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Studies of High Power Edge-localized Wave Propagation in Novel Materials; Opportunities to Provide Advanced Antenna and Circuit Capabilities

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REGENTS OF THE UNIVERSITY OF COLORADO

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14. ABSTRACT
The PI's research efforts in physically based nonlinear wave propagation with emphasis in non-linear optics is extremely active. There have been a number of important research contributions carried out as part of the effort. During the period 15 November 2015-14 November, 2018, nine papers were published in refereed journals, fourteen invited lectures were given and one paper/preprint is under review. The PI also received a number of honors during this period. The key results and research directions are described in the section on accomplishments/new findings. Full details can be found in our research papers.

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OBJECTIVES

To carry out fundamental and wide ranging research investigations involving high power wave propagation, edge waves and topologically protected modes in nonlinear optics. Experiments have demonstrated that photonic honeycomb lattices (HCL) with helical structure written in the propagating direction and magneto-optical square and honeycomb lattices admit a class of localized edge waves. It has been found that energy can be directed and transported effectively near boundary edges and these waves can propagate around defects without back scatter. Existence of such edge waves is related to underlying topological structure in the underlying spectral domain; i.e. the Brillouin zone. Motivated by these experiments the PIs research involves investigations of a range of optical lattices including honeycomb lattices, Lieb, Kagome and magneto-optical lattices. The objectives were to describe the experiments in terms of reduced/discrete models, find associated topological waves and show they were immune from backscatter. The PI, postdoctoral associate and consultant are investigating novel micro-circuits, sensors and antennas.

STATUS OF EFFORT

The PI's research efforts in physically based nonlinear wave propagation with emphasis in nonlinear optics is extremely active. There have been a number of important research contributions carried out as part of the effort . During the period 15 November 2015 – 14 November, 2018, nine papers were published in refereed journals, fourteen invited lectures were given and one paper/preprint is under review. The PI also received a number of honors during this period.

The key results and research directions are described in the section on accomplishments/new findings. Full details can be found in our research papers which are listed at the end of this report.

Photonic and magneto-optical lattices are central to our studies. The lattice nonlinear Schrödinger equation, which is derived from Maxwell's equations under the paraxial approximation, is the key equation which governs these investigations of lattice photonics. Optical lattices analyzed includes honeycomb, Lieb, Kagome and staggered square lattices. The linear index of refraction is associated with the lattice background; the nonlinearity originates due to the inclusion of cubic nonlinearity. The study of magneto-optics involves Maxwell's equations with magnetic materials; in this case the underlying lattice structure is rectangular. The cases studied here have two-dimensional periodicity.

When the underlying potential is taken large, one can employ an approximation, called the tight binding limit. Within this approximation nonlinear discrete systems are derived. These reduced equations are more amenable to detailed analysis than the original equations. Furthermore, numerical simulations on reduced equations are considerably more accessible than would be the case if the governing nonlinear equations, were analyzed directly. Exponentially localized edge waves are shown to satisfy the discrete systems in accordance with experiments and large scale computations on the governing equations. The base, or unperturbed problem, involves electromagnetic wave propagation without active ferrites. In photonic systems the optical lattices are perturbed by writing helical structure in the longitudinal direction. In magneto-optical systems the perturbation is due to the addition of a magnetic field. The perturbation breaks time reversal symmetry. In the unperturbed problem the edge waves are stationary while in the perturbed problem the propagate unidirectionally. Remarkably the edge waves propagate with little/no backscatter. This effect is traced to topological effects (nontrivial Chern numbers) in the underlying spectral domain or Brillouin zone. These materials are often referred to as topological insulators.

Novel continuous and discrete systems closely related to the classical nonlinear Schrödinger equation which exhibit parity time symmetry were solved exactly via the inverse scattering transform. They have have the property that the nonlinear induced potential simultaneously depends on current and reflected spatial locations. Large amplitude waves, termed rogue waves, were found/analyzed in optical media. Motivated by experiments at the University of Colorado and physical observations dispersive shock waves (DSWs) were also investigated. DSWs are found in many applications including nonlinear optics, fluid dynamics and Bose Einstein condensation. DSWs with two-dimensional structure were also studied.

ACCOMPLISHMENTS/NEW FINDINGS

Nonlinear photonic lattices

Background

Light wave propagation in periodic optical photonic lattices is an active area of research. This is partially due to the realization that photonic lattices can be constructed on extremely small scales. Manipulation and navigation of localized nonlinear optical light pulses have been investigated in a variety of situations.

One dimensional optical periodic backgrounds, often referred to as waveguide arrays, can be constructed in various ways. For example they can be fabricated mechanically by etching techniques such as those comprised of AlGaAs materials cf. [1, 2] or all-optically via interference of two or more plane waves [3, 4] or imprinting waveguides in the propagating direction by femtosecond laser writing techniques cf. [5].

In general an input optical beam will diffract in the lateral or horizontal direction unless it is balanced by nonlinearity. The first theoretical prediction of nonlinear wave propagation in an optical waveguide array was reported in [6]. The wave phenomena was shown to satisfy the well known discrete nonlinear Schrödinger (DNLS) equation. This equation has localized solutions [6], usually called discrete solitons. Subsequently theoretical studies of discrete solitons in waveguide arrays reported switching, steering and a variety of other interesting properties cf. [7, 8]. Discrete bright and dark solitons were also found in quadratic ($\chi^{(2)}$) media [9], and in some cases, their properties differ from their Kerr counterparts [10]. After about a decade Discrete solitons in a nonlinear waveguide array was experimentally observed [1, 2]. These experimental observations and diffraction management [11] motivated considerable interest in discrete solitons in nonlinear lattices.

In [12] we considered the possibility of moving discrete solitons; we found that as discretization effects become important the solitons slowed down. This work motivated our studies of more complicated situations in which light beams can propagate in a discrete medium with alternating diffraction. A natural question which arises is: What happens if we launch a high power laser beam into a diffraction managed waveguide array ? We found modes which "breathe" periodically upon propagation [13]. During propagation the peak amplitude, FWHM, and phase of the beam evolve. For example, during each map period, the FWHM of the beam oscillates between a nonzero minimum and a maximum value while maintaining conservation of power.

By designing the diffraction properties of a linear waveguide array, additional new phenomena are found to occur. In [14] we showed that by appropriately tailoring the diffraction properties of a waveguide array, the interaction between discrete solitons can be altered to achieve better tunability and control over the collision outcome. For example, by colliding vector discrete solitons, remarkably the interaction picture involves beam shaping, fusion and fission. Interestingly it was also shown that discrete solitons in two-dimensional networks of nonlinear waveguides can be used to realize intelligent functional operations cf. [15].

The propagation of intense, paraxial light beams in Kerr media is well-known to be governed by a nonlinear Schrödinger (NLS) type equation given in normalized form by

$$i\psi_z + \nabla^2\psi - V(\mathbf{r}, z)\psi + \gamma|\psi|^2\psi = 0 \tag{1}$$

where $\psi(\mathbf{r}, z)$ corresponds to the slowly-varying complex amplitude of the electric field in the $\mathbf{r} = (x, y)$ plane, propagating along the z direction, $V(\mathbf{r}, z)$ represents the linear index of refraction which can vary in the longitudinal direction z and γ corresponds to the nonlinear index.

When the potential is only a function of the horizontal coordinate: $V = V(\mathbf{r})$ then if $\psi(\mathbf{r}, z) = \varphi(\mathbf{r})e^{-i\mu z}$ with $|\varphi| \ll 1$ we have the time independent linear Schrödinger equation with periodic coefficients:

$$(\nabla^2 - V(\mathbf{r}))\varphi = -\mu\varphi \tag{2}$$

Further if $V(\mathbf{r})$ is a 2-d periodic potential with lattice vectors: $\mathbf{v}_1, \mathbf{v}_2$ on the lattice $\mathbb{P} = \{m\mathbf{v}_1 + n\mathbf{v}_2 : m, n \in \mathbb{Z}\}$. Well-known results from Bloch theory tell us that solutions φ of equation (2), called Bloch modes, have the form $\varphi = \varphi(\mathbf{r}; \mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}}U(\mathbf{r}; \mathbf{k})$ where $U(\mathbf{r}; \mathbf{k})$ is periodic in $\mathbf{r}$ with the same periodicity as $V(\mathbf{r})$ and $\varphi(\cdot, \mathbf{k}), \mu = \mu(\mathbf{k})$ are periodic in $\mathbf{k}$ with 'dual' lattice vectors:

$\mathbf{k}_1, \mathbf{k}_2$ where $\mathbf{v}_m \cdot \mathbf{k}_n = 2\pi\delta_{mn}$; $\mu = \mu(\mathbf{k})$ is called the dispersion relation. Due to periodicity in $\mathbf{k}$ we can represent $\varphi(\mathbf{r};\mathbf{k})$ as a Fourier series:

$$\varphi(\mathbf{r};\mathbf{k}) = \sum_{\mathbf{v}} \phi_{\mathbf{v}}(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{v}} \tag{3}$$

where the $\sum_{\mathbf{v}}$ is over all values in the lattice $\mathbb{P}$. The Fourier coefficients $\phi_{\mathbf{v}}(\mathbf{r})$ are sometimes termed 'Wannier functions'. As Fourier coefficients they are given by

$$\phi_{\mathbf{v}}(\mathbf{r}) = \frac{1}{|\Omega|} \int_{BZ} e^{-i\mathbf{k}\cdot\mathbf{v}} \varphi(\mathbf{r};\mathbf{k}) d\mathbf{k} \tag{4}$$

where $|\Omega|$ is the area in the Brillouin zone (BZ).

Due to the periodicity it can be shown that Wannier functions are related through translational shifts: $\phi_{\mathbf{v}}(\mathbf{r}) = \phi_{\mathbf{0}}(\mathbf{r} - \mathbf{v})$. For equations such as (2) there are bounded wave-like solutions corresponding to a denumerable set n of dispersion surfaces; the dispersion surface $\mu(\mathbf{k})$ is said to have n spectral bands. Equations (3)-(4) hold for any band number n.

We mention the above because Wannier functions are used as the basis for constructing tight binding models in photonic and magneto-optical systems.

In general one cannot find explicit analytic solutions for Bloch/Wannier functions. But when the potential is large, i.e. $|V| \gg 1$, which is termed the tight binding (TB) approximation, the Wannier functions are strongly localized and one can approximate the Wannier functions. For simple lattices (i.e. when all sites can be constructed from one site) Wannier functions, in say the lowest band, are approximated by "orbitals" via:

$$\left(\nabla^2 - V_0(\mathbf{r})\right) \phi(\mathbf{r}) = -E\phi(\mathbf{r})$$

where $V_0(\mathbf{r})$ is an approximating potential; $\phi(\mathbf{r}), E$ are the corresponding eigenfunction (orbital) and eigenvalue. The approximation involves finding $\phi(\mathbf{r}), E$ in the limit $|V| \gg 1$. It is often useful to use $V_0(\mathbf{r}) \approx D_0(x^2 + c^2y^2)$ locally near its minima; in this case at the origin.

For simple lattices we can find discrete evolution equations for the envelope associated with the lattice NLS equation (1) by looking for solutions of the form

$$\psi \sim \sum_{\mathbf{v}} a_{\mathbf{v}}(z)\phi(\mathbf{r}-\mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}}$$

where $a_{\mathbf{v}}(z)$ represents the longitudinally varying Bloch wave envelope at the site $S_{\mathbf{v}}$. Substituting this approximation into the equation, multiplying by $\phi(\mathbf{r}-\mathbf{p})e^{-i\mathbf{k}\cdot\mathbf{p}}$, integrating over all space, then after some manipulation and rescaling, a two dimensional discrete NLS equation at general values of **k** is obtained

$$i\frac{da_{\mathbf{p}}}{dz} + \sum_{<\mathbf{v}>} a_{\mathbf{p}+\mathbf{v}}C_{\mathbf{v}}e^{i\mathbf{k}\cdot\mathbf{v}} + \gamma g|a_{\mathbf{p}}|^2 a_{\mathbf{p}} = 0$$

where $C_{\mathbf{v}}$, the dispersion and nonlinear coefficient g respectively, are found in terms of integrals over the orbitals and potential, and $< \mathbf{v} >$ means we take nearest neighbors to point **p**. This is a generalization of the well-known one dimensional DNLS equation extended to two dimensional square and triangular lattices [16]; cf. also [17]. In the continuum limit a two dimensional NLS equation for each **k** point in the Brillouin zone can be found [16].

In the case of a simple square two dimensional lattice, the analog experiment described earlier in the context of waveguide arrays, was explored. At low power an input beam diffracts; whereas at high power a two dimensional lattice soliton was observed and associated computations were carried out [18]. Subsequently there have been many studies which have investigated a variety of localized soliton solutions in two-dimensional simple lattices cf. [4, 19, 20, 21]

Next we consider background HCLs that are uniform in the longitudinal direction such as that depicted in Figure 1. This type of photonic lattice was considered by Segev's group cf. [22, 23] where the honeycomb lattice (HCL) $V(\mathbf{r})$ was constructed from the interaction of three plane waves.

A typical dispersion relation $\mu(\mathbf{k})$ for the lowest two bands is depicted in Figure 2. It is seen that the bands touch (cf. [24, 25]) at special points, called Dirac points, which in turn form a honeycomb lattice. In the material graphene Dirac points play an important role in terms of the novel phenomena found such as strong conductance and strength. Similarly, in optical applications interesting phenomena including nonlinear effects are found. In optics this research direction is sometimes referred to as photonic graphene or optical graphene.

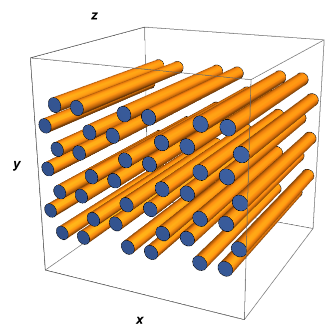

Figure 1: HCL: uniform in the longitudinal direction

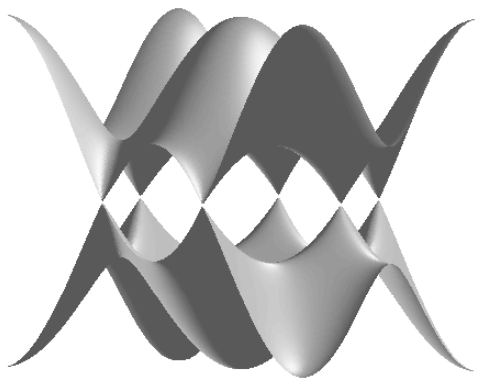

Figure 2: First two bands of the dispersion relation $\mu(\mathbf{k})$ associated with a typical honeycomb lattice; the bands touch at Dirac points

In earlier research we found novel discrete equations associated with honeycomb lattices via the tight binding approximation [26, 27]. Since the HCL is comprised of two triangular sublattices the solution of the lattice NLS (1) equation, in say the lowest two bands, is assumed to be of the form

$$\psi \sim \sum_{\mathbf{v}} a_{\mathbf{v}}(z)\phi_A(\mathbf{r}-\mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}} + \sum_{\mathbf{v}} b_{\mathbf{v}}(Z)\phi_B(\mathbf{r}-\mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}} \quad (5)$$

where the sum $\mathbf{v}$ takes all values $m, n \in \mathbb{Z}$ associated with the A, B sublattice sites (minima in each cell) respectively and

$$\left(\nabla^2 - V_j(\mathbf{r})\right)\phi_j(\mathbf{r}) = -E_j\phi_j(\mathbf{r}); \quad j = A, B$$

where $V_j(\mathbf{r})$ is the underlying approximating potential; $\phi_j(\mathbf{r}), j = A, B$, are Wannier functions.

Substituting ψ into the lattice NLS equation after some manipulation, normalization, maximal balance (slow evolution, dispersion, nonlinearity all of the same order) the following discrete HCL system, **at a general location k**:

$$
\begin{aligned}
i\frac{da_{\mathbf{p}}}{dz} + \mathcal{L}^{-}b_{\mathbf{p}} + \sigma|a_{\mathbf{p}}|^2 a_{\mathbf{p}} = 0 \\
i\frac{db_{\mathbf{p}}}{dz} + \mathcal{L}^{+}a_{\mathbf{p}} + \sigma|b_{\mathbf{p}}|^2 b_{\mathbf{p}} = 0
\end{aligned}
\tag{6}
$$

where

$$
\mathcal{L}^{-}b_{\mathbf{p}} = b_{\mathbf{p}} + \rho b_{\mathbf{p}-\mathbf{v}_1}e^{-i\mathbf{k}\cdot\mathbf{v}_1} + \rho b_{\mathbf{p}-\mathbf{v}_2}e^{-i\mathbf{k}\cdot\mathbf{v}_2}
$$

$$
\mathcal{L}^{+}a_{\mathbf{p}} = a_{\mathbf{p}} + \rho a_{\mathbf{p}+\mathbf{v}_1}e^{i\mathbf{k}\cdot\mathbf{v}_1} + \rho a_{\mathbf{p}+\mathbf{v}_2}e^{i\mathbf{k}\cdot\mathbf{v}_2}
$$

where σ depends on the original coefficient of nonlinearity (γ), integrals over the orbitals (g) and ρ is deformation parameter which depends on given lattice parameters; ρ an be related to η in equation (**??**) [27]. When $\rho = 1$ we have perfect hexagonal HCL. The effect of ρ in physical space is to squeeze certain hexagonal sides of the HCL when $\rho < 1$ and push them apart when $\rho > 1$.

We can consider the continuum limit where i.e. $a_{\mathbf{p}+\mathbf{v}} \sim (1 + \nu\mathbf{v}\cdot\nabla + \cdots)a,\ 0 < \nu \ll 1$ here the envelopes can be written in the form $a(\mathbf{R}, Z)$ and $b(\mathbf{R}, Z)$ where $\mathbf{R} = \nu\mathbf{r}$ represents the spatial coordinate of the slowly varying envelope. If we assume the wave is propagating with **k** near a Dirac point we find the following continuous system of equations, termed a nonlinear Dirac system

$$
\begin{aligned}
i\frac{\partial a}{\partial z} + \tilde{\partial}_{-}b + \sigma_\nu|a|^2 a = 0 \\
i\frac{\partial b}{\partial z} - \tilde{\partial}_{+}a + \sigma_\nu|b|^2 b = 0
\end{aligned}
\tag{7}
$$

where $\tilde{\partial}_{\pm} = \partial_x \pm i\zeta\partial_y,\ \sigma_\nu = \frac{\sigma}{\nu}$ and $\zeta = \sqrt{\frac{4\rho^2-1}{3}}$. If $\sigma_\nu = 0$, we find a linear system which can be transformed to the familiar 2 dimensional wave equation.

The first experimental studies of honeycomb lattices dealt with localized inputs where boundary effects could be ignored [22]. Our research [26, 27] showed that experiment and the above theory were in agreement and that the continuous theory (7) is a good approximation of the discrete theory

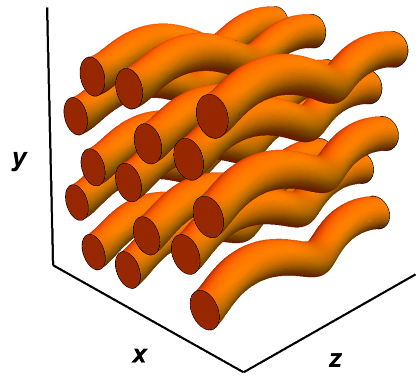

Figure 3: HCL with typical variation along the longitudinal direction.

(6). Our research showed that for small nonlinear coefficient in the discrete lattice Dirac system (6) the resulting pattern is nearly conical. Indeed this behavior seems to be rather general when there is underlying hexagonal structures; e.g. it was found in a hexagonally packed lattice of spherical particles that interact nonlinearly via point contacts cf. [28].

When the deformation parameter $\rho \neq 1$ the nonlinear Dirac system (7) admits an elliptical diffraction pattern. When $\rho \to 1/2$ the elliptical diffraction pattern tends to become more elongated; i.e straight line diffraction.. In this limit a new class of nonlinear wave equations was found [29].

The potential is assumed uniform along the propagation direction z, although non-uniform potentials in z could also be considered in a manner analogous to diffraction or dispersion management, and more recently has been employed in a novel way in honeycomb lattices [30].

Topological modes

In [30] it was shown experimentally that certain honeycomb lattices admit linear localized traveling edge waves. The key idea was to introduce longitudinally varying waveguides; cf. Fig.3. Waveguides varying in the longitudinal direction were written into fused silica via femtosecond laser technology [5].

The helical variation in the z-direction corresponds to introducing a suitable 'path function': $x' = x - h_1(z),\ y' = y - h_2(z),\ z' = z.$

Using this, after transforming variables the original lattice NLS equation (1) takes the form (after dropping primes),

$$i\psi_z = -(\nabla + i\mathbf{A_p}(z))^2\psi + V(\mathbf{r})\psi - \gamma\,|\psi|^{\mathbf{2}}\,\psi. \tag{8}$$

Equation (8) is a lattice NLS equation with an additional potential $\mathbf{A_p}(z) = -\mathbf{h}'(z)$. In quantum mechanics this correspond to adding an additional magnetic field. In honeycomb lattice applications we refer to $\mathbf{A_p}(z)$ as a pseudo field (i.e. a pseudo magnetic field). Substituting the TB approximation eq. (5) into the above equation and looking for solutions proportional to $e^{im\omega}$ (i.e. taking a discrete Fourier transform in m) yields the following normalized coupled mode equations

$$i\frac{da_n(z)}{dz} + e^{i\mathbf{d}\cdot\mathbf{A_p}}\left(b_n + \rho\tilde{\eta}^* b_{n-1}\right) + \sigma|a_n|^2 a_n = 0 \tag{9}$$

$$i\frac{db_n(z)}{dz} + e^{-i\mathbf{d}\cdot\mathbf{A_p}}\left(a_n + \rho\tilde{\eta} a_{n+1}\right) + \sigma|b_n|^2 b_n = 0 \tag{10}$$

where, $\mathbf{d}$ is a given vector between adjacent horizontal sites, $\tilde{\eta} = \tilde{\eta}(\omega, \mathbf{A_p})$ is known, $*$ represents complex conjugate, ρ corresponds to a lattice deformation and for convenience we take $\mathbf{k} = 0$.

In [31], motivated by the experiments in [30], we developed an asymptotic theory assuming a relatively fast helical change; namely we assume $\mathbf{A_p}$ periodic and rapidly varying on a semi-infinite lattice: $\mathbf{A_p} = \mathbf{A_p}(\zeta),\ \zeta = z/\epsilon, |\epsilon| \ll 1$. A typical path function and pseudo field are given by:

$$\mathbf{h_1}(\zeta) = \kappa(\cos\zeta, \sin\zeta),\ \ \mathbf{A_p}(\zeta) = \kappa(-\sin\zeta, \cos\zeta), \tag{11}$$

where κ is constant. Without the pseudo field the edge modes are stationary (cf. [32]). Asymptotic theory leads to explicit formulae.

Applying a fixed boundary condition on a 'Zig-Zag' edge of an HCL we find an exponentially decaying edge mode given asymptotically ($|\epsilon| \ll 1$) by

$$a_n(z,\zeta) \sim 0,\ \ b_n(z,\zeta) \sim C(Z,\omega)r^n,\ |r| = |r(\omega,\rho)| < 1$$

where $r(\omega, \rho)$ is found analytically. For the linear problem we find

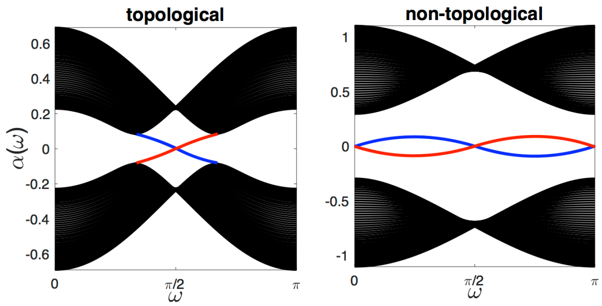

Figure 4: Typical dispersion relations; the blue curves correspond to edge mode modes on the left edge and red curves are edge modes on the right edge. The left figure is when $\rho = 1$: this case corresponds to a topological mode; The right figure is when $\rho = 0.4$: this case is not topological.

$$C(Z, \omega) = C(0)\,\text{exp}\left(-i\alpha(\omega)Z\right), Z = \epsilon z$$

$\alpha(\omega)$: is called the edge mode dispersion relation. Since $\mathbf{A_p}$ is periodic $\alpha(\omega)$ corresponds to the so called Floquet coefficient, which in this asymptotic limit can also be obtained explicitly.

Typical dispersion relations are plotted in Fig. 4. Bulk or nondecaying modes fall within the upper and lower black zones. The blue curves correspond to edge mode modes on the left edge, the red curves are edge modes on the right edge. Of the two dispersion relations it can be shown that the case which corresponds to nontrivial topology in its respective Brillouin zone is the left figure. The right figure correspond to edge modes which have trivial topology in their respective Brillouin zone. The linear problem corresponding to the left figure in Fig. 4 was studied in [30].

A typical edge mode is depicted in the Fig. 5. Notice that the edge mode decays rapidly in the n direction and has an envelope in the m direction. The mode evolves along the edge in the m-direction.

If we add nonlinearity and allow the envelope C to vary slowly with respect to the lattice (scale $|\nu| \ll 1$) in the transverse direction, and take the continuum limit ($\nu m \to y$) so that $C = C(Z, y)$, we find that C satisfies the following nonlinear Schrödinger (NLS) equation

$$iC_Z = \alpha_0 C - i\alpha_0' \nu C_y - \frac{\alpha_0''}{2}\nu^2 C_{yy} + i\frac{\alpha_0'''}{6}\nu^3 C_{yyy} - \alpha_{nl,0}|C|^2 C + \cdots \tag{12}$$

In the above equation ω_0 is the central frequency associated with $\alpha(\omega)$ and $\alpha_0 = \alpha(\omega_0)$ etc. When

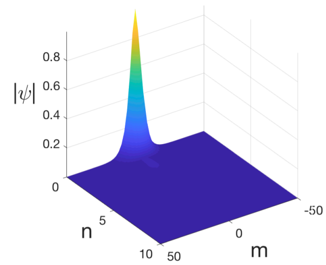

Figure 5: A typical edge mode: ψ as a function of m, n.

$\alpha_0'' \neq 0$ and dropping the α_0''' term, we can transform this equation to a standard NLS equation. This means that the nonlinear edge mode satisfies the NLS equation. Our studies have shown that this NLS equation and its linear counterpart provides a very good approximation to the discrete tight binding equations.

In [33] the above theory was developed for the case when the pseudo-field $\mathbf{A_p}$ varies slowly. While there are edge states that evolve unidirectionally we find that they can evolve out of the gap into the bulk region and therefore the edge state will no no longer decaying. The nonlinear system is governed by an NLS equation like that in equation (12) but with coefficients that vary with respect to the longitudinal coordinate. In this case the cofficients can vary in such a way that the NLS equation originally is in the focusing region and admits edge solitons, but at subsequent values of longitudinal coordinate the NLS equation can be defocusing.

In [34] and recent studies on topological HCLs we have investigated the propagation of topological modes around a bounded rectangle in a perfect honeycomb lattice ($\rho = 1$). In Fig. 6 we show the amplitudes of the linear wave propagation and the nonlinear wave propagation. The initial conditions (ICs) for the linear waves are chosen to minimize dispersion ($\omega = \pi/2$) and for the nonlinear waves the nonlinear coefficient (σ) is chosen to balance linear dispersion so that the NLS equation models the propagation along a straight side; see [31]. As indicated in Fig. 6, remarkably we see that nonlinearity can significantly improve wave transmission characteristics. Note that the linear unidirectional wave suffers from dispersion whereas in the presence of nonlinearity this dispersive effect is essentially eliminated; the end result in the nonlinear case is a coherent soliton-like structure. The dispersive decay of the linear wave agrees with stationary phase calculations. The

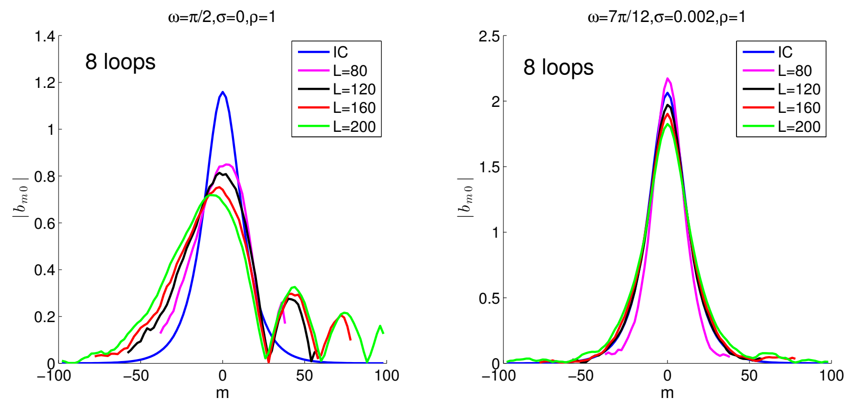

Figure 6: Mode propagation around a bounded HCL domain; the results of the evolution are shown after propagation with different rectangular perimeters; Left: Linear; Right: Nonlinear.

effect of corners on nonlinear modes appears to be weak; only a modest phase shift on the wave.

ii) General longitudinally driven lattices: honeycomb and staggered square lattices

We have seen above that honeycomb lattices support topological modes when both sublattices have the same helically varying structure imprinted in the longitudinal direction. In recent work it was shown that staggered square lattices which are out of phase with each other can also support topological modes [35, 36]. Subsequently we have developed a method to derive tight-binding equations that describe beam propagation in general longitudinally varying lattices with either simple or non-simple lattice configurations [37]. As typical representatives we analyze honeycomb and staggered square lattices, though more complex lattices such as Lieb and Kagome lattices can be considered [38]. In doing so, we are able to study a wide range of lattice dynamics including periodic waveguides with different helical radii, different frequencies, lattices with phase differences (e.g. out of phase sublattices) and counter rotating sublattices. We can find the underlying Floquet bands and from the band structure we can readily see when there are topological modes. Indeed we have found a large class of new topological modes. These modes complement the well-known HCL traveling edge states present when the lattice waveguides rotate in-phase with each other such as those discussed above [30, 31].

Typical path functions of the sublattices are given by

$$\mathbf{h}_i(z) = \eta_i \left(\cos\left(\Omega_i z + \tilde{\chi}_i\right), \sin\left(\Omega_i z + \tilde{\chi}_i\right)\right) \;,\; i = 1, 2 \tag{13}$$

where $\eta_i \geq 0$ denote the radii (ratio of the helix radius to the distance between adjacent lattice sites), Ω_i are the helix rotation frequencies, and $\tilde{\chi}_i$ are phase shifts. This is a significant generalization of what was done previously (e.g.–see equation (11)).

We have developed the tight binding analysis where each of the two sublattices can vary independently [37]. The Floquet analysis for different radii, different frequencies and counter rotation, was carried out. Examples of Floquet HCL bands are presented in Fig. 7. In these figures the black regions are bulk bands. Between the top and bottom bulk bands is a gap. In the gap the blue curve represent edge modes on the left zig-zag side and red correspond to edge mode on the right zig-zag side. These modes have one crossing from top bulk region to bottom bulk region; they correspond to topological modes cf. [31]. In [37] we developed the tight binding framework for a staggered square lattice configuration [37] and studied the evolution of the edge modes.

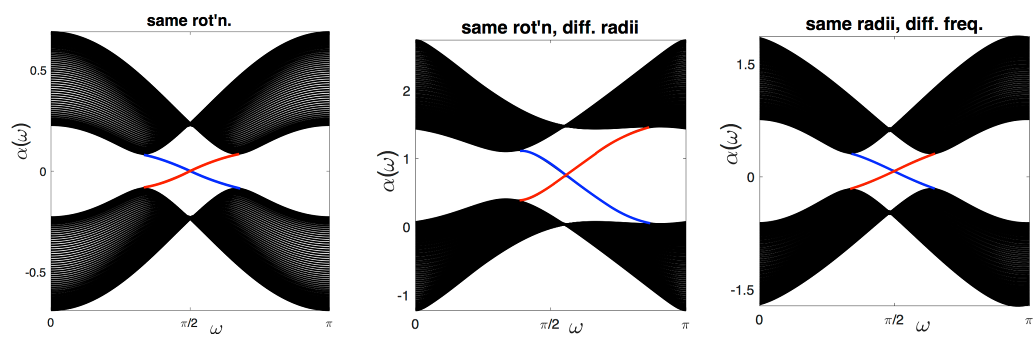

Figure 7: Floquet HCL bands: Left figure: same rotation frequency, same radius; center figure: same frequency, differing radii: $\eta_2 = 0.6\eta_1$; right figure: same radii, different frequency: $\Omega_2 = 2\Omega_1$

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Magneto-Optical systems

The first experimental discovery of topological insulators in an optical regime was related to magneto-optics (MO) on a square/rectangular lattice [39, 40]; both computation and experiment were carried out. The associated computations employed COMSOL which is a large electromagnetics wave software package. Without an external magnetic field the underlying band structure was calculated. It was found that the 2nd and 3nd band had a point where the bands touched. An external magnetic field was employed to create ferro-magnetic permeability, this broke time reversal symmetry and the 2nd and third bands split apart to created a gap region between the bands.

In this region it was found that there were edge waves. Further due to the fact that the 2nd and third bands had non trivial Chen numbers. In turn this system supported topologically protected edge waves. These waves were shown to be resistant to back-scatter. Namely when a source was introduced on the edge, waves propagated towards an introduced defect. The defect did not cause backscatter. An antenna downstream measured wave energy but antennas upstream did not.

In our current studies we are developing tight binding models from the maxwell equations that describe this MO system; to date no one has developed such tight binding systems for this MO problem. Such a tight binding system will admit semi-analytic approaches and allow investigations of the MO system conveniently/rapidly without need for specialized software.

In [39, 40] Maxwell's equations are reduced exactly to a scalar equation; it has the form

$$(\nabla^2 + B \cdot \nabla + \tilde{\epsilon}\omega^2)\psi = 0 \tag{14}$$

where ψ, ω are the scalar electromagnetic field and frequency respectively; B is related to the certain coefficients of the permeability tensor, $\tilde{\epsilon}$ is related to the permittivity as well as certain coefficients in the permeability tensor.

In current work we are employing the methods described in [41, 42]; this involves minimizing a certain functional related to the spread of the Wannier functions in physical space. This process yields localized Wannier functions. We do this for the lowest few bands associated with equation (14) with $B = 0$. When $|B| \ll |\tilde{\epsilon}|$ we are using these Wannier functions to create approximating orbitals. These orbitals are used to find a tight binding system via methods similar to those discussed earlier in this proposal. We have found that these Wannier functions give very good approximations to the band structure, with $B = 0$. This band structures agree with the band structures given in [39, 40] which was obtained using the electromagnetic package: COMSOL.

We are using these computed localized Wannier functions to develop tight binding equations to approximate equation (14) but now with a nonzero perturbed magnetic term: $B \neq 0$. By incorporating time dependence into equation (14) this leads us to coupled mode equations which apply when MO waves propagate along edges as well as in the interior. We find nontrivial Chern numbers associated with edge modes. Further studies of edge mode evolution indicate that they do not suffer from backscatter. This work is continuing.

Nonlinear PT-Symmetric systems

Another area of current interest in optics is so called $\mathcal{PT}$ optics. $\mathcal{PT}$ systems are made by introducing gain and loss in just such a way that the equations describing the propagation of fields which are invariant under the combined action of spatial inversion, $\mathcal{P}$, and time reversal, $\mathcal{T}$. $\mathcal{PT}$-symmetric optical systems represent a novel class of optical metamaterials potentially allowing for greater control of light. Phenomena such as unusual beam diffraction patterns and unidirectional invisibility have been observed in $\mathcal{PT}$ systems . Wave propagation in PT symmetric coupled waveguides/photonic lattices has been experimentally observed in optics cf. [43]. All of this phenomena depends on the fundamental question of whether spectra is real or if a 'phase transition' [43] has caused spectra with non-zero imaginary part to appear. Establishing whether phase-transitions do or do not occur has been studied in many contexts cf. [44]. In [45], starting from a linear Schrödinger equation of the type discussed above, see eq. (1) with a PT symmetric lattice potential, we develop a necessary and sufficient condition which establishes when phase transitions can or can not occur in $\mathcal{PT}$ symmetric honeycomb lattices for small $\mathcal{PT}$ perturbations. We we show that honeycomb potentials with added symmetry allow us to find $\mathcal{PT}$ perturbations which satisfy this condition. Via numerical experiments, it is seen that $\mathcal{PT}$-symmetric lattices satisfying the analytic condition do not exhibit phase transitions for a range of parameter regimes, sometimes even for relatively large $\mathcal{PT}$ perturbations. This goes beyond the standard theory, and shows with added symmetry that carefully designed $\mathcal{PT}$ symmetric lattices can be robust against phase transitions.

In a related direction we have found that a 'simple looking' nonlocal PT-symmtric equation is integrable. The equation takes the form

$$iq_t(x,t) = q_{xx}(x,t) \pm 2q(x,t)q^*(-x,t)q(x,t) \tag{15}$$

where $*$ denotes complex conjugation and $q(x,t)$ is a complex valued function of the real variables x and t. Eq. (15) admits a linear (Lax) pair formulation and possesses an infinite number of conservation laws; hence it is an integrable system. Using the the inverse scattering transform, corresponding to rapidly decaying initial data, we can linearize the equation and obtain solutions to Eq. (15) including soliton solutions [46, 47]. Some of the important properties of the nonlocal NLS equation are contrasted with the classical NLS equation where the nonlocal nonlinear term $q^*(-x,t)$ is replaced by $q^*(x,t)$. Indeed we note that both equation (15) and the classical NLS share the symmetry that when $x \to -x,\ t \to -t$ and a complex conjugate is taken, then the equation remains invariant. Thus, the new nonlocal equation is PT symmetric which, in the case of classical optics, amounts to the invariance of the induced potential $V(x,t) = q(x,t)q^*(-x,t)$ under the combined action of parity and time reversal symmetry. This theory has been extended to many other equations including discrete systems [48, 49].

Dispersive Shock Waves

Shock waves in compressible fluids is a classically important field in applied mathematics and physics, whose origins date back to the work of Riemann. Such shock waves, which we refer to as classical or viscous shock waves (VSWs), are characterized by a localized steep gradient in fluid properties across the shock front. Without viscosity one has a mathematical discontinuity; when viscosity is added to the equations, the discontinuity is "regularized" and the solution is smooth. An equation that models classical shock wave phenomena is the Burgers equation

$$u_t + uu_x = \nu u_{xx} \tag{16}$$

If $\nu = 0$, we have the inviscid Burgers equation which admits wave breaking. When the underlying characteristics cross a discontinuous solution, i.e. a shock wave, is introduced which satisfies the Rankine-Hugoniot jump conditions which, in turn, determines the shock speed. Thus the mathematical discontinuity is regularized when viscosity ν is introduced.

Another type of shock wave is a so-called dispersive shock wave (DSW). Gurevich and Pitaevskii [50] studied DSWs in the Korteweg-deVries (KdV) equation; DSWs correspond to the small dispersion limit. They obtained an analytical representation of a DSW. As opposed to a localized shock as in the viscous problem, the description of a DSW is one with a sharp front with an expanding, rapidly oscillating rear tail. The Korteweg-deVries (KdV) equation with small dispersion is given by

$$u_t + uu_x = \epsilon^2 u_{xxx} \tag{17}$$

where $|\epsilon| \ll 1$ regularizes the discontinuity that otherwise would be present. The mathematical technique used to analyze DSWs relies on wave averaging, often referred to as Whitham theory. Whitham theory is used to construct equations for the parameters associated with slowly varying wavetrains; it provides an analytical basis for DSW dynamics. The solution and details are quite different from viscous shock waves.

A viscous shock wave which occurs in Burgers equation is depicted below in Fig. 8–in red; a typical DSW associated with the KdV equation is illustrated in blue. For the KdV DSW one finds that there are two speeds associated with a DSW: one is the speed associated with the frontal wave which is a soliton (located at x_s in the figure), and the other speed corresponds to the group velocity of near linear trailing waves on the rear end (depicted by x_l in the figure) of the DSW. This is very different from the classical or viscous shock wave (located at x_c in the figure) associated with the Burgers equation. Interestingly, the structure of the KdV DSW is strikingly similar to the original plasma observations.

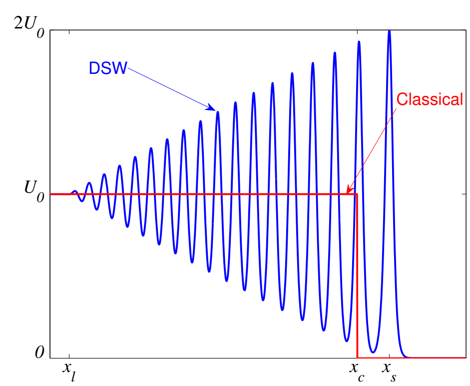

Figure 8: Left figure: typical DSW satisfying the KdV eq. (17) and a classical shock wave satisfying Burgers eq. (16)
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Experiments in Bose-Einstein condensates (BEC) and nonlinear optics have enhanced interest in DSWs. The BEC experiments, originally performed in the Physics Department at the University of Colorado, motivated our studies [51]. Experiments in nonlinear optics have also observed similar blast waves and other interesting DSW phenomena [52].

Recently we have investigated multi-dimensional extensions of this phenomena; we analyzed a reduction of a two-dimensional generalization of the KdV equation often referred to as the the Kadomtsev-Petviashvili equation [53]

$$(u_t + uu_x + \epsilon^2 u_{xxx})_x + 3\sigma u_{yy} = 0, \;\; \sigma + \pm 1 \tag{18}$$

We found that reductions of this equation to a cylindrical KdV equation which in turn has interesting DSW solutions. In terms of the KP equation the DSW has curved DSW structure which in turn has been observed in nature.

Dispersive shock waves are an interesting and developing area of research which we believe will play an increasingly important role in nonlinear optics applications and other areas of physics.

Rogue Waves

Abnormally large waves have been observed in the ocean cf. (wikipedia/rogue waves), in optics [54] and many other areas of physics. They appear to be universal structures in nature.

We often say a wave is a rogue wave, sometimes called a freak wave, if the amplitude of the wave is two-three times the size of typical background wave. Such large waves, often termed "rogue" waves can be extremely dangerous e.g. to large ships and structures.

A wave that caught the attention of the scientific community was the measurement of the so-called Draupner wave which was a rogue wave at the Draupner platform in the North Sea on January 1, 1995. It had a wave height of 25.6 meters (84 ft) with peak elevation of 18.5 metres [61 ft]) . During that event, minor damage was also inflicted on the platform, far above sea level, indicating that the reading was valid.

In many cases the NLS equation and a special rational soliton solution: the Pergrine solito,n are central to the understanding of the phenomena and they have been observed in laboratory settings [55]. But sometimes the underlying equations describing the phenomena are not the NLS equation and they are unlikely to be integrable. Yet there appears to be similarity in the solutions and rogue wave structures. Understanding this situation has been a topic of our recent work [56, 57].

It should be noted that the description of rogue wave phenomena related to one dimensional models such as the one dimensional NLS equation. However in water waves it is known that in deep water the relevant equation is a two-dimensional NLS equation whose dispersive term is hyperbolic, not elliptic; this is often terms the HNLS equation. Unlike the one dimensional NLS equation the HNLS equation does not appear to have solitary wave solutions. Furthermore the HNLS equation admits a universal regime where the similarity solution describes its behavior [58].

PERSONNEL SUPPORTED

- Faculty: Mark J. Ablowitz
- Post-Doctoral Associate: J.T. Cole
- Other (please list role) R. Alanbese, consultant

PUBLICATIONS

- **ACCEPTED/PUBLISHED**
  - **Journals–refereed**

1. Adiabatic dynamics of edge waves in photonic graphene, MJ Ablowitz, CW Curtis and Y-P Ma, *2D Materials* **2** (2015) 024001
2. Strong transmission and reflection of edge modes in bounded photonic graphene, (2015), M.J. Ablowitz and Y-P Ma, *Opt. Lett.* **40** (2015), 4671-4674
3. Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation, Mark J. Ablowitz and Ziad H. Musslimani, *Nonlinearity* **29** (2016) 915-946
4. Conical Wave Propagation and Diffraction in 2D Hexagonally Packed Granular Lattices, C. Chong, P.G. Kevrekidis, M.J. Ablowitz, Y-P Ma, *Phys. Rev E* **93** (2016) 012909
5. Dispersive shock waves in the Kadomtsev-Petviashvili and Two Dimensional Benjamin-Ono equations, Mark J. Ablowitz, Ali Demirci, Yi-Ping Ma, *Physica D* **333** (2016) 84-98
6. Rogue waves in birefringent optical fibers: elliptical and isotropic fibers, Mark J Ablowitz, Theodoros P Horikis, *J. Optics* **19** (2017), 065501
7. Rogue waves in nonlocal media, Theodoros P. Horikis and Mark J Ablowitz, *Phys. Rev. E* **95** (2017), 042211

8. A universal asymptotic regime in the hyperbolic nonlinear Schrödinger equation, M.J. Ablowitz Y-P Ma and I. Rumanov, *SIAM J. Appl. Math.* **77** (2017) 1248
9. Tight-binding methods for general longitudinally driven photonic lattices: Edge states and solitons, M.J. Ablowitz and J.T. Cole *Phys Rev A* **96** (2017) 043868

– **SUBMITTED**

1. Tight-binding equations for longitudinally driven waveguides: Lieb and Kagome lattices, Mark J. Ablowitz and Justin T. Cole, 2018, arXiv preprint arXiv:1804.09880v1, (2018) Submitted Phys Rev.A

– **Conferences, Seminars**

– **INTERACTIONS/TRANSITIONS**

1. Invited lecture: "Photonic Graphene and Wave Progagtion", SIAM National Meeting, Boston Mass, July 11, 2016 (Meeting July 11-15, 2016)
2. AF Contractor meeting: "Wave Propagation Across Edges and Corners in Bounded Photonic Graphene", AF Workshop Nonlinear Optics, Arlington VA, Sept 27, 2016
3. Invited lecture: "Wave Propagation in Photonic Graphene and Rogue Waves", Inst. for Math and Applications (IMA), U. Minn., Oct. 31, 2016 (Meeting Oct. 31-Nov 4, 2016)
4. Invited lecture: "Nonlinear Waves: Solitons at age 50", Physics Dept Colloquium, U. Mich., Nov. 8, 2016
5. Invited lecture: "Solitons and (Many) Nonlocal Integrable Nonlinear Equations", Applied Math Seminar, U. Mich., Nov. 10, 2016
6. Invited lecture: "Solitons At Age 50", Plenary Lecture, Canadian Math. Society, Dec. 5, 2016
7. Invited lecture: "Solitons at 50", Connecticut Valley Symposium, Math Dept., University of Mass., Amherst, Mass, April 6, 2017

8. AF Contractor meeting: "Edge Mode Dynamics in Rapidly and Adiabatically Varying Photonic Graphene", AF Workshop Nonlinear Optics, Arlington VA, March 8, 2017
9. Invited lecture: "Edge Mode Dynamics in Rapidly and Adiabatically Varying Photonic Graphene" May 29, 2017, Nonlinear Waves in Israel, Rosh Pinna, Israel, May 29-30, 2017
10. Invited lecture: "Solitons and (Many) Integrable Nonlocal Nonlocal Equations", June 19, 2017, Physics and Mathematics of Nonlinear Phenomena 2017: 50 years of I.S.T., Gallipoli, Italy, June 18-23, 2017
11. Invited lecture: "Integrable Nonlocal Nonlinear Equations and Solitons", Nonlinear days, University of Colorado, Colorado Springs, Nov 11, 2017
12. AF Contractor meeting: "Tight Binding Models for Longitudinally Driven Linear/Nonlinear Lattices", AF Workshop Nonlinear Optics, Arlington VA, March 8, 2018
13. Invited lecture: "Wave Dynamics in Linear/Nonlinear Photonic Lattices and Topological Insulators", Waves in Random Media, May 21-25, 2018, Colorado State University, May 23, 2108
14. Invited lecture: "New Classes of Integrable Nonlocal Nonlinear Equations and Solitons", Dynamics Days Europe Sept 1-6, 2018, Loughborough Univ., Sept 2, 2018

– **Consultative and Advisory Functions to Other Laboratories and Agencies:** None

NEW DISCOVERIES, INVENTIONS, OR PATENT DISCLOSURES: None

HONORS/AWARDS:

– Two special issues of the Journal Studies in Applied Mathematics **137**, 2016, Issues 1,2 were dedicated to the PI, in honor of his research

– Doctor of Science, Honoris Causa, July 2018, University of Kent, Canterbury, UK

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