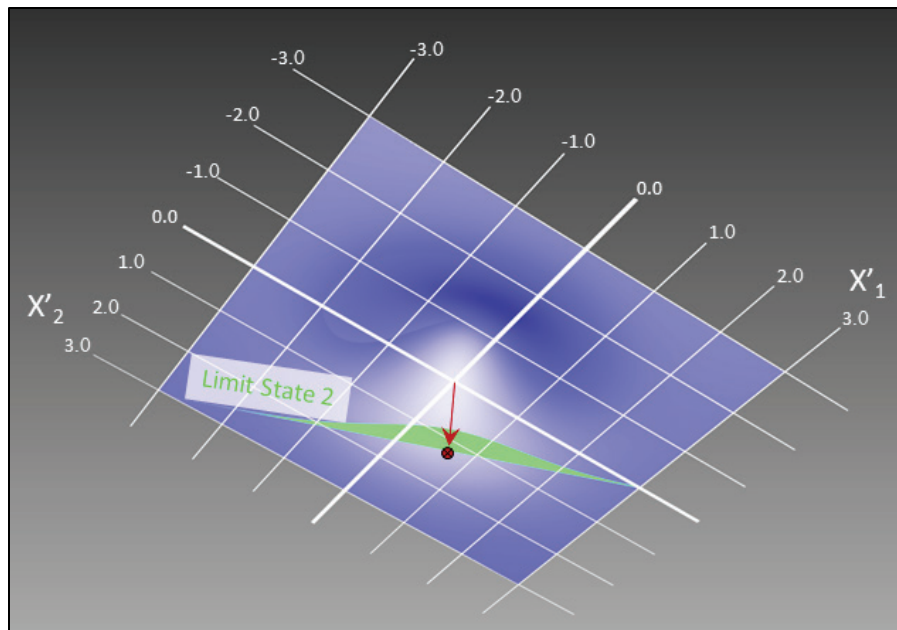
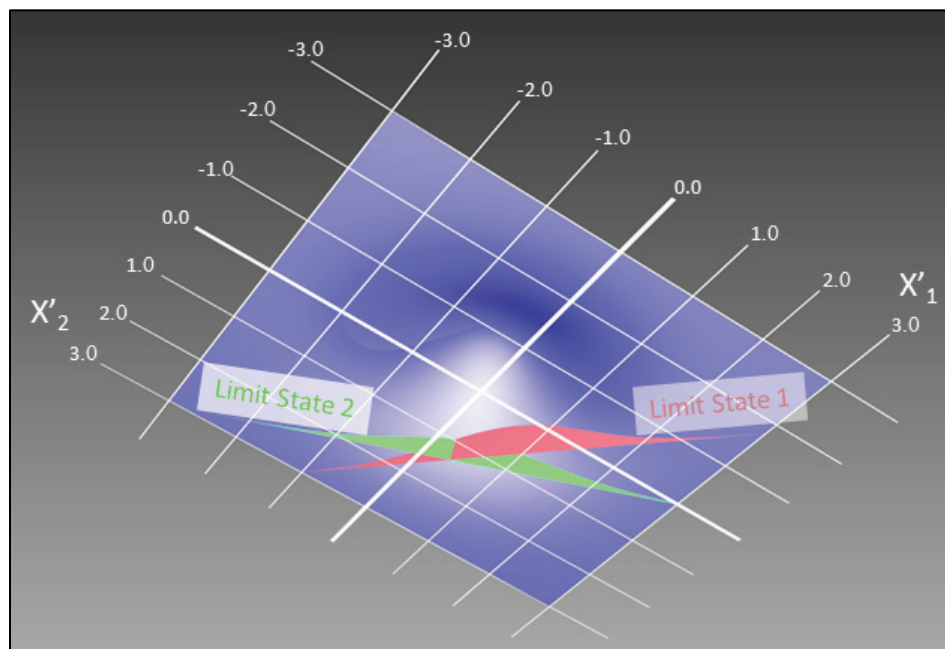


Figure 2-14 shows the cross sections along the position of the two limit state response surfaces through the PDF distribution in the 3-D PDF rendering. The viewpoint is made from below.

Figure 2-14. Cross sections along the position of the two limit state response surfaces through the PDF distribution in a 3-D rendering – view from underneath.



Multiple failure modes and the computation of PUP: The computation of the *PUP* value for this two-variable problem with two ULSs is best determined in the transformed basic variate space. Returning to the plan view of the PDF distribution in Figure 2-10, the *PUP* value is equal to the integral of the *PUP* volume up and to the right of both limit states. This region is shaded in light brown, green, and olive green. The Reliability Value is equal to the integral of the *PUP* volume for the region shaded in blue in this figure. The sum of the reliability and *PUP* values is 1.0 and equal to the volume of the entire PDF.

2.8 Implication of multiple limit states on the computed *PUP*

There are limited technical publications reporting on probabilistic investigations of multiple geotechnical limit states and multiple failure modes. Fan et al. (2014) discuss the reliability analysis of piles in spatially varying soils considering multiple failure modes. Specifically, they summarize their research into the performance state of vertical drilled shafts under combined lateral and axial loading using a reliability analysis. The performance of the vertical pile is defined in terms of the displacements induced by external loading. They consider three performance limit states: (1) lateral pile deflection, (2) angular pile distortion, and (3) axial pile movement at the top of shaft. The unsatisfactory performance event is said to occur if the induced displacements at the top of the pile are greater than the corresponding allowable displacements. Lateral pile response is computed using the *p-y* method of pile-to-soil interaction analysis, and vertical displacements are computed using a *t-z* model for pile-to-soil interaction.¹ Uncertainties in soil deformation and strength properties are considered in their reliability analysis. Each of the soil property variables used in the analytical model is characterized by values of mean, standard deviation, and distribution type (e.g., normal, lognormal). With the statistical properties defined, a Monte-Carlo simulation method is used to conduct the reliability analysis. They apply a conventional reliability methodology of computing a *PUP* value for each of the three displacement limit states as well as a *PUP* value for the system.

One of the important findings of the Fan et al. (2014) study is that the *PUP* for the system is greater than for any of the three individual displacement limit state modes and is less than the sum of the *PUP* values of the three

¹ Examples of the USACE CASE programs that perform lateral *p-y* analysis and axial *t-z* analysis are COM624G and CAXPILE, respectively.

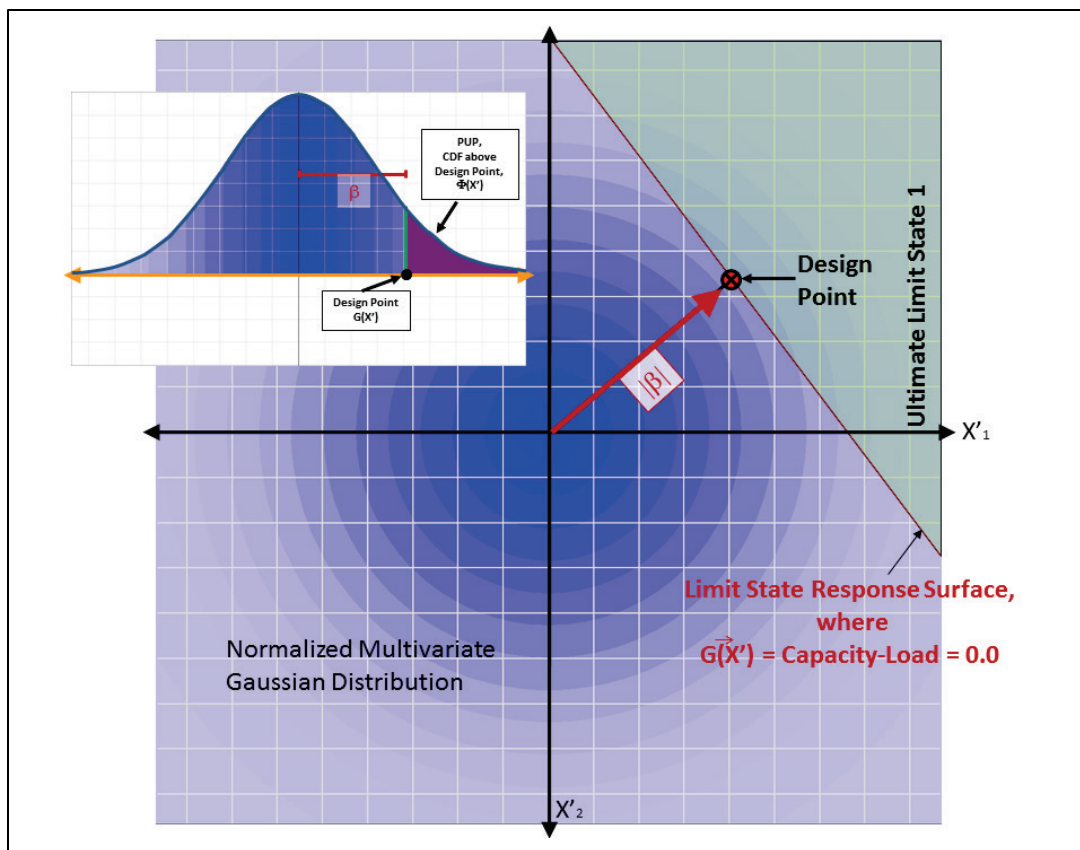
individual limit state modes if assuming the statistical independence among the three limit state modes. Fan et al. (2014) emphasize that the value for *PUP* will be underestimated if multiple displacement limit state modes are not considered simultaneously while the *PUP* value for the system will be overestimated if the dependence among the different displacement limit states is not considered.

The USACE hydraulic structures are designed to resist multiple limit states, which are listed in Section 2.2. When a hydraulic structure is viewed in terms of a reliability methodology, Fan's research indicates that the interrelationship between multiple performance limit states and a system *PUP* value is an important consideration. This is one of the research tasks investigated by the authors of this report, with the findings to-date in this Phase 1 portion of the two-phase research study discussed in Chapter 4 and Appendix A.

Accommodating multiple limit states and/or failure modes given multiple variables in the *PUP* calculation has been shown by the Fan et al. (2014) study to be important part of a reliability analysis. Therefore, Appendix A describes different methods that have been researched and/or devised by the authors of this report to estimate the overall *PUP* values for systems with multiple limit state response surfaces.

For a single limit state with a hyperplanar Limit State Response Surface, as shown in Figure 2-15, the calculation of *PUP* is simple and accurate. Because the variables are mapped to a normalized multivariate Gaussian distribution space, the integration of the area of unsatisfactory performance on one side of the hyperplane of distance β (at the design point) from the mean origin is the same as the area for the single variable normalized Gaussian distribution beyond the design point distance β .

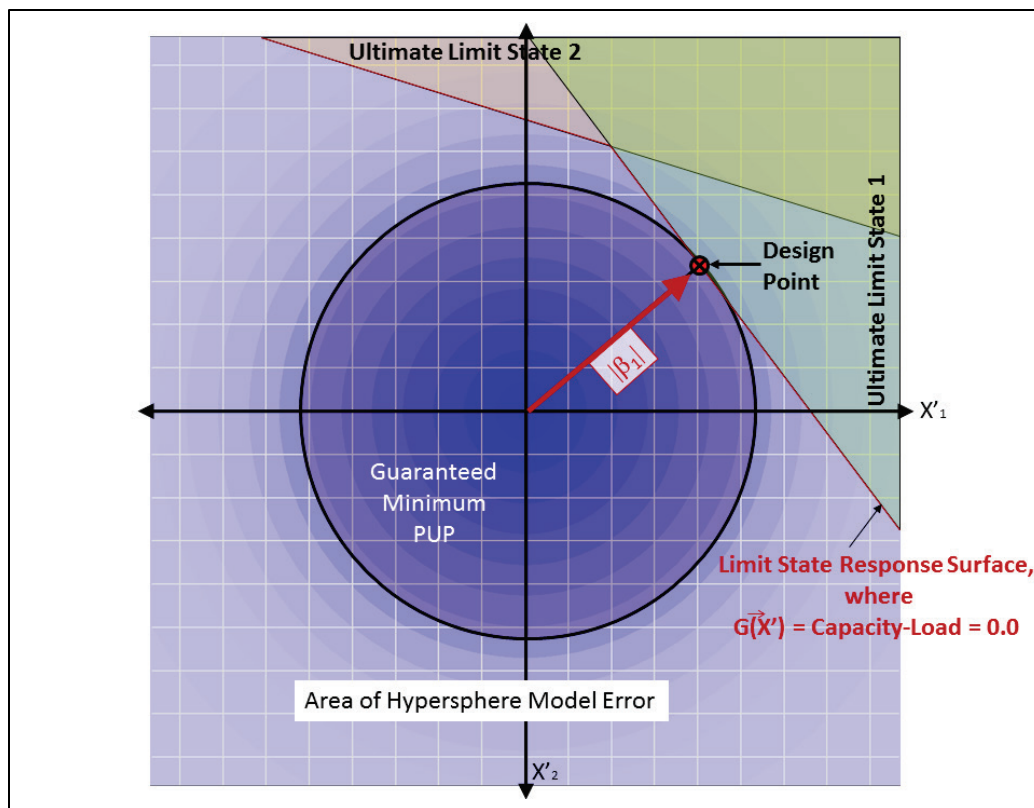
Figure 2-15. Mapping the integrated PUP from the multivariate hyperplane to a single Gaussian distribution.



This situation becomes more complicated as non-hyperplanar limit state response surfaces are considered. Figure 2-16 shows how two hyperplanar limit states combine to create a non-hyperplanar limit state response surface. In this case, the integral of the probabilities includes the probabilities under the second limit state that does not overlap the first limit state probabilities. The probabilities of the overlap of the second limit state and the first limit state only need to be accounted for once.

Because the design point exists at the point where the limit state response surface is nearest the origin of the normalized multivariate Gaussian distribution space, a way to conservatively estimate the *PUP* so that satisfactory performance is guaranteed is the based on the probability of the hypersphere of radius β , as shown in Figure 2-16. This estimate integrates the probabilities outside of the hypersphere to determine *PUP*. This value is conservative because the hypersphere is guaranteed to be completely in the area of satisfactory performance but is not guaranteed to encompass all of the satisfactory performance region.

Figure 2-16. Mapping the integrated PUP from the multivariate hypersphere to a single Gaussian distribution.



More accurate range finding *PUP* estimation methods provide a minimal range for the upper and lower bounds of the *PUP* using logical combination constraints given the limit state response surfaces, their number, and possible overlap information. The minimal range refers to the fact that the bounds are at the closest distance from each other based on the logical combinations. Choosing a value at the midpoint of the range guarantees an estimate that is accurate to within half the range width. Ang and Tang (1984) provide methods for finding the bounds of the overlap of regions based on Bayesian logic. These estimations are based on non-correlated variables, so they involve perpendicular response surfaces. The overlap regions involve the combination of comparing each limit state with respect to each other, as shown in the two limit state, 2-D example of Figures 2.17 and 2.18. The upper and lower bounds for the overall *PUP* would be the different summations and removals of probabilities based on the Bayesian combinations of non-correlated values as described in subsection A.3.1 of Appendix A. The difference in the bounds is described by the third overlap region between overlap regions A and B from Figures 2.17 and 2.18. Note that the bounds can be found for multi-variate, multi-limit state problems.

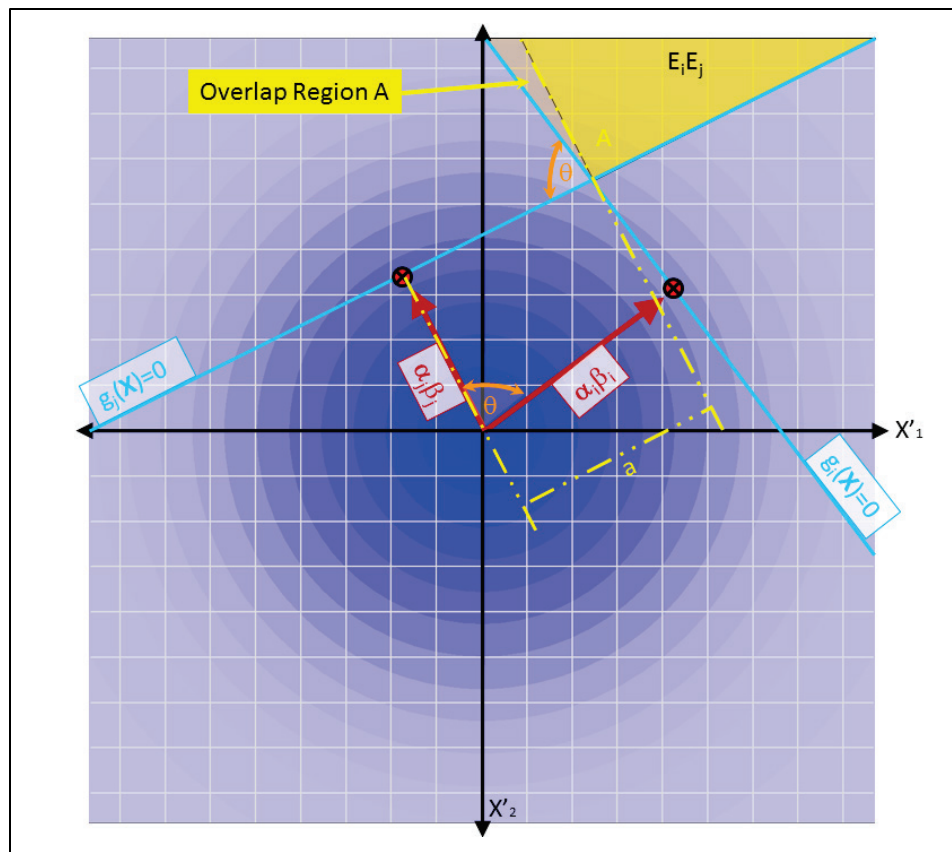
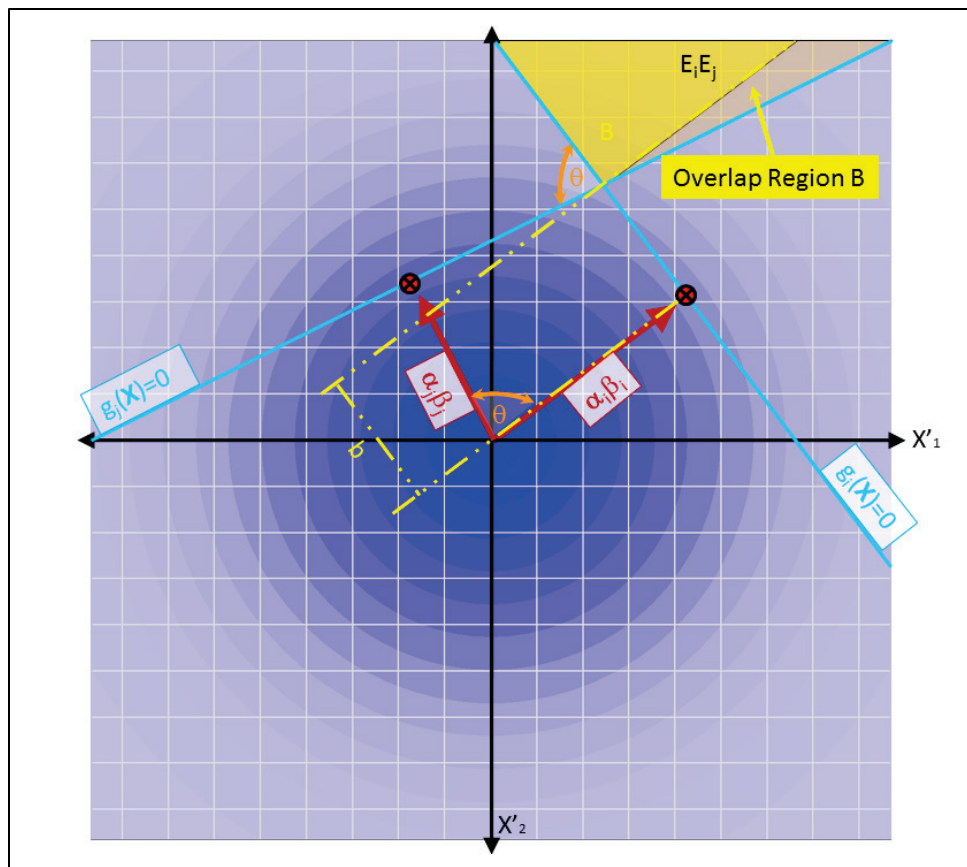
Figure 2-17. Finding the overlap region A given limit state I with respect to limit state J .

Figure 2-18. Finding the overlap region B given limit state J with respect to limit state I .

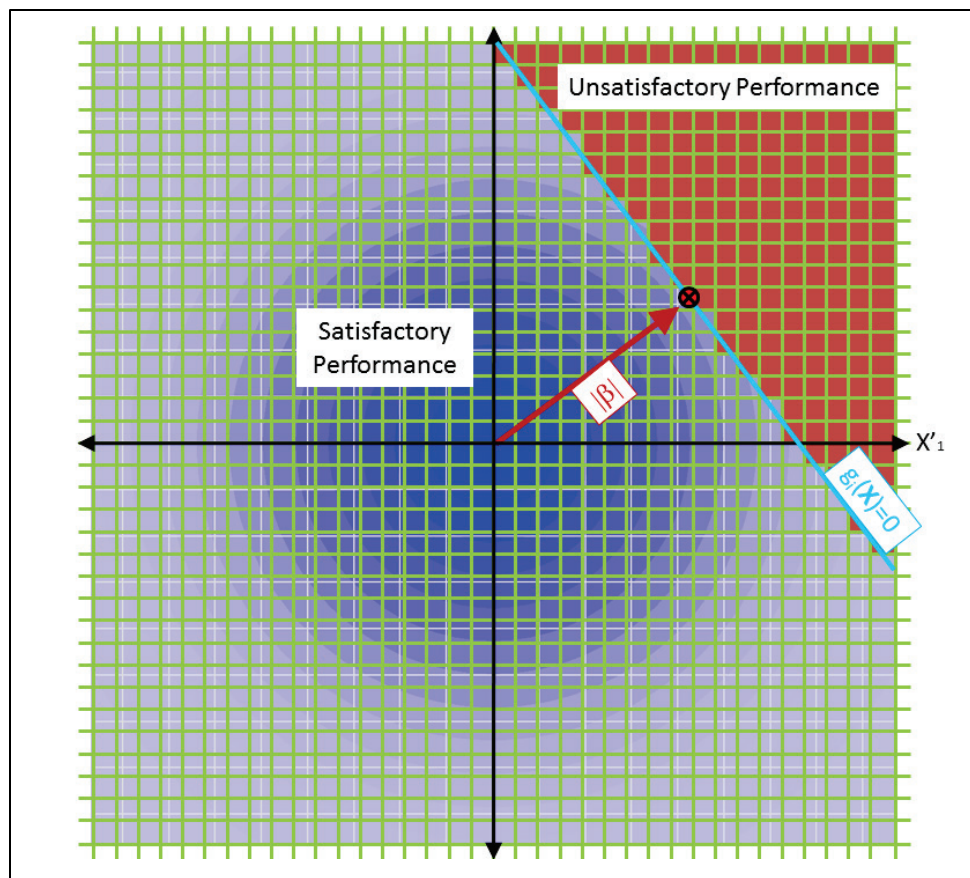
The range-finding methods were developed at a time (e.g., Ang and Tang, 1984) when computational power was not available to perform the more accurate numerical integration techniques that handle correlation between limit state response surfaces. This computational power is now available for a limited number of variables. However, range-finding methods are still useful in that they provide valuable information that bounds the range in *PUP* value with minimal computational effort.

Monte Carlo simulation methods can also be used to estimate *PUP*. Samples would be taken at random from the variable space according to each variable's distribution; then, the unsatisfactory or satisfactory result of the sample would be computed. Many samples would be taken, and the ratio of unsatisfactory results to the overall number of samples would give the *PUP* estimate. For a full Monte Carlo simulation, a large number of samples (>10,000) would be required. The Latin Hypercube sampling method reduces the number of samples required by guaranteeing that samples are not repeatedly chosen in the same regions. Also, Importance Sampling lowers the error in calculation of the probability, by sampling

about the design point, which is the point of the lowest β distance for all of the limit states (Figure 2-16), on the response surface for the combined limit states. To perform Importance Sampling, the design point, which gives the point on the closest limit state to the origin of the normalized Gaussian distribution, needs to be found. This can be done using the ASM techniques. A major concern for the simulation methods is that there are no guarantees that sample points will be generated in all of the limit state unsatisfactory performance regions, when multiple limit states exist. This concern is mitigated by the fact that the limit state regions that are not covered will typically have very low probabilities.

Numerical integration can also be used to estimate *PUP* but can quickly grow to be unsolvable in a timely manner, even on the most advanced computers. The authors believe that for a small number of variables, this method may provide results in a timely manner. Figure 2-19 shows an integration method for estimating the *PUP* value. The space is divided into a hypergrid, and each grid cell probability would be added to the *PUP* if the variable values would lead to unsatisfactory performance. The two sources of error for this calculation would be the resolution of the hypergrid and the range of the calculations. By having the range vary from -5 to 5 standard deviations from the mean for each variable, the error can be minimized. An improvement to the integration method utilizes the fact that the probability of a normalized Gaussian distribution stays the same at a given radius; therefore, a polar hypergrid could be used so that the probability calculations would be quicker. These numerical integration methods are describe in Appendix A subsections A.3.2 and A.3.4. Because all of the variable spaces are mapped for integration and the solution looks for any unsatisfactory performance in any grid cell (voxel, or volume element), this solution method works for multi-variate, multi-limit state problems.

Figure 2-19. Spatial subdivision for numerical integration and computation of a PUP value with two variables with Gaussian distributions given a single hyperplanar limit state.



2.9 Published design values for reliability index β and/or PUP

The USACE hydraulic structures are designed to resist multiple limit states, as listed in Section 2.2. Some of these design limit states fall within the sphere of influence of the structural engineering community while others are shared between the geotechnical and structural community. In general, the structural community both inside as well as outside the USACE has embraced the usage of β values whereas the geotechnical community has embraced the use of Probability of Failure, P_f , values and the use of PUP values. It is the viewpoint of the authors of this R&D report that this distinction has more to do with how reliability theory is being applied by the two disciplines of civil engineers when dealing with the different and distinct limit state evaluations being conducted during the design process.

The application of reliability theory to the first three design limit states listed in Section 2.2 have been dominated in the civil engineering technical

literature by studies and decisions made by civil engineers within the structural community. These mechanisms are (1) flexural hinging of individual structural members, (2) shear failure of individual structural members, and (3) buckling failure of individual structural members. For the reinforced concrete hydraulic structures being designed by the USACE, these three limit states dictate the size of the concrete members, the required amount of reinforcement steel to be placed within the individual reinforced concrete members, and the minimum compressive strength of the concrete as well as the yield strength of the steel reinforcement. In the application of reliability theory in structural limit state design, the structural engineering community tend to use a β value rather than a P_f or PUP value. One example taken from outside of the USACE technical literature is in the application of LRFD to the design of bridges: The bridge community of structural engineers has developed load and resistance factors so that reliability indices for bridges designed using the new (bridge design) code would be close to the a target Reliability Index β_{Target} of 3.5 referenced to a target period of 75 years (service life) by the discussions found on page 283 of Nowak and Collins (2013), on pages 48–49 in Ellingwood¹, and on page 245 in Fenton et al. (2016). The authors of this R&D report observe it is important to recognize that when expressing reliability-based design criteria in terms of a β design value, the value for β will need to be changed when the design life is changed. Different types of structures, as well as USACE projects, have different values for design life. The length of design life (usually 50, 75, or 100 years) is associated with the type of structure. In general, the longer the design life the more vital the structure is to society and frequently, the more costly it is to design and build.

The major USACE hydraulic structures are a significant financial investment. Consequently, these have a 100-year design life with an anticipated retrofit of superstructure, especially surfaces features and mechanical systems, mid-way in time during their design life. Another detail for consideration is that once in place, the substructure foundations, and especially those for hydraulic structures, are costly to retrofit relative to the superstructure. There are often accessibility issues to the foundation. The items will need to be considered when selecting a β value for use in the design of the superstructure and for the foundation substructure. The authors of this R&D report pose this question: Should

¹ Ellingwood, B. R. 2015. *Procedures for Developing Reliability-Based Load Criteria for Hydraulic Steel and Concrete Structures*. Final letter report submitted to the URS Group, Inc. U.S. Army Corps of Engineers.

the design β values used in the design of the superstructure and for the foundation substructure be the same?

Regarding reliability theory, the geotechnical engineering community is most involved with the application of the theory to the limit states of sliding, bearing, and overturning within the foundation soil (or rock). In the past, a significant portion of the geotechnical reliability community has been making use of the probability of failure, P_f , rather than focusing on the value of β . β is determined from the P_f value by applying reliability theory to a project feature's design for these limit states and for a specified design life and PDF distribution.

In their report on a reliability-based analysis of sliding and bearing limit states of a retaining structure, Sleep and Duncan (2014) observe that estimating the probability of failure is a major motive for applying probability theory in geotechnical engineering. They also comment that most geotechnical engineers prefer to use P_f rather than β because P_f has an easily understood relationship to likely performance and β does not.¹ The authors of this report comment that this viewpoint may not be universal, especially when project feature designs involve the mixing of structural and geotechnical disciplines. For example, Fenton et al. (2016) provide their reliability-based target design criteria in terms of both target failure probabilities and β values when describing reliability-based geotechnical design in the 2014 Canadian highway bridge design code.

Recall that the PUP or P_f value is uniquely related to β by Equation 2.16, given an assumption for the distribution of the FoS (e.g., normal, lognormal). It is not unusual to see a lognormal or a normal distribution being assumed for the FoS against sliding failure in proposed reliability based, retaining wall design procedures with a focus on sliding stability (e.g., Sleep and Duncan 2014).

The authors of this R&D report observe that P_f and/or PUP values will need to be converted into load and resistance factors for use in LRFD design with consideration of individual and multiple limit states for the USACE hydraulic structures. Converting the PUP values to load and resistance factors will provide the means for a combined LRFD. It is anticipated that this computational process will involve the conversion of

¹The authors of this report observe that β is a simplified probabilistic description of Reliability given a normal distribution (in a Gaussian variate space).

P_f and/or PUP values into reliability indices for each of the involved limit states. This can be done using reliability transformation relationships given in previous sections. For example, Equation 2.37 serves to demonstrate the conversion of a β value into a value for the load factor. In preparation for this action, this section collects reliability data from the non-USACE technical literature for geotechnical limit states: sliding, bearing, and/or overturning failures. Most of these data are expressed in terms of P_f with some also reporting β values. When unavailable, P_f values have been converted by the authors of this report into β values, assuming a normal distribution.

2.9.1 1994 Reliability analysis of proposed dykes of James Bay hydroelectric project in Quebec, Canada

In 1994, Christian, Ladd, and Baecher¹ described a formal calculation of reliability and risk to the design of 50 kilometers (km) of dikes for a James Bay hydroelectric project. This slope stability assessment was used to help to decide among the alternatives to new dyke construction on a soft clay foundation. These alternatives were construction of the embankment in a single stage, to either a height of 6 or 12 meters (m) with one berm. The second alternative involved the construction of the dyke in multiple stages to allow the soft clay foundation to dissipate the excess pore water pressures generated during berm construction: Following the installation of vertical drains within the clay foundation, the first stage would consist of two berms built to a height of 12 m. Then, the zone with the drains would be allowed to consolidate to 80% of primary consolidation, and the second stage would be built to a height of 23 m. Their analysis formally considered the uncertainty in soil strength parameters in the reliability-based slope stability, culminating in the calculation of a P_f for slope failure for each dyke construction alternative. This study was special because of the significant geotechnical data base of soil properties and strength data for the project as well as the fact that Drs. Christian and Baecher are pioneers in the application of reliability theory to solve geotechnical foundation problems. Dr. Ladd is recognized as an expert in characterizing the strength of soft clays, which was a key aspect of the design challenge for the construction of these dykes at this project. Today, this study is considered a landmark reliability-based study by the geotechnical community even though the project was not built.

¹ Christian et al. (1994)

Target probability of failure: The design approach used for the dykes started with a probabilistic-based interpretation of the historical data on earthen dam failure since slope failure is one of the categories of earthen dam failures. Baecher et al. (1980) determined that earth dams seem to have probabilities of failure of approximately 10^{-4} per dam-year for a wide range of locations and times of construction. Their study suggests that approximately one-third of the failures are due to slides (with another third due to overtopping and the final third due to the category of internal erosion, piping, or seepage). Using this historical data, Christian et al. (1994) computed an overall probability of slope failure for a 100-year life, equal to 10^{-2} ($= 10^{-4}$ times 100 years), in the absence of specific attention during design and construction to mitigate the hazards. With less than one-third of the historical failures being due to slope instability, Christian et al. suggested a target probability of failure of 10^{-3} ($= 1/3$ times $10^{-2} = 3.3 \times 10^{-3}$, rounded down to 10^{-3}) as being reasonable for design purposes and consistent with past experience. These target reliability data are summarized in Table 2-2. The risk evaluation stage of the dyke construction assessment introduced the consequences of failure into the evaluation and is summarized next.

Table 2-2. Target probability of slope failure and reliability index for a dyke with a life of 100 years – based on the historical rate of failure of dams.

Project and Location	Historical Rate of Failure of Dams (per dam year)	$P_{f,100}$ (per dam life of 100 years)	Proportion of Sliding Failures to All Types of Dam Failures	Intermediate Calculation of $P_{f,100}$ (per dam life of 100 years)	Target $P_{f,100}$ for Design Purposes (per dam life of 100 years)	Reliability Index, β
Proposed dykes of James Bay hydroelectric project (Quebec, Canada); 1994	1×10^{-4}	1×10^{-2}	1/3	3.3×10^{-3}	1×10^{-3}	3.09*

*Computed by the authors of this report and assuming a normal distribution for the PDF of the FoS.

Consideration of risk and consistent factors of safety: Christian et al. (1994) recognized that a slope failure during dyke construction would primarily be associated with the costs of reconstruction. However, a slope failure during operation of the hydroelectric project would involve much larger economic costs as well as other public consequences. Thus, Christian, et al. introduced different target reliability indices or probability of failures for the different

stages in the life of the project by informal consideration of project risk through the assignment of what they refer to as “consistent” factors of safety. This information is summarized in Table 2-3.

Table 2-3. Consistent factors of safety associated with target probability of slope failures and reliability Indices for a dyke with a life of 100 years.

Project and Location	Case	Target $P_{f,100}$ for Design Purposes (per dyke life of 100 years)	Consistent FoS (without model error)	Reliability Index, β
Proposed Dykes of James Bay hydroelectric project (Quebec, Canada); 1994	Single stage, H = 6 m, circular failure surface	1×10^{-2}	1.63	2.325 *
	Single stage, H = 12 m, circular failure surface	1×10^{-3}	1.53	3.09 *
	Multistage, H = 23 m, wedge failure surface	1×10^{-4}	1.43	3.718 *

*Computed by the authors of this report and assuming a normal distribution for the PDF of the FoS.

The consistent FoS listed in Table 2-3 was obtained by rearranging the Christian et al. (1994) Equation 9 into the form

$$E[F] = \frac{1.0}{1.0 - \beta * COV[F]} \quad (2.59)$$

β values are listed in Table 2-3. Values for the coefficient of variation of the FoS $COV[F]$ are obtained from the data contained in the Christian et al. (1994) Table 7 variance in FoS for each of the three dikes. This variance in FoS comes from using Christian et al. (1994) Equation 13 (a first-order, second moment method) in conjunction with slope stability analyses. An example of the application of the variance decomposition technique they used is outlined in their Table 5.

Note that once the design probability of failure is defined for the project feature and thus the value for the Reliability Index β , then by Equation 2.59, one only needs the value for coefficient of variation in FoS $COV[F]$ to define the value for the consistent FoS. This equation bridges the reliability/statistical data of β and $COV[F]$ values back into an expected FoS value, $E[F]$.

Because of the consequences of failure of a major portion of the completed multistage, 23 m high embankment are expected to be greater than those for failure of a 12 m high embankment and because experience with staged construction is less fully evaluated for this project, the target probability for the multistage case was reduced from 10^{-3} (for the 12 m high embankment) to 10^{-4} . With this P_{f-100} value, a target FoS equal to 1.43 was determined from the reliability-based slope stability data for the multistage construction by Christian et al. (1994).

Christian et al. rationalized that the consequences of failure of a portion of the smallest dike (6 m) might be only one-tenth as great as the 12 m high dyke, so the target P_{f-100} value was increased from 10^{-3} to 10^{-2} . With a P_{f-100} value of 10^{-2} , a target FoS equal to 1.63 was determined by them from the reliability-based slope stability data for the 6 m high dyke using Equation 2.59.

Christian et al. (1994) reasoned that because experience with staged construction is less fully evaluated and the consequences of failure of a major portion of the completed multistage embankment are expected to be greater than those of the failure of a 12 m high embankment, the target probability for the multistage case was reduced by a factor of ten to 1×10^{-4} . This value contrasts with the P_{f-100} equal to 1×10^{-3} value for the 12 m high dyke (Table 2-3).

The Christian et al. (1994) reliability-based slope stability results led the committee to recommend that a FoS equal to 1.5 be adopted for feasibility studies and preliminary cost estimates for both single and multistage dykes. In closing comments, they observed that a prior committee had recommended using an FoS equal to 1.4 for the minor dykes and 1.7 for the major, multistage dykes.

2.9.2 Revisited reliability calculations of the proposed dykes for the James Bay hydroelectric project in Quebec, Canada

El-Ramly et al. (2002a) revisited the reliability-based calculation for the proposed design of the 50 km of dikes for a James Bay hydroelectric project. Only one design was re-evaluated in their study: the single stage construction of a 12 m high embankment with a slope angle of 18.4 degrees (3H:1V) and a 56 m wide berm at mid-height. Three probabilistic procedures of analysis were used: (1) an integrated slope stability analysis based on the use of a Monte Carlo simulation technique, (2) an FOSM

approach in which the PDF distribution for the FoS against sliding failure was assumed to be either normal or lognormal (to the base e), and (3) a simplified analysis that ignored the spatial variability of soil properties. This latter analysis was made to contrast with the results of the first two analyses in which the spatial variability of soil properties was accounted for by using a variance reduction function and an autocorrelation distances (each with an corresponding scale of fluctuation distance) for soil strength along the slip plane within each of the soil layers. Their probabilistic slope stability assessment made use of the Bishop method of slices for circular slip surfaces. Note that this type of stability analysis cannot accommodate wedge-shaped and curved noncircular surfaces constructed using planar segments. The probabilistic results for the El-Ramly et al. (2002a) re-evaluation is summarized in Table 2-4 for the single stage construction of a 12 m high embankment for their FOSM reliability analysis results. Recall the Christian et al. (1994) target P_{f-100} for a 100-year design life is equal to 1×10^{-3} . It is included in this table for reference. The El-Ramly Table 2-4 results using an updated reliability based, procedure of slope stability analysis show that their expected FoS values are somewhat lower than the Christian et al. results and that the annual probability of failure values is much higher. Additionally, their reliability index values are lower than the Christian et al. (1994) target value.

Table 2-4. Computed expected factors of safety, annual probability of failure and reliability indices for the proposed single stage construction of a 12 m high embankment dyke at the proposed James Bay project.

Reference	Assumed Distribution for FoS	Probability of Failure, P_f	Expected FoS $E[FS]$	Standard Deviation for the FoS $\sigma[FS]$	Coefficient of Variation for the FoS COV_{FS}	Reliability Index, β
Christian et al. (1994)		1×10^{-3}	1.53 *			3.09 **
El-Ramly et al. (2002a)	Normal	7.74×10^{-3} ***	1.46	0.19	0.13	2.42
Duncan et al. (2003)	Normal	1.9×10^{-1}	1.17	0.194 **	0.166	0.877 **
El-Ramly et al. (2002a)	Lognormal	2.15×10^{-3} ***	1.43	0.19	0.13	2.855 ***
Duncan et al. (2003)	Lognormal	1.9×10^{-1}	1.17	0.194 **	0.166	0.872 **

*Consistent FoS.

**Computed by the authors of this report using the PDF listed in the second column of this table.

*** Recomputed by the authors of this report to correct for computational errors made in the original paper.

Table 2-4 also includes results from the Duncan et al. (2003) re-evaluation of the El-Ramly et al. (2003a) results using yet another type of reliability based procedure of slope stability analysis. One important feature of their slope stability evaluation was their ability to analyze wedged-shaped and curved noncircular slip plane surfaces constructed using planar segments. They found their critical slip planes to be geometrically different from the critical circular slip planes found in the El-Ramly et al. (2002a) and Christian et al. (1994) analyses. The Duncan et al. results indicate a significantly lower value for the expected FoS value of 1.17 as compared to both the Christian et al. result of 1.53 and the El-Ramly et al. results of 1.46 and 1.43. As would be expected in the case of comparable values for the standard deviation of the FoS, the Duncan et al. computed probability of failure value of 1.9×10^{-1} is much higher than the Christian et al. (1994) target probability of failure of 1×10^{-3} . Similarly, the Duncan et al. reliability index value of 0.87 is much lower than the (approximate) target value of 3. A subsequent El-Ramly et al. (2003b) noncircular wedge slip surface evaluation using Spencer's method of slope stability analysis result of $P_{f-I} = 1.2 \times 10^{-1}$ and $\beta = 1.136$ corroborated the Duncan et al. (2002) probabilistic results of a high probability of failure for the critical failure mechanism.

The revisited reliability-based calculations for the proposed design of the 50 km of dikes for a James Bay hydroelectric project made by El-Ramly et al. (2002, 2003b) and made by Duncan et al. (2002) provides the basis for an important observation: Both teams of engineers comment that it is absolutely essential to identify the critical failure mechanism. If the critical slip surface is not located and used in the probabilistic analysis, then no probabilistic methodology can overcome this deficiency, and the computed value for the probability of failure will be meaningless.

The James Bay dyke system provided a case study used for conducting a reliability based designs/analyses for a geotechnical structural feature whose limit state is dominated by a sliding slope failure mechanism. Recall that this dyke system was not constructed. Therefore, the link between design target β or conversely, *PUP* values, and field performance is not established in this example. The next subsection discusses a case study that demonstrates this link using the example of a tailings dyke constructed to retain tailings sands from an oil sands mine operation in Canada.

2.9.3 Probabilistic slope stability analysis calculations of a Syncrude Tailings Dyke on pre-sheared clay-shale in Fort McMurray, Canada

In 2003, El-Ramly (2003a) performed a 2-D probabilistic slope stability analysis of a 44 m (144 ft) high Syncrude Tailings Dyke on a pre-sheared clay-shale foundation in Fort McMurray, Canada. The project's target FoS was 1.3. Tailings cell 23 (out of 30 cells) was completed to final design elevation in 1993. The maximum lateral movement recorded was 0.437 m (1.4 ft) in this tailings cell. The movement has been negligible since 1996, and the performance of the dyke has been deemed adequate by El-Ramly et al. An El-Ramly et al. (2002a) spread-sheet-based Monte Carlo probabilistic simulation procedure of analysis was applied to this tailings cell. This procedure of solution has its origins in the Vanmarcke (1977a, 1977b, 1980, 1983, and 2010) random field-based reliability analysis as applied to slope stability. Specifically, El-Ramly et al. (2003a) uses the variability of local averages over adjacent segments of the slip surface versus modeling the point-to-point variability of soil properties along the slip surface, which is difficult to obtain in practice. The portion of the slip surface within each soil layer is divided into segments of a length equal to the scale of fluctuation of that material. The varying soil property along an individual segment is modeled by one random variable representing the

spatial average of the property over the segment length.¹ The noncircular wedge slope stability analysis used included consideration of variability in soil strength and pore water pressures through a spatial averaging technique in which point estimates are combined with variance reduction of random fields approach (e.g., using a [horizontal] autocorrelation distance r_o of 33 m in the pre-sheared clay-shale). By Equation D.6, an r_o of 33 m corresponds to a scale of fluctuation value of 66 m (216 ft). They observed that the analytical performance of the dyke was governed by uncertainties about geotechnical material properties and pore water pressures. The spreadsheet-based Monte Carlo simulation analysis resulted in a *PUP* value of 1.6×10^{-3} . The results for the El-Ramly (2003a) probabilistic study is summarized in Table 2-5.

Table 2-5. Values of the expected value and standard deviation of the FoS (FS) against sliding failure, *PUP*, reliability index and corresponding Syncrude Tailings Dyke performance in terms of horizontal deformation.

Project	Project Location	E[FS]	σ [FS]	<i>PUP</i>	Reliability Index	Horizontal Deformation (m)	Reference
Syncrude Tailings Dyke, 44 m high	Fort McMurray, Canada	1.31	0.14	1.6×10^{-3}	2.31	0.437	El-Ramly et al. (2003a)

The β value of 2.31 was computed by El-Ramly (2003a) using the relationship

$$\beta = \frac{E[FS] - 1.0}{\sigma[FS]} \quad (2.60)$$

These probabilistic results indicate that the 0.437 m (1.43 ft) horizontal deformation of this 44 m high tailings dyke founded on pre-sheared clay-shale, with an expected value for FoS against sliding of 1.31, has a *PUP* equal to 1.6×10^{-3} and a corresponding reliability index of 2.31. Again, considering this type of structure, the dyke performance is deemed adequate in their study.

A parametric study conducted by El-Ramly (2003a) showed that by ignoring the spatial correlation of the earth data and relying only on the means, standard deviations, and probability distribution functions resulted in a *PUP* equal to 1.6×10^{-2} . This value is one order of magnitude

¹ By assigning slip plane segmental lengths that are equal to the scale of fluctuation, the variance reduction due to spatial averaging can be ignored, and the correlation coefficients between most of the variables representing the local averages approach zero, as per Vanmarcke (1977a).

higher than the probabilistic analysis that considered spatial correlation (Table 2-5). This is an important detail: Christian and Baecher (2011) observe that uncertainty in soil properties is being overestimated, helping to explain why failures are less frequent than reliability studies have been predicting. Refer to Christian and Baecher (2011) for their discussion of 10 unresolved problems in geotechnical risk and reliability.

2.9.4 Reliability-based target design criteria in the 2014 Canadian highway bridge design code (CHBDC)

In 2014, a revised version of the CHBDC was published. It is of importance to this R&D effort because reliability-based criteria and procedures were used to establish the design criteria and LRFD design values. Fenton et al. (2016) provides detailed discussion of the code's reliability-based target design criteria in terms of both target failure probabilities and β values when describing reliability-based geotechnical design in the 2014 CHBDC.

Under the CHBDC LRFD approach and using Fenton et al. (2016) notation, Canadian bridge designs must satisfy

$$\phi_g * \hat{R} = \sum_i I_i * \eta_i * \alpha_i * \hat{F}_i \quad (2.61)$$

where:

ϕ_g = the geotechnical resistance factor with the subscript referring to *geotechnical* or *ground*

$\hat{R}$ = the characteristic geotechnical resistance based on characteristic ground parameters

I_i = is a structure importance factor for the i^{th} characteristic load effect $\hat{F}_i$

η_i = a load combination factor

α_i = the load factor

$\hat{F}_i$ = the i^{th} characteristic load effect.

Fenton et al. (2016) note that the word *characteristic* is used because it suggests a value that characterizes a design parameter that is uncertain, (e.g., a random load or resistance). They feel that the words *nominal* or *design* do not convey the underlying randomness of the design parameter. In some Eurocodes, there is a specific statistical definition of the word *characteristic*, often as being the 5th or 95th percentile, whichever leads to the highest probability of failure. Eurocode 7 (CEN 2004) defines a

characteristic parameter in geotechnical design as the value selected such that the probability of occurrence of the associated limit state does not exceed 5% (i.e., a reliability of 95% with respect to the performance of the structure). Hicks (2013) provides a detailed explanation of this code's characteristic values.

2.9.4.1 ULS

For the ULS under the CHBDC LRFD approach, the geotechnical design for Canadian bridges will need to satisfy

$$\psi * \phi_{gu} * \hat{R} = \sum_i I_i * \eta_i * \alpha_{iu} * \hat{F}_{iu} \quad (2.62)$$

in which the notation is the same as for Equation 2.61 but with the addition of a subscript u to designate the ULS. The consequence factor ψ , listed in Table 2-6, accounts for failure consequences. Note this consequence factor is applied to the resistance term (versus the demand/load term).

**Table 2-6. The 2014 CHBDC ULS and SLS
consequence factor values
(after Fenton et al. 2016).**

Consequence Level	Consequence factor, ψ
High	0.9
Typical	1.0
Low	1.1

The targeted theoretical maximum 75-year-lifetime bridge ultimate (limit state) failure probability, p_{f-n} with n equal to 75 years (p_m in Fenton et al. [2014] notation) is 1/5,000 for a typical Canadian bridge foundation. Fenton et al. (2016) describes the basic idea of the consequence factor is to adjust the maximum acceptable factor of the design down (e.g., to 1/10,000) for high failure consequences (relative to a typical consequence case) or up (e.g., to 1/1,000) for low failure consequences. Table 2-7 lists the corresponding β_{f-75} values of the foundation ULS for the three consequence levels.

Table 2-7. The 2014 CHBDC geotechnical design ULS consequence factor values and reliability criteria (after Fenton et al. 2016).

Consequence Level	p_{f75}	Reliability Index, β_{f75}
High	1/10,000	3.7
Typical	1/5,000	3.5
Low	1/1,000	3.1

By Melchers (1999) Equation 6.17, the annual probability of limit state failure, p_{f-1} , can be (approximately) determined from the $n = 75$ years lifetime (limit state) failure probability p_{f-75} using

$$p_{f-1} = \frac{p_{f-n}}{n} \quad (2.63)$$

Using Equation 2.63, the annual probability of (ULS) failure for the 2014 CHBDC criteria is computed, and the results reported in Table 2-8 in the column labeled “ p_{f-1} .” The USACE hydraulic structures are designed for an n equal to 100 years (ER 1110-2-8159; HQUSACE [1997]). It is of interest to see what an equivalent CHBDC-type probabilistic criteria would be for a 100-year lifetime instead of its 75-years lifetime. In an n equal to 100-year lifetime, the probability of limit state failure, p_{f-100} , is determined by using Melchers (1999) Equation 6.19,

$$p_{f-n} = 1 - (1 - p_{f-1})^n \quad (2.64)$$

Computations were made, and the results reported in Table 2-8 column labeled p_{f-100} . Its corresponding annual β_{f-1} is also computed and listed in this table. Values for p_{f-100} and the β_{f-100} are computed using the Excel statistical function NORM.INV[(1- p_{f-100}), $\mu = 0$, $\sigma = 1$] and are listed in this table.¹

¹ μ is the Greek symbol representing a mean value, and σ represents the standard deviation.

Table 2-8. Expanded table of 2014 CHBDC geotechnical design ULS consequence factor values and reliability criteria.

CHBDC (2014) Document				Calculated			
Consequence Level	p_{f75}		Reliability Index, β_{f75}	Pf-1 (Annual)	Annual Reliability Index, β_{f1}	p_{f100}	Reliability Index, β_{f100}
High	1/10,000	1.0×10^{-4}	3.7 (3.72)*	1.33×10^{-6}	4.69	1.33×10^{-4}	3.65
Typical	1/5,000	2.0×10^{-4}	3.5 (3.54)*	2.67×10^{-6}	4.55	2.67×10^{-4}	3.46
Low	1/1,000	1.0×10^{-3}	3.1 (3.09)*	1.33×10^{-5}	4.2	1.33×10^{-3}	3.0

* β_{f75} value is recalculated and is reported with two decimal places for greater accuracy when used in comparisons with β_{fn} values for other design lives of n years.

Observe the inverse relationship between values of β_{f-n} and p_{f-n} in the Table 2-8 data for each n years: As β_{f-n} goes up in value, the value of p_{f-n} goes down. This is consistent with the characteristics of a normal distribution curve. Recall that β is the number of standard deviations from the variable's PDF mean value to the variable's value x and that p_{f-n} is the PDF's area from negative infinity up to value x (and equal to the CDF value within this x -range).

β_{f-75} ranges from a low of 3.1 to a high of 3.7 for a 75-year design life for each of the three Canadian bridge design consequence levels. The Table 2-8 data calculated for a 100- year design life shows β_{f-100} ranges from a low of 3.0 to a high of 3.65. The two columns of data show that the design life n -value increases, the value for β_{f-n} decreases. However, the effect of a change in design life from 75 to 100 years is minor; β_{f-n} decreases in value by 2.9% or less for each of the three design consequence levels with this 25-year design life increase (Table 2-9).

Table 2-9. Comparisons of reliability indices for the 2014 CHBDC geotechnical design ULS with a 75-year design life and for a 100-year design life.

Consequence Level	Reliability Index, β_{f75}	Reliability Index, β_{f100}	$(\beta_{f100} - \beta_{f75})$	β_{f100}/β_{f75}
High	3.72	3.65	-0.07	0.98
Typical	3.54	3.46	-0.08	0.98
Low	3.09	3	-0.09	0.97

By the data contained in the second to the last column of Table 2-9, the 2014 CHBDC geotechnical design ULS-based reliability criteria would translate into a reduction in the β value for a 100-year (foundation) design life that is between 0.07 and 0.09 lower than the β value for a 75-year Canadian bridge design life.

The annual β_{f-1} ranges from a low of 4.2 to a high of 4.69 for a 75-year design life for each of the three Canadian bridge design (failure) consequence levels (Table 2-8). The difference between the annual reliability index value and that for a 75-year design life is significant, ranging in value from 21 to 26% lower values for the three design (failure) consequence levels.

2.9.4.2 Time variability of both loads and resistance and a theoretical reliability target value

Fenton et al. (2016) provide a thought-provoking discussion of the appropriate reliability index value and conclude with a basis for selection of a theoretical reliability target value. Summarizing their discussion on pages 244–245: Citing two clauses in an earlier (2006) version of the CHBDC, it specifies (in Clause 3.5.1) that calibration of load and resistance factors shall be based on a minimum annual reliability index of 3.75 for CL-625 loading (in accordance with Clause 3.8.3). If each year is considered to be independent, then for the 75-year design life, the annual β_{f-1} value would equal 3.75 for a Canadian bridge of typical consequence (failure) level. This would correspond to β_{f-75} equal to 2.5.¹ In actuality, each year is not actually independent. Consequently, it is unlikely that the actual lifetime reliability index is as low as 2.5. To determine the actual lifetime reliability index, Fenton et al. suggest consideration of the time variability of both loads and resistance. They observe that both structural and geotechnical systems often exhibit substantial degradation over a Canadian bridge's design life (of 75 years). They cite the continual degradation of geotechnical systems by pore-pressure variations, freeze-thaw effects, erosion, seismic motion, local liquefaction, etc. Even if the annual reliability index of 3.75 is achieved in the first year after construction, the 75-year reliability index would be a

¹ A value of ULS Reliability Index β_{f-75} equal to 2.5 (i.e., 2.478 before rounding) is obtained as follows: The value for p_{f-1} is derived from an annual Reliability Index set equal to 3.75 for a Canadian bridge of typical consequence level. With β_{f-1} set equal to 3.75, its corresponding p_{f-1} value would be 8.84×10^{-5} as calculated using Excel with the equation and statistical function of [1-NORM.S.DIST($\beta=3.75$,TRUE)]. For an $n = 75$ -year design life and using Equation 2.64 with p_{f-1} equal to 8.84×10^{-5} , p_{f-75} is calculated to be 6.608×10^{-3} . Its corresponding Reliability Index β_{f-75} is calculated to be equal to 2.478 using Excel with the statistical function of NORM.INV[(1- p_{f-75}), $\mu = 0$, $\sigma = 1$].

lower value for these reasons. Fenton et al. suggest that the actual 75-year reliability index will be somewhere between 2.5 (based upon assumed independence between years) and 3.75 (assuming neither degradation nor fluctuation in loads). They recognize that further research is needed on this complex issue of fluctuations and degradation with time and conclude that it is reasonable to assume that the actual reliability index will be somewhere between 2.5 (deemed a conservative estimate) and 3.75, with a recommended theoretical target of 3.5 for a 75-year lifetime. This CHBDC ULS target value is listed in Table 2-7 for a typical consequence level Canadian bridge. They feel it is reasonable to assume the lifetime reliability index will fall between 3 and 3.5 and would conservatively correspond to an annual reliability index of 3.75 for Canadian bridge (of typical consequence level). Until further research and development advances reliability methodologies and the required data on time variability of both loads and resistance to the point that it is robust enough to be formally included in design guidance (such as the CHBDC), the Fenton et al. logic described in this paragraph for a fixed target β value provides for a practical alternative.

2.9.4.3 Serviceability Limit State (SLS)

For the SLS under the CHBDC LRFD approach, the geotechnical design for bridges will need to satisfy

$$\psi * \phi_{gs} * \hat{R} = \sum_i I_i * \eta_i * \alpha_{is} * \hat{F}_{is} \quad (2.65)$$

in which the notation is the same as for Equation 2.61 but with the addition of a subscript s to designate the SLS. According to the CHBDC, the SLS corresponds to behavior of the ground that causes unacceptable conditions, such as deformations, that restrict the intended use of the supported structure and/or geotechnical system. The serviceability geotechnical resistance is the load the ground can support at the SLS, usually at a predefined deformation in the ground or structure, estimated using characteristic geotechnical parameters. Table 2-10 lists the corresponding β_{s-n} values of the foundation SLS for the three consequence levels.

Table 2-10. The 2014 CHBDC SLS consequence factor values and reliability criteria (after Fenton et al. 2016).

Consequence Level	p_{s-75}	Reliability Index, β
High	1/1,000	3.1
Typical	1/500	2.9
Low	1/100	2.3

Using Equation 2.63, the annual probability of exceeding the SLS for the 2014 CHBDC is computed and the results reported in Table 2-11, along with its corresponding annual serviceability β_{s-1} . Values for p_{s-100} and the β_{s-100} are also given.

Table 2-11. Expanded table of 2014 CHBDC SLS consequence factor values and reliability criteria.

CHBDC (2014) Document				Calculated			
Consequence Level	p_{s-75}		Reliability Index, β_{s-75}	P_{s-1} (Annual)	Annual Reliability Index, β_{s-1}	p_{s-100}	Reliability Index, β_{s-100}
High	1/1,000	1.0×10^{-3}	3.1 (3.09)*	1.33×10^{-5}	4.2	1.33×10^{-3}	3.0
Typical	1/500	2.0×10^{-3}	2.9 (2.88)*	2.67×10^{-5}	4.04	2.66×10^{-3}	2.79
Low	1/100	1.0×10^{-2}	2.3 (2.33)*	1.33×10^{-4}	3.65	1.32×10^{-2}	2.22

* β_{s-75} value is recalculated and is reported with two decimal places for greater accuracy when used in comparison with a β_{s-n} value for another design live of n years (e.g., $n = 100$).

The Table 2-11 data show the same inverse relationship observed in the Table 2-8 data between values of β_{f-n} and p_{f-n} for each n years: As β_{s-n} goes up in value, the value of p_{s-n} goes down.

β_{s-75} ranges from a low of 2.3 to a high of 3.1 for a 75-year design life for each of the three Canadian bridge design consequence levels (Table 2-11). These are lower reliability indices than for the failure limit state β_{f-75} values. Recall from Table 2-8 that β_{f-75} ranges from a low of 3.1 to a high of 3.7 for a 75-year design life for each of the three Canadian bridge design consequence levels. This corresponds to a 16% to 26% decrease in the three sets of β_{f-75} to β_{s-75} comparisons. In terms of probabilities, there is a

factor of 10 difference in the three sets of p_{f-75} to p_{s-75} comparisons. The SLS is more probable than the ULS.

The Table 2-11 reliability data calculated for a 100-year design life shows β_{f-100} ranges from a low of 2.22 to a high of 3.0. The two columns of 75- and 100-year data show that as the design life n -value increases, the value for β_{f-n} decreases. This was the same trend seen in the Table 2-8 data. The effect of a change in design life from 75 to 100 years is again shown to be minor; β_{f-n} decreases in value by approximately 3% for each of the three design consequence levels with this 25-year design life change.

By the data contained in the second-to-last column of Table 2-12, the 2014 CHBDC geotechnical design SLS based reliability criteria would translate into a reduction in the β value for a 100-year (foundation) design life that is between 0.09 and 0.11 lower than the β value for a 75 year Canadian bridge design life.

Table 2-12. Comparisons of reliability indices for the 2014 CHBDC geotechnical design SLS with a 75-year design life and for a 100-year design life.

Consequence Level	Reliability Index, β_{s75}	Reliability Index, β_{s100}	$(\beta_{s100} - \beta_{s75})$	β_{s100}/β_{s75}
High	3.09	3	-0.09	0.97
Typical	2.88	2.79	-0.09	0.97
Low	2.33	2.22	-0.11	0.95

The annual β_{s-1} ranges from a low of 3.65 to a high of 4.2 for a 75-year design life for each of the three Canadian bridge design (serviceability) consequence levels (Table 2-11). The difference between the annual reliability index value and that for a 75-year design life is significant, ranging in value from 26% to 37% lower values for the three design (serviceability) consequence levels.

The three annual SLS β_{s-1} values (Table 2-11) are smaller than the three ULS β_{f-1} values (Table 2-8). Specifically, the β_{s-1} values are 10% to 13% less than the β_{f-1} values.

2.9.5 Reliability-based target design criteria in the American Society of Civil Engineers (ASCE) design code for buildings

The ASCE ASCE-7-10 (Table C.1.3.1a) acceptable reliability for non-earthquake events is listed in Tables 2.13–2.16 for the risk categories associated with four different building system categories and three building (ultimate) limit states. The ASCE criteria were developed for building systems with a 50-year design life. The column labeled “ p_{f-1} ” is the ASCE annual probability of limit state failure. This value is the basis for all subsequent calculations. Also included in these tables are reliability calculations made by the authors of this report using Equation 2.64 with the design life, n , set equal to 50 years and to 100 years, and using the statistical function of $\text{NORM.INV}[(1-p_{f-100}), \mu = 0, \sigma = 1]$ defined in Excel. These calculations extend the ASCE annual probability of building limit state failure criteria (i.e., p_{f-1}) to a design life of 100 years, which is the design life required of the foundation systems used for the USACE hydraulic structures (ER 1110-2-8159; HQUSACE 1997).

Table 2-13. ASCE-7-10 acceptable reliability for non-earthquake – Risk Category I, temporary storage.

Limit State	P_{f1} (annual)	P_{f50}	β_{f50}^*	P_{f100}	β_{f100}	$(\beta_{f100} - \beta_{f50})$	β_{f100}/β_{f50}
Benign	1.25×10^{-4}	6.231×10^{-3}	2.5 (2.5)	1.24×10^{-2}	2.24	-0.26	0.9
Sudden or Severe	3×10^{-5}	1.499×10^{-3}	2.97 (3)	3×10^{-3}	2.75	-0.22	0.93
Sudden and Severe	5×10^{-6}	2.5×10^{-4}	3.48 (3.5)	5×10^{-4}	3.29	-0.19	0.95

*ASCE 10-7 values have been rounded and are reported in parentheses.

Table 2-14. ASCE-7-10 acceptable reliability for non-earthquake – Risk Category II, normal.

Limit State	P_{f1} (annual)	P_{f50}	β_{f50}^*	P_{f100}	β_{f100}	$(\beta_{f100} - \beta_{f50})$	β_{f100}/β_{f50}
Benign	3×10^{-5}	1.499×10^{-3}	2.97 (3)	3×10^{-3}	2.75	-0.22	0.93
Sudden or Severe	5×10^{-6}	2.5×10^{-4}	3.48 (3.5)	5×10^{-4}	3.29	-0.19	0.95
Sudden and Severe	7×10^{-7}	3.5×10^{-5}	3.98 (4)	7×10^{-5}	3.81	-0.17	0.96

*ASCE 10-7 values have been rounded and are reported in parentheses.

Table 2-15. ASCE-7-10 acceptable reliability for non-earthquake – Risk Category III, assembly.

Limit State	P_{f1} (annual)	P_{f50}	β_{f50}^*	P_{f100}	β_{f100}	$(\beta_{f100} - \beta_{f50})$	β_{f100}/β_{f50}
Benign	1.25×10^{-5}	6.248×10^{-4}	3.23 (3.25)	1.25×10^{-3}	3.02	-0.2	0.94
Sudden or Severe	2×10^{-6}	1×10^{-4}	3.72 (3.75)	2×10^{-4}	3.54	-0.18	0.95
Sudden and Severe	2.5×10^{-7}	1.25×10^{-5}	4.21 (4.25)	2.5×10^{-5}	4.06	-0.16	0.96

*ASCE 10-7 values have been rounded and are reported in parentheses.

Table 2-16. ASCE-7-10 acceptable reliability for non-earthquake – Risk Category IV, hospitals.

Limit State	P_{f1} (annual)	P_{f50}	β_{f50}^*	P_{f100}	β_{f100}	$(\beta_{f100} - \beta_{f50})$	β_{f100}/β_{f50}
Benign	5×10^{-6}	2.5×10^{-4}	3.48 (3.5)	5×10^{-4}	3.29	-0.19	0.95
Sudden or Severe	7×10^{-7}	3.5×10^{-5}	3.98 (4)	7×10^{-5}	3.81	-0.17	0.96
Sudden and Severe	1×10^{-7}	5×10^{-6}	4.42 (4.5)	1×10^{-5}	4.26	-0.15	0.97

*ASCE 10-7 values have been rounded and are reported in parentheses.

When considering the USACE hydraulic structures, the ASCE 7-10 reliability criteria for normal, assembly and/or hospital building systems would be of more interest than that for temporary storage structures. Therefore, by the data contained in the second to the last column of Tables 2.14 through 2.16, the ASCE-based reliability criteria would translate into a reduction in the β value for a 100-year (foundation) design life that is between 0.15 and 0.22 lower than the β value for a 50-year building design life. Recall that as the value for reliability index decreases, the probability of failure increases, given a specified design life of n years. This is observed when comparing the p_{f-100} values to the p_{f-50} values in these tables: They differ by a factor of two according to the data contained in these tables and by Equation 2.62 for n equal to 100 and 50 years, respectively.

2.9.6 Reliability-based target design criteria in the 2010 American Association of State Highway and Transportation Officials (AASHTO) LRFD bridge design specifications

In 2010, the fifth edition of the AASHTO LRFD bridge design specifications was published. Although not dealing with hydraulic structures, it is of importance to this R&D effort because reliability-based criteria and procedures were used to establish the design criteria and LRFD design values. The AASHTO LRFD bridge design specification was developed for American bridges with a 75-year design life, on which the statistical derivation of transient loads is based.

Under the AASHTO LRFD specification for extreme event limit states, bridge designs must satisfy

$$\phi_n * R_n \geq \sum_i \eta_i * \gamma_i * Q_i \quad (2.66)$$

where:

ϕ_n = the resistance factor

R_n = the nominal resistance

η_i = a load modifier factor: a factor relating ductility, redundancy, and operational classification

γ_i = the load factor: a statistically based multiplier applied to force effects

Q_i = the force effect.

During code development, the influence of η on the bridge girder β was estimated by observing its effect on minimum values of β calculated in a database of girder-type bridges. For discussion purposes, the girder bridge data used in the calibration of these specifications were modified by multiplying the total factored loads by $\eta = 0.95, 1.0, 1.05$, and 1.10 . The resulting minimum values of β for 95 combinations of span, spacing, and type of construction were determined to be approximately 3.0, 3.5, 3.8, and 4.0, respectively. Thus, for conventionally design-detailed bridges consisting of typical steel, reinforced concrete, or pre-stressed concrete bridge girders, β is 3.5. Generally speaking and focusing on the operational importance aspect of bridge design, the two higher β values of 3.8 and 4.0 are associated with critical or essential bridges while the lower β value of 3.0 is associated with less important bridges.

Ellingwood¹ provides insight into and discussion of the AASHTO LRFD Bridge Design Specifications reliability index. On pages 48–49 of his letter report to the USACE, Dr. Ellingwood observes that the structural actions – moments and shears in girders are based on a quasi-system analysis in which traffic loads are distributed to the individual girders through “girder distribution factors,” but the target reliabilities are comparable to those for buildings, and indeed, the reliability indices for buildings were used to benchmark the bridge girder reliabilities in the early 1990s when the LRFD method was developed and implemented for bridges. He further notes that in contrast to building structures, the bridge engineering research community believed that the service life of a typical bridge would be on the order of 75 years and chose that as a convenient period of reference for determining the maximum bridge live load and the target bridge girder reliability index. Recall that for buildings, the building community referenced their target reliability index to 50 years. Assuming (1) the load process is stationary (from a statistical point of view) and (2) no structural deterioration occurs during the service life, the four AASHTO target reliability index values of 3.0, 3.5, 3.8, and 4.0 translate to the β_n -values for $n = 50$ years and $n = 100$ years as listed in Table 2-17.² Again, it is the data for the three AASHTO target reliability index values of 3.5, 3.8, and 4.0 that are of most interest.

Table 2-17. AASHTO 2010 LRFD Bridge Design Specification reliability criteria extended to 50-year and 100-year β values – ULS.

β_{F75}	P_{F75}	P_{F1} (annual)	P_{F50}	β_{F50}	P_{F100}	β_{F100}	$(\beta_{F50}$ - $\beta_{F75})$	$(\beta_{F100}$ - $\beta_{F75})$
3.0	1.35×10^{-3}	1.80×10^{-5}	9.0×10^{-4}	3.12	1.80×10^{-3}	2.91	0.12	-0.09
3.5	2.33×10^{-4}	3.10×10^{-6}	1.55×10^{-4}	3.61	3.10×10^{-4}	3.42	0.11	-0.08
3.8	7.24×10^{-5}	9.65×10^{-7}	4.82×10^{-5}	3.9	9.65×10^{-5}	3.73	0.1	-0.07
4.0	3.17×10^{-5}	4.22×10^{-7}	2.11×10^{-5}	4.1	4.22×10^{-5}	3.93	0.1	-0.07

¹ Ellingwood, B. R. 2015. Procedures for Developing Reliability-Based Load Criteria for Hydraulic Steel and Concrete Structures. Final letter report submitted to the URS Group, Inc. U.S. Army Corps of Engineers.

² β_{F-n} for n years is calculated according to the following steps: With β_{F75} set equal to 3.0, 3.5, 3.8 and 4.0, their corresponding p_{F75} values are calculated using Excel with the equation and statistical function of $1-[0.5+\text{GAUSS}(\beta_{F75})]$ for an AASHTO $n = 75$ years design life. Next, the annual probability of the ULS, p_{F1} , is calculated using Equation 2.63. The p_{F1} value serves as the basis for converting the 75 year AASHTO reliability criteria to $n = 50$ years and $n = 100$ years: The p_{F-n} value is computed using Equation 2.64 for each $n = 50$ years and 100 years value. Last, the corresponding β_{F-n} value is computed using Excel with the equation and statistical function of $\text{NORM.INV}([1-p_{F-n}], 0, 1)$.

2.9.7 Comparison of CHBDC, AASHTO, and ASCE reliability criteria for a 100-year design life

Table 2-18 compares the CHBDC Table 2-8 reliability criteria data for typical bridges of typical consequence with the Table 2-17 AASHTO reliability criteria data for a standard bridge and with Table 2-14 ASCE reliability criteria for a normal building structure. These reliability data are for a ULS. For a 100-year design life, the reliability criteria for typical/standard bridges and normal buildings provides for a range in β values of between 3.29 and 3.46. A tighter range in 100-year β values of 3.42 to 3.46 is observed for the two bridge criteria alone.

Table 2-18. Comparison of reliability criteria data for conventionally designed detailed bridges with typical consequence level to that for normal buildings – ULS.

Type of Structure	Design Code	$\beta_{\#50}$	$\beta_{\#75}$	$\beta_{\#100}$	$(\beta_{\#100} - \beta_{\#50})$	$(\beta_{\#100} - \beta_{\#75})$
Typical bridge	CHBDC (2014)	-	3.54	3.46*	-	-0.08
Standard bridge	AASHTO (2010)	3.61*	3.5	3.42*	-0.19	-0.08
Normal building	ASCE 7-10 (2010)	3.48	-	3.29*	-0.19	-

*Calculated β value.

By the data contained in the last column of Table 2-18, the CHBDC- and AASHTO-based reliability criteria for typical and standard bridges, respectively, would translate into a reduction in the β value for a 100-year (foundation) design life that is 0.08 lower than the β value for a 75-year bridge design life. Also, the ASCE-based reliability criteria for a normal building would translate into a reduction in the β value for a 100-year design life that is 0.19 lower than the β value for a 50-year building design life. Recall that the design life for the foundation of a USACE hydraulic structure is 100 years (HQUSACE 1997; ER 1110-2-8159) with superstructure rehabilitation after 50 years of service.

3 Variance Reduction in Soil Shear Strength Due to Spatial Averaging

3.1 Variability of geotechnical design soil properties

The direct measurement from a geotechnical test is not always directly applicable to design. A transformation model is frequently needed to relate the geotechnical test measurement to the required geotechnical design property. This is especially true of in situ tests like SPT which generate N -values from the blow count measurements and cone penetration tests (CPT) that generate tip resistance and skin friction measurements. Measured N -values and CPT tip resistance values are transformed to Mohr-Coulomb soil strength parameters, pile capacities, etc., through geotechnical-based correlations. During the course of characterizing geotechnical variability, Phoon and Kulhawy (1999a,b) recognized that geotechnical design property variability can be attributed to three distinct sources of uncertainty:

- inherent soil variability
- measurement error
- transformation uncertainty.

Inherent soil variability pertains to the variability of the soil profile at the site. Inherent soil variability, caused primarily by the natural geologic processes that are involved in soil formation, is discussed in detail in section D.1 of Appendix D. Measurement error pertains to the degree of equipment and procedural control maintained during testing. Transformation uncertainty pertains to the precision of the correlation model used to transform the test-result measurement into the geotechnical engineering property characterizing the soil being tested.

The relationships given in this chapter are discussed in detail in Phoon and Kulhawy (1999b). The reader is referred to this paper if further details are needed.

3.2 Calculation of the variability of soil strength and variance reduction due to spatial averaging

Each of the three sources contributing to the variability in design shear strength assigned to a soil type is assessed separately in the Phoon and

Kulhawy (1999a,b) procedure of probabilistic analysis. This subsection summarizes their approach to the characterization of design shear strength variability for a soil.

El-Ramly et al. (2002a,b) and others observe that by taking the spatial average of the geotechnical design property variable ξ_d [e.g., $\xi_d(z) = S_u(z)$] over the whole area of interest, such as along the slip surface in a limit equilibrium sliding stability analysis, positive and negative random errors at different locations within the averaging domain tend to cancel out. As a result, the random error variance associated with the averaged quantity is largely reduced over point estimates (Vanmarcke 1977a,b, 1980). This reduction in variance over a point estimate's variance is why it is so important to account for variance reduction due to spatial correlation of soil properties in probabilistic geotechnical engineering analyses. The variance reduction function developed by Vanmarcke (1977a,b) is a fundamental part of this approach.

Phoon and Kulhawy (1999a,b) and others suggest that inherent soil variability be modeled as a random field, which can be described concisely by the COV. Recall that the COV is equal to the standard deviation for the variable divided by its mean value ($= \sigma/\mu$). Phoon and Kulhawy (1999b) express the total coefficient of variation for COV_{ξ_a} of the spatial average variable (ξ_a) of the soil geotechnical design property as

$$COV_{\xi_a}^2 = \Gamma^2(L) * COV_w^2 + COV_e^2 + COV_\varepsilon^2 \quad (3.1)$$

where the three components of variance are determined based on values of (1) the COV_w of inherent soil variability, (2) COV_e of measurement error and (3) COV_ε of transformation uncertainty. L is the averaging length (e.g., along the potential slip plane in a sliding stability analysis). A variance reduction function $\Gamma^2(L)$ is also used in this computation. The value of $\Gamma^2(L)$ ranges from a low of zero to a high of 1 and depends on the length of the averaging interval, L . Chapter 4 of this technical report, for example, discusses an averaging interval being made along the potential slip plane in a sliding stability analysis for a hydraulic structure. In the Phoon and Kulhawy (1999b) examples, the averaging interval L was taken to be 5 m in the vertical direction. The authors of this technical report speculate that this is because Phoon and Kulhawy focused their attention in their paper on the interpretation of soil strength data from laboratory and in situ soil strength tests and did not associate the testing program with a specific

geotechnical limit state (e.g., sliding, bearing capacity, or overturning) nor with a specific geotechnical/structural system. Additionally, it is observed that site investigations involving soil specimen recovery for laboratory strength testing or in situ strength testing are nearly always executed along a vertical axis. Therefore, the variance reduction relationships cited by Phoon and Kulhawy (1999b) use lengths measured solely in the vertical direction. This technical report presents a broader perspective in Chapters 4 and 5 that is associated with specific geotechnical limit state(s) for a geotechnical/structural system of interest to USACE, with supporting data and relationships given in Appendices D and E, respectively.

The approach used by Phoon and Kulhawy (1999a,b) to define the variance reduction term applied to the Equation 3.1 COV_w term makes use of the following Vanmarcke (1983, 2010) approximations but applied in the vertical direction: For the scale of fluctuation δ set equal to the vertical scale of fluctuation δ_v greater than or equal to an averaging length L that is set equal to an L_v value of 5 m in the vertical direction:

$$\Gamma^2(L) = 1 \quad (3.2)$$

While, with $\delta = \delta_v$ less than $L (= L_v)$,

$$\Gamma^2(L) = \frac{\delta}{L} \quad (3.3)$$

δ_v is a measure of the vertical spatial extent within which a soil property shows a strong correlation (subsection D.1.5), and L_v was assigned 5 m in the vertical direction (from Phoon and Kulhawy 1999a,b).

Chapter 4 and Appendix D show that, for a potential sliding limit state in which a weak horizontal layer dominates the stability model, δ_h will be substituted for δ in the qualifying criteria for Equation 3.2 and in Equation 3.3 and L is measured in the horizontal direction (L_h). For potential slip plane orientations that are neither perfectly vertical nor horizontal, please refer to Appendix E and procedures used to compute an equivalent scale of fluctuation, δ_E . The distance of the slip plane of any orientation is given as L .

3.2.1 Undrained shear strength by laboratory measurement

In the direct laboratory measurement of undrained shear strength, Phoon and Kulhawy (1999b) express S_u using symbolic functional letter notation $S_{u-direct}$. Their symbolic letter notation has $S_{u-direct}$ as equal to the sum of the trend function t for S_u , the soil variability w about the trend function in S_u , and the measurement error e about the trend function in S_u :

$$\xi_d = S_{u-direct} = t + w + e \quad (3.4)$$

This symbolic functional letter notation does not follow traditional geotechnical engineering notation. The mean and variance of the design property S_u is then given as

$$\overline{\xi_d} = t \quad (3.5)$$

and

$$(\sigma_{\xi_d})^2 \cong (\sigma_w)^2 + (\sigma_e)^2 \quad (3.6)$$

respectively. By Equation 3.6, the standard deviation of the design property S_u is

$$\sigma_{\xi_d} \cong \sqrt{(\sigma_w)^2 + (\sigma_e)^2} \quad (3.7)$$

with the coefficient of variation of the design property S_u given by

$$COV_{\xi_d}^2 \cong (COV_w)^2 + (COV_e)^2 \quad (3.8)$$

and

$$COV_w = \frac{\sigma_w}{t} \quad (3.9)$$

$$COV_e = \frac{\sigma_e}{t} \quad (3.10)$$

Phoon and Kulhawy (1999b) point out that structural loads and resistances usually can be modeled adequately as lumped parameters in a reliability calibration process. In contrast, they point out that it is no longer realistic to model geotechnical capacities as lumped parameters because of spatial correlation of materials. Therefore, Phoon and Kulhawy (1999a,b) took care

to distinguish inherent variability of strength properties for COV *point estimates* from those pertaining to *spatial average* COV values. Table 3-1 lists the typical range in point estimate COV_w values for of inherent variabilities in design parameter S_u for three types of geotechnical laboratory tests: (1) the unconfined compression test (UC), (2) the unconsolidated-undrained triaxial compression test (UU), (3) and the consolidated isotropic undrained triaxial compression test (CIUC).

Table 3-1. Approximate guidelines for design soil property variability – point estimates from direct laboratory measurements (after Phoon and Kulhawy 1999b [Table 5]).

Design Strength Parameter	Test	Soil Type	Point COV_w (%)
S_u (UC)	Direct (lab)	Clay	20-55
S_u (UU)	Direct (lab)	Clay	10-35
S_u (CIUC)	Direct (lab)	Clay	20-45

The COV of the spatial average, COV_{S_u-ave} , of the design undrained shear strength S_u (by laboratory measurement) is given by

$$COV_{\xi_a}^2 = \Gamma^2(L) \cdot (COV_w)^2 + (COV_e)^2 \quad (3.11)$$

With the variance reduction function $\Gamma^2(L)$ given by either Equation 3.2 or Equation 3.3, dependent upon the value of the averaging length L along the potential slip plane relative to the value of equivalent scale of fluctuation, δ_E . Procedures to compute δ_E for potential slip planes that are neither exactly vertical or horizontal are summarized in Appendix E. Phoon and Kulhawy (1999b) suggest that the COV of measurement error, COV_e , for undrained tests is between 5% and 15%. Typical values for the scale of fluctuation in the vertical and horizontal direction for different soil types are collected and listed in Tables D.1 in Appendix D. Tables D.2 and D.3 provide generalizations made by other authors on the anticipated magnitude of the vertical and horizontal scale of fluctuation distances of select geotechnical engineering soil properties characterizing the strength of soils.

3.2.2 Undrained shear strength from in situ vane shear test measurement

The undrained shear strength S_u of a soil may be estimated using the in situ vane shear test (VST). However a correction factor is needed to account for strain-rate effects, drainage, and anisotropic vertical and

horizontal shear strengths (Phoon and Kulhawy 1999b). Baecher and Ladd (1985) described these correction factors in this equation:

$$\xi_d = (m_\mu + \varepsilon)(t + w + e) \quad (3.12)$$

where:

ξ_d = the corrected S_u from the vane shear test

m_μ = the mean correction factor

ε = the uncertainty in the correction factor

$S_u(VST) = (t + w + e)$ is the undrained shear strength from the in-situ test results (as in Equation 3.4).

The mean of the design property ξ_d can be approximated by

$$m_{\xi_d} \cong m_\mu t \quad (3.13)$$

The variance for the design property distribution (the square of the standard deviation) is given by

$$\sigma_{\xi_d}^2 \cong m_\mu^2(\sigma_w^2 + \sigma_e^2) + t^2 \sigma_\varepsilon^2 \quad (3.14)$$

The COV of the design property can be expressed by

$$COV_{\xi_d}^2 = COV_{S_u}^2 = COV_w^2 + COV_e^2 + COV_\varepsilon^2 \quad (3.15)$$

Where COV_ε is the COV of transformation uncertainty

$$COV_\varepsilon = \frac{\sigma_\varepsilon}{m_\mu} \quad (3.16)$$

For soft clays, COV_ε was estimated to be between 7.5% and 15% (Baecher and Ladd 1985).

The COV of the spatial average, COV_{ξ_a} , of the design undrained shear strength S_u is given by

$$COV_{\xi_a}^2 = \Gamma^2(L) \cdot (COV_w)^2 + (COV_e)^2 + (COV_\varepsilon)^2 \quad (3.1 \text{ bis})$$

Phoon and Kulhawy (1999b) present that from the vane shear test, the COV of the measurement error (COV_e) was determined to be in the range of 10% to 20%, and the COV of the inherent variability (COV_w) was determined to be in the range of 10% to 40%. Phoon and Kulhawy (1999b)

comment that the vertical scale of fluctuation was between 2 and 6 m. The data collected by the authors of this technical report and contained in Appendix D support this observation. Table 3-2 has the results of second moment statistical analysis of point samples for the design strength parameter S_u , based on VST data.

Table 3-2. Approximate guidelines for design soil property S_u variability – point estimates from VST correlations (after Phoon and Kulhawy 1999b [Table 5]).

Design Strength Parameter	Test	Soil Type	Point COV_w (%)
S_u (field)	VST	Clay	15-50

3.2.3 Undrained shear strength by correlation with corrected cone tip resistance

Equation 3.17 gives the equation for the correlation between corrected cone tip resistance and undrained shear strength, according to Phoon and Kulhawy (1999b).

$$\frac{S_u}{\sigma_{vo}} = D_K \frac{(q_T - \sigma_{vo})}{\sigma_{vo}} = (m_{DK} + \varepsilon) \frac{(q_T - \sigma_{vo})}{\sigma_{vo}} \quad (3.17)$$

where :

q_T = the test measurement of the corrected cone tip resistance

D_K = the uncertain model slope

m_{DK} = the mean of D_K

σ_{vo} = the total overburden stress

$\overline{\sigma_{vo}}$ = the effective overburden stress

ε = a zero mean random variable representing transformation uncertainty.

Phoon and Kulhawy (1999b) provide a statistical summary of values for D_K , depending upon S_u test type. This data are summarized in Table 3-3.

Table 3-3. Statistics for D_K (after Kulhawy et al. 1992).

S_u test type	Mean D_K	COV of D_K %
Consolidated isotropic undrained triaxial compression test, CIUC	0.0789	35
Unconsolidated-undrained triaxial compression test, UU	0.0512	29
Vane shear test, VST	0.0906	40

The corrected cone tip resistance is calculated as

$$q_T = q_c + (1 - a)u_{bt} \quad (3.18)$$

where:

q_c = the cone tip resistance

a = the net area ratio

u_{bt} = the porewater pressure behind the cone tip.

According to Phoon and Kulhawy (1999b), Equation 3.17 can be expressed as

$$\xi_d = s_u = (m_{DK} + \varepsilon)(t + w + e - \sigma_{vo}) \quad (3.19)$$

where $q_T = (t+w+e)$.

The mean of the design property ξ_d can be approximated by

$$m_{\xi_d} \cong m_{DK}(t - \sigma_{vo}) \quad (3.20)$$

The variance for the design property distribution (the square of the standard deviation) is given by

$$\sigma_{\xi_d}^2 \cong m_{DK}^2(\sigma_w^2 + \sigma_e^2) + (t - \sigma_{vo})^2 \sigma_\varepsilon^2 \quad (3.21)$$

The COV of the design property can be expressed by

$$COV_{\xi_d}^2 \cong \frac{(COV_w^2 + COV_e^2)}{(1 - \frac{\sigma_{vo}}{t})^2} + COV_\varepsilon^2 \quad (3.22)$$

Where COV_ε is the COV of transformation uncertainty:

$$COV_\varepsilon = \frac{\sigma_\varepsilon}{m_{DK}} \quad (3.16 \text{ bis})$$

The coefficient of variation of the spatial average, COV_{Su-ave} , of the design undrained shear strength S_u is given by

$$COV_{\xi_a}^2 \cong \frac{[\Gamma^2(L)COV_w^2 + COV_e^2]}{(1 - \frac{\sigma_{vm}}{t})^2} + COV_\varepsilon^2 \quad (3.23)$$

According to Phoon and Kulhawy (1999b), for Clay material, $COV_s(UU)$ was determined to be 29% and $COV_s(CIUC)$ was determined to be 35%. From the CPT, the COV of the measurement error (COV_e) was estimated to be in the range of 5% to 15%, and the COV of the inherent variability (COV_w) was estimated to be less than 20%. The vertical scale of fluctuation was less than 0.5 m. Table 3-4 has the results of second moment statistical analysis of point samples for the design strength parameter S_u .

Table 3-4. Approximate guidelines for design soil property variability – point estimates from cone penetration correlations (after Phoon and Kulhawy, 1999b [Table 5]).

Design Strength Parameter	Test	Soil Type	Point COV_w (%)
S_u (UU)	q_T	Clay	30-40
S_u (CIUC)	q_T	Clay	35-50

3.2.4 Undrained shear strength by correlation with SPT N value

If a correlation is performed with only one type of geology, undrained shear strength can reasonably be correlated with a SPT N value (Kulhawy and Mayne 1990). According to Phoon and Kulhawy (1999b), the equation for the correlation give probability is

$$\log_{10}(S_u/p_a) = \log_{10}(0.29) + 0.72 \log_{10} N + \varepsilon \quad (3.24)$$

where:

s_u = the undrained shear strength from UU tests

p_a = the atmospheric pressure.

In exponential form,

$$\frac{s_u}{p_a} = 0.29 N^{0.72} 10^\varepsilon \quad (3.25)$$

In terms of the corrected value,

$$\xi_d = s_u = 0.29 p_a (t + w + e)^{0.72} 10^\varepsilon \quad (3.26)$$

In which $N = (t+w+e)$.

The mean of the design property ξ_d can be approximated by

$$m_{\xi_d} \cong 0.29t^{0.72} \quad (3.27)$$

The variance for the design property distribution (the square of the standard deviation) is given by

$$\sigma_{\xi_d}^2 \cong m_{\xi_d}^2 [0.72^2 (COV_w^2 + COV_e^2) + (\log_e 10)^2 \sigma_\varepsilon^2] \quad (3.28)$$

The COV of the design property can be expressed by

$$COV_{\xi_d}^2 \cong 0.72^2 (COV_w^2 + COV_e^2) + (\log_e 10)^2 \sigma_\varepsilon^2 \quad (3.29)$$

The coefficient of variation of the spatial average, COV_{Su-ave} , of the design undrained shear strength S_u is given by

$$COV_{\xi_a}^2 \cong 0.72^2 [\Gamma^2(L) COV_w^2 + COV_e^2] + (\log_e 10)^2 \sigma_\varepsilon^2 \quad (3.30)$$

According to Phoon and Kulhawy (1999b), from data analyzed by Hara et al. (1974), a first-order estimate of σ_ε can be given as 0.15. From the CPT, the COV of the measurement error (COV_e) was estimated to be in the range of 15% to 45% and the COV of the inherent variability (COV_w) was estimated to be in the range of 25% to 50%. The vertical scale of fluctuation was not available. A comparison of clay and sands showed that the effect of soil type was negligible. Vanmarcke (1977a) reported that the scale of fluctuation for sandy soils was 2.4 in one study. Table 3-5 has the results of second moment statistical analysis of point samples for the design strength parameter S_u .

Table 3-5. Phoon and Kulhawy (1999b; Table 5).
Approximate guidelines for design soil property variability –
point estimates from SPT N correlations (after Phoon and
Kulhawy 1999b; Table 5).

Design Strength Parameter	Test	Soil Type	Point COV_w (%)
S_u (UU)	N	Clay	40-60

3.2.5 Undrained shear strength by correlation with plasticity index

Chandler (1988) provided the correlation equations for the VST to the plasticity index (PI) of the soil with the undrained shear strength S_u given as follows,

$$\frac{s_u(VST)}{\bar{\sigma}_p} = 0.11 + 0.0037 * PI \quad (3.31)$$

In which $\bar{\sigma}_p$ is the preconsolidation stress. The transformation uncertainty was not specified, but the author said that the accuracy of equation 3.30 would be on the order of +/- 25%. In a design, a correction factor should be applied to $s_u(VST)$. According to Phoon and Kulhawy (1999b), the COV of the correction value should be between 7.5% and 15%. The preconsolidation stress has a variability between 12% and 39% in a study by Vanmarcke and Fuleihan (1975).

A probabilistic model for the correlation is

$$\xi_d = m_{\sigma p}(1 + \varepsilon)[0.11 + 0.0037(t + w + e)] \quad (3.32)$$

where:

$m_{\sigma p}$ = the mean preconsolidation stress

$PI = (t+w+e)$

E = the inaccuracy in Equation 3.31.

The mean of the design property ξ_d can be approximated by

$$m_{\xi_d} \cong m_{\sigma p}(0.11 + 0.0037t) \quad (3.33)$$

The variance for the design property distribution (the square of the standard deviation) is given by

$$\sigma_{\xi_d}^2 \cong m_{\xi_d}^2 \left[\frac{(\sigma_w^2 + \sigma_e^2)}{(29.73+t)^2} + \sigma_\varepsilon^2 \right] \quad (3.34)$$

The COV of the design property can be expressed by

$$COV_{\xi_d}^2 \cong \frac{(\sigma_w^2 + \sigma_e^2)}{(29.73+t)^2} + \sigma_\varepsilon^2 \quad (3.35)$$

The $COV_{\xi d}$ does not have a simple solution for this case, because

$$\frac{\partial \xi_d}{\partial w} = 0.0037 m_{\sigma p} \quad (3.36)$$

and $m_{\sigma p}$ varies with depth.

Phoon and Kulhawy (1999a) estimated the standard deviation of the inherent variability (σ_w) of PI to be between 3% and 12% and the standard deviation of the measurement error (σ_e) to be between 2% and 6%.

The range of σ_e was determined to be in the range of 0.28 to 0.50. Given a typical mean PI of 30%, the total $COV_{\xi d}$ was between 29% and 55% (Phoon and Kulhawy 1999b). Table 3-6 has the results of this second moment statistical analysis of point samples for the design strength parameter S_u .

Table 3-6. Approximate guidelines for design soil property variability – point estimates from PI correlations (after Phoon and Kulhawy (1999b; Table 5).

Design Strength Parameter	Test	Soil Type	Point COV_w (%)
S_u (field)	PI	Clay	30-55

3.2.6 Effective stress friction angle by laboratory measurement

The equation to determine the effective stress friction angle, ϕ' , from direct laboratory measurements is

$$\xi_d = \phi' = t + w + e \quad (3.37)$$

The mean and variance of the design property ϕ' is then given as

$$\bar{\phi}' = t \quad (3.38)$$

and

$$(\sigma_{\phi'})^2 = (\sigma_w)^2 + (\sigma_e)^2 \quad (3.39)$$

respectively. By Equation 3.6, the standard deviation of the design property ϕ' is

$$\sigma_{\phi'} = \sqrt{(\sigma_w)^2 + (\sigma_e)^2} \quad (3.40)$$

with the coefficient of variation of the design property ϕ' given by

$$COV_{\xi_d}^2 = (COV_{\phi'})^2 = (COV_w)^2 + (COV_e)^2 \quad (3.41)$$

and

$$COV_w = \frac{\sigma_w}{t} \quad (3.42)$$

$$COV_e = \frac{\sigma_e}{t} \quad (3.43)$$

The coefficient of variation of the spatial average, COV_{ϕ} , of the effective stress friction angle ϕ' is given by

$$COV_{\xi_a}^2 = \Gamma^2(L) \cdot (COV_w)^2 + (COV_e)^2 \quad (3.11 \text{ bis})$$

According to Phoon and Kulhawy (1999b), from the direct laboratory measurements, the COV of the measurement error (COV_e) was determined to be in the range of 5% to 15%, and the COV of the inherent variability (COV_w) was determined to be in the range of 5% to 15%. The vertical scale of fluctuation was not available but assumed (by Phoon and Kulhawy 1999b) to be between 2 and 6 m. Table 3-7 has the results of second moment statistical analysis of point samples for the design strength parameter ϕ' .

Table 3-7. Approximate guidelines for design soil property variability – point estimates from direct laboratory measurements (after Phoon and Kulhawy 1999b; Table 5).

Design Strength Parameter	Test	Soil Type	Point COV_w (%)
ϕ'	Direct (lab)	Clay,sand	7-20

3.2.7 Effective stress friction angle by correlation with cone tip resistance

Kulhawy and Mayne (1990) examined several correlations between cone tip resistance q_c and the triaxial compression effective stress friction angle, ϕ' . The following equations are specifically for sands.

$$\phi' = 17.6 + 11.0 \log_{10} \left(\frac{q_c/p_a}{\sqrt{\bar{\sigma}_{v0}/p_a}} \right) + \varepsilon \quad (3.44)$$

with p_a being the atmospheric pressure and $\bar{\sigma}_{v0}$ being the effective overburden pressure.

In this correlation, the standard deviation of ε is 2.8d (Kulhawy and Mayne 1990). Letting $(t + w + e) = q_c$, the design property becomes

$$\xi_d = \phi' = 17.6 + 11.0 \log_{10} \left(\frac{(t+w+e)/p_a}{\sqrt{\bar{\sigma}_{v0}/p_a}} \right) + \varepsilon \quad (3.45)$$

The mean of the design property ξ_d can be approximated by

$$m_{\xi_d} \cong 17.6 + 11.0 \log_{10} \left(\frac{t/p_a}{\sqrt{\bar{\sigma}_{v0}/p_a}} \right) \quad (3.46)$$

The variance for the design property distribution (the square of the standard deviation) is given by

$$\sigma_{\xi_d}^2 \cong 22.8(COV_w^2 + COV_e^2) + \sigma_\varepsilon^2 \quad (3.47)$$

The COV of the spatial average is

$$COV_{\xi_d}^2 \cong \frac{22.8(COV_w^2 + COV_e^2) + \sigma_\varepsilon^2}{m_{\xi_d}^2} \quad (3.48)$$

The COV of the spatial average, $COV_{\phi'-ave}$, of the design triaxial compression effective stress friction angle ϕ' is given by

$$COV_{\xi_d}^2 \cong \frac{22.8(\Gamma^2(L)COV_w^2 + COV_e^2) + \sigma_\varepsilon^2}{m_{\xi_d}} \quad (3.49)$$

According to Phoon and Kulhawy (1999b), from the cone tip resistance correlation in sand, the COV of the measurement error (COV_e) was estimated to be in the range of 5% to 15%, and the COV of the inherent variability (COV_w) was determined (by Phoon and Kulhawy 1999a) to be in the range of 20% to 60%. The standard deviation of error (σ_ε) for this correlation is 2.8 degrees (Kulhawy and Mayne 1990). The vertical scale of fluctuation for q_c was found to be less than 1 m. The mean friction angle (m_{ξ_d}) for most soils is between 20 and 40 degrees, and a typical value assigned is 30 degrees. Table 3-8 has the results of second moment statistical analysis of point samples for the design strength parameter, ϕ' .

Table 3-8. Approximate guidelines for design soil property ϕ' variability – point estimates from cone tip resistance (after Phoon and Kulhawy 1999b; Table 5).

Design Strength Parameter	Test	Soil Type	Point COV_w (%)
ϕ' (TC)	q_T	Sand	10-15

3.2.8 Effective stress friction angle by correlation with plasticity index

Mitchell (1976) provided the following equation to correlate the PI to a constant-volume effective stress friction angle ϕ'_{cv}

$$\sin \phi'_{cv} = 0.8 - 0.094 \log_e PI \quad (3.50)$$

Accounting for spread about the data, the probabilistic equation becomes

$$\sin \phi'_{cv} = 0.8 - 0.094 \log_e PI + \varepsilon \quad (3.51)$$

Determining the design property (in degrees)

$$\xi_d = \phi'_{cv} = \sin^{-1}[0.8 - 0.094 \log_e(t + w + e) + \varepsilon] \quad (3.52)$$

The mean of the design property ξ_d can be approximated (in degrees) by

$$m_{\xi_d} \cong \sin^{-1}[0.8 - 0.094 \log_e(t)] \quad (3.53)$$

The variance for the design property distribution (the square of the standard deviation, in degrees²) is given by

$$\sigma_{\xi_d}^2 \cong \left(\frac{180}{\pi}\right)^2 \left[\frac{0.094^2 (COV_w^2 + COV_e^2) + \sigma_e^2}{1 - (0.8 - 0.094 \log_e t)^2} \right] \quad (3.54)$$

For the spatial average,

$$COV_{\xi_a}^2 \cong \left(\frac{180}{\pi}\right)^2 \left[\frac{0.094^2 [\Gamma^2(L) COV_w^2 + COV_e^2] + \sigma_e^2}{m_{\xi_d}^2 [1 - (0.8 - 0.094 \log_e t)^2]} \right] \quad (3.55)$$

According to Phoon and Kulhawy (1999b), for the plasticity index, the standard deviation of the measurement error (σ_e) was determined to be in the range of 3% to 12% and the standard deviation of the inherent variability (σ_w) was determined to be in the range of 2% to 6%. The standard deviation of error (σ_e) for this correlation was not given but was given a first-order estimate of 0.07. Phoon and Kulhawy (1999b) assumed a typical mean PI of 30%, resulting in a COV_{ξ_d} between 16% and 19%. The vertical scale of fluctuation was assumed (Phoon and Kulhawy 1999a) to lie between 2 and 6 m. Table 3-9 has the results of second moment statistical analysis of point samples for the design strength parameter ϕ_{cv}' .

Table 3-9. Approximate guidelines for design soil property variability – point estimates from plasticity index (after Phoon and Kulhawy 1999b: Table 5).

Design Strength Parameter	Test	Soil Type	Point COV_w (%)
ϕ_{cv}'	PI	Clay	15-20

3.3 Conclusions

The point estimate COV values for soil strength parameters given in this chapter are used in conjunction with a variance reduction relationship to define spatial COV values for the soil strength parameters. The data summary presented in this chapter clearly show that different point estimate COV values are obtained from both direct laboratory measurements than from the values obtained from correlations with basic soil property tests or with the different types of in situ (soil) tests.

The application of a variance reduction factor applied to COV point estimates for soil strength is an important aspect of a reliability analysis that includes geotechnical limit state(s). Resulting values for COV spatial

estimates are less than COV point estimates due to spatial averaging of the engineering strength parameters of the soil along specific planes. These planes are usually selected by the engineer to be coincident with likely potential slip planes within the soil mass during a reliability analysis considering geotechnical limit states.

Chapter 4 discusses different reliability analysis techniques that make use of the point data and soil strength relationships described in this chapter as well as different types of geotechnical reliability analysis approaches that are used to address geotechnical limit states.

4 A Structural System's Probability of Unsatisfactory Performance (PUP_{system}) for a Hydraulic Structure

4.1 Introduction

The USACE hydraulic structures are subjected to different sources of loads beyond just the water loads associated with the surface water of navigable waterways. For example, it is common for lock walls to not only form the chamber of the lock but to also serve as a retention structure to a compacted backfill landside of the chamber. Due to its contact with the retained earth, the majority of the USACE hydraulic structures face SSI issues when responding to loading(s) associated with a hydraulic structure, which therefore mandates that a portion of the design loads result from the earth pressures imposed on the heel-side of the hydraulic structure by the retained earth.¹ Due to the height of these walls (i.e., tens of feet) it is not uncommon that the retained soil becomes the source of a destabilizing force imposed on the hydraulic structure as this force attempts to cause the lock wall to slide and/or to rotate away from the retained soil and into the lock chamber. Some of the USACE hydraulic structures are not just resting on but are embedded within a rock (or earth) foundation. In these cases where the base of the hydraulic structure is below ground surface, an earthen mass forms a buttress to the toe of a hydraulic structure. This soil (or rock) buttress develops a stabilizing force that adds to stability by helping the hydraulic structure to resist sliding, overturning, and foundation bearing failure of the hydraulic structure.

Recall from subsection 2.2 that there are seven limit states to be considered during the design/analysis of the USACE hydraulic navigation structures. The first three are evaluated on a structural member-by-member basis: (1) flexural hinging, (2) shear failure, and (3) buckling failure of members due to axial loading. Another limit state (4) is

¹ SSI is sometimes described in general terms as the process in which the response of the soil influences the structural movement, and this structural movement in turn influences the soil's response. In the case of the USACE massive gravity lock walls and lighter and flexible U-frame locks, the retained soil-to-hydraulic structure-to-foundation rock (or soil) interaction effects becomes a prominent aspect of the definition of loads acting along the sides, with base of hydraulic structure foundation movements playing a prominent role. Complete SSI analyses demonstrating these observations can be found in the Ebeling and Wahl (1997) analysis of a rock-founded gravity lock wall retaining earth and in the Ebeling and Mosher (1996) SSI analysis of a U-frame lock.

associated with a limit placed on the displacement of the structural system in its entirety. The latter three are commonly associated with geotechnical aspects of the application of reliability theory to a project feature's design. They are (5) global sliding failure, (6) bearing failure of the foundation, and (7) overturning or rotational failure of the structural system about its foundation in its entirety. This chapter focuses on assessing the latter three limit states for the USACE hydraulic structures because reliability-based LRFD design methodologies are not as advanced on geotechnical related issues as they are on structural related issues. Other sections of this report provide details in support of the reliability engineering process discussed in this chapter and are cited accordingly.

4.2 Three stages to load factor γ_L development using reliability theory

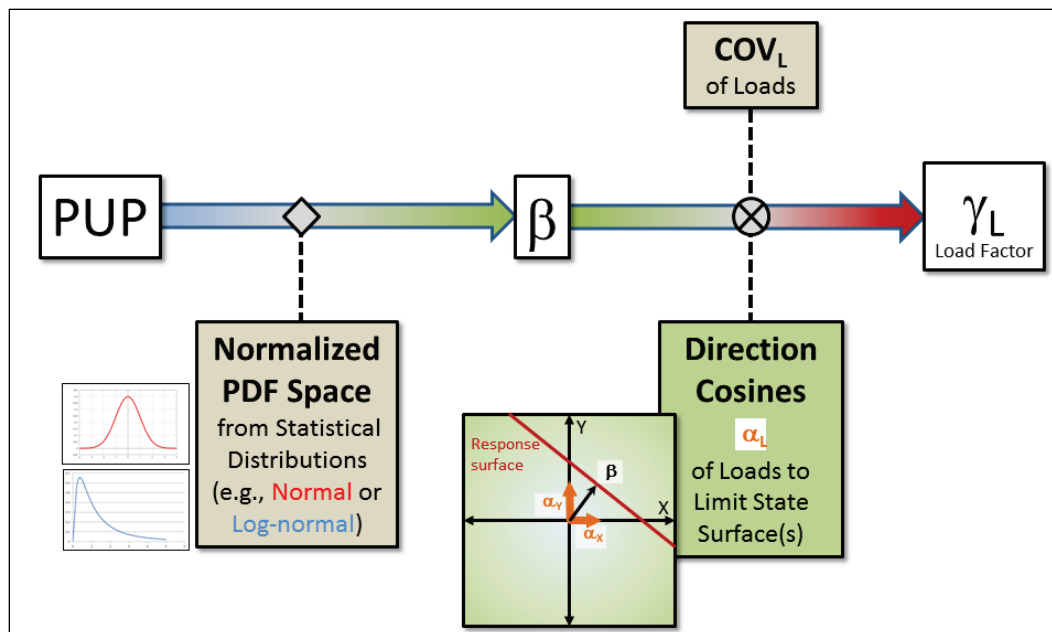
Subsection 2.5 outlined the steps leading to the definition of the load factor γ_L using reliability theory. From this subsection, γ_L is given by

$$\gamma_L = (1 + \alpha_L * \beta * COV_L) \quad (2.37 \text{ bis})$$

where α_L is the directional cosine for load L , β is the reliability index (a scaler value), and COV_L is the COV ($= \sigma/\mu$) of the load L .

The authors of this report recognize that three stages of analysis are needed to define a value for the load factor γ_L via reliability analysis. These three stages of development are summarized in Figure 4-1. The process starts with the calculation of the *PUP* value. By reliability theory (e.g., Equations 2.27a or 2.27b), the *PUP* value leads directly to the calculation of a value for the reliability index β . As discussed in subsection 2.7, it is important that the seven limit states listed in subsection 2.2 must be taken into account when computing the *PUP* value. This calculation, using numerical methods, will be discussed further in subsequent subsections in this chapter. To state it succinctly, several of the seven limit states are expected to interact with one another to influence the magnitude of the computed *PUP* value. This has been shown to be true for pile foundation analysis via reliability theory made by Fan et al. (2014).

Figure 4-1. Three stages to load factor development using Reliability analysis



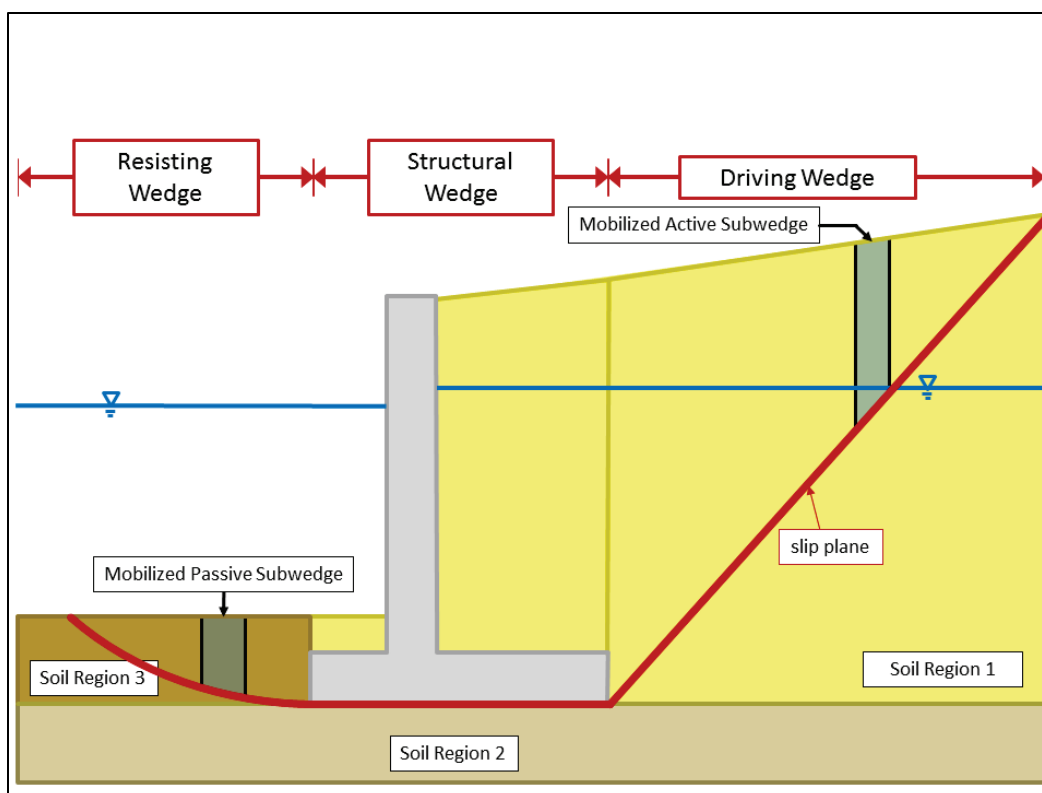
The last stage of the Figure 4-1 computational process is to use values for α_L , β , and COV_L to compute γ_L by Equation 2.37. The procedure of reliability analysis leading to the calculation of a value for load factor γ_L are outlined in this chapter.

4.3 Three-wedge model of hydraulic structures retaining and buttressed by earthen features

With the focus of this chapter on a geotechnical aspects of reliability-based LRFD design methodologies, it is convenient to devise an example retaining wall structural problem containing the three limit state features of interest: (1) a global sliding failure, (2) a foundation bearing failure, and (3) an overturning or rotational failure of the structural system about its foundation but not be so complex as to bog down the analysis with geologic details such as those stemming from patterns of rock jointing, etc. Therefore, the example retaining structure selected for study is shown in Figure 4-2 and consists of a T-Wall retaining a soil backfill behind its heel, founded on a geologically formed soil and with a soil buttress in front of its toe that is also geologically formed. It possesses a pool of water to the LHS side of the T-Wall and above the soil buttress. A water table is contained within the retained soil on the RHS of the T-Wall. A differential head condition exists for this T-Wall due to the existence of a higher head on the retained side of the wall relative to the pool elevation on the LHS of the wall. There are three soil regions identified in this figure, designated as soil

regions 1, 2, and 3. Soil region 1 is engineer-selected backfill material placed by earth moving equipment and compacted to engineering specification after construction of the T-Wall. Soil region 2 is naturally occurring soil and is assumed to be *undisturbed* by the construction of the T-Wall and backfill placement above it. Soil region 3 is also assumed to be a naturally occurring soil that was undisturbed by the construction of the T-Wall. A temporary vertical soil face was assumed during a short-term, T-Wall site construction at the interface of the structural wedge with the resisting wedge for soil region 2. Soil backfill was placed and compacted above the T-Wall shortly after T-Wall construction and (reinforced) concrete curing concluded.

Figure 4-2. Three-wedge model of a combined geological and structural system and a potential slip-plane for sliding stability analysis of a T-Wall as per EM 1110-2-2502 (HQUSACE 1989)



The height of the retained soil relative to the left-hand soil and pool elevation dictates the RHS to be the driving load side. The RHS retained soil attempts to move the retaining structure towards the pool. A potential sliding plane is depicted in red in this Figure 4-2 cross-section. This postulated retaining structure possesses all but one of the seven subsection

2.2 limit state features that will need to be addressed during a combined LRFD design process.

In preparation for engineering evaluation, the Figure 4-2 retaining wall system is divided into the driving wedge, the structural wedge, and the resisting wedge. This three-global-wedges problem division is consistent with the USACE EM 1110-2-2502 (HQUSACE 1989) design procedure of retaining wall analysis.

On the RHS of the retaining structure is the driving wedge, as labeled in Figure 4-2. Gravity causes the soil mass located above the potential slip plane in this wedge to slide downward along the (red) potential slip surface. This, in turn, generates lateral movement of the structural wedge towards the pool and results in *mobilized active* earth pressures acting against along the driving wedge to structural wedge interface.

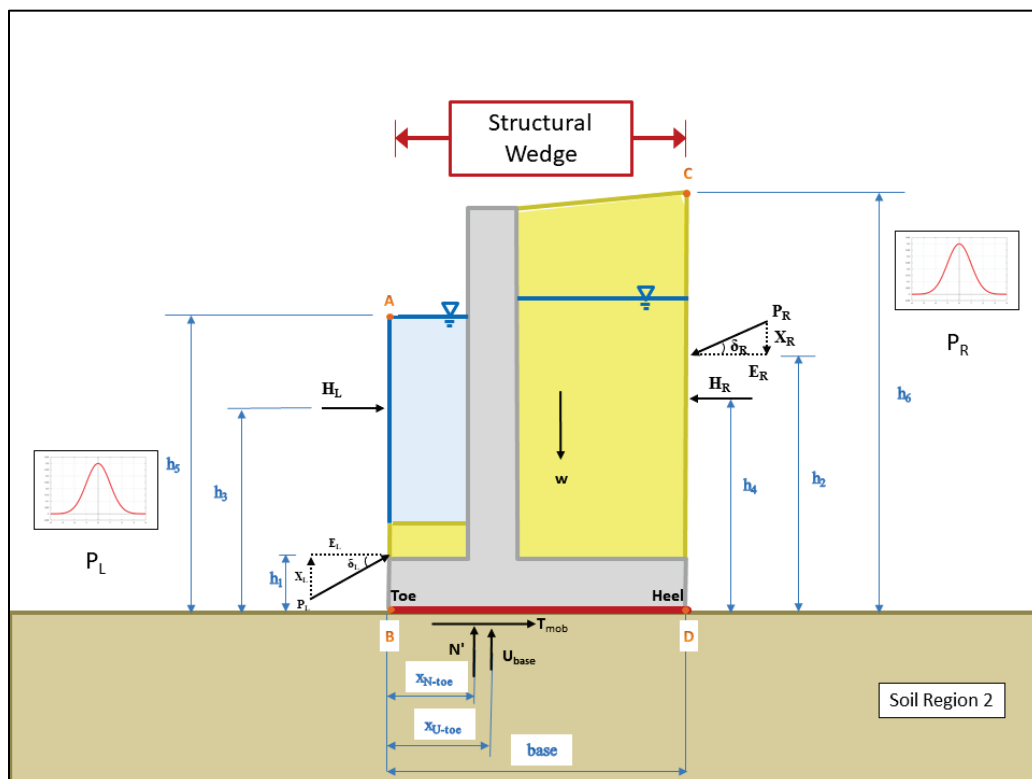
In the center of this figure is the structural wedge. The structural wedge consists of the retaining structure and any other soil fill material that lies contained within an area defined by the width of the structure. For this example, the potential slip plane is contained within the foundation soil (region 2) below the reinforced concrete structure.

On the LHS of the structure is the resisting wedge. In this wedge, the weight of the soil and other (i.e., pool of water) loads/weights are providing *mobilized passive* earth pressure against the structural wedge, therefore resisting sliding of the structural wedge. The pool region extending above the top of the resisting wedge's ground surface also exerts a hydrostatic water pressure force acting horizontally on the structural wedge's LHS interface. This triangular hydrostatic water pressure distribution is not depicted in this figure. The slip plane in this region is typically more complicated than in the active pressure region. In the case of Figure 4-2, the slip plane of the resisting wedge is modeled using a spiral curve (in contrast to the planar surface of the mobilized active driving wedge).

At the heart of a reliability analysis is the desire to formally account for the uncertainty in the loads and resistance forces (and moments) and design based on these uncertainties through a reliability analysis (as discussed in subsection 2.3). Figure 4-3 shows an idealization of a normal distribution for the PDF distributions of LHS and RHS forces P_L and P_R , respectively,

acting on a structural wedge. Note that for these and many other variables, the subscript L designates the LHS wedge (or subwedge) interface and subscript R designates the RHS.

Figure 4-3. PDF distributions of left and RHS forces acting on a structural wedge, P_L and P_R .



The list of symbols used in this structural wedge are defined as follows:

- E_L = the horizontal shear force component of P_L acting on the left side of the wedge
- E_R = the horizontal shear force component of P_R acting on the right side of the wedge
- h_1 = the height from point B to E_L
- h_2 = the height from point D to E_R
- h_3 = the height from midpoint elevation of the base to H_L
- h_4 = the height from midpoint elevation of the base to H_R
- h_5 = the height from point B to the top of the LHS structural wedge free body diagram
- h_6 = the height from point D to the top of the RHS structural wedge free body diagram
- H_L = the horizontal resultant force exerted by pore water pressure acting on the left side of the wedge

- H_R = the horizontal resultant force exerted by pore water pressure acting on the right side of the wedge
 N' = the effective force acting normal to the base of the wedge
 P_L = the resultant interslice force acting on the left side of the wedge
 P_R = the resultant interslice force acting on the right side of the wedge
 T_{mob} = the mobilized shear resistance force acting along the base of the wedge
 U_{base} = the resultant water pressure force acting normal to the base calculated using the average water pressure acting normal to wedge base
 W = the (total) weight of the wedge
 X_L = the vertical shear force component of P_L acting on the left side of the wedge
 X_R = the vertical shear force component of P_R acting on the right side of the wedge
 x_{N-toe} = the distance from the Toe (point B) to the effective resultant normal force N'
 x_{U-toe} = the distance from the Toe (point B) to the resultant uplift force U_{base} acting normal to wedge base
 x_{W-toe} = the horizontal distance from the Toe (point B) to the center of gravity for the structural wedge, where the weight of the wedge is applied.

The Greek character symbols are defined as follows:

- α = the angle the base of the planar wedge as measured from horizontal; $\alpha = 0$ degrees in this example and is therefore not included in this figure
 δ_L = the angle of interslice friction on the left side of the wedge
 δ_R = the angle of interslice friction on the right side of the wedge.

The resultant mobilized active earth pressure force labeled P_R in Figure 4-3 is imposed on the RHS of the structural wedge by the driving wedge. Similarly, the resultant mobilized passive earth pressure force labeled P_L in Figure 4-3 is imposed on the LHS of the structural wedge by the resisting wedge.

The Figure 4-3 PDFs for P_R and P_L are each characterized by a mean value, a standard deviation, and a distribution type (e.g., normal, logarithmic normal, uniform, triangular). A convenient form of expressing the magnitude of dispersion in each force variable such as the load force P_R is by using a COV. Recall the COV is the standard deviation σ of the force P_R distribution divided by the mean force value μ of the P_R distribution.

In EM 1110-2-2502 (HQUSACE 1989) retaining wall model sliding wedge analysis, a key contribution to uncertainty is in the random variable dispersion (i.e., variance or standard deviation) in Mohr-Coulomb shear strength parameter value(s) about a mean value. For the Mohr-Coulomb, geotechnical engineering material parameters these variables would be cohesion and angle of internal soil friction within each of the three Figure 4-2 soil regimes. Using the *generalized limit equilibrium method* formulation, as described in Appendix G of Ebeling et al. (2018) and first devised for USACE I-Wall Version 2.0 to solve for the passive and active soil forces acting on the wall. The uncertainties in the Mohr-Coulomb shear strength parameters for the soils manifest themselves in the Figure 4-3 PDF distributions for P_L and P_R . Chapter 5 discusses three methodologies to compute the driving wedge force P_L and the buttressing, resisting wedge force P_R imposed on the Figure 4-3 structural wedge in the context of these probabilistic, reliability analyses.

4.4 Three reliability models

The authors of this report have devised three means for computing the Figure 4-3 resultant earth pressure force statistical distributions for the P_L and P_R forces acting on the structural wedge. These reliability methodologies are the following:

1. A sliding stability assessment based upon the wedge solution method using subwedge divisions with the Appendix E Equivalent Scale of Fluctuation lengths combined with a simulation methodology to transfer uncertainty in the Mohr-Coulomb soil shear strength parameters to the Figure 5-1 P_L and P_R resultant forces acting on the structural wedge.
2. An earth pressure coefficient-based method of analysis combined with a simulation methodology to transfer uncertainty in the Mohr-Coulomb soil shear strength parameters directly to the Figure 5-1 P_L and P_R resultant forces acting on the structural wedge. This procedure is also restricted to homogeneous soil in the areas of the driving and

- resisting wedges. Note that the resisting wedge soil need not be the same in the driving wedge. The orientation of the slip planes are estimated using the Mohr-Coulomb soil shear strength parameters for determining the magnitude of the Variance Reduction Factor.
3. A sliding stability assessment with a wedge solution method that accounts for the reduction of correlation due to distances greater than the Scale of Fluctuation within the driving wedge combined with a simulation methodology to transfer uncertainty in the Mohr-Coulomb soil shear strength parameters to the Figure 5-1 P_L and P_R resultant forces acting on the structural wedge. This procedure minimizes the number of wedges in the solution by using the Variance Reduction Factor. Its application requires homogeneous material within the driving wedge. This procedure can also be applied to the foundation soil below the structural wedge as well as to the resisting wedge, so long as each is homogeneous as well. Each of these three (wedge) regions may be comprised of different soil types.

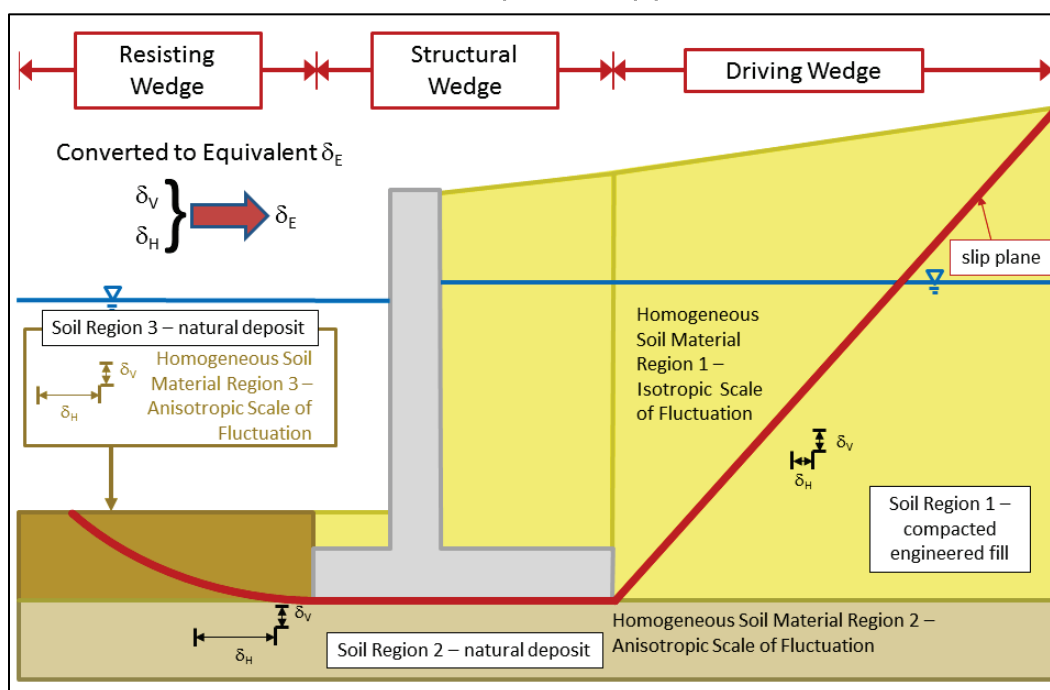
The soil wedge solution scheme used in the first and third approaches for computing P_R and P_L acting on the structural wedge are discussed in the subsections 4.4.1 and 4.4.2. Of these two wedge-based methodologies, the subwedge division model is more versatile. It can accommodate nonplanar slip planes and/or heterogeneous materials. The Coefficient Method solution scheme used in the second approach for computing P_R and P_L acting on the structural wedge is discussed in subsection 4.4.3.

4.4.1 Reliability Methodology #1: Multiple point simulations with subwedge divisions using δ_E slip plane distance

The geotechnical strength data for natural, soil deposits formed over geologic time, clearly demonstrate anisotropic scale of fluctuation values in the horizontal and vertical directions: δ_V and δ_H . Appendix D provides a data base of values for δ_V and δ_H , as well as the ratio of δ_H/δ_V . These data were obtained from the statistical processing of shear strength data with corresponding vertical and horizontal distances in various soil types. Over the last 20 years, engineers researching the spatial correlation of engineering soil properties have collected and studied the vertical and horizontal scale of fluctuation (or autocorrelation) engineering soil property data and some have made generalized observations. Several of these generalizations that have been published in the technical literature have been collected in Appendix D as well.

Figure 4-4 for the study T-Wall of interest envisions three regions of soil types, as discussed previously. For soil region 1 (in yellow) that consists of select engineered-compacted soil, isotropic scales of fluctuation in the vertical and horizontal directions are hypothesized for the backfill. This contrasts with the other two soil region material types; they are assumed to be naturally occurring soil deposits that have been formed through geologic means and on a geologic time scale. For these two soil regions 2 and 3, δ_H is assumed to be greater than δ_V . According to the data in Appendix D, the trend may be on the order of a factor of between 9 to 13 (subsection D.1.10).

Figure 4-4. Anisotropic scale of fluctuations and equivalent scale of fluctuation along a user-defined potential slip plane.



After assigning the vertical and horizontal scales of fluctuation to each of the three Figure 4-4 soil regions, the next step is to assign a potential slip surface through the driving, structural, and resisting wedges. This is shown in red in Figure 4-4.

Next, an equivalent scale of fluctuation, δ_E , is computed along each segment of the potential slip plane within each of the three Figure 4-4 soil regions. The value for δ_E is computed based on the values for (1) δ_H , (2) δ_V , and the orientation of the potential slip plane. Any of the procedures outlined in Appendix E may be used to compute δ_E . Two observations are

made at this point: (1) δ_E is soil-region specific and (2) δ_E may vary within a homogeneous soil region if the orientation of the slip plane changes. This latter issue will be the case for the Figure 4-4 soil region 3.

Recall that δ_E is the distance along the slip plane where the material properties along that slip plane are correlated. Appendix E contains methods for calculating δ_E given anisotropic scales of fluctuation for angled slip planes. One such method is found in Equation E.1, which gives an elliptical method of calculation diagrammed in Figure E-1. This means that for each distance δ_E along the slip plane, it is entirely reasonable to generate a material instance by random samples of the material property point estimate distributions to apply to the slip plane wedge. Also, starting at the end of that region and proceeding exactly distance δ_E , the next region is internally correlated but uncorrelated from the previous region. This property has been leveraged by Vanmarcke (1977b) and El-Ramly et al. (2003a).

For linear slip planes, δ_E does not change. This is shown in the driving wedge and structural wedge slip planes of Figure 4-5. For non-linear slip planes, such as the resisting wedge of Figure 4-5, δ_E must be calculated numerically as the slope changes for non-isotropic materials. The numerical procedure to do the nonlinear estimation of δ_E was provided by Vanmarcke (1977b) and has been elaborated in Appendix E.

Unfortunately, slip planes rarely have a distance that is an integral number of δ_E distances. When this happens, the last segment is shorter than the δ_E distance. Because δ_E is greater than the length of the final segment distance fsd , this implies that there is an overlap between the last two segments that is $(\delta_E - fsd)$ wide that has high correlation between the next to final and final segment. This overlap segment is a region of correlation between the final two segments, whereas all the previous segments with exact distance δ_E are uncorrelated. This relationship is shown in Figure E-3 of Appendix E.

This correlation can be computed from the Variance Reduction Factor, $\Gamma^2(\delta_Z)$, which can be estimated using the scale of fluctuation. This estimation is given in Equations 3.2 and 3.3 as well as Equation E.15 in Appendix E. Equation 4.1 (based on work by El-Ramly et al. 2003a) gives the equation to determine the correlation, ρ , between any two segments

based on the distance between the segments and width of each segment. Using the notation of Appendix E, their correlation is given by

$$\rho(X_{\Delta z}, X'_{\Delta z'}) = \frac{Z_0^2 \Gamma^2(Z_0) - Z_1^2 \Gamma^2(Z_1) + Z_2^2 \Gamma^2(Z_2) - Z_3^2 \Gamma^2(Z_3)}{2 \Delta z \Delta z' \sqrt{\Gamma^2(\Delta z) \Gamma^2(\Delta z')}} \quad (4.1)$$

Figure 4-5. Subwedge division using the equivalent scale of fluctuation δ_E per each material type.

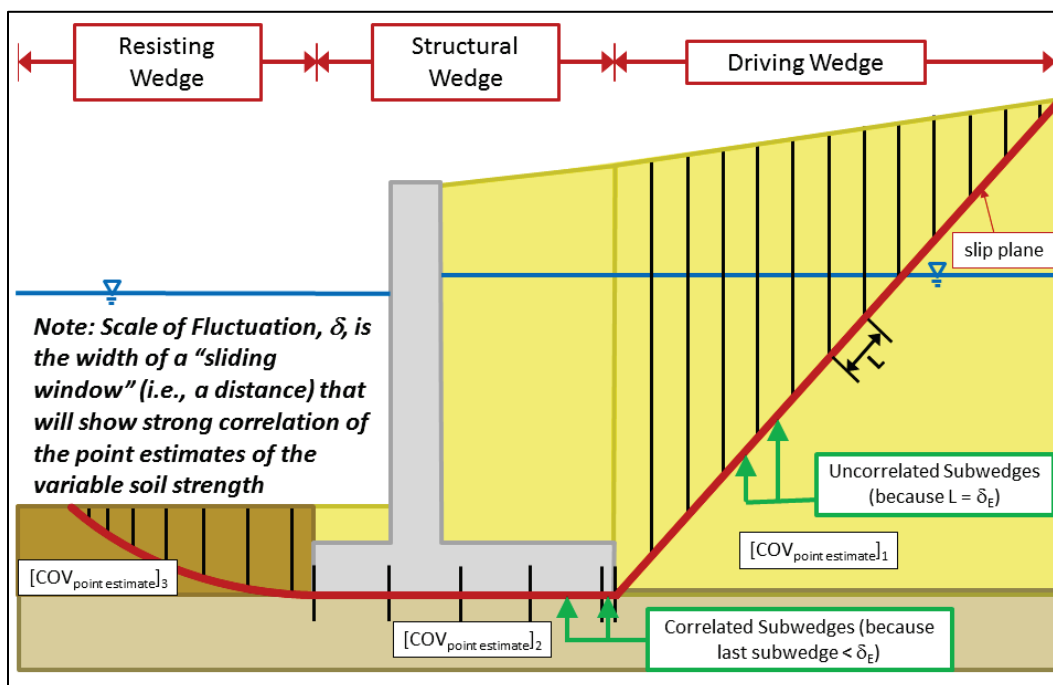
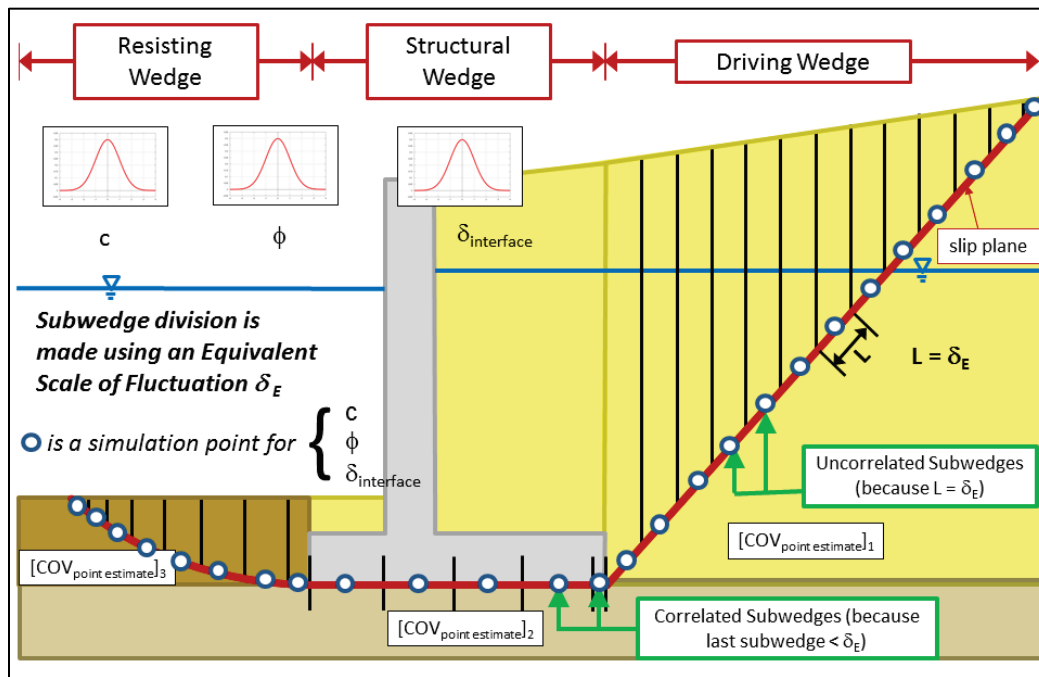


Figure 4-6 shows the simulation model that is used based on the calculations discussed above and in Appendix E. A wedge model is created based on subwedges that primarily have width L equal to δ_E with exceptions for the final subwedges of the model.

Figure 4-6. Multiple point c , ϕ and δ simulations with subwedge divisions using δ_E slip plane distance.



To calculate δ_E and determine the peak forces acting on the structure, the angle of the driving wedge needs to be determined that gives the maximum active force acting against the structural wedge. In the same manner, the resisting wedge shape needs to be determined to find the minimal passive force acting against the structural wedge.

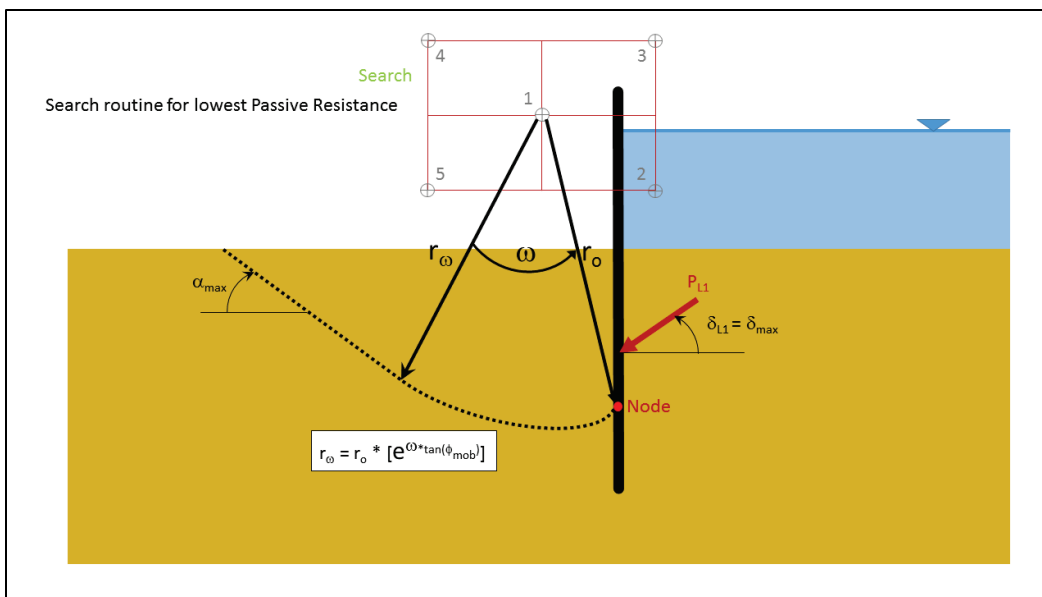
The angle of the driving wedge is determined using a sweep search method (see Appendix A in either Ebeling and White [2006] or Ebeling et al. [2007]). Using the mean values for the material properties and small wedge angle α , from which δ_E can be determined, the wedge solution method determines the earth forces acting in the driving and resisting wedges from the subwedge that is farthest from the wall inward, with equal and opposite forces acting at the interface between subwedges. The farthest subwedge does not have a force along one interface, so it is solved to determine the interslice force for the next subwedge. The next subwedge then has equal and opposite solved interslice forces on the side away from the structural wedge, and the solution continues until the forces are collected at the structural wedge interface. This force calculation method is discussed in detail in subsection 5.2 of this technical report.

This procedure is repeated with increasing values of potential slip plane angle α until the value of α that generates the maximum force acting

against the structural wedge can be determined. From this α , the δ_E and the value of L can be computed for the active wedge.

Similarly, the passive log-spiral shape needs to be determined to find the minimum force that can act against the structural wedge from the resisting wedge. In this case, a numerical method would need to be utilized to determine the log-spiral curve with the least resisting force. Ebeling et al. (2018) describe a method to do this based on soil depth of embedment and beginning spiral center point. In this case, the embedment depth is given by the structural wedge base. The spiral center point is assigned an estimated starting position, and the wedges are constructed based on the log-spiral generated from those values until the slip plane subwedge reaches an angle of $45^\circ - (\phi/2)$ where this slip plane orientation is maintained at this constant angle and the slip plane intersects the ground surface (Figure 5-6), and the resisting force is returned. Figure 4-7 indicates that this same process is repeated to generate a log-spiral center point with four corner points at ΔX and ΔY distance from the original center point. If one of the corner points has the least force acting against the structural wedge, the same procedure is performed around that new center point. If the original center point has the least force acting against the structural wedge, the distances ΔX and ΔY are halved. If these ΔX and ΔY values still exceed a tolerance distance, the procedure is done again at the center point and the new corner points. If these ΔX and ΔY values do not exceed the tolerance distance, then the appropriate log-spiral surface has been found. In this way, the passive slip-plane with the least resistance can be discovered.

Figure 4-7. Searching for the minimum resistance passive region defined by a log-spiral slip plane.



For the nonlinear subwedges of the log-spiral wedge, Section E.2.3 provides a procedure based on Vanmarcke (1977b) to integrate the values of length based on how δ_E varies as the slope of the slip plane for a subwedge changes. Because δ_E changes, the length of the slip plane L no longer maps to δ_E lengths directly.

The material properties for the subwedge that have an effect along the slip plane (e.g., Mohr-Coulomb shear strength parameters c and ϕ) are chosen at the simulation point for the subwedge without correlation from a distribution for all but the last two wedges. The last two subwedge properties are drawn from the distributions, but their material properties are correlated according to Equation 4.1. The value for $\delta_{Interface}$, the wedge interface friction between the structural wedge and the driving and resisting wedges, is also drawn from its distribution.

When all of the subwedge properties are assigned for a single simulation run, the wedge solution method determines the earth forces acting in the driving and resisting wedges from the subwedge that is farthest from the wall inward, with equal and opposite forces acting at the interface between subwedges. The farthest subwedge does not have a force along one interface, so it is solved to determine the interface force for the next subwedge. The next subwedge then has equal and opposite solved interface forces on the side away from the structural wedge, and the solution continues until the forces are collected at the structural wedge

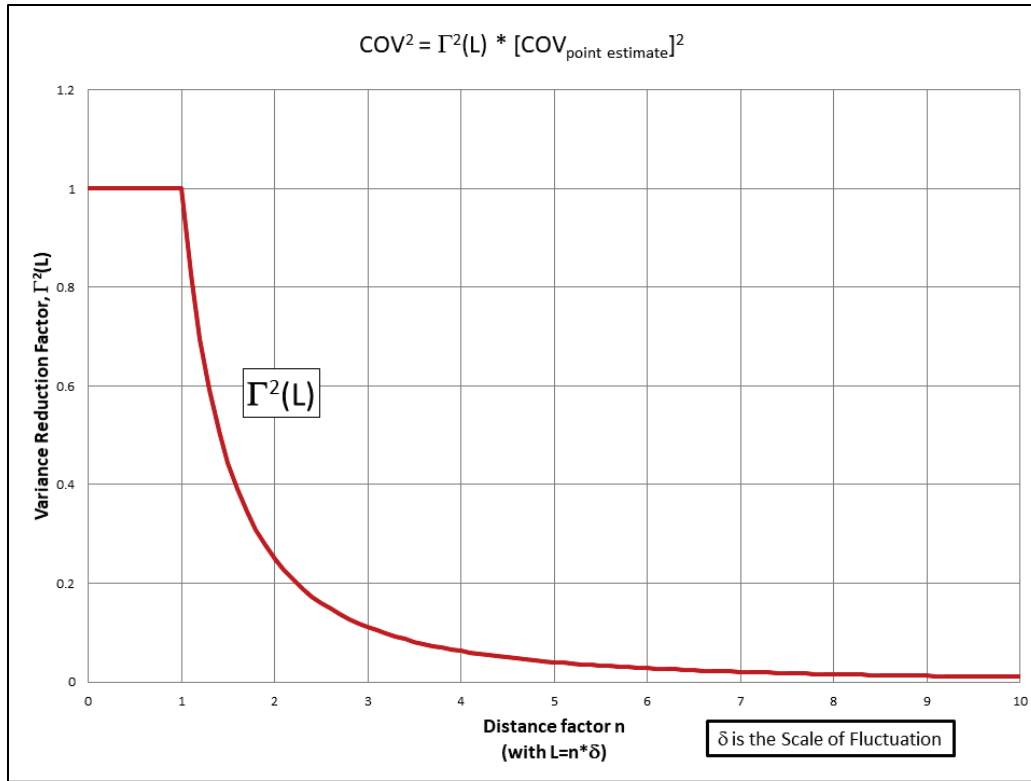
interface. This force calculation method is discussed in detail in subsection 5.2 of this technical report.

4.4.2 Reliability Methodology #2: Homogeneous subwedge simulations with scaled distributions using Variance Reduction Factors

The Variance Reduction Factor $\Gamma^2(L)$ is used in the Subwedge Division model to calculate the correlation between the final subwedges. This occurs because the last subwedge had a slip plane length that was less than the scale of fluctuation, as depicted in Figure 4-6. However, the Variance Reduction Factor has other uses. When materials are chosen from a distribution in an uncorrelated fashion, the effects of those materials tend to counteract each other in an averaging fashion, as the values are chosen about the mean. This averaging effect can be approximated using $\Gamma^2(L)$ for anisotropic materials with distances that are longer than the scale of fluctuation.

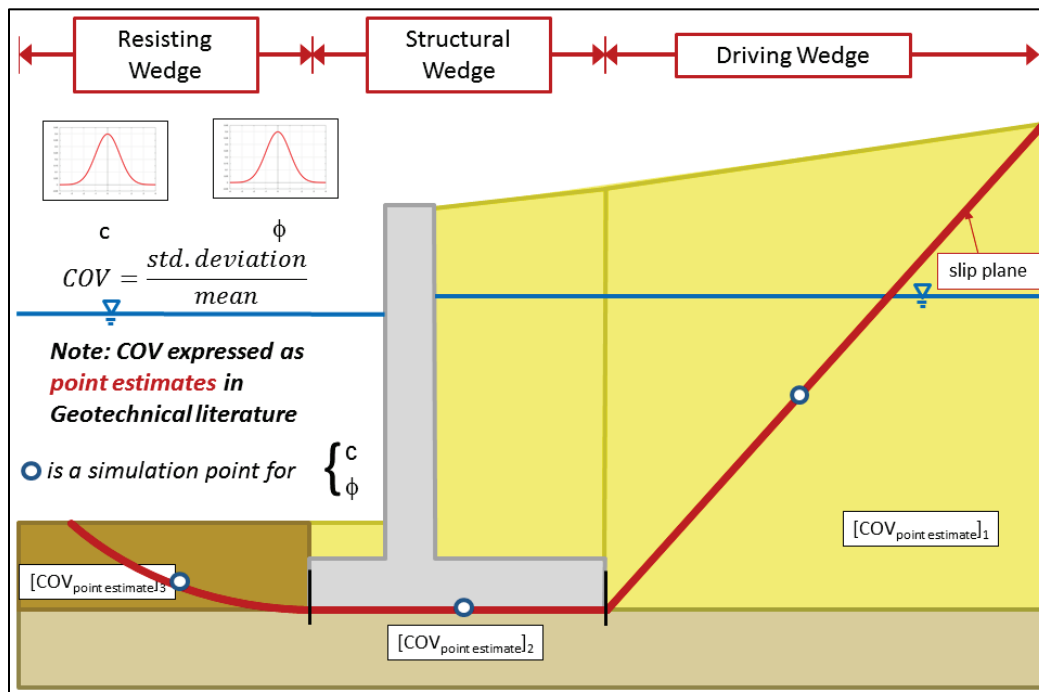
Figure 4-8 shows the relationship of variance to distance L , where L is a factor n times the scale of fluctuation δ , according to Equations 3.2 and 3.3. Normalizing Equations 3.2 and 3.3 by $1/\delta$ results in the curve shown. For scaled wedge slip plane distances n greater than 1.0, the point estimate distribution variance can be scaled by $1/n$ (by the normalized Equation 3.3). This results in a distribution that has the same mean but has a smaller variance due to the averaging effects of the loss of spatial correlation.

Figure 4-8. Normalized Variance Reduction Factor.



In a modeling framework, this means that homogeneous materials can be formed into complete larger subwedges with longer slip planes, but the distribution of the variables is changed to represent the effect of loss of spatial correlation. In Figure 4-9, three homogeneous materials are used for the resisting, structural, and driving wedges, respectively. In this example, only three wedges are needed for the calculation of forces because the potential slip planes are orientated at a constant angle (α from horizontal) within two of the three material regions. The calculations for driving and resisting wedge forces acting against the structural wedge are described in subsection 5.5, and Figure 5-7 shows the definition of the length of slip plane for the single driving wedge model where L is set equal to L_{Active} in this figure. The length of the Figure 4-9 slip plane for the resisting wedge and for the structural wedge is similarly defined. Using the $COV_{\text{point estimate}}$ values for each of the three material regions along with the three slip plane lengths, the spatial COV are generated for each region using the equation given in Figure 4-8 and using the prescribed potential slip plane orientations for each of the three Figure 4-9 wedges.

Figure 4-9. Single point c , ϕ and $\delta_{Interface}$ simulations with adjusted distribution ranges for slip plane distances $> \delta_E$.



For each wedge, the equivalent scale of fluctuation δ_E needs to be calculated. Recall that Appendix E contains methods for calculating δ_E given anisotropic scales of fluctuation for inclined slip planes. For linear slip planes such as the driving wedge of Figure 4-9, δ_E is a single calculation. The calculation for the multiple n is therefore L / δ_E , where L is the length of its planar slip plane. For nonlinear wedges, subsection E.2.3 provides a procedure based on Vanmarcke (1977b) to integrate the value of the multiple n based on how δ_E varies as the slope changes. Because δ_E changes, the length of the slip plane L no longer maps to the multiple n directly.

The same initial process of determining the a slope for the driving wedge and the assumed logarithmicspiral surface for the resisting wedge from the mean material properties needs to be done here as in subsection 4.4.1. These calculations fix δ_E and the Normalized Variance Reduction Factor based on the multiple n .

Once a value n has been determined for each homogeneous material subwedge, the distributions for simulation variable selection are changed from the point estimate distributions to the scaled point estimate distributions with the same mean but with the variance modified by the normalized Variance Reduction factor based on n . For a simulation run,

each subwedge has its material property variable values chosen at random from its modified distribution at the simulation point.

When all of the subwedge properties are assigned for a single simulation run, the wedge solution method determines the earth forces acting in the driving and resisting wedges from the subwedge that is farthest from the wall inward, with equal and opposite forces acting at the interface between subwedges. The farthest subwedge does not have a force along one interface, so it is solved to determine the interface force for the next subwedge. The next subwedge then has equal and opposite solved interface forces on the side away from the structural wedge and the solution continues until the forces are collected at the structural wedge interface. This procedural method is discussed in detail in section 5.2 of this technical report.

Of course, the number of subwedges is greatly reduced from the Subwedge Division method because subwedge boundaries only need to be applied when a homogeneous soil material boundary is crossed by the computed slip plane. This method would greatly speed performance.

4.4.3 Reliability Methodology #3: Point simulations with Coefficient Method

The Coefficient Method for solving for forces acting against the structural wedge is shown in Figure 4-10. This method makes certain assumptions so that active and passive earth forces can be computed directly for the interfaces with the structural wedge. The assumptions for this model are that the homogeneous materials are applied in horizontal layers, except the ground surface, which must have a consistent slope. The other methods will allow for more generic geometry since they are not bound by these assumptions.

The Coefficient Method for force calculation is given in subsection 5.3. Of course, before simulations can take place, the scale of fluctuation δ_E and the slip plane length L need to be determined. Because subwedges are not being created, *equivalent* logarithmic spiral slip plane area converted to planar slip plane area calculations (based on the slip plane angle α as measured from horizontal) also need to be performed to derive an approximate L value for an equivalent planar slip surface wedge. The authors of this report recognize that this expedient modeling step is an approximate solution in the computational process.

There is a slip plane angle α for both the driving and resisting wedge. The resisting wedge no longer follows a log-spiral curve, but it has properties derived from using a log-spiral approximation. Therefore, a single angle is used to describe and measure the surface for both driving and resisting wedges.

The search for the angle α is performed in the same manner as in subsection 4.4.1, where mean values are assigned to all of the variables and α is determined using the sweep search method with the appropriate earth pressure approximation calculation method. The approximate earth pressure calculation methods are used with the mean values and are based on the δ values for interface friction, as shown in Table 4-1 and described in subsection 5.4. Active pressure calculations are used for the driving wedge, and passive pressure calculations are used for the resisting wedge.

Table 4-1. Earth pressure solution methods based on interface friction coefficient at the structural wedge boundaries.

Active		
$\delta=0$	$\delta>0$	
Rankine	Coulomb	
Passive		
$\delta=0$	$0<\delta<\phi/2$	$\delta>\phi/2$
Rankine	Coulomb	Log-Spiral

The geometry of the wedge is determined from an assigned slip plane angle α and the angle the ground surface makes from the end of the structural wedge θ . This geometry includes the area of the wedge and length of the slip plane L . This geometry is solution is derived in Appendix F. The angle α is used to determine δ_E using one of the methods of Appendix E.

For the driving wedge, the sweep search method is applied using the active earth pressure force equations of subsection 5.3 with the appropriate planar base wedge approximation calculations to find the driving wedge slip surface orientation α with the maximum force acting against the structural wedge. For the resisting wedge, the sweep search method is applied using the passive pressure and force equations with the appropriate earth pressure force approximation calculations of subsection

5.4 to find the resisting wedge α with the minimum force acting against the structural wedge.

When the driving and resisting wedge α 's have been approximated, with their derived geometries, then the model can be simulated. Because the slip plane length L for either wedge can be longer than δ_E for either wedge, the Normalized Variance Reduction Function is used with L/δ_E to modify the variable distributions as described in Subsection 4.4.2 for either wedge.

The log-spiral slip plane approximation for determining passive soil pressures relies on a reduction term described in Subsection 5.4.5. This non-linear reduction term transforms the resisting wedge's slip surface orientation α and material properties to accurately approximate the log-spiral interface for homogeneous materials.

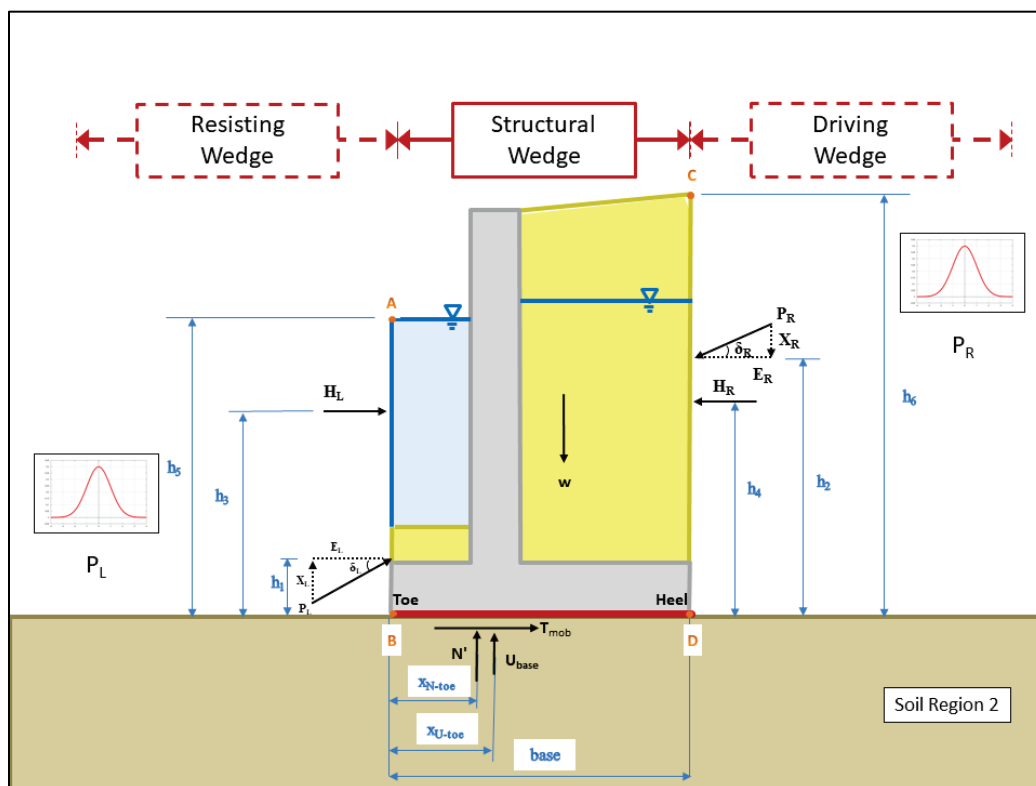
Subsection 5.4 describes how the variable values for c , ϕ , and δ are used to determine the active and passive pressures and therefore the forces acting at the structural wedge boundaries. Because these variables are distributed according to fixed distributions, the results are pressures that are a combined distribution as shown in Figure 4-10, where the distributions of c and ϕ yield a distribution for the horizontal earth pressure coefficient K_H . Likewise, transforming the pressure distribution K_H yields the force distribution E_R . These distributions can be derived from existing data, and performing a simulation varying c , ϕ , and δ can give these distributions as well as the *PUP* value for the structural wedge for the earth limit states.

5 Calculating the Two Resultant Earth Pressure Thrust Forces of P_L and P_R That Are Imposed on the Structural Wedge by the Driving and Resisting Wedges

5.1 Introduction

The thrust force P_R imposed by the driving wedge upon the structural wedge is shown in Figure 5-1. Due to variability in the Mohr-Coulomb soil shear strength parameters (i.e., c and ϕ for total stress analysis or c' and ϕ' for effective stress analysis), there will be a PDF associated with this P_R force. This P_R PDF distribution will be computed using one of the three reliability methodologies described in Chapter 4 as a function of the varying soil shear strength parameters and the geological depositions. Similarly, there will be a PDF associated with the P_L force imposed on the structural wedge by the resisting wedge.

Figure 5-1. Driving wedge and resisting wedge forces imposed on a structural wedge, P_R and P_L .



The authors of this report have devised three means for computing the Figure 5-1 resultant earth pressure force statistical distributions for the P_L and P_R forces acting on the structural wedge:

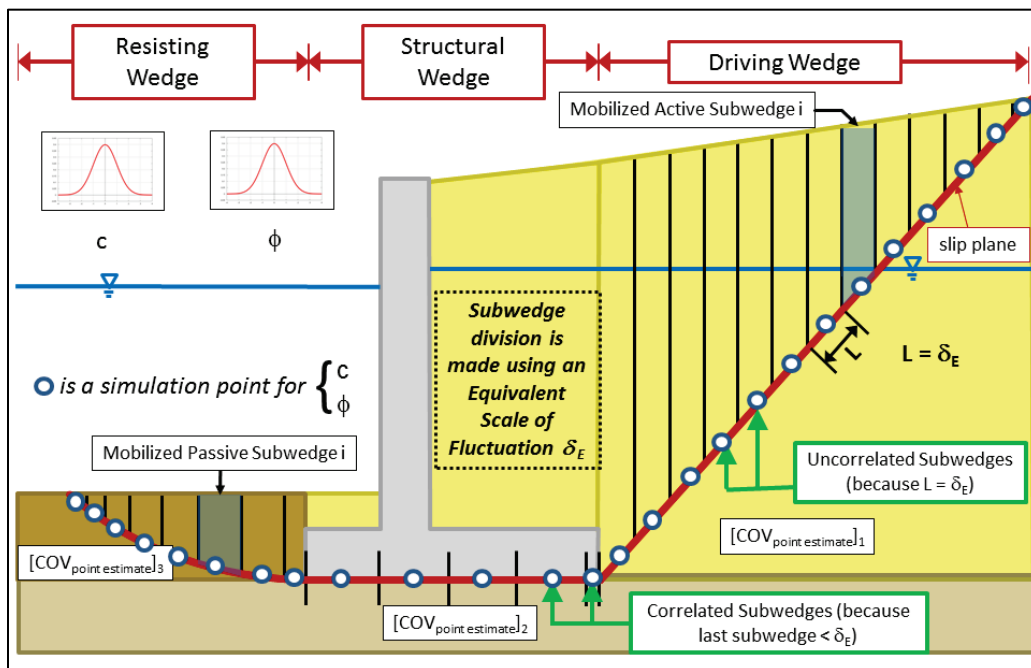
1. A sliding stability assessment based upon the wedge solution method using subwedge divisions with the Appendix E Equivalent Scale of Fluctuation widths combined with a simulation methodology to transfer uncertainty in the Mohr-Coulomb soil shear strength parameters to the Figure 5-1 P_L and P_R resultant forces acting on the structural wedge.
2. An earth pressure coefficient based method of analysis combined with a simulation methodology to transfer uncertainty in the Mohr-Coulomb soil shear strength parameters directly to the Figure 5-1 P_L and P_R resultant forces acting on the structural wedge. This procedure is also restricted to homogeneous soil in the areas of the driving and resisting wedges. Note that the resisting wedge soil need not be the same in the driving wedge. The orientation of the slip planes are estimated using the Mohr-Coulomb soil shear strength parameters for determining the magnitude of the Variance Reduction Factor.
3. A sliding stability assessment with a wedge solution method that accounts for the reduction of correlation due to distances greater than the Scale of Fluctuation within the driving wedge combined with a simulation methodology to transfer uncertainty in the Mohr-Coulomb soil shear strength parameters to the Figure 5-1 P_L and P_R resultant forces acting on the structural wedge. This procedure minimizes the number of wedges in the solution by using the Variance Reduction Factor. Its application requires homogeneous material within the driving wedge. This procedure can also be applied to the foundation soil below the structural wedge as well as to the resisting wedge, so long as each is homogeneous as well. Each of these three (wedge) regions may be comprised of different soil types.

The soil wedge solution scheme used in the first and third approaches for computing P_R and P_L acting on the structural wedge are discussed in the subsections 5.2 and 5.5. Of these two wedge-based methodologies, the subwedge division model is more versatile. It can accommodate nonplanar slip planes and/or heterogeneous materials. The Coefficient Method solution scheme used in the second approach for computing P_R and P_L acting on the structural wedge is discussed in subsections 5.3 and 5.4.

5.2 Wedge solution method using a subwedge division model for the mobilized active and mobilized passive subwedges

In a subwedge solution scheme formulation, the driving and the resisting wedges for the potential slip plane (in red) are subdivided into subwedges. Figure 5-2 identifies one of the *mobilized active* subwedges (in gray) within the driving wedge and one of the *mobilized passive* subwedges (in gray) within the resisting wedge. Three soil material regions are designated in this figure: (1) the soil region containing the driving wedge, (2) the foundation soil region below the structural wedge, and (3) the resisting soil region. The subwedge divisions are made using an equivalent scale of fluctuation for each of the three soil material regions. Each soil region is designated with a different color in this figure. The equivalent scale of fluctuation for soil material region 1 of the driving wedge dictates the length of subwedge as measured along the planar potential slip plane (in red). A similar calculation is made for soil material regions 2 and 3 based on their respective equivalent scales of fluctuation. Recall from Appendix E that in an anisotropic scale of fluctuation soil layer, the equivalent scale of fluctuation depends upon the orientation of the slip plane relative to the values for the vertical and horizontal scales of fluctuation of that soil material region.

Figure 5-2. Driving-structural-resisting subwedge model of a combined geological and structural system and a potential slip-plane for sliding stability analysis of a T-Wall as per EM 1110-2-2502 (HQUSACE 1989).

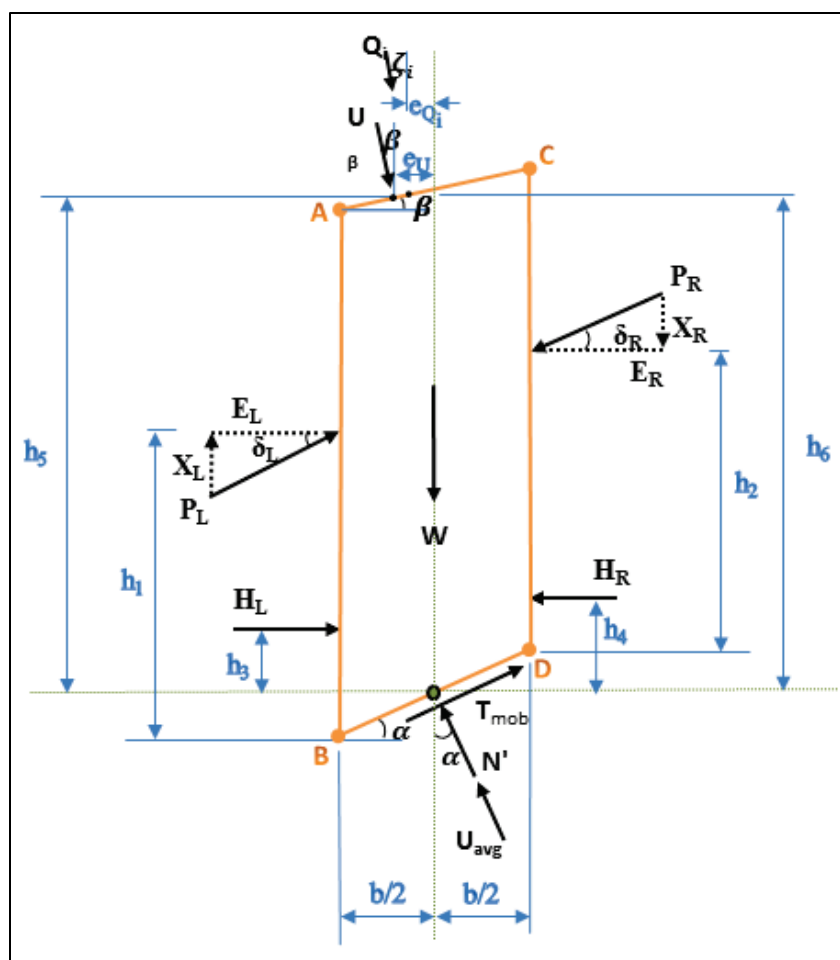


The following subsections describe the subwedge-based models used for computing the driving to structural wedge interslice force P_R and the resisting to structural wedge interslice force P_L , given a user-defined potential slip surface.

5.2.1 Subwedge models for the mobilized active subwedges

In the subwedge solution scheme formulation, the driving wedge is subdivided into subwedges, as depicted in Figure 5-2. A mobilized active subwedge i is identified within the driving wedge in this figure with gray shading. It is one of n mobilized active subwedges within this driving wedge. Figures 5-3 shows the forces acting on this one mobilized active subwedge in an effective stress analysis. Note the planar base representation from points B to D of the soil subwedge in this figure.

Figure 5-3. Forces acting on a typical individual mobilized active soil subwedge i – effective stress.



The list of symbols used in this mobilized active subwedge i are defined as follows:

- B = the width of the subwedge
- E_L = the horizontal shear force component of P_L acting on the left side of the subwedge
- E_R = the horizontal shear force component of P_R acting on the right side of the subwedge
- e_Q = the eccentricity of the surface surcharge force Q_i
- e_u = the eccentricity of the surface water force U_β
- h_1 = the height from point B to E_L
- h_2 = the height from point D to E_R
- h_3 = the height from midpoint of the base to H_L
- h_4 = the height from midpoint of the base to H_R
- h_7 = the height from midpoint of the base to the top of subwedge where the resultant surface water force U_β acts at an angle β from vertical
- h_8 = the height from midpoint of the base to the top of subwedge where the surface surcharge force Q_i acts at an angle ζ_i from vertical
- H_L = the horizontal resultant force exerted by pore water pressure acting on the left side of the subwedge plus any external horizontal pool induced, boundary water pressures
- H_R = the horizontal resultant force exerted by pore water pressure acting on the right side of the subwedge plus any external horizontal pool induced, boundary water pressures
- N' = the effective force acting normal to the base of the subwedge
- P_L = the resultant interslice force acting on the left side of the subwedge
- P_R = the resultant interslice force acting on the right side of the subwedge
- Q_i = the surcharge force acting on top of the subwedge i
- T_{mob} = the mobilized shear resistance force acting along the base of the subwedge
- U_β = the surface water force acting normal to the top of the subwedge
- U_{ave} = the resultant water pressure force acting normal to the base calculated using the average water pressure midpoint along the subwedge base
- W = the (total) weight of the subwedge

X_L = the vertical shear force component of P_L acting on the left side of the subwedge

X_R = the vertical shear force component of P_R acting on the right side of the subwedge.

With the Greek character symbols defined as follows:

α = the angle the base of the planar subwedge as measured from horizontal

β = the slope of the top of the planar subwedge

δ_L = the angle of interslice friction on the left side of the subwedge

δ_R = the angle of interslice friction on the right side of the subwedge.

The Figure 5-2 structural retaining wall wedge is moving away from the driving soil wedge. With this type of movement, the driving soil subwedges are therefore labeled as mobilized active. Consequently, for each mobilized active soil subwedge, the shear force required for equilibrium T along the base of subwedge i acts in the direction shown in Figure 5-3 and counter to the direction of soil subwedge movement. The shear force required for equilibrium along the base of each subwedge, T , is set equal to and designated by the mobilized shear force T_{mob} in this figure. For an effective stress analysis, T_{mob} is given by

$$T_{mob} = c'_{mob-Active} * \frac{b}{\cos(\alpha)} + N' * \tan(\phi'_{mob-Active}) \quad (5.1)$$

with the mobilized effective cohesion $c'_{mob-Active}$ and mobilized effective angle of internal friction $\phi'_{mob-Active}$ given by

$$c'_{mob-Active} = \frac{c'}{FS_{Active}} \quad (5.2)$$

and

$$\tan(\phi'_{mob-Active}) = \frac{\tan(\phi')}{FS_{Active}} \quad (5.3)$$

For total stress, T_{mob} is

$$T_{mob} = c_{mob-Active} * \frac{b}{\cos(\alpha)} + N * \tan(\phi_{mob-Active}) \quad (5.4)$$

with the mobilized cohesion $c_{mob-Active}$ and mobilized angle of internal friction $\phi_{mob-Active}$ given by

$$c_{mob-Active} = \frac{c}{FS_{Active}} \quad (5.5)$$

and

$$\tan(\phi_{mob-Active}) = \frac{\tan(\phi)}{FS_{Active}} \quad (5.6)$$

In a total stress analysis, there would not be uplift pressures acting normal to the subwedge base and thus, no U_{avg} force. N' becomes the total force N acting normal to the base of the Figure 5-3 mobilized active subwedge. Additionally, the horizontal resultant force exerted by pore water pressure acting on the left side and right side of the subwedge would be zero for their contributions to the forces H_L and H_R .

5.2.1.1 Generalized limit equilibrium method

The *generalized limit equilibrium method* formulation is used to solve for the active and passive soil forces acting on the Figure 5-2 structural wedge. The authors of this report first devised and implemented this solution procedure in the USACE_I-Wall Version 2.0 software (Ebeling et al. 2018). The roots of this procedure of analysis lay in slope stability. The *generalized limit equilibrium method* is an extension of Spencer's (1973) procedure, which has been generalized by Chugh (1986). Subsection 6.9.4 in Abramson et al. (1996) provides a summary of the wedge equations. They provided a check to the derivations made independently by the authors of this report during the USACE_I-Wall Version 2.0 formulation development. The following subsections summarize the details of the analysis.

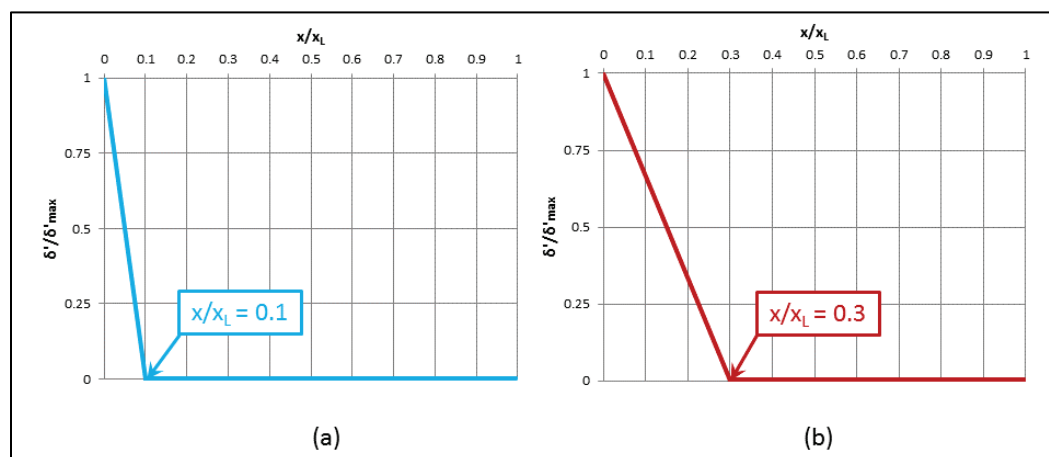
5.2.1.2 Normalized interslice friction versus normalized lateral distance from the wall

The *generalized limit equilibrium method* has the advantage of allowing the assignment of a variable interslice friction force angles δ_L and δ_R along the faces of each of the Figure 5-3 soil subwedges with distance from the structural wedge interface with the Figure 5-2 driving wedge. This is a significant advantage when applied to retaining wall problems, as discussed in Ebeling et al. (2018). The accuracy of the computations were found during that research and development effort to be affected by the

distribution of interslice friction values with position of a particular soil subwedge in the sequence of soil subwedges comprising the soil mass.

The general framework of specifying the distribution of interslice friction angle is the use of a normalized interslice friction distribution function versus normalized distance (from the structural wedge interface location). The optimum interslice angle of friction distribution with mobilized active subwedge distance from the structural wedge interface is shown in Figure 5-4(a). The ordinate is the interslice friction δ at a lateral point away from the structural wedge interface of distance x , divided by the interslice friction value at this interface, designated δ_{max} . The Figure 5-2 planar driving wedge slip plane intersects the ground surface at a lateral distance of x_L away from the driving to structural wedge interface. Consequently, the lateral distance x to any interslice position is normalized by the value of x_L in this procedure.

Figure 5-4. An example normalized interslice friction δ/δ_{max} distribution versus normalized distance (x/x_L) relationship from the driving to structural wedge interface, (a) active wedge, (b) passive wedge.



According to Figure 5-4(a), the interslice friction force at any soil subwedge interslice up to a distance of 0.1 times x_L will be assigned an interslice friction angle of δ_{max} , where δ_{max} is the interslice friction at the driving wedge to structural wedge interface. For interslices located less than a distance of 0.1 times x_L , the value of δ will be less than δ_{max} , by the linearly decreasing relationship with normalized distance given in this figure. The interslice friction at the wall δ_{max} is set equal to the mobilized interface is shear friction at the structural to driving wedges interface.

An extensive series of parametric studies were conducted by Ebeling et al. (2018) to determine an appropriate distribution for normalized interslice friction distribution versus normalized distance from the structural wedge interface. The results from this study are summarized in Appendix H in Ebeling et al. (2018) for assumed planar slip plane of the active driving wedge and for a log-spiral slip plane of the passive resisting wedge. Figure 5.4 shows example relationships.

In an effective stress analysis, the mobilized effective angle of interface friction for the structural to driving active wedges interface is given by

$$\tan(\delta'_{mob-Active}) = \frac{\tan(\delta')}{FS_{Active}} \quad (5.7)$$

and the mobilized effective angle of interface friction for the structural to resisting passive wedges interface is given by

$$\tan(\delta'_{mob-Passive}) = \frac{\tan(\delta')}{FS_{Passive}} \quad (5.8)$$

In a total stress analysis, a total stress angle δ is substituted into Equations 5.7 and 5.8. This results in

$$\tan(\delta_{mob-Active}) = \frac{\tan(\delta)}{FS_{Active}} \quad (5.9)$$

for the mobilized angle of interface friction for the structural to driving active wedges interface. The mobilized angle of interface friction for the structural to resisting passive wedges interface is

$$\tan(\delta_{mob-Passive}) = \frac{\tan(\delta)}{FS_{Passive}} \quad (5.10)$$

5.2.1.3 Solution of each P_L force acting on the LHS of mobilized active soil subwedge interslice i face

Figure 5-3 shows the forces acting on active subwedge i . A right-hand coordinate system is assumed and is used in the following equations (e.g., positive is up and to the right in this figure). Note that a positive angle is associated with a counter-clockwise rotation (e.g., angle α is shown acting in a positive orientation in Figure 4-4. In an effective stress analysis, the mobilized active subwedge interslice forces include the LHS active soil subwedge force P_L is given by

$$P_L = \frac{\left\{ \begin{array}{l} P_R [\cos(\delta_R - \alpha) - \sin(\delta_R - \alpha) \tan \phi'_{mob}] \\ - [c'_{mob} b \sec \alpha - W \sin \alpha + (W \cos \alpha - U_{avg}) \tan \phi'_{mob}] \\ - (H_L - H_R) [\cos \alpha + \sin \alpha \tan \phi'_{mob}] \\ - U_\beta [\cos(\beta - \alpha) \tan \phi'_{mob} + \sin(\beta - \alpha)] \\ - Q [\cos(\zeta - \alpha) \tan \phi'_{mob} + \sin(\zeta - \alpha)] \end{array} \right\}}{\cos(\delta_L - \alpha) - \sin(\delta_L - \alpha) \tan \phi'_{mob}} \quad (5.11)$$

with

$$h_1 = \frac{\left\{ \begin{array}{l} P_R \cos(\delta_R) \left[h_2 - \frac{b}{2} [\tan(\delta_R) - \tan \alpha] \right] \\ - U_\beta \cos \beta (h_5 \tan \beta - e_U) \\ - Q \cos \zeta (h_6 \tan \zeta - e_Q) \\ - (H_L h_3 - H_R h_4) \end{array} \right\}}{P_L \cos(\delta_L)} - \frac{b}{2} [\tan(\delta_L) - \tan \alpha] \quad (5.12)$$

and

$$N' = \left[\begin{array}{l} P_R \sin(\delta_R - \alpha) - P_L \sin(\delta_L - \alpha) \\ - H_L \sin \alpha + H_R \sin \alpha + W \cos \alpha \\ + U_\beta \cos(\beta - \alpha) + Q \cos(\zeta - \alpha) - U_{avg} \end{array} \right] \quad (5.13)$$

Note that the subscript “mob-Active” has been shortened to “mob” in these three equations. The LHS and RHS interslice forces angle of friction are varied with position among the soil wedges according to a relationship like that shown in Figure 5-4(a).

In a total stress analysis, there would not be uplift pressures acting normal to the subwedge base (i.e., no U_{avg} force), nor would there be internal water pressures included within the soil regime to act as boundary water forces normal to the vertical faces of the subwedge. All active mobilized cohesion as well as soil friction and interface friction angles would be expressed a total stress strength parameters: c_{mob} , ϕ_{mob} , and δ_{mob} in these three equations. In Equation 5.13, N replaces N' .

5.2.1.4 Solution of the generalized limit equilibrium method relationships for the Figure 5-1 P_R force for each Figure 5-2 LHS mobilized active soil subwedge interslice face for n active soil subwedges

The equations given in the preceding sections in the form of recursive relationships for the variable interslice force inclination are used to compute the Figure 5-1 mobilized active resultant force P_R (or the passive resultant force P_L) imposed on the structural wedge by the Figure 5-2

driving wedge. The solution always starts with the farthest wedge from the wall, for the subwedge designated number n . The value of P_{R-n} is set equal to zero in the triangular shaped, last soil wedge and solve for the value of P_{L-n} . The next step is to apply P_{L-n} as a known value, equal to $P_{R-(n-1)}$, on the right interslice face of soil wedge $(n-1)$ and solve for the LHS interface value of $P_{L-(n-1)}$. This sequential solution process proceeds back through each wedge and to soil wedge 1 for which the last calculation is made for the value of P_{L-1} . The active soil subwedge 1 interfaces with the structural wedge. In an active force computation, the calculated P_{L-1} ($=P_R$ in Figure 5-3) force acting on soil wedge 1 is equal to the mobilized P_A value that the driving wedge imposes upon the RHS structural wedge interface.

5.2.2 Subwedge model for the mobilized passive subwedges

In the subwedge solution scheme formulation, the resisting wedge is subdivided into subwedges, as depicted in Figure 5-2. A mobilized passive subwedge i is identified within the resisting wedge in this figure with gray shading. It is one of n mobilized passive subwedges within this resisting wedge. Figure 5.5 shows the forces acting on this one mobilized passive subwedge in an effective stress analysis. Note the planar base representation from points B to D of the soil subwedge in this figure.

with the mobilized cohesion $c'_{mob-Passive}$ and mobilized friction $\phi'_{mob-Passive}$ given by

$$c'_{mob-passive} = \frac{c'}{FS_{Passive}} \quad (5.15)$$

and

$$\tan(\phi'_{mob-Passive}) = \frac{\tan(\phi')}{FS_{Passive}} \quad (5.16)$$

For total stress, T_{mob} is

$$T_{mob} = c_{mob-Passive} * \frac{b}{\cos(\alpha)} + N * \tan(\phi_{mob-Passive}) \quad (5.17)$$

with the mobilized cohesion $c_{mob-Active}$ and mobilized angle of internal friction $\phi_{mob-Active}$ given by

$$c_{mob-Passive} = \frac{c}{FS_{Passive}} \quad (5.18)$$

and

$$\tan(\phi_{mob-Passive}) = \frac{\tan(\phi)}{FS_{Passive}} \quad (5.19)$$

In a total stress analysis, there would not be uplift pressures acting normal to the subwedge base and thus no U_{avg} force. N' becomes the total force N acting normal to the base of the Figure 5-5 mobilized passive subwedge. Additionally, the horizontal resultant force exerted by pore water pressure acting on the left side and right side of the subwedge would be zero for their contributions to the forces H_L and H_R . The LHS and RHS interslice forces angle of friction are varied with position among the soil wedges according to a relationship like that shown in Figure 5-4(b).

Note that Figure 5-5 contrasts with Figure 5-3 by the direction of the shear force acting along the base of the wedge, T_{mob} . This is due to the direction of movement of the Figure 5-2 retaining wall system from right to left for both mobilized active and passive subwedges. The symbols used to describe the geometry and the forces acting on the Figure 5-5 mobilized passive subwedge are the same as those used for the mobilized active subwedge of Figure 5-3.

5.2.2.1 Solution of each P_R force acting on the RHS of mobilized passive soil subwedge interslice i face

Figure 5-5 shows the forces acting on passive subwedge i of n subwedges in an effective stress analysis. Again, a right-hand coordinate system is assumed and is used in the following equations (e.g., positive is up and to the right in this figure). Because a positive angle is associated with a counter-clockwise rotation, the value assigned to angle α in Figure 5-5 will be negative. The passive wedge interslice forces include the RHS passive soil wedge force P_R is given by

$$P_R = \frac{\left\{ \begin{aligned} &P_L [\cos(\delta_L + \alpha) - \sin(\delta_L + \alpha) \tan \phi'_{mob}] \\ &+ [c'_{mob} b \sec \alpha + W \sin \alpha + (W \cos \alpha - U_{avg}) \tan \phi'_{mob}] \\ &- (H_R - H_L) [\cos \alpha - \sin \alpha \tan \phi'_{mob}] \\ &+ U_\beta [\cos(\beta - \alpha) \tan \phi'_{mob} - \sin(\beta - \alpha)] \\ &+ Q [\cos(\zeta - \alpha) \tan \phi'_{mob} - \sin(\zeta - \alpha)] \end{aligned} \right\}}{\cos(\delta_R + \alpha) - \sin(\delta_R + \alpha) \tan \phi'_{mob}} \quad (5.20)$$

with

$$h_2 = \frac{\left\{ \begin{aligned} &P_L \cos(\delta_L) \left[h_1 + \frac{b}{2} [\tan(\delta_L) + \tan \alpha] \right] \\ &- U_\beta \cos \beta (h_7 \tan \beta - e_U) \\ &- Q \cos \zeta (h_8 \tan \zeta - e_Q) \\ &- (H_R h_4 - H_L h_3) \end{aligned} \right\}}{P_R \cos(\delta_R)} + \frac{b}{2} [\tan(\delta_R) + \tan \alpha] \quad (5.21)$$

and

$$N' = \left[\begin{aligned} &P_R \sin(\delta_R + \alpha) - P_L \sin(\delta_L + \alpha) \\ &+ H_R \sin \alpha - H_L \sin \alpha + W \cos \alpha \\ &+ U_\beta \cos(\beta - \alpha) + Q \cos(\zeta - \alpha) - U_{avg} \end{aligned} \right] \quad (5.22)$$

Note that the subscript “mob-Passive” has been shortened to “mob” in these three equations. The LHS and RHS interslice forces angle of friction are varied with position among the soil wedges according to a relationship like that shown in Figure 5-4(b).

In a total stress analysis, there would not be uplift pressures acting normal to the subwedge base (i.e., no U_{avg} force) nor would there be internal water pressures included within the soil regime to act as boundary water forces normal to the vertical faces of the subwedge. All active mobilized cohesion as well as soil friction and interface friction angles would be expressed as

total stress strength parameters: c_{mob} , ϕ_{mob} , and δ_{mob} in these three equations. In Equation 5.22, N replaces N' .

5.2.2.2 Solution of the generalized limit equilibrium method relationships for the Figure 5-1 P_L force for each Figure 5-2 RHS mobilized passive soil subwedge interslice face for n passive soil subwedges

The solution of the generalized limit equilibrium method relationships for the Figure 5-1 P_L force for each Figure 5-5 RHS passive soil subwedge interslice face for n passive soil subwedges follows the procedure described in subsection 5.2.3.4. It consists of solving a series of recursive relationships for the calculated P_{R-1} ($=P_L$ in Figure 5-1) force acting on passive soil wedge 1 and is equal to the mobilized P_p value that the resisting wedge imposes upon the LHS structural wedge interface. The solution starts with the subwedge number n that is farthest from the resisting wedge to structural wedge interface. Note that for each mobilized passive subwedge i of the resisting wedge, the RHS P_{R-i} force is the dependent variable in the equation. This contrasts with the case of the subsection 5.2.3.4 active subwedges of the driving wedge in which P_{L-i} is the unknown, LHS force in each solution step for each active subwedge i .

5.3 Coefficient method procedure for computing the active and passive earth pressure forces acting on the structural wedge

This subsection describes the *coefficient method* procedure for computing the P_L and P_R forces acting on the Figure 5-1 structural wedge. This methodology is an alternative to using the *generalized limit equilibrium method*, recursive relationships for n mobilized active soil subwedges and n mobilized passive subwedges to solve for the values for the Figure 5-1 P_R and P_L force values.

5.3.1 Introduction

Earth pressure coefficients, which are a function of Mohr-Coulomb shear strength parameters for the retained soils, is an approach that may be used to compute the distribution of earth pressure with depth along each of the imaginary vertical sections made through the Figure 5-1 structural wedge heel and toe. Recall that the location of these two imaginary sections correspond to the structural to driving wedges interface and the structural to resisting wedges interface. The integration of the earth pressure distribution with depth along an imaginary section made at the structural to

driving wedges interface results in the computation of the resultant earth force P_R . A similar integration of the earth pressure distribution with depth along an imaginary section made at the structural to resisting wedges interface results in the computation of the resultant earth force P_L . Resultant water pressure distributions (and their resultant forces) acting along these two imaginary sections are derived from the pool and internal pore water pressure distributions acting normal to each of the imaginary sections. In a total stress analysis, internal pore water pressures are not considered and hence, not included as a boundary water pressure.

Two different methods may be used to compute the magnitude of the earth pressure forces acting on the Figure 5-1 structural wedge: total or effective stress analysis. The earth pressure distribution relationships and earth pressure coefficient relationships will be described subsequently and within separate subsections. They will further be distinguished by mobilized active or passive earth pressure relationships. The former relationships are used for the Figure 5-1 structural to driving wedges interface while the latter are used for the structural to resisting wedges interface.

5.3.2 Total stress – mobilized active earth pressure

For either a total stress or effective stress analysis, the computation of the horizontal earth pressure acting normal to the structural to driving wedges interface starts with a calculation of total vertical stress at a given elevation, designated σ_v . For each soil region within the driving wedge that is designated with a total stress material property, the mobilized active earth pressure is computed at a given elevation node using the relationship

$$\sigma_{mob-Active} = K_{mob-Active} \cdot \sigma_v - 2 \cdot c_{mob-Active} \cdot \sqrt{K_{mob-Active}} \quad (5.23)$$

According to Jumikis (1984), consideration of stress condition in soil shows that cohesion of a cohesive-frictional $\{c, \phi\}$ soil does not affect the position of the rupture surface. Because of this, Jumikis concludes that the earth pressure of a cohesive-frictional $\{c, \phi\}$ soil can be approximately determined by the method used for cohesionless (i.e., noncohesive) soil as described in his book (on page 380). Jumikis extended this procedure to $\{c, \phi\}$ soils for Coulomb active and passive earth pressure-based calculations. By this Jumikis approximation as well as by a similar procedure presented in EM-1110-2-2502 (HQUSACE 1989) of Coulomb wedge results, Coulomb active and passive earth pressure coefficients of a $\{c, \phi\}$ soil (computed using the relationships given in subsections 5.3.2 and

5.3.3) may be used in Equations 5.23 and 5.25 to compute the active and passive soil pressure distributions. This same logic holds true for the effective stress-based earth pressure relationships given by Equations 5.29 and 5.31 for mobilized active and passive earth pressures, respectively.

The mobilized active earth pressure (by Equation 5.23) acts at an angle of $\delta_{mob-Active}$ from horizontal. The horizontal component of $\sigma_{mob-Active}$ given by

$$\sigma_{mob-Active-horizontal} = \sigma_{mob-Active} \cdot \cos(\delta_{mob-Active}) \quad (5.24)$$

The value of the (mobilized) active earth pressure coefficient $K_{mob-Active}$ is computed using one of two relationships given in a subsequent subsection. The computation of the total vertical stress σ_v is made at each elevation that the mobilized active stress is desired to construct a distribution of earth pressure with depth along the imaginary section of interest. The mobilized cohesion $c_{mob-Active}$ and mobilized interface friction $\delta_{mob-Active}$ are given by Equations 5.5 and 5.9, respectively. Because each soil region may be assigned different strength parameters, there can be a discontinuity in the active earth pressure at each interface between adjoining soil regions. Thus, the need for maintaining an upper and a lower earth pressure computation per elevation point node to accommodate this possibility.

5.3.3 Total stress – mobilized passive earth pressure

For each soil region within the resisting wedge that is designated with a total stress material property, the mobilized passive earth pressure is computed at elevation node using the relationship

$$\sigma_{mob-Passive} = K_{mob-Passive} \cdot \sigma_v + 2 \cdot c_{mob-Passive} \cdot \sqrt{K_{mob-Passive}} \quad (5.25)$$

This mobilized passive earth pressure acts at an angle of $\delta_{mob-Passive}$ from horizontal. The horizontal component of $\sigma_{mob-Passive}$ given by

$$\sigma_{mob-Passive-horizontal} = \sigma_{mob-Passive} \cdot \cos(\delta_{mob-Passive}) \quad (5.26)$$

The value of the passive earth pressure coefficient $K_{mob-Passive}$ is computed using one of three relationships given in a subsequent subsection. The computation of the total vertical stress σ_v is made to construct a distribution of earth pressure with depth along the imaginary section of interest. The mobilized cohesion $c_{mob-Passive}$ and mobilized interface friction $\delta_{mob-Passive}$ are given by Equations 5.18 and 5.10, respectively.

Because each soil region may be assigned different strength parameters, there can be a discontinuity in the earth pressure at each interface between adjoining soil regions. Thus, the need for maintaining an upper and a lower earth pressure computation per elevation point node to accommodate this possibility.

5.3.4 Effective stress

Within an effective stress-designated soil region, the total vertical stress σ_v acting in the soil adjacent to each elevation node is combined with the soil regions pore water pressure u (if any) to compute the vertical effective stress σ'_v . Terzaghi's principle of effective stress is expressed as

$$\sigma_v = \sigma'_v + u \quad (5.27)$$

Rearranging terms,

$$\sigma'_v = \sigma_v - u \quad (5.28)$$

The effective vertical stress is then used to compute the mobilized active earth pressure acting on each of the elevation nodes used to develop the distribution of earth pressures with depth along the imaginary section.

5.3.5 Effective stress – mobilized active earth pressure

For each soil region within the driving wedge that is designated with an effective stress material property, the mobilized effective active earth pressure is computed at each node using the relationship

$$\sigma'_{mob-Active} = K_{mob-Active} \cdot \sigma'_v - 2 \cdot c'_{mob-Active} \cdot \sqrt{K_{mob-Active}} \quad (5.29)$$

This mobilized effective active earth pressure (by Equation 5.29) acts at an angle of $\delta'_{mob-Active}$ from horizontal. The horizontal component of $\sigma'_{mob-Active}$ given by

$$\sigma'_{mob-Active-horizontal} = \sigma'_{mob-Active} \cdot \cos(\delta'_{mob-Active}) \quad (5.30)$$

The value of the active earth pressure coefficient $K_{mob-Active}$ is computed using one of two relationships given in a subsequent subsection. The mobilized effective cohesion $c'_{mob-Active}$ and mobilized effective interface friction $\delta'_{mob-Active}$ are given by Equations 5.2 and 5.7, respectively. In an effective stress-based analysis, in addition to the mobilized active earth

pressure, the pore-water pressure also acts normal to the node on structural to driving wedges imaginary section. Last, there can be a discontinuity in the active earth pressure at each soil region because each soil region may be assigned different strength parameters.

5.3.6 Effective stress – mobilized passive earth pressure

For each soil region within the resisting wedge that is designated with an effective stress material property, the mobilized effective passive earth pressure is computed at each elevation node using the relationship

$$\sigma'_{mob-Passive} = K_{mob-Passive} \cdot \sigma'_v + 2 \cdot c'_{mob-Passive} \cdot \sqrt{K_{mob-Passive}} \quad (5.31)$$

This mobilized effective passive earth pressure acts at an angle of $\delta'_{mob-Passive}$ from horizontal. The computation of the effective vertical stress σ'_v is made to construct a distribution of earth pressure with depth along the imaginary section of interest. The horizontal component of $\sigma'_{mob-Passive}$ given by

$$\sigma'_{mob-Passive-horizontal} = \sigma'_{mob-Passive} \cdot \cos(\delta'_{mob-Passive}) \quad (5.32)$$

The value of the passive earth pressure coefficient $K_{mob-Passive}$ is computed using one of three relationships given in a subsequent subsection. The mobilized effective cohesion $c'_{mob-Passive}$ and mobilized effective interface friction $\delta'_{mob-Passive}$ are given by Equations 5.15 and 5.8, respectively. In an effective stress-based analysis, in addition to the mobilized passive earth pressure, the pore water pressure also acts normal to the node on the sheet-pile wall. Last, there can be a discontinuity in the passive earth pressure at each soil region because each soil region may be assigned different strength parameters.

5.4 Earth pressure coefficient methods used for computing earth pressures

Earth pressure coefficient methods may be used to compute the distribution of earth pressure with depth along each of the imaginary vertical sections made through the Figure 5-1 structural wedge heel and toe, as described in subsection 5.1. In general, there are two methods to compute a value for the active earth pressure coefficient: Rankine or Coulomb. To compute a value for the passive earth pressure coefficient, there are three options available: Rankine, Coulomb, or a passive earth pressure coefficient computed using a logarithmic spiral based method. The three failure mechanisms are

illustrated in Figure 5-6, with active, at-rest, and passive earth pressure coefficients designated as K_A , K_o , and K_P , respectively.

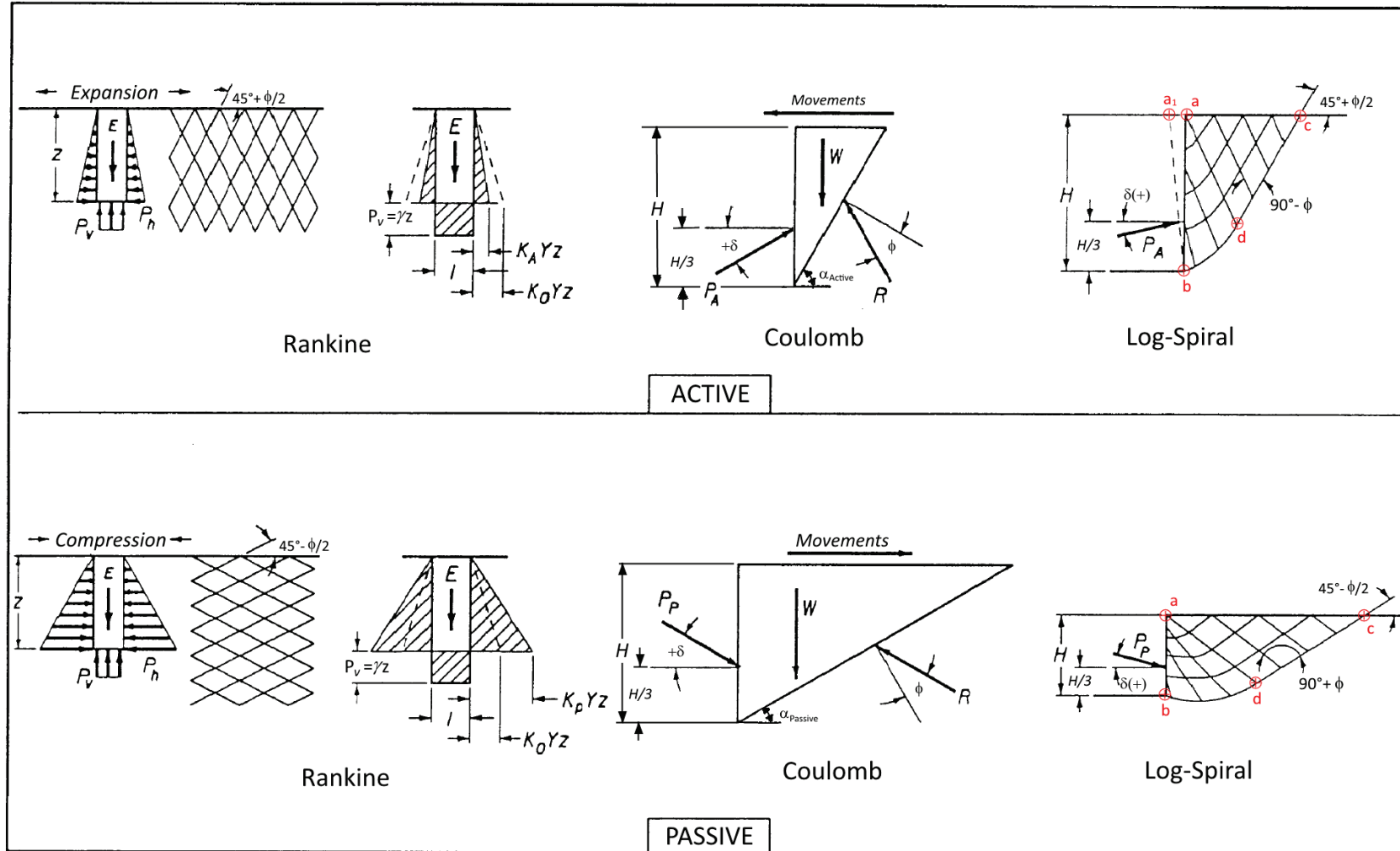
The Rankine theory of active and passive earth pressures (Rankine 1857) determines the state of stress within a semi-infinite (soil) mass that, because of expansion or compression of the (soil) mass, is transformed from an elastic state to a state of plastic equilibrium. The orientation of the linear slip lines within the (soil mass) are also determined in the analysis. The shear stress at failure within the soil is defined by a Mohr-Coulomb shear strength relationship. The resulting failure surfaces within the soil mass and the corresponding Rankine active and passive earth pressures are shown in Figure 5-6 for a cohesionless soil.

The wedge theory, as developed by Coulomb (1776), looks at the equilibrium of forces acting upon a soil wedge without regard to the state of stress within the soil mass. This wedge theory assumes a linear slip plane within the backfill and the full mobilization of the shear strength of the soil along this plane. Interface friction between the wall and the backfill may be considered in the analysis.

Numerous authors have developed relationships for active and passive earth pressure coefficients based upon an assumption of a logarithmic failure surface, as illustrated in Figure 5-6. One of the most commonly used sets of coefficients was tabulated by Caquot and Kerisel (1948). Representative K_{Active} and $K_{passive}$ values from that effort are illustrated in Table 5-1 and discussed in subsection 5.4.5. Department of Navy NAVFAC developed nomography from the Caquot and Kerisel efforts and are provided in the NAVFAC DM 7.2 (1982) and reproduced in Ebeling and Morrison (1992).

Rankine's theory, Coulomb's wedge theory, and the logarithmic spiral procedure result in similar values for active and passive thrust when the interface friction between the wall and the backfill is equal to zero. For interface friction angles greater than zero, the wedge method and the logarithmic spiral procedure result in nearly the same values for active thrust. The logarithmic spiral procedure results in accurate values for passive thrust for all values of interface friction between the wall and the backfill. The accuracy of the passive thrust values computed using the wedge method diminishes with increasing values of interface friction because the boundary of the failure block becomes increasingly curved.

Figure 5-6. Three earth pressure theories for computing active and passive earth pressures (after Terzaghi and Peck 1967).



Assuming a horizontal ground surface, Rankine earth pressures always act normal to the imaginary section. Consequently, δ in a total stress soil or δ' in an effective stress soil is set equal to zero degrees for the active and passive earth pressure relationships of Equations 5.24, 5.26, 5.30, and 5.32.

5.4.1 Rankine's active earth pressure coefficients

Total stress: For a horizontal ground surface, Rankine's active earth pressure coefficient of a total stress designated soil region, $K_{mob-Active}$ is

$$K_{mob-Active} = \tan^2 \left(45^\circ - \frac{\phi_{mob-Active}}{2} \right) \quad (5.33)$$

where $\phi_{mob-Active}$ is the mobilized angle of internal friction of the soil. The value for $\phi_{mob-Active}$ is given by Equation 5.6.

Rankine's earth pressure theory for a $\{c, \phi\}$ soil (level ground) assumes the earth pressure acts normal to the structural to driving wedges imaginary section. When the engineer selects $K_{mob-Active}$ to be calculated by Rankine earth pressure theory, the interface friction δ is set equal to zero in Equation 5.9, and thus Equation 5.23 describes the earth pressure normal to the imaginary section.

For homogeneous frictional soils (i.e., $c = 0$) with a horizontal ground surface, Rankine's theory gives the orientation of the planar slip plane, as measured from horizontal, to be

$$\alpha_{mob-Active} = \left(45^\circ + \frac{\phi_{mob-Active}}{2} \right) \quad (5.34)$$

Ebeling and Morrison (1992) report that for a frictional soil with a ground surface inclined at angle θ from horizontal, $K_{mob-Active}$ is

$$K_{mob-Active} = [\cos\theta] * \frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi_{mob-Active}}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi_{mob-Active}}} \quad (5.35)$$

With the limitation that θ is less than or equal to $\phi_{mob-Active}$. Equation 5.23 for $\sigma_{mob-Active}$ still applies but is inclined at an angle θ from horizontal.

Effective stress: For an effective stress soil region below a horizontal ground surface, Rankine's mobilized effective active earth pressure coefficient $K_{mob-Active}$ is

$$K_{mob-Active} = \tan^2 \left(45^\circ - \frac{\phi'_{mob-Active}}{2} \right) \quad (5.36)$$

where $\phi'_{mob-Active}$ is the mobilized effective angle of internal friction of the soil. The value for $\phi'_{mob-Active}$ is given by Equation 5.3.

Rankine's earth pressure theory for a $\{c', \phi'\}$ soil assumes the earth pressure acts normal to the structural to driving wedges imaginary section. Therefore, when the engineer selects $K_{mob-Active}$ to be calculated by Rankine earth pressure theory, the interface friction δ' is set equal to zero in Equation 5.7 and thus, Equation 5.29 describes the earth pressure normal to the imaginary section.

For homogeneous frictional soils (i.e., $c' = 0$) with a horizontal ground surface, Rankine's theory gives the orientation of the planar slip plane, as measured from horizontal, to be

$$\alpha_{mob-Active} = \left(45^\circ + \frac{\phi'_{mob-Active}}{2} \right) \quad (5.37)$$

Ebeling and Morrison (1992) report that for a frictional soil with a ground surface inclined at angle θ from horizontal, $K_{mob-Active}$ is

$$K_{mob-Active} = [\cos\theta] * \frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi'_{mob-Active}}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi'_{mob-Active}}} \quad (5.38)$$

With the limitation that θ is less than or equal to $\phi'_{mob-Active}$. Equation 5.29 for $\sigma'_{mob-Active}$ still applies but is inclined at an angle θ from horizontal.

5.4.2 Rankine's passive earth pressure coefficients

Total stress: For a horizontal ground surface, Rankine's passive earth pressure coefficient of a total stress designated soil region, $K_{mob-Passive}$ is

$$K_{mob-Passive} = \tan^2 \left(45^\circ - \frac{\phi_{mob-Passive}}{2} \right) \quad (5.39)$$

where $\phi_{mob-Passive}$ is the mobilized angle of internal friction of the soil. The value for $\phi_{mob-Passive}$ is given by Equation 5.19.

Rankine's earth pressure theory for a $\{c, \phi\}$ soil assumes the earth pressure acts normal to the structural to resisting wedges imaginary section. When the engineer selects $K_{mob-Passive}$ to be calculated by Rankine earth pressure theory, the interface friction δ is set equal to zero in Equation 5.10 and thus, Equation 5.25 describes the earth pressure normal to the imaginary section.

For homogeneous frictional soils (i.e., $c = 0$) with a horizontal ground surface, Rankine's theory gives the orientation of a planar slip plane, as measured from horizontal, to be

$$\alpha_{mob-Passive} = \left(45^\circ - \frac{\phi_{mob-Passive}}{2} \right) \quad (5.40)$$

Ebeling and Morrison (1992) report that for a frictional soil with a ground surface inclined at angle θ from horizontal, $K_{mob-Passive}$ is

$$K_{mob-Passive} = [\cos\theta] * \frac{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi_{mob-Passive}}}{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi_{mob-Passive}}} \quad (5.41)$$

With the limitation that θ is less than or equal to $\phi_{mob-Passive}$. Equation 5.25 for $\sigma_{mob-Passive}$ still applies but is inclined at an angle θ from horizontal.

Effective stress: For an effective stress soil region below a horizontal ground surface, Rankine's mobilized effective passive earth pressure coefficient $K_{mob-Active}$ is

$$K_{mob-Passive} = \tan^2 \left(45^\circ - \frac{\phi'_{mob-Passive}}{2} \right) \quad (5.42)$$

where $\phi'_{mob-Passive}$ is the mobilized effective angle of internal friction of the soil. The value for $\phi'_{mob-Passive}$ is given by Equation 5.16.

Rankine's earth pressure theory for a $\{c', \phi'\}$ soil assumes the earth pressure acts normal to the structure to resisting wedges imaginary section. When the engineer selects $K_{mob-Passive}$ to be calculated by Rankine earth pressure theory, the interface friction δ' is set equal to zero in Equation 5.8, and thus Equation 5.31 describes the earth pressure normal to the imaginary section.

For homogeneous frictional soils (i.e., $c' = 0$) with a horizontal ground surface, Rankine's theory gives the orientation of a planar slip plane, as measured from horizontal, to be

$$\alpha_{mob-Passive} = \left(45^\circ - \frac{\phi'_{mob-Passive}}{2}\right) \quad (5.43)$$

Ebeling and Morrison (1992) report that for a frictional soil with a ground surface inclined at angle θ from horizontal, $K_{mob-Passive}$ is

$$K_{mob-Passive} = [\cos\theta] * \frac{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi'_{mob-Passive}}}{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi'_{mob-Passive}}} \quad (5.44)$$

with the limitation that θ is less than or equal to $\phi'_{mob-Passive}$. Equation 5.31 for $\sigma'_{mob-Passive}$ still applies but is inclined at an angle θ from horizontal.

5.4.3 Coulomb's active earth pressure coefficients

Total stress: For a ground surface inclined at a constant angle θ from horizontal, Coulomb's active earth pressure coefficient (Ebeling and Morrison 1992) for a total stress designated soil region, $K_{mob-Active}$ is

$$K_{mob-Active} = \frac{\cos^2(\phi_{mob-Active})}{\cos(\delta_{mob-Active}) \left[1 + \sqrt{\frac{\sin(\phi_{mob-Active} + \delta_{mob-Active}) * \sin(\phi_{mob-Active} - \theta)}{\cos(\delta_{mob-Active}) * \cos(\theta)}} \right]^2} \quad (5.45)$$

where $\phi_{mob-Active}$ is the mobilized angle of internal friction of the soil and $\delta_{mob-Active}$ is the mobilized interface angle of friction along the structure to driving wedges imaginary section. The values for $\phi_{mob-Active}$ and $\delta_{mob-Active}$ are given by Equations 5.6 and 5.9, respectively. For level backfills θ is set equal to 0 degrees.

The orientation of a planar slip plane (Ebeling and Morrison 1992), as measured from horizontal, is

$$\alpha_{mob-Active} = \phi_{mob-Active} + \tan^{-1} \left[\frac{-\tan(\phi_{mob-Active} - \theta) + c_{T1}}{c_{T2}} \right] \quad (5.46)$$

where

$$c_{T1} = \sqrt{[\tan(\phi_{mob-Active} - \theta)] * [\tan(\phi_{mob-Active} - \theta) + \cot(\phi_{mob-Active})] * term_{T1}} \quad (5.47)$$

with

$$term_{T1} = [1 + \tan(\delta_{mob-Active}) * \cot(\phi_{mob-Active})] \quad (5.48)$$

and

$$c_{T2} = 1 + \{[\tan(\delta_{mob-Active})] * [\tan(\phi_{mob-Active} - \theta) + \cot(\phi_{mob-Active})]\} \quad (5.49)$$

To compute a length of slip plane for the driving wedge, the value used for $\alpha_{mob-Active}$ must be greater than θ by a few degrees. This minimum lower bound angle is also required when computing the Figure 5-8 height h_3 used in the Equation 5.69 computation of the equivalent unit weight for non-level ground.

Effective stress: For a ground surface inclined at a constant angle θ from horizontal, Coulomb's active earth pressure coefficient (Ebeling and Morrison 1992) for an effective stress designated soil region, $K_{mob-Active}$ is

$$K_{mob-Active} = \frac{\cos^2(\phi'_{mob-Active})}{\cos(\delta'_{mob-Active}) \left[1 + \sqrt{\frac{\sin(\phi'_{mob-Active} + \delta'_{mob-Active}) * \sin(\phi'_{mob-Active} - \theta)}{\cos(\delta'_{mob-Active}) * \cos(\theta)}} \right]^2} \quad (5.50)$$

where $\phi'_{mob-Active}$ is the mobilized effective angle of internal friction of the soil and $\delta'_{mob-Active}$ is the mobilized soil to sheet-pile interface angle of friction. The values for $\phi'_{mob-Active}$ and $\delta'_{mob-Active}$ are given by Equations 5.3 and 5.7, respectively. For level backfills θ is set equal to 0 degrees.

The orientation of a planar slip plane (Ebeling and Morrison 1992), as measured from horizontal, is

$$\alpha_{mob-Active} = \phi'_{mob-Active} + \tan^{-1} \left[\frac{-\tan(\phi'_{mob-Active} - \theta) + c_{E1}}{c_{E2}} \right] \quad (5.51)$$

where

$$c_{E1} = \sqrt{[\tan(\phi'_{mob-Active} - \theta)] * [\tan(\phi'_{mob-Active} - \theta) + \cot(\phi'_{mob-Active})] * term_{E1}} \quad (5.52)$$

with

$$term_{E1} = [1 + \tan(\delta'_{mob-Active}) * \cot(\phi'_{mob-Active})] \quad (5.53)$$

and

$$c_{E2} = 1 + \{[\tan(\delta'_{mob-Active})] * [\tan(\phi'_{mob-Active} - \theta) + \cot(\phi'_{mob-Active})]\} \quad (5.54)$$

Recall that to compute a length of slip plane for a planer active of the driving wedge, the value used for $\alpha_{mob-Active}$ must be greater than θ by a few degrees. This minimum lower bound angle is also required when computing the Figure 5-8 height h_3 used in the Equation 5.69 computation of the equivalent unit weight for non-level ground.

5.4.4 Coulomb's passive earth pressure coefficients

Total stress: For a ground surface inclined at a constant angle θ from horizontal, Coulomb's passive earth pressure coefficient (Ebeling and Morrison 1992) for a total stress designated soil region, $K_{mob-Passive}$ is

$$K_{mob-Passive} = \frac{\cos^2(\phi_{mob-Passive})}{\cos(\delta_{mob-Passive}) \left[1 - \sqrt{\frac{\sin(\phi_{mob-Passive} + \delta_{mob-Passive}) * \sin(\phi_{mob-Passive} + \theta)}{\cos(\delta_{mob-Passive}) * \cos(\theta)}} \right]^2} \quad (5.55)$$

where $\phi_{mob-Passive}$ is the mobilized angle of internal friction of the soil and $\delta_{mob-Passive}$ is the mobilized interface angle of friction along the structure to resisting wedges imaginary section. The values for $\phi_{mob-Passive}$ and $\delta_{mob-Passive}$ are given by Equations 5.19 and 5.10, respectively. For level backfills θ is set equal to 0 degrees.

The orientation of a planar slip plane (Ebeling and Morrison 1992), as measured from horizontal, is

$$\alpha_{mob-Passive} = -\phi_{mob-Passive} + \tan^{-1} \left[\frac{\tan(\phi_{mob-Passive} + \theta) + c_{T3}}{c_{T4}} \right] \quad (5.56)$$

where

$$c_{T3} = \sqrt{[\tan(\phi_{mob-Passive} + \theta)] * [\tan(\phi_{mob-Passive} + \theta) + \cot(\phi_{mob-Passive})] * term_{T2}} \quad (5.57)$$

with

$$term_{T2} = [1 + \tan(\delta_{mob-Passive}) * \cot(\phi_{mob-Passive})] \quad (5.58)$$

and

$$c_{T4} = 1 + \{[\tan(\delta_{mob-Passive})] * [\tan(\phi_{mob-Passive} + \theta) + \cot(\phi_{mob-Passive})]\} \quad (5.59)$$

The application of this Coulomb $K_{mob-Passive}$ relationship is to soils with $\delta_{mob-Passive}$ less than or equal to $\phi_{mob-Passive}/2$, as discussed in subsection 3.3.4.1 of Ebeling and Morrison (1992) or Section 4.6 of Chen and Liu (1990). This is due to the inappropriateness of a planar slip surface assumption used in defining the resultant passive earth pressure force relationship from which the Coulomb passive earth coefficient is derived. A curved slip surface is more appropriate when $\delta_{mob-Passive}$ is greater than $\phi_{mob-Passive}/2$.

Computations made using Equation 5.56 in support of an approximate length computation of the passive resisting wedge slip plane $L_{Passive}$ have restricted the minimum value of $\alpha_{mob-Passive}$ to be equal to 5 degrees for this level ground problem of a fully submerged resisting wedge. In this case the computation of the equivalent unit weight by Equation 5.69 for level ground is not influenced by $\alpha_{mob-Passive}$, as is the case for the driving wedge discussed above.

Effective stress: For a ground surface inclined at a constant angle θ from horizontal, Coulomb's passive earth pressure coefficient (Ebeling and Morrison 1992) for an effective stress designated soil region, $K_{mob-Passive}$ is

$$K_{mob-Passive} = \frac{\cos^2(\phi'_{mob-Passive})}{\cos(\delta'_{mob-Passive}) \left[1 - \sqrt{\frac{\sin(\phi'_{mob-Passive} + \delta'_{mob-Passive}) * \sin(\phi'_{mob-Passive} + \theta)}{\cos(\delta'_{mob-Passive}) * \cos(\theta)}} \right]^2} \quad (5.60)$$

where $\phi'_{mob-Passive}$ is the mobilized effective angle of internal friction of the soil and $\delta'_{mob-Passive}$ is the mobilized effective interface angle of friction along the structure to resisting wedges imaginary section. The values for $\phi'_{mob-Passive}$ and $\delta'_{mob-Passive}$ are given by Equations 5.16 and 5.8, respectively. For level backfills, θ is set equal to 0 degrees.

The orientation of a planar slip plane (Ebeling and Morrison 1992), as measured from horizontal, is

$$\alpha_{mob-Passive} = -\phi'_{mob-Passive} + \tan^{-1} \left[\frac{\tan(\phi'_{mob-Passive} + \theta) + c_{E3}}{c_{E4}} \right] \quad (5.61)$$

where

$$c_{E3} = \sqrt{[\tan(\phi'_{mob-Passive} + \theta)] * [\tan(\phi'_{mob-Passive} + \theta) + \cot(\phi'_{mob-Passive})] * term_{E2}} \quad (5.62)$$

with

$$term_{E2} = [1 + \tan(\delta'_{mob-Passive}) * \cot(\phi'_{mob-Passive})] \quad (5.63)$$

and

$$c_{E4} = 1 + \{[\tan(\delta'_{mob-Passive})] * [\tan(\phi'_{mob-Passive} + \theta) + \cot(\phi'_{mob-Passive})]\} \quad (5.64)$$

The application of this Coulomb $K_{mob-Passive}$ relationship is to soils with $\delta'_{mob-Passive}$ less than or equal to $\phi'_{mob-Passive}/2$, as discussed in subsection 3.3.4.1 of Ebeling and Morrison (1992) or Section 4.6 of Chen and Liu (1990). This is due to the inappropriateness of a planar slip surface assumption used in defining the resultant passive earth pressure force relationship from which the Coulomb passive earth coefficient is derived. A curved slip surface is more appropriate when $\delta'_{mob-Passive}$ is greater than $\phi'_{mob-Passive}/2$.

Computations made using Equation 5.61 in support of an approximate length computation of the passive resisting wedge slip plane $L_{Passive}$ have restricted the minimum value of $\alpha_{mob-Passive}$ to be equal to 5 degrees for this level ground problem of a fully submerged resisting wedge. In this case the computation of the equivalent unit weight by Equation 5.69 for level ground is not influenced by $\alpha_{mob-Passive}$, as is the case for the driving wedge discussed above.

5.4.5 Logarithmic-spiral method for determination of passive earth pressure coefficients

The logarithmic spiral earth pressure theory (sometimes referred to as the logarithmic spiral method) is described by Terzaghi (1943), Chen and Liu (1990), and others. A significant portion of the slip surface in a retained soil regime is approximated by a logarithmic spiral. Under this condition and using different sets of engineering formulations, values for passive earth coefficients were produced as a function of internal friction values for cohesionless soils (e.g., tabulations given in Caquot and Kerisel [1948]; Kerisel and Absi [1990]; Chen and Liu [1990]; Sokolovskii [1965]; and in

the Figure 5 chart in Chapter 3 of NAVFAC DM 7.2 [Department of the Navy 1982]). The Figure 5 chart given in Chapter 3 in NAVFAC DM 7.2 (1982) provides values for $K_{Passive}$ for δ equal to ϕ as a nonlinear curve that is a function of ϕ . Using this information, a tabulation of $K_{Passive}$ values for level ground (i.e., $\theta = 0$) was assembled (Table 1). To determine an appropriate value for $K_{Passive}$ in those cases when δ is less than ϕ , a reduction factor, R , is applied to the Table 5-1 $K_{Passive}$ for level ground value.

$$K_{Passive}(\delta) = K_{Passive} \cdot R(\delta, \phi) \quad (5.65)$$

Values for R are provided in Table 5-2 as a function of the value of ϕ and the value of the ratio δ/ϕ . For intermediate values, interpolation of the Tables 5.1 and 5.2 are appropriate.

Table 5-1. Passive earth pressure coefficient $K_{Passive}(K_p)$ values for $-\delta$ equal to ϕ and a solution based on assuming a failure surface composed of logarithmic spiral portion and a planar surface portion – level ground.

ϕ	$K_p(-\delta=\phi)$
0	1
5	1.28
10	1.64
15	2.19
20	3.01
25	4.29
30	6.42
35	10.2
40	17.5
45	33.5
50	74.3
51	90
52	110
53	130
54	160
55	204
60	782

Table 5-2. Reduction factor, R , values for various ratios of $-\delta/\phi$.

ϕ (deg)	δ/ϕ										
	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
0	1	1	1	1	1	1	1	1	1	1	1
5	1	0.996	0.995	0.989	0.981	0.973	0.9645	0.956	0.949	0.9405	0.932
10	1	0.991	0.989	0.978	0.962	0.946	0.929	0.912	0.898	0.881	0.864
15	1	0.986	0.979	0.961	0.934	0.907	0.881	0.854	0.83	0.803	0.775
20	1	0.983	0.968	0.939	0.901	0.862	0.824	0.787	0.752	0.716	0.678
25	1	0.980	0.954	0.912	0.86	0.808	0.759	0.711	0.666	0.62	0.574
30	1	0.980	0.937	0.878	0.811	0.746	0.686	0.627	0.574	0.52	0.467
35	1	0.980	0.916	0.836	0.752	0.674	0.603	0.536	0.475	0.417	0.362
40	1	0.980	0.886	0.783	0.682	0.592	0.512	0.439	0.375	0.316	0.262
45	1	0.979	0.848	0.718	0.6	0.5	0.414	0.339	0.276	0.221	0.174
50	1	0.975	0.797	0.638	0.506	0.399	0.313	0.242	0.185	0.138	0.102
55	1	0.966	0.731	0.543	0.401	0.295	0.215	0.153	0.108	0.0737	0.0492
60	1	0.948	0.647	0.434	0.29	0.193	0.127	0.0809	0.0505	0.0301	0.0178

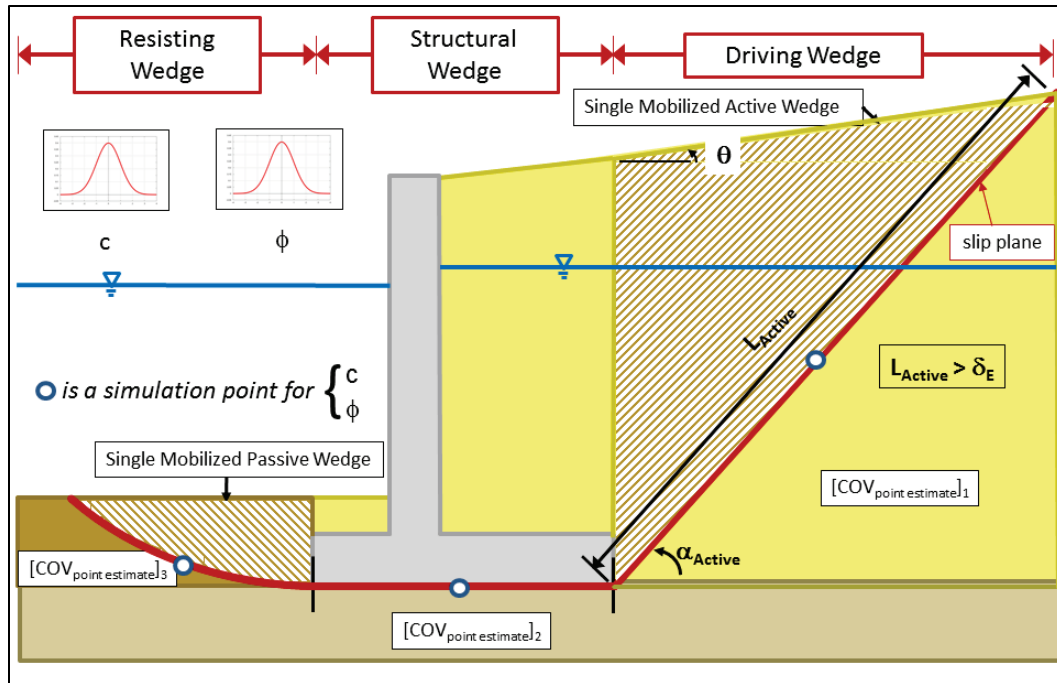
To use the Tables 5-1 and 5-2 data in total stress designated materials, the value for mobilized interface friction, $\delta_{mob-Passive}$ is substituted for δ , and the value for mobilized angle of internal friction $\phi_{mob-Passive}$ is substituted for ϕ . A similar substitution is made for effective stress designated materials. These tables of values may be used for soils designated as either total or effective stress materials. Last, due to the sign convention used to describe the passive earth pressure force based on the data contained in the Figure 5 chart in Chapter 3 of NAVFAC DM 7.2 (Department of the Navy 1982), δ is expressed as a negative value when using Tables 5-1 and 5-2.

In cases of non-level ground (i.e., $\theta > 0$ degrees) the passive earth pressure coefficient for $\delta = \phi$ would be obtained not from Table 5-1 but from the data contained in the Figure 5 chart in Chapter 3 of NAVFAC DM 7.2 (Department of the Navy 1982) or Figure 3-12 in Ebeling and Morrison (1992). Values for the correction factor R would be continue to be obtained from Table 5-2.

5.5 Single wedge solution method for the mobilized active and mobilized passive single wedge models

In a single wedge solution scheme formulation, the driving and the resisting wedges for the potential slip plane are each modeled as a single wedge. Figure 5-7 shows one mobilized active driving wedge and one of the mobilized passive resisting wedge buttressing the structural wedge. These two single wedges are crosshatched in this figure. Application of the single wedge model requires homogeneous material within the wedge in which it is being applied.

Figure 5-7. Driving-structural-resisting single wedge models of a combined geological and structural system and a potential slip-plane for sliding stability analysis of a T-Wall as per EM 1110-2-2502 (HQUSACE 1989).



Three soil material regions are designated in Figure 5-7: (1) the soil region containing the driving wedge, (2) the foundation soil region below the structural wedge, and (3) the resisting soil region. Each soil region is designated with a different color in this figure. Each of these three material regions will have its own equivalent scale of fluctuation δ_E value (Chapter 4). Recall from Appendix E that in an anisotropic scale of fluctuation soil layer, the equivalent scale of fluctuation δ_E depends upon the orientation of the slip plane relative to the values for the vertical and horizontal scales of fluctuation of that soil material region.

A sliding stability assessment with a wedge solution method that accounts for the reduction of correlation due to distances greater than the Scale of Fluctuation within the driving wedge combined with a simulation methodology to transfer uncertainty in the Mohr-Coulomb soil shear strength parameters to the Figure 5-1 P_L and P_R resultant forces acting on the structural wedge. This procedure minimizes the number of wedges in the solution by using the Variance Reduction Factor with a value less than 1.0. Its application requires homogeneous material within the driving wedge, and it can also be applied to the foundation soil below the structural wedge as well as to the resisting wedge.

The following subsections describe the single wedge based models used for computing the driving to structural wedge interslice force P_R (Figure 5-1) for the homogeneous driving wedge and the resisting to structural wedge interslice force P_L for the homogeneous resisting wedge, given a user defined potential slip surface.

5.5.1 Single wedge model for a homogeneous, mobilized active subwedge

Driving soil wedge: Recall from Chapter 4 that the value for the equivalent scale of fluctuation δ_E for Figure 5-7 soil material region 1 of the driving wedge is dictated by the orientation of the planar potential slip plane (in red). The mobilized active soil wedge, labeled as the driving wedge in Figure 5-7, is shown oriented at the angle α_{Active} from horizontal. This soil wedge corresponds to the mobilized active soil stress state. The length of its slip plane is designated L_{Active} in this figure.

5.5.1.1 Total stress analysis

In a total stress analysis, the value of angle α_{Active} depends upon the angle of internal friction, ϕ , for the retained soil and the value for structural to driving wedge interslice friction, δ_{Active} . (In an effective stress analysis, α_{Active} depends upon the value of ϕ' and δ'_{Active} .) For frictional backfills with δ_{Active} equal to zero, the value for α_{Active} is given by Equation 5.34 in a total stress analysis and by Equation 5.37 in an effective stress analysis. Should the value of δ_{Active} be greater than zero, the value for α_{Active} is given by Equation 5.46 in a total stress analysis and by Equation 5.51 in an effective stress analysis. All four α_{Active} relationships cited are for a planar slip surface within a homogeneous retained soil. Knowing the value for α_{Active} allows for the computation of L_{Active} of the Figure 5-7 driving wedge as well as the equivalent scale of fluctuation δ_E (Appendix E) for the planar slip plane oriented at angle α_{Active} .

By relationships discussed in Chapter 3 (e.g., Equation 3.1) and Appendix D, the $COV_{spatial}$ for soil shear strength parameters is the product of the square of the variance reduction factor, $\Gamma(L)^2$, times the square of the soil shear strength parameter $COV_{point\ estimate}$ plus the square of the measurement error $COV_{measurement\ error}$ for the soil test,

$$[COV]_{spatial}^2 = \Gamma(L_{Active})^2 * [COV_{point\ estimate}]^2 + [COV_{measurement\ error}]^2 \quad (5.66)$$

Equations 3.2 and 3.3 provide for the definition of the Variance Reduction Factor, and Appendix D provides a data base of $COV_{point\ estimate}$ values of shear strength parameters for different soil types.

For a homogeneous, cohesionless retained soil, the P_A thrust force that the driving wedge imposes on the structural wedge is designated as P_R in Figure 5-1 and computed using

$$P_A = K_{Active} * \frac{1}{2} * \gamma_e * H^2 \quad (5.67)$$

with

K_{Active} = the active earth pressure coefficient by the Rankine or Coulomb relationships given in subsections 5.4.1 or 5.4.3, respectively

γ_e = the equivalent unit weight

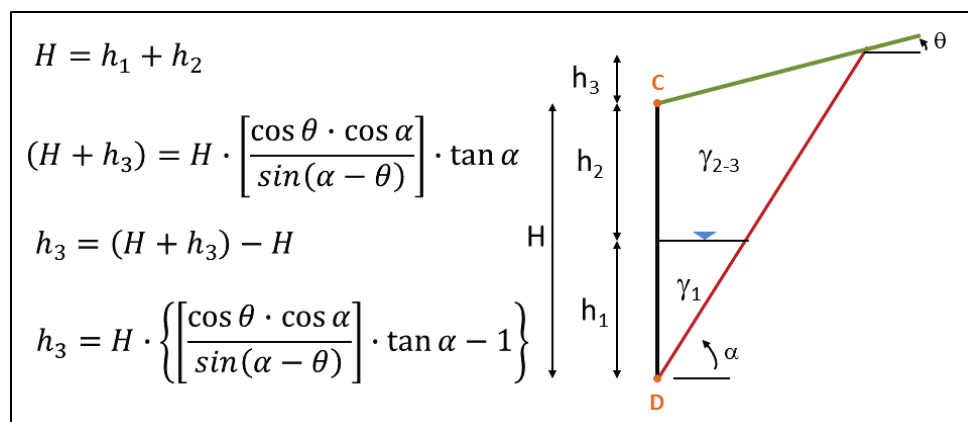
H = the vertical height of the structural to driving wedge interface as measured from the Heel of the T-Wall (section C-D in Figure 5-1).

An equivalent unit weight γ_e is used in Equation 5.67 to accommodate a water table within the retained soil of the driving wedge with a planar slip surface. In a total stress analysis, a moist soil unit weight is assigned above the water table, and a saturated unit weight is assigned to the soil below the water table. Appendix F summarizes the derivation of the relationship for the equivalent unit weight γ_e :

$$\gamma_e = \gamma_1 * \left[\frac{h_1}{(h_1+h_2)+h_3} * \frac{h_1}{(h_1+h_2)} \right] + \gamma_{2-3} * \left[1 - \frac{h_1}{(h_1+h_2)+h_3} * \frac{h_1}{(h_1+h_2)} \right] \quad (5.68)$$

for the Figure 5-8 driving wedge possessing a planar slip plane oriented at an angle α from horizontal, with a saturated unit weight γ_{sat} assigned to γ_1 for soil area 1 below the water table and a moist unit weight γ_{moist} assigned to γ_{2-3} for soil areas 2 and 3 above the water table. The ground surface is oriented at an angle of θ from horizontal.

Figure 5-8 Geometrical relationships used to define the height terms in the equivalent unit weight relationship for a partially submerged soil wedge delineated by a planar slip surface



When the backfill is horizontal, the height h_3 is set equal to zero and the angle of θ is zero. In this situation, the Equation 5.68 equivalent unit weight relationship simplifies to

$$\gamma_e = \gamma_1 * \left[\left(\frac{h_1}{(h_1 + h_2)} \right)^2 \right] + \gamma_{2-3} * \left[1 - \left(\frac{h_1}{(h_1 + h_2)} \right)^2 \right] \quad (5.69)$$

A saturated unit weight γ_{sat} is assigned to γ_1 for the Figure 5-8 soil area 1 below the water table and a moist unit weight γ_{moist} is assigned to γ_{2-3} for soil areas 2 and 3 above the water table in a total stress analysis. This relationship agrees with the Ebeling and Morrison (1992) Figure 4-13 relationship for a driving wedge with a planar slip surface and level ground. In this 1992 technical report, the term *effective unit weight* was used to describe γ_e , the LHS of Equation 5.69. Its name has been updated in this report to *equivalent unit weight* to better reflect its usage in Equation 5.67 for either effective or total stress analysis.

5.5.1.2 Effective stress analysis

Hydrostatic: In an effective stress analysis with hydrostatic pore water pressures below the water table within the driving wedge, the equivalent unit weight γ_e is computed using a moist unit weight γ_{moist} assigned to γ_{2-3} for soil areas 2 and 3 above the water table and with a buoyant unit weight γ_{buy} assigned to γ_1 for soil area 1 below the water table in Equation 5.68. The buoyant unit weight γ_{buy} is defined as

$$\gamma_{buy} = \gamma_{sat} - \gamma_{water} \quad (5.70)$$

with γ_{water} designating the unit weight of water.

Seepage: When there is a differential head of water existing from one side of the structural wedge to the other, there will be flow of water within the soil regime. For the differential head example shown in Figure 5-1, water will flow from the high head side of the driving wedge side, through the foundation and up into the low head side of the resisting wedge. In an effective stress analysis and assuming steady-state flow in this example, the buoyant unit weight γ_{buy} is not assigned to γ_1 for the Figure 5-8 soil area 1 below the water table. To account for the effect of steady state seepage acting downward on the saturated volume of soil contained within the driving wedge, γ_1 for soil area 1 is

$$\gamma_1 = \gamma_{buy} + \gamma_{water} * i_{y-DW} \quad (5.71)$$

with i_{y-DW} being the downward, vertical gradient of steady-state seepage through the saturated soil volume of the driving wedge, and for the saturated volume of soil contained within the resisting wedge, γ_1 for its soil area 1 below the water table is computed using

$$\gamma_1 = \gamma_{buy} - \gamma_{water} * i_{y-RW} \quad (5.72)$$

with i_{y-RW} being the upward, vertical gradient of steady-state seepage through the saturated soil volume of the resisting wedge. Note the sign change on the gradient terms between Equations 5.71 and 5.72, reflecting the influence of the downward and upward seepage, respectively. A steady-state seepage analysis may be conducted to determine these seepage gradients. Alternatively, Appendix G outlines a rapid-calculation, approximate procedure named the Line of Seepage (a.k.a., the Line of Creep) for computing the two needed vertical seepage gradients within the driving and resisting soil wedges, respectively.

If the resisting soil wedge is on the high head side of the structural wedge and the driving wedge is on the low head side, to account for the effect of steady state seepage acting upward on the saturated volume of soil contained within the driving wedge, γ_1 for the submerged soil area 1 is given by

$$\gamma_1 = \gamma_{buy} - \gamma_{water} * i_{y-DW} \quad (5.73)$$

with i_{y-DW} being the upward, vertical gradient of steady-state seepage through the saturated soil volume of the driving wedge, and for the

saturated volume of soil contained within the resisting wedge, γ_1 for the submerged soil area 1 is computed using

$$\gamma_1 = \gamma_{buoy} + \gamma_{water} * i_{y-RW} \quad (5.74)$$

with i_{y-RW} being the downward, vertical gradient of steady-state seepage through the saturated soil volume of the resisting wedge. Again, there is a sign change on the gradient terms between Equations 5.73 and 5.74, reflecting the influence of the upward seepage in the driving wedge and the downward seepage within the resisting wedge.

For a homogeneous, cohesionless retained soil, the P_A thrust force that the driving wedge imposes on the structural wedge is designated as P_R in Figure 5-1 and computed using Equation 5.67. Note that in an effective stress analysis, boundary water pressures are also applied to the structural to driving wedges interface in an effective stress analysis. These boundary water pressures will come from either a steady-state seepage analysis or from the approximate, line of seepage analysis described in Appendix G.

5.5.2 Single wedge model for a homogeneous, mobilized passive subwedge

Resisting soil wedge: The value for the equivalent scale of fluctuation δ_E (Chapter 4) for the Figure 5-7 soil material region 3 of the resisting wedge is dictated by the orientation of the planar potential slip plane (in red). The slip plane of this mobilized passive soil wedge, labeled as the resisting wedge in Figure 5-7, is shown to be nonlinear, more of a parabolic or a logarithmic spiral based orientation in this figure. For the purposes of the simplified computational procedure to determine an approximate length of slip plane, as discussed in subsection 5.5.1, a passive soil wedge with a planar slip plane may be fitted through the Figure 5-7 mass such that the same soil volume is being contained approximates that contained within the parabolic slip plane. The length of this approximate-fit, planar slip plane oriented at the angle $\alpha_{Passive}$ from horizontal (not shown) with a length of slip plane designated $L_{Passive}$. This approximate-fit planar soil wedge thrust is intended to correspond to the mobilized passive soil stress state. This $L_{Passive}$ length is intended for use in Equations 3.2 and 3.3 ($L = L_{Passive}$) and as an expedient, less rigorous approximation to the Appendix E equivalent length computational procedure for a logarithmic spiral slip plane.

5.5.2.1 Total stress analysis

In a total stress analysis, the value of angle $\alpha_{Passive}$ depends upon the angle of internal friction, ϕ of the soil within the resisting wedge and the value for structural to resisting wedge interslice friction, $\delta_{Passive}$. (In an effective stress analysis, $\alpha_{Passive}$ depends upon the value of ϕ' and $\delta'_{Passive}$.) For frictional backfills with $\delta_{Passive}$ equal to zero, the value for $\alpha_{Passive}$ is given by Equation 5.40 in a total stress analysis and by Equation 5.43 in an effective stress analysis. Should the value of $\delta_{Passive}$ be greater than zero, the value for $\alpha_{Passive}$ is given by Equation 5.56 in a total stress analysis and by Equation or 5.61 in an effective stress analysis. All four $\alpha_{Passive}$ relationships cited are for a planar slip surface within a homogeneous soil. Knowing the value for $\alpha_{Passive}$ allows for the computation of $L_{Passive}$ of the planar-fit slip surface defined, resisting wedge approximation as well as the equivalent scale of fluctuation δ_E (Appendix E) for the slip plane oriented at angle $\alpha_{Passive}$. The planar slip surface approximation with its constant angle $\alpha_{Passive}$ facilitates the definition of a single equivalent scale of fluctuation δ_E value for the passive wedge.

By relationships discussed in Chapter 3 (e.g., Equation 3.1) and Appendix D, the $COV_{spatial}$ for soil shear strength parameters is the product of the square of the Variance Reduction Factor, $\Gamma(L)^2$, times the square of the soil shear strength parameter $COV_{point\ estimate}$ plus the square of the measurement error $COV_{measurement\ error}$ for the soil test,

$$[COV]_{spatial} = \Gamma(L_{Passive})^2 * [COV_{point\ estimate}]^2 + [COV_{measurement\ error}]^2 \quad (5.75)$$

Equations 3.2 and 3.3 provide for the definition of the Variance Reduction Function, and Appendix D provides a data base of $COV_{point\ estimate}$ values of shear strength parameters for different soil types.

For a homogeneous, cohesionless buttressing soil, the P_P thrust force that the resisting wedge imposes on the structural wedge is designated as P_L in Figure 5-1 and computed using

$$P_P = K_{Passive} * \frac{1}{2} * \gamma_e * H^2 \quad (5.76)$$

where:

$K_{Passive}$ = the passive earth pressure coefficient from Rankine, Coulomb or Logarithmic Spiral relationships/tabulated data given in subsections 5.4.2, 5.4.4 or 5.4.5, respectively

γ_e = the equivalent unit weight

H = the vertical height of the structural to resisting wedge interface as measured from the Toe of the T-Wall (section A-B in Figure 5-1).

Computation of the equivalent unit weight γ_e is discussed in subsequent paragraphs.

Total stress analysis: An equivalent unit weight is used in Equation 5.76 to accommodate a water table within the retained soil of the resisting wedge. The relationship was derived based on a planar slip surface approximation (Appendix F). Consequently, it will be an approximation for the parabolic slip plane of the resisting wedge shown in Figure 5-7. In a total stress analysis, a moist soil unit weight is assigned above the water table, and a saturated unit weight is assigned to the soil below the water table. The equivalent unit weight γ_e is given by

$$\gamma_e = \gamma_1 * \left[\frac{h_1}{(h_1+h_2)+h_3} * \frac{h_1}{(h_1+h_2)} \right] + \gamma_{2-3} * \left[1 - \frac{h_1}{(h_1+h_2)+h_3} * \frac{h_1}{(h_1+h_2)} \right] \quad (5.68 \text{ bis})$$

for the Figure 5-8 wedge with its planar slip plane oriented at an angle α from horizontal, with a saturated unit weight γ_{sat} assigned to γ_1 for soil area 1 below the water table and a moist unit weight γ_{moist} assigned to γ_{2-3} for soil areas 2 and 3 above the water table in a total stress analysis. The ground surface is oriented at an angle of θ from horizontal.

When the buttressing soil is horizontal, the Figure 5-8 height h_3 is set equal to zero, and the angle of θ is zero. In this situation, the Equation 5.68 equivalent unit weight relationship is given by

$$\gamma_e = \gamma_1 * \left[\left(\frac{h_1}{(h_1+h_2)} \right)^2 \right] + \gamma_{2-3} * \left[1 - \left(\frac{h_1}{(h_1+h_2)} \right)^2 \right] \quad (5.69 \text{ bis})$$

with a moist unit weight γ_{moist} assigned to γ_{2-3} for soil area 2 above the water table in a total stress analysis. A saturated unit weight γ_{sat} assigned

to γ_1 , as usual, and if the resisting wedge is fully submerged, the Figure 5-8 height h_2 is equal to zero, and Equation 5.69 simplifies to

$$\gamma_e = \gamma_1 = \gamma_{sat} \quad (5.77)$$

5.5.2.2 Effective stress analysis

In an effective stress analysis with hydrostatic pore water pressures below the water table in the resisting wedge, the equivalent unit weight γ_e is computed using a moist unit weight γ_{moist} assigned to γ_{2-3} for soil areas 2 and 3 above the water table and with a buoyant unit weight γ_{buoy} assigned to γ_1 for soil area 1 below the water table in Equation 5.68 with a ground surface oriented at an angle of θ from horizontal. When the buttressing soil is horizontal, the Figure 5-8 height h_3 is set equal to zero, and the angle of θ is zero. In this situation, equivalent unit weight is computed using Equation 5.69 with a moist unit weight γ_{moist} assigned to γ_{2-3} for soil area 2 above the water table and a buoyant unit weight γ_{buoy} assigned to γ_1 for soil area 1 below the water table.

When there is a differential head of water existing from one side of the structural wedge to the other, there will be flow of water within the soil regime. Subsection 5.5.1.2 discussed the computation of the submerged soil γ_1 value for use in the equivalent unit weight γ_e by Equation 5.68 for sloped ground or Equation 5.69 for level ground in an effective stress analysis. A key step is the computation of the vertical seepage gradient to be used when computing γ_1 . Equation 5.72 is used to compute γ_1 for upward seepage within the resisting wedge and Equation 5.74 is used to compute γ_1 for downward seepage. For a homogeneous, cohesionless buttressing soil, the P_P thrust force that the resisting wedge soil imposes on the structural wedge is designated as P_L in Figure 5-1 and computed using Equation 5.76. Note that in an effective stress analysis, boundary water pressures are also applied to the structural to resisting wedges interface. These boundary water pressures will come from either a steady-state seepage analysis, a transient seepage analysis or from the approximate, line of seepage analysis described in Appendix G.

6 Structural Variables with Respect to Limit States

6.1 Introduction

As stated earlier in Chapter 2, in reliability analysis the probability that a structure will fail is defined by the passive Capacity (or Resistance) and the active Demand (or Load). When the Demand exceeds the Capacity of a structural element, that structural element will have unsatisfactory performance. A response state function (sometimes referred to as a $g(\mathbf{X})$ function), returns the status of the system performance. The $g(\mathbf{X})$ function is thus defined as

$$g(\mathbf{X}) = R - L \quad (2.11 \text{ bis})$$

where R is the Resistance (Capacity) and L is the Load (Demand). In this case, the limit state Response Surface is defined by the point where the Resistance equals the Load, or

$$g(\mathbf{X}) = (R - L) = 0 \quad (2.17 \text{ bis})$$

It is important to find how to calculate the Capacity from variable engineering properties given these relationships to calculate the probability that the Capacity will be exceeded. This implies that the Resistance is a similar function taking a vector of variable properties drawn from distributions of the form

$$R = R_m(\mathbf{X})\lambda_R = R_m(f_y, f_c, E, b, c, d \dots)\lambda_R \quad (6.1)$$

in which $R_m(\dots)$ is a mechanics-based expression of strength, with probabilistic material properties ($f_y, f_c, \dots$), and λ_R is the factor representing uncertainties because of approximations in structural analysis and simplifications in the constitutive model of the material behaviors. The factor is called the resistance bias factor in subsection 2.1.2 and is defined as the ratio of the mean and nominal resistances.

$$\lambda_R = \frac{R_m}{R_n} \quad (2.4 \text{ bis})$$

6.2 Introductory observations for three structural limit states

Limit state criteria is needed for both reinforced concrete hydraulic structures as well as for the steel piles that sometimes provide its foundation. This subsection provides introductory observations for the three structural limit states of

- flexural capacity
- shear capacity
- buckling capacity.

The assessment provided is not complete due to a time limitation requirement for publication of this ERDC technical report. For example, not all the structural limit state criteria for reinforced concrete structures and for the steel used in the pile foundations of hydraulic structures have been gathered. The information that has been gathered does allow for an initial organizational effort of the engineering criteria and identification of the type of information that has not yet been researched for reinforced concrete hydraulic structures as well as for the steel when pile foundations are used for hydraulic structures.

Flexural capacity: The first structural limit state of interest to reinforced concrete hydraulic structures is the flexural capacity of reinforced concrete members. Equation 6.2 gives an example of a resistance equation for flexural capacity of a reinforced concrete member, M_n :

$$R_m(A_s, F_y, d, f'_c, b) = M_n = A_s F_y \left[d - \frac{\frac{A_s F_y}{0.85 f'_c b}}{2} \right] \quad (6.2)$$

where:

- A_s = area of reinforcing steel
- F_y = yield strength of reinforcing steel
- D = distance from compression face to center of reinforcing steel
- f'_c = concrete compression strength
- b = width of beam or design section.

Shear capacity: The second structural limit state of interest to reinforced concrete hydraulic structures is the shear capacity of reinforced concrete members. Equation 6.3 gives an example of a resistance equation for shear

capacity, V_c , computed using the simplified American Concrete Institute (ACI) equation for beam capacity (ACI 318-14 Equation 22.5.5.1):

$$R_m(b, d, f'_c) = V_c = 2\sqrt{f'_c}bd \quad (6.3)$$

where:

b = section width

d = distance from compression face to center of reinforcing steel

f'_c = concrete compression strength.

Table 6-1 shows representative distributions for certain material properties given construction practice in the United States (Ellingwood¹). These material properties, as well as others not listed, are used in the specified $R_M(\mathbf{X})$ functions to calculate capacity in a probabilistic fashion.

Table 6-1. Probabilistic models for concrete and steel reinforcement strength* (after Ellingwood¹).

Variable	Distribution Type	Nominal (pounds per square inch [psi])	Mean (psi)	COV
E_c (4,000 psi concrete)**	Normal	3.6×10^6	3.95×10^6	0.18
E_c (6,000 psi concrete)**	Normal	4.4×10^6	4.84×10^6	0.18
f_c (4,000 psi concrete)	Normal	4,000	4,800	0.15
f_c (6,000 psi concrete)	Normal	6,000	7,200	0.12
f_t (4,000 psi concrete)	Normal	405	443	0.20
f_t (6,000 psi concrete)	Normal	496	543	0.20
f_y (deformed bar)	LogNormal	60,000	66,000	0.08
f_u (deformed bar)	LogNormal	90,000	104,000	0.07
E_s	Normal	29×10^6	29×10^6	0.04
Effective depth of reinforcement	Normal	D	D	$0.5/d^{***}$
Reinforcement cover	Normal	c	c	$0.5/c^{***}$
Placement of formwork	Normal	b, h	b, h	$0.25/b^{***}$

*1 psi = 6.9 megapascals

**Unit weight of concrete = 145 pounds per cubic foot

***COV equals standard deviation in placement, σ , divided by nominal dimension.

¹ Ellingwood, B. R. 2015. *Procedures for Developing Reliability-Based Load Criteria for Hydraulic Steel and Concrete Structures*. Final letter report submitted to the URS Group, Inc. U.S. Army Corps of Engineers.

The probabilistic capacities can be computed for various limit states given variable material properties can be calculated using the specified $R_m(\mathbf{X})$ capacity equation (e.g., flexural capacity of beams). Table 6-2 shows the results of running Monte Carlo simulations to determine distributions for several $R_m(\mathbf{X})$ capacity equations. The $R_m(\mathbf{X})$ capacity equations were performed by assigning an input vector $\mathbf{X}$ with values drawn from the appropriate material distributions and then a distribution was mapped to the results. This work was performed by MacGregor et al. (1983), Ellingwood and Hwang (1985), Nowak and Szerszen (2003), and others. The final statistics take into account the bias of the mechanics-based expression (λ_R).

Table 6-2. Resistance models (after Ellingwood¹).

Limit State	Distribution Type	Mean	COV
Flexure Capacity (beams)	LogNormal	1.10 Mn	0.12
Flexure Capacity (one way slabs)	LogNormal	1.12 Mn	0.14
Shear Capacity (beams)	LogNormal	1.15 Vn	0.18
Shear Capacity (walls)	LogNormal	1.20 Vn	0.18
Axial load and flexure capacity (short columns, tension failure)	LogNormal	1.05 Rn	0.12
Axial load and flexure capacity (short columns, compression failure)	LogNormal	0.98 Rn	0.14
Compression Capacity (plain concrete)	LogNormal	1.20 Pn	0.15

For the resistance models in Table 6-2, the variable capacity (R) for reliability analysis can be derived simply by drawing from the distribution. Using Equation 2.11, if the capacity (R) exceeds the demand (L), then the $g(\mathbf{X})$ function is greater than 0, indicating failure.

Buckling capacity: The third structural limit state of interest to reinforced concrete hydraulic structures is the buckling capacity of reinforced concrete members that are loaded in compression. This is a complicated limit state that is frequently analyzed under compressive loading and for elastic (buckling) stability. This is because the (axial) compressive strength

¹ Ellingwood, B. R. 2015. *Procedures for Developing Reliability-Based Load Criteria for Hydraulic Steel and Concrete Structures*. Final letter report submitted to the URS Group, Inc. U.S. Army Corps of Engineers.

of the pile and the moment strength are not independent of each other. The resulting axial and moment column/pile capacity is expressed through an interaction diagram. Even without a moment applied to the top of a pile, any eccentricity in the axial load placement from the neutral axis (often coincident with the centerline) of the pile induces a moment within the pile that affects the (axial) compressive strength of the pile. For this reason, a single, independent resistance-only equation $R_m(\mathbf{X})$ cannot be generated for columns/piles as it can be for flexural and shear capacities (Equations 6.2 and 6.3, respectively). With variability, an axial load applied to the column/pile affects the axial resistance of the column/pile due to the introduction of moment within the pile resulting in eccentricity or from an external moment boundary condition applied to the end of the structural member.

Figures 6-1 and 6-2 show the configuration of interaction diagrams for steel and reinforced concrete piles/columns, respectively. The capacity of the pile is the area below the interaction curve for compression and to the left of the curve under moment. Notice that as the moment increases for the column/pile, the compressive strength decreases, although there is a break point for the reinforced concrete problem when both the concrete and reinforcement have been exceeded.

Figure 6-1. Interaction diagram for a steel pile/column
(after Ebeling et al. 2012).

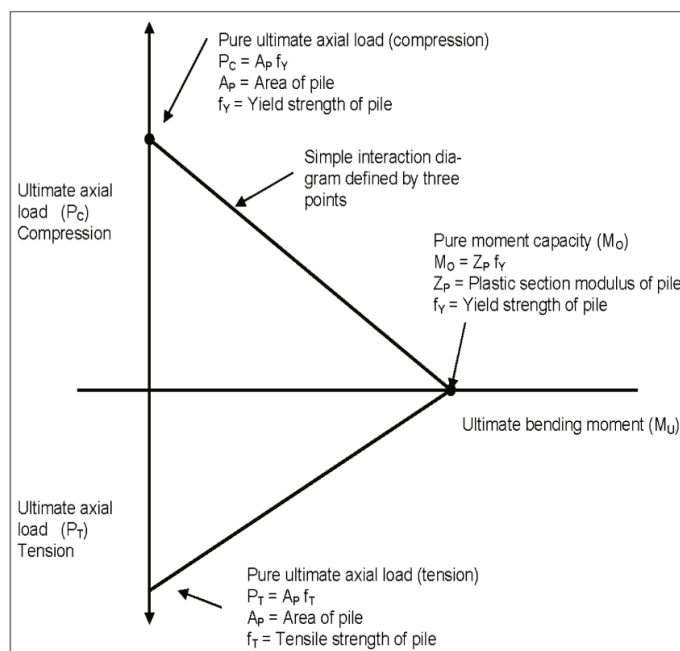
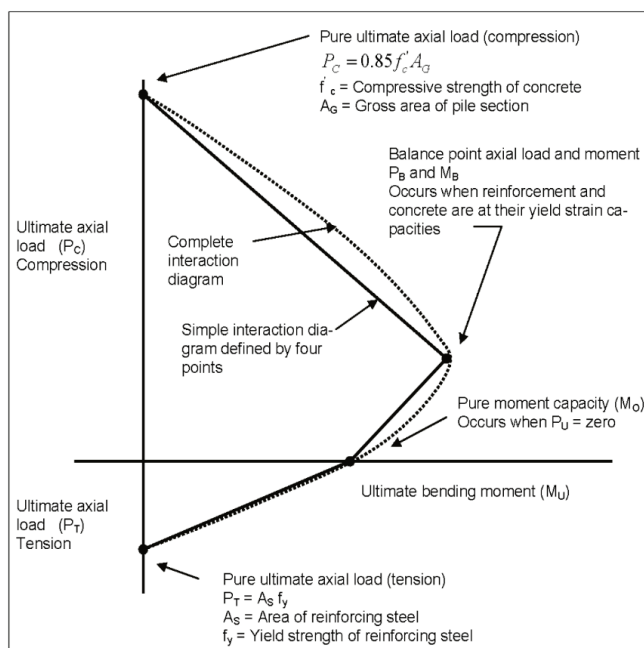


Figure 6-2. Interaction diagram for a reinforced concrete pile/column (after Ebeling et al. 2012).



The interaction diagrams shown are for the case of a perfectly straight column/pile and frame system Powell (2010). In the figures, the structural members' behavior is not elastic-perfectly-plastic. The interaction surface is the ultimate surface.

Axially loaded columns/piles have different failure methods depending on the slenderness of the column/pile with respect to the length (L) and the axis with the lowest radius of gyration (r). The radius of gyration is defined as

$$r = \sqrt{\frac{I}{A}} \quad (6.4)$$

where I is the second moment of area¹ (i.e., moment of inertia) and A is the total cross-sectional area.

The slenderness value is necessary because the probability that a beam is purely homogeneous and that loads are exactly parallel to the axis of the structural column/pile and centered on the member are extremely rare in the real world. The slenderness term also takes into account how the beam is connected to the system (i.e., the boundary conditions imposed at the

¹ Wikipedia: [second moment of area](#)

The slenderness ratio (Hibbeler 2014) is defined as

$$\frac{kL}{r} \quad (6.5)$$

with L being the unsupported length and r the radius of gyration. The higher the value for the slenderness ratio the more slender the structural element is. Generalizing, the slenderness ratio for steel compression members is restricted to a value of less than 200 to avoid buckling.

The value of the slenderness determines which of three conditions will cause the column/pile under axial loading to fail. The three conditions and associated column compression member classifications are the following:

1. the compressive failure under the compression strength limit state with a material failure in compression (i.e., crushing) for a short column
2. buckling failure due to the inelastic stability limit for an intermediate/medium column
3. buckling failure due to the elastic stability limit (i.e., Euler buckling) for a long column.

The slenderness values for these limit states are both material dependent and geometry dependent, and representative values are provided in Table 6-4. The geometry dependence comes about through the column/pile sections' I value. Recall by Equation 6.4 that r is proportional to I : As the value for I increases, the value of r also increases and vice versa. Also, by Equation 6.5, as the value for r increases, the value for the slenderness ratio decreases. For anisotropic cross sections, it is the weak axis cross section that establishes the buckling capacity for the structural member for it possesses the lower I value of the two principal axes (i.e., strong and weak axes), the lower r value for the two axes and the higher slenderness ratio. Figure 6-3 gives a representation of the relationship of critical compressive stress to the slenderness ratio, and reveals the reasoning for dividing the columns/piles into three categories based on the slenderness ratio (i.e., short, intermediate/medium and long columns). The Computer-Aided Structural Engineering (CASE) Library program CPGA (Hartman et al. 1989) assumes that most piles are equivalent to long columns, so it does not deal with the inelastic stability limit state. Note that the Adluri data reported in Table 6-4 is delineated by material type (i.e., steel, aluminum, and wood). Criteria for reinforced concrete members are not included in this table.

Table 6-4. Defining behavior of columns using the slenderness ratio (after Adluri 2017).

Material	Short Column (strength limit)	Intermediate Column (inelastic stability limit)	Long Column (elastic stability limit)
Structural Steel	$kL/r < 40$	$40 < kL/r < 150$	$kL/r > 150$
Aluminum Alloy AA 6061 – T6	$kL/r < 9.5$	$9.5 < kL/r < 66$	$kL/r > 66$
Aluminum Alloy AA 2014 – T6	$kL/r < 12$	$12 < kL/r < 55$	$kL/r > 55$
Wood	$kL/r < 11$	$11 < kL/r < (18\sim30)$	$(18\sim30) < kL/r < 50$
CPGA Pile	$kL/r < 120$		$kL/r > 120$

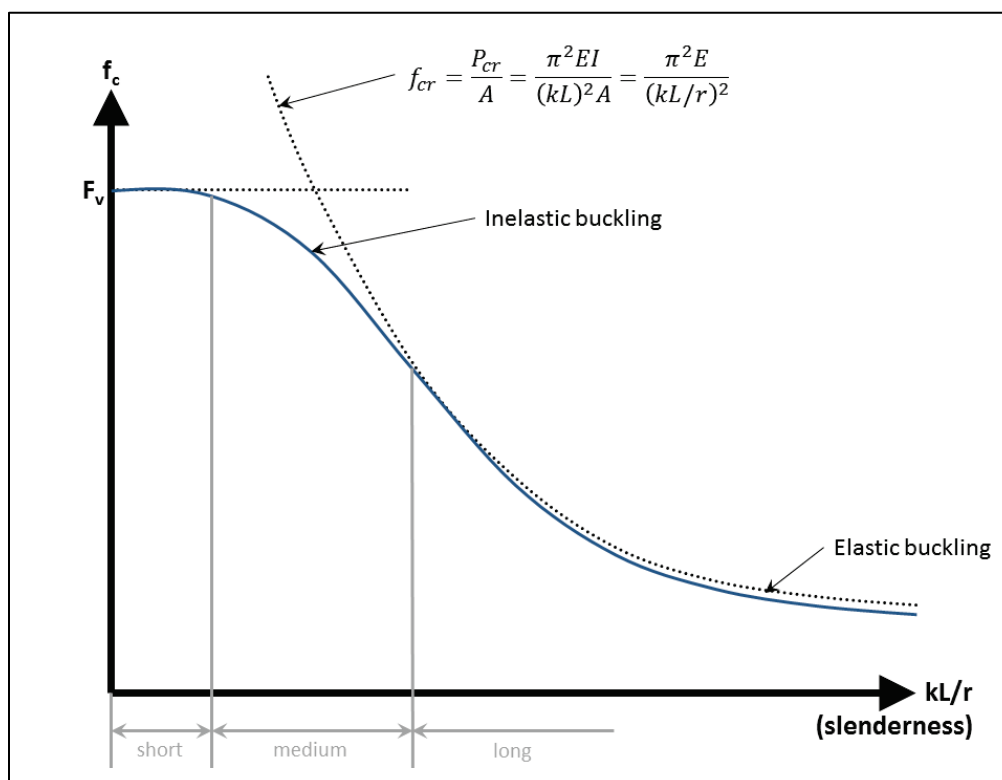
Figure 6-3. Critical axial stress (f_c) versus slenderness ratio (after Powell 2010 and Beer and Johnston 1981).

Figure 6-3 shows the axial compression capacity f_c of a structural member decreases with increasing value in slenderness ratio. In this figure, F_v is the compressive yield stress, plotted at kL/r equal to zero.

For anisotropic columns/piles, the axis with the larger slenderness ratio governs the buckling limit state for the structural compression member.

This weak axis will typically possess smaller I and r values than those of the strong axis.

For a short column/pile (referencing the slenderness ratio), there is very little to no bending of the column. This means that a hinge forms when the load (axial force) is close to the capacity (axial compressive/yield force). The effects of geometric nonlinearity is negligible, and material nonlinearity dominates (Powell 2010). The capacity (R) of a short reinforced concrete column/pile under only axial loading is determined by the pure ultimate axial compressive capacity.

$$R_m(f'_c, A_G) = P_c = 0.85f'_c A_G \quad (6.6)$$

where:

f'_c = compressive strength of concrete
 A_G = gross area of the pile section.

The capacity (R) of a short steel column/pile under only axial compressive loading is given as

$$R_m(f_y, A_p) = P_c = f_y A_p \quad (6.7)$$

where:

f_y = yield strength of the steel pile
 A_p = gross cross-section area of the column/pile.

For a long column/pile (referencing the slenderness ratio), frame and individual column/pile deflection/deformations drive the performance of the column, leading to buckling. This is because the column/pile will develop larger initial deformations and moments that are amplified by the column/pile length. The small displacements bending moment for failure with axial force P is $P^*\delta_o$, where δ_o is the transverse deflection of the neutral axis of the column/pile at the location of the applied axial load P . The moment acting on the column at failure is magnified by the length of the column and can be much greater than $P^*\delta_o$ (Powell 2010).

The Euler buckling load is therefore the ultimate capacity of the reinforced concrete column/pile. Equation 6.8 gives an approximate capacity of the long column under axial loads.

$$R_m(E_c, I, k, L) = P_e = \frac{\pi^2 E_c I}{(kL)^2} \quad (6.8)$$

where:

- E_c = elastic modulus of concrete
- I = moment of inertia about the minor radius axis
- K = effective length factor (Table 6-3)
- L = length of the column.

For steel piles, replace E_c with E_s , in Equation 6.8. E_s is the Young's modulus for steel. Observe that this equation does not include the American Institute of Steel Construction FoS of 1.92 (=23/12) for steel columns (Hibbeler 2014). This is because reliability analysis requires the use of limit states without incorporation of factors of safety. The basis for and possible need to apply an FoS to the Euler buckling equation will be a point for further research during the Phase 2 study.

For steel piles/columns, Hibbeler (2014) defines the lowest value of slenderness ratio for which the Euler (elastic) buckling relationship of Equation 6.8 (with E_c replaced by E_s) as

$$\left(\frac{kL}{r}\right)_c = \sqrt{\frac{2\pi^2 E_s}{F_v}} \quad (6.9)$$

Recall that F_v is the yield stress for steel (in compression) in this equation. In the case of steel piles/columns possessing a slenderness ratio less than the value given by Equation 6.9, a more involved equation for the allowable stress to avoid buckling is given by Hibbeler (2014) with his Equation 13-23 (not shown). This equation incorporates an FoS of 1.92. The basis for and possible need to apply an FoS to the allowable stress buckling equation for slenderness ratio values less than that given by Equation 6.9 will be a point for further research during the Phase 2 study because of its intended application in a reliability analysis.

Inelastic buckling: Application of the Euler equation requires that the stress in the steel column/pile remain below the material's yield point (i.e.,

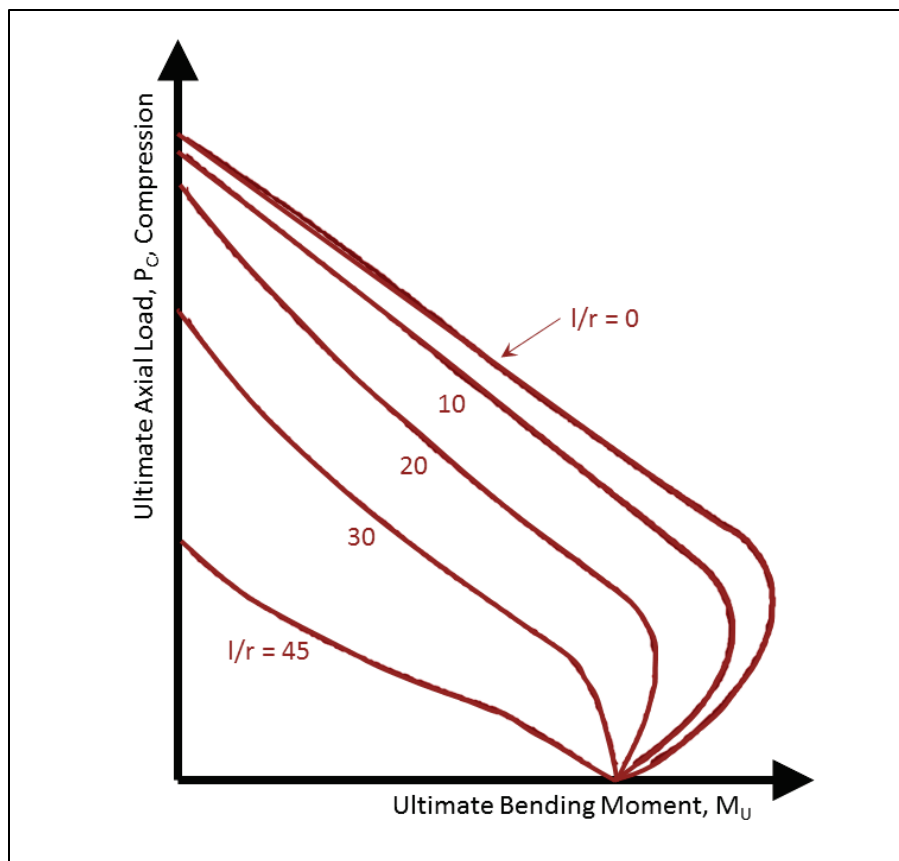
proportional limit) when the column/pile buckles. If the steel column/pile has a slenderness ratio less than the value given by Equation 6.9, then the critical stress in the steel column/pile must be greater than the proportional limit. In this case, the use of the elastic steel modulus is no longer valid since the some of the steel is starting to yield across the columns cross section. This phenomena helps explain the need to restrict the application of Euler's elastic buckling equation for slenderness ratios in steel columns/piles that are less than the value given by Equation 6.9.

The Beer and Johnston (1981) Figure 11.23 (not shown) for a steel column ($F_v = 36$ kips per square inch [ksi]; $E_s = 29 \times 10^6$ ksi) shows that eccentricity of the compressive load location affects axial load capacity of the column. The Euler Equation is accurate if ec/r^2 is equal to 0 and with a slenderness ratio value of 90 and above, where e is the distance from the neutral axis to the axial load, c is the distance from the neutral axis to the extreme fiber, and r is the radius of gyration. For greater values of ec/r^2 (i.e., greater eccentricity), the axial compression capacity of the steel column diminishes. This reduced capacity will be an item for research during the companion LRFD research and development effort for steel pile foundations.

For intermediate columns (referencing the slenderness ratio), the capacity (R) of the column under axial loading is more difficult to determine. In this case, the moment acting on the column/pile is magnified by a ratio of compressive capacity to the buckling capacity. This has an effect of reducing the interaction diagrams shown in Figures 6.1 and 6.2 to create a slender column interaction curve. Figure 6-4 was constructed considering a reinforced concrete column and shows how the interaction diagram changes with respect to the slenderness ratio. The slenderness ratio kL/r (Equation 6.5) is sometimes expressed as l/r . This later length l is referred to as an effective length (Hibbeler 2014) and is defined as

$$l = k * L \quad (6.10)$$

Figure 6-4. The effects of slenderness ratio to the column/pile axial capacity interaction diagram with the same cross section (after MacGregor 1992).



The length l used in Figure 6-4 is the effective length of Equation 6.10. Curves are shown in this figure for slenderness ratios of 0, 10, 20, 30, and 45. Observe how the axial capacity of the interaction diagram diminishes the slenderness ratio increases. For a given fixity condition (refer to Table 6-3) for the structural compression member, the slenderness ratio increases with a reduction in the weak-axis moment of inertia I and/or with an increase in unsupported length L .

Moment magnifier design: According to MacGregor (1992), CalTrans (2015), AASHTO (2012), and ACI (1978), for a factored load P_u , the magnified factored moment for reinforced concrete columns/piles is given by

$$M_c = \delta_b M_{2b} + \delta_s M_{2s} \quad (6.11)$$

where M_{2b} is the larger factored end moment on a reinforced concrete compression member due to loads causing minimal lateral deflections

$(l/1500)$ and M_{2s} is the larger factored end moment on a compression member due to loads causing appreciable lateral deflections. These are moments from a conventional first-order elastic frame analysis:

$$\delta_b = \frac{1}{1 - \frac{P_u}{\phi P_e}} \geq 1.0 \quad (6.12)$$

and

$$\delta_s = \frac{1}{1 - \frac{\sum P_u}{\phi \sum P_e}} \geq 1.0 \quad (6.13)$$

and P_e is the Euler Buckling load of Equation 6.8. With multiple limit states, the response state function $g(\mathbf{Z})$, given load L , becomes

$$g(\mathbf{Z}) = \min(R_{m1}(\mathbf{X}_1), R_{m2}(\mathbf{X}_2), \dots, R_{mN}(\mathbf{X}_N)) - L \quad (6.14)$$

In terms of unsatisfactory performance, $G(\mathbf{Z})$ can be expressed as the logical union of the unsatisfactory performance ($G(\mathbf{Z}) < 0$) states

$$G(\mathbf{Z}) = \begin{cases} -1: ((R_{m1}(\mathbf{X}_1) - L < 0) \cup (R_{m2}(\mathbf{X}_2) - L < 0) \cup \dots \cup (R_{mN}(\mathbf{X}_N) - L < 0)) \\ 1: otherwise \end{cases} \quad (6.15)$$

6.3 Recommendations for the Phase 2 research study on three structural limit states

Structural limit state criteria are needed for both reinforced concrete hydraulic structures as well as for the steel piles that sometimes provide for its foundation. This chapter provides introductory observations to the three structural limit states of

- flexural capacity
- shear capacity
- buckling capacity.

There are numerous gaps in the details regarding criteria for these three structural limit states for both reinforced concrete hydraulic structures as well as for the steel piles that are sometimes used as the foundation to a hydraulic structure and within the context of a reliability analysis. Some of these engineering criteria and reliability relationship gaps include

- Engineering criteria for (1) an axial compression load and (2 and 3) applied moments for a reinforced concrete member about the strong and weak axes of reinforced concrete columns (i.e., biaxial loading). This includes a need for precast, prestressed concrete piles as well.
- Engineering criteria for (1) an axial compression load and (2 and 3) applied moments at the pile cap about the strong and weak axes of steel piles (i.e., biaxial loading).
- Complete the specifics in the formulation of the flexural capacity, shear capacity, and buckling capacity of reinforced concrete structures and steel piles for hydraulic structures in terms of reliability relationships.
- Devise the reliability relationship needed for expressing the limit state criteria in a reliability based relationship format for the interaction diagrams of Figures 6.1 and 6.2, which express axial compression capacity as a function of moment.
- The basis for and possible need to apply an FoS to the Euler buckling equation for steel pile members in a reliability analysis.
- The basis for and possible need to apply an FoS to the allowable stress buckling equation of steel for slenderness ratio values less than that given by Equation 6.9.
- Slenderness ratio limits for reinforced concrete members in compression and buckling capacity criteria.

7 Summary, Conclusions, and Research Recommendations

7.1 Summary

This report focuses on the work that will be required to perform an LRFD for complete hydraulic structures. Complete hydraulic structural analysis involves the structure and geotechnical limit states as well as the SSI induced by loading. A total of seven limit states needs to be incorporated in the engineering model's reliability analysis of a hydraulic structure. These are

- flexural hinging of individual structural members
- shear failure of individual structural members
- buckling failure of individual structural members subjected to axial compressive loading
- excessive displacement of the structural system in its entirety
- global sliding failure of the structural system in its entirety
- bearing failure of the foundation to the structural system in its entirety
- overturning or rotational failure of the structural system about its foundation in its entirety.

The geotechnical reliability research results of Fan et al. (2014) demonstrate that all seven limit states need to be considered concurrently and not independently, for all variables in the reliability analysis to compute an accurate *PUP* value, and ultimately, accurate reliability results for use in the computation of Load Factors.

Combined LRFD, involving structural as well as geotechnical limit states, is an area that has not been addressed by the USACE. While research has been conducted for structural LRFD where the materials are more uniform, geotechnical research has been complicated by spatial variability of engineering material properties for the soil(s). In that vein, this technical report has focused primarily on building engineering models that capture soil-structure interaction and are readily adaptable to reliability analysis. This holds true with respect to the model for first engineering limit discussed, which centers on the sliding stability analysis of a T-Wall. A T-Wall structure was chosen to lay out the engineering and reliability analysis capabilities needed because it possesses the attributes of all three wedges in its engineering model: (1) a driving wedge, (2) a structural

wedge, and (3) a resisting wedge. The engineering and reliability principles proposed for use are broad enough to be used for overturning and bearing failure limit states as well. Note that due to time requirement for publication of this Phase 1 study, these two limit states have not yet been developed for incorporation in a reliability analysis of a hydraulic structure. This research will occur during the follow-on Phase 2 study starting in 2018. Last, note that the engineering model can be run deterministically as well as stochastically.

Modeling variability in soils is complicated by the fact that even *homogeneous* soil engineering material properties change with relative distance. With enough difference in distance, the soil strength properties can be completely unrelated to the distribution at the original location. This led to the development of three different models in this Phase 1 study that tried to break up the slip plane in a homogeneous material region into (1) wedges with totally correlated data (based on point samples), (2) larger wedges considering the effect of non-correlated regions using a Variance Reduction Factor (which reduces the standard deviation of the point samples to account for averaging of samples), and (3) adjusting the values with the Variance Reduction Factor and used in an Earth Pressure Coefficient form of geotechnical engineering analysis. These models and their mathematical generation and solution methods are described in Chapters 4 and 5, respectively.

These engineering models can be used in a reliability analysis to determine the *PUP* and reliability index (β) for the geotechnical/structural system. A reliability analysis can compute a value for *PUP* in a few different ways. Early methods researched and discussed had to deal with the ASM method and Monte Carlo simulation techniques. The ASM method by itself only works for a single hyperplanar limit state for estimating *PUP*, which does not serve the geotechnical/structural systems in this report because the *PUP* value would be underestimated due to lack of consideration of multiple the limit states concurrently. However, the ASM method does determine the closest approach of the limit state response surface, giving a good β value and directional cosines (α) to define a design point on the response surface.

Monte Carlo simulation techniques can achieve greater accuracy than the ASM method with enough simulation samples. The number of simulation samples can be reduced by using the Latin Hypercube sampling method,

and the accuracy can be increased by using Importance Sampling guided by an ASM design point. Monte Carlo techniques return a *PUP* value but do not give an accurate β or α value unless an ASM search for a design point has been performed. It is also possible, given random selection of samples, that not every limit states contribution will be included in the *PUP* calculation.

Other methods have been developed for estimating *PUP* and have been discussed in the main body of this technical report and in Appendix A. These methods include using Bayesian Logic between limit states to bound the possible range of values where the actual *PUP* value can exist. Using an ASM approach for each limit state, each limit state's response surface plane equation is calculated, and then Bayesian Logic can determine the minimum and maximum each limit state area can overlap, assuming that the limit states are uncorrelated. Assuming the *PUP* value to lie at the center of the range means that the greatest error for the *PUP* value would be half the range width.

The results from this Phase 1 study lead the authors of this report to conclude that to get a more accurate value for *PUP*, a numerical integration needs to be performed of the normalized Gaussian multivariate space with respect to *PUP*, where the limit state function $G(\vec{z}) < 1.0$. Because of the necessity to have high β values for stability of structures, having this type of accuracy may be necessary for LRFD analysis.

The accuracy of numerical integration depends on the resolution of the spatial subdivision. To reduce error, a large number of model runs will need to be performed in the multivariate space. This numerical research effort, involving an example problem, will occur during the Phase 2 study. Appendix A provides alternatives for faster integration based on properties of the normalized Gaussian multivariate space. It is suggested that numerical integration be done on high-performance computing machinery. The U.S. Army Engineer Research and Development Center (ERDC), High Performance Computing Center (HPCC), platform is ideal for conducting these computationally intensive numerical calculations and for assessing the accuracy of the numerical method being devised during the Phase 2 study.

The methods for bounding the *PUP* value and numerical integration provide a direct *PUP* value. These methods do not provide a β or

information for the derivation of directional cosines (α). Methods for determining β from *PUP* values for normalized Gaussian multivariate space will be approached during the Phase 2 follow-on study.

In section A.4 of Appendix A, preliminary research is proposed for generating load factors in a multiple limit state system. This method takes the β 's and α 's for each limit state (which defines the hyperplane equation of the limit state) to find scale factors for capacity that can be applied to guarantee a specified probability of satisfactory performance. The viability of this approach will be assessed during the Phase 2 study.

In Appendix H, a method is introduced for calculating partial load and resistance safety factors given distributions for load and resistance variables and a target β . This method is based on an LRFD bridge design procedure as outlined by Nowak and Collins (2013). In this method, the limit state is constrained to be a single case where the resistance must exceed the combined loads. The method involves having at most two variables with unknown means (although the COV are all known). The other variables are defined by their relationships (i.e., ratios) to one of the unknown means. With that information, the relationship between the two unknown means can be determined for variables that may have non-normal distributions. This relationship with the target β can be used to compute the load and resistance safety factors. Additionally, a method to compute the β for specified distributions (possibly non-normal) is introduced to act as a verification step for the previous procedure. Creating a distribution with mean values computed based on the derived partial load and resistance safety factors, this second procedure should calculate a β that is equivalent to the initial target β . An example problem taken from a 2016 USACE LRFD study (unpublished) is included and executed using both procedures. Important differences between the results from the Appendix H numerical procedure and the 2016 unpublished LRFD study are explained.

7.2 Conclusions

For this Phase 1 research effort, there have been developed three reliability models of hydraulic structures founded on soils and retaining soils. These models were specially developed because of spatial variability of the engineering soil properties. The spatial variability is described by the scale of fluctuation, which give the distance at which soil engineering properties have changed enough to no longer be highly correlated. The Variance

Reduction Function equations of Subsection 3.2 (Equations 3.2 and 3.3) provide an estimate of the loss of correlation between two distant point samples in the soil. The Variance Reduction Factor is a significant consideration when performing a reliability analysis involving geotechnical limit state(s). The three reliability models mathematically model the spatial variation in the soil by (1) dividing the wedge into subwedges with low correlation (so that the subwedge slip-plane is less than the scale of fluctuation distance), (2) adjusting the variance of the probabilistic engineering soil properties using the length of the slip-plane with the Variance Reduction Factor for a wedge solution, or (3) adjusting the variance of the probabilistic engineering soil properties using the length of the slip-plane with the variance reduction factor for a coefficient solution.

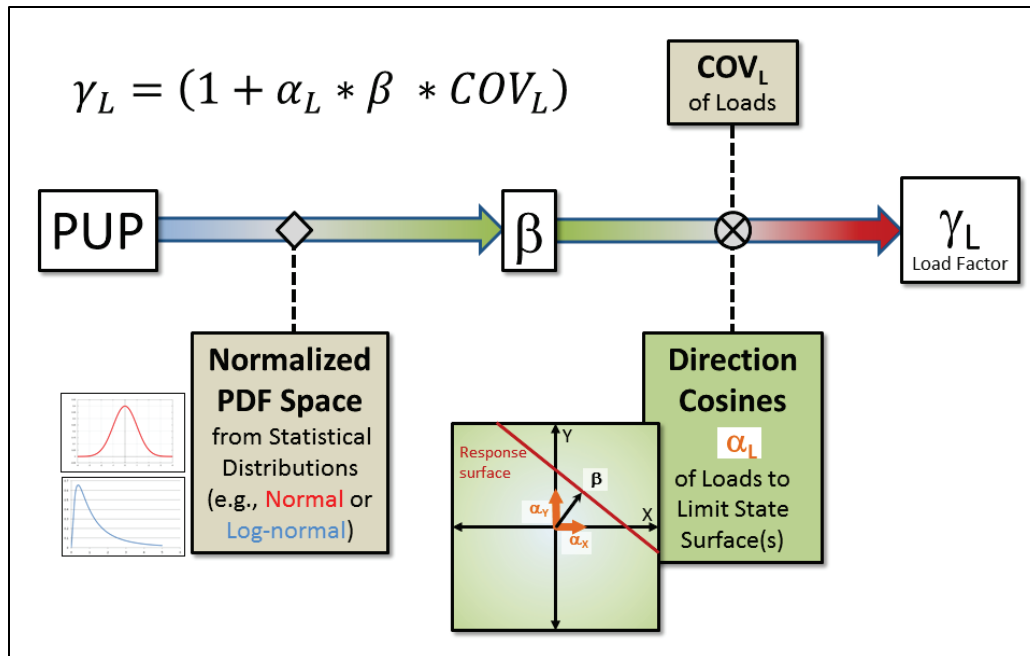
Because the Variance Reduction Factor and scale of fluctuation plays such a significant role in the modeling of the engineering soil strength properties, it is important that reliability methods are as accurate as numerically possible. This is why this report discusses more accurate means of calculating the *PUP* values, such as the probability integration model (Appendix A).

7.3 Phase 2 follow-on research recommendations

7.3.1 Overview of the Phase 2 research into *PUP* to β to load factor research

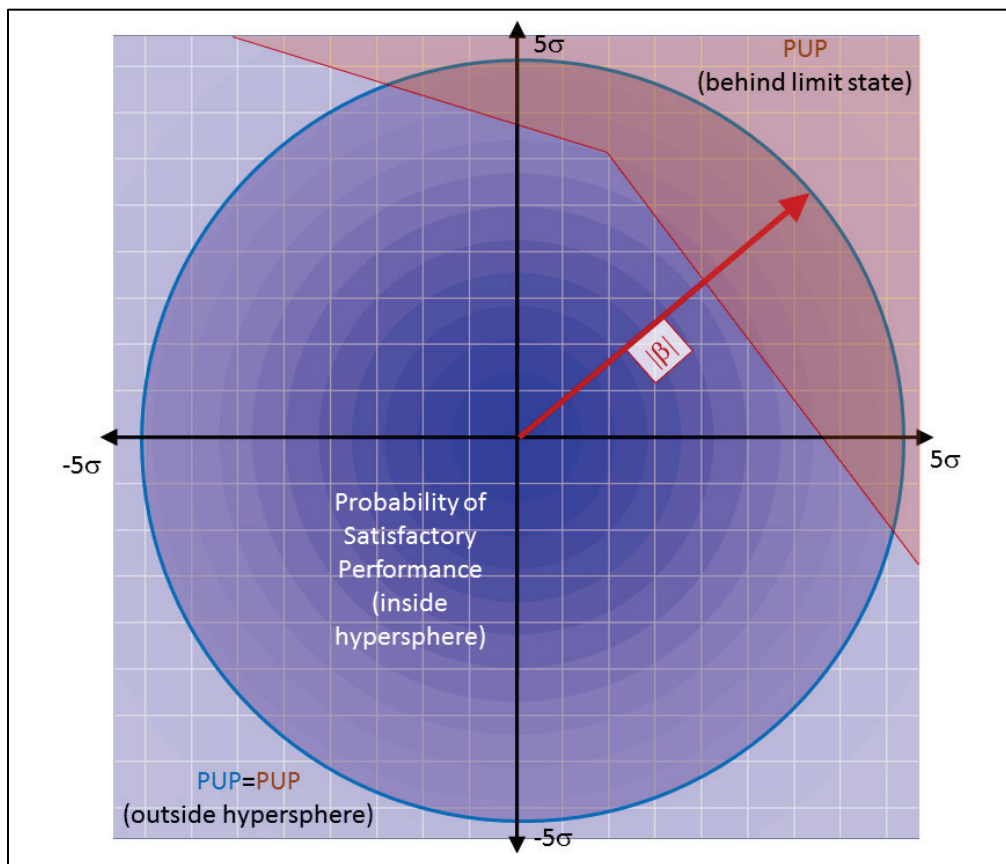
Reliability analysis: Figure 7-1 shows the three stages to reliability analysis to determine factors for LRFD. Methods have been discussed that allow for the collection of *PUP* for hydraulic structures founded on soils and retaining soils. For a single hyperplanar limit state, deriving the β , α , and *PUP* values is simple. This situation becomes much more complicated with multiple limit states, especially when trying to achieve accuracy in the *PUP* evaluation.

Figure 7-1. Three stages to load factor development via reliability analysis.



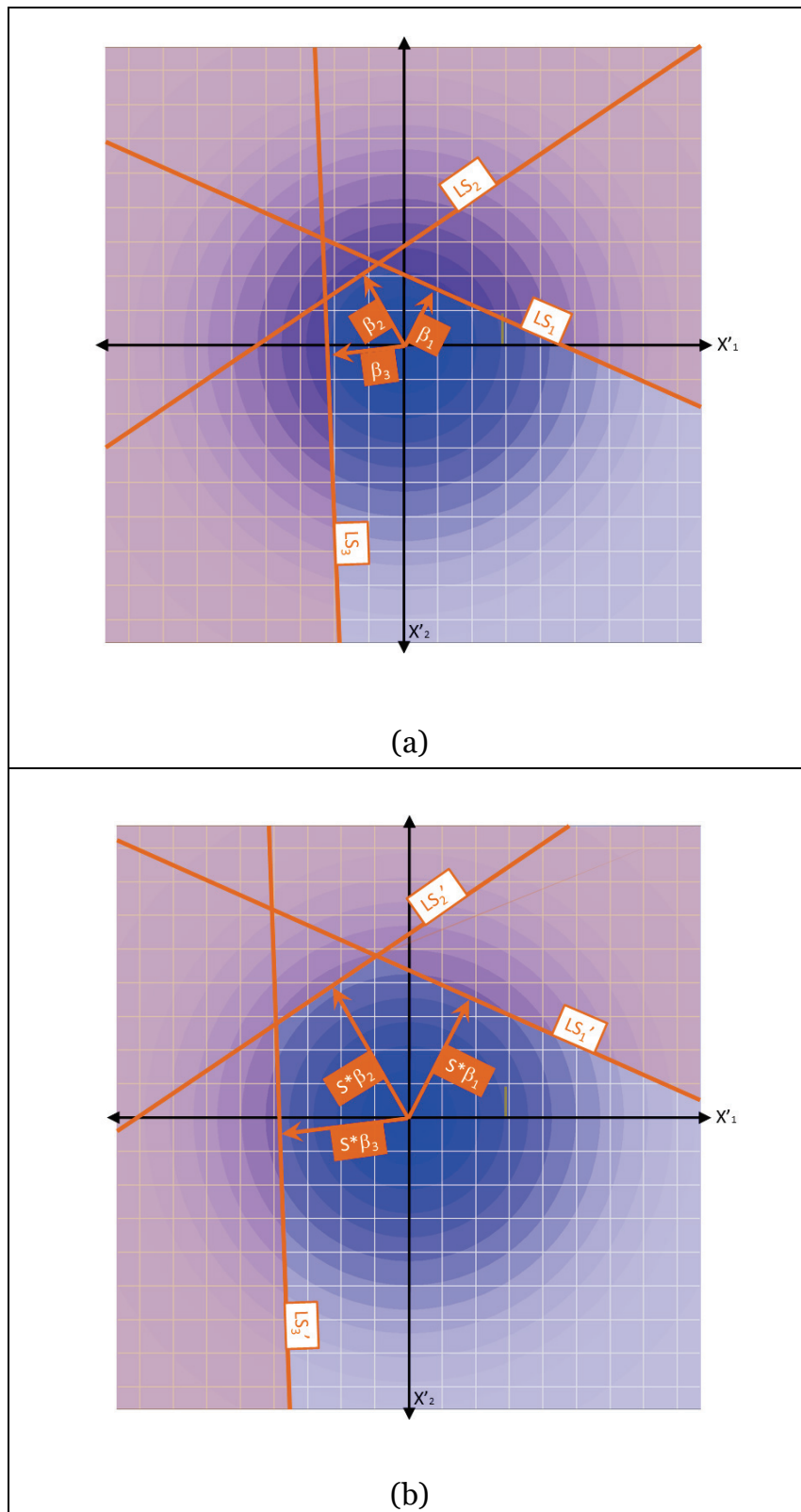
For the numerical integration methods, which will have the highest accuracy, and for the Monte Carlo simulation techniques (including Latin Hypercube and Importance Sampling), the *PUP* value can be calculated without the actual discovery β or α values, which are shown to be necessary in Figure 7-1. This is an area that needs to be researched starting in 2018 during the follow-on Phase 2 study for methods that will give appropriate values. Figure 7-2 shows a proposed method for estimating β using equal probability areas. The actual area that delineates the *PUP* for the limit state response surface is given in the red-ish area at the upper right of the figure. A hypersphere with radius β encircles the Probability of Satisfactory Performance. With the correct value for β , the integration of the probability outside of the hypersphere will be equal to the actual *PUP* in the red-ish area. This β can then be used for the equation in Figure 7-1. An ASM run can be performed to find the design point for the closest limit state to retrieve α .

Figure 7-2. Estimation of β from PUP using equivalent probability areas (not to scale).



The bounding *PUP* value based on the work of Ang and Tang (1984) required that all of the limit states have their design points found, which means that β and α are available for all the limit states. In this case, a resistance scale can be found directly, by applying a scale factor S to the β values for all the limit states, while the α values are static. This scale factor can be set so that the integration of the probability space for the *PUP* can reach an arbitrary value. This procedure is shown in Figure 7-3.

Figure 7-3. (a) Determining multiple limit state β s and (b) scaling them to match desired PUP.



Of course, the numerical models specified in the report need to be tested using high-performance computers. All of these reliability-based steps in an analysis will be investigated during the follow-on Phase 2 research study starting in 2018. A hydraulic structure example problem will be used during the course of this Phase 2 study to flush out the reliability analysis based details.

7.3.2 Summary of the Phase 2 research into limit state analyses

Due to Phase 1 required time for publication, reliability methods of analysis for two geotechnical limit states were not researched and summarized. They are

- global sliding failure of the structural system in its entirety
- bearing failure of the foundation to the structural system in its entirety.

These limit states will be cast in a reliability methodology for use in the multiple limit state, reliability analysis of hydraulic structures. They will build upon the formulations discussed in Chapters 3, 4, and 5.

Various details in the three of the structural limit state formulations need to be completed to be able to conduct a multiple limit state reliability analysis. They are discussed in depth in section 6.3 of Chapter 6.

The most important contribution of the Phase 2 R&D technical report will be a step-by-step reliability and engineering analysis method for developing load factors that will be used for reinforced concrete, hydraulic structures. When completed, this procedural method and developed research tools will form the basis for the district engineer to revise USACE guidance documents to incorporate an LRFD methodology, as well as to update documents like EM 1110-2-2104, *Strength Design for Reinforced Concrete Hydraulic Structures* (HQUSACE 2016).

These items identified above will be researched during the Phase 2 follow-on study starting in 2018 for use in the multiple limit state, reliability analysis of hydraulic structures.

References

- AASHTO. 2012. *AASHTO LRFD Bridge Design Specifications*. 6th Edition. Washington, DC: American Association of State Highway and Transportation Officials.
- Abramson, L. W., T. L. Lee, S. Sharma, and G. M. Boyce. 1996. *Slope Stability and Stabilization Methods*. New York: John Wiley & Sons Inc.
- ACI Committee 340. 1978. *Design Handbook in Accordance with the Strength Design Method of ACI 318-77, Volume 2, Columns*. ACI Publication SP-17A(78). Detroit, MI: American Concrete Institute.
- Adluri, S. 2017. *Structural Steel Design: Compression Members*. Memorial University, Newfoundland and Labrador, Canada
<https://www.engr.mun.ca/~adluri/courses/steel/ppt%20files1/Topic%20-Compression.pdf>
- Allen, T. M. 2013. *AASHTO Geotechnical Design Specification Development in the USA, in Modern Geotechnical Design Codes of Practice; Implementation, Application and Development*. Edited by P. Arnold, G. A. Fenton, M. A. Hicks, T. Schweckendiek, and B. Simpson. Amsterdam, Netherlands: IOS Press BV under the imprint Millpress.
- Alonso, E. E., and R. J. Krizek. 1975. "Stochastic Formulation of Soil Properties." *Proc. Of the 2nd International Conference Applications of Statistics and Probability in Soil and Structural Engineering*. Aachen, F. R. Germany. Vol. 2, 9–32.
- American Association of State Highway and Transportation Officials (AASHTO). 2010. *AASHTO LRFD Bridge Design Specifications*. Fifth edition, Parts I and II. Washington, DC: American Association of State Highway and Transportation Officials.
- American Concrete Institute (ACI) Committee 318. 2014. *Building Code Requirements for Structural Concrete (ACI 318-14) and the Commentary (ACI 318R-14)*. Farmington Hills, MI: American Concrete Institute.
- American Society of Civil Engineers (ASCE) Standard. 2010. *Minimum Design Loads for Buildings and Other Structures*. ASCE/SEI 7-10. Reston, VA.
- Ang, A. H-S., and W. H. Tang. 1984. *Probability Concepts in Engineering Planning and Design: Decision, Risk and Reliability*. Volume II. New York: John Wiley & Sons.
- Asaoka, A., and D. A-Grivas. 1982. "Spatial Variability of the Undrained Stength of Clays." *Journal of the Geotechnical Engineering Division, ASCE* 108(5): 743–756.
- Babu, G. L. S. 2014. "Spatial Variability Using Random Fields." Chapter 3. *Reliability Engineering*. Madras, India: National Programme on Technology Enhanced Learning (NPTEL).
- Baecher, D. E. 1996. "Limit States Design for Foundations. Part I. An Overview of the Foundation Design Process." *Canadian Geotechnical Journal* 32: 956–983.

- Baecher, G. B., and J. T. Christian. 2003. *Reliability and Statistics in Geotechnical Engineering*. West Sussex, England: John Wiley & Sons, Ltd.
- Baecher, G. B., M.-E. Pate, and R. de Neufville. 1980. "Risk of Dam Failure in Benefit-Cost Analysis." *Water Resource Research* 16(3): 449–456.
- Baecher, G., and J. Christian. 2006. "The Influence of Spatial Correlation on the Performance of Earth Structures and Foundations." *GeoCongress 2006: Geotechnical Engineering in the Information Technology Age*, American Society of Civil Engineers, 1–6.
- Baecher, G. B., and C. C. Ladd. 1985. *Reliability Analysis of the Stability of Embankments of Soft Clays*. Special Summer Course 1.60s. Cambridge, MA: Massachusetts Institute of Technology.
- Bao, C. G., W. F. Huang, and Q. N. Chang. 1993. "Reliability Analysis of Bearing Capacity for a Gravity Wharf Foundation in Random Field Theory." *Proceedings of the Conference on Probabilistic Methods in Geotechnical Engineering, Canberra, Australia*, 291–294.
- Bathurst, R. J., T. M. Allen, Y. Miyata, and B. Huang. 2013. *Lessons Learned from LRFD Calibration of Reinforced Soil Wall Structures, in Modern Geotechnical Design Codes of Practice; Implementation, Application, and Development*. Edited by P. Arnold, G. A. Fenton, M. A. Hicks, T. Schweckendiek, and B. Simpson. Amsterdam, Netherlands: IOS Press BV under the imprint Millpress.
- Beer, F.P., and E. R. Johnston. 1981. *Mechanics of Materials*. New York, NY: McGraw-Hill Book Company.
- Bergado, D. T., B. C. Patron, W. Youyongwatana, J. C. Chai, and Yudhbir. 1994. "Reliability-Based Analysis of Embankment on Soft Bangkok Clay." *Structural Safety* 13: 247–266.
- Burlington, R. S., and D. C. May. 1970. *Handbook of Probability and Statistics with Tables*, 2nd Edition. New York, NY: McGraw-Hill Book Company.
- Cafaro, F., and C. Cherubini. 2002. "Large Sample Spacing in Evaluation of Vertical Strength Variability of Clayey Soil." *Journal of Geotechnical and Geoenvironmental Engineering, ASCE* 128(7): 558–568.
- Caltrans. 2015. *Bridge Design Practice*. California Department of Transportation. Sacramento, CA.
- Canadian Standards Association (CSN). 2006. *Canadian Highway Bridge Design Code*. CAN/CSA-S6-06. Mississauga, Ont.
- Caquot, A., and J. Kerisel. 1948. *Tables for the Calculation of Passive Pressure, Active Pressure and Bearing Capacity of Foundations*. Paris: Gauthier-Villars.
- CEN 2004. *Eurocode 7: Geotechnical Design. Part I: General Rules*. EN 1997-1. European Committee for Standardization (CEN).

- Chandler, R. J. 1988. "In-Situ Measurement of Undrained Shear Strength of Clays Using Field Vane." *Vane Shear Strength Testing in Soils: Field and Laboratory Studies*. Special Technical Publication 1014, ASTM. Philadelphia, PA: American Society for Testing and Materials.
- Chen, W. F., and X. L. Liu. 1990. *Limit Analysis in Soil Mechanics. Developments in Geotechnical Engineering*, Vol. 52. New York: Elsevier Science Publishers, Amsterdam.
- Cherubini, C. 1997. "Data and Consideration on the Variability of Geotechnical Properties of Soils." In *Proceedings of the International Conference on Safety and Reliability (ESREL) 97, Lisbon*. Vol. 2, 1583–1591.
- Cherubini, C. 2000. "Probabilistic Approach to the Design of Anchored Sheet Pile Walls." *Computers and Geotechnics* 26(3–4): 309–330.
- Cherubini, C., and G. Vessia. 2004. "A Stochastic Approach to Manage Variability From In Situ Test Data." *Proceedings, 9th ASCE Joint Specialty Conference on Probabilistic Mechanics and Structural Reliability, July 26-28, Albuquerque, New Mexico*.
- Cherubini, C., and G. Vessia. 2006. "Silt Mixture CPT Characterization for a Reliability Evaluation of Pile Total Stress Bearing Capacity." *Proceedings of the 13th Danube-European Conference on Geotechnical Engineering, Ljubljana, May 29-31*.
- Cherubini, C., G. Vessia, and W. Pula. 2006. "Statistical Soil Characterization of Italian Sites for Reliability Analyses." In *Characterization and Engineering Properties of Natural Soils, Proceedings of the Second International Workshop on Characterization and Engineering Properties of Natural Soils. Singapore, November 29 - December 1, 2006*. National University of Singapore: Taylor & Francis.
- Chiasson, P., J. Lafluer, M. Soulie, and K. T. Law. 1995. "Characterizing Spatial Variability of a Clay by Geostatistics." *Canadian Geotechnical Journal* 32: 1–10.
- Christian, J. T., and G. B. Baecher. 2001. "Factors of Safety and Reliability in Geotechnical Engineering by J. M. Duncan." *ASCE, Journal of Geotechnical and Geoenvironmental Engineering* 127(8): 700–712.
- Christian, J. T., and G. B. Baecher. 2011. "Unsolved Problems in Geotechnical Risk and Reliability." *GeoRisk 2011: Geotechnical Risk Assessment and Management*. American Society of Civil Engineers. [http://dx.doi.org/10.1061/41183\(418\)3](http://dx.doi.org/10.1061/41183(418)3)
- Christian, J. T., C. C. Ladd, and G. B. Baecher. 1994. "Reliability and Probability in Stability Analysis." *Journal of Geotechnical Engineering, ASCE* 120: 1071–1111.
- Chugh, A. K. 1986. "Variable Interslice Force Inclination in Slope Stability Analysis, Soils and Foundations." *Japanese Society of Soil Mechanics and Foundation Engineering* 26(1): 115–121.
- Coulomb, C. A. 1776. "Essai sur Une Application des R~gles des Maximis et Minimis A Quelques Problmes de Statique Relatifs ~ l'architecture." *M&m. Acad. Rov. Pr+s Divers Savants*, Vol. 7. Paris: France.

- CSN. 2014. *Canadian Highway Bridge Design Code*. CAN/CSA-S6-14. Mississauga, Ont.
- Dai, S. H., and M. O. Wang. 1992. *Reliability Analysis in Engineering Applications*. New York: Van Nostrand Reinhold.
- Dawkins, W. P. 1984. *User's Guide: Computer Program for Soil-Structure Interaction Analysis of Axially Loaded Piles (CAXPILE)*. Instruction Report K-84-4. Vicksburg, MS: U.S. Army Engineer Waterways Experiment Station.
- DeGroot, D. J., and G. B. Baecher. 1993. "Estimating Autocovariance of In-Situ Soil Properties." *ASCE, Journal of Geotechnical Engineering* 119(1): 147–165.
- Department of the Navy. 1982. *Foundations and Earth Structures*. NAVFAC Design Manual DM 7.2. Alexandria, VA: Naval Facilities Engineering Command.
- Ditlevsen, O. 1981. *Uncertainty Modeling with Applications to Multidimensional Civil Engineering Systems*. New York: McGraw-Hill.
- Doherty, P., and K. Gavin. 2010. "Statistical Review of CPT Data and Implications for Pile Design." *2nd International Symposium on Cone Penetration Testing: Volume 3. 2nd International Symposium on Cone Penetration Testing (CPT10), Huntington Beach, California, USA, May 9-11, 2010*.
- Duncan, J. M. 2000. "Factors of Safety and Reliability in Geotechnical Engineering, ASCE." *Journal of Geotechnical and Geoenvironmental Engineering* 126(4): 307–316.
- Duncan, J. M., M. Navin, and T. F. Wolff. 2003. "Probabilistic Slope Stability Analysis for Practice. Edited by H. El-Ramly, N. R. Morgenstern, and D. Cruden. *Canadian Geotechnical Journal* 40: 848–850.
- Duncan, J. M., and S. G. Wright. 2005. *Soil Strength and Slope Stability*. Hoboken, NJ: John Wiley & Sons, Inc.
- Duncan, J. M., S. G. Wright, and T. L. Brandon. 2014. *Soil Strength and Slope Stability*. 2nd edition. Hoboken, NJ: John Wiley & Sons, Inc.
- Duncan, J. M., C. K. Tan, R. M. Barker, and K. B. Rojiani. 1989. "Load and Resistance Factor Design of Bridge Structures." *Proceedings of the Symposium on Limit States Design in Foundation Engineering, Toronto, Canadian Geotechnical Society, Southern Ontario Section*, 47–63.
- Ebeling, R. M., A. Chase, and B. C. White. 2007. *Translational Response of Toe-Restrained Retaining Walls to Earthquake Ground Motions Using CorpsWallSlip (CWSlip)*. ERDC/ITL TR-07-1. Vicksburg, MS: U.S. Army Engineer Research and Development Center.
- Ebeling, R. M., and E. E. Morrison. 1992. *The Seismic Design of Waterfront Retaining Structures*. WES Technical Report ITL-92-11. Vicksburg, MS: U.S. Army Engineer Waterways Experiment Station.
- Ebeling, R. M., and R. L. Mosher. 1996. "Red River U-Frame Lock No. 1 Backfill-Structure-Foundation Interaction." *ASCE Journal of Geotechnical Engineering* 122(3): 216–225.

- Ebeling, R. M., B. C. White, and K. Abraham. 2012. *Simplified Analysis Procedures for Flexible Approach Wall Systems Founded on Groups of Piles and Subjected to Barge Train Impact*. ERDC/ITL TR-12-3. Vicksburg, MS: U.S. Army Engineer Research and Development Center.
- Ebeling, R. M., and R. E. Wahl. 1997. *Soil-Structure-Foundation Interaction Analysis of New Roller-Compacted Concrete North Lock Wall at McAlpine Locks*. Technical Report ITL-97-5. Vicksburg, MS: U.S. Army Engineer Waterways Experiment Station.
- Ebeling, R. M., and B. C. White. 2006. *Rotational Response of Toe-Restrained Retaining Walls to Earthquake Ground Motions*. ERDC/ITL TR-06-2. Vicksburg, MS: U.S. Army Engineer Research and Development Center.
- Ebeling, R. M., B. C. White, and M. T. Fong. 2013. *Simulation and Advanced Second Moment Reliability Analyses of Pile Groups using CPGA-R; Formulation, User's Guide, and Example Problems*. ERDC/ITL TR-13-2. Vicksburg, MS: U.S. Army Engineer Research and Development Center.
- Ebeling, R. M., B. C. White, and M. T. Fong. 2018. *Analysis of a River Bank, Levee, or Floodplain I-Wall Embedded in Stratified Soil Layers During Flood Events Using Corps_I-Wall (CI-Wall) Version 2.0*. ERDC/ITL TR-18-2. Vicksburg, MS: U.S. Army Engineer Research and Development Center.
- Ellingwood, B., and H. Hwang. 1985. "Probabilistic Descriptions of Resistance of Safety Related Structures in Nuclear Plants." *Nuclear Engineering and Design* 88(2): 169–178.
- Ellingwood, B., T. V. Galambos, J. G. MacGregor, and C. A. Cornell. 1980. *Development of a Probability-Based Load Criterion for American National Standard A58 – Building Code Requirements for Minimum Design Loads in Buildings and Other Structures*. Special Publication 577. Washington, DC: National Bureau of Standards.
- Ellingwood, B., J. G. MacGregor, T. V. Galambos, and C. A. Cornell. 1982. "Probability Based Load Criteria: Load Factors and Load Combinations." *Journal of the Structural Division, ASCE* 108(ST5) Proc. Paper 17068: 978–997.
- El-Ramly, H. 2001. *Probabilistic Analysis of Landslide Hazards and Risks: Bridging Theory and Practice*. Doctor of Philosophy in Geotechnical Engineering Thesis. University of Alberta, Edmonton, Canada.
- El-Ramly, H., N. R. Morgenstern, and D. Cruden. 2002a. "Probabilistic Slope Stability Analysis for Practice." *Canadian Geotechnical Journal* 39: 665–683.
- EL-Ramly, H., N. R. Morgenstern, and D. M. Cruden. 2002b. "Probabilistic Stability Analysis of Lodalen Slide." *Proc., 55th Canadian Geotechnical Conf., Niagara Falls*, 1053–1060.
- El-Ramly, H., N. R. Morgenstern, and D. Cruden. 2003a. "Probabilistic Stability Analysis of a Tailings Dyke on Presheared Clay–Shale." *Canadian Geotechnical Journal* 40: 192–208.

- El-Ramly, H., N. R. Morgenstern, and D. Cruden. 2003b. "Probabilistic Slope Stability Analysis for Practice." *Canadian Geotechnical Journal* 40: 851–855.
- Fan, H., Q. Huand, and R. Liang. 2014. "Reliability Analysis of Piles in Spatially Varying Soils Considering Multiple Failure Modes." *Computers and Geotechnics* 57: 97–104.
- Federal Highway Administration (FHWA). 2001. *Load and Resistance Factor Design (LRFD) for Highway Bridge Substructures; Reference Manual and Participant Workbook*. Publication No. FHWA HI-98-032. NHI Course No. 132068.
- Fenton, G. A., and D. V. Griffiths. 2008. *Risk Assessment in Geotechnical Engineering*. Hoboken, NJ: John Wiley & Sons, Inc.
- Fenton, G. A., F. Naghibi, D. Dundas, R. J. Bathurst, and D. V. Griffiths. 2016. "Reliability-Based Geotechnical Design in 2014 Canadian Highway Bridge Design Code." *NRC Research Press, Canadian Geotechnical Journal* 53: 236–251.
- Fenton, G. A., and E. H. Vanmarcke. 1991. "Spatial Variation in Liquefaction Risk Assessment." *Geotechnical Engineering Congress 1991, Boulder, Colorado, 10–12 June*. Geotechnical Special Publication No. 27. Vol. 1. Reston, VA: American Society of Civil Engineers.
- Foye, K. C., R. Salgado, and B. Scott. 2006. "Assessment of Variable Uncertainties for Reliability-Based Design of Foundations." *Journal of Geotechnical and Geoenvironmental Engineering, ASCE* 132(9): 1197–1207.
- Gao, A. Z., and J. P. Li. 1993. "Reliability Analysis on Pile Bearing Capacity." *Proc. Of the Conference on Probability Methods in Geotechnical Engineering*. Canberra, 295–301.
- Hara, A., T. Ohta, M. Niwa, S. Tanaka, and T. Banno. 1974. "Shear Modulus and Shear Strength of Cohesive Soils." *Soils and Foundations* 14(3): 1–12.
- Hartman, J. P., J. J. Jaeger, J. J. Jobst, and D. K. Martin. 1989. *User's guide: Pile group analysis (CPGA)*. Technical Report ITL-89-3. Vicksburg, MS: U.S. Army Engineer Waterways Experiment Station.
- Hasofer, A. M., and N. C. Lind. 1974. "Exact and Invariant Second Moment Code Format." *Journal of the Eng. Mech. Division, ASCE* 100(EM1): 111–121.
- Headquarters, U.S. Army Corps of Engineers (HQUSACE). 1989. *Engineering and Design: Retaining and Flood Walls*. EM 1110-2-2502. Washington, DC.
- HQUSACE. 1997. *Life Cycle Design and Performance*. ER 1110-2-8159. Washington, DC.
- HQUSACE. 2005. *Stability Analysis of Concrete Structures*. EM 1110-2-2100. Washington, DC.
- HQUSACE. 2016. *Strength Design for Reinforced Concrete Hydraulic Structures*. EM 1110-2- 2104. Washington, DC.
- Hibbeler, R. C. 2014. *Mechanics of Materials*, 9th Edition. Boston, MA: Prentice Hall.

- Hicks, M. A. 2013. "An Explanation of Characteristics Values of Soil Properties in Eurocode 7." *Modern Geotechnical Design Codes of Practice; Implementation, Application and Development*. Edited by P. Arnold, G. A. Fenton, M. A. Hicks, T. Schweckendiek, and B. Simpson. Amsterdam: IOS Press,
- Hicks, M. A., and K. Samy. 2002. "Influence of Heterogeneity on Undrained Clay Slope Stability." *Quarterly Journal of Engineering Geology and Hydrology* 35: 41–49.
- Hicks, M. A., and W. A. Spencer. 2010. "Influence of Heterogeneity on the Reliability and Failure of a Long 3D Slope." *Computers and Structures* 37: 948–955.
- Honjo, Y., and K. Kuroda. 1991. "A New Look at Fluctuating Geotechnical Data for Reliability Design." *Soils and Foundations* 31: 110–120.
- Honjo, Y., and Y. Otake. 2013. "A Simple Method to Assess the Effects of Soil Spatial Variability on the Performance of a Shallow Foundation." *Foundation Engineering in the Face of Uncertainty*, Honoring Fred H. Kulhawy. ASCE Geotechnical Specialty publication no. 229. Edited by J. L. Withiam, K-K Phoon, and M. H. Hussein.
- Jaska, M. B., P. L. Brooker, and W. S. Kaggwa. 1997. "Inaccuracies Associated with Estimating Random Measurement Errors." *ASCE, Journal of Geotechnical and Geoenvironmental Engineering* 23(5): 393–401.
- Jaksa, M. B., W. S. Kaggwa, and P. I. Brooker. 2000. "Experimental Evaluation of the Scale of Fluctuation of a Stiff Clay." *8th International Conference on the Application of Statistics and Probability*. Edited by R. E. Melchers, M. G. Stewart and A. A. Sydney. Rotterdam: Balkema.
- Journal, A. G., and Ch. J. Huijbregts. 1978. *Mining Geostatistics*. London: Academic Press, Harcourt Brace & Company.
- Jumikis, A. R. 1984. *Soil Mechanics*. Malabar, FL: Robert E. Krieger Publishing Company, Inc.
- Keaveny, J. M., F. Nadim, and S. Lacasse. 1989. "Autocorrelation Functions for Offshore Geotechnical Data." In *Proceedings of the 5th International Conference on Structural Safety and Reliability, San Francisco, Calif., 7–11 August*. Edited by A.H.-S. Ang, M. Shinozuka, and G.I. Schuëller. American Society of Civil Engineers, New York, NY.
- Kerisel, J., and E. Absi. 1990. *Active and Passive Earth Pressure Tables*, 3rd edition. Rotterdam/Brookfield: A. A. Balkema.
- Kulatilake, P. H. S. W., and A. Ghosh. 1988. "An Investigation into Accuracy of Spatial Variation Estimation Using Static Cone Penetrometer Data." In *Proceedings of the 1st International Conference on Penetration Testing, Orlando, FL*, 815–821.
- Kulhawy, F. H., B. Birgisson, and M. D. Grigoriu. 1992. *Reliability-Based Foundation Design for Transmission Line Structures: Transformation Models for In Situ Tests*. Report EL-5507(4). Palo Alto, CA: Electric Power Research Institute.

- Kulhawy, F. H., and P. W. Mayne. 1990. *Manual on Estimating Soil Properties for Foundation Design*. Report EPRI-EL-6800. Palo Alto, CA: Electric Power Research Institute.
- Lacasse, S., and J. Y. de Lamballerie. 1995. "Statistical Treatment of CPT Data." In *Proceedings of the International Symposium on Cone Penetration Testing, 4–5 October, Linköping, Sweden*. Swedish Geotechnical Society.
- Lacasse, S., and F. Nadim. 1996. "Uncertainties in Characterizing Soil Properties." In *Proceedings, Uncertainty '96. American Society of Civil Engineers, Reston, VA*. Geotechnical Special Publication No. 58, Vol. 1, 49-75.
- Li, K. S. 1992. "Some Common Mistakes in Probabilistic Analysis of Slopes." *Landslides*, Vol.1. Edited by David Bell, Geology Department, University of Canterbury, Christchurch, New Zealand. In *Proceedings of the Sixth International Symposium 10-14 February, 1992, Christchurch, New Zealand*. Rotterdam, Brookfield: A. A. Balkema,
- Li, K. S., and I. K. Lee. 1991. "The Assessment of Geotechnical Safety." *Selected Topics in Geotechnical Engineering—Lumb Volume*. Edited by K. S. Li. Kensington, Australia: University of New South Wales.
- MacGregor, J. G. 1992. *Reinforced Concrete: Mechanics and Design*, 2nd Edition. Englewood Cliffs, NJ: Prentice Hall.
- MacGregor, J. G., S. A. Mirza, and B. Ellingwood. 1983. "Statistical Analysis of Resistance of Reinforced and Prestressed Concrete Members." *J. American Concrete Institute* 80(3): 167–176.
- Matheron, G. 1962. *Traite de Geostatistique Appliquee*, Vol. 1. Paris: Technip.
- Matsuo, M., and A. Asaoka. 1977. "Probability Models of Undrained Shear Strength of Marine Clay Layer." *Soils and Foundations* 17(3): 53–68.
- McCormac, J. C., and S. F. Csernak. 2012. *Structural Steel Design Fifth Edition*. Boston, MA: Prentice Hall.
- Melchers, R. E. 1999. *Structural Reliability: Analysis and Prediction*, 2nd Edition. Chichester, UK: John Wiley and Sons, Ltd.
- Meyerhof, G. G. 1976. "Bearing Capacity and Settlement of Pile Foundations." *Journal of Geotechnical Engineering, ASCE* 102(GT3): 197–227.
- Mitchell, J. K. 1976. *Fundamentals of Soil Behavior*. New York: Wiley.
- Mollon, M. G., K.-K. Phoon, D. Dias, and A.-M. Soubra. 2011. "Influence of the Scale of Fluctuation of the Friction Angle on the Face Stability of a Pressurized Tunnel in Sands, ASCE." *Georisk 2011 - Geotechnical Risk Assessment & Management*. Atlanta, GA.
- Nowak, A. S., and K. R. Collins. 2000. *Reliability of Structures*. Boston: McGraw-Hill.
- Nowak, A. S., and K. R. Collins. 2013. *Reliability of Structures*, 2nd Edition. Boca Raton, FL: CRC Press, Taylor and Francis Group.

- Nowak, A. S., and M. M. Szerszen. 2003. "Calibration of Design Codes for Buildings (ACI 318): Part 1 – Statistical Models for Resistance." *ACI Structural J.* 100: 377–382.
- Onsiphorou, C. 2000. *Stochastic Analysis of Saturated Soils Using Finite Elements*. PhD thesis. University of Manchester, UK.
- Phoon, K. K., and F. H. Kulhawy. 1999a. "Characterization of Geotechnical Variability." *Canadian Geotechnical Journal* 36: 612–624.
- Phoon, K. K., and F. H. Kulhawy. 1999b. "Evaluation of Geotechnical Property Variability." *Canadian Geotechnical Journal* 36: 625–639.
- Popescu, R., J. H. Prevost, and G. Deodatis. 1997. "Effects of Spatial Variability on Soil Liquefaction: Some Design Recommendations." *Geotechnique* 47: 1019–1036.
- Powell, G. H. 2010. *Modeling for Structural Analysis: Behavior and Basics*. Berkeley, CA: Computers and Structures, Inc.
- Rackwitz, R., and B. Fiessler. 1976. "Note on Discrete Safety Checking When Using Nonnormal Stochastic Model for Basic Variables." *Loads Project Working Session*. Cambridge, MA: MIT.
- Rackwitz, R., and B. Fiessler. 1978. "Structural Reliability under Combined Random Load Sequences." In *Computer and Structures* 9: 489–494. Oxford: Pergamon Press.
- Rankine, W. J. M. 1857. On the Stability of Loose Earth, Trans. Royal Society London, Vol. 147.
- Reese, L. C., L. A. Cooley, and N. Radhakrishnan. 1984. *Laterally Loaded Piles and Computer Program COM624G*. Technical Report K-84-2. Vicksburg, MS: U.S. Army Engineer Waterways Experiment Station.
- Rosenblatt, M. 1952. "Remarks on a Multivariate Transformation." *Annals of Mathematical Statistics* 23(3): 470–472.
- Schneider, H. R. and M. A. Schneider. 2013. "Dealing with Uncertainties in EC7 with Emphasis on Determination of Characteristic Soil Properties. In *Modern Geotechnical Design Codes of Practice; Implementation, Application and Development*. Edited by P. Arnold, G. A. Fenton, M. A. Hicks, T. Schweckendiek, and B. Simpson. Amsterdam, Netherlands: IOS Press BV under the imprint Millpress,
- Singh, V. P., S. K. Jain, and A. Tyagi. 2007. *Risk and Reliability Analysis: A Handbook for Civil and Environmental Engineers*. Reston, VA: ASCE Press.
- Sleep, M., and J. M. Duncan. 2014. *Manual for Geotechnical Engineering Reliability Calculations*, 2nd Edition. CGPR #76. Blacksburg, VA: Virginia Polytechnic Institute and State University, Center for Geotechnical Practice and Research.
- Sokolovskii, V. V. 1965. *Static of Granular Media*. New York: Pergamon Press.
- Soulie, M., P. Montes, and V. Silvestri. 1990. "Modelling Spatial Variability of Soil Parameters." *Canadian Geotechnical Journal* 27: 617–630.

- Spangler, M. G., and R. L. Handy. 1973. *Soil Engineering*, 3rd Edition. New York: Educational Publishers.
- Spencer, E. 1973. "Thrust Line Criterion in Embankment Stability Analysis." *Geotechnique* 23: 85–100.
- Tang, W. H. 1979. "Probabilistic Evaluation of Penetration Resistances." *Journal of the Geotechnical Engineering Division, ASCE* 105: 1173–1191.
- Terzaghi, K. 1943. *Theoretical Soil Mechanics*. New York: John Wiley and Sons.
- Terzaghi, K., and R. B. Peck. 1967. *Soil Mechanics in Engineering Practice*, 2nd Edition. Hoboken, NJ: Wiley.
- Tippett L. H. C. 1925. "On Extreme Individuals and a Range of Samples Taken from a Normal Distribution." *Biometrika* 17(3/4): 364–387.
- Uzielli, M., G. Vannucchi, and K. K. Phoon. 2005. "Random Field Characterization of Stress-Normalized Cone Penetration Test Parameters." *Geotechnique* 55(1): 3–20.
- Vanmarcke, E. H., and N. F. Fuleihan. 1975. "Probabilistic Prediction of Levee Settlements." In *Proceedings of the 2nd International Conference on Applications of Statistics and Probability in Soil and Structural Engineering, Aachen, Vol. 2*, 175–190.
- Vanmarcke, E. 1977a. "Probabilistic Modeling of Soil Profiles." *ASCE Journal of the Geotechnical Division* 103(GT11): 1227–1246.
- Vanmarcke, E. 1977b. "Reliability of Earth Slopes." *ASCE Journal of the Geotechnical Division* 103(GT11): 1247–1265.
- Vanmarcke, E. 1980. "Probabilistic Stability Analysis of Earth Slopes." *Engineering Geology* Vol. 16, Special Issue Mechanics of Landslides and Slope Stability: 29–50.
- Vanmarcke, E. 1983. *Random Fields: Analysis and Synthesis*. Cambridge, MA: MIT Press.
- Vanmarcke, E. 2010. *Random Fields: Analysis and Synthesis*. Revised and expanded newer edition. Singapore: World Scientific Publishing Co.
- Vanmarcke, E. 2011. "Risk of Limit-Equilibrium Failure of Long Earth Slopes: How It Depends on Length." GeoRisk 2011, keynote lecture. In *Proceedings of GeoRisk International conference, Atlanta, GA, June 2011*. ASCE Geotechnical Specialty publication no. 224, June 2011, 1–29.
- Vanmarcke, E., and Y. Otsubo. 2013. "Reliability Analysis of Long Multi-Segment Earth Slopes." *Foundation Engineering in the Face of Uncertainty*. Honoring Fred H. Kulhawy. ASCE Geotechnical Specialty Publication No. 229. Edited by J. L. Withiam, K.-K. Phoon, and M. H. Hussein, 520–529. Reston, VA.

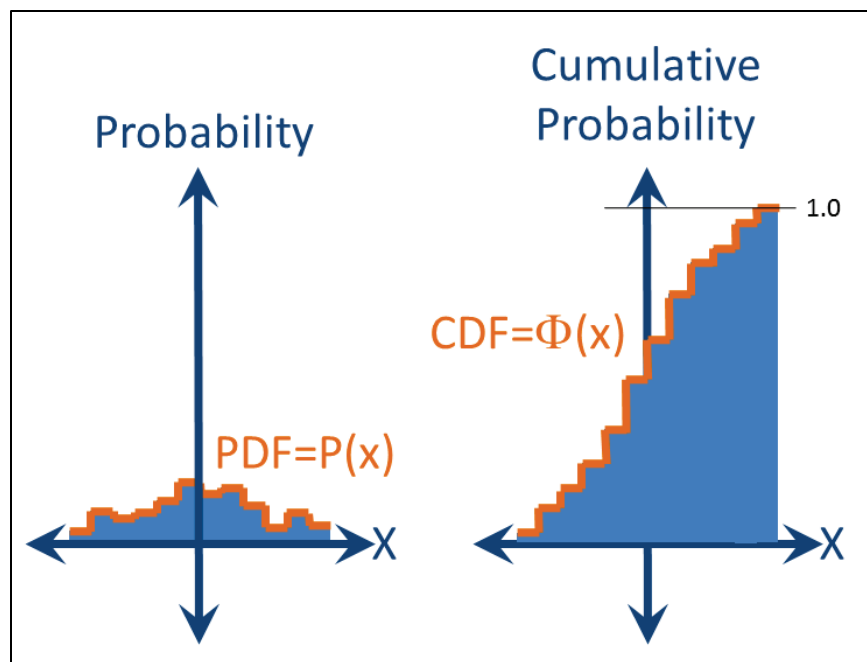
- Wickremesinghe, D., and R. G. Campanella. 1993. "Scale of Fluctuation as a Descriptor of Soil Variability." *Probabilistic Methods in Geotechnical Engineering*. Edited by K. S. Li and S. C. R. Lo. Rotterdam: Balkema.
- Withiam, J. L., E. P. Voytko, R. M. Barker, J. M. Duncan, B. C. Kelly, S. C. Musser, and V. Elias. 1997. *Load and Resistance Design (LRFD) for Highway Bridge Structures*. FHWA Report DTFH61-94-C-00098. Washington, DC: Federal Highway Administration.
- Wu, T. H. 1974. "Uncertainty, Safety, and Decision in Soil Engineering." *Journal of the Geotechnical Engineering Division, ASCE* 100: 329–348.
- Yang, B., and J. M. Duncan. 2002. *Retaining Wall Stability 2.05 – An Excel Workbook and Documentation*. Blacksburg, VA: Virginia Polytechnic Institute and State University, Center for Geotechnical Practice and Research, CGPR.

Appendix A: Computing Probabilities

A.1 The Gaussian multivariate spaces

While design and analysis procedures of structures are well defined in theory, in reality the constituent parameters have many variations, which is the reason for LRFD. Material properties (e.g., compressibility, x position) are typically chosen from a range of values that are pulled from a distribution with certain probabilities. An example distribution may be a discrete histogram as shown in Figure A-1. Each bin in the histogram presents how many times that bin has been selected from a sample data set. Dividing those values by the total number of samples drawn gives a normalized distribution, where the probability of selection of any given bin (PDF) is a fraction of the total set and integrating the area under the PDF gives the CDF. For the entire curve, the total area is 1.

Figure A-1. PDF and CDF of histogram data.



A Gaussian (also referred to as Normal) distribution is given by the equation

$$f(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (\text{A.1})$$

This function provides numerous advantages when dealing with multiple orthogonal variables in LRFD, so a means to create a Gaussian distribution that approximates other distributions was established by Hasofer-Lind (1974) based on a distribution transform method by Rakowicz and Fiessler (1976, 1978). The Hasofer-Lind procedure does not find the simple mean and standard deviation of another distribution but instead maps the CDF of the non-normal distribution to the CDF of a normal distribution to get the best approximate Gaussian distribution with its own mean and standard deviation (Figures A-2 and A-3).

Figure A-2. Histogram data and the Gaussian (normal) distribution.

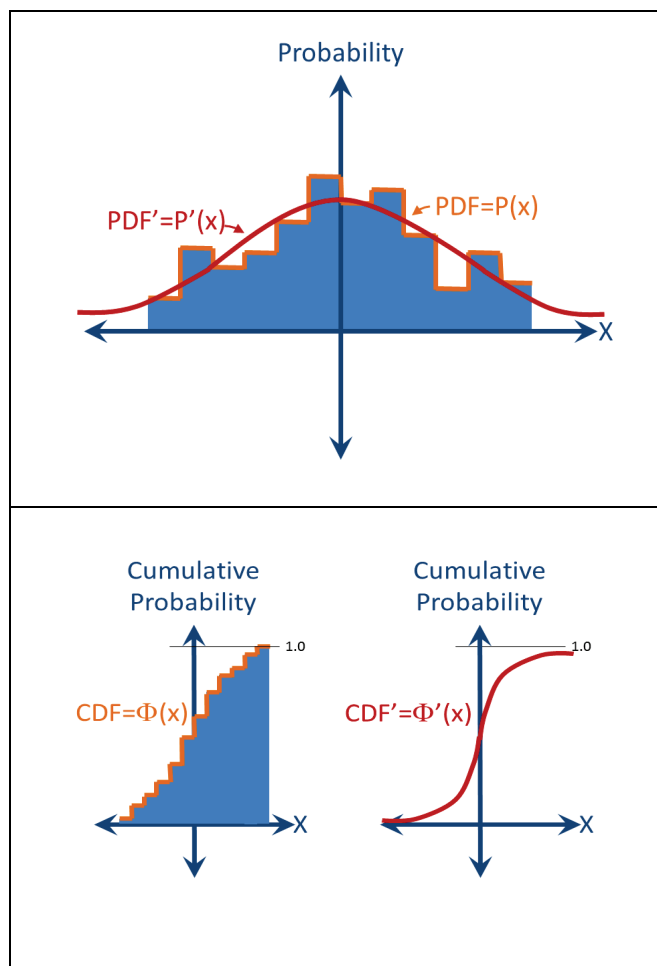
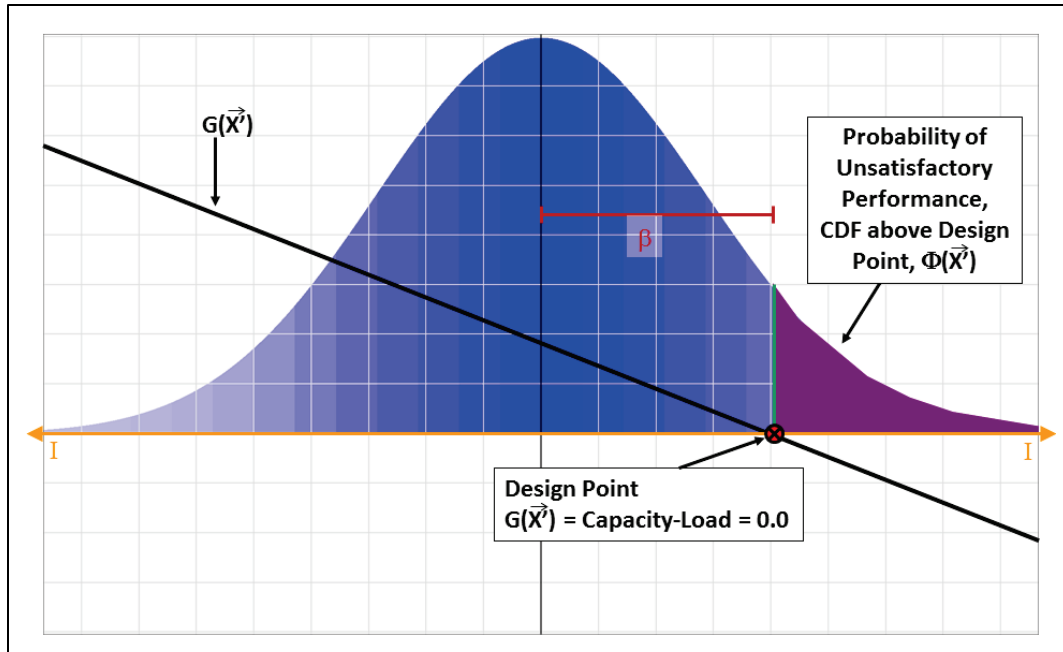


Figure A-3. The $G(\vec{X})$ function is used to determine PUP.

With N uncorrelated and orthogonal variables, an N -dimensional multivariate space can be formed from the distributions for those variables. With Gaussian distributions, a transformed space can be created that has a unit distance of 1σ . The following equation is used to transform the X value of i^{th} variable of the distribution to the transformed coordinates:

$$X'_i = \frac{(X_i - \mu_i)}{\sigma_i} \quad (\text{A.2})$$

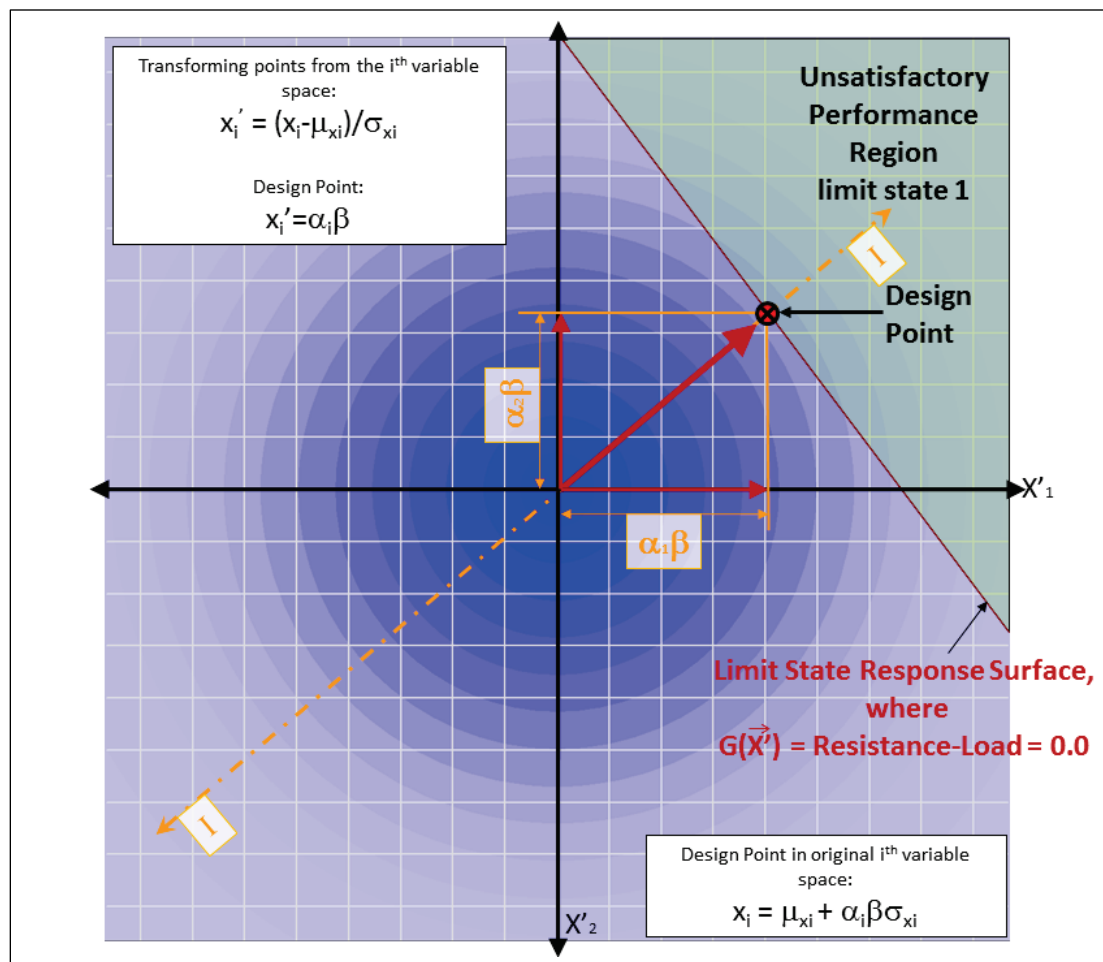
and conversely,

$$X_i = X'_i \sigma_i + \mu_i \quad (\text{A.3})$$

A hyperplane can partition the multivariate Gaussian transformed space into half-spaces. The N -space hyperplane has the equation

$$0.0 = a_1 * X_1 + a_2 * X_2 + \dots + a_N * X_N + c \quad (\text{A.4})$$

This hyperplane can also be described by a vector from the origin of the multivariate space to the nearest point on the hyperplane. This vector is perpendicular to the surface of the hyperplane. The vector has length β and has unit direction component vector α , with N^{th} axis component of α_N (Figure A-4).

Figure A-4. A 2D example of the *PUP* region and the hyperplane description using α and β .

This multivariate, Gaussian transformed space has several properties that make it attractive for determining limit state probabilities, such as is done for reliability analysis and LRFD. The most important property is that, for any hyperplane in the N -dimensional space partitions the space into two half-spaces, the integrated volume of the half-space gives the same result as the $CDF(\beta)$ for a normal distribution along the hyperplane's defining vector.

A hyperplane perfectly describes the response surface of a limit state that has linear effects across multiple variable axes. For the purposes of engineering evaluation, it is assumed that each limit state has linear properties and that failure occurs because of non-linear, or plastic, response.

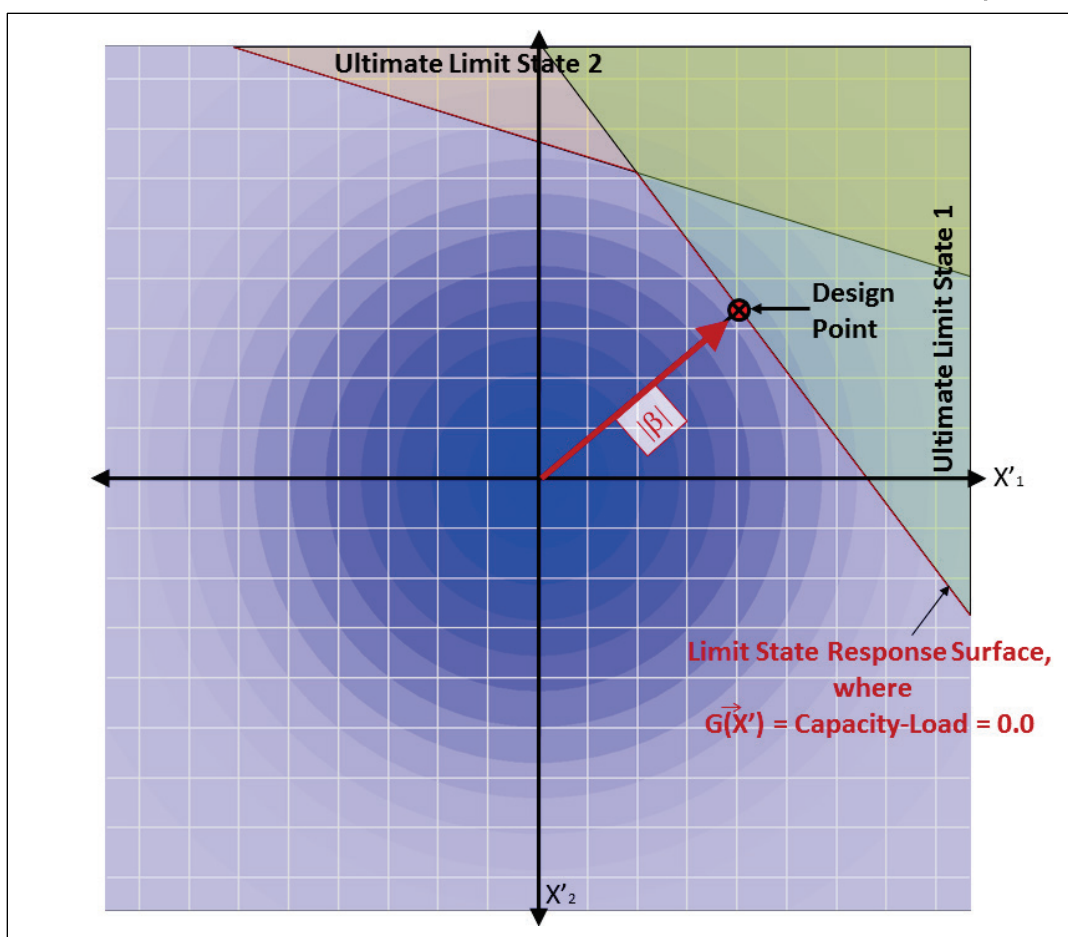
A.2 Limit states and *PUP*

For engineering analysis with properties that vary, described by multivariate space, a particular limit state (e.g., foundation sliding) can be described by the multivariate function

$$g(\mathbf{X}) = \text{Capacity} - \text{Load} \quad (\text{A.5})$$

This function has positive values for satisfactory performance, negative values for unsatisfactory performance, and a value of 0.0 at the response surface, which separates the performance regions (Figure A-5). For probabilities, as discussed above, the response surface is assumed to be hyperplanar.

Figure A-5. A 2D example of multiple limit states with closest approach of length $|\beta|$.



An indicator function can be created with the express purpose of providing a multiplicative identity for $\mathbf{X}$ vector values where there is unsatisfactory performance, and a value of 0.0 at the response surface of the limit state or less.

$$i(\mathbf{X}) = \begin{cases} 0: g(\mathbf{X}) \geq 0.0 \\ 1: g(\mathbf{X}) < 0.0 \end{cases} \quad (\text{A.6})$$

The function to calculate PUP for the Gaussian multivariate space $\mathbf{X}$ is

$$p_U = \Phi(\mathbf{X}) = \int_{\mathbb{R}^N} i(\mathbf{X}) h(\mathbf{X}) d\mathbf{X} \quad (\text{A.7})$$

where $h(\mathbf{X})$ is the probability density function for the space and is given by

$$h(\mathbf{X}) = \prod_{n=1}^N h(X_n) \quad (\text{A.8})$$

As noted before, for a single hyperplanar surface in a Gaussian multivariate space, the PUP is the same as the 1-D CDF along the line through the origin that is perpendicular to the plane. For line $l = \alpha\beta$, where α is the directional cosine vector, $p_U = \Phi(\beta)$, with $\Phi(\beta)$ being the CDF for the normal Gaussian distribution.

Unfortunately, it is seldom that only one limit state is possible with engineering structures. In this case, a family of limit states needs to be assessed, with each limit state having a different $g(\mathbf{X})$ function. In this case, combining the limit state functions gives a new, nonlinear, overall response surface, described by the boundary of the union of the half-space volumes for each limit state response surface hyperplane. For L limit states

$$p_U = \bigcup_{n=1}^L \Phi_n(\mathbf{X}) \quad (\text{A.9})$$

It is important to realize that these half-space sets of data are guaranteed to intersect and overlap if the hyperplane equations are not parallel and may subsume each other if the equations are parallel. This makes it difficult to get an accurate determination of the PUP for multiple limit states.

A.3 Estimating *PUP* for multiple limit states

A.3.1 Bounds method

Ang and Tang (1975) suggested an alternative method for estimating the *PUP* with multiple limit states based on finding the minimum and maximum bounds for the value that the *PUP* can assume. This estimation is not as accurate as the numerical Euclidean methods but can be found quickly with methods to determine a design point for each limit state and therefore the equation for the hyperplane and the β distance to the hyperplane, resulting in the *PUP*.

$$PUP_{LSi} = \Phi(\beta_i) \quad (A.10)$$

Each limit state hyperplane divides the Gaussian multivariate space into half-spaces, with one half-space being the satisfactory performance region and the other half-space being the unsatisfactory performance region for that limit state. When there are multiple limit states, these regions in the Gaussian multivariate space overlap and combine. The probability of these combined regions having unsatisfactory performance is the probability of failure given the union of all the regions.

$$PUP \cong p_u[R_1 \cup R_2 \cup \dots \cup R_N] \quad (A.11)$$

However, as it has been previously described, the union of the regions contains the overlapping volumes of unsatisfactory performance probabilities. This complicates the calculation of the overall *PUP*.

Because each limit state is described by a hyperplane with shortest distance β_I from the origin, the lowest bound of the overall *PUP* can be proven to be the greatest of the individual hyperplane *PUPs* of the limit states. This is true because, for a set of limit states with parallel hyperplanes, the greatest individual *PUP* subsumes the volumes of the other individual *PUPs*.

$$\max_i (\Phi(\beta_i)) \leq PUP \quad (A.12)$$

Parallel hyperplanes can have non-intersecting half-planes of unsatisfactory performance. For two such hyperplanes that do not overlap, the probability of satisfactory performance is the multiplied probabilities of satisfactory performance for each limit state. This implies that an upper

limit to the *PUP* can be found by subtracting the combined probability of satisfactory performance for multiple limit states from 1.0.

$$PUP \leq 1 - \prod_i (1 - \Phi(\beta_i)) \quad (A.13)$$

Equations A.13 and A.14 establish absolute lower and upper bounds for the *PUP*, but these bounds are for limit states that are not correlated (i.e., have parallel hyperplanes in unimodal sets). In reality, the hyperplanes are seldom parallel and have complex overlapping regions. Therefore, correlation techniques can be used between pairs of hyperplanes (bimodal) to tighten the bounds about the *PUP*. With more than two hyperplanes (multimodal), the overlapping regions become much more complicated.

According to a decomposition of the union of sets using deMorgan's rule as described in Ang and Tang (1975), the bounding values for *PUP* using bimodal probabilities is

$$PUP \geq \max[\sum_i \{PUP_{LSi} - \sum_{j=1}^{i-1} P(E_i E_j)\}; 0.0] \quad (A.14)$$

and

$$PUP \leq \sum_{i=1}^N PUP_{LSi} - \sum_{i=2}^N \max_{j < i} (P(E_i E_j)) \quad (A.15)$$

Given hyperplane equations for the *i*th and *j*th limit state equations (in two dimensions for simplicity):

$$g_i(\mathbf{X}) = a_0 + a_1 X_1 + a_2 X_2 \quad (A.16)$$

$$g_j(\mathbf{X}) = b_0 + b_1 X_1 + b_2 X_2 \quad (A.17)$$

The limit state response surface is determined by setting these equations to 0.0. According to Ang and Tang (1975) in Gaussian multivariate space, the correlation coefficient can be found as the cosine of the dihedral angle (θ) between these hyperplanes.

$$\rho_{ij} = \frac{Cov(g_i, g_j)}{\sigma_{g_i} \sigma_{g_j}} = \cos \theta \quad (A.18)$$

The dihedral angle between the planes is equivalent to the angle between the vectors from the origin to the respective design points because each

hyperplane is perpendicular to its design point vector. The cosine of this angle can be expressed, from the normalized design point vectors α_i and α_j , as

$$\rho_{ij} = \cos\theta = \alpha_i \cdot \alpha_j \quad (\text{A.19})$$

From Figures A-6 and A-7, it is observed that $E_i E_j \supset A$ and $E_i E_j \supset B$. The probability of $E_i E_j$ is therefore bounded by

$$\max[P(A), P(B)] \leq P(E_i E_j) \leq P(A) + P(B) \quad (\text{A.20})$$

and, by reason of orthogonality,

$$P(A) = \Phi(-\beta_i)\Phi(-a) = \Phi(-\beta_i)\Phi\left(-\frac{\beta_j - \rho\beta_i}{\sqrt{1-\rho^2}}\right) \quad (\text{A.21})$$

and

$$P(B) = \Phi(-\beta_j)\Phi(-b) = \Phi(-\beta_j)\Phi\left(-\frac{\beta_i - \rho\beta_j}{\sqrt{1-\rho^2}}\right) \quad (\text{A.22})$$

Using the relationships of Equations A.21 and A.22, the lower bound for $P(E_i E_j)$, given as

$$P(E_i E_j) = P(A) + P(B) \quad (\text{A.23})$$

should be used in Equation A.14 for each pair of limit states, and the upper bound, given as

$$P(E_i E_j) = \max[P(A), P(B)] \quad (\text{A.24})$$

should be used in Equation A.15 for each pair of limit states.

Figure A-6. Finding the contribution of area A to overlapping probabilities of $g(X)$ and $g(X)$.

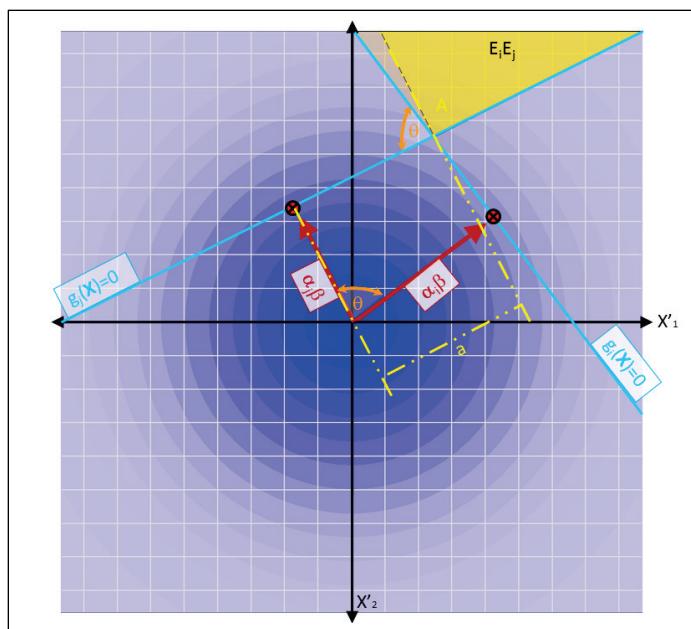
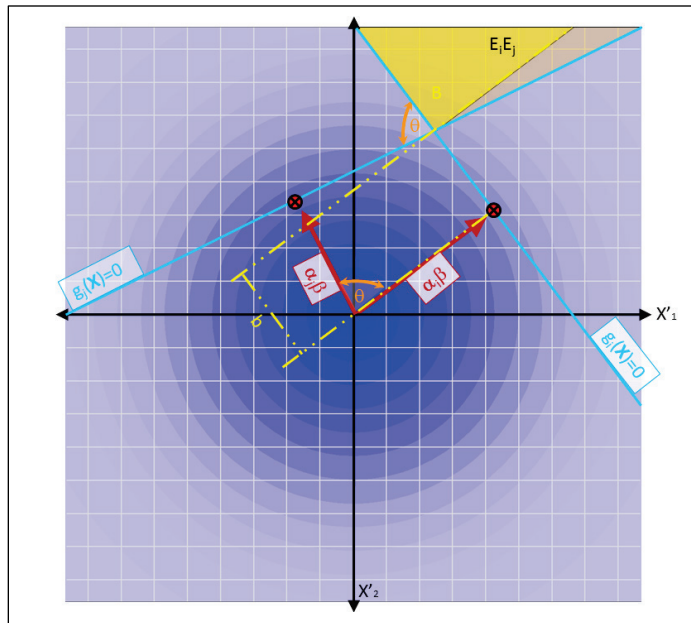


Figure A-7. Finding the contribution of area A to overlapping probabilities of $g(X)$ and $g(X)$.



The results from the application of these equations is a minimum and maximum range for the overall *PUP* given multiple limit states using a bimodal analysis, using the bounds of each bimodal probabilities. The *PUP* is guaranteed to be in these ranges, so a reasonable estimate of the actual

value of the *PUP* is the average of the two bounding values, giving a maximum error of half the range.

A.3.2 Euclidean numerical method

For multiple limit states, it is still possible to integrate the probabilities the spaces using Equation A.7, but there needs to be an adjustment to the indicator function specified in Equation A.6. That indicator function is defined for a single limit state function $g(\mathbf{X})$. To work with Z multiple limit states, the overall indicator function must return the multiplicative identity if any of the Z single limit state indicator functions returns with a value of 1.

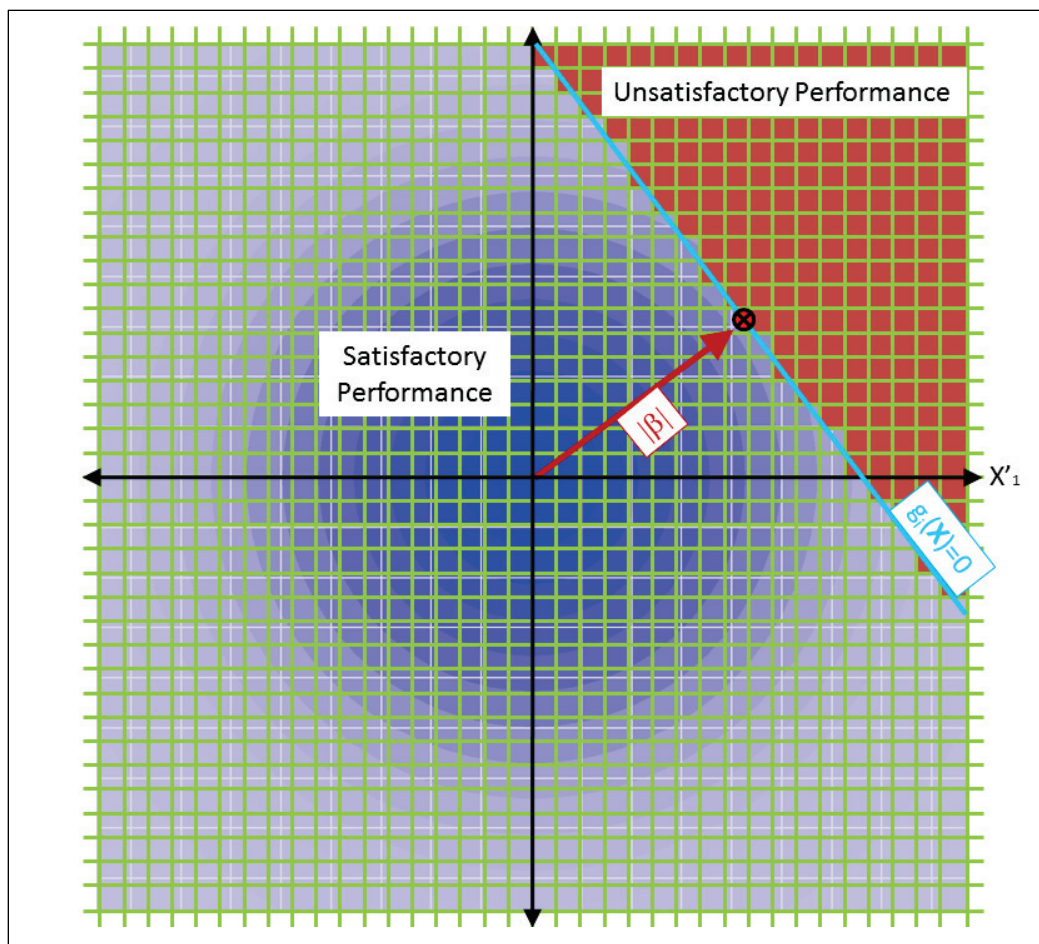
$$i_Z(\mathbf{X}) = 1 - \prod_{z=1}^Z (1 - i_z(\mathbf{X})) \quad (\text{A.25})$$

$$PUP = \Phi(\mathbf{X}) = \int_{X_1} \int_{X_2} \dots \int_{X_N} i_Z(X_1, X_2, \dots, X_N) h(X_1, X_2, \dots, X_N) dX_1 dX_2 \dots dX_N \quad (\text{A.26})$$

It would be possible to approximate the value of the *PUP* using a Euclidean method. The Midpoint Rule Euclidean method is the one most people remember from early calculus courses, but there are other Euclidean methods (i.e., the Trapezoid Rule and Simpson's Rule) that could give a better estimation and require fewer sub-intervals.

Of course, the Euclidean methods require that the space be divided into a number of segments for each dimension (Figure A-8). Because the Gaussian multivariate space is orthogonal and normalized, the same number of intervals should be applied to each dimension. The greatest rate of change of the function occurs near the origin, so for efficiency, the subdivisions can be placed closer together at the origin than at the extremes. However, since most of the limit states are applied near the tails of the distributions, and therefore at extreme distances ($>3.0\sigma$), care should be taken to not spread the subdivisions too far so that accuracy and precision will be maintained. Because the probabilities become extreme at distances of greater than 3σ , the numerical solution should be satisfactory in the range of -5σ to 5σ for each variable.

Figure A-8. Gridding the Gaussian multivariate space to determine the *PUP* based on a single limit state response surface.



An issue with using the Euclidean approximations is the number of analyses that will need to be run. To evaluate an individual subdivision point, each of the limit state indicator functions will need to be found. If the analysis function is written correctly, these values could be collected on a single analysis execution. However, for a number of sub-divisions (S) and a number of variables (N), the total number of runs that will need to be performed is N^S . The computational complexity of this method means that it will rarely be tractable for anything less than a high-performance computational platform. Another issue with the numerical integration is that the combined PDF for multiple variables, which are multiplied together, make numbers so small that they outstrip the precision of even double-precision floating point values in programming languages.

A.3.3 Hypersphere numerical method

So far, all of the methods explored have been based on hyperplanes because there is a simple method of determining the CDF of half-spaces divided by the hyperplane in the Gaussian multivariate space. There is another property of the Gaussian multivariate space that seems to be overlooked. For any hypersphere (also known as an n -sphere) in the space that is centered at the origin and has radius R , all of the points on the surface of the hypersphere have the same probability density function value, as shown in Figures A-9 and A-10.

$$p(\mathbf{X}_R) = p(R_{X1}) \times p(R_{X2}) \times \dots \times p(R_{XN}) = p(0)^{N-1}p(R) \quad (\text{A.27})$$

Figure A-9. Because of symmetry, the probability for any thin shell is the same at a set distance from the origin.

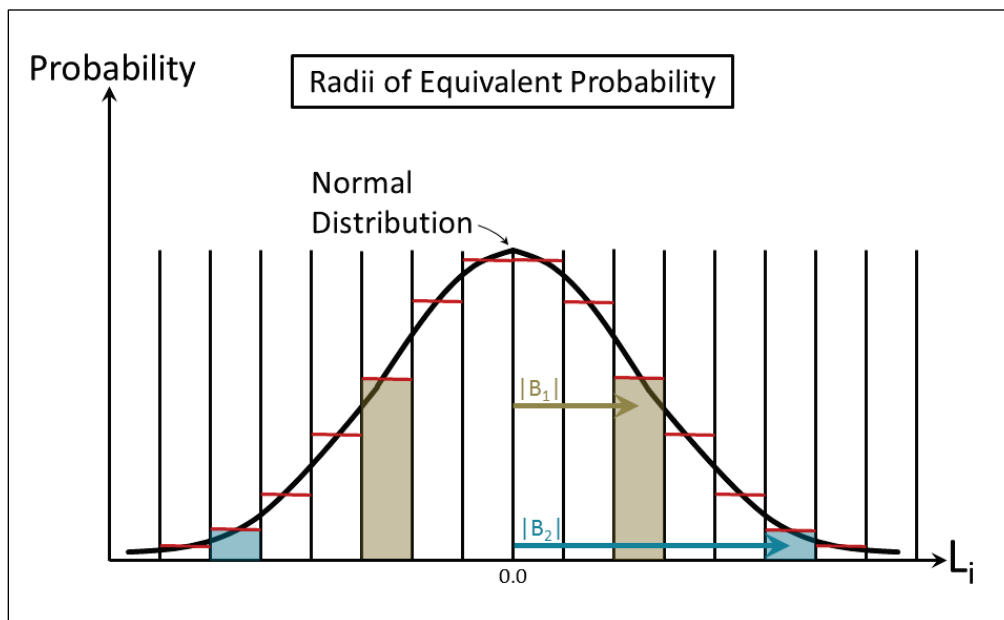
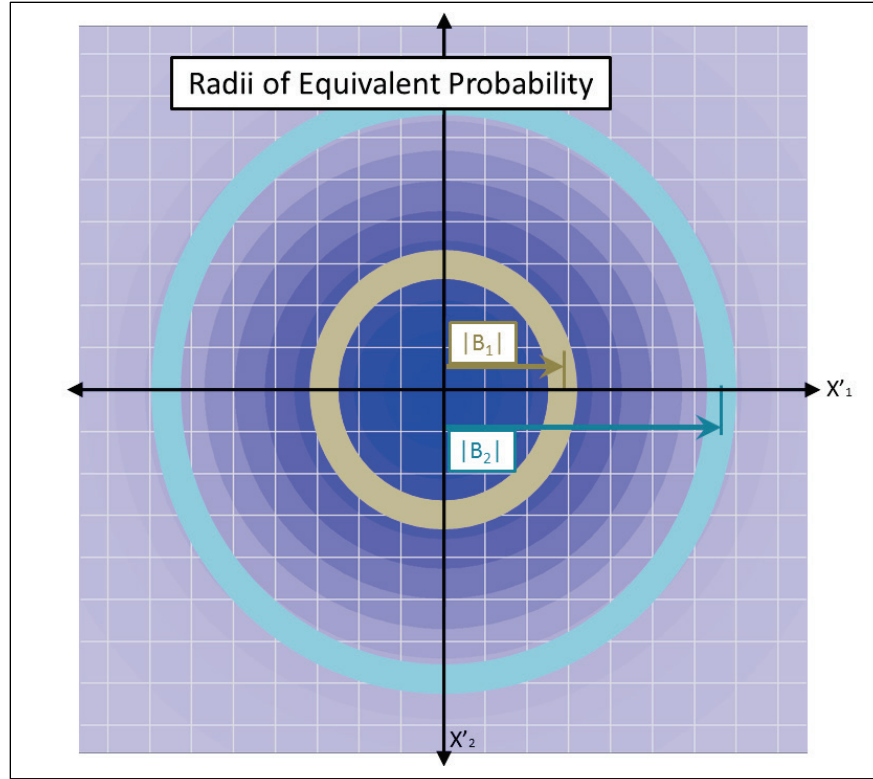


Figure A-10. Symmetry and the properties of the Gaussian Normal curve means that the probability at any distance in the multivariate space is the same at all points on the hypersphere surface.



Given that the closed form equation of the surface area for the unit hypersphere with dimension N is

$$S_N = \begin{cases} \frac{2^{\frac{N+1}{2}} \pi^{\frac{N-1}{2}}}{(N-2)!!} & \text{for odd } N, \text{ and} \\ \frac{2\pi^{\frac{N}{2}}}{(\frac{1}{2}N-1)!} & \text{for even } N \end{cases} \quad (\text{A.28})$$

Using concentric shells, the probability of satisfactory performance of the N -dimensional hypersphere (h) with radius R in Gaussian multivariate space (Figure A-11) is

$$P_h(R) = \int_{r=0}^R S_N r^2 p(0)^{N-1} p(r) dr \quad (\text{A.29})$$

which seems much more tractable for a numerical solution, the factorial divisors in Equation A.27 can grow quite large as the number of dimensions increases. For a *PUP* given hyperplanar limit states and β radius, it follows that

$$PUP \leq (1 - P_h(\beta)) \quad (\text{A.30})$$

Unfortunately, this approach overestimates the *PUP*, and the error increases as the radius β increases. The error increases because the hyperplane *PUP* decreases at such a rate that any error is exacerbated at the tails of the distribution. Also, the accuracy of the hypersphere *PUP* is greatly influenced by the resolution of numerical procedure to determine its values, with greater error occurring at greater β values. Of course, higher resolution numerical solutions also become intractable.

Table A-1 shows a quick calculation of error for a two-variable space, with hyperplanar *PUP* calculations and the numerical approximation of the integral of the hypersphere with different resolution step sizes. Recall that for a two-variable space, the hyperplane is a line and the hypersphere is a circle. Table A-2 shows the hypersphere estimation error as a multiple of the actual linear *PUP* value. Notice that as the hypersphere b increases, the error becomes greater. This is because the dropoff of the hyperplanar limit state CDF ($\Phi(1-\beta)$) drops off more rapidly than the CDF of the area outside the circular area in the opposite half-plane.

Figure A-11. A 2-D demonstration of the hypersphere calculation of PUP.

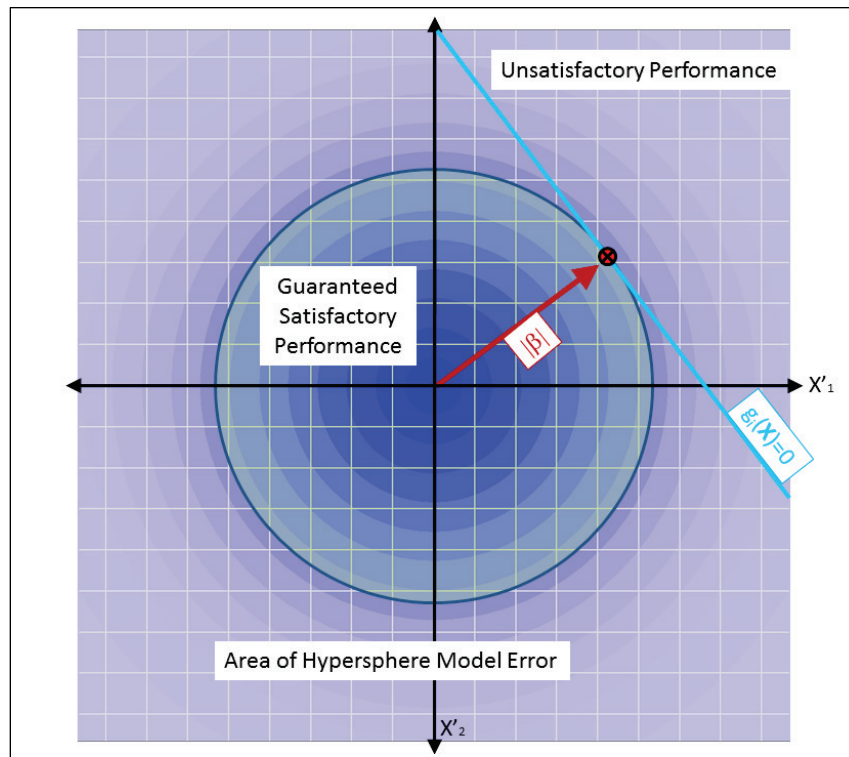


Table A-1. PUP values for hyperplanar limit state and hypersphere *PUP* estimations with increasing accuracy.

β	Hyperplanar Linear <i>PUP</i> (<i>IPUP</i>)	Hypersphere <i>PUP</i> (<i>hPUP</i>)	Hypersphere <i>PUP</i> (<i>hPUP</i>)	Hypersphere <i>PUP</i> (<i>hPUP</i>)	Hypersphere <i>PUP</i> (<i>hPUP</i>)
		(0.001 interval)	(0.0001 interval)	(0.00001 interval)	(0.000001 interval)
0.5	0.30853754	0.88207477	0.88245471	0.8824971	0.88249648
1	0.15865525	0.60604896	0.60648246	0.60653191	0.60653018
1.5	0.0668072	0.32446544	0.32463372	0.32465059	0.32465228
2	0.02275013	0.1355278	0.1353545	0.13533991	0.13533548
2.5	0.00620967	0.04439116	0.04398235	0.04394257	0.04393739
3	0.0013499	0.01168391	0.01116649	0.01111475	0.01110957
3.5	0.00023263	0.00280223	0.00224898	0.00219364	0.00218811
4	0.00003167	0.00095991	0.00039792	0.00034172	0.00033609
4.5	0.0000034	0.00066628	0.0001027	0.00004633	0.00004069
5	0.00000029	0.00063019	0.00006639	0.00000999	0.00000435

Table A-2. Hypersphere *PUP* estimate error as a multiple of the actual hyperplanar *PUP*.

β	hPUP0.001/IPUP	hPUP0.0001/IPUP	hPUP0.00001/IPUP	hPUP0.000001/IPUP
0.5	2.858889618	2.860121041	2.860258431	2.860256421
1	3.81991116	3.822643499	3.822955181	3.822944277
1.5	4.856743585	4.859262475	4.859514992	4.859540289
2	5.957231893	5.949614354	5.948973039	5.948778315
2.5	7.148714827	7.082880411	7.076474273	7.07564009
3	8.655389288	8.272086821	8.233758056	8.229920735
3.5	12.04586683	9.667626703	9.429738211	9.405966556
4	30.30975687	12.56457215	10.7900221	10.61225134
4.5	195.9647059	30.20588235	13.62647059	11.96764706
5	2173.068966	228.9310345	34.44827586	15

A.3.4 Hybrid hyperspherical-polar numerical method

The hypersphere numerical method works quickly to determine the probabilities of values occurring within a hypersphere of a defined radius and can be run to generate tables for many different radii hyperspheres for quick lookup. This method could be used to generate a quick estimate of

the value to the closest design point for the most likely to occur limit state but will result in a large error region.

The authors of this report have looked at a Euclidean numerical method in Cartesian space for the determination of probabilities with respect to multiple limit states. This concept can be expanded to a polar coordinate space, using

$$i_z(r, \theta) = 1 - \prod_{z=1}^Z (1 - i_z(r, \theta)) \quad (\text{A.31})$$

$$PUP = \Phi(r, \theta) = \int_{\theta} \int_{r=0}^{\infty} i_z(r, \theta_1, \theta_2, \dots, \theta_N) h(r, \theta_1, \theta_2, \dots, \theta_N) dr d\theta \quad (\text{A.32})$$

A numerical solution could transform the polar coordinates to Cartesian coordinates for simplification of probability calculation. This means that the $i_z()$ function can be determined from the Cartesian equation given in Equation A.25. The probability function $h()$ is the same as Equation A.8. The rest of the equation deals with the approximation of volume occupied by a polar coordinate described shell element of dr thickness and all of its sweep angles. Because dimensionless angles are used for these sweep angles in multiple dimensions and the angles interact within polar coordinates, the definition of $d\theta$ needs to be re-derived for each number of dimensions. The following two equations show the polar integration equations for two and three dimensions, respectively:

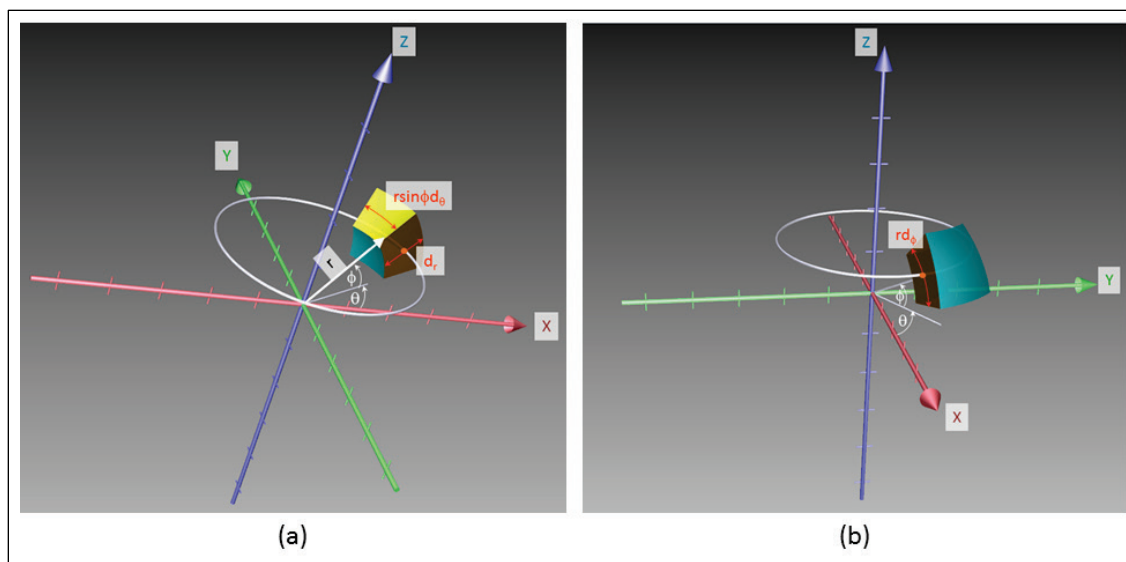
$$PUP = \Phi(r, \theta) = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} i_z(r, \theta) h(r, \theta) dr (rd\theta) \quad (\text{A.32a})$$

$$PUP = \Phi(r, \theta_1, \theta_2) = \int_{\theta_2=0}^{2\pi} \int_{\theta_1=0}^{\pi} \int_{r=0}^{\infty} i_z(r, \theta_1, \theta_2) h(r, \theta_1, \theta_2) dr (rd\theta_1) (r \sin(d\theta_1) d\theta_2) \quad (\text{A.32b})$$

The length term r needs to be multiplied by each angular delta to express the shell element arc-distance for that angular sweep. For the 3-D Equation A.32b, the total sweep of angle θ_1 is reduced to make the functions one-to-one, much like latitude and longitude are described on a sphere. Figure A-12 (a),(b) show why the reduction of distance for the $d\theta_2$ term is required for polar coordinates. These figures use more conventional notation for 3-D axes and angles with X , Y , and Z axes representing X_1 , X_2 , and X_3 variable space axes, respectively, and ϕ and θ representing the θ_1 and θ_2 variable space angles, respectively. The equation

derivations for Equation A.32 become more involved as more variable dimensions are included.

Figure A-12. Calculating dimensions for integrating spherical coordinate 3-D space.



A full numerical solution along these lines would have the same issues of exponential time growth as the Euclidean method. Recall that the numerical solution could be considered complete when r equals 5σ because the probabilities are extremely small past that limit.

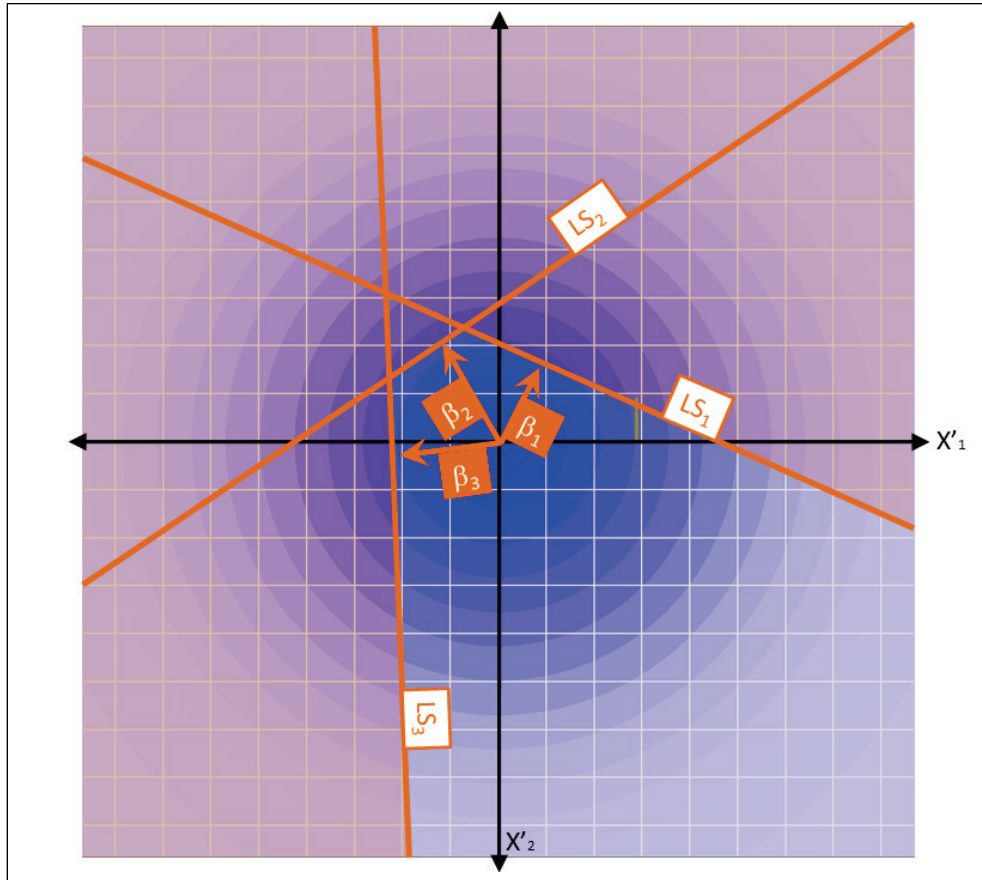
It is certainly possible to minimize calculation time by marrying the hypersphere calculation method with the polar coordinate method for calculating the probabilities of multiple limit states in the Gaussian multivariate space. Given the design point for the highest probability limit state defined by $\beta\alpha$, the probability of satisfactory performance would be the hypersphere probability of radius β combined with 1 minus the polar PUP with r ranging from β to 5, thus limiting the polar search to a somewhat manageable area. Of course the PUP is 1 minus the probability of satisfactory performance.

A.4 Using an estimation of PUP to establish LRFD factors

Because the structural probability of usability is based on the lifetime of the structure, the PUP can be used to establish the design limits for these structures. Under applied load cases, the design points for several limit states can be determined as a distance (in standard deviations) and direction from the mean characteristics of the structure, $\beta\alpha$. An example

of this is shown in Figure A-13. The expected overall *PUP* value can be set to a limit (e.g., $\Phi(3.5)$) which guarantees the structure's lifetime integrity probability.

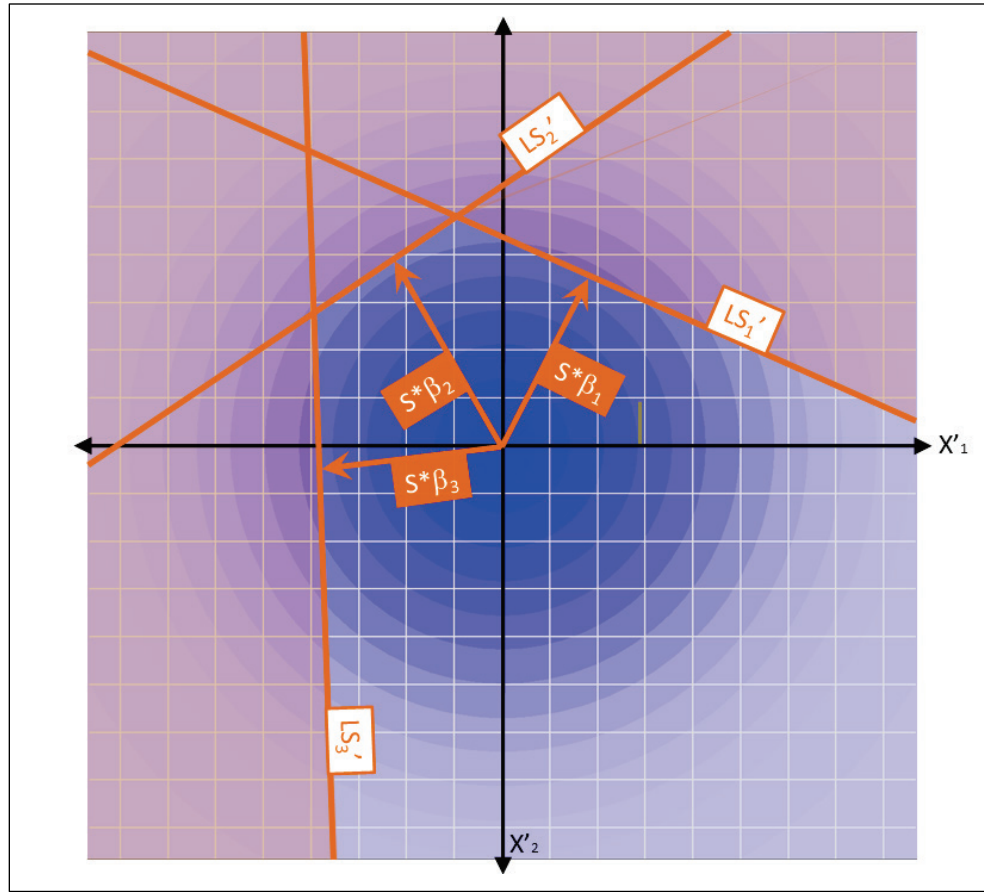
Figure A-13. Example limit states and β s for unfactored loads and resistances.



There are two ways that a scale factor can be applied to the structure for structural design under a load. The first method assumes that the load and response directions do not change. In this case, each variable axis has the same universal scale applied (Figure A-14). Because the factor is applied uniformly, the limit states maintain their relative positions and angles to each other in the Gaussian multivariate space. Once the limit states have been determined for the unscaled design load, the system *PUP* can be considered to be a function of the single applied scale factor (*S*) to the system.

$$PUP = f(S) \quad (A.33)$$

Figure A-14. A uniform scale factor (S) is applied to the β s for the limit state design points.



For this equation, $f(S)$ is a result of one of the multivariate, multi-limit state estimation techniques discussed in the previous section. The purpose is to make the calculated *PUP* approximately equivalent to the established limit (e.g., $\Phi(3.5)$). Each limit state response surface is described by

$$\beta_i \alpha_{i-original} = (\beta_{i-original} \times S) \alpha_{i-original} \quad (A.34)$$

Since the *PUP* can be described by a function of a single variable (S), a Newtonian or secant-based numerical method can be used to adjust the design points (by changing the scale) until the solution approaches the limit set for the overall system *PUP*, in an efficient way.

The second method to apply scale factors is more complicated but can give a better overall design due to the fact that all of the variables are considered independently. To balance the design to the load and intended probability of failure, the initial attempt at the design must find the hyperplanar definition vectors for all of the limit states, $\beta_{i-original} \alpha_{i-original}$

(Figure A-13). When these hyperplanar descriptions are discovered, factors can be applied to the minimum β value limit state that extends the distance from the origin in the Gaussian multivariate space to the limit state hyperplane, thus lowering the *PUP* for that limit state. However, this factor affects all of the limit states to varying degrees as each variable axis is extended to varying degrees by the minimum limit state's directional cosines (e.g., $\alpha_{i-original\_X1}$). This is shown in Figure A-15. In this case, Equation A.34 is interpreted in a different way:

$$\beta_{i-original} \alpha_i = \beta_{i-original} (\alpha_{i-original} \times S) \quad (A.35)$$

In, this case, each variable axis has its own scale factor

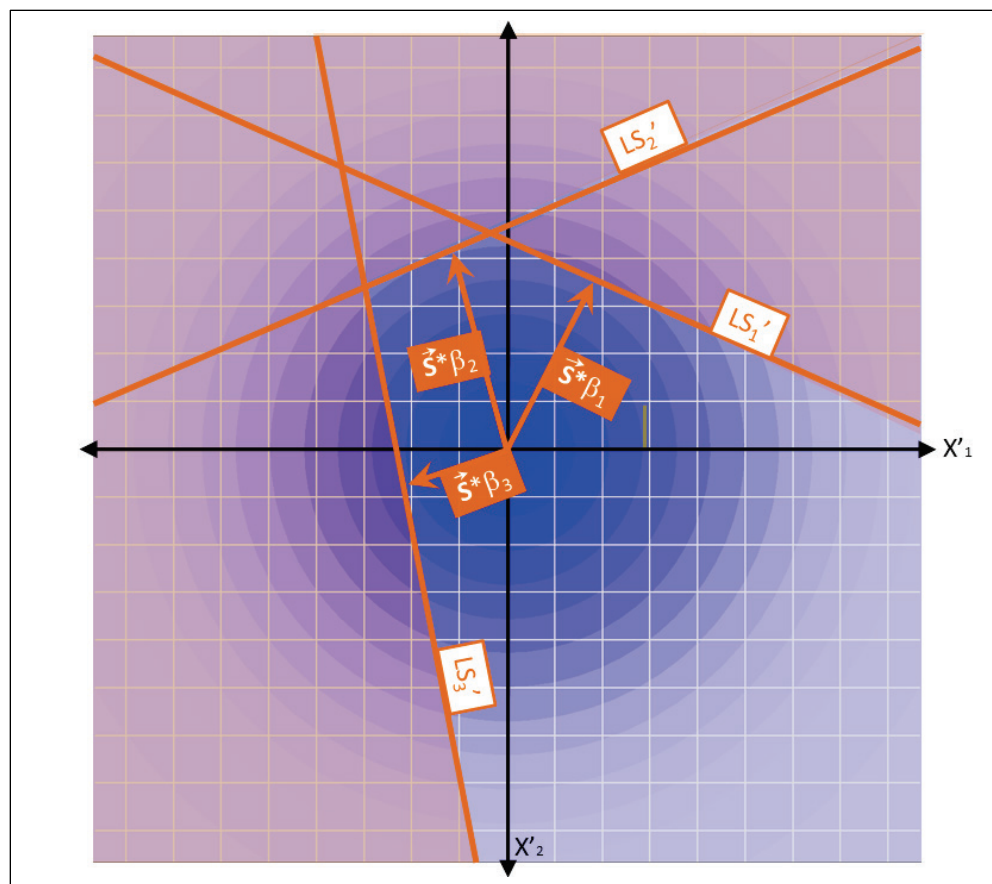
$$S_{Xi} = \alpha_{i-original\_Xi} \times S \quad (A.36)$$

For this situation, the *PUP* is a function of the scale factors for each of the variables.

$$PUP = f(S_{X1}, S_{X2}, \dots, S_{XN}) = f(\mathbf{S}) \quad (A.37)$$

Because of correlation between the limit state response surfaces, these functions can lead to local minima and maxima in a root finding because of the new slope calculations. Testing will need to be performed to find if these issues confound root-finding numerical algorithms.

Figure A-15. Applying variable axis scale factors to adjust probabilities. A scale factor applied to limit state 1 causes slope changes in the response surfaces for limit state 2 and 3.

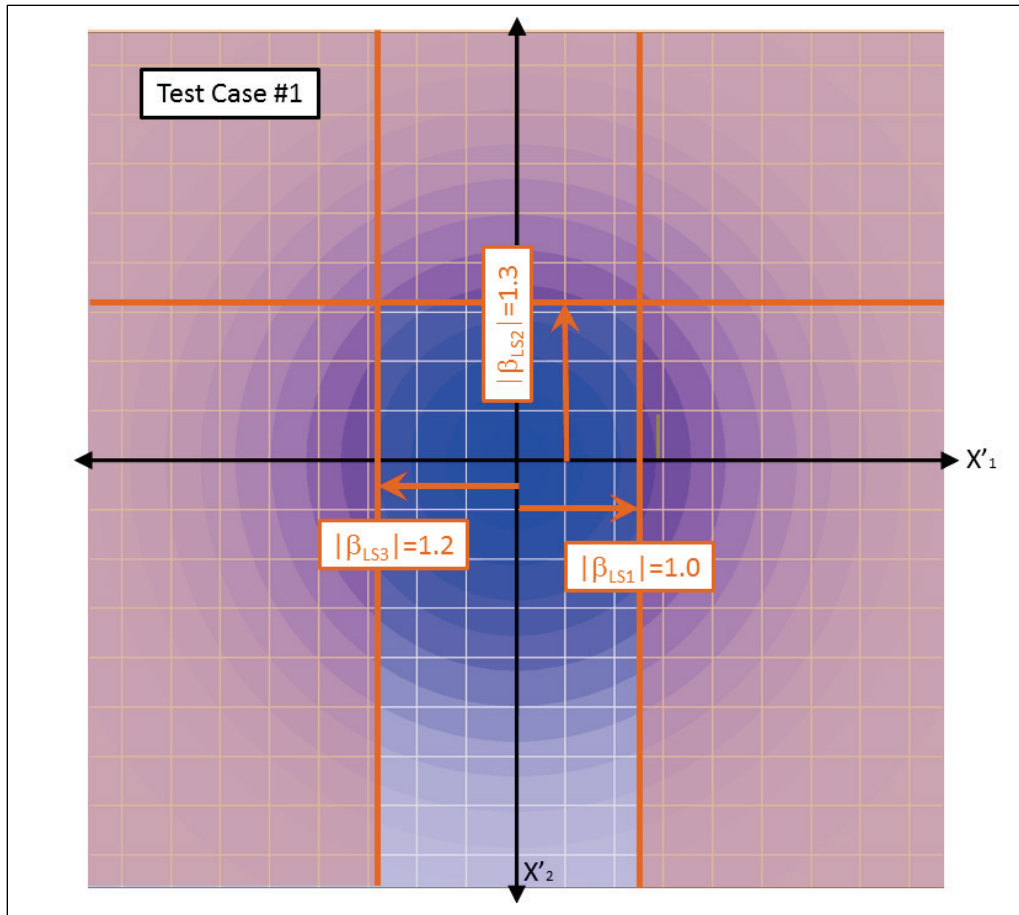


A.5 Validating *PUP* numerical estimation methods

To validate the models for *PUP* estimation, evaluate the inputs that will lead to an increase in accuracy, and to determine the resources (namely, time) that will be needed to do these estimations, simple test cases were created. These test cases were performed with two variables and had three overlapping limit states. These tests were all performed on a Dell Precision T7610 computer using Python 2.6 language for purposes of timing.

The first test case had two limit states that were perpendicular to the X_1 variable axis at distance 1.0 and -1.2 along the axis. The third limit state was perpendicular to the X_2 variable axis at distance 1.3 along the axis. For all three limit states, the unsatisfactory performance was in the half space away from the origin of the Gaussian multivariate space. This can be seen in Figure A-16.

Figure A-16. The first test case for PUP estimation.



Because the limit states are perpendicular or parallel, the limit states are completely uncorrelated. This simplifies the calculation of the actual *PUP*. The probability of the overlapping area of two limit state half spaces is the multiplication of each limit states' *PUP*. This result is shown in Equations A.21 and A.22, when $\rho = 0.0$. This means that

$$P(E_i E_j) = \Phi(-\beta_i) \Phi(-\beta_j) \quad (\text{A.38})$$

Therefore, for the first simple test case, the *PUP* is given by

$$PUP = \Phi(-\beta_{LS1}) + \Phi(-\beta_{LS2}) + \Phi(-\beta_{LS3}) - \Phi(-\beta_{LS1})\Phi(-\beta_{LS3}) - \Phi(-\beta_{LS2})\Phi(-\beta_{LS3}) = 0.34402870$$

Listings A.1 and A.2 show the result of performing a simple numeric integration across the test case from -5 to 5 in each dimension of the Gaussian multivariate space with step resolutions of 0.01 and 0.001, respectively. The first few lines show the calculation of the *PUP* as

discussed previously. The estimated value of *PUP* with resolution 0.01 from the integration is 0.3414388654 and took 154 seconds to integrate. This value is within 0.2% of the actual calculated *PUP* value. The estimated value of *PUP* with resolution 0.001 from the integration is 0.3437685994 and took 16,127 seconds to integrate. This value is with 0.05% of the actual calculated *PUP* value. While the additional accuracy can make slight changes at the tails of the distribution, the solution time means that the 0.01 resolution numerical integration is preferred.

Listing A.1. Output from Euclidean numerical integration with step size 0.01.

```
===== Direct Calculation =====
PUP(LS1) = 0.15865525
PUP(LS2) = 0.11506967
PUP(LS3) = 0.09680048
E(LS1,LS3) = 0.01535791
E(LS2,LS3) = 0.01113880

PUP(Overall) = PUP(LS1)+PUP(LS2)+PUP(LS3)-E(LS1,LS3)-
E(LS2,LS3) = 0.34402870

===== Direct Numerical Integration Calculation =====
Input Parameters: Resolution=0.0100000000
numsteps=500
Solution Time=152
PUP(Overall) = 0.3414388654
```

Listing A.2. Output from Euclidean numerical integration with step size 0.001.

```

===== Direct Calculation =====
PUP (LS1) = 0.15865525
PUP (LS2) = 0.11506967
PUP (LS3) = 0.09680048
E (LS1,LS3) = 0.01535791
E (LS2,LS3) = 0.01113880

PUP (Overall) = PUP (LS1)+PUP (LS2)+PUP (LS3) -E (LS1,LS3) -
E (LS2,LS3) = 0.34402870

===== Direct Numerical Integration Calculation =====
Input Parameters: Resolution=0.0010000000
numsteps=5000
Solution Time=16127
PUP (Overall) = 0.3437685994

```

Listing A.3 and A.4 show the results of pure polar integration for the same test case. Recall that polar integration can take advantage of the fact that any point on the hypersphere has the same probability in the Gaussian multivariate space for a significant speed-up. The integration is performed from the center to a hypersphere of radius 5. The angle step is 0.01 for both runs, and the radius changes by steps of 0.01 and 0.001, respectively. The first few lines show the calculation of the *PUP* as discussed previously. The estimated value of *PUP* with radius resolution 0.01 from the integration is 0.3438739669 and took 217 seconds to integrate. This value is within 0.05% of the actual calculated *PUP* value and has better accuracy than the Euclidean numerical integration does with 0.001 step resolution. The estimated value of *PUP* with resolution 0.001 from the integration is 0.3440204733 and took 2,036 seconds to integrate. This value is within 0.003% of the actual calculated *PUP* value. While the additional accuracy can make slight changes at the tails of the distribution, the solution time means that the 0.01 resolution polar numerical integration is preferred and is better than the Euclidean numerical integration.

Listing A.3. Output from polar numerical integration with radius step size 0.01 and angle step size 0.001.

```
===== Direct Calculation =====
PUP(LS1) = 0.15865525
PUP(LS2) = 0.11506967
PUP(LS3) = 0.09680048
E(LS1,LS3) = 0.01535791
E(LS2,LS3) = 0.01113880

PUP(Overall) = PUP(LS1)+PUP(LS2)+PUP(LS3)-E(LS1,LS3)-
E(LS2,LS3) = 0.34402870

===== Polar Numerical Integration Calculation =====
Input Parameters: Radius Step=0.0100000000 Angle
Step=0.0010000000
Solution Time=217
PUP(Overall) = 0.3438739669
```

Listing A.4. Output from polar numerical integration with radius step size 0.001 and angle step size 0.001.

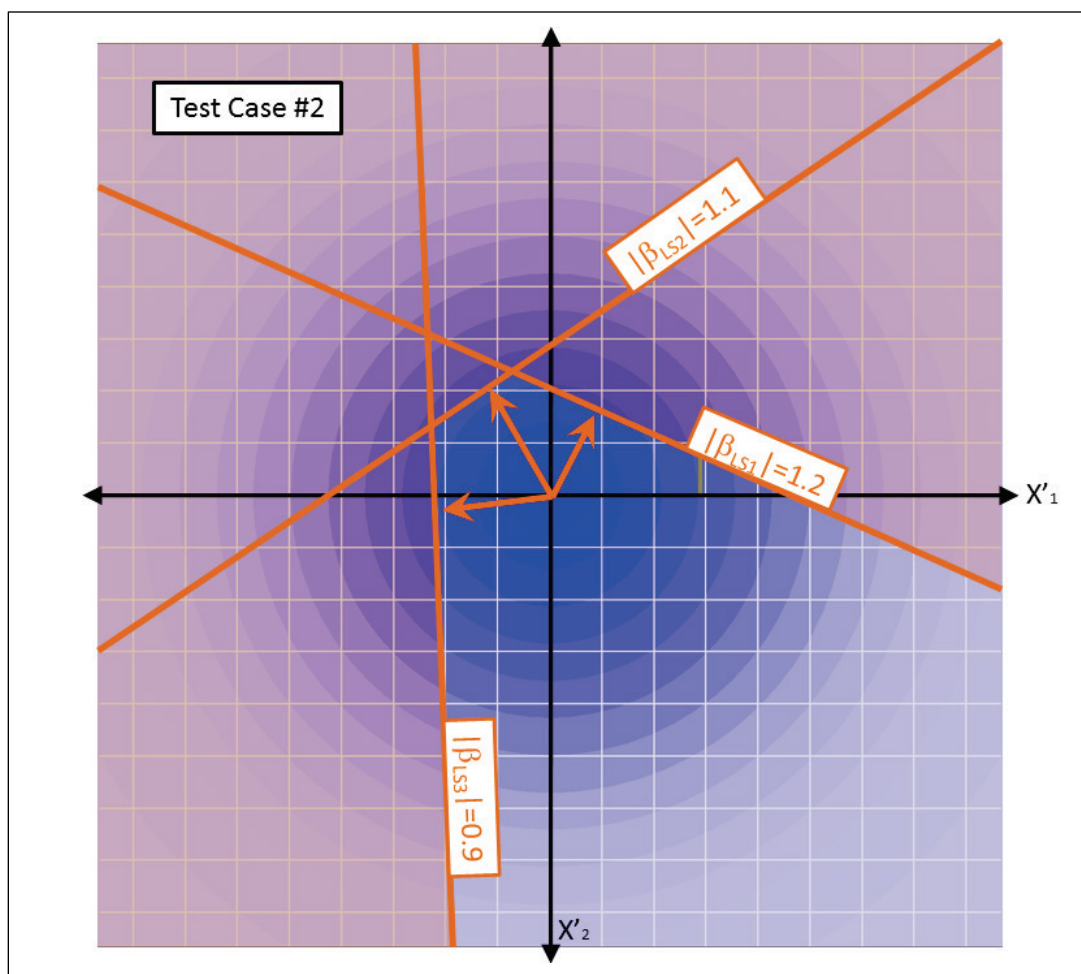
```
===== Direct Calculation =====
PUP(LS1) = 0.15865525
PUP(LS2) = 0.11506967
PUP(LS3) = 0.09680048
E(LS1,LS3) = 0.01535791
E(LS2,LS3) = 0.01113880

PUP(Overall) = PUP(LS1)+PUP(LS2)+PUP(LS3)-E(LS1,LS3)-
E(LS2,LS3) = 0.34402870

===== Polar Numerical Integration Calculation =====
Input Parameters: Radius Step=0.0010000000 Angle
Step=0.0010000000
Solution Time=2036
PUP(Overall) = 0.3440204733
```

The second test case (Figure A-17) was designed to test the Ang and Tang (1975) bounds method with correlated limit states. In this case, a design point was chosen for three hyperplanar limit states. The design points were (0.5,1.1), (-0.6,0.9), and (-0.9,-0.1) for limit states 1, 2, and 3, respectively. The unsatisfactory performance regions were in the half-planes for the limit state response surfaces away from the origin.

Figure A-17. The second test case for *PUP* estimation.



Listing A.5 shows the input design points along with the calculated b values and a vectors for the limit states. The absolute bounds for the *PUP* are calculated from Equations A.12 and A.13, resulting in $0.182590 \leq PUP \leq 0.376573$, with an average value of 0.279582. Using the pairwise correlation relationships in Equations A.14 and A.15, with estimated values from Equations A.21–A.24, a tighter bound was established as $0.229335 \leq PUP \leq 0.344048$, with an average value of 0.286692. Because of the correlation, an actual value was not available, so a polar

integration was performed with a radius step of 0.01 and an angular step of 0.01, which resulted in a *PUP* of 0.329675.

Listing A.5. Output from Ang and Tang (1975) bounds estimation compared to polar integration method.

```
===== Input Parameters =====
Limit State 1 (LS1) Design Point (0.500000 1.100000)
Beta=1.208305 Alpha = (0.413803 0.910366)
Limit State 2 (LS2) Design Point (-0.600000 0.900000)
Beta=1.081665 Alpha = (-0.554700 0.832050)
Limit State 3 (LS3) Design Point (-0.900000 -0.100000)
Beta=0.905539 Alpha = (-0.993884 -0.110432)

===== Ang and Tang Limits =====
PUP(LS1) = 0.113465
PUP(LS2) = 0.139701
PUP(LS3) = 0.182590

Absolute Bounds: 0.182590 <= PUP <= 0.376573 Average =
0.279582

Tighter Bounds: 0.229335 <= PUP <= 0.344048 Average =
0.286692

===== Numerical Analysis =====
Input Resolution: Radius step size = 0.010000 Angle
step size = 0.001000
PUP = 0.329675
```

While the Bounds Estimation Method was simple to compute, the range of the values is too big for estimation in the tails of the Gaussian normal distribution. The averaged values also were not accurate enough for the rigorous application that would be required by an LRFD solution method.

Appendix B: Observations Regarding the Sleep and Duncan (2014) Procedure for Geotechnical Engineering Reliability Calculations

B.1 Overview

In 2014, Sleep and Duncan published an updated manual (i.e., second edition) on the topic of Geotechnical Engineering Reliability Calculations through the Virginia Tech Center for Geotechnical Practice and Research. This manual summarizes the advantages of applying reliability analysis in geotechnical practice. They observe that reliability calculations provide an analytical means of introducing uncertainty into the variables for stability computations of a retaining wall and measuring the effects of this uncertainty. This allows the engineer to evaluate conditions where data are uncertain and ascertain whether the probabilities of system stability are particularly low or high. Two of the geotechnical limit states for the foundation of an example retaining wall were investigated: (1) the sliding limit states and (2) the bearing capacity for a T-Wall founded on clay.

Each limit state has associated with it a value for the probability of failure, P_f . Sleep and Duncan point out that their reliability analysis concepts can be applied in simple ways that preclude the need to gather more geotechnical data and preclude the need to expend more time or effort than are common in geotechnical engineering practice. They demonstrate their simplified reliability analysis by analyzing the two limit states, stated above, for the foundation for a T-Wall retaining structure.

The Sleep and Duncan (2014) manual outlines the calculation steps for four methods of reliability analysis:

- The Monte Carlo Simulation Method that computes a value for P_f for each limit state.
- A Simplified Hasofer-Lind Method that computes a value for β and for P_f for each limit state.
- An FOSM reliability method. This is referred to as a point estimate method in their manual because it finds the design point on the limit state response surface that has the highest probability.

- A Taylor Series method with an assumed normal or log-normal distribution for FoS.

For the latter two reliability analysis methods, the distribution for the FoS is assumed and then used to compute a P_f value.

If normal and log-normal distributions are assumed for the FoS, tabulated values of P_f are presented as a function of two parameters: (1) the most likely value (i.e., the mean value) for the FoS and (2) the COV for the FoS. Recall that the COV is equal to the standard deviation divided by the mean value for the FoS. The Sleep and Duncan (2014) data show that a higher P_f value results from assuming a normal distribution for FoS than assuming a lognormal distribution in all cases with the exception of low mean FoS values and high COV values.

Three variables are considered in the manual's reliability analysis of a T-Wall example problem:

- unit weight of the backfill
- shear resistance at or below the base of the T-Wall (for the sliding limit state)
- undrained shear strength of the clay foundation immediately below the T-Wall (for the bearing capacity limit state).

When conducting reliability analyses of their retaining wall example problem, the Sleep and Duncan (2014) manual makes use of an Excel spreadsheet to facilitate these computations. This Excel workbook was developed in an earlier Virginia Tech Center for Geotechnical Practice effort to facilitate these computations (Yang and Duncan 2002). For their Monte Carlo simulations, they use the macro @Risk™ to conduct these simulations. The manual recommends a minimum required number of simulations of 5,000 or greater to obtain accurate results.

The relationship between reliability index and probability of failure is carefully summarized in this manual for normal or log-normal distributions of the FoS. A wide range in β and corresponding P_f values are presented in tabulated form. The meaning of *failure* in the context of a retaining wall was also discussed. They point out that it is preferable to use descriptions such as “probability of retaining wall sliding” or “probability

of foundation movement through full mobilization of its bearing capacity,” etc., for specificity.

B.2 Two reliability methods recommended by the manual

Among the four reliability computational methods examined in Sleep and Duncan (2014), they found the most accurate to be the Monte Carlo Simulation Method and the simplified Hasofer-Lind Method. This was demonstrated through computations made using their T-Wall example. They endorsed the use of either of these two methods of reliability analysis.

B.3 The manual's commentary on the accuracy of reliability computations as they pertain to geotechnical engineering

Sleep and Duncan (2014) make several important observations regarding the application of reliability analysis to geotechnical engineering using the methodology laid out in their updated (2014) manual. These include the following:

- The accuracy of the means (referred to as the most likely values in this manual) and the standard deviations of the variables considered in the reliability problem model dictate the precision with which P_f can be computed.
- Considering the often sparse geotechnical engineering data and challenge in estimating the standard deviations of these data, computed values of P_f cannot be expected to be highly precise.
- Without sufficient data collection, values of standard deviation of geotechnical engineering parameters can be estimated using available data, combined with engineering judgement.
- Application of expertise with informed judgement is an important part of the process used to devise values of standard deviations and mean values for each of the geotechnical engineering, material-characterization variables (e.g., unit weights, shear strength parameters).
- The great advantage of using reliability analysis to compute a value for P_f is that P_f is a single value that provides a measure of uncertainty from all random variables considered in the analysis model. Recall that P_f is the probability that the FoS could be less than or equal to 1.0.

B.4 Observations regarding USACE needs and the manual's formulations

Some simplified details contained in the manuals' reliability formulation and retaining wall model that differ from USACE needs are the following:

- The reliability model is formulated using uncorrelated variables only. Correlated variables are outside the scope of this formulation, although key engineering materials are often correlated. For example, effective stress-based Mohr-Coulomb shear strength parameters of cohesion and angle of internal friction, c' and ϕ' , respectively, are often strongly correlated (typically inversely correlated with a negative correlation coefficient, on the order of -0.7).
- The four reliability formulations that were devised assume that all variables are either normal or log-normally distributed. USACE experience has demonstrated the need for truncated distributions describing variables.
- Only the structural wedge of the T-Wall is modeled. Driving or resisting soil wedges (as described in EM 1110-2-2502 [HQUSACE 1989]) are not considered in the model formulation described in this manual.
- An equivalent fluid pressure methodology (Terzaghi and Peck 1967) is used to characterize the earth pressure force destabilizing the T-Wall structural wedge. This approach is not used in USACE design.
- Because of computational limitations at the time, Sleep and Duncan (2014) did not consider numerical integration techniques to capture P_f values for combined limit states.
- The Sleep and Duncan (2014) manual on the application of reliability analysis in geotechnical practice was not extended to address LRFD.

Appendix C: Using Sparse Data to Determine Geotechnical Soil Property Distributions/Statistics

C.1 Introduction

It is often the case in geotechnical work that a scarcity of material property data makes it difficult to describe the range and statistical distribution of values that characterize the soil material. This appendix discusses one approach used by geotechnical engineers to develop mean and standard deviations values for engineering material parameters describing soils, particularly with regard to shear strength parameters. In the absence of enough data to describe a distribution, a bell-shaped distribution (e.g., the Gaussian Normal distribution) can be used to estimate the parameter spread of values.

This approach makes use of the range in values for a variable, where range is defined as the difference between the lowest conceivable value (LCV) and the highest conceivable value (HCV) for the data.

$$\text{Range} = \text{HCV} - \text{LCV} \quad (\text{C.1})$$

Because of the sparsity and the magnitude of scatter in geotechnical engineering data, this characterization is found to be especially helpful when assigning a standard deviation value. Baecher and Christian (2003) observe that the dispersion from a small number of tests can be made from the sample range. In statistics, variance (equal to the square of the standard deviation, σ^2) is the measure of dispersion.

C.2 The importance of accurate standard deviation values in a reliability analysis

Sleep and Duncan (2014) make several important observations regarding the application of reliability analysis to geotechnical engineering and the importance of providing accurate mean and standard deviation values. When considering the challenge the engineer faces in developing values of standard deviation for the soil strength variables given sparse data, these points are worth repeating:

- The accuracy of the means and the standard deviations of the variables considered in the Duncan and Sleep (2014) reliability problem model dictate the precision with which P_f can be computed. Considering the interrelationship between P_f and β , this conclusion extends to β as well.
- Considering the often sparse geotechnical engineering data and challenge in estimating the standard deviations of these data, computed values of P_f cannot be expected to be highly precise.
- Values of standard deviation of geotechnical engineering parameters can be estimated using available data, combined with engineering judgement.
- Application of expertise with informed judgement is an important part of the process used to devise values of standard deviations and mean values for each of the geotechnical engineering, material-characterization variables (e.g., unit weights, shear strength parameters).

Estimating material parameter distributions with the least amount of error directly impacts the accuracy of reliability analyses. The cumulative effect of material properties drawn from distributions affects capacity estimates and limit state(s) calculations, which directly impact P_f , β and LRFD values.

C.3 A quick estimate for the mean value

Baecher and Christian (2003) suggest that a quick and often good estimate of the central tendency is the median value. For data that are symmetrically distributed about its central value and for small numbers of clustered data, the sample median is a good estimate of the mean.¹ Data that are normally distributed is an example of a symmetrical distribution.

A second shortcut suggested by Baecher and Christian (2003) is that in some instances, the mean value μ can be approximated using

$$\mu = \frac{HCV + LCV}{2} \quad (C.2)$$

They caution that this estimate requires expertise to determine the smallest conceivable range. It is recognized that experts have a tendency to underestimate the range, as discussed in subsection C.4.2. Therefore,

¹ The mean is the arithmetic average of a set of numbers, or distribution. The median is described as the numeric value separating the higher half of a sample, a population, or a probability distribution, from the lower half.

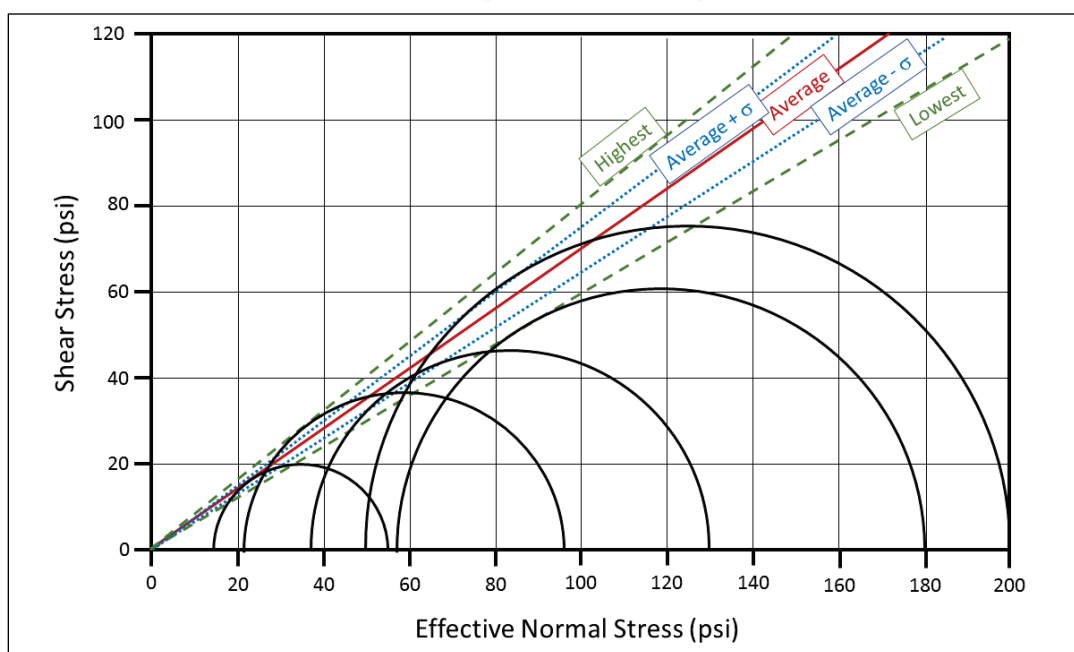
Equation C.2 should be used with caution. Ellingwood (2015) also cites this relationship for a *bell-shaped* PDF (e.g., a normally distributed variable) and when 95% of the values lie between the LCV and HCV values.

If the PDF is skewed in the positive direction, as is natural for geotechnical engineering material parameters that rarely have values that are less than 0.0, Ellingwood (2015) suggests a mean estimate can be based on

$$\mu = 0.65 * LCV + 0.35 * HCV \quad (C.3)$$

Graphical N_σ approach: In the graphical approach described in Duncan et al. (2014) and Sleep and Duncan (2014) for interpreting a plot of sparse data (e.g., consisting of undrained shear strength S_u versus depth [or pressure]), a straight line or a curve is drawn through the data that represent the most likely variation of the S_u with depth (or pressure). In the case of triaxial test results, a straight line is drawn through the data of Mohr-circles at failure for each of the triaxial test data that represent the most likely variation of a Mohr-Coulomb shear strength envelope with pressure for triaxial tests, which are usually run at different confining pressures (Figure C-1).

Figure C-1. Development of average, average plus one standard deviation, and average minus one standard deviation strength envelopes using the 2- σ rule ($N_\sigma = 4$).



Application of expertise with informed engineering judgement is an important part of the graphical N_σ process. This is because of the influence of sample disturbance, the sensitivity of test data to the quality of the overall execution of the laboratory (or in situ in the field) soil testing process, differences in measured strength results for different types of soil strength tests due to stress anisotropy, etc. Sample disturbance is a *catch-all* term used in geotechnical site characterization to reflect the effect of disturbance on an engineering material characterization parameter (e.g., S_u) that can occur during the drilling and sampling process, during sample extraction from the field and transport to the soil testing laboratory, and during sample storage and handling of the sample in the laboratory.

C.4 A quick estimate for the value of the standard deviation

C.4.1 A statistical procedure using collected data

Because of sparse geotechnical data, one approach cited by Baecher and Christian (2003), Foye et al. (2006), Sleep and Duncan (2014), and Duncan et al. (2014) is what is sometimes referred to as the **N_σ -rule** to estimate the standard deviation σ of geotechnical engineering material parameter (e.g., S_u). Citing the work of Tippett (1925), Foye et al. (2006) state this rule as

$$\sigma = \frac{Range}{N_\sigma} \quad (C.4)$$

for a normally distributed variable. The value for N_σ varies with the number of sample measurements for the parameter (e.g., S_u); from a low of 1.128 for 2 samples to a high approaching 6.5 for 1,000 samples. Table C-1 summarizes N_σ values for different sample sizes according to Tippett (1925) and Burlington and May (1970).

Table C-1. Values of sample range (N_σ) for normally distributed variable with $\sigma = 1$ and n sample measurements (after Tippett 1925 and Burington and May 1970).

Sample Size, n	N_σ
2	1.128
3	1.693
4	2.059
5	2.326
6	2.534
7	2.704
8	2.847
9	2.970
10	3.078
11	3.173
12	3.258
13	3.336
14	3.407
15	3.472
16	3.532
17	3.588
18	3.640
19	3.689
20	3.735
30	4.09
50	4.5
75	4.81
100	5.02
200	5.49
300	5.756
400	5.936
500	6.073
1,000	6.483

C.4.2. A procedure using expert judgement

It is not uncommon that collected geotechnical data either have too few samples for statistical processing or have values that experienced geotechnical engineers can readily identify as outliers that lie beyond the actual variance of the data trends. Again, the application of expertise with informed engineering judgement is an important part of this process. This is because of the influence of sample disturbance, the sensitivity of test data to the quality of the overall execution of the laboratory (or in situ in the field) soil testing process, differences in measured strength results for different types of soil strength tests due to stress anisotropy, etc. When this occurs, geotechnical expertise is required to specify the actual (e.g., S_u) values for HCV and LCV. The following rules provide a qualified estimate for the probabilities of occurrence at the expert-provided HCV and LCV, therefore assigning a standard deviation.

Dai and Wang (1992), Withiam et al. (1997), Foye et al. (2006), and Professor Duncan in his earlier 2000 publication (Duncan 2000) on the topic of reliability analysis in geotechnical engineering all recommended using a value of 6 for N_σ . The basis for the N_σ equal to 6, also called the *three-sigma rule* (Duncan 2000), lies in the fact that for a normally distributed variable, three standard deviations in either direction from the mean of a normal distribution encompasses 99.74% of all possible values.

Baecher and Christian (2001) suggested changing the value of N_σ from 6 to 4 after reviewing the Duncan (2000) paper with the following realizations:

- First, if the person making the the estimate for standard deviation uses sparse data with sample sizes of 20 to 30 values, then a better estimate would be made using N_σ equal to 4 rather than 6. This can be seen in the Table C-1 information which interprets the data statistically.
- Second, Duncan and Wright (2005), Duncan, et al. (2014), Sleep and Duncan (2014), and Christian and Baecher (2001) discuss studies that show there is a tendency for people, including statisticians as well as experienced geotechnical engineers, to be overconfident in their ability to estimate the range from LCV to HCV. In other words, their estimated value for the range (Equation C.1) between LCV and HCV is typically too small, resulting in distributions that overestimate values close to the mean.

For geotechnical engineers working with sparse data, it is the opinion of the authors of this ERDC report that the second point is especially relevant to geotechnical material characterization. In summary, these two points encourage using a value of 4 for N_σ when data are scarce for the determination of engineering material parameter distributions for soils that will be used in geotechnical reliability applications.

Duncan and Wright (2005), Sleep and Duncan (2014), Duncan, et al. (2014) have since recommended assigning a value of N_σ equal to 4. Professor Duncan refers to this as the “two-sigma rule,” which reflects that the range value encompasses 2 times σ above the mean value and 2 times σ less than the mean. Ellingwood (2015) also recommends the use of N_σ equal to 4 for a PDF that is bell-shaped or is skewed in the positive direction.

Graphical N_σ approach: Duncan and Wright (2005), Duncan et al. (2014), Sleep and Duncan (2014) provide examples of applying the N_σ equal to 4, two-sigma rule graphically to sparse data consisting of undrained shear strength S_u versus depth data or Mohr-circles at failure for triaxial data with different confining pressures that are used to define the angle of internal friction. First, a straight line or a curve is drawn through the data that represent the most likely variation of the parameter with depth (or with pressure). Next, straight lines or curves are drawn through the data that represent the highest conceivable bound and lowest conceivable bounds on the data. These become the two-sigma lines, with N_σ equal to 4. Finally, straight lines or curves that represent the average plus one standard deviation and the average minus one standard deviation are drawn halfway between the average and the lowest and highest conceivable lines. Figure 14 in Sleep and Duncan (2014) and Figure 13-1 in Duncan et al. (2014) demonstrate this procedure using undrained shear strength S_u data with depth. This graphical approach is also demonstrated in Figure C-1 for triaxial test data in this appendix.

Appendix D: Values for Coefficient of Variation and Scale of Fluctuation for Soils

D.1 Introduction

Geotechnical site investigations are designed with two objectives in mind: (1) to identify the soil layers contained in the foundation for the project and (2) to characterize the geotechnical engineering soil properties of each of the soil layers. During the site investigation, soil specimens are recovered from each of the foundation soil layers. Some specimens are used to identify the soil type(s) while other specimens are selected for transportation to a laboratory where they are subjected to a variety of geotechnical tests, especially strength data. These soil tests generate geotechnical engineering data that are used to characterize the strength and/or stiffness of the various soil layers from which high quality *undisturbed* samples were recovered. Some of the geotechnical laboratory tests that are conducted include unconfined compression tests, various types of triaxial tests, direct shear tests, direct simple shear tests, ring shear tests, and 1-D consolidation tests. The site investigation may also involve the use of a variety of in situ tests conducted at the project site that result in measurements used to indicate and/or correlate to engineering shear strength and stiffness parameters of the soils within the layers of foundation soils. Examples of these in-situ tests include the SPT, various types of CPT, VST, as well as the less frequently used dilatometer tests and pressuremeter tests. The measurements and/or interpreted geotechnical engineering material properties generated for each of the foundation soil layers from the laboratory and/or in situ test results inherently possess *scatter*. Geotechnical engineers have found that reliability-based statistical procedures are one means for quantifying this scatter and formally incorporating it in the engineering analysis. Calculation of an *average trend function* that characterizes the spatial average of the geotechnical engineering data within each of the soil layers, along with the calculation of the data's *statistical* variance about this trend function is an approach to statistically interpret data possessing scatter.

The COV of a geotechnical engineering property for a particular type of soil is an important parameter when formulating a resistance factor in LRFD format, as evidenced by its presence in the LRFD resistance factor, ϕ_R , Equation 2.36 (and Equation 2.39). Recall, COV is the standard deviation divided by the mean value for a specified parameter. The COV is

frequently used as an index for the variance of an engineering parameter. Recall, variance is the square of the standard deviation, σ^2 . This appendix summarizes COV values obtained from geotechnical articles whose COV data were reviewed for pertinence to the geotechnical portion of this USACE LRFD research effort. The focus of this geotechnical parameters COV data collection is on shear strength parameters for different soil categories (e.g., clay, sand). COV data are further grouped according to the type of laboratory or field test used to generate the strength data. Some of the selected geotechnical publications present these data as a range in COV values while others as approximate guidelines for design soil property variability.

D.1.1 Geotechnical data in the vertical and horizontal directions

A geotechnical site investigation consisting of drilled bore holes and in situ tests (e.g., CPT) usually are performed in the vertical direction. The foundation soil types and layering are identified using these data. The in situ measurements and laboratory soil test data are processed and interpreted for each of the soil layers. This data collection and soil type interpretation is conducted in the vertical direction, using the data recovered from different boreholes and/or in situ measurements made across the site. Soil profiles are then constructed across the site. The typical geologic site formation process for soils allows for a site characterization process that is oriented in the vertical and horizontal directions within each of the soil layers.

D.1.2 Sparse data and typical COV values

Phoon and Kulhawy (1999a) and others in the geotechnical community observe that ideally, appropriate resistance factors (in an LRFD-based design procedure) should be based on the variability of the soil data collected at the project site. However, when these data are not yet available or are too limited for meaningful statistical analysis, guidelines on the likely range of soil property COV are useful. The COV data contained in this appendix are intended to bridge this gap in knowledge. These data also serve the purpose of allowing geotechnical engineers to develop an appreciation for the probable range in COV values and to identify unusual values for soil variance, as expressed in a COV value.

D.1.3 Point estimate versus spatial average and variance reduction

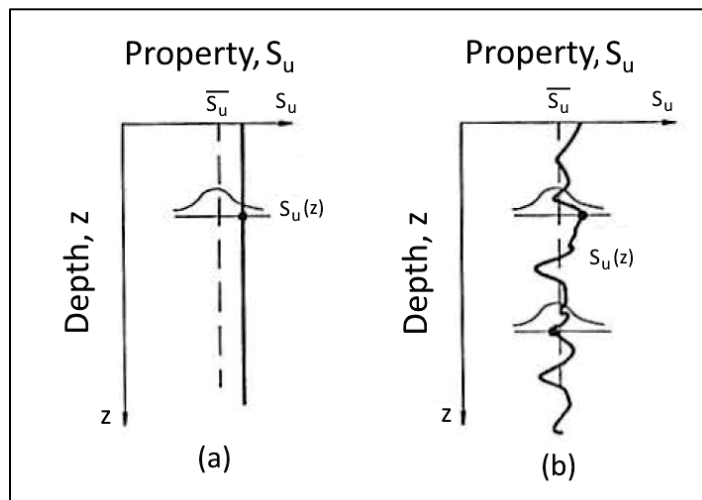
Vanmarcke (1977b) points out that probabilistic distributions are primarily needed along a potential failure surface in a clay foundation for each spatially averaged geotechnical engineering soil parameter, like the undrained shear strength S_u . A full description of these parameters is provided by a PDF from which the mean and standard deviation can be calculated. These spatial averages will have narrower PDFs than the corresponding point estimate property of a statistically homogeneous soil because the longer the distance over which this spatial averaging occurs, the uncorrelated fluctuations in S_u tend to cancel during the process of spatial averaging. This tends to cause a reduction in the standard deviation (and variance) in S_u as the size of the averaging length increases. This attribute is referred to as variance reduction within the zone of influence for the structure. Thus the statistical characterization of the variance in S_u , which can be expressed in terms of its COV value for S_u , needs to account for this variance reduction that occurs when spatial averaging estimates are made for soils properties within a zone of influence (e.g., along a potential slip plane) rather than for a statistical characterization of S_u at discrete points/locations (El-Ramly et al. 2003a,b).

Phoon and Kulhawy (1999a) make an important observation that relates to the fact that there are different methods to quantify COV for geotechnical engineering parameters for soils. Their observation is that soil property statistics reported in a number of pre-1999 geotechnical publications were determined from total variability analyses that implicitly assume a uniform source of uncertainty, as reflected in their method used to gather and statistically process the geotechnical data. Because of this, they conclude that these earlier results are not suitable for use in a geotechnical reliability analysis. Variance estimates of soil properties based on spatial average are needed. With an intended use in geotechnical reliability analysis, Phoon and Kulhawy take care in their 1999 studies to distinguish a point estimate value of COV in a particular soil layer from the COV value pertaining to spatial average in that layer. The authors of this report recognize the importance of these observations to reliability computation for the USACE hydraulic structures which are in contact with the earth, especially those that retain earth.

Li (1992) provides an explanation supported by informative graphics (Figure D-1(a),(b) to distinguish a point estimate for a geotechnical engineering soil property in what Li refers to as a uniform soil layer from

the spatial average for that soil property in that layer. Li notes that in “early works” on probabilistic analysis, soil properties were commonly modeled using what he referred to as a single-random-variable (SRV) approach in which the random variation of the soil property (x) within the entire soil layer is represented by one single random variable. Figure D-1(a) illustrates that this SRV approach which implies that all realizations (i.e., simulations) of the soil property (x) are the same at every depth (z) although the exact magnitude of the realization for (x) remains a random variable. A normal distribution is assumed, as shown in this figure at the depth z in which the simulation occurs. Li observes that this approach is inappropriate for geotechnical engineering site characterization because it infers that the actual magnitude of the soil parameter (x) at all depths in the soil layer can be determined with absolute certainty by measuring the value of the soil property using one sample taken anywhere within the soil layer. Modeling the engineering soil property (x) within the soil layer as a random field (Vanmarcke 1977a, 1983) is considered to be more realistic by Li (1992), Phoon and Kulhawy (1999a,b), and others. In this approach, the values of the soil property (x) at different depths z are treated as separate random variables, as depicted in Figure D-1(b). The actual mean value for the soil property of undrained shear strength (S_u) within the soil layer is shown as a constant labeled $\bar{x}$ and designated by a dashed line in these two figures.

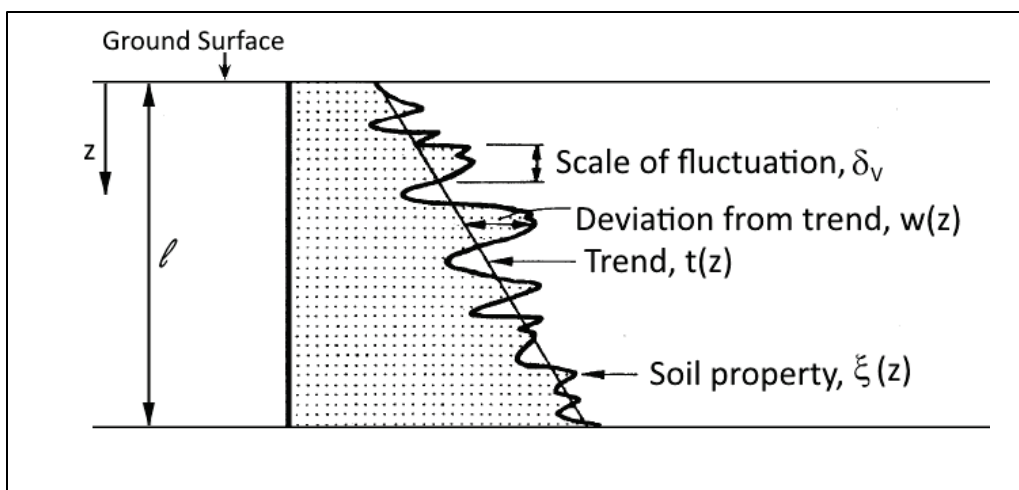
Figure D-1. Inherent soil variability of the geotechnical engineering soil property undrained shear strength (S_u) at depth z ; (a) point estimate; (b) random field (after Li 1992).



D.1.4 The Phoon and Kulhawy procedure for computing spatial average for a soil layer

The Phoon and Kulhawy approach for defining variances, standard deviations and COV values based on spatial average is best explained using Figure D-2. This figure displays an engineering soil property they designate as ξ , which varies with depth, z . This Figure D-2 engineering soil property (ξ) is equivalent to Li's engineering soil property (x) shown in Figure D-1. ξ could be the undrained shear strength, S_u , of the soil, for example. In Figure D-2 for soil layer j , ξ is shown to vary with depth z below the ground surface.

Figure D-2. Graphical definition of inherent soil variability (w) with depth, z , within a soil layer (after Phoon and Kulhawy 1996a or 1999a).



This figure depicts the soil property $\xi(z)$ varying about a trend function $t(z)$, with $t(z)$ assumed to be a linear function with elevation in Figure D-2.¹ This $\xi(z)$ deviation from the trend function $t(z)$ is designated $w(z)$ and is given by the relationship

$$\xi(z) = t(z) + w(z) \quad (D.1)$$

The fluctuating component $w(z)$ represents the inherent soil variability within soil layer j .

¹ Note that the trend function $t(z)$ does not have to be a linear trend function as depicted in Figure D-2. For example, Jaksa et al. (1997) used a quadratic trend function when interpreting CPT tip resistance data. A quadratic trend function was also used by Doherty and Gavin (2010) in their interpretation of CPT data for use in pile design.

Inherent soil variability: Rearranging, Equation D.1, the inherent soil variability $w(z)$ is defined mathematically as the difference between $\xi(z)$ and $t(z)$:

$$w(z) = \xi(z) - t(z) \quad (D.2)$$

It is important to the resulting statistics that the soil property data from different soil layers are not mixed. This requirement is also portrayed in Figure D-2. Additionally, the application of Equation D.2 ensures that deterministic trends in the soil property data are removed by subtracting the trend line function $t(z)$. The inherent soil variability $w(z)$ is modeled as random field (Vanmarcke 1983, 2010) and is considered to statistically homogeneous for a homogeneous soil layer when two statistical criteria are met for the data: Phoon and Kulhawy (1999a) state that these criteria are met if (1) the mean and variance of w does not change with depth and (2) the correlation between the deviations at two different depths is a function only of their separation distance, rather than their absolute positions. The standard deviation of the inherent soil variability $w(z)$ is designated as SD_w and evaluated in the usual manner using

$$SD_w = \sqrt{\frac{1}{n-1} * \sum_{i=1}^n [w(z_i)]^2} \quad (D.3)$$

with $w(z_i)$ being the fluctuation at depth z_i about the trend function. The coefficient of variation of inherent soil variability $w(z)$ is

$$COV_w = \frac{SD_w}{t} \quad (D.4)$$

with t being the mean soil property trend for the soil layer in this equation.

D.1.5 Scale of fluctuation; autocorrelation distance; characteristic length

Another statistical parameter used to describe inherent soil variability in a soil layer is the autocorrelation distance, designated r_o by DeGroot and Baecher (1993) and Keaveny et al. (1989). Mollon et al. (2011) observe that an infinite autocorrelation length corresponds to a homogeneous field, while an autocorrelation length equal to zero corresponds to a non-correlated field (i.e., a soil in which the value of the geotechnical engineering soil parameter at each point is totally uncorrelated with the ones of surrounding points). The autocorrelation distance r_o is also

referred to as the characteristic length. For a simple and commonly used exponentially decaying autocorrelation function, ρ ,

$$\rho(r) = b * e^{-(|r|/r_o)} \quad (D.5)$$

the Vanmarcke (1977a) scale of fluctuation, δ , is given by¹

$$\delta = 2 * r_o \quad (D.6)$$

In equation D.5, r is separation (lag) distance between points and b is a value between 0 and 1, depending upon measurement noise (Keaveny et al. 1989). The scale of fluctuation, δ , is related to the autocorrelation distance, r_o , but differs in numeric value by a constant associated with the model used (e.g., equation D.5) to define the autocorrelation function with distance.

The scale of fluctuation represents the length of a range where all values should show correlation whereas the autocorrelation distance is the distance from a certain location until properties are uncorrelated (much like a diameter and radius). Both parameters r_o and δ possess units of length. The value for r_o or δ is a measure of the spatial extent within which a soil property shows a strong correlation. Vanmarcke (1977a) comments that at two points that lie within a distance δ , the corresponding values of the soil parameter are likely to be either both above or both below the soil layers trend function within this region. A large value for r_o (or δ) implies that the soil property is highly correlated over a large spatial extent, resulting in a smooth variation within the soil profile about a mean trend function and a small value of r_o (or δ) implies the fluctuation of the soil property is large, giving rise to a rapid spatial variation of soil property about a mean trend function (Li and Lee 1991; Wickremesinghe and Campanella 1993).

Because of soil anisotropy, a distinction is made between the scale of fluctuation in the vertical and in the horizontal direction. Note that the vertical direction is designated by the subscript v in this appendix and the horizontal direction is designated by the subscript h .

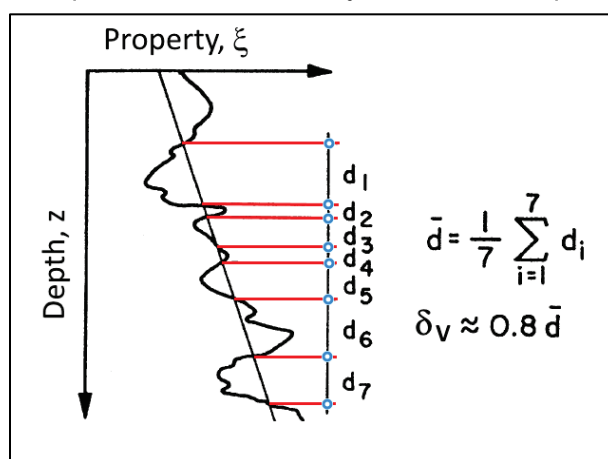
¹ For other mathematical options that may be used to describe the autocorrelation function with distance, refer to Table 1 of Keaveny et al. (1989); Figure 2 of DeGroot and Baecher (1993); Table 2 of Li and Lee (1991); Table 1 of Jaksa et al. (2000); Section 5.1 in Vanmarcke (1983, 2010); or Table 1 of Vanmarcke (1977a).

A simplified approach for assessing the value for δ_v is described in Vanmarcke (1977a) and outlined using Figure D-3. Given sufficient geotechnical material property data ξ with depth z within a soil layer, as depicted in Figure D-3, a simple approximation for the vertical scale of fluctuation δ_v is given by Vanmarcke (1977a) to be

$$\delta_v = 0.8 * \bar{d} \quad (\text{D.7})$$

in which $\bar{d}$ is the average distance between intersections of the fluctuating soil property ξ and its trend function $t(z)$.

Figure D-3. Estimation of vertical scale of fluctuation (after Phoon and Kulhawy 1996a or 1999a).



D.1.6 Geology's influence on horizontal and vertical scales of fluctuation

Fenton and Vanmarcke (1991) and others suggest that the scale of fluctuation is largely dependent on the geological process of the soil layer deposition rather than the specific soil property being studied. During a review of the geotechnical literature on autocorrelation distance of in situ soils (versus manmade soil layer(s)), the authors of this report observed that the values for autocorrelation distance (i.e., characteristic length) r_o and for the scale of fluctuation δ differ in the vertical direction from the value in the horizontal direction, within the same soil layer. This is logical from a geologic point of view when the depositional environment for the soil layer is considered: For example, Baecher and Christian (2006) comment that for fluvial soils the correlation distance might be small while for lacustrine soil the correlation distance might be large. El-Ramley et al. (2003a) also encourage approaching the assignment of r_o and δ values from a geologic viewpoint: They cite the example of a

marine sediment tending to be homogeneous with continuous patterns of variability and large autocorrelation distances whereas tills and residual soils tend to be more erratic with notable variabilities over small distances and with small autocorrelation distances. By the observations cited in this paragraph, the authors of this report recognize the importance of geologic depositional characteristics to r_o and δ values and subsequently, their impact on a reliability computation for the USACE hydraulic structures in contact with earth.

D.1.7 Anisotropy in δ_v and δ_h values: site-specific data

Table D-1 is a collection of values of the vertical and horizontal autocorrelation distances and/or vertical and horizontal scales of fluctuation for different engineering tests made in different soil types. Nearly all of the soils cited are naturally formed soils. Two exceptions are the hydraulic fill sand sites that were described in Onisiphorou (2000), as cited in Hicks and Samy (2002) and in Popescu et al. (1997). The soil data reported in this table focus almost entirely on strength tests results. The data in this table support the observation that as a result of the geologic process and within a given layer, soil properties tend to be more variable in the vertical direction than in the horizontal direction, as reflected in smaller r_o (and δ) values in the vertical direction as compared to those for the horizontal direction.

Table D-1. Summary of autocorrelation and scale of fluctuation distances of some geotechnical engineering soil properties characterizing the strength of soils.

Test Type	Soil Property	Soil Type	Autocorrelation Distance, r_o (m)		Scale of Fluctuation, δ (m)		Horizontal Vertical	Reference
			Vert.	Horiz.	Vert.	Horiz.		
Unconfined compressive strength	q_u	Chicago clay	0.4		0.79			Wu (1974)
CPT	q_c	Clay	0.52		1.04			Alonzo and Krizek (1975)
Vane	S_u	New Liskeard Clay			5	46	9.2	Vanmarcke (1977b)
Unconfined compressive strength	S_u	Nagoya Port Marine clay, Japan; NC			1.25			Matsuo and Asaoka (1977)
Unconfined compressive strength	S_u	Nagoya Port Marine clay, Japan; NC			2.5			Matsuo and Asaoka (1977)
Unconfined compressive strength	S_u	Horikawa bridge Marine clay, Japan; NC			1.82			Matsuo and Asaoka (1977)
CPT	q_c	North Sea clay		30				Tang (1979)
Vane	S_u	Organic soft clay, New York City	1.2					Asaoka and A-Grivas (1982)
Vane	S_u	Organic soft clay, New York City	3.1					Asaoka and A-Grivas (1982)
Unconsolidated Undrained	S_u	Offshore soil	3.6		7.1			Keaveny et al. (1989)
Direct Simple Shear	S_u	Offshore soil	1.4		2.8			Keaveny et al. (1989)
CPT	q_c	Offshore soil; CPT at 7m depth		13.9		24.6		Keaveny et al. (1989)
CPT	q_c	Offshore soil; CPT at 9m depth		37.5		66.5		Keaveny et al. (1989)
Vane	S_u	Marine clay Quebec, Canada	3	30			10	Soulie et al. (1990)
Unconfined compressive strength	q_u	Soft clay	2	40			20	Honjo and Kuroda (1991)
CPT	q_c	Muddy silty clay; Shanghai, China			0.4 to 0.8			Gao and Li (1993)
CPT	q_c	Muddy soft clay; Shanghai, China			1.0 to 1.5			Gao and Li (1993)
CPT	q_c	Greyish silty clay; Shanghai, China			0.6 to 1.2			Gao and Li (1993)
CPT	q_c	Dark green clay; Shanghai, China; (OCR>1)			0.5 to 0.8			Gao and Li (1993)

Test Type	Soil Property	Soil Type	Autocorrelation Distance, r_o (m)		Scale of Fluctuation, δ (m)		Horizontal Vertical	Reference
			Vert.	Horiz.	Vert.	Horiz.		
Vane	S_u	Clay		21.4				Christian et al. (1994)
Triaxial and direct shear	SHANSEP, S_u	Clay		7				Christian et al. (1994)
Vane	S_u	Very soft Bangkok clay	1.1	22.1			20	Bergado et al. (1994)
Piezocone and Vane	q_c	Sensitive clay, Saint-Hilaire, Canada	2					Chiasson et al. (1995)
CPT	q_c	Silty clay	1					Lacasse and de Lamballerie (1995)
CPT	q_c	Laminated clay		9.6				Lacasse and Nadim (1996)
CPT	q_c	Kelwick Clay; homogeneous; (OCR > 1)	0.88	0.97	-	-	1.1	Jaksa et al. (1997)
CPT	q_c	Kelwick Clay; homogeneous; (OCR > 1)			0.15	0.14 to 2	0.93 to 13.3	Jaksa et al. (2000)
CPT	q_c	Upper (yellow) clay; clayey silt to silty clay; Taranto, Italy; (OCR > 1); five tests			0.33 ave. (0.2 to 0.44)			Cafaro and Cherubini (2002); Cherubini et al. (2006)
		Lower (gray) clay; clayey silt to silty clay; Taranto, Italy; (OCR > 1); five tests			0.4 ave. (0.19 to 0.72)			
CPT	q_{c1N}	Clay			0.32			Uzielli et al. (2005)
CPT	q_{c1N}	Silty clay and clayey silt; Brindisi, Italy; (OCR > 1)			1.6			Cherubini and Vessia (2006)
CPT	q_c	Clean, uniform sand; Eglin Field, FL	1.1		2.2			Alonzo and Krizek (1975)
CPT	Friction ratio	Clean, uniform sand; Eglin Field, FL	0.625		1.25			Alonzo and Krizek (1975)
CPT	q_c	Clean sand	1.6					Kulatilake and Ghosh (1988)
CPT	q_c	Fine sand; Shanghai, China; Medium density			0.25 to 0.6			Gao and Li (1993)
CPT	q_c	Dense sand		37.5				Lacasse and Nadim (1996)

Test Type	Soil Property	Soil Type	Autocorrelation Distance, r_o (m)		Scale of Fluctuation, δ (m)		Horizontal Vertical	Reference
			Vert.	Horiz.	Vert.	Horiz.		
CPT	q_{c1N}	Sand			0.7			Uzielli et al., (2005)
CPT	q_{c1N}	Silty sand			0.35			Uzielli et al., (2005)
CPT	q_{c1N}	Silt			0.76			Uzielli et al., (2005)
CPT	q_c	Sand; three tests per each of two sites				0.3		Wickremesinghe and Campanella (1993)
		Hydraulic sandfill slope			0.7			Onisiphorou (2000), as cited in Hicks and Samy (2002)
CPT	q_c	North Sea soil		13.9				Keaveny et al. (1989)
CPT	q_c	North Sea soil		37.5				Keaveny et al. (1989)
CPT	q_c	Sand, hydraulic fill artificial island	0.95	12.1	-	-	12.7	Popescu et al., (1997)
CPT	q_c	Dense Sand, Wicklow, Ireland; four tests	0.25		0.4 to 0.83			Doherty and Gavin (2010)
Wave velocity tests; Down-Hole	P-wave	Alluvial Deposit; Fivizzano Village in Tuscany, Italy; two tests			1.8 to 2			Cherubini and Vessia (2006)
	SH-wave				1.4 to 2.6			

Notes

CPT is the cone penetration test, q_c is its tip resistance, and q_{c1N} is a normalized tip resistance.

In clays, silty clays, clayey silts, etc., variability in CPT indices as q_c and q_{c1N} correspond to variability of undrained shear resistance S_u .

OCR is the overconsolidation ratio; $OCR > 1$ means the soil is overconsolidated. NC means the soil is normally consolidated.

δ may be approximated as 2 times r_o for an exponentially decaying autocorrelation function.

D.1.8 Anisotropy in δ_v and δ_h values – generalizations published in the technical literature

Over the last 20 years, engineers researching the spatial correlation of engineering soil properties have collected and studied the vertical and horizontal scale of fluctuation (or autocorrelation) engineering soil property data and some have made generalized observations. Tables D-2 and D-3 list some of their generalized observations. For example, based on her (limited) study of values cited in the geotechnical literature on vertical and horizontal scale of fluctuation for CPT and undrained shear strength data for natural

(versus manmade) soils, Cherubini (2000) concludes that the values of the vertical scale of fluctuation, δ_v , are generally found between 0.5 and 2 m while for the horizontal scale of fluctuation, δ_h , are generally on the order of a 12 m. The Phoon and Kulhawy (1999a) data suggest values of r_o and δ in the horizontal direction to be more than one order of magnitude greater than the values in the vertical direction. Babu (2014) citing a Uzielli et al. (2005) vertical random field modeling of soil variability assessment of 40 CPT profiles in mostly sandy soils that were collected at five regional sites, makes this same conclusion.

Differences are to be expected among the (Table D-2) generalizations due to consideration of the amount and type(s) of engineering soil property data available and collected at the time each generalization is made. However, these generalizations appear to be consistent. Among the researchers, the Phoon and Kulhawy studies contained the largest soil test results data base(s). The Table D-1 data are for the most part for soil strength tests. These data collected by the authors of this report support these Table D-2 and D-3 generalized observations.

Table D-2. Generalizations on the magnitude of the vertical and horizontal scale of fluctuation distances of select geotechnical engineering soil properties characterizing the strength of soils.

Reference	Range in Vertical Scale of Fluctuation, δ_v (m)	Range in Horizontal Scale of Fluctuation, δ_h (m)	Notes
Cherubini (1997)	0.5 to 2 m	10 to 30 m	
Cherubini (2000)	0.5 to 2 m	On the order of 12 m	
Phoon and Kulhawy (1999a)	1 to 2 m	40 to 60 m	Laboratory measurement of undrained shear strength
	<1 m		Cone tip resistance
	<0.5 m		Corrected cone tip resistance
	2 to 6 m		Vane shear and standard penetration tests

Table D-3. Generalizations on the magnitude of the vertical and horizontal autocorrelation distances of select geotechnical engineering soil properties characterizing the strength of soils.

Reference	Range in Vertical Autocorrelation Distance, $(r_o)_v$ (m)	Range in Horizontal Autocorrelation Distance, $(r_o)_h$ (m)	Notes
El-Ramly (2001)	1 to 3 m	20 to 40 m	Generalized observations based on data reported in the geotechnical literature.
El-Ramly et al. (2002b, 2003a, 2003b)	1 to 3 m	10 to 40 m	

Note: δ may be approximated as 2 times r_o for an exponentially decaying autocorrelation function.

D.1.9 Is the value of δ (and r_o) a property of the soil mass?

Citing the study of different (engineering) soil properties of the same soil made by Bao et al. (1993), Cherubini (1997) asserts that δ would seem an attribute of a soil mass. During the technical review for this report, the authors noted that Bao, et al. (1993) article does not present supporting data but cites a 1992 Bao-authored article (published in Chinese) as the basis for this observation. However, the soil property data obtained from different laboratory engineering strength and in situ vane and CPT tests that were reported in Phoon and Kulhawy (1999a) do appear to support the assertion that δ would seem an attribute of a soil mass.

D.1.10 Value of the anisotropy factor, δ_h/δ_v

Hicks and Samy (2002) conclude that the clay data of Vanmarcke (1977b) supports an anisotropy factor ($= \delta_h/\delta_v$) of 9; the Soulie et al. (1990) data support a factor of 10; and the Phoon and Kulhawy (1999a) data support a factor of 13 (using mean VST results). Using CPT profiles taken from a hydraulic sandfill, Popescu et al., (1997) found δ_h to be on average 12 times greater than δ_v . Note that although the hydraulic sandfill structure was manmade, the value for the resulting anisotropy factor is consistent with those from natural geologic depositional environments that were discussed in this paragraph. One exception to these trends is the lower end estimate made by Jaksa et al. (1997) and Jaksa et al. (2000) for a homogeneous, stiff, overconsolidated Kelswick Clay of Adelaide, Australia, that indicates an anisotropy factor between 0.93 and 13.3 (using vertical and horizontal CPT profiles). These limited data on clay suggests that the degree of overconsolidation can influence the magnitude of the anisotropy factor; higher OCRs reduce the value for the anisotropy factor.

D.1.11 Impact of directionality of the scale of fluctuation on foundations

Because of the significant difference in δ_h and δ_v values, El-Ramley et al. (2003a) and others point out the importance to identify the direction in which the variability of geotechnical engineering properties has a dominant impact on the stability model used in the analysis. Three examples can be cited:

- The earth pressure on a retaining structure being controlled by the variability of the coefficient of earth pressure with depth.
- The soil-to-pile resistance of vertical or batter pile foundations.

In these first two examples, the spatial variability in the vertical direction is likely to be more important to the reliability analysis.

- The stability of slopes in presheared or jointed formations with shallow dip angles is controlled by the variability of strength parameters along a long, near-horizontal rupture surface. This example comes from a probabilistic stability analysis of a tailings dyke on pre-sheared clay shale made by El-Ramly et al. (2003a). Autocorrelation distances in horizontal directions are thus more relevant in this case.

In this third example of a sliding stability analysis, the spatial variability in the horizontal direction may be more important to the reliability analysis of system consisting of driving, structural, and resisting wedges if the structural wedge dominates the stability calculation.

D.1.12 Equivalent scale of fluctuation distance

Within a given layer, as a result of the geologic process, soil properties tend to be anisotropic and specifically, more variable in the vertical direction than in the horizontal direction. This is borne out by the Table D-1 data. Therefore, for sliding stability computations, the challenge is to account for scale of fluctuation anisotropy along the (potential) slip planes of driving and resisting soil wedges when these slip planes are not oriented along a plane that is neither vertical nor horizontal but is inclined (relative to these two orientations). Appendix E discusses four methodologies presented in the engineering literature for dealing with anisotropic scales of fluctuation. Each of these methodologies determine an equivalent scale of fluctuation.

D.1.13 Scale of fluctuation dependent upon the type of engineering soil test and engineering parameter

Although the general trend of the Table D-1 strength data is in support of Cherubini's (1997) assertion that δ would seem an attribute of a soil mass, it was also observed in the published data that the exact values for r_o and δ will differ among the geotechnical engineering soil parameters according to the type of soil test conducted (e.g., strength, stiffness, index properties, water contents).

D.1.14 Sampling interval

Lesser known but also important is the DeGroot and Baecher (1993) and Phoon and Kulhawy (1999a) observation that the r_o and δ values are strongly influenced by the sampling interval. Jaska et al. (1997) and Jaksa et al. (2000) also demonstrated this fact as well as the dependence of r_o on the degree of trend (i.e., order of polynomial) removed from the data using vertical and horizontal CPT data obtained from a relatively homogeneous, overconsolidated clay known as Keswick clay. It possesses properties similar to London clay.

D.2 Variability of design soil properties

To evaluate geotechnical soil property variability on a general basis that facilitates the use of reliability-based geotechnical design procedures, Phoon and Kulhawy (1999b) recognized that it is necessary to describe total soil variability in terms of

- inherent soil variability (w)
- soil test measurement error (e)
- transformation uncertainty (ε).

Each source of variability is assessed separately in the Phoon and Kulhawy procedure of probabilistic analysis. Inherent soil variability, caused primarily by the natural geologic processes that are involved in soil formation, was discussed in the previous section. It is depicted as $w(z)$ in Figure D-2 and labeled "deviation from trend." Soil test measurement error is due to the fact that the process of measurement introduces variability into the soil data. It arises from equipment, procedural-operator, and random testing effects (Phoon and Kulhawy 1999a). The direct measurement from a geotechnical test typically is not directly

applicable to the geotechnical engineering design property. Instead, a transformation model is needed to relate the test measurement to an appropriate design property. One such example is the uncertainty in the correlation (i.e., the transformation model) between the SPT field test measurement and the unconfined compressive soil strength parameter, q_u , (Table 45.2 in Terzaghi and Peck [1967]) for clays. S_u is equal to $q_u/2$. The SPT-N value measured in the field is not a geotechnical engineering soil property but the unconfined compressive soil strength value is.

D.2.1 Spatial average

Considering pile foundation design, Phoon and Kulhawy (1999b) suggest the use of a spatial average of design geotechnical property ξ_d (e.g., $\xi_d(z) = S_u(z)$) over some (vertical) length l_v rather than the design property at a point. The spatial average of variable ξ_d is defined as

$$\xi_a = \frac{1}{l_v} * \int_{l_v} \xi_d(z) dz \quad (D.8)$$

In the design of a pile embedded in a homogeneous soil (i.e., a single soil layer), l_v would be the pile's embedment depth.

For embankments, slopes, retaining structures, hydraulic structures, etc., founded on a soil stratum consisting of weak horizontal layer(s) where more horizontal-type sliding surfaces are of concern, the length used in Equation D.8 becomes l_h , a near-to-horizontal distance along the potential slip surface within each homogeneous soil layer. Examples of reliability studies of this type are discussed in detail for the proposed Canadian James Bay dyke project in Christian et al. (1994) and for a tailings dyke founded on a presheared clay–shale in El-Ramly et al. (2003a).

D.2.2 COV_{ξ_a} of spatial average

Phoon and Kulhawy (1999b) express the total COV for COV_{ξ_a} of the spatial average (ξ_a) of the soil property as

$$COV_{\xi_a}^2 = \Gamma^2(l_v) * COV_w^2 + COV_e^2 + COV_\varepsilon^2 \quad (D.9)$$

where the three components of variance are determined based on values of the COV_w of inherent soil variability, COV_e of measurement error and COV_ε of transformation uncertainty. A variance reduction function $\Gamma^2(l_v)$

is also used in this computation. The value of $\Gamma^2(l_v)$ ranges from a low of zero to a high of 1 and depends on the length of the averaging interval, l_v . The approach used to define the variance reduction term applied to the Equation D.9 coefficient of variation term makes use of the following Vanmarcke (1983, 2010) approximations: For the vertical scale of fluctuation δ_v greater than or equal to the vertical averaging length l_v ,

$$\Gamma^2(l_v) = 1 \quad (\text{D.10})$$

while

$$\Gamma^2(l_v) = \frac{\delta_v}{l_v} \quad (\text{D.11})$$

with δ_v less than l_v . For a potential foundation sliding limit state in which horizontal layering is of concern, δ_h is substituted for δ_v in the qualifying criteria for Equation D.10 and in Equation D.11, and l_v is measured in the horizontal direction and now designated as l_h . For directions that are neither perfectly vertical nor horizontal, please refer to Appendix E and procedures used to compute an equivalent scale of fluctuation, δ_E . The distance of the slip plane of any orientation is given as L ($L = L_{Active}$ in Figure 5-7). The results from research by the authors of this report into this topic for hydraulic structures is discussed in Chapters 4 and 5.

Observation

For the approach suggested by Phoon and Kulhawy (1999a,b), inherent soil variability is modeled as random field (Vanmarcke 1983), with its COV_w value adjusted by a variance reduction function $\Gamma^2(l_v)$ in Equation D.9. The significance of Equation D.11 is that the COV_ξ of the spatial average is smaller in comparison to the COV_w of inherent soil variability when the averaging length l_v is greater than the (vertical) scale of fluctuation δ_v value.

The scale of fluctuation value δ_v reflects the autocorrelation distance of the design soil property. Consequently, the authors of this report find it useful to recall that the scale of fluctuation is approximately twice the autocorrelation distance (i.e., $\delta = 2 r_o$) for the exponentially decaying autocorrelation function, whose value of δ reflects the correlation distance of the soil's engineering design property.

Inherent soil variability is caused primarily by the natural geologic processes that are involved in soil formation. Depending on the type of laboratory or field soil test involved, each type of test will have a unique COV due to measurement errors for that testing procedure.

D.2.3 COV values for soil strengths compared to those for steel resistances

As pointed out by Phoon and Kulhawy (1999b) and others, the difference between geotechnical and structural variabilities is significant and is accounted for differently in a reliability analysis. Most structural COV are fairly small compared to geotechnical COV. They point out that, for example, the COV of undrained shear strength (S_u) of clays can vary from 10% to 60%, depending upon its form of measurement (e.g., direct laboratory measurement or correlated with a field measurement). In contrast to geotechnical materials and citing the landmark study made by Ellingwood et al. (1980), they observe that uncertainties in structural resistances typically fall within a range of 10% to 20% for a wide range of materials (e.g., concrete, steel, aluminum) and resistance model (e.g., tension, flexure, shear). Consequently, the differences between geotechnical and structural variability, as reflected in COV values, impact the development of reliability-based design procedures for geotechnical engineering, in contrast to those developed for structural engineering.

D.2.4 Point and spatial average COV values from the Phoon and Kulhawy (1999b) soil strength data assessment

Phoon and Kulhawy (1999b) point out that structural loads and resistances usually can be modeled adequately as lumped parameters in a reliability calibration process. In contrast, they point out that it is no longer realistic to model geotechnical capacities as lumped parameters because of spatial correlation of materials. Therefore, Phoon and Kulhawy (1999a,b) took care to distinguish inherent variability of strength properties for COV point estimates from those pertaining to spatial average COV values¹. Table D-4 summarizes Phoon and Kulhawy (1999b) approximate guidelines for design soil property variability in terms of the

¹ Chapter 3 addresses variance reduction techniques that are used to quantify vertical and horizontal spatial average COV values. Application of variance reduction techniques of the type suggested by Phoon and Kulhawy (1999a,b) lower the magnitude of COV values for use in geotechnical engineering applications from point estimate COV values by as much as 50%. Vanmarcke expressed this general trend in his 1977a reference on random fields that it is not soil parameter values at a point but the average over a certain size of line, area or volume (i.e., the local average) that controls geotechnical limit state(s) performances.

two COV values for different soil types. Note that they used vertical spatial average COV values in contrast to horizontal spatial average values. Additionally, the vertical spatial average was computed over a vertical distance (L) of 5 m.

Table D-4. A summary of Phoon and Kulhawy (1999b; Table 5) approximate guidelines for design soil property variability.

Design Strength Parameter ^a	Test b	Soil Type	Point COV (%)	Vertical Spatial Average COV (%) with $L = 5$ m
S_u (UC)	Direct (lab)	Clay	20-55	10-40
S_u (UU)	Direct (lab)	Clay	10-35	7-25
S_u (CIUC)	Direct (lab)	Clay	20-45	10-30
S_u (field)	Correlated to a VST (field)	Clay	15-50	15-50
S_u (UU)	Correlated to a cone tip resistance (q_T)	Clay	30-40	30-35
S_u (CIUC)	Correlated to a cone tip resistance (q_T)	Clay	35-50	35-40
S_u (UU)	Correlated to a SPT blow count N value (field)	Clay	40-60	40-55
S_u (field)	Correlated to a dilatometer horizontal stress index K_D value	Clay	30-55	30-55
S_u (field)	Correlated to a PI value (for UU, UC and VST S_u values)	Clay	30-55	-
ϕ'	Direct (lab)	Clay, sand	7-20	6-20
ϕ' (TC)	Correlated to a cone tip resistance (q_T)	Sand	10-15	10
ϕ'_{CV}	Correlated to a PI	Clay	15-20	15-20

Note: ^a S_u , undrained shear strength; UC, unconfined compression test; UU, unconsolidated undrained triaxial test; CIUC, consolidated isotopically undrained triaxial test – compression; ϕ' , effective angle of internal friction; ϕ'_{CV} , constant-volume effective angle of internal friction.

Appendix E discusses four techniques used to quantify vertical and horizontal spatial average COV values into an equivalent scale of fluctuation. Application of variance reduction techniques of the type suggested by Phoon and Kulhawy (1999a,b) lower the magnitude of COV values for use in geotechnical engineering applications from point estimate COV values by as much as 50%. Vanmarcke expressed this general trend in his 1977a reference on random fields that it is not soil parameter values at a point but the average over a certain size of line, area or volume (i.e., the local average, that controls geotechnical limit state(s) performances).

D.3 The width of 3-D slope failures

A method of probabilistic 3-D limit equilibrium analysis stability of long earth slopes was first discussed in Varmarcke (1977b, 1980). This method accounts for the spatial variability of the shear strength. The spatial variability takes into account the correlation effects of soils near the initial failure cross section, so this model predicts that slope failure events involving very long or very short widths of the failure zone are highly improbable. The contribution of 3-D end resisting moment effects, the value for the horizontal scale of fluctuation, δ_h , and the variance reduction function (into the third dimension) play important roles in 3-D probabilistic slope stability analyses and likely slope failure widths. Vanmarcke (2011) matured this probabilistic engineering methodology vision in his 2011 paper on the risk of limit-equilibrium failure of long earth slopes and how it depends on length. Again, the value of the horizontal scale of fluctuation in the third dimension (of slope width) and the correlation structure of the resisting moment/force as represented by the variance reduction function in the third dimension (i.e., along the slope) plays a dominant role in the dictating the width along the slope axis of the 3-D slope failure in the analysis. A third Vanmarcke paper published in 2013 (Vanmarcke and Otsubo 2013), extended the Vanmarcke 3-D reliability analysis to long multi-segment earth slopes.

Appendix E: Methodologies for Dealing with Anisotropic Scales of Fluctuation in Sliding Stability Analyses

E.1 Introduction

As a result of the geologic process and within a given layer, soil properties tend to be anisotropic and specifically, more variable in the vertical direction than in the horizontal direction. Anisotropy of soil engineering properties is typically found during geotechnical engineering site investigations and interpretation of the in situ test measurements and the subsequent geotechnical laboratory tests conducted on soil specimens recovered from the site. The engineering strength anisotropy in the vertical and horizontal direction results in anisotropic scales of fluctuation in different soil types. This is borne out by the Table D-1 data. Therefore, for sliding stability computations, the challenge is to account for scale of fluctuation anisotropy along the (potential) slip planes of driving and resisting soil wedges when these slip planes are oriented along a plane that is neither vertical nor horizontal but is inclined (relative to these two orientations). This appendix discusses four methodologies presented in the engineering literature for dealing with anisotropic scales of fluctuation.

E.2 The Vanmarcke (1980) numerical procedure for calculating the variance of the integral of a random process which depends on a spatial coordinate

E.2.1 Introduction

Vanmarcke (1980) introduced methods for calculating variance in soil strength properties along segmented intervals that follow the path of a potential slip plane using the scale of fluctuation, δ , of the soil region. Recall that the scale of fluctuation is related to autocorrelation distance in random field theory. And the autocorrelation distance, r_o , is the distance that can be traveled from a point sample until the variance properties change significantly with a correlation limit of $1/e$ (i.e., 0.37). This limit establishes the correlation coefficient between neighboring soil points. The scale of fluctuation is typically defined as 2 times the autocorrelation distance for exponential autocorrelation models (Equation D.6) and 1.7 times the autocorrelation distance for Gaussian autocorrelation models. References to other relationships are listed in subsection D.1.5. For the

interval of distance δ , the variable soil properties have a high degree of correlation. Points at a greater distance are assumed to be uncorrelated.

This section deals with calculating the spatial variance of the strength parameters for intervals in isotropic and anisotropic homogeneous soil regimes. Numerical methods for integrating interval contributions are introduced for determining variance on a piece-wise non-linear slip plane. The intended usage is for a USACE-type retaining system comprising driving, structural, and resisting wedges. Methods will also be introduced to calculate the inter-correlation between intervals.

E.2.2 Computing an equivalent scale of fluctuation for anisotropic materials with eccentric slip planes

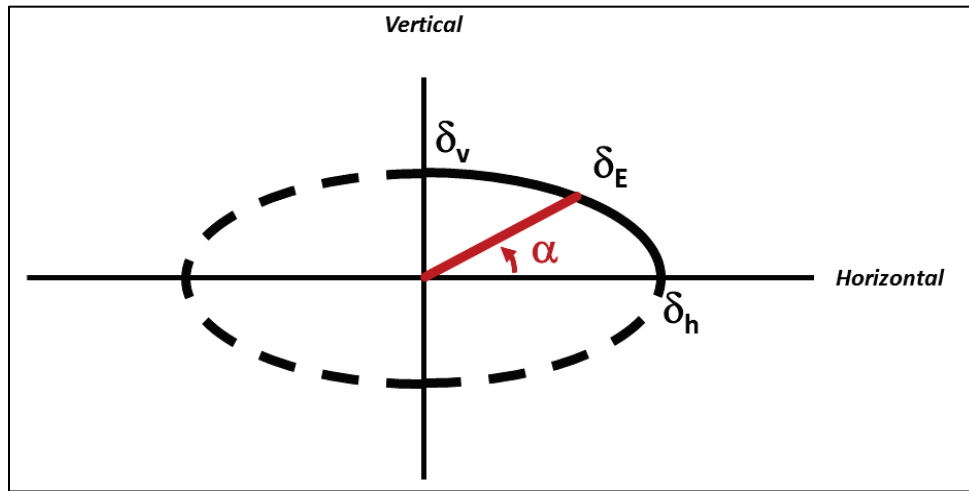
In 1980, Vanmarcke introduced a methodology for accounting for the spatial variability of the shear strength in soils for use in 3-D probabilistic stability analysis of earth slopes. This procedure explicitly accounts for different scales of fluctuation values in the horizontal and vertical directions (i.e., using Vanmarcke's [1980] coordinate notation, $\delta_z < [\delta_x = \delta_y]$).

Figure E-1 shows an elliptical representation of how an equivalent scale of fluctuation for a slip plane along the angle α , δ_E , can be estimated in 1980 by Vanmarcke from the horizontal scale of fluctuation, $\delta_h = [\delta_x = \delta_y]$, and the vertical scale of fluctuation, $\delta_v = \delta_z$. The equation to calculate this estimate is

$$\delta_E = \sqrt{\delta_v^2 \sin^2 \alpha + \delta_h^2 \cos^2 \alpha} \quad (\text{E.1})$$

This equivalent scale of fluctuation, δ_E , will always be a positive value. Because of this, the graphical representation of Equation E.1 is represented by the red line in quadrant I in Figure E-1.

Figure E-1. Graphical description of Vanmarcke's 1980 procedure used to define the equivalent scale of fluctuation δ_E for a base of wedge slip plane at angle α from horizontal.



E.2.3 Calculating variance of the integral of a random process

In the appendix of Vanmarcke (1980), Vanmarcke introduced a method to determine the variance along a continuous failure arc, or slip-plane. This procedure is summarized in this subsection. Since Vanmarcke was working with a circular failure arc, his description was in arc-lengths. The length of the entire failure arc was given as l . The failure arc was broken into subdivisions where subdivision i 's length was designated as length l_i and

$$\sum_{i=1}^n l_i = l \quad (\text{E.2})$$

The mean, variance, and scale of fluctuation could vary across the subdivisions. For subdivision i , the variance at the midpoint is denoted by $(\tilde{S}_i)^2$ for a standard deviation designated as $\tilde{S}_i$ and for a scale of fluctuation denoted δ_i . For a circular slip-plane arc example used throughout the Vanmarcke (1980) paper, the line tangent to the midpoint of subdivision i is parallel to the line through the endpoints of the arc, so angle α_i is simple to compute for the equivalent scale of fluctuation δ_i in Equation E.1. Geometrically, the slip plane becomes the cord length of each i^{th} subdivision. This is an appropriate approximation with low error for any continuous slip-plane, given enough subdivisions n . For linear slip-planes, there should be no reason to subdivide further.

Recall that the scale of fluctuation describes a distance over which there is a distinct correlation between material properties. With δ_i changing because of anisotropy in the horizontal and vertical space using the

approximation of Equation E.1, the distance traveled along the continuous slip-plane before the values change significantly must be calculated. This calculation can be accomplished by summing the ratios of the subdivision lengths to the subdivision scale of fluctuation until the value of 1.0 is exceeded at subdivision k .

$$\sum_{i=1}^k \frac{l_i}{\delta_i} \leq 1.0 \quad (\text{E.3})$$

If Equation E.3 is greater than 1.0, then the excess must be estimated and removed from the summation. This excess is in the k^{th} contribution. Keeping the δ_k value, the excess length $\dot{l}_k$ is computed

$$\frac{\dot{l}_k}{\delta_k} = \sum_{i=1}^k \frac{l_i}{\delta_i} - 1.0 \quad (\text{E.4})$$

and the residual $\dot{l}_k = l_k - \dot{l}_k$ so that

$$\sum_{i=1}^{k-1} \frac{l_i}{\delta_i} + \frac{\dot{l}_k}{\delta_k} = 1.0 \quad (\text{E.5})$$

The total length where the values are correlated is therefore

$$\Delta l = \left(\sum_{i=1}^{k-1} l_i \right) + \dot{l}_k \quad (\text{E.6})$$

Vanmarcke estimates the contribution to the variance due to this Equation E.6 length as

$$\left(\left(\sum_{i=1}^k (\tilde{S}_i)^2 l_i \right) + (\tilde{S}_k)^2 \dot{l}_k \right) \Delta l \quad (\text{E.7})$$

This equation is especially useful for analysis of the resisting wedge with a nonlinear slip plane surface and with anisotropic scales of fluctuation. The next contributing length is calculated using the same method with the new start point

$$\frac{\ddot{l}_k}{\delta_k} + \sum_{i=k+1}^{j-1} \frac{l_i}{\delta_i} + \frac{\dot{l}_j}{\delta_j} = 1.0 \quad (\text{E.8})$$

Equations E.5 and E.6 must be altered to accommodate the new distances. This procedure is followed until a value of 1.0 cannot be reached at the end of the span.

$$\frac{\ddot{l}_k}{\delta_k} + \sum_{i=k+1}^n \frac{l_i}{\delta_i} < 1.0 \quad (\text{E.9})$$

For a single soil deposition region with similar properties and anisotropic variation, this procedure can be used to break the slip-plane into different segments with each segment having a single variable distribution, where there is minimal correlation coefficient between variables, and the variance is described in Equation E.7.

For a homogeneous material region ($\tilde{S}_i = \tilde{S}$) possessing a slip-plane with a length that is shorter than the scale of fluctuation, δ_r , for the reach, the variance for the distribution across the entire reach is entirely correlated and is given by

$$\left(\sum_i (\tilde{S}_i)^2 \Delta l_i \right) l \quad (\text{E.10})$$

which reduces to

$$(\tilde{S}_r)^2 (l_r)^2 \quad (\text{E.11})$$

where l_r is the length of the reach. For anisotropic scale of fluctuations, the equivalent scale of fluctuation δ_E can be computed via Equation E.1 and substituted into Equation E.5 to compute the length of correlation. Recall that this Equation (E.5) defines the length of correlation for anisotropic scale(s) of fluctuation. For large scales of fluctuation in a homogeneous soil, this equation may be applied to the structural wedge possessing a *short* length of slip plane relative to the scale of fluctuation. It is also relevant when a short slip plane crosses through homogeneous soil layers.

For a homogeneous material region ($\tilde{S}_i = \tilde{S}$) with an isotropic scale of fluctuation (where $\delta_{s,I} = \delta_r$) and with a length that is longer than the scale of fluctuation, δ_r , for the reach, the variance for the distribution across the entire reach, which must take the loss of correlation into account, is given by

$$(\tilde{S}_r)^2 \delta_r l_r \quad (\text{E.12})$$

where l_r is the length of the reach. This is especially useful for a horizontal slip plane below the structural wedge because δ_h is the only scale of fluctuation in use for this special case.

For a homogeneous material region ($\tilde{S}_i = \tilde{S}$) with a linear slip-plane, the equivalent scale of fluctuation, δ_E is constant and $\delta_{s,E} = \delta_E$. For this situation, if the length of the slip-plane is longer than the equivalent scale of fluctuation, $\delta_{r,E}$, for the reach, the variance for the distribution across the entire reach, which must take the loss of correlation into account, is given by

$$(\tilde{S}_r)^2 \delta_{r,E} l_r \quad (\text{E.13})$$

where l_r is the length of the reach. This is especially useful for analysis of the driving wedge with a (constant) linear slope for the slip plane. But it can also be used in the case of a structural wedge with a constant but inclined slip plane.

In 1977 Vanmarcke (1977a,b) published a pair of landmark papers introducing a methodology for performing probabilistic modeling of soil profiles and for the reliability analysis of earth slopes. Vanmarcke (1977a) started with the characterization of spatial, inherent uncertainty of an engineering soil property (e.g., undrained shear strength) about a spatial average with distance, area, or volume. Using the example of engineering parameter $\underline{u}(z)$ that varies with elevation z , the larger the length z of consideration over which $u(z)$ is averaged, the more the fluctuation of u tend to cancel in the process of spatial averaging. This trend tends to cause a reduction in the standard deviation as the size of the averaging length (or volume) increases. The dimensionless ratio between the bounded region standard deviation (about the true spatial average) divided by the standard deviation for the entire data set (i.e., a point estimate) is defined as the ratio

$$\Gamma_u(\Delta z) = \frac{\tilde{u}_{\Delta z}}{\bar{u}} \quad (\text{E.14})$$

Vanmarcke observes that when $\Delta z = 0$, the spatial average $u_{\Delta z}$ becomes just a point value of u , and therefore $\Gamma_u(0) = 1$. At larger Δz , the function $\Gamma_u(\Delta z)$ measures the decay of the standard deviation, $\tilde{u}_{\Delta z}$, relative to the point root mean square value $\bar{u}$. The square of this reduction factor is the variance reduction function, $\Gamma^2(\Delta z)$.

The scale of fluctuation, δ , can also be defined by the ratio $n_e = \Delta z / \delta$, which should equal the number of uncorrelated observations within the Δz range. Using this approximate relationship for large Δz , the variance $(\tilde{u}_{\Delta z})^2$ is

equal to $(\tilde{u})^2$ divided by n_e . For small values of Δz , the number of uncorrelated values within the range becomes exceedingly small, implying complete correlation. Vanmarcke (1977a) and Vanmarcke (1983, 2010) with his Case 2 exponential correlation function, provided an approximation of the variance reduction function in terms of scale of fluctuation as

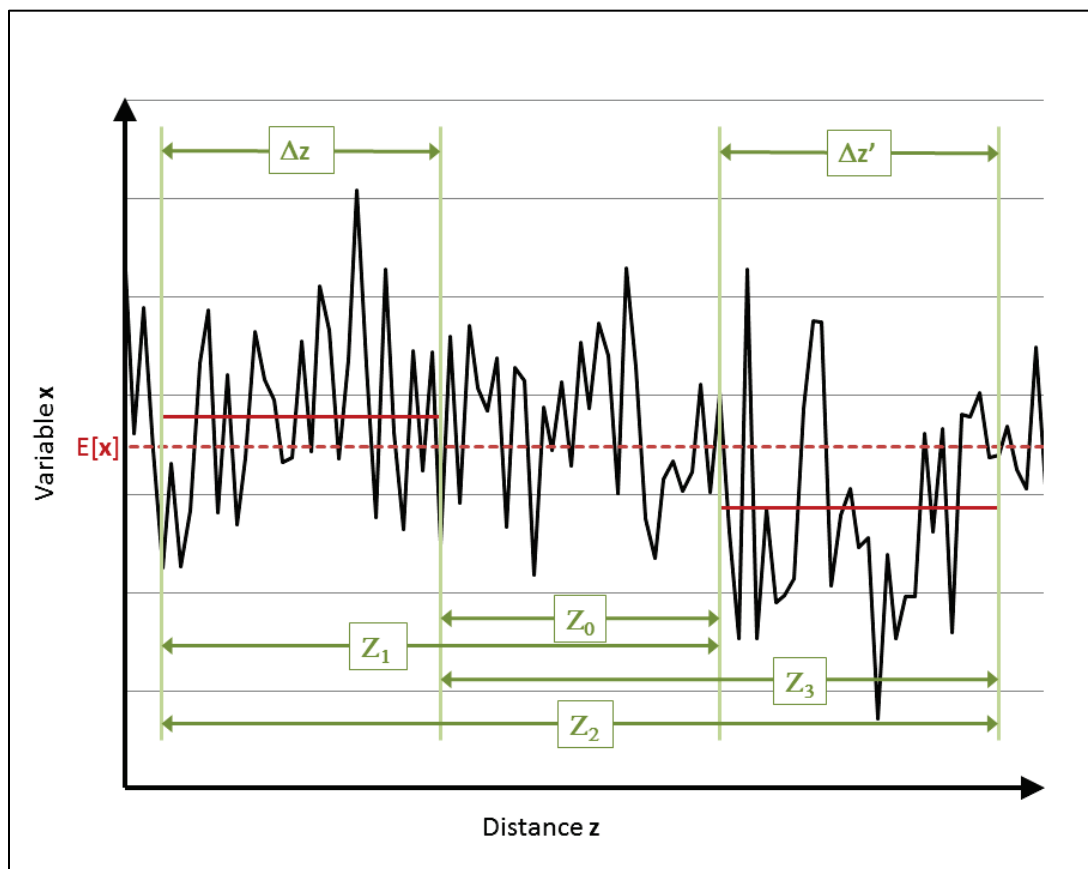
$$\gamma(\Delta z) = \Gamma^2(\Delta z) = \begin{cases} 1 & \Delta z \leq \delta \\ \frac{\delta}{\Delta z} & \Delta z \geq \delta \end{cases} \quad (\text{E.15})$$

Recall from subsection D.1.5 that for an exponentially decaying autocorrelation function, the scale of fluctuation is twice the autocorrelation distance r_o (Equation D.6). Equation E.15 requires that a scale of fluctuation is derived from a set of point samples (at distances less than the scale of fluctuation). The first conditional of Equation E.15 is an estimate for the point set taking into account that the aggregation of data and its processing will provide no significant difference for distances less than the scale of fluctuation. Equation E.15 is graphically depicted in Figure 5-2 of Vanmarcke (2010) along with variance reduction functions for other autocorrelation functions. The variance reductions function approximation(s) can be used to approximate the spatial correlation between aggregations of point data, as discussed subsequently.

Figure E-2 shows a set of point samples for variable $\mathbf{x}$ in a random field with two intervals of length Δz and $\Delta z'$. With point estimates, it is easy to see that the local variations in these two intervals differ from the overall variation, and from each other.

These local variations for the two intervals are spatially correlated. Equation E.16 (El-Ramly et al. 2002a) computes the correlation between the two intervals given a distance, Z_0 , between the two intervals. This equation takes into account the variance reduction factor of Equation E.15, reducing correlation significantly as the distance between the intervals (Z_0) grows.

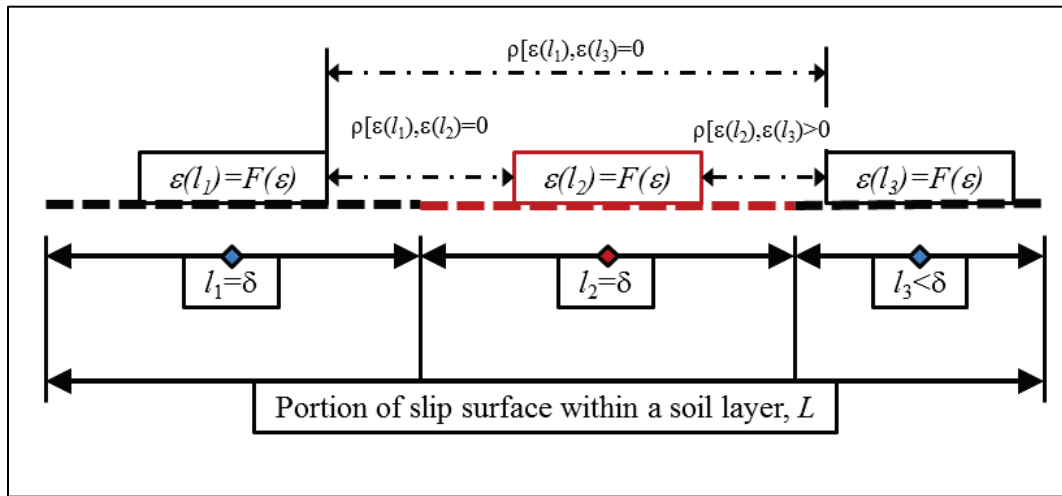
Figure E-2. Point samples from a random field with a mean of $E[x]$, and with local groupings with intervals of length Δz and $\Delta z'$ (after EL-Ramly et al. 2002).



$$\rho(X_{\Delta z}, X'_{\Delta z'}) = \frac{Z_0^2 \Gamma^2(Z_0) - Z_1^2 \Gamma^2(Z_1) + Z_2^2 \Gamma^2(Z_2) - Z_3^2 \Gamma^2(Z_3)}{2\Delta z \Delta z' \sqrt{\Gamma^2(\Delta z) \Gamma^2(\Delta z')}} \quad (\text{E.16})$$

Figure E-3 shows the results of applying this interval method for a continuous surface (such as a slip plane). The procedure, attributed to Vanmarcke (1983, 2010), models the 1-D random field spatial variability (for a single isotropic material) of the strength parameter along a failure surface. In this case, the surface is divided into intervals of length equal or less than the scale of fluctuation, δ . This reveals that using intervals based on the scale of fluctuations will provide the minimal amount of variables required and also has the lowest correlation.

Figure E-3. The effects of dividing a surface region in a random field into sections with length equal to the scale of fluctuation, δ (after EL-Ramly et al. 2002).



This paragraph discusses the correlation within an interval and between intervals, when the interval lengths are constrained by the scale of fluctuation. Equation E.16 can be simplified by the appropriate selection of interval lengths so that the variance reduction function approximation of Equation E.15 returns a value of 1.0 for intervals Δz and $\Delta z'$. This means that the internal correlation within each of the intervals (Δz and $\Delta z'$) is 1.0. This occurs when the Δz and $\Delta z'$ are less than or equal to the scale of fluctuation δ . The minimal number of intervals in the slip plane with internal interval correlation of 1.0 would then be the total length of the slip plane divided by the scale of fluctuation (δ) with a final interval length that can be less than δ . For this situation, Z_0 is always given as a multiple of δ . When the multiple of Z_0 is greater than 0, Equation E.16 returns a value for the inter-interval correlation (ρ) of 0.0. For the situation where $Z_0 = 0$ and $\Delta z = \Delta z' = \delta$ in Figure E-2, the inter-interval correlation coefficient (ρ) = 0.0. It is only when $Z_0 = 0$ and $\Delta z = \delta > \Delta z'$ that the inter-correlation between the intervals becomes greater than 0.0. All of this is illustrated in Figure E-3 using three intervals, with the first two intervals of length l_1 and l_2 are set equal to δ and the third interval length l_3 is less than δ . Therefore, breaking the slip plane into regions of length equal to the scale of fluctuation (δ) creates the minimum number of inner-correlated variables for simulation and reliability studies that also have the minimal amount of spatial inter-correlation.

E.2.4 El-Ramly simplification of the Vanmarcke method at James Bay Dykes

In 1994, Christian et al. described a formal calculation of reliability and risk with respect to the design of dykes along 50 km for a James Bay hydroelectric project. This slope stability assessment was used to help decide among the alternatives to new dyke construction on a soft clay foundation. The first alternative was construction of the embankment in a single stage, to either a height of 6 or 12 m with one berm. The second alternative involved the construction of the dyke in multiple stages to allow the soft clay foundation to dissipate the excess pore water pressures generated during berm construction: Following the installation of vertical drains within the clay foundation, the first stage would consist of two berms built to a height of 12 m. Then the zone with the drains would be allowed to consolidate to 80% of primary consolidation, and the second stage would be built to a height of 23 m. Their analysis formally considered the uncertainty in soil strength parameters in the reliability-based slope stability, culminating in the calculation of a probability for slope failure, P_f , for each dyke construction alternative. This study was special because of the collection of a significant amount of geotechnical data for a database of soil properties and strength data as well as the fact that Drs. Christian and Baecher are pioneers in the application of reliability theory to solve geotechnical foundation problems. Also, Dr. Ladd is recognized as an expert in characterizing the strength of soft clays, which was a key aspect of the design challenge for the construction of these dykes at this project. Today, this study is considered a landmark reliability-based study by the geotechnical community even though the project was not built.

In 2002, El-Ramly et al. revisited the reliability-based calculation for the proposed design of the 50 km of dykes for a James Bay hydroelectric project. Only one design was re-evaluated in their study: the single stage construction of a 12 m high embankment with a slope angle of 18.4 degrees (3H:1V) and a 56 m wide berm at midheight. Three probabilistic procedures of analysis were used: (1) an integrated slope stability analysis based on the use of a Monte Carlo simulation technique, (2) an FOSM approach in which the PDF distribution for the FoS against sliding failure was assumed to be either normal or lognormal (to the base e), and (3) a simplified analysis that ignored the spatial variability of soil properties. This latter analysis was made to contrast with the results of the first two analyses in which the spatial variability of soil properties was accounted for by using a variance reduction function and an autocorrelation

distances (each with an corresponding scale of fluctuation distance) for soil strength along the slip plane within each of the soil layers. Their probabilistic slope stability assessment made use of the Bishop method of slices for circular slip surfaces. Note that this type of stability analysis cannot accommodate wedge-shaped and curved noncircular surfaces constructed using planar segments. The probabilistic FOSM results for the El-Ramly (2002) re-evaluation are summarized in Table 2-4 given the single stage construction of a 12 m high embankment. Recall from Chapter 2 that the Christian et al. (1994) target Pf -100 for a 100-year design life is equal to 1×10^{-3} (Table 2-3). It is included in this table for reference. The El-Ramly Table 2-4 results using an updated reliability based procedure of slope stability analysis show that their expected FoS values are somewhat lower than the Christian et al. results; the annual probability of failure values are much higher. Additionally, their reliability index values are lower than the Christian et al. (1994) target value.

Using strength and unit weight data collected for this proposed dyke project from Christian et al. (1994), El-Ramly et al. (2002a) divided the slip plane into variable reaches using the material region boundaries and scales of fluctuations for the each material. The variable reaches were created to have a high inner correlation and very low inter-correlations between reaches, as discussed in Section E.2.3.

The only material region with a significant slip-plane length that would exceed the scale of fluctuation was the bottom, lacustrine clay region. Because the height of this material region was less than the vertical scale of fluctuation, it was assumed that horizontal variability would have the greatest effect on the slip-plane calculations. The slip-plane was therefore divided into reaches using the horizontal scale of fluctuation for the lacustrine clay region by El-Ramly et al. (2002a). In the marine clay, located directly above the lacustrine clay layer, there was a driving wedge region and a resisting wedge region along the investigated slip plane. The lengths of these two slip plane reaches in the marine clay layer were less than its scale of fluctuation.

All in all, there were five reaches used in the strength simulations via the Monte Carlo method of slope stability analysis as formulated by El-Ramly et al. (2002a) and shown in their Figure 6. There were two marine clay regions and three Lacustrine clay regions.

The authors of this report express concern over the use of a horizontal scale of fluctuation for use in the three arc lengths of the potential slip surface contained within the lacustrine clay. A more accurate approach would be to compute an equivalent scale of fluctuation with vertical and horizontal scales of fluctuations using one of the four methodologies described in this appendix.

E.3 The Vanmarcke (1977b) equivalent scale of fluctuation distance δ_L

In 1977, Vanmarcke (1977b) suggested the computation of an equivalent scale of fluctuation distance, δ_L , as one approach to account for the vertical and horizontal scale of fluctuation anisotropy.¹ Referring to the case of anisotropic strength data and using Vanmarcke's (1977b) coordinate notation, he proposed an equivalent scale of fluctuation distance δ_L given by

$$\frac{1}{\delta_L} = \frac{1}{\delta_v} * \left[\frac{\Delta z_1 + \Delta z_2}{L} \right] + \frac{1}{\delta_h} * \left[\frac{\Delta x}{L} \right] \quad (\text{E.17})$$

In the anisotropic case (i.e., when $\delta_v \neq \delta_h$) it is considered reasonable to approximate the failure arc by segments Δz_1 , Δx , and Δz_2 , for which the sum $\Delta z_1 + \Delta x + \Delta z_2$ is equal to the arc length L . Professor Vanmarcke defines the equivalent number of independent contributions along the failure arc as $[(\Delta z_1 + \Delta z_2)/\delta_v] + (\Delta x/\delta_h)$, a quantity that equals L/δ_L in the isotropic case.

E.4 Transformation of the anisotropic scale of fluctuation into an isotropic model

E.4.1 Introduction

In recent years, select researchers within the geotechnical community have been conducting research that combines Random Field (RF) theory with the Finite Element Method (FEM) of analysis to account for spatial variability of geotechnical parameters for soils. Honjo and Otake (2013) provide an overview of three categories of representative methods being used to generate multidimensional random fields and examples of the

¹ Vanmarcke's δ_L refers to an equivalent scale of fluctuation which was referred δ_E notation used earlier in this report.

methodologies being used in each of the three categories. These categories are the following:

- frequency domain technique
- turning bands method
- local average sub-division.

Researchers like Professors G. A. Fenton and D. V. Griffiths have published an extensive list of research results for investigations using the local average sub-division method of analysis applied to the soil regime(s). Fenton and Griffiths (2008) published *Risk Assessment in Geotechnical Engineering* in which they discuss the application of random finite element method in their geotechnical engineering based response analyses of soil regimes. Several of their publications are in the United States and Canadian technical literature, and Honjo and Otake (2013) provide a list of their publications. In Europe, engineers like Professor M. A. Hicks and his research associates have been conducting RF-FEM studies of a similar nature and have been publishing their results in the European technical literature. One of the technical issues faced in RF-FEM analyses is how to account for anisotropic scales of fluctuation in the finite element models of soil regimes. Hicks and Spencer (2010) discuss a two-step process for a 3-D problem of initially generating a random field for scales of fluctuations in all three directions set equal to the same value, the maximum value among the three axes for the site. Random fields of spatial variability are generated using the statistics of undrained shear strength of the soil and a 3D algorithm based on local average subdivision. Next, an anisotropic random field is produced by post-processing the isotropic field of step 1 based on the ratios of the scales of fluctuation in the other two dimensions, compared to the largest scale of fluctuation.

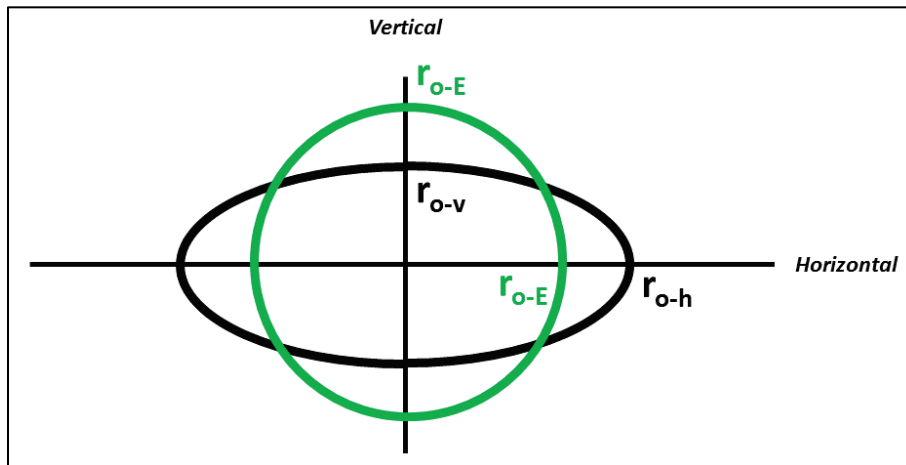
Other approaches can involve a step in which there is a transform of the statistical data from an anisotropic soil regime into an *equivalent* geologically homogeneous regime. Honjo and Otake (2013), among others, discuss the assumptions made to simplify and idealize the problem. This step is explored further in the next subsection.

E.4.2 Equivalent isotropic model

One approach for dealing with anisotropic scales of fluctuation (i.e., $\delta_v \neq \delta_h$) in probabilistic engineering analyses is to transform this space into an isotropic spatial structure with an equivalent isotropic autocorrelation

distance r_{o-E} . Recall from Equation D.6 that the scale of fluctuation δ value is twice the autocorrelation distance r_o , for an exponentially decaying autocorrelation function, ρ (Equation D.5).¹ One procedure is as follows: In El-Ramly's 2001 doctoral dissertation research study of probabilistic analyses of landslide hazards and risks and citing Journel and Huijbregts (1978), horizontal isotropy is assumed, and the variation of the autocorrelation distance in the vertical plane is represented by an ellipse. The radius of the ellipse in the horizontal direction is the horizontal autocorrelation distance r_{o-h} , and the radius of the ellipse in the vertical direction is the vertical autocorrelation distance r_{o-v} . El-Ramly states that where vertical and horizontal spatial structures are deemed equally important, soil variability is roughly approximated by an isotropic spatial structure with an equivalent isotropic autocorrelation distance r_{o-E} (Figure E-4).

Figure E-4. Estimation of an approximate isotropic autocorrelation distance from an anisotropic random field by equivalent areas.



El-Ramly assumes that an estimate of r_{o-E} can be obtained by approximating the ellipse defining the variation of the autocorrelation distance with direction by a circle of equal area:

$$Area_{ellipse} = \pi \cdot (r_{o-h} \cdot r_{o-v}) \quad (E.18)$$

$$Area_{circle} = \pi \cdot (r_{o-E})^2 \quad (E.19)$$

$$r_{o-E} = \sqrt{r_{o-h} \cdot r_{o-v}} \quad (E.20)$$

¹ The value for r_o or δ is a measure of the spatial extent within which a soil property shows a strong correlation, as discussed in subsection D.1.5.

Introducing Equation D.6 for each of the three terms, Equation E.5 becomes

$$\delta_E = \sqrt{\delta_h \cdot \delta_v} \quad (\text{E.21})$$

In some problems, the soil stratigraphy is such that a particular geometric orientation dominates the problem and the statistical characterization of soil strength properties for that particular stability problem. Examples of these types of horizontal (soil) structure dominated stability problems include the El-Ramly et al. (2002a) probabilistic slope stability analysis of dykes of the James Bay hydroelectric project in Quebec, Canada, and the El-Ramly et al. (2003a) probabilistic stability analysis of a tailings dyke on a near-horizontal layer of presheared clay–shale. El-Ramly (2001) also shares the following opinions on this subject:

- The earth pressure in retaining structure is controlled by the variability of the coefficient of earth pressure with depth, and that spatial variability in the vertical direction is likely to be more important to the analysis.
- Failure surfaces in flat cohesionless slope tend to be superficial with a shallow dip. Since stability is controlled by the average shear strength along the failure surface, the horizontal spatial structure is likely to be more critical to the analysis.

E.5 Transformation of the anisotropic scale of fluctuation into an isotropic model in mining geostatistics

The first complete technical reference work on the topic of mining geostatistics¹ was published by Journel and Huijbregts (1978). The decision process procedure they discuss is intended for use in the mining industry to decide, for example, if geometrically defined blocks of excavated rock within an open pit select mining operation are to be sent for ore extraction at the processing plant or to waste. The block-by-block decision estimator model is based on mineral/ore content data obtained from a sparsely gridded core drillings and on the chemical analyses of blast holes from a much finer grid. The mineralized phenomenon is characterized by the spatial distribution (in 3-D space) of a certain number of measurable quantities that are referred to as regionalized variables by

¹ Journel and Huijbregts (1978) state that the term *geostatistics* designated the statistical study of natural phenomena and that they retain the Matheron (1962) definition that it is the application of the formalism of random functions to the reconnaissance and estimation of natural phenomena.

Journal and Huijbregts. Autocorrelation of grades $z(x)$ and $z(x + h)$ of a given metal (i.e., mineral) at points x and $(x + h)$ are considered and depends on both the vector h which separates the two points as well on the mineral type itself. The variability between the two quantities of $z(x)$ and $z(x + h)$ is characterized by the variogram function $2\gamma(x, h)$, which is defined as the expectation of the random variable $[Z(x) - Z(x + h)]^2$. That is,

$$2\gamma(x, h) = E\{[Z(x) - Z(x + h)]^2\} \quad (\text{E.22})$$

The variogram model acts as a quantified summary of all the available structural information, which is then channeled into the various procedures of (mineral) resource and reserve evaluation. A variogram will give a measure of how much two samples taken from the mining area will vary in mineral (e.g., copper) percentage depending on the distance between those samples. Samples taken far apart will vary more than samples taken close to each other. Also, $\gamma(x, h)$ is referred to as a semi-variogram. Journal and Huijbregts (1978) discuss a method by which anisotropies are represented that reduces them to the isotropic case either by linear transformation of the rectangular coordinates (h_u, h_v, h_w) of the vector h in the case of general anisotropy or by representing separately each of the directional variabilities concerned in the case of zonal anisotropy. RFs that have the same spatial variability in all directions are referred to as isotropic models. In 3-D space, the isotropic notation $\gamma(h) = \gamma(r)$ denotes

$$\gamma\left(\sqrt{(h_u^2 + h_v^2 + h_w^2)}\right) \quad (\text{E.23})$$

where (h_u, h_v, h_w) are the three coordinates of the vector h . The phenomenon of anisotropy occurs when variability is not the same in every direction. A semi-variogram $\gamma(h_u, h_v, h_w)$ has a geometric anisotropy when the anisotropy can be reduced to isotropy by a mere linear transformation of coordinates:

$$\gamma(h_u, h_v, h_w) = \gamma'\left(\sqrt{(h'_u{}^2 + h'_v{}^2 + h'_w{}^2)}\right) \quad (\text{E.24})$$

anisotropic isotropic

with

$$h'_u = a_{11}h_u + a_{12}h_v + a_{13}h_w \quad (\text{E.25})$$

$$h'_v = a_{21}h_u + a_{22}h_v + a_{23}h_w \quad (\text{E.26})$$

$$h'_w = a_{31}h_u + a_{32}h_v + a_{33}h_w \quad (\text{E.27})$$

and a_{ij} ($i = 1$ to $3; j = 1$ to 3) are the directional cosines of coordinate transformation.

Three steps are used to transform an ellipse into a circle and reduce the anisotropy to isotropy (Journel and Huijbregts 1978):

1. The first step is to rotate the coordinate axes by the angle φ so that they become parallel to the main axes of the ellipse (Figure E-5). The new coordinates (y_1, y_2) resulting from the rotation are

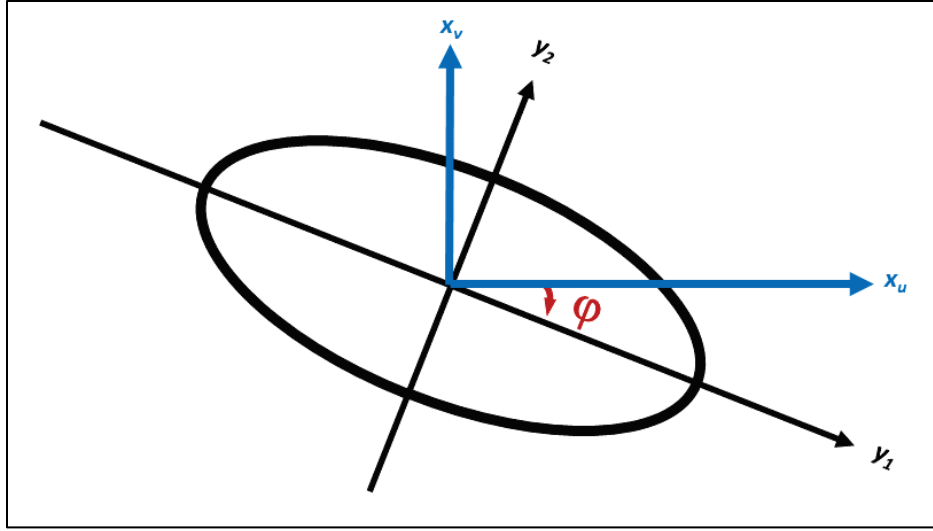
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [R_\varphi] \begin{bmatrix} x_u \\ x_v \end{bmatrix} \quad (\text{E.28})$$

with

$$[R_\varphi] = \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix} \quad (\text{E.29})$$

where $[R_\varphi]$ is the matrix of rotation through the angle φ .

Figure E-5. Correction of 2-D geometric anisotropy.



2. The second step is to transform the ellipse into a circle with a radius equal to the major range of the ellipse. This is achieved by multiplying the coordinate y_2 by the ratio of anisotropy λ ($\lambda = y_1/y_2$), with $\lambda > 1$. The new coordinates (y'_1, y'_2) are then deduced from the coordinates (y_1, y_2). The relationship is given in matrix form as

$$\begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} = [\lambda] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (\text{E.30})$$

with

$$[\lambda] = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \quad (\text{E.31})$$

3. This third step restores the initial orientation of the coordinate system. The initial orientation of the coordinate system can be restored by rotating through the angle $-\phi$, and the final transformed coordinates (x'_u, x'_v) are re-established using

$$\begin{bmatrix} x'_u \\ x'_v \end{bmatrix} = [R_{-\phi}] \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} \quad (\text{E.32})$$

Introducing Equations E.30, E.31, E.29, Equation E.32 becomes

$$\begin{bmatrix} x'_u \\ x'_v \end{bmatrix} = [A] \begin{bmatrix} x_u \\ x_v \end{bmatrix} \quad (\text{E.33})$$

with

$$[A] = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \quad (\text{E.34})$$

and

$$a = \cos^2 \varphi + \lambda \sin^2 \varphi \quad (\text{E.35})$$

$$b = \sin^2 \varphi + \lambda \cos^2 \varphi \quad (\text{E.36})$$

$$c = (1 - \lambda) \sin \varphi \cos \varphi \quad (\text{E.37})$$

If h is any vector in the 2-D space with initial coordinates (h_u, h_v) , then to obtain the value of the anisotropic semi-variogram $\gamma(h) = \gamma(h_u, h_v)$, the transformed coordinates (h'_u, h'_v) are first calculated using

$$\begin{bmatrix} h'_u \\ h'_v \end{bmatrix} = [A] \begin{bmatrix} h_u \\ h_v \end{bmatrix} \quad (\text{E.38})$$

Then, these coordinates are substituted in the isotropic model $\gamma'(h')$, which has the range equal to the major range of the directional ellipse by Equation E.24 but in 2-D,

$$\gamma(h_u, h_v) = \gamma' \left(\sqrt{(h'_u)^2 + (h'_v)^2} \right) \quad (\text{E.39})$$

anisotropic isotropic

Baecher and Christian (2003) observe that in mining geostatistics, the importance of autocorrelation for estimating mineral/ore reserves and that the variogram function, which is related to autocovariance, is used to express the spatial structure of the data: The autocovariance is defined as the expected value of the product of two observations while the variogram 2γ is the expected value of the squared difference.

The mining geostatistical approach described in this subsection relies on the use of the variogram function to express the spatial structure of the field data for assessing mineral/ore deposits within a rock regime. Dealing with geometric anisotropy is an important aspect of the analysis process for a mining site. The transformation of an ellipse into a circle to reduce

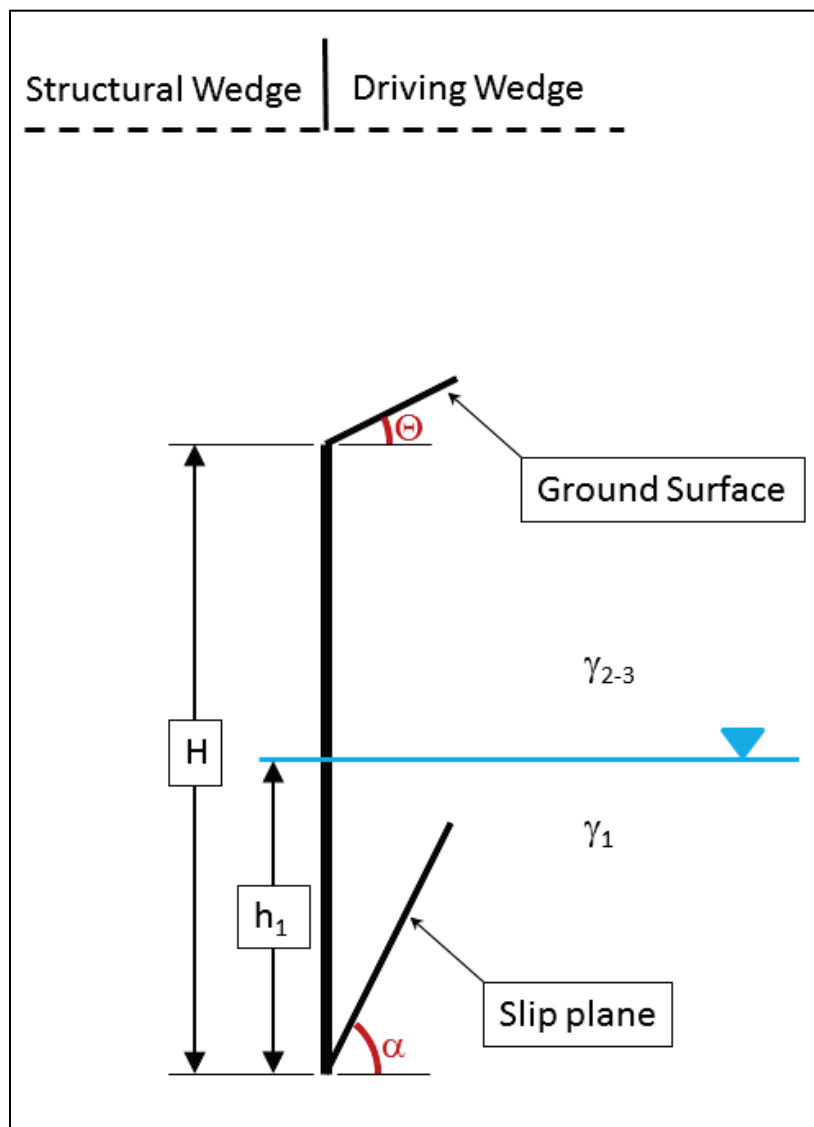
the anisotropy to isotropy is an important step in the Journel and Huijbregts (1978) mining geostatistical methodology. Although this use of variograms has not been applied to the statistical processing of engineering property data of soils discussed earlier in this appendix, the Journel and Huijbregts (1978) method for dealing with site anisotropy is insightful. Recall that El-Ramly (2001) cited Journel and Huijbregts (1978) in his approach to transform an anisotropic soil regime into anisotropic regime (see subsection E.4.2).

Appendix F: Determining the Equivalent Unit Weight of Soil for a Sloping Surface Coefficient Equation

F.1 Overview

In Section 5.5, a single wedge solution scheme formulation for modeling the driving and the resisting wedges for the potential slip plane is discussed. In this procedure of analysis, a single soil wedge is used in the driving and/or resisting wedge models. The Section 5.4 Coefficient Method for calculating distributions of earth pressures and their corresponding resultant forces acting against a structural wedge relies on being able to perform a calculation with few material properties and very little geometry information. For a homogeneous, cohesionless retained soil, the P_A thrust force that the driving wedge imposes on the structural wedge is calculated using Equation 5.66. Besides the earth pressure coefficient (e.g., K_A for the driving wedge), a value for the equivalent unit weight γ_e of the (single) retained soil wedge is also required. For a homogeneous, cohesionless buttressing soil, the P_P that the resisting wedge imposes on the structural wedge is computed using Equation 5.74. This Appendix derives simple equations based on calculable heights for a structural wedges boundaries, which has a defined height H , with a given surface angle (from horizontal) Θ and a slip-plane angle α , as well as unit weights γ_{2-3} and γ_1 , to determine the equivalent unit weight of the driving (or resisting) wedge γ_e . Notice that for a total stress material assignment, γ_1 will be assigned the saturated soil unit weight and γ_{2-3} will be assigned the moist soil unit weight. Additionally, the solution discussed below will work with either total or effective stress soil material regions, as described in Section 5.5. It is also assumed that the height of saturation (h_1) is known. Figure F-1 summarizes the required information.

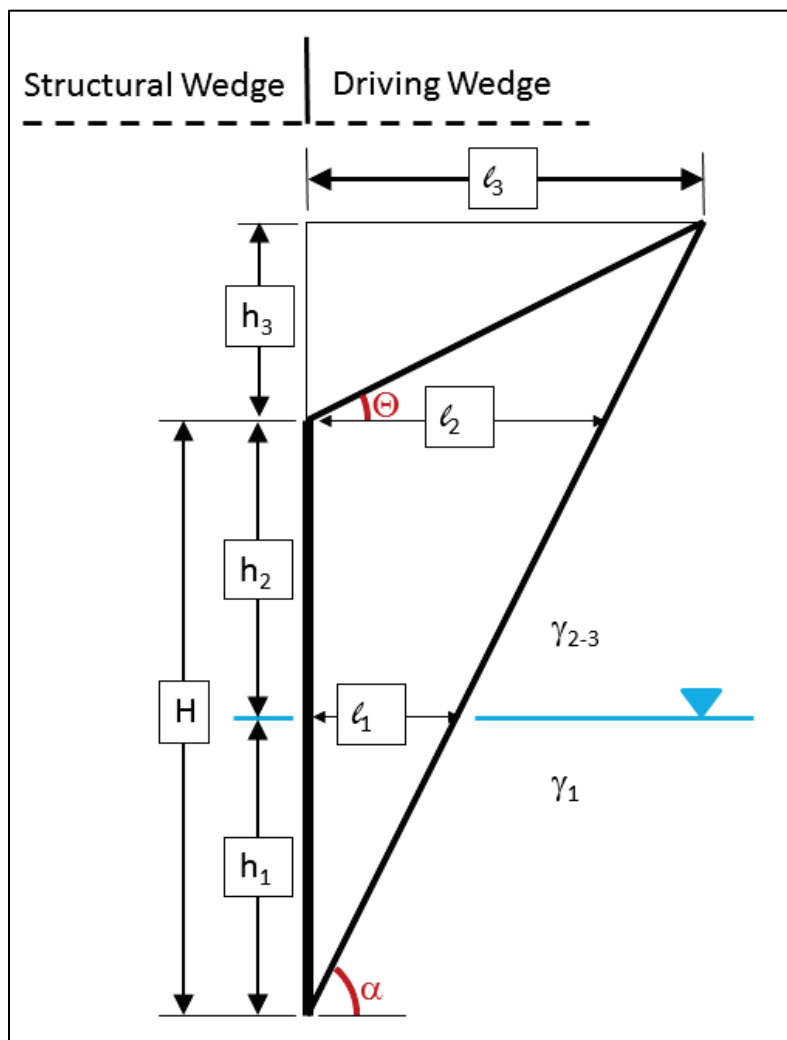
Figure F-1. Input information for determining the equivalent unit weight γ_e for the driving wedge.



F.2 Formulation

Figure F-2 shows the extension of the slip plane and the ground surface until the intersection is found. The intersection only occurs when the slip plane angle (from horizontal) α is greater than the ground surface angle (also from horizontal) Θ . The intersection point is at a distance l_3 from the structural wedge interface and height h_3 above the top of the structural wedge interface. The height of the structural wedge above the height of saturation h_1 is designated as h_2 .

Figure F-2. Geometrical input and derived information for determining the equivalent unit weight γ_e .



The interior angle to the right triangle at the relative position $(l_3, H + h_3)$ from the slip plane to horizontal is α . Using the definition of the tangent,

$$(H + h_3) = l_3 \tan \alpha \quad (\text{F.1})$$

with

$$h_3 = l_3 \tan \Theta \quad (\text{F.2})$$

Introducing Equation F.2 into Equation F.1 yields

$$(H + l_3 \tan \Theta) = l_3 \tan \alpha \quad (\text{F.3})$$

Combining the l_3 terms

$$H = l_3 \tan \alpha - l_3 \tan \Theta \quad (\text{F.4})$$

$$H = l_3 (\tan \alpha - \tan \Theta) \quad (\text{F.5})$$

$$H = l_3 \left(\frac{\sin \alpha}{\cos \alpha} - \frac{\sin \Theta}{\cos \Theta} \right) \quad (\text{F.6})$$

$$H = l_3 \left(\frac{\sin \alpha \cos \Theta - \sin \Theta \cos \alpha}{\cos \Theta \cos \alpha} \right) \quad (\text{F.7})$$

$$H = l_3 \left(\frac{\sin(\alpha - \Theta)}{\cos \Theta \cos \alpha} \right) \quad (\text{F.8})$$

Solving for l_3

$$l_3 = H \left(\frac{\cos \Theta \cos \alpha}{\sin(\alpha - \Theta)} \right) \quad (\text{F.9})$$

The Law of Sines states that

$$\frac{l_1}{h_1} = \frac{l_2}{H} = \frac{l_3}{H + h_3} \quad (\text{F.10})$$

The area of the triangle with sides l_3 and $H + h_3$ is

$$Area_{total} = 0.5(H + h_3)l_3 \quad (\text{F.11})$$

The area of the triangle with side l_1 and h_1 is

$$Area_1 = 0.5(h_1)l_1 \quad (\text{F.12})$$

The area of the trapezoidal region 2 is given by

$$Area_2 = 0.5(H)l_2 - Area_1 \quad (\text{F.13})$$

$$Area_2 = 0.5(H)l_2 - 0.5(h_1)l_1 \quad (\text{F.14})$$

The area of the additional air region (with sides l_3 and h_3) is

$$Area_{Air} = 0.5(h_3)l_3 \quad (\text{F.15})$$

The area of the obtuse triangular region 3 becomes

$$Area_3 = Area_{total} - [Area_1 + Area_2 + Area_{Air}] \quad (F.16)$$

$$Area_3 = 0.5(H + h_3)l_3 - [0.5(h_1)l_1 + 0.5(H)l_2 - 0.5(h_1)l_1 + 0.5(h_3)l_3] \quad (F.17)$$

$$Area_3 = 0.5(H)l_3 + 0.5(h_3)l_3 - [0.5(h_1)l_1 + 0.5(H)l_2 - 0.5(h_1)l_1 + 0.5(h_3)l_3] \quad (F.18)$$

$$Area_3 = 0.5(H)l_3 + 0.5(h_3)l_3 - [0.5(H)l_2 + 0.5(h_3)l_3] \quad (F.19)$$

$$Area_3 = 0.5(H)l_3 - 0.5(H)l_2 \quad (F.20)$$

$$Area_3 = 0.5(l_3 - l_2)H \quad (F.21)$$

The total equivalent unit weight γ_e of the soil wedge is based on the proportions of the moist and saturated areas to total area multiplied by their respective unit weights γ_{2-3} and γ_1 .

$$\gamma_e = \gamma_{2-3} \left(\frac{Area_2 + Area_3}{Area_{total} - Area_{Air}} \right) + \gamma_1 \left(\frac{Area_1}{Area_{total} - Area_{Air}} \right) \quad (F.22)$$

Substituting the Area equations

$$\gamma_e = \gamma_{2-3} \left(\frac{0.5(H)l_2 - 0.5(h_1)l_1 + 0.5(l_3 - l_2)H}{0.5(H + h_3)l_3 - 0.5(h_3)l_3} \right) + \gamma_1 \left(\frac{0.5(h_1)l_1}{0.5(H + h_3)l_3 - 0.5(h_3)l_3} \right) \quad (F.23)$$

Reducing the terms

$$\gamma_e = \gamma_{2-3} \left(\frac{0.5(H)l_3 - 0.5(h_1)l_1}{0.5(H)l_3} \right) + \gamma_1 \left(\frac{0.5(h_1)l_1}{0.5(H)l_3} \right) \quad (F.24)$$

$$\gamma_e = \gamma_{2-3} \left(\frac{(H)l_3 - (h_1)l_1}{(H)l_3} \right) + \gamma_1 \left(\frac{(h_1)l_1}{(H)l_3} \right) \quad (F.25)$$

Because $H = h_1 + h_2$

$$\gamma_e = \gamma_{2-3} \left(\frac{(h_1 + h_2)l_3 - (h_1)l_1}{(h_1 + h_2)l_3} \right) + \gamma_1 \left(\frac{(h_1)l_1}{(h_1 + h_2)l_3} \right) \quad (F.26)$$

$$\gamma_e = \gamma_{2-3} \left(1 - \frac{(h_1)l_1}{(h_1 + h_2)l_3} \right) + \gamma_1 \left(\frac{(h_1)l_1}{(h_1 + h_2)l_3} \right) \quad (F.27)$$

Applying the Law of Sines

$$\frac{l_1}{l_3} = \frac{h_1}{h_1+h_2+h_3} \quad (\text{F.28})$$

Substituting Equation F.28 into Equation F.26 yields the equation listed in Chapter 5

$$\gamma_e = \gamma_{2-3} \left(1 - \frac{h_1}{(h_1+h_2)} \times \frac{h_1}{h_1+h_2+h_3} \right) + \gamma_1 \left(\frac{h_1}{(h_1+h_2)} \times \frac{h_1}{h_1+h_2+h_3} \right) \quad (5.67\text{bis})$$

The input information of Figure F-2 can be used to solve for l_3 (Equation F.9). Using l_3 and the input information can be used to solve for h_3 (Equation F.2). Given h_3 and the input information, Equation F.29 can be used to determine the equivalent unit weight γ_e of the entire wedge acting against the wall as shown in Figure F-2.

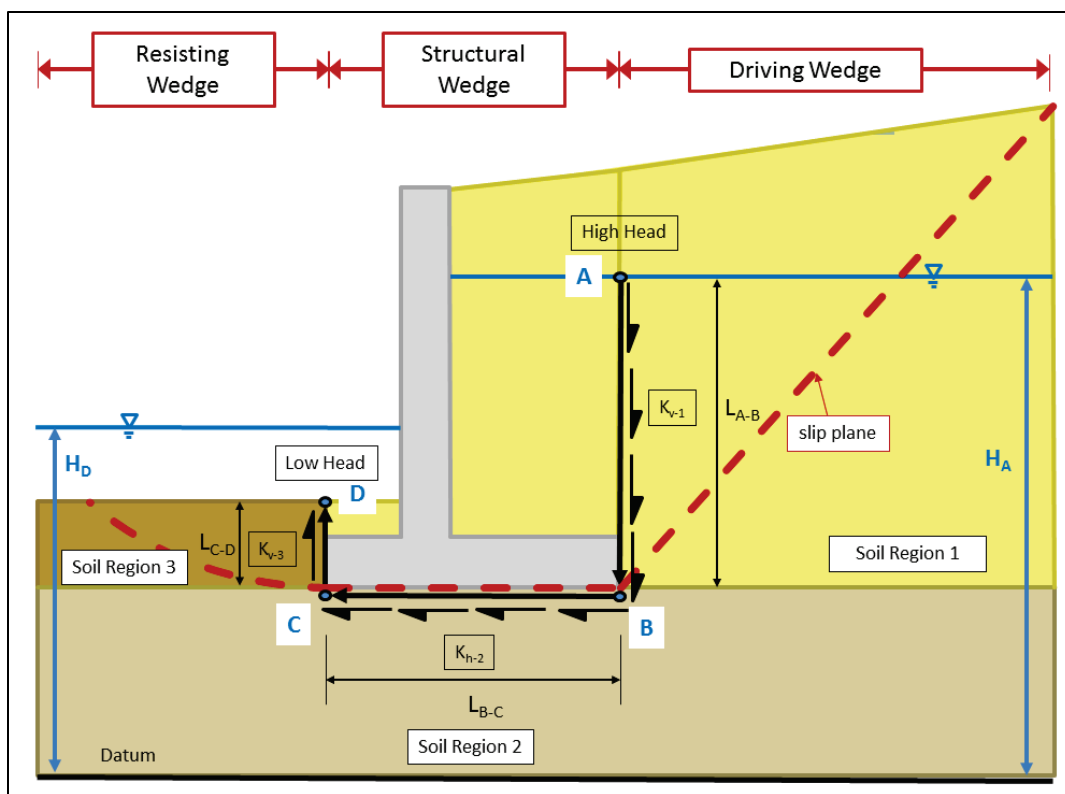
Subsection 5.5.1.2 discusses the determination of γ_1 for use in an effective stress analysis using Equation 5.67.

Appendix G: Line of Seepage

Line of seepage (a.k.a., line of creep) constructs a 1-D seepage path about a structural wedge that is affected by differing head pressures on either side of the structural wedge and the soil material region hydraulic conductivity. This path is analyzed in a method where the actual length of seepage about the structural wedge is determined first. From the length of seepage, transformed by the soil conductivity, a uniform seepage gradient is calculated for the transformed length. The inverse transform of this gradient space gives the actual seepage values along the line of seepage. The results from this rapid calculation, simplified method of analysis are then used for determining water pressures acting within the soil adjacent to the structural wedge and acting normal to both landside and flood side embedded faces as well as normal to the planar slip lane below this T-Wall.

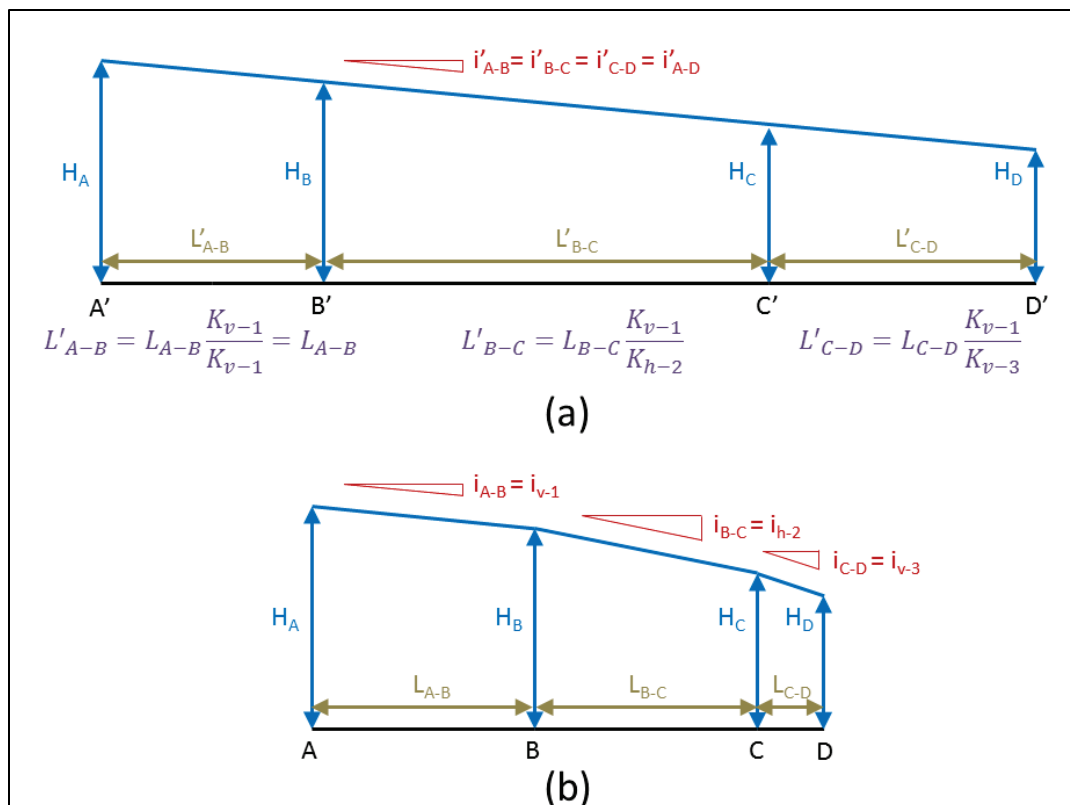
Water will travel from high (total) head (i.e., H_A in Figure G-1) to low (total) head (H_D), dictating the direction of flow within the soil regions. Consequently, the side with the highest (total) head dictates that the direction of flow is downward along that side of the T-Wall. A vertical streamline, or flow line, is assumed within the soil and adjacent to the structural wedge. Water flows within the soil, downward and parallel to the interface between the structural and driving wedges, to the Heel of the T-Wall (i.e., from point A to point B in Figure G-1). At the T-Wall Heel (point B), the vertical flow of water rotates (by 90 degrees) to flow along the horizontal slip plane through the foundation soil to point C at the Toe of the T-Wall. At point C the horizontal flow of water rotates (by 90 degrees) to flow upward along the interface between the structural and resisting wedges.

Figure G-1. Seepage path for gradient determination.



The transformation procedure for the line of seepage method is for 1-D flow along a singular path consisting of 1-D flow paths connected in series from one soil region on into the next soil region that are aligned along the three faces of the structural wedge of a T-Wall. These flow paths contained within the soil regions may possess different values for the saturated hydraulic conductivity of the soil. For example, Figure G-1 contains three unique soil regions. Each flow line contained within each soil region has a unique value for its saturated hydraulic conductivity, K . The hydraulic conductivity of the geologically formed soil Regions 2 and 3 (K_2 and K_3) are smaller than the hydraulic conductivity of the soil contained within soil Region 1 (K_1), which is comprised of engineered, select porous fill. Consequently, a transformation is required to first construct a 1-D flow regime down the retained side of the T-Wall interface from points A to B, along the base of the structural wedge from points B to C and up the opposite face of the structural wedge from points C to D to represent a 1-D flow regime of constant hydraulic conductivity of value K_1 .

Figure G-2. 1-D seepage path. (a) Constant slope in total head with distance along the transformed length of line of seepage in a homogeneous, 1-D seepage path through all K_1 material. (b) Variation of slope in total head with distance along the three soil regions of the line of seepage.



The transformed horizontal axis is first constructed (Figure G-2(a)) using the two transformed lengths L'_{B-C} and L'_{C-D} by means of the equations given in the figure. The transformed length L'_{A-B} equals the vertical length L_{A-B} . This means that positions A and B are at the same relative coordinates as A' and B'. Note that starting with the second 1-D flow path (of transformed length L'_{B-C}), there is a transformed position denoting the end of 1-D flow path B to C, designated as point C'. With K_{h-2} being less than K_{v-1} , the length of the transformed segment L'_{B-C} is longer than the physical distance from points B to C, L_{B-C} . Similarly for 1-D flow Path 3 of transformed length L'_{C-D} , its starting and ending transformed points are C' and D' respectively. With K_{v-3} being less than K_{v-1} , the length of the transformed segment L'_{C-D} is greater than the physical distance from points C to D, L_{C-D} . Because the transformed 1-D line of flow model now possesses a constant hydraulic conductivity of value K_{v-1} , the variation in total head along the transformed section is linear (see Figure 12-1 in Spangler and Handy 1973). There exists a linear variation in total head from point A to point D' as denoted in Figure G1(a) for this transformed 1-D line of flow. The gradient, designated i'_{A-D} , is

defined as equal to the change in total head from point A to point D' (i.e., $= H_A - H_D$) divided by the total length of the transformed 1-D line of flow ($= L_{A-B} + L'_{B-C} + L'_{C-D}$). Also at points B and C' of Figure G-2(a) the value for total head is obtained and is designated as H_B and H_C , respectively, computed by taking advantage of the key feature of the Figure G2(a), which shows that the hydraulic gradient is the known constant of value i_{A-D} . The total head values H_B and H_C are easily obtained from a linear relationship in total head with the transformed distance between Figure G-2(a) points A and D'.

Figure G-2(b) shows the original axis for the problem using the actual dimensions (labeled L_{A-B} , L_{B-C} and L_{C-D}) for the entire length of the 1-D streamline contained within the three soil regions and along the wet perimeter of the structural wedge of the T-Wall. The values for total head are mapped from points A, B', C', and D' from Figure G-2(a), to points A, B, C, and D in Figure G-2(b). Observe that the transformed Figure G-2(a) is required to compute the head values H_B , H_C from H_A and H_D at the transformed position points of B, C' and D', respectively. Figure G-2(b) shows these computed heads of H_B , H_C and actual heads H_A and H_D are mapped back to their actual position points A, B, C, and D.

Further observe in Figure G-2(b) that the slope of total head with distance along each of the three 1-D flow line segments is no longer a single constant, as is the case for the transformed horizontal axis plot of the Figure G-2(a) streamline segments. Here, the hydraulic gradient is constant along the segment of the 1-D flow line contained within each of the three Figure G-1 soil regions. Recall that each 1-D flow line possesses a different value for saturated hydraulic conductivity, resulting in a different hydraulic gradient for each of the three Figure G-2(b) streamline segments: i_{v-1} , i_{h-2} and i_{v-3} , respectively. In the Section 5.5 single wedge solution scheme for an effective stress based analysis, the downward acting vertical seepage gradient value of i_{v-1} within the driving wedge is used in Equation 5.70, and the upward acting vertical seepage gradient value of i_{v-3} is used within the resisting wedge Equation 5.71.

If the high and low head sides are reversed from that shown in Figure G-1, then the upward acting vertical seepage gradient value of i_{v-1} within the driving wedge is used in Equation 5.72, and the downward acting vertical seepage gradient value of i_{v-3} is used within the resisting wedge Equation 5.73.

Values of water pressures at points A, B, C, and D of Figure G-2(b) are calculated using Bernoulli's equation by determining the total head at each point. For point A, the total head value is established by the boundary condition stipulated at point A in Figure G2, with a value of H_A ¹. Additionally, according to Bernoulli's equation the total head (H_A) at this point can be expressed as

$$H = h_e + h_p + h_v \quad (G.1)$$

where h_e is the elevation head, h_p is the pressure head, and h_v is the velocity head at any point. Because groundwater seeps and flows slowly for finer materials than open-pored, coarse gravels, the velocity head is considered negligible. Total head is frequently calculated at a point as the sum of the pressure head and elevation head, and the pressure head is the difference between the total head and the elevation head.

$$h_p = H - h_e \quad (G.2)$$

Once the pressure head at a point is determined at a point, it can be multiplied by the unit weight of water to determine the pore water pressure (u)

$$u = h_p * \gamma_w \quad (G.3)$$

Equations G.2 and G.3 are used to compute the pressure head for points along the line of seepage and the pore water pressure at that point based on the pressure head. Figure G-2(b) provides the total head value for all points along the line of seepage.

¹ A similar boundary condition exists at point D as stipulated in Figure G.1; the total head is equal to H_D .

Appendix H: Calibrating Partial Load and Resistance Safety Factors Given a Target β

H.1 Introduction: Assumptions and requirements for calculating load and resistance factors

This appendix outlines the steps in a numerical procedure for calibrating partial load and resistance safety factors given a target reliability index (β) value. The numerical procedure described is adapted to this R&D effort from that outlined in Section 8.5 of Nowak and Collins (2013) for bridge design. A numerical procedure to calculate β given variable load and resistance with distributions, which may have partial load and resistance safety factors applied is presented to be used as a verification step. To verify the computation of partial safety factors, this procedure is used by calculating β from the partial load and resistant factors applied to the load and resistance distributions. This value should then match the target β value. This second numerical procedure is also adapted to this R&D effort from a procedure described in Nowak and Collins (2013).

There are certain conditions that affect the calculation of partial safety factors for loads and resistances, γ and ϕ , respectively. For a set of uncorrelated variables described by normal distributions, the procedure is straightforward. Conversion of the normal distributions into a Gaussian Normal space allows for the determination of a design point for the limit state function $g(\mathbf{x}^*)$. The closest point from the means to the limit state response surface where $g(\mathbf{x}^*) = 0$ is the design point. In the Gaussian Normal space, the unit distance corresponds to a change of 1 standard deviation from the mean origin. Therefore, the distance to the design point gives the β distance to the response surface from which PDF (ϕ) and CDF (Φ) values can be obtained. The vector from the mean origin of each of the i variable, μ_i , to the design point x_i^* on the limit state response surface after being normalized to unit length gives the directional cosine components (α) for the uncorrelated variables. Recall σ_i is the standard deviation for the i^{th} variable. The design point for $g(\mathbf{x}^*) = 0$ in variable space is given by the equation

$$x_i^* = \mu_i + \alpha_i \beta \sigma_i \quad \text{H.1}$$

Substituting the target β_{target} for the design point β projects the response surface to β_{target} . Then the factor that gives that value is calculated as

$$\gamma_i = \frac{x_i^*}{x_i^N} \quad \text{H.2}$$

Where γ_i is the factor and X_i is the component value given for β_{target} , and x_i^N is the nominal value for the variable (load or resistance), which can change through the iterative procedure. For resistance, γ_i becomes ϕ_i in Equation H.2.

The next section describes the numerical procedure for performing this same set of steps when some or all of the variables possess non-normal distributions.

H.2 Procedure for calculating load and resistance factors given variables with non-normal distributions

The following steps outline the procedure for calculating load and resistance factors given variables with non-normal distributions.

1. Formulate the limit state function and the design equation. These equations will work with as many random variables as possible, with their distributions and parameters. It is assumed that every random variable will have a COV or standard deviation. At most, only two unknown means will be allowed. Typically, the resistance variable accounts for one unknown mean with the other unknown mean corresponding to one of the load terms. Load ratios set the relationships of the mean values of the loads to the unknown values. For example, in LRFD application to bridge design it is common for the live load to be set equal to one-third the dead load (Nowak and Collins 2013) when applying this numerical procedure. Initializing for the first iteration, the limit state equation where $g(X) = 0$ is evaluated to determine the relationship between the unknown mean values.
2. An initial design point X is obtained by assuming values for $n-1$ of the random variables, with mean values being a good assumption. Solve the limit state equation where $g(X) = 0$ to determine the remaining variable on the failure boundary.
3. For non-normal axis design point values, determine the equivalent distribution (μ^e_X and σ^e_X) using Equation H.3 and Equation H.4. These values create a distribution at x^* that has an equivalent CDF and PDF at the design point x^* . If the axis of a design point variable is on a normal distribution, then μ^e_X and σ^e_X correspond to the already existent

distribution. This step may not be possible if the normal mean is one of the unknowns.

$$\mu_x^e = x^* - \sigma_x^e [\Phi^{-1}(F_x(x^*))] \quad \text{H.3}$$

$$\sigma_x^e = \frac{1}{f_x(x^*)} \phi\left(\frac{x^* - \mu_x^e}{\sigma_x^e}\right) = \frac{1}{f_x(x^*)} \phi[\Phi^{-1}(F_x(x^*))] \quad \text{H.4}$$

4. Calculate the partial derivatives of the limit state function with respect to the reduced variates. The column vector $\{G\}$ is comprised of these derivatives:

$$\{G\} = \begin{Bmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \\ G_n \end{Bmatrix}, \text{ where } G_i = \left. \frac{\partial g}{\partial z_i} \right|_{\text{evaluated at the design point}} \quad \text{H.5}$$

5. From the matrix of correlation coefficients $[\rho]$, calculate the directional cosine column vector $\{\alpha\}$:

$$\{\alpha\} = \frac{[\rho]\{G\}}{\sqrt{\{G\}^T [\rho] \{G\}}} \quad \text{H.6}$$

with the correlation matrix for a two-variable problem given as

$$[\rho] = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \quad \text{H.7}$$

The diagonals of the correlation matrix are set to unity, and ρ_{12} is typically equal to ρ_{21} . For independent variables X_1 and X_2 , there is no correlation (i.e., $\rho_{12} = \rho_{21} = 0$) and the diagonal terms are unity.

6. The new design point in reduced variates for $n-1$ variables is computed using the target β , β_{target} .

$$z_i^* = \alpha_i \beta_{target} \quad \text{H.8}$$

7. Return the new design point in the original coordinate space for the $n-1$ variables using the expected distribution for the variables

$$X_i^* = \mu_{X_i}^e + z_i^* \sigma_{X_i}^e \quad \text{H.9}$$

8. At this point, the remaining random value can be solved for by solving the limit state function so that $g(x^*) = 0$. The relationship between the two unknown mean values may have changed, so the relationship can be updated by assuming that the nominal $x_i^N = \mu_{x_i}$.

$$\gamma_i = \frac{x_i^*}{\mu_{x_i}} = \frac{\mu_{x_i} + z_i^* \sigma_{x_i}}{\mu_{x_i}} = 1 + z_i^* COV_{x_i} = 1 + \alpha_i \beta_{target} COV_{x_i} \quad H.10$$

Therefore,

$$\mu_{x_i} = \frac{x_i^*}{1 + \alpha_i \beta_{target} COV_{x_i}} \quad H.11$$

9. Repeat steps 3 to 8 until $\{\alpha\}$ converges.
 10. Use Equation H.2 with the converged values to calculate each design factor.

Example – Dead load bending moment for a beam

A trial design was performed of a singly reinforced beam with an unfactored dead load bending moment, M_{DL} , set equal to 200 kip-feet (kip-ft). This example only deals with a dead-load condition. The beam with a width of 12 inches (in.) and a depth (d) to center of reinforcement of 28 in. was analyzed in a previous USACE LRFD study (unpublished). A concrete compressive strength, f'_c , of 4,000 psi and a reinforcement yield strength, f_y , of 60,000 psi were used. The required area of steel, using a resistance factor, ϕ_{ACI} , of 0.9 from ACI 318-14, is 2.37 in.².

The load factor for dead load, when not combined with other loads, is 1.4 in ASCE 7-10 and is designated γ_{ASCE} in this example. Ellingwood (2015) states that the PDF (or described functionally as $\phi()$) for dead loads applied to buildings is normally distributed with a bias factor, λ_L , of 1.05 and a COV_L of 0.1. The mean dead load, μ_{DL} , is computed to be 210 kip-ft by

$$\mu_{DL} = \lambda_{DL} * M_{DL} \quad H.12$$

The safety margin relationship for LRFD for this example is

$$\phi_{ACI} * M_n = \gamma_{ASCE} * M_{DL} \quad H.13$$

Rearranging for M_n , this equation becomes

$$M_n = \frac{\gamma_{ASCE} * M_{DL}}{\phi_{ACI}} \quad \text{H.14}$$

Introducing the ASCE dead load factor, γ_{ASCE} , of 1.4, the dead load moment, M_{DL} , of 200 kip-ft and the ACI flexural resistance factor, ϕ_{ACI} , of 0.9, the nominal moment is computed as

$$M_n = \frac{1.4 * 200}{0.9} = 311.11 \text{ kip-ft} \quad \text{H.15}$$

By the following two relationships for the nominal flexural capacity of a singly reinforced beam equal to 311.11 kip-ft, the required reinforcement steel cross-sectional area is verified (for the unpublished 2016 USACE LRFD document) to be 2.37 in.² for the signally reinforced beam using the relationships

$$M_n = A_f * f_y \left[d - \frac{a}{2} \right] \quad \text{H.16}$$

with

$$a = \frac{A_f * f_y}{0.85 * f'_c * b} \quad \text{H.17}$$

where:

- A_s = area of reinforcing steel
- f_y = yield strength of reinforcing steel
- d = distance from compression face to center of reinforcing steel
- a = height of equivalent Whitney stress block in reinforced concrete
- f'_c = concrete compression strength
- b = width of beam or design section.

The mean resistance, μ_R , is computed to be 348.44 kip-ft by

$$\mu_R = \lambda_R * M_n \quad \text{H.18}$$

where the bias factor on flexural resistance, λ_R , is defined as 1.12 by Ellingwood (2015) for a one-way slab. Recall M_n is the nominal resistance computed using Equation H.14. Ellingwood (2015) also suggests a lognormal distribution for flexural capacity of one-way slabs with a COV_R of 0.14 in his document.

Following the procedural steps for calculating load and resistance factors given variables with non-normal distributions for the first iteration:

1. There are two variables in this example: the resistance variable (R) and the load variable (L). The design equation is defined by $g(\mathbf{x}^*) = R - L$, and the limit state occurs when $g(\mathbf{x}^*) = 0.0$, which occurs when $R = L$. The distribution for the load L is the same as the normal distribution, so the $COVL = 0.1$. It is assumed that both mean load and resistance are unknown. The unknown mean load μ_L is used as the basis for calculations, with all of the other values relating to its value through ratios. This relationship is assumed to be $\mu_R = 1 \mu_L$ for $g(\mathbf{x}^*) = 0.0$.
2. An initial design point is calculated by assuming $n-1$ variable values. For the two-variable problem the design point value for load (l^*) is assumed to be the mean value, $1\mu_L$. Solving the $g(\mathbf{x}^*) = 0.0$ limit state response surface equation gives the design point for the resistance (r^*) as $1 \mu_L$.
3. R is the only variable with a non-normal distribution, so its equivalent normal distribution values need to be determined using Equation H.2 and Equation H.3. Some additional information is needed from the original log-normal distribution values. From the original log-normal distribution (and these values are constant through the iterations):

$$\sigma_{\ln R}^2 = \ln \left(1 + \frac{\sigma_R^2}{\mu_R^2} \right) = \ln \left(1 + \frac{48.78^2}{348.44^2} \right) = 0.1941 \rightarrow \sigma_{\ln R} = 0.139$$

$$\mu_{\ln R} = \ln(\mu_R) - \frac{\sigma_{\ln R}^2}{2} = 5.844$$

The closed form solutions of these equations for a log-normal distribution are

$$\sigma_R^e = r^* \sigma_{\ln R} = 0.139 * \mu_L$$

$$\mu_R^e = r^* [1 - \ln(r^*) + \mu_{\ln R}] = 6.844 * \mu_L$$

4. Calculating $\{G\}$ from the derivatives:

$$G_1 = - \frac{\partial g}{\partial R} \sigma_R^e \Big|_{design\ point} = -\sigma_R^e = -0.139 * \mu_L$$

$$G_2 = - \frac{\partial g}{\partial L} \sigma_L^e \Big|_{design\ point} = +\mu_L = COV_L * \mu_L = 0.1 * \mu_L$$

5. From the matrix Equation H.6:

$$\{\alpha\} = \begin{Bmatrix} -0.812 \\ 0.583 \end{Bmatrix}$$

Note that since the variables R and L are independent variables, the correlation matrix $[\rho]$ is an identity matrix, with values of 1 along the major term diagonals and off-diagonal terms of 0.

6. The new design point in Gaussian Normal coordinates for the load is determined, with $\beta_{target} = 3.0$,

$$z_L^* = \alpha_L \beta_{target} = 0.583 * 3.0 = 1.75$$

7. Returning to the original coordinate system,

$$l^* = \mu_L + z_L^* \sigma_L = \mu_L (1 + z_L^* COV_L) = 1.175 * \mu_L$$

8. Using $g(\mathbf{x}^*)=0.0$, then,

$$r^* = l^* = 1.175 * \mu_L$$

An updated value of the estimated mean value of R ,

$$\mu_R = \frac{r^*}{1 + \alpha_R \beta COV_R} = \frac{1.175 * \mu_L}{1 + (-0.812)(3)(0.14)} = 1.783 * \mu_L$$

9. Table H-1 summarizes how the values change for each iteration of the procedure. The iterative computations for this example were made in an Excel spread sheet. While four iterations were performed, in actuality, a threshold for convergence would be established when implemented within an algorithm.

Table H-1. Iterations of step 3–8 as the design point and mean resistance change.

	ORIGINAL	ITERATION 1	ITERATION 2	ITERATION 3	ITERATION 4
r^*	1	1.174933	1.156396	1.158209	1.15803
l^*	1	1.174933	1.156396	1.158209	1.15803
μ_R	1	1.783459	1.802396	1.800834	1.800991

10. Using Equation H.2, the resistance factor is calculated as

$$\gamma_R = 0.72$$

Recall, γ_R is also designated as ϕ_R or simply ϕ in the main body of this report.

The load factor is calculated as

$$\gamma_L = 1.216.$$

Recall that by applying this numerical procedure outlined in this example guarantees minimal partial resistance and load safety factors that, when multiplied by the nominal resistance and its resistance bias and nominal load and its load bias, will compute a design point on the limit state response surface at distance β_{Target} .

H.3 Supplemental procedure: Calculation of the partial load safety factor given a specified partial resistance safety factor and the procedurally defined partial safety factor ratio

The limit state $g(X)$ function is defined for i resistances and j loads as

$$\sum_i \left(\gamma_{R_i} * \frac{\mu_{R_i}}{\lambda_{R_i}} \right) \geq \sum_j \left(\gamma_{L_j} * \frac{\mu_{L_j}}{\lambda_{L_j}} \right) \quad \text{H.19}$$

For the single load and resistance problem and because $g(X) = 0.0$ at the limit state response surface where $R-L = 0.0$,

$$\gamma_R * \frac{\mu_R}{\lambda_R} = \gamma_L * \frac{\mu_L}{\lambda_L} \rightarrow \frac{\mu_R}{\mu_L} * \frac{\lambda_L}{\lambda_R} = \frac{\gamma_L}{\gamma_R} \quad \text{H.20}$$

This implies that the ratio of the factors is a constant for the single load and resistance problem because the ratio of the means is a constant at the limit state response surface. Recall from example H.1 the values for γ_R and γ_L are 0.72 and 1.216. In keeping with the specified flexural resistance factor 0.9 of ACI 318-14, and maintaining the same relationship between the load and resistance factor to satisfy the $g(X) = 0$ limit state response surface equation (assuming the mean load and resistance values do not change), then the load factor according to an ACI 318-14 ϕ_{ACI} value of 0.9 should be the following:

$$\frac{\gamma_{ACI}}{\phi_{ACI}} = \frac{\gamma_L}{\gamma_R} \rightarrow \gamma_{ACI} = 0.9 \left(\frac{1.216}{0.72} \right) = 1.52$$

This γ_{ACI} load factor value shows a greater than 15% difference from the 1.4 value that was calculated and reported in the USACE unpublished write-up of LRFD procedures for this singly reinforced beam example problem.

This supplemental scaling procedure only works for the single load and resistance problem as outlined in this subsection. Keeping the same ratio of load factor to resistance factor guarantees that the factored load will be equivalent to the factored resistance for a design point on the limit state response surface. Recall that the factors were calculated so that the factored resistance and factored load satisfy the $g(\mathbf{x}^*) = 0$ equation for a design point at β_{Target} distance from the mean values for load and resistance.

H.4 Verification: Procedure for calculating design point β at the limit state for given variables with non-normal distributions

The verification procedure for determining the design point β for $g(\mathbf{x}^*) = 0$ given non-normal distributions for the variable space is similar to the process for determining load and resistance factors as outlined in subsection H.2. This follows because the latter procedure is derived from the former (Nowak and Collins 2013), as described in their subsection 5.4.1. This means that the procedure to calculate the β for the design point can act as verification of the load and resistance factors, by using factored loads and resistances and checking to make sure the calculated β is indeed equal to the target value.

The $g(X)$ function provides limit state satisfactory values when the factored load and factored resistance satisfy the following inequality:

$$\sum_i \left(\gamma_{R_i} * \frac{\mu_{R_i}}{\lambda_{R_i}} \right) \geq \sum_j \left(\gamma_{L_j} * \frac{\mu_{L_j}}{\lambda_{L_j}} \right) \quad \text{H.19 bis}$$

Recall γ_R is also designated as ϕ_R or simply ϕ in the main body of this report. Observe that this relationship is expressed in terms of mean values. The limit state response surface occurs when both sides are equivalent.

The formulation steps for the two independent variables problem are the following:

1. Formulate limit state function $g(\mathbf{x}^*)$ and probability distributions. These values should match the values used in the load and resistance factor calculations.
2. Guess an initial design point. A good initial guess would be an average of the mean values for load and for resistance.
3. Determine the equivalent normal distribution parameters for the non-normal distributions at the design point. For instance, for a log-normal distribution of variate X_i ,

$$\sigma_{\ln X_i}^2 = \ln \left(1 + \frac{\sigma_{X_i}^2}{\mu_{X_i}^2} \right) \quad \text{H.21}$$

$$\mu_{\ln X_i}^2 = \ln(\mu_{X_i}) - 0.5 \sigma_{\ln X_i}^2 \quad \text{H.22}$$

Nowak and Collins (2013) derived the following equivalent normal distribution values from Equations H.2 and H.3 for the log-normal distribution, with design point x^* :

$$\sigma_{X_i}^e = x_i^* \sigma_{\ln X_i} \quad \text{H.23}$$

$$\mu_{X_i}^e = x_i^* [1 - \ln(x_i^*) + \mu_{\ln X_i}] \quad \text{H.24}$$

Nowak and Collins (2013) also provide other equations for other non-normal distributions.

4. Transform the variates to the reduced Gaussian normalized space.

$$z_i^* = \frac{x_i^* - \mu_{X_i}^e}{\sigma_{X_i}^e} \quad \text{H.25}$$

5. Determine the partial derivatives of the slope of the $g(\mathbf{x}^*)$ function for each variate and store in a $\{\mathbf{G}\}$ vector. For a limit state that only has a load and resistance and $g(\mathbf{x}^*) = X_1 - X_2$ (where X_1 corresponds to the resistance and X_2 corresponds to the load). From Equation H.4:

$$G_1 = - \left. \frac{\partial g}{\partial X_1} \right|_{x_1^*} = -1 \sigma_{X_1}^e \quad \text{H.26}$$

$$G_2 = - \left. \frac{\partial g}{\partial X_2} \right|_{x_2^*} = +1 \sigma_{X_2}^e \quad \text{H.27}$$

Observe the negative sign in front of the partial in both of these equations.

6. From the reduced variates and the vector $\{G\}$, estimate β .

$$\beta = \frac{\{G\}^T \{z^*\}}{\sqrt{\{G\}^T \{G\}}} \quad \text{H.28}$$

For this uncorrelated two-variable problem, the Equation H.7 correlation matrix $[\rho]$ is an identity matrix, which by definition does not change the vector it is multiplied against. In this case, the transposed matrix Equation H.6 times the vector $\{z^*\}$ simplifies to the matrix equation form given here to determine the scalar value of β .

7. The directional cosine values α are then calculated using Equation H.6. It is generally assumed that the variables are uncorrelated.
8. The Gaussian normalized design point is calculated for the variables using α and β values for $n-1$ variables, which are on the response surface based on the equivalent normal distribution.

$$z_i^* = \alpha_i \beta \quad \text{H.29}$$

9. Convert the Gaussian normalized design point back to the variable space for the $n-1$ variables.
10. For the unknown variable, determine the value such that the position in non-normalized variable space satisfies the limit state function where $g(\mathbf{x}^*) = 0$.
11. Iterate until the value of β and the design point x^* converge. Because the variable spaces have non-normal distributions, this convergence step needs to occur even if the limit state function is linear.

Example H.2 – Verifying Example H.1 results

This section is a continuation of the Example H.1 problem, including its results.

1. This step has been performed in Example H.1.
2. The mean load μ_L is specified as 210 kip-ft based on the nominal value equal to 200 kip-ft and a load bias factor λ_L equal to 1.05. The load $COV_L = 0.1$. The log-normal resistance distribution has a $COV_R = 0.14$

(Ellingwood¹) and has a resistance bias factor λ_R equal to 1.12. The load and resistance factors γ_L and γ_R are drawn from the previous example (H.1) with values of 1.216 and 0.72, respectively. The mean resistance μ_R is calculated (using the equivalency expressed in the relationship of Equation H.20) as

$$\mu_R = \left(\frac{\mu_L \gamma_L}{\gamma_R} \right) * \frac{\lambda_R}{\lambda_L} = 378.2081$$

Therefore, the initial guess of the design point gives a value of

$$r^* = 294.104$$

Given $g(\mathbf{x}^*) = R - L = 0$,

$$l^* = 294.104$$

3. The equivalent normal parameters can be calculated in the same manner as Example H.1, resulting in

$$\sigma_{\ln R}^2 = 0.139$$

$$\mu_{\ln R} = 5.926$$

And an equivalent normal distribution for resistance at the design point given by

$$\sigma_R^e = r^* \sigma_{\ln R} = 40.975$$

$$\mu_R^e = r^* [1 - \ln(r^*) + \mu_{\ln R}] = 365.22$$

The equivalent normal distribution for the load at the design point is the existing distribution,

$$\sigma_L^e = 21$$

$$\mu_L^e = 210$$

¹ Ellingwood, B. R. 2015. *Procedures for Developing Reliability-Based Load Criteria for Hydraulic Steel and Concrete Structures*. Final letter report submitted to the URS Group, Inc. U.S. Army Corps of Engineers.

4. The reduced variate design point is calculated with Equation H.25:

$$z_L^* = \frac{l^* - \mu_L^e}{\sigma_L^e} = \left(\frac{294.104 - 210}{21} \right) = 4.005$$

$$z_R^* = \frac{r^* - \mu_R^e}{\sigma_R^e} = \left(\frac{294.104 - 365.22}{40.975} \right) = -1.736$$

5. Determining the $\{G\}$ vector

$$G_1 = -\left. \frac{\partial g}{\partial R} \right|_{r^*} = -1\sigma_R^e = -40.975$$

$$G_2 = -\left. \frac{\partial g}{\partial L} \right|_{l^*} = +1\sigma_R^e = 21$$

6. Estimate β using Equation H.28, $\beta = 3.371$.
 7. The $\{\alpha\}$ vector is determined using Equation H.6:

$$\alpha = \begin{Bmatrix} -0.890 \\ 0.456 \end{Bmatrix}$$

with the correlation coefficient matrix $[\rho]$ defined as an identity matrix for the pair of independent variables R and L .

8. The new reduced design point for the resistance term is calculated as

$$z_R^* = \alpha_R \beta = -0.89 * 3.371 = -3.00014$$

9. Converting back from the Gaussian Normal space,

$$r^* = \mu_R^e + z_R^* \sigma_R^e = 365.22 + 3.00014 * 40.975 = 355.106$$

10. From the limit state response surface equation $g(X) = 0.0$, $l^* = r^* = 355.106$.
 11. Table H-2 shows the results of iterating steps 3 through 10 four times. The calculations were all made in an Excel spread sheet. This gives a clear idea of the convergence pattern that is approaching a β of 2.98, or a difference of less than 1% from the β_{Target} values of 3. The loss of precision may be attributed to the small number of iterations (four) that were performed using the Taylor series approximation of Example H.1 as well as the fact that higher-order terms were neglected in the Taylor series expansion formulation.

Table H-2. Iterations of step 3–10 as the design point and β values change.

	ORIGINAL	ITERATION 1	ITERATION 2	ITERATION 3	ITERATION 4
r^*	294.1041	355.106	365.4663	366.2033	366.2448343
l^*	294.1041	355.106	365.4663	366.2033	366.2448343
β	3.37121	3.052147	2.985671	2.980849	2.980576254

Notice that in step 2, the factored mean resistance is calculated based on the original mean load per the relationship of Equation H.29. This proves that the calculation of the proportional γ_L values from the ACI γ_R values of step 10 of Example 1 will provide the same values of Table H-2.

Table H-3 shows the calculation of β using the verification procedure for four iterations using the load and resistance factors ($\gamma_L=1.4$ and $\gamma_R=0.9$, respectively) as calculated in the 2016 USACE unpublished write-up of LRFD procedures.

Table H-3. Iterations of step 3-10 as the design point and β values change for the 2016 USACE unpublished writeup of LRFD procedures.

	ORIGINAL	TRIAL1	TRIAL2	TRIAL3	TRIAL4
r^*	279.2222	330.9341	338.344	338.7603	338.7790902
l^*	279.2222	330.9341	338.344	338.7603	338.7790902
β	2.903375	2.660407	2.616295	2.613765	2.613651168

The results reveal that the actual β given these factors converges to 2.61, which is over 10% less than the desired β_{Target} value of 3. This observation demonstrates the need to follow the numerical procedure outlined in section H.1 to determine the partial resistance and load safety factor values that are consistent with the β_{Target} value and allow for the scaling of the load factor based on any other user prescribed resistance factor value using the supplemental procedure of section H.3.

H.5 Summary and conclusions

A numerical procedure is outlined in the first part of this appendix for calibrating partial load and resistance safety factors given a target reliability index value, β_{Target} . A numerical procedure for calculating a Design Point β at the Limit State, given variables with non-normal distributions, was also included in this appendix as a complete, second numerical procedure.

Two example problems were included based upon a trial design of a singly reinforced beam with an unfactored dead load bending moment, M_{DL} , set equal to 200 kip-ft for a beam with a width of 12 in. and a depth (d) to center of reinforcement of 28 in. This example was taken from a 2016 USACE LRFD study (unpublished). This appendix outlines the complete numerical procedure to follow to obtain appropriate values for load and resistance factors for a prescribed β_{Target} value as well as a supplemental procedure to compute a load factor consistent with the ACI prescribed resistance factor ϕ value maintaining factor relationships computed for a prescribed β_{Target} value, so long as the mean load value remain unaltered. The consistency between computed γ and ϕ values in Example H.1 for the specified β_{Target} value of 3.0 was verified by the second numerical procedure in Example H.2. This Example H.1 calculation of ϕ and γ resulted in different values than the values reported from the 2016 USACE LRFD study (unpublished). In Example H.2, the calculated values were shown to be more accurate than the values from the 2016 USACE LRFD study (unpublished).

Unit Conversion Factors

Multiply	By	To Obtain
atmosphere (standard)	101.325	kilopascals
cubic feet	0.02831685	cubic meters
cubic inches	1.6387064 E-05	cubic meters
cubic yards	0.7645549	cubic meters
degrees (angle)	0.01745329	radians
feet	0.3048	meters
foot-pounds force	1.355818	joules
gallons (U.S. liquid)	3.785412 E-03	cubic meters
inches	0.0254	meters
inch-pounds (force)	0.1129848	newton meters
microinches	0.0254	micrometers
microns	1.0 E-06	meters
miles (nautical)	1,852	meters
miles (U.S. statute)	1,609.347	meters
mils	0.0254	millimeters
pounds (force)	4.448222	newtons
pounds (force) per foot	14.59390	newtons per meter
pounds (force) per inch	175.1268	newtons per meter
pounds (force) per square foot	47.88026	pascals
pounds (force) per square inch	6.894757	kilopascals
pounds (mass)	0.45359237	kilograms
pounds (mass) per cubic foot	16.01846	kilograms per cubic meter
pounds (mass) per cubic inch	2.757990 E+04	kilograms per cubic meter
pounds (mass) per square foot	4.882428	kilograms per square meter
pounds (mass) per square yard	0.542492	kilograms per square meter
slugs	14.59390	kilograms
square feet	0.09290304	square meters
square inches	6.4516 E-04	square meters
square miles	2.589998 E+06	square meters
square yards	0.8361274	square meters
tons (force)	8,896.443	newtons
tons (force) per square foot	95.76052	kilopascals
tons (long) per cubic yard	1,328.939	kilograms per cubic meter
tons (2,000 pounds, mass)	907.1847	kilograms

Multiply	By	To Obtain
tons (2,000 pounds, mass) per square foot	9,764.856	kilograms per square meter
yards	0.9144	meters

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