



The Role Of Kinetic Fluctuations in Laminar-Turbulent Transition of High-Speed Boundary-Layer Flows

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Receptivity of High-Speed Boundary Layers to Kinetic Fluctuations

A. Tumin, L. D. Edwards, K. Luna

November 27, 2018

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Chapter 1

Introduction

Laminar-turbulent transition in boundary-layer flows essentially depends on receptivity to free-stream disturbances like acoustic waves and/or body-induced disturbances such as roughness and vibrations [Mor69, Res76]. For hypersonic flight in the high-altitude atmosphere, particularly relevant disturbances include vorticity/entropy fluctuations and atmospheric particulates including volcanic dust or the products of rocket exhaust. However, at certain conditions, such as those of the HiFIRE-1 flight test [LCC⁺11], the aforementioned sources of external forcing may be ruled out as likely causes of transition [Fed15]. In such ‘clean’ cases, it is possible that kinetic fluctuations (KF) may play a role in transition. This hypothesis was tested by Fedorov who showed that, at the flight conditions for HiFIRE-1, KF could lead to rms wall pressure perturbations at the experimentally observed transition location on the order of that associated with transition in quiet wind tunnel experiments [Fed15].

All studies of receptivity to KF prior to the work in this report have considered a calorically perfect gas model [ZR84, Luc08a, Luc10, FA10, Fed12, Fed15]. However, for the study of conditions relevant for future hypersonic aircraft, it is critical to understand the role of real gas effects on the receptivity phenomenon. This involves the use of a more realistic model for air in the mean flow computations. It also requires the inclusion of additional forcing terms associated with species diffusion and chemical reactions in the fluctuating hydrodynamics framework. This more advanced modeling approach allows for the first detailed studies of KF-induced transition onset for realistic chemically reacting flows.

In this report, we present the key results produced out of the research conducted under the present AFOSR grant. We begin with a brief review of *fluctuating hydrodynamics*, which forms the basis for all of the work presented herein. A matched asymptotic (MA) solution to receptivity to distributed forcing by KF, used extensively in the works to follow, is then presented in detail. This is followed by three studies of receptivity to KF for cases relevant to realistic hypersonic flight conditions. The first explores real gas effects on the receptivity phenomenon through comparison of the results stemming from two different mean flow models: a calorically perfect gas model and a 5-species model for air. The next considers the role of KF in transition for the Reentry-F flight experiment using the aforementioned air mean flow model. In the third, a random species diffusion and a random chemical source term are introduced into the formulation in order to study chemically reacting flows. The final

section of this report details a method of multiple scales (MMS) solution to the receptivity problem which provides greater resolution of the excitation region and is applicable in cases where the MA approach is not.

This work was supported by AFOSR Grant FA9550-15-1-0369, monitored by Dr. Ivett Leyva. The results have been reported in 2 journal papers (1 published, 1 in preparation), 3 conference publications, and 3 presentations at the Annual Meeting of the American Physical Society Division of Fluid Dynamics (APS DFD).

Journal Papers

1. A. Fedorov, and A. Tumin, Receptivity of High-Speed Boundary Layers to Kinetic Fluctuations, *AIAA J*, Vol. 55, No. 7, 2017.
2. L.D. Edwards and A. Tumin, A Model of Distributed Receptivity to Kinetic Fluctuations in High-Speed Boundary Layers, *In preparation*

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3. K. Luna and A. Tumin, Receptivity of High-Speed Boundary Layers in Binary Mixture of Gases to Kinetic Fluctuations, 71st Annual Meeting of the APS Division of Fluid Dynamics, Atlanta GA, 2018.

Chapter 2

Fluctuating Hydrodynamics and the Boundary Layer Disturbance Equations

2.1 Fluctuating hydrodynamics

Before formulating the receptivity problem, we outline underlying principles of *fluctuating hydrodynamics*. One can find theoretical background of the fluctuating thermodynamics in Refs.[LL80, LP80]. More details including updated list of references are presented in books by de Groot and Mazur[dM84], Ortiz de Zarate and Sengers[OS06], and Kondepudi and Prigogine[KP15].

The first key step is derivation of the entropy balance equation because it introduces how dissipative fluxes and conjugate thermodynamic forces are involved into the entropy generation. The entropy balance equation includes a source term $\dot{S}$ (entropy-generation-rate density)[dM84, OS06]. Using the thermodynamic dissipation function, $\Psi = T\dot{S}$, one can find for one-component fluid in absence of body forces:

$$\Psi = -\mathbf{Q} \cdot \frac{\nabla T}{T} + \Pi : (\nabla \mathbf{v})^{(s)}, \quad (2.1)$$

where $\mathbf{Q}$ is the heat flux, T is temperature, Π is the deviatoric stress tensor[Bat67], and $(\nabla \mathbf{v})^{(s)}$ is the symmetric part of the tensor $(1/2)(\partial v_i / \partial x_j + \partial v_j / \partial x_i)$. The dissipative fluxes Π and $\mathbf{Q}$ are accompanied by the conjugate thermodynamic forces $(\nabla \mathbf{v})^{(s)}$ and $-\nabla T / T$, respectively. At an equilibrium state, the fluxes and forces are vanished. Equation (2.1) can be presented in the general form:[dM84, OS06, KP15]

$$\Psi = \sum_{\alpha} J_{\alpha}(\mathbf{r}, t) X_{\alpha}(\mathbf{r}, t), \quad (2.2)$$

where J_{α} and X_{α} stand for dissipative fluxes and thermodynamic forces, respectively, index

α stands for various thermodynamic forces.¹

The next step is to determine phenomenological relations between J_α and X_α using the linear approximation:

$$J_\alpha(\mathbf{r}, t) = \sum_{\beta} M_{\alpha\beta}(\mathbf{r}, t) X_\beta(\mathbf{r}, t), \quad (2.3)$$

where $M_{\alpha\beta}(\mathbf{r}, t)$ are the Onsager coefficients. The random molecular motions manifest themselves in fluctuating hydrodynamics through the fluctuating dissipative fluxes $\delta J_\alpha(\mathbf{r}, t)$ in the phenomenological relations (2.3):

$$J_\alpha(\mathbf{r}, t) = \sum_{\beta} M_{\alpha\beta}(\mathbf{r}, t) X_\beta(\mathbf{r}, t) + \delta J_\alpha(\mathbf{r}, t). \quad (2.4)$$

This assumption leads to the Navier-Stokes equations with random (or Langevin) forces associated with δJ_α in the momentum and the energy equations. By the definition, the ensemble average of δJ_α is zero

$$\overline{\delta J_\alpha(\mathbf{r}, t)} = 0. \quad (2.5)$$

For an equilibrium state, the fluctuation-dissipation theorem leads to the correlations between the random fluxes

$$\overline{\delta J_\alpha(\mathbf{r}, t) \cdot \delta J_\beta(\mathbf{r}', t')} = 2k_B T M_{\alpha\beta}(\mathbf{r}, t) \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (2.6)$$

where δ is the Dirac's delta function. The fluctuating hydrodynamics assumes a local equilibrium so that the fluctuation-dissipation theorem (2.6) can be used for shear layers and other flows which are inhomogeneous at macroscopic (or hydrodynamic) scales. The validity of such a local version of the fluctuation-dissipation theorem has been established in the statistical physics[SC85a, SC85b, ZM83]. In this framework, Landau and Lifshitz[LL57] (see also Refs.[LP80, OS06]) derived the correlations for fluctuating heat flux δQ_i and fluctuating stress tensor $\delta \Pi_{ij}$:

$$\overline{\delta Q_i(\mathbf{r}, t) \cdot \delta Q_j(\mathbf{r}', t')} = 2k_B \lambda T^2 \delta_{ij} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (2.7)$$

$$\overline{\delta Q_i(\mathbf{r}, t) \cdot \delta \Pi_{jk}(\mathbf{r}', t')} = \overline{\delta \Pi_{ij}(\mathbf{r}, t) \cdot \delta Q_k(\mathbf{r}', t')} = 0, \quad (2.8)$$

$$\begin{aligned} \overline{\delta \Pi_{i,j}(\mathbf{r}, t) \cdot \delta \Pi_{kl}(\mathbf{r}', t')} = \\ 2k_B T \left[\mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \left(\mu_2 - \frac{2}{3} \mu \right) \delta_{ij} \delta_{kl} \right] \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \end{aligned} \quad (2.9)$$

where μ and λ are viscosity and thermal conductivity, respectively; and μ_2 is the bulk viscosity.

¹We should notice that thermodynamic forces in Refs.[dM84, KP15] are introduced through the entropy-generation-rate density $\dot{S}$, whereas we follow notation of Ref. [OS06].

2.2 Governing Equations

Consider an unsteady three-dimensional compressible flow having velocity $u^* = (u_1^*, u_2^*, u_3^*)$, density ρ^* , pressure p^* and temperature T^* , which are functions of time t^* and coordinates r_1^*, r_2^*, r_3^* . Hereafter, asterisks denote dimensional quantities. This flow is governed by the Navier-Stokes equations[LL87]

$$\frac{\partial \rho^*}{\partial t^*} + \text{div} \rho^* \mathbf{u}^* = 0, \quad (2.10)$$

$$\rho^* \left[\frac{\partial u_i^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) u_i^* \right] = -\frac{\partial p^*}{\partial r_i^*} + \frac{\partial \Pi_{ik}^*}{\partial r_k^*}, \quad (2.11)$$

$$\rho^* T^* \left(\frac{\partial s^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) s^* \right) = \frac{1}{2} \Pi_{ik}^* \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) - \frac{\partial Q_i^*}{\partial r_i^*}, \quad (2.12)$$

$$p^* = \rho^* R_g^* T^*. \quad (2.13)$$

Here summation is performed over the repeated subscripts, the stress tensor Π_{ij}^* and the heat flux vector components Q_i^* are determined as

$$\Pi_{ik}^* = \mu^* \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) + \left(\mu_2^* - \frac{2}{3} \mu^* \right) \delta_{ik} \text{div} u^* + \delta \Pi_{ik}^*, \quad (2.14)$$

$$Q_i^* = -\lambda^* \frac{\partial T^*}{\partial r_i^*} + \delta Q_i^*, \quad (2.15)$$

s^* is entropy per unit mass, R_g^* is gas constant per unit mass, $\delta \Pi_{ik}^*$ and δQ_i^* are Langevin source terms. Substitution of Eqs. (7.4) and (7.5) into Eqs. (7.2) and (7.3) and manipulations with the energy equation (7.3) lead to the equations

$$\begin{aligned} \rho^* \left[\frac{\partial u_i^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) u_i^* \right] = \\ -\frac{\partial p^*}{\partial r_i^*} + \frac{\partial}{\partial r_k^*} \left[\mu^* \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) + \left(\mu_2^* - \frac{2}{3} \mu^* \right) \delta_{ik} \text{div} \mathbf{u}^* \right] + \frac{\partial \delta \Pi_{ik}^*}{\partial r_k^*}, \end{aligned} \quad (2.16)$$

$$\begin{aligned} \rho^* c_p^* \left(\frac{\partial T^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) T^* \right) = \\ \left(\frac{\partial p^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) p^* \right) + \frac{\partial}{\partial r_i^*} \left(\lambda^* \frac{\partial T^*}{\partial r_i^*} \right) + \Phi^* + \frac{1}{2} \delta \Pi_{ik}^* \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) - \frac{\partial \delta Q_i^*}{\partial r_i^*}, \end{aligned} \quad (2.17)$$

where c_p^* is specific heat per unit mass at constant pressure,

$$\Phi^* = \frac{1}{2} \left[\mu^* \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) + \left(\mu_2^* - \frac{2}{3} \mu^* \right) \delta_{ik} \text{div} \mathbf{u}^* \right] \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) \quad (2.18)$$

is the dissipation function.

Coordinates, velocity, time, pressure and temperature are made nondimensional using a length-scale Δ^* which is of the order of the boundary-layer thickness, free-stream velocity U_∞^* , time-scale Δ^*/U_∞^* , doubled dynamic pressure $\rho_\infty^* U_\infty^{*2}$ and temperature T_∞^* , respectively. The correlations (2.7) and (6.10) can be written as

$$\overline{\delta Q_i^*(\mathbf{r}^*, t^*) \delta Q_j^*(\mathbf{r}', t'^*)} = \frac{\lambda^* k_B T_\infty^{*2} U_\infty^*}{\Delta^{*4}} 2T^2 \delta_{ij} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (2.19)$$

$$\begin{aligned} \overline{\delta \Pi_{ik}^*(\mathbf{r}^*, t^*) \delta \Pi_{lm}^*(\mathbf{r}', t'^*)} = \\ \frac{k_B T_\infty^* \mu_\infty^* U_\infty^*}{\Delta^{*4}} 2T \left[\mu (\delta_{il} \delta_{km} + \delta_{im} \delta_{kl}) + \left(\mu_2 - \frac{2}{3} \mu \right) \delta_{ik} \delta_{lm} \right] \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \end{aligned} \quad (2.20)$$

These relations suggest the following nondimensional forms of the Langevin terms

$$\delta Q_i^* = \left[\frac{\lambda^* k_B T_\infty^{*2} U_\infty^*}{\Delta^{*4}} \right]^{1/2} \delta Q_i, \quad \delta \Pi_{ik}^* = \left[\frac{k_B T_\infty^* \mu_\infty^* U_\infty^*}{\Delta^{*4}} \right]^{1/2} \delta \Pi_{ik}, \quad (2.21)$$

so that the nondimensional correlations are expressed as

$$\overline{\delta Q_i(\mathbf{r}, t) \delta Q_j(\mathbf{r}', t')} = 2T^2 \delta_{ij} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (2.22)$$

$$\begin{aligned} \overline{\delta \Pi_{ik}(r, t) \delta \Pi_{lm}(r', t')} = \\ 2T \mu \left[\left(\delta_{il} \delta_{km} + \delta_{im} \delta_{kl} - \frac{2}{3} \delta_{ik} \delta_{lm} \right) + \frac{\mu_2}{\mu} \delta_{ik} \delta_{lm} \right] \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \end{aligned} \quad (2.23)$$

Then the nondimensional governing equations (7.1), (2.16)-(2.18) and (7.6) are

$$\frac{\partial \rho}{\partial t} + \text{div} \rho \mathbf{u} = 0, \quad (2.24)$$

$$\begin{aligned} \rho \left[\frac{\partial u_i}{\partial t} + (\mathbf{u} \cdot \nabla) u_i \right] = \\ - \frac{\partial p}{\partial r_i} + \frac{1}{R} \frac{\partial}{\partial r_k} \left[\mu \left[\left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right) + \left(\frac{\mu_2}{\mu} - \frac{2}{3} \right) \delta_{ik} \text{div} \mathbf{u} \right] \right] + \varepsilon_L \frac{\partial \delta \Pi_{ik}}{\partial r_k}, \end{aligned} \quad (2.25)$$

$$\begin{aligned} \rho \left(\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T \right) = \\ (\gamma - 1) M^2 \left(\frac{\partial p}{\partial t} + (\mathbf{u} \cdot \nabla) p \right) + \frac{1}{R \text{Pr}} \frac{\partial}{\partial r_i} \left(\mu \frac{\partial T}{\partial r_i} \right) + \frac{(\gamma - 1) M^2}{R} \Phi + \\ \varepsilon_L (\gamma - 1) M^2 \frac{1}{2} \delta \Pi_{ik} \left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right) - \varepsilon_L \left[\frac{(\gamma - 1) M^2}{\text{Pr}} \right]^{1/2} \frac{\partial \delta Q_i}{\partial r_i}, \end{aligned} \quad (2.26)$$

$$p = \frac{1}{\gamma M^2} \rho T, \quad (2.27)$$

$$\Phi = \frac{\mu}{2} \left[\left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right) + \left(\frac{\mu_2}{\mu} - \frac{2}{3} \right) \delta_{ik} \operatorname{div} \mathbf{u} \right] \left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right). \quad (2.28)$$

Here $R = U_\infty^* \rho_\infty^* \Delta^* / \mu_\infty^*$ is Reynolds number, $\operatorname{Pr} = \mu^* c_p^* / \lambda^*$ is Prandtl number, $M = U_\infty^* / a_\infty^*$ is Mach number, and

$$\varepsilon_L = \frac{1}{R^2} \left(\frac{k_B T_\infty^* U_\infty^* \rho_\infty^{*2}}{\mu_\infty^{*3}} \right)^{1/2} \quad (2.29)$$

is a small parameter related to kinetic fluctuations. This parameter can be written in the form separating the medium properties from the flow properties:

$$\varepsilon_L = \frac{M^{1/2}}{R^2} \varepsilon_{kin}, \quad \varepsilon_{kin} = \left(\frac{k_B T_\infty^* a_\infty^* \rho_\infty^{*2}}{\mu_\infty^{*3}} \right)^{1/2}. \quad (2.30)$$

Note that $\varepsilon_{kin} = \rho_\infty^* f(T_\infty^*)$ is proportional to the free-stream density.

2.3 Equations for disturbances

The unsteady flow field is expressed as a superposition of a steady basic flow and an unsteady disturbance induced by kinetic fluctuations

$$\Psi(x, y, z, t) = \Psi_s(x, y, z) + \varepsilon_L \psi'(x, y, z, t), \varepsilon_L \ll 1. \quad (2.31)$$

Hereafter we use the standard notations: $(r_1, r_2, r_3) = (x, y, z)$, $(U_1, U_2, U_3) = (U_s, V_s, W_s)$, $(u'_1, u'_2, u'_3) = (u, v, w)$, $T' = \theta$, $p' = p$. It is assumed that the basic flow is a laminar boundary layer, which is weakly nonparallel in x -direction and it does not depend on the spanwise coordinate z . This could be the attached (non-separated) boundary layer on an airfoil, on a flat plate, on a body of revolution at zero angle of attack or on a swept wing. Then the basic-flow profiles are approximated as

$$\begin{aligned} U_s &= U_s(x_1, y) + O(\varepsilon), \quad V_s = \varepsilon V_0(x_1, y) + O(\varepsilon^2), \\ W_s &= W_s(x_1, y) + O(\varepsilon), \quad P_s = P_s(x_1) + O(\varepsilon^2), \end{aligned} \quad (2.32)$$

where $\varepsilon = R^{-1}$ is a small parameter characterizing the nonparallel flow effect and $x_1 = \varepsilon x$ is a slow variable. Linearizing Eqs. (7.10)-(7.14) and using Eq. (7.17) we obtain the linear system of equations in the leading-order approximation with respect to ε

$$\frac{\partial \rho}{\partial t} + \rho_s \frac{\partial u}{\partial x} + U_s \frac{\partial \rho}{\partial x} + \frac{\partial}{\partial y}(\rho_s v) + \rho_s \frac{\partial w}{\partial z} + W_s \frac{\partial \rho}{\partial z} = 0, \quad (2.33)$$

$$\begin{aligned} \rho_s \left(\frac{\partial u}{\partial t} + U_s \frac{\partial u}{\partial x} + v \frac{\partial U_s}{\partial y} + W_s \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \\ \frac{1}{R} \left\{ \mu_s \frac{\partial}{\partial x} \left(r \frac{\partial u}{\partial x} + m \frac{\partial v}{\partial y} + m \frac{\partial w}{\partial z} \right) + \frac{\partial}{\partial y} \left[\mu_s \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \mu \frac{\partial U_s}{\partial y} \right] + \mu_s \frac{\partial}{\partial z} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right\} + \\ \left(\frac{\partial \delta \Pi_{xx}}{\partial x} + \frac{\partial \delta \Pi_{xy}}{\partial y} + \frac{\partial \delta \Pi_{xz}}{\partial z} \right), \quad (2.34) \end{aligned}$$

$$\begin{aligned} \rho_s \left(\frac{\partial v}{\partial t} + U_s \frac{\partial v}{\partial x} + W_s \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \\ \frac{1}{R} \left\{ \mu_s \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\partial \mu}{\partial x} \frac{\partial U_s}{\partial y} + \frac{\partial}{\partial y} \left[\mu_s \left(m \frac{\partial u}{\partial x} + r \frac{\partial v}{\partial y} + m \frac{\partial w}{\partial z} \right) \right] + \mu_s \frac{\partial}{\partial z} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) + \frac{\partial \mu}{\partial z} \frac{\partial W_s}{\partial y} \right\} + \\ \left(\frac{\partial \delta \Pi_{yx}}{\partial x} + \frac{\partial \delta \Pi_{yy}}{\partial y} + \frac{\partial \delta \Pi_{yz}}{\partial z} \right), \quad (2.35) \end{aligned}$$

$$\begin{aligned} \rho_s \left(\frac{\partial w}{\partial t} + U_s \frac{\partial w}{\partial x} + v \frac{\partial W_s}{\partial y} + W_s \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \\ \frac{1}{R} \left\{ \mu_s \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) + \frac{\partial}{\partial y} \left[\mu_s \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) + \mu \frac{\partial W_s}{\partial y} \right] + \mu_s \frac{\partial}{\partial z} \left(m \frac{\partial u}{\partial x} + m \frac{\partial v}{\partial y} + r \frac{\partial w}{\partial z} \right) \right\} + \\ \left(\frac{\partial \delta \Pi_{zx}}{\partial x} + \frac{\partial \delta \Pi_{zy}}{\partial y} + \frac{\partial \delta \Pi_{zz}}{\partial z} \right), \quad (2.36) \end{aligned}$$

$$\begin{aligned} \rho_s \left(\frac{\partial \theta}{\partial t} + U_s \frac{\partial \theta}{\partial x} + v \frac{\partial T_s}{\partial y} + W_s \frac{\partial \theta}{\partial z} \right) = (\gamma - 1) M^2 \left[\frac{\partial p}{\partial t} + U_s \frac{\partial p}{\partial x} + W_s \frac{\partial p}{\partial z} + \frac{1}{R} \Phi' \right] + \\ \frac{1}{R \text{Pr}} \left[\mu_s \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial}{\partial y} \left(\mu_s \frac{\partial \theta}{\partial y} + \mu \frac{\partial T_s}{\partial y} \right) + \mu_s \frac{\partial^2 \theta}{\partial z^2} \right] - \\ \left(\frac{(\gamma - 1) M^2}{\text{Pr}} \right)^{1/2} \left(\frac{\partial \delta Q_x}{\partial x} + \frac{\partial \delta Q_y}{\partial y} + \frac{\partial \delta Q_z}{\partial z} \right) + \\ \frac{(\gamma - 1) M^2}{2} \left[(\delta \Pi_{xy} + \delta \Pi_{yx}) \frac{\partial U_s}{\partial y} + (\delta \Pi_{yz} + \delta \Pi_{zy}) \frac{\partial W_s}{\partial y} \right], \quad (2.37) \end{aligned}$$

$$\frac{p}{P_s} = \frac{\theta}{T_s} + \frac{\rho}{\rho_s}, \quad (2.38)$$

where Φ' is the linearized dissipation function (its explicit form is given by Nayfeh[Nay80]). The coefficients r and m in Eqs. (7.19) - (7.21) depend on the bulk viscosity ratio $\mu_2/\mu = 2e/3$ as $m = 2(e - 1)/3$, $r = 2(e + 2)/3$. Note that the system of Eqs. (7.18)-(7.23) represents the linearized Navier-Stokes equations for 3D disturbances in the local-parallel approximation of the boundary-layer flow. The only difference is the presence of Langevin terms associated with the random forcing produced by kinetic fluctuations.

Chapter 3

Receptivity of High-Speed Boundary Layers to Kinetic Fluctuations

Nomenclature

A	Eigenvector of the direct stability problem
a	Speed of sound
B	Eigenvector of the adjoint stability problem
c	Amplitude of the unstable mode generated at the lower neutral curve
c_r	Phase velocity
c_p	Specific heat at constant pressure
F	Dimensionless frequency parameter
$\hat{\mathbf{G}}_L$	Fourier component of the Langevin vector
J_α	Dissipative flux
k_B	Boltzmann constant
M	Mach number
$M_{\alpha\beta}$	Onsager coefficients
N	$= -\int_{x_0}^x \alpha_i dx$, N-factor
P	Pressure (p - pressure perturbation)
Pr	Prandtl number
R	Reynolds number
R_g	Gas constant
T	Temperature
t	Time
u_i	Velocity component
X_α	thermodynamic force
x	Streamwise coordinate
x_1	$= \varepsilon x$
y	Distance from the wall
z	Spanwise coordinate

α	$= \alpha_r + i\alpha_i$, streamwise wave number (complex)
β	Spanwise wave number
γ	Specific heats ratio
Δ^*	Boundary layer thickness scale
$\delta\Pi_{ij}$	Fluctuating shear stress
δQ_i	Fluctuating heat flux
ε	$= R^{-1}$, small parameter
ε_{kin}	Amplitude factor in Langevin terms
θ	Temperature perturbation
λ	Thermal conductivity
μ	Viscosity coefficient
μ_2	Bulk viscosity coefficient
ν	Kinematic viscosity coefficient
ξ	$= \varepsilon^{-1}(x_1 - x_{10})$, inner variable in the vicinity of the lower neutral curve
ρ	Density
σ	$= -\alpha_i$
σ_x	$= -d\alpha_i/dx$
ω	Angular frequency
<i>Superscript</i>	
(L)	Langevin forcing terms
*	Dimensional parameter (or complex conjugate)
<i>Subscript</i>	
c	Critical layer location
e	Boundary layer upper edge condition
i	Variable number
rms	Root-mean-square
max	downstream location of the wave packet observation
s	Mean flow parameter
w	Parameter at the wall condition
β	Derivative of N with respect to β
ω	Derivative of N with respect to ω
0	Neutral point
∞	Free stream flow parameter

3.1 Introduction

3.1.1 Background

Laminar-turbulent transition in boundary-layer flows essentially depends on receptivity to free-stream disturbances and/or body-induced disturbances such as roughness and vibrations[Mor69, Res76, Sar02]. The free-stream disturbances generally include[Bus90]: (1) vorticity fluctua-

tions, (2) entropy disturbances or temperature spottiness, (3) acoustic noise, (4) fluctuations of species concentrations, and (5) fluctuations induced by particulates. The internalization of such disturbances can be quite different depending upon body geometry and speed. Receptivity to wall-induced disturbances has been studied in Refs.[Cho93, FK02, Fed03] and many other papers. It was found that the body-surface vibrations in the instability frequency band are extremely effective in excitation of the boundary-layer modes. Small roughness elements and especially distributed roughness are also very effective in excitation of stationary cross-flow (CF) instabilities in 3-D boundary layers[Fed12]. Nevertheless, the body-generated disturbances can be reduced, at least in principle, to negligible levels through carefully designed vehicle surfaces. Numerous receptivity studies have been performed to understand mechanisms by which the free-stream disturbances excite unstable modes in the boundary-layer flow (see, for example, Refs.[Sar02, Bal09, BK11]). These disturbances can be partially controlled in ground facilities. For example, in quiet hypersonic wind tunnels[WAC92, Sch08], laminarization of the boundary layer on the nozzle walls suppresses the free-stream acoustic noise. With special filtering it is feasible to diminish the concentration and size of dust particles and thereby to avoid the particle-induced transition.

In hypersonic flight the situation is quite different. There are no acoustic disturbances in high-altitude atmosphere, while vorticity and entropy fluctuations as well as particulates (associated with ice clouds, volcanic and other terrestrial dust, cosmic dust and the products of rocket exhausts from previous flights) are indispensable. It is commonly believed that atmospheric turbulence is the major source of receptivity. However, the analysis[Fed15] of the HIFiRE-1 flight data[LCC⁺11] led to the different conclusion. Transition on the HIFiRE-1 conical forebody is driven by the second-mode instability of high frequency (~ 500 kHz). In this instability frequency band the length scales of turbulent fluctuations are very small (of the order of few millimeters). Even under severe turbulence conditions, they lie well below the upper boundary of the viscous dissipation region[Hoc85], where the kinetic energy spectrum drops exponentially. Therefore, the turbulent fluctuations are much smaller than kinetic fluctuations associated with thermal noise.

Another source of external forcing is relevant to particulates. The mechanisms by which particulate matter can affect transition are manifold and include[Bus90]: (1) impacting/striking on the surface leading to surface cratering; (2) vortex/vorticity shedding while the particle is immersed in the boundary layer; (3) particle rotation and consequent fluid motions; (4) production of reverse shocklets after passage through the vehicle-induced shock; (5) rebound after impact on the vehicle surface and subsequent dynamic interaction with the bow shock causing formation of embedded shear layers and jets. In high-altitude (stratospheric) flights, particulates are quite small - their diameter is, typically, less than 10 micron[Tur92]. Such minute particles cannot trigger turbulent patches, while they can generate the boundary-layer instabilities of small initial amplitudes via a receptivity mechanism associated with the dynamic interaction of particles with the boundary-layer flow[FK11, Fed13]. Because of inertia the particle crosses the bow shock, decelerates in the shock layer and ultimately hits the wall. Moving through the boundary layer the particle induces a local force, which, in turn, generates the unstable wavepacket. The theoretical

analysis[Fed13] of this receptivity mechanism for supersonic boundary layers on a sharp wedge showed that spherical particles of radius $\sim 10 \mu m$ and density $1 g/cm^3$ (close to water density) can trigger unstable wavepackets, which can reach the critical amplitude at the amplification factors $N \sim 10$. However, particulates of such size have quite low concentration in stratosphere — less than 10^{-4} particles per cm^3 [Tur92]. Therefore, their penetration into the boundary layer is a low-frequency event. For example, Fedorov[Fed15] found that frequency of 10 micron particle collisions with the upstream half of the HIFiRE-1 conical forebody is less than the global flow frequency U_e/L , where L is the cone length and U_e is the flow speed at the upper boundary-layer edge. In this case, particulates generate very rare patches of instability, which do not form the transition onset line.

These findings led Fedorov[Fed15] to the assumption that in the clean cases of the HIFiRE-1 flight, transition may be induced by kinetic fluctuations (KF) associated with thermal noise. Although these fluctuations are quite small (their energy is of the order of $k_B T$, where k_B is the Boltzmann constant and T is local temperature of the mean flow), they may effectively excite unstable perturbations, which grow downstream and ultimately cause transition. This assumption was confirmed by comparisons of the theoretical prediction with the HIFiRE-1 transition data.

For particular configuration, there is an upper bound of the transition Reynolds number Re_{tr} relevant to absolutely quiet free stream. In this perfectly clean case, there are indispensable kinetic fluctuations or thermal noise. Betchov[Bet61] was the first who presumed that thermal noise could trigger boundary-layer instabilities in sufficiently quiet environments. Zavolskii and Reutov[ZR84] developed a first-cut theoretical model describing excitation of Tollmien-Schlichting (TS) waves by kinetic fluctuations in the incompressible boundary layer on a flat plate. Luchini[Luc08b, Luc10, Luc17] considered the temporal problem for the parallel incompressible boundary layer and estimated the KF-induced transition onset at low subsonic speeds. Fedorov and Averkin[FA10] analyzed KF-receptivity for compressible boundary layers of perfect gas and predicted the KF-induced transition onset for the first-mode and second-mode dominated transition at high speeds. These studies as well as Ref. [Fed15] show that the experimentally observed transition onset points in quiet free streams are not very far from the theoretically predicted upper bound of Re_{tr} associated with kinetic fluctuations.

Besides the fundamental interest, predictions of the transition onset upper bound have practical meaning. On the preliminary design stage it is important to estimate the lower and upper bounds of the aerothermal performance of a hypersonic configuration. These bounds essentially depend on the transition locus. The lower bound can be estimated with the assumption that transition occurs just at the nose or leading edge of a hypersonic body (fully turbulent boundary-layer flow). The upper bound can be evaluated with the assumption that in the perfectly clean case transition is induced by kinetic fluctuations.

In the present paper we provide a detailed theoretical approach to receptivity of high-speed boundary layers to kinetic fluctuations used in Refs.[FA10, Fed12, Fed15] and perform parametric studies clarifying the effect of flow parameters on the KF receptivity mechanism. It is shown that the irregular dynamics of molecules at a microscopic scale can trigger a

macroscopic phenomenon by laminar-turbulent transition associated with excitation and amplification of instabilities in the boundary-layer flow.

3.2 Receptivity analysis

Introduce the disturbance vector function

$$\mathbf{F}(x, y, z, t) = (u, \frac{\partial u}{\partial y}, v, p, \theta, \frac{\partial \theta}{\partial y}, w, \frac{\partial w}{\partial y})^T \quad (3.1)$$

and its Fourier transform versus t and z

$$\hat{\mathbf{F}}(x, y; \beta, \omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{F}(x, y, z, t) \exp(i\omega t - i\beta z) dt dz. \quad (3.2)$$

The amplitude function $\hat{\mathbf{F}}$ satisfies a system of partial differential equations resulting from the Fourier transform of Eqs. (7.18)-(7.23). This system can be written in the matrix-operator form 8×8

$$\mathbf{H}(y, \partial_y, x_1, \varepsilon \partial_{x_1}, \omega, \beta) \hat{\mathbf{F}} = \hat{\mathbf{G}}^{(L)}. \quad (3.3)$$

The second equation in Eq. (6.25) is the x -momentum equation, the third equation is the continuity equation, the fourth equation is the y -momentum equation, the sixth equation is the energy equation, and the eighth equation is the z -momentum equation. The other equations have the role of definitions of the components in vector $\mathbf{F}$. The vector $\hat{\mathbf{G}}^{(L)}$ represent Fourier components of the Langevin terms.

The standard boundary conditions on the wall $(u, v, w, \theta) = 0$ and the boundedness condition in the outer inviscid flow give

$$\hat{F}_1 = \hat{F}_3 = \hat{F}_5 = \hat{F}_7 = 0 \text{ at } y = 0, \quad (3.4)$$

$$\left| \hat{F}_1, \hat{F}_3, \hat{F}_5, \hat{F}_7 \right| < \infty \text{ at } y \rightarrow \infty. \quad (3.5)$$

The problem (6.25)-(6.27) is similar to the problem of receptivity to wall vibrations, which was solved by Fedorov and Khokhlov[FK02] using asymptotic techniques. Following Ref.[FK02] we consider a partial solution of the problem (6.25)-(6.27), which comprises a normal mode $\hat{\mathbf{F}}_m(x_1, y)$ of the discrete spectrum (for example, the Mack second mode) with the eigenvalue $\alpha_m(x_1, \beta, \omega)$ and a forced disturbance $\hat{\mathbf{F}}^{(L)}(x, y)$. It is assumed that the mode m is unstable. At the neutral point $x_{10}(\beta, \omega)$, its eigenvalue $\alpha_m(x_{10}, \beta, \omega) = \alpha_0$ is real while its growth rate $\sigma = -\text{Im}(\alpha_m)$ satisfies the condition $\frac{\partial \sigma}{\partial x_1}(x_{10}) > 0$. The dominant excitation of this mode occurs in the region $|x_1 - x_{10}| = O(\varepsilon^{1/2})$, where the forcing waves have wavenumbers $\alpha = \alpha_0 + \varepsilon^{1/2}\gamma$, $\gamma = O(1)$ is real parameter characterizing detuning of the resonance condition $\alpha = \alpha_0$. Introduce the inner variable $\xi = \varepsilon^{-1/2}(x_1 - x_{10})$ and approximate the forced disturbance by the local Fourier integral

$$\hat{\mathbf{F}}^{(L)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathbf{F}}_\alpha^{(L)}(y) e^{i\alpha(x-x_0)} d\alpha = \frac{\varepsilon^{1/2}}{2\pi} e^{i\varepsilon^{-1/2}\alpha_0\xi} \int_{-\infty}^{+\infty} \hat{\mathbf{F}}_\alpha^{(L)}(y) e^{i\gamma\xi} d\gamma. \quad (3.6)$$

The local solution can be expressed in the form

$$\hat{\mathbf{F}}(\xi, y) = \frac{1}{2\pi} e^{i\varepsilon^{-1/2}\alpha_0\xi} \int_{-\infty}^{+\infty} [c(\xi, \gamma) \mathbf{A}_0(y) + \varepsilon^{1/2} \mathbf{A}^{(L)}(y, \gamma) e^{i\gamma\xi}] d\gamma. \quad (3.7)$$

Here $\mathbf{A}_0(y) = \mathbf{A}_m(x_{10}, y, \alpha_0)$ is the eigenfunction of mode m , which is a solution of the local-parallel stability problem

$$\begin{aligned} \left(\frac{\partial}{\partial y} - \mathbf{H}_0 \right) \mathbf{A} &= 0, \\ A_1 = A_3 = A_5 = A_7 &= 0 \quad \text{at } y = 0, \\ A_1, A_3, A_5, A_7 &\rightarrow 0 \text{ at } y \rightarrow \infty. \end{aligned} \quad (3.8)$$

The matrix $\mathbf{H}_0 = \mathbf{H}_0(x_1, \alpha, \beta, \omega)$ of dimension 8×8 is calculated at $x_1 = x_{10}$ and $\alpha = \alpha_0$; its explicit form $\mathbf{H}_0$ is given in Appendix 6.6.1. Expanding the matrix-operator $\mathbf{H}$ in Eq. (6.25) near the point $x_1 = x_{10}$, we obtain the inner problem for $\mathbf{A}^{(L)}$

$$\begin{aligned} \left(\frac{\partial}{\partial y} - \mathbf{H}_0(x_{10}, \alpha_0) \right) \mathbf{A}^{(L)} e^{i\gamma\xi} &= -i \frac{dc}{d\xi} \frac{\partial \mathbf{H}_0}{\partial \alpha}(x_{10}, \alpha_0) \mathbf{A}_0 + \\ c\xi \frac{\partial \mathbf{H}_0}{\partial x_1}(x_{10}, \alpha_0) \mathbf{A}_0 &+ \hat{\mathbf{G}}_\alpha^{(L)}(x_{10}, \alpha_0) e^{i\gamma\xi}, \\ A_1^{(L)} = A_3^{(L)} = A_5^{(L)} = A_7^{(L)} &= 0 \text{ at } y = 0, \\ A_1^{(L)}, A_3^{(L)}, A_5^{(L)}, A_7^{(L)} &\rightarrow 0 \text{ at } y \rightarrow \infty. \end{aligned} \quad (3.9)$$

Here $\hat{\mathbf{G}}_\alpha^{(L)}$ is the local Fourier transform of the vector $\hat{\mathbf{G}}^{(L)}$

$$\hat{\mathbf{G}}_\alpha^{(L)} = \int_{-\infty}^{+\infty} \hat{\mathbf{G}}^{(L)} e^{-i\alpha(x-x_0)} dx, \quad (3.10)$$

its explicit form is given in Appendix 6.6.1. The problem (3.9) has a non-trivial solution, if its right-hand side is orthogonal to the eigenfunction $B_0 = B_m(x_{10}, y, \alpha_0)$ of the adjoint problem

$$\begin{aligned} \left(\frac{\partial}{\partial y} + \mathbf{H}_0^T \right) \mathbf{B} &= 0, \\ B_2 = B_4 = B_6 = B_8 &= 0 \text{ at } y = 0, \\ B_2, B_4, B_6, B_8 &\rightarrow 0 \text{ at } y \rightarrow \infty. \end{aligned} \quad (3.11)$$

The solvability condition leads to the equation for $c(\xi, \gamma)$

$$\frac{dc}{d\xi} - ib\xi c = q \exp(i\gamma\xi), \quad (3.12)$$

$$b = \frac{\partial \alpha_m}{\partial x_1}(x_{10}), \quad (3.13)$$

$$q = -i \frac{\langle \hat{\mathbf{G}}_\alpha^{(L)}, \mathbf{B}_0 \rangle}{\langle \frac{\partial \mathbf{H}_0}{\partial \alpha}(x_{10}, \alpha_0) \mathbf{A}_0, \mathbf{B}_0 \rangle}, \quad (3.14)$$

where the inner products are defined as

$$\langle \mathbf{A}, \mathbf{B} \rangle = \int_0^\infty \sum_{j=1}^8 A_j B_j dy. \quad (3.15)$$

Equations (3.12)-(3.14) are derived in Appendix 6.6.2. They are similar to the equations derived by Fedorov and Khokhlov[FK02] for receptivity to the vibration wave. Solving Eq. (3.12) with the boundary condition $c(-\infty) = 0$ and using the steepest descent method one can find[FK02]

$$c(\xi, \gamma) \rightarrow q \sqrt{\frac{2\pi i}{b}} \exp\left(\frac{i\gamma^2}{2b} + \frac{ib\xi^2}{2}\right), \quad \xi \rightarrow +\infty. \quad (3.16)$$

Considering the outer limit of the inner solution (6.29) we obtain the amplitude of mode m propagating downstream from the neutral point

$$\hat{\mathbf{F}}_m(x_1, y; \beta, \omega) = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} c_\infty(\gamma) d\gamma \right) \mathbf{A}_m(x_1, y, \alpha_m) e^{S_m}, \quad x_1 > x_{10}, \quad (3.17)$$

$$c_\infty(\gamma) = q \sqrt{\frac{2\pi i}{b}} \exp\left(\frac{i\gamma^2}{2b}\right), \quad (3.18)$$

$$S_m = i\varepsilon^{-1} \int_{x_{10}(\beta, \omega)}^{x_1} \alpha_m(x_1; \beta, \omega) dx_1. \quad (3.19)$$

Since the forcing is random, we should proceed with a statistical analysis of the induced mode m .

3.3 Statistical analysis

Using the solution (3.17) we can calculate the amplitude $\hat{f}_m(x_1; \beta, \omega)$ of any physical quantity associated with the mode m at a certain point y . The correlation of this quantity at a point x_1 can be expressed as

$$\overline{f(x, z, t) f(x - x', z - z', t - t')} = \frac{1}{(2\pi)^6} \int_{-\infty}^{+\infty} K_f e^{i\alpha x + i\beta t - i\omega t - i\alpha' x' - i\beta' t' + i\omega' t'} d\alpha d\alpha' d\beta d\beta' d\omega d\omega', \quad (3.20)$$

where K_f is the correlation spectral density defined as

$$K_f \equiv \overline{f(\alpha, \beta, \omega) f^*(\alpha', \beta', \omega')} = (f^2)_{\alpha, \beta, \omega} (2\pi)^3 \delta(\alpha - \alpha') \delta(\beta - \beta') \delta(\omega - \omega'), \quad (3.21)$$

$$(f^2)_{\alpha, \beta, \omega} = \int_{-\infty}^{+\infty} \overline{f^2(x, z, t)} e^{i\omega t - i\alpha x - i\beta z} d\alpha d\beta d\omega. \quad (3.22)$$

In this section ‘*’ denotes a complex conjugate quantity. With this definition, the correlation spectral density of the amplitude coefficient c_∞ in Eq. (3.17) is expressed as

$$K_c = \frac{1}{(2\pi)^2} \frac{2\pi}{|b|} \overline{\left[\int_{-\infty}^{+\infty} q \exp\left(\frac{i\gamma^2}{2b}\right) d\gamma \right]} \left[\int_{-\infty}^{+\infty} q \exp\left(\frac{i\gamma'^2}{2b}\right) d\gamma' \right]^* (2\pi)^3 \delta(\beta - \beta') \delta(\omega - \omega'). \quad (3.23)$$

Integrating over γ and γ' , performing averaging and using the correlations (2.22)-(2.23), we can reduce (3.23) to the compact form

$$K_c = \sqrt{\frac{\pi}{\sigma_x(x_{10})}} (2\pi)^2 (q^2)_{\alpha,\beta,\omega} \delta(\beta - \beta') \delta(\omega - \omega'), \quad (3.24)$$

where $\sigma_x(x_{10}) = d\sigma/dx$ is the growth-rate gradient at the resonance point x_{10} , and the spectral density $(q^2)_{\alpha,\beta,\omega}$ is expressed in the analytical form given in Appendix 6.6.3.

The mean square of disturbance $f(x_1, y, z, t)$ at a certain point x_1 is determined as

$$\begin{aligned} \overline{f^2}(x_1) = & \frac{\varepsilon_{kin}^2}{(2\pi)^4} \frac{M}{R^4} \int_{-\infty}^{+\infty} \sqrt{\frac{\pi}{\sigma_x(x_{10})}} (q^2)_{\alpha,\beta,\omega}(x_{10}) \hat{f}_m(x_1, y; \beta, \omega) \hat{f}_m^*(x_1, y; \beta', \omega') \times \\ & \exp[S_m(\beta, \omega) + S_m^*(\beta', \omega') + i(\beta - \beta')z - i(\omega - \omega')t] (2\pi)^2 \delta(\beta - \beta') \delta(\omega - \omega') d\beta d\beta' d\omega d\omega'. \end{aligned} \quad (3.25)$$

Integration over β' and ω' gives

$$\overline{f^2}(x_1) = \frac{\varepsilon_{kin}^2}{2\pi^2} \frac{M}{R^4} \int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} \sqrt{\frac{\pi}{\sigma_x(x_{10})}} (q^2)_{\alpha,\beta,\omega}(x_{10}) \left| \hat{f}_m(x_1, y; \beta, \omega) \right|^2 e^{2N(x_1; \beta, \omega)} d\beta, \quad (3.26)$$

where N is the amplification factor

$$N(x_1; \beta, \omega) = \varepsilon^{-1} \int_{x_{10}(\beta, \omega)}^{x_1} \sigma_m(x_1; \beta, \omega) dx_1. \quad (3.27)$$

Using the steepest descent method[BH86, Won01] for the double integral in Eq. (3.27), we can find frequency $\omega_{\max}$ and wavenumber $\beta_{\max}$ at which the mean square amplitude of instability has maximum. They are determined from the conditions

$$\frac{\partial N}{\partial \omega}(x_1; \beta_{\max}, \omega_{\max}) = \frac{\partial N}{\partial \beta}(x_1; \beta_{\max}, \omega_{\max}) = 0. \quad (3.28)$$

The corresponding mean-square amplitude is estimated as[BH86, Won01]

$$\overline{f^2}(x_1) \approx \frac{\varepsilon_{kin}^2}{\pi} \frac{M}{R^4} \sqrt{\frac{\pi}{\sigma_x(x_{10})}} \frac{(q^2)_{\alpha,\beta,\omega}(x_{10})}{\sqrt{|N_{\beta\beta} N_{\omega\omega} - N_{\beta\omega}^2|}} \left| \hat{f}_m(x_1, y; 0, \omega_{\max}) \right|^2 e^{2N(\beta_m, \omega_m; x)}, \quad (3.29)$$

where

$$N_{\beta\beta} = \frac{\partial^2 N}{\partial \beta^2}(x_1; \beta_{\max}, \omega_{\max}), \quad N_{\omega\omega} = \frac{\partial^2 N}{\partial \omega^2}(x_1; \beta_{\max}, \omega_{\max}), \quad \text{and} \quad N_{\beta\omega} = \frac{\partial^2 N}{\partial \beta \partial \omega}(x_1; \beta_{\max}, \omega_{\max}).$$

For the Mack second mode in a high-speed boundary layer and TS waves at small Mach numbers, the most unstable are plain waves: $\beta_{\max} = 0$ at any $\omega_{\max}$. Therefore $N_{\beta\omega} = 0$, and the mean-square amplitude is estimated as

$$\overline{f^2}(x_1) = \frac{\varepsilon_{kin}^2}{2\pi} \frac{M}{R^4} \sqrt{\frac{\pi}{\sigma_x(x_{10})}} \frac{(q^2)_{\alpha,\beta,\omega}(x_{10})}{\sqrt{|N_{\beta\beta}N_{\omega\omega}|}} \left| \hat{f}_m(x_1, y; 0, \omega_{\max}) \right|^2 e^{2N(x_1; 0, \omega_{\max})}. \quad (3.30)$$

Thus, propagating downstream the unstable wave packets are narrowed down in (ω, β) space due to selective amplification of their components. In the far field, the central frequency ω_{max} and wavenumber β_{max} depend on the observation point x_1 , while the rms amplitude of wave-packet maximum grows exponentially with the integral amplification factor N .

Note that the solutions (3.29) and (3.30) do not contain any empirical quantities. The mean-square amplitudes of instability can be calculated using a standard stability code with the computational time being of the order of time required for e^N computations.

3.4 Results

3.4.1 Numerical method

The numerical results were obtained using two sets of computational tools developed independently at the Moscow Institute of Physics and Technology (MIPT) and at the University of Arizona (UA). The fundamental solutions of the direct and adjoint eigenvalue problems (Eqs. (6.30) and (3.11), respectively) were found by numerical integration of the equations from the upper boundary y_{max} toward the wall with the known analytical asymptotic solutions outside the boundary layer using the Gramm-Schmidt orthonormalization procedure and the fourth order Runge-Kutta algorithm. Number of grid points was varied across the boundary layer (up to 1201) to check accuracy of the computations. Eigenfunctions of the direct and adjoint problems are obtained as a linear combination of the fundamental solutions with coefficients being determined from the boundary conditions on the wall. To satisfy the boundary conditions using the four fundamental solutions for modes of the discrete spectrum, the eigenvalue $\alpha_m(x_1; \beta, \omega)$ is found using the Newton-Raphson iterations. About 600 grid points equally-spaced across the boundary layer are needed to compute α_r with 6 significant digits and α_i with 4 significant digits. More details of the numerical method can be found in Ref. [Tum07].

Test case: Low-speed flow, $M = 0.04$

To verify the both sets of computational tools, we use the low speed results of Ref. [ZR84, FA10] for the Blasius boundary layer on a flat plate. The free-stream parameters are $M = 0.04$, $U_\infty^* = 15$ m/s, $T_\infty^* = 300$ K, $\rho_\infty^* = 1.176$ kg/m³, and $\nu_\infty^* = 1.458 \times 10^{-5}$ m²/s (in this case $\varepsilon_{kin} = 1.76 \times 10^{-2}$). The wall is adiabatic so that the flow temperature is constant across the boundary layer. Figure 3.1 shows the distribution of $u_{rms} = (\overline{u^2})^{1/2}$ (at y corresponding to maximum u -velocity disturbance) versus $R_{max} = (U_\infty^* x^* / \nu_\infty^*)^{1/2}$ at the

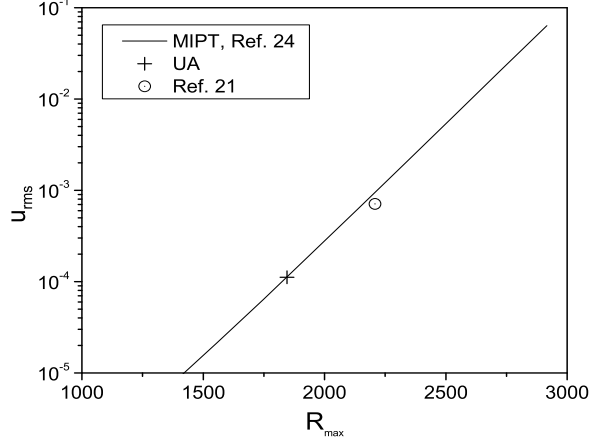


Figure 3.1: Root-mean-square amplitudes of TS waves as a function of R_{max} for the low-speed case.

observation point x^* for TS waves excited by kinetic fluctuations. The computations are performed using Eq. (3.30). The results obtained with the help of the UA code and the MIPT code are particularly identical, while u_{rms} predicted by Zavolskii and Reutov[ZR84] is slightly smaller. Using the transition onset criterion $u_{rms} = 1\%$ we get the transition Reynolds number $Re_{tr} = U_{\infty}^* x_{tr}^* / \nu_{\infty}^* \approx 6.8 \times 10^6$, which is not so far from $Re_{tr} \approx 5.6 \times 10^6$ measured on a flat plate at low free-stream turbulence.[Wel67]

Test case: High-speed flow, $M = 6$

For the high-speed flow test, the boundary layer over a flat plate is considered at the free-stream parameters: $M = 6$, $Pr = 0.72$, $\gamma = 1.4$, $T_{\infty}^* = 300$ K, $T_w^*/T_{\infty}^* = 4$, $U_{\infty}^* = 2.084$ km/s, $\rho_{\infty}^* = 0.4127$ kg/m³, $\nu_{\infty}^* = 4.473 \times 10^{-5}$ m²/s. The Sutherland's law was used for viscosity $\mu^*(T^*) = 1.458 \times 10^{-6} T^{3/2} / (T^* + 110.4)$ kg m⁻¹ s⁻¹ (in this case $\varepsilon_{kin} = 6.24 \times 10^{-3}$). It is assumed that the viscous-inviscid interaction is negligible and the flow parameters at the upper boundary-layer edge equal to the free-stream parameters: $(U_e^*, P_e^*, T_e^*, \rho_e^*) = (U_{\infty}^*, P_{\infty}^*, T_{\infty}^*, \rho_{\infty}^*)$. The mean-flow profiles $U(y)$ and $T(y)$ are computed using the compressible Blasius solutions. They are shown in Fig. 3.2 where $\Delta^* = (\nu_e^* x^* / U_e^*)^{1/2}$.

The root-mean-square amplitudes of the wall pressure disturbance $(p_w^*/P_e^*)_{rms}$ are calculated using Eq. (3.30) for the Mack second mode of $\beta = 0$ and frequency ω_{max} corresponding to the frequency parameter $F = 2\pi f^* \mu_{\infty}^* / (\rho_{\infty}^* U_{\infty}^{*2}) = 36.117 \times 10^{-6}$. Comparisons of the results obtained by the different codes are presented in Table 3.1, where subscript '0' stands for the neutral point x_0^* , and subscript 'max' stands for the observation point. Derivatives $\sigma_{x0} = \sigma_x(x_{10})$, $N_{\omega\omega}$, and $N_{\beta\beta}$ are made nondimensional using the length scale Δ^* at the neutral point.

The test case illustrate that the rms value of the wall pressure perturbation $(p_w^*/P_e^*)_{rms,0}$ generated by kinetic fluctuations near the neutral point (the most efficient location for the

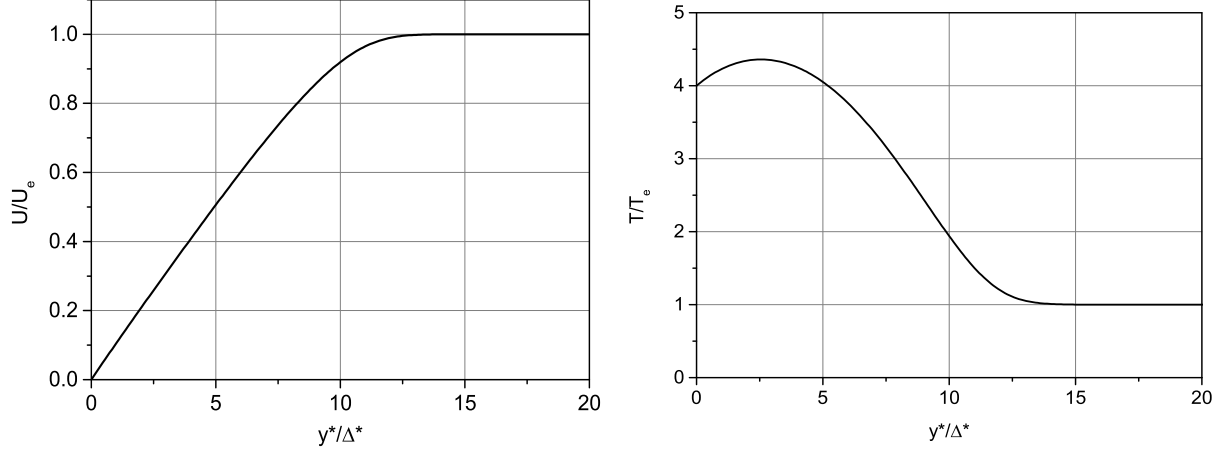


Figure 3.2: Mean velocity and temperature profiles in the test case of $M = 6$.

Table 3.1: Test case: Flat plate, $M = 6$.

	R_0	R_{max}	N_{max}	σ_{x0}	$N_{\omega\omega}$	$N_{\beta\beta}$	$(p_w^*/P_e^*)_{rms,0}$	$(p_w^*/P_e^*)_{rms,max}$
MIPT	4227.7	5447.8	11.41	1.48E-5	-6.79E+4	-1.18E+3	4.53E-8	4.10E-3
UA	4227.6	5461.7	11.44	1.48E-5	-6.88E+4	-1.18E+3	4.51E-8	4.19E-3

KF receptivity) is very small. However, the instability goes through a strong amplification (about 10^5) and becomes significant downstream at the observation point R_{max} .

It is interesting to explore details of KF receptivity using this example. The spectral density $(q^2)_{\alpha,\beta,\omega}$ in Eq. (6.31) is expressed in Appendix 6.6.3 in terms of integrals J_{km} having their origin in the Langevin source terms in Eqs. (7.19) - (7.22). The nonzero integrands $j_{km}(y)$ normalized with the denominator

$|\langle \partial H_0 / \partial \alpha \mathbf{A}, \mathbf{B} \rangle|^2$ are shown in Fig. 3.3a (note that $\beta_{max} = 0$ for the Mack second mode in the test case, and the terms having an index ‘8’ are zero). The dominant contribution to $(q^2)_{\alpha,\beta,\omega}$ is provided by $j_{22}(y)$ which is related to the Langevin terms in the x -momentum equation (7.19). The function j_{22} has a strong maximum in the critical layer formed near the point $y = y_c$ where $U(y_c) = c_r = \omega / \alpha_r$. Figure 3.3b shows that the other components are noticeable in the near-wall (Stokes) layer and in the critical layer. However, their contributions to the spectral density $(q^2)_{\alpha,\beta,\omega}$ are negligible. Analysis of components of the adjoint eigenvector $\mathbf{B}$ and their derivatives revealed that they also maximal in the critical layer. In other words, the boundary layer is most sensitive to kinetic fluctuations in the critical layer with the main effect being produced by the fluctuating shear stress in the x -momentum equation. This observation has an important physical implication: because the critical layer of the Mack second mode at high Mach numbers are close to the upper boundary-layer edge (in our case $c_r = 0.95$), the KF receptivity mechanism is dominated by the edge mean-flow parameters.

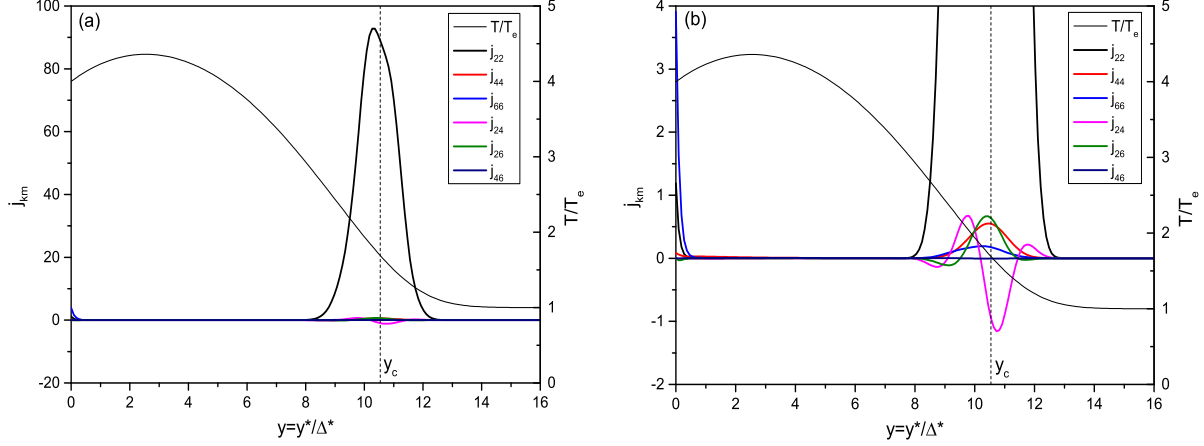


Figure 3.3: Normalized nonzero integrands $j_{km}(y)$.

3.4.2 Example: $M = 7.9$, “Wind tunnel” conditions

In this example we consider a boundary flow of zero pressure gradient at Mach number $M = 7.9$. The specific heat ratio and the Prandtl number are $\gamma = 1.4$ and $\text{Pr} = 0.72$, respectively. The other dimensional flow parameters are: $T_\infty^* = 78.73$ K, $\rho_\infty^* = 0.0376$ kg/m³, $\nu_\infty^* = 1.4322 \times 10^{-4}$ m²/s, $U_\infty^* = 1.4053$ km/s (in this case $\varepsilon_{kin} = 1.32 \times 10^{-3}$). The wall temperature is $T_w^* = 300$ K. Although we refer to a boundary layer over a flat plate, the free stream parameters actually correspond to the boundary-layer edge flow parameters in the experiment[MML⁺15] on a 7-deg sharp cone at Mach number 9.6 (Run 3743). The mean velocity and temperature profiles across the boundary layer are shown in Fig. 3.4. Figure 3.5 shows amplitudes of the wall pressure perturbation $(p_w^*/P_e^*)_{rms}$ in the wave packet generated by kinetic fluctuations at the neutral and corresponding Reynolds number $R_{neutral} = (U_e^* x_0^*/\nu_e^*)^{1/2}$ as functions of the local Reynolds number at the point of observation R_{max} . Assuming that the nonlinear breakdown occurs at $(p_w^*/P_e^*)_{rms} = 0.1$, the transition onset Reynolds number is $R_{tr} \approx 9000$ that corresponds to the N -factor $N_{tr} \approx 17$. Apparently such quiet conditions are not achievable in conventional hypersonic wind tunnels where N_{tr} is about 5-6. However, the KF-induced transition may be relevant to hypersonic quiet tunnels. For example, the boundary layer on a sharp flared cone tested at zero angle of attack in the NASA Langley Mach 6 Quiet Tunnel (M6QT) at Texas A&M remained laminar at N -factors up to 16 [KPO⁺13].

3.4.3 Example: $M = 7.9$, “Flight” conditions

In order to emulate “flight” conditions, we consider a zero pressure gradient flow at Mach number $M = 7.9$ and the dimensional flow parameters: $T_\infty^* = 327$ K, $\rho_\infty^* = 0.04032$ kg/m³, $\nu_\infty^* = 4.8885 \times 10^{-4}$ m²/s, $U_\infty^* = 2.8641$ km/s (in this case $\varepsilon_{kin} = 5.89 \times 10^{-4}$). The wall temperature is $T_w^* = 1200$ K. The mean velocity and temperature profiles across the boundary layer are shown in Fig. 3.6. The receptivity results are shown in Fig. 3.7. With

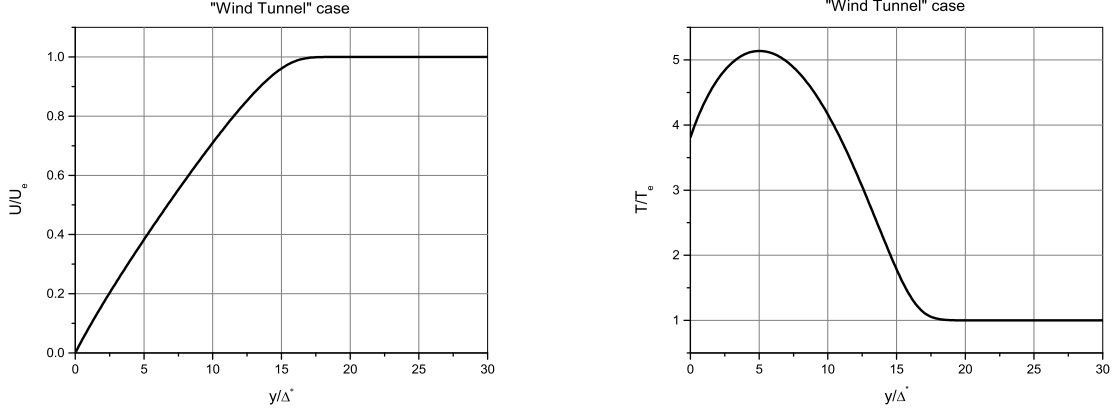


Figure 3.4: Mean velocity and temperature profiles in the "wind tunnel" case.

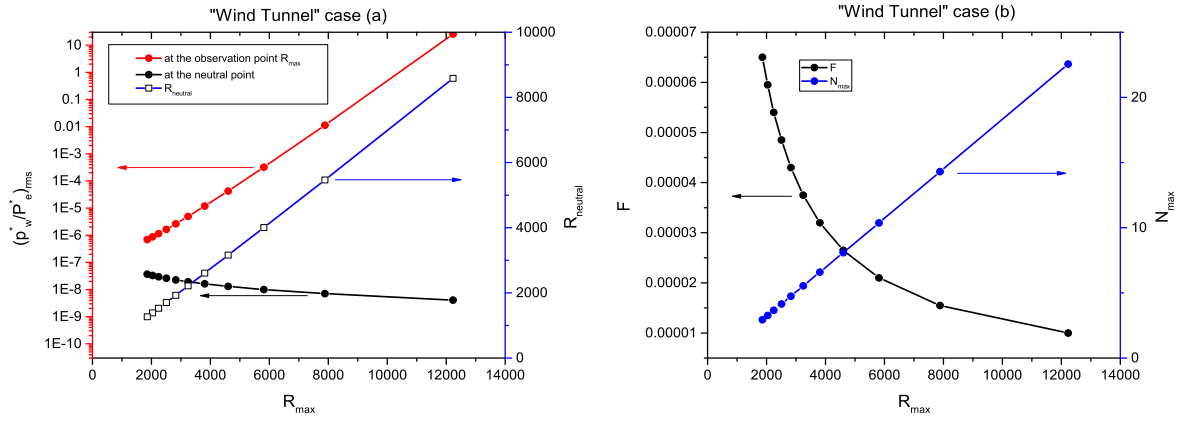


Figure 3.5: "Wind Tunnel" case: (a) The wall pressure perturbation $(p_w^*/P_e^*)_{rms}$ in the wave packet generated by KF fluctuations; (b) Frequency parameter F corresponding to the maximum input at the observation point R_{max} and the N -factor.

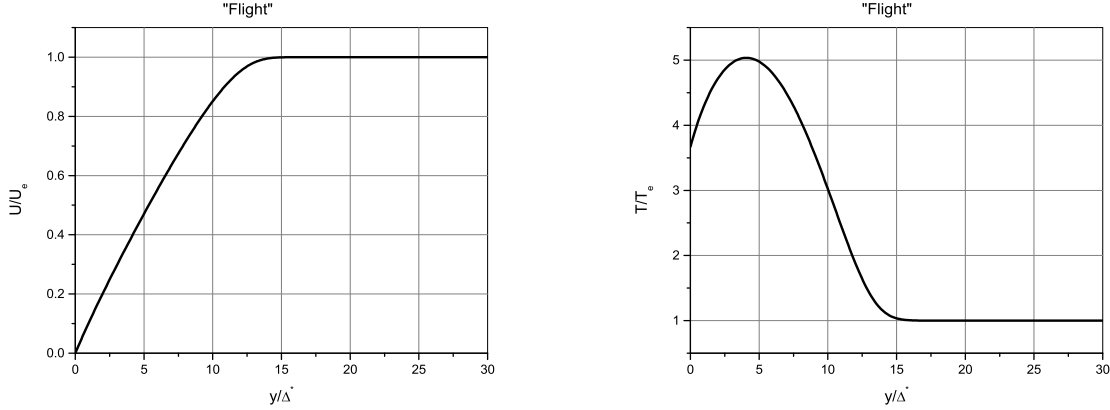


Figure 3.6: Mean velocity and temperature profiles in the "flight" case.

the transition onset criterion $(p_w^*/P_e^*)_{rms} = 0.1$ the transition Reynolds number is $R_{tr} \approx 900$ that corresponds to the N -factor $N_{tr} \approx 17.5$.

In atmosphere, variation of mean temperature with altitude is about 100 K, while mean density varies on orders of magnitude. Density ρ_∞^* decreases approximately twice as the altitude increases 5 km. Therefore, the main effect of the flight altitude on the rms amplitude of instability comes through free-stream density in ε_{kin} (see equation (6.20)). To emulate the altitude effect, we consider the flow densities $\rho_\infty^* = 2\rho_{0\infty}^*$ and $\rho_\infty^* = 4\rho_{0\infty}^*$, where $\rho_{0\infty}^* = 0.04032 \text{ kg/m}^3$ with the free-stream temperature being fixed and equal to $T_\infty = 327 \text{ K}$. Figure 3.8a shows that the density change leads to the curves shift in the logarithmic scale as it follows from equation (6.20) for ε_{kin} . The rms of the wall pressure amplitude as a function of the dimensional coordinate x_0^* is shown in Fig. 3.8b. This trend can be understood from the approximation of $\log_{10}(p_w^*/P_e^*)_{rms}$ in Fig. 3.8a as a linear function $a + bR_{max}$. The density effect from ε_{kin} will appear as the density logarithm, and the main effect of transformation from R_{max} to x^* is $x^* \sim 1/\rho_\infty^*$.

3.5 Concluding remarks

This paper provides all details of the KF receptivity analysis for the compressible boundary-layer flows. The final solutions for the rms amplitude of KF-induced instabilities are expressed in a compact analytical form, which can be incorporated in a standard linear stability code. As contrasted to other receptivity mechanisms, these solutions do not contain any empirical quantities. Their numerical implementation requires the computational resources comparable with those for the e^N method. This opens up an opportunity to perform quick estimates of the upper bound of transition Reynolds number in the framework of physics-based amplitude method.

The results, which were obtained using the two codes developed independently at the Moscow Institute of Physics and Technology and at the University of Arizona, are almost

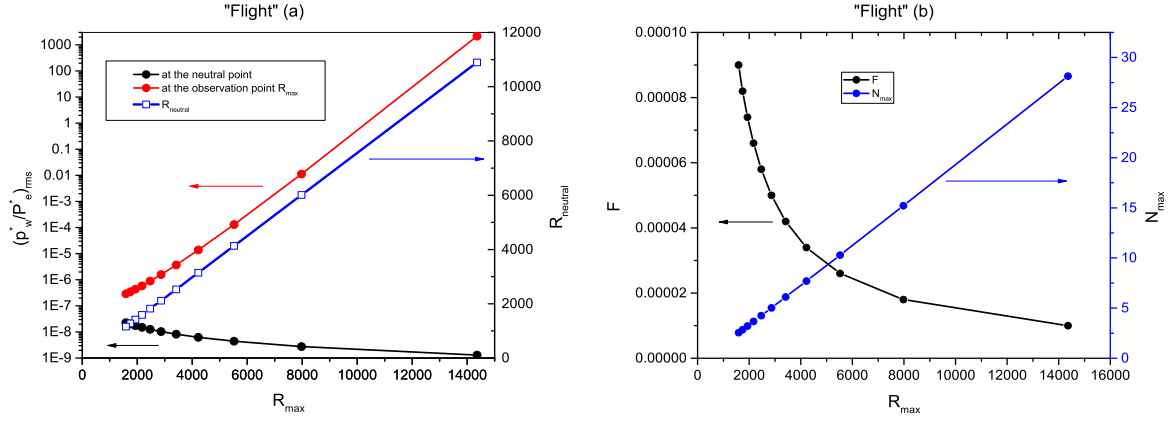


Figure 3.7: "Flight" case: (a) The wall pressure perturbation $(p_w^*/P_e^*)_{rms}$ in the wave packet generated by KF fluctuations; (b) Frequency parameter F corresponding to the maximum input at the observation point R_{max} and the N -factor.

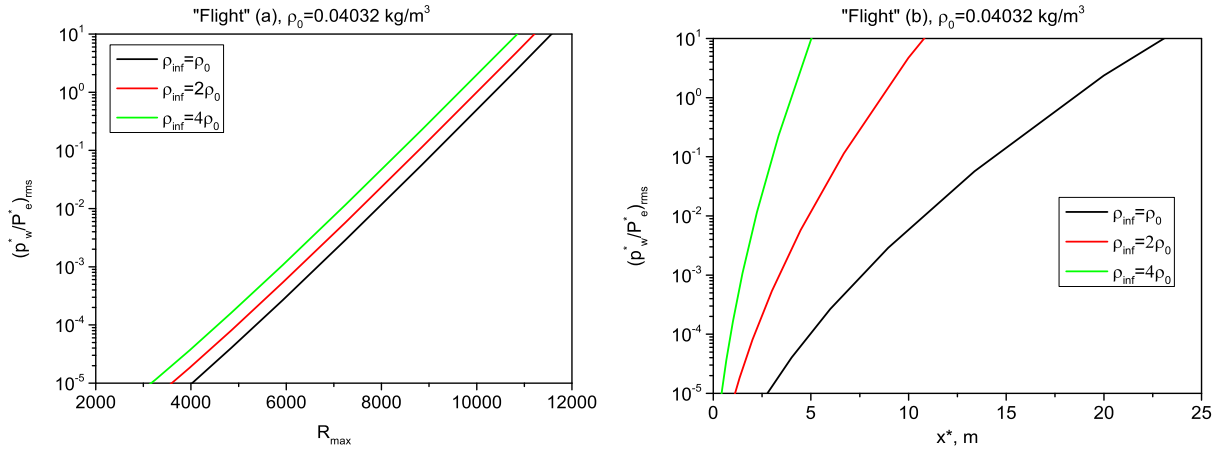


Figure 3.8: "Flight" case: (a) The density ρ_∞ effect on the wall pressure perturbation $(p_w^*/P_e^*)_{rms}$ in the wave packet generated by KF fluctuations; (b) The density effect on the dimensional coordinate x^* versus the wall pressure perturbation $(p_w^*/P_e^*)_{rms}$ in the wave packet generated by KF fluctuations.

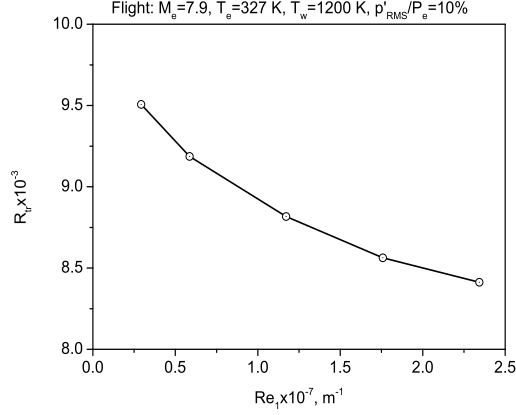


Figure 3.9: The KF-induced transition onset Reynolds number as a function of unit Reynolds number for the flight case.

identical. They also agree with the first-cut theoretical assessments[ZR84] of transition on a flat plate in the incompressible boundary layer.

It was shown that thermal fluctuations excite random wave packets of TS or Mack second-mode instability in the vicinity of the lower neutral branch. The most effective excitation occurs in the critical layer, where fluctuations of the shear stress involved in the streamwise momentum equation are dominant. Propagating downstream the unstable wave packets are narrowed down in (ω, β) space due to selective amplification of their components. Although the KF-induced instabilities are originally small, their downstream growth can be sufficiently large to reach a threshold level for the nonlinear breakdown. These findings are consistent with the previous theoretical results [ZR84, Luc08b, Luc10, FA10, Fed15] indicating that kinetic fluctuations at a microscopic scale can trigger a macroscopic phenomenon – transition from laminar to turbulent flow.

Unfortunately it is not feasible to perform direct measurements of receptivity to kinetic fluctuations because of very low levels of the excited instabilities. However, we can exploit indirect evidences of the KF-induced transition. Let fix the Reynolds number, Mach number, and the wall-temperature ratio and increase the unit Reynolds number Re_1 by increasing free-stream density ρ_∞^* . Since ε_{kin} is proportional to ρ_∞^* , the initial level of instability increases and, therefore, the transition onset Reynolds number decreases with Re_1 . This dependence is shown in Fig. 3.9 for the “flight” case of Section 3.4.3 with the assumption that the transition onset corresponds to the threshold level $(p_w^*/P_e^*)_{rms} = 0.1$. Similar trends were observed for transition on a flat plate and a sharp cone in the Langley Mach 3.5 quiet tunnel[CMB89]. If transition is due to acoustic noise or free-stream turbulence, the trend is, as a rule, opposite – the transition Reynolds number increases with Re_1 .

The foregoing results for “wind tunnel” and “flight” cases demonstrate that kinetic fluctuations are not relevant to transition experiments in noisy wind tunnels like in Ref.[MML⁺15] However, in free flights the KF receptivity mechanism can be a realistic option. Presumably,

the KF-induced transition may be observed in quiet hypersonic wind tunnels, where the boundary layer on sharp cones remains laminar at the integral amplification N larger than 16. The present study raises a question about feasibility to develop a tool for extrapolation of wind-tunnel transition data to flight where transition is associated with excitation and amplification of unstable modes - path A in the terminology of Morkovin, Reshotko, and Herbert[MRH94]. The main obstacle is the lack of reliable data on the disturbance sources (discussed in Introduction) responsible for generation of unstable modes. If levels of environmental perturbations (vorticity and entropy disturbances, acoustic noise, concentration fluctuations, and disturbances induced by particulates) become low, indispensable kinetic fluctuations (or thermal noise) should be taken into account.

At the present stage, our studies of the KF receptivity mechanism including real gas effects (dependence of specific heat on temperature, dissociation, etc.) are at the beginning. It is expected that they may provide new perspectives on laminar-turbulent transition in high-speed flights under quiet environmental conditions.

3.6 Appendices

3.6.1

Components of vector $\hat{\mathbf{G}}^{(L)}(x, y; \beta, \omega)$ in Eq. (6.25)

$$\begin{aligned}\hat{G}_1^{(L)} &= \hat{G}_3^{(L)} = \hat{G}_5^{(L)} = \hat{G}_7^{(L)} = 0 \\ \hat{G}_2^{(L)} &= -\frac{R}{\mu_s} \left(\frac{\partial \delta \hat{\Pi}_{xx}}{\partial x} + \frac{\partial \delta \hat{\Pi}_{xy}}{\partial y} + i\beta \delta \hat{\Pi}_{xz} \right) \\ \hat{G}_4^{(L)} &= \frac{\partial \delta \hat{\Pi}_{yx}}{\partial x} + \frac{\partial \delta \hat{\Pi}_{yy}}{\partial y} + i\beta \delta \hat{\Pi}_{yz} \\ \hat{G}_6^{(L)} &= \frac{R \text{Pr}}{\mu_s} \left(\frac{(\gamma - 1) M^2}{\text{Pr}} \right)^{1/2} \left(\frac{\partial \delta \hat{Q}_x}{\partial x} + \frac{\partial \delta \hat{Q}_y}{\partial y} + i\beta \delta \hat{Q}_z \right) - \\ &\quad \frac{R \text{Pr} (\gamma - 1) M^2}{\mu_s} \left[\left(\delta \hat{\Pi}_{xy} + \delta \hat{\Pi}_{yx} \right) \frac{\partial U_s}{\partial y} + \left(\delta \hat{\Pi}_{yz} + \delta \hat{\Pi}_{zy} \right) \frac{\partial W_s}{\partial y} \right] \\ \hat{G}_8^{(L)} &= -\frac{R}{\mu_s} \left(\frac{\partial \delta \hat{\Pi}_{zx}}{\partial x} + \frac{\partial \delta \hat{\Pi}_{zy}}{\partial y} + i\beta \delta \hat{\Pi}_{zz} \right).\end{aligned}$$

The matrix $\mathbf{H}_0$ of the local-stability problem is expressed as follows. The subscript ‘0’ is omitted for brevity. With the definitions $D = d/dy$, $\mu' = d\mu_s/dT_s$, $m = 2(e - 1)/3$, $r = 2(e + 2)/3 = m + 2$, $\mu_2/\mu = \frac{2}{3}e$; $\hat{\omega} = \omega - \alpha U_s - \beta W_s$, $\chi = \left[\frac{R}{\mu_s} - ir\gamma M^2 \hat{\omega} \right]^{-1}$, the non-zero

elements are

$$\begin{aligned}
H_{12} &= H_{56} = H_{78} = 1; \quad H_{21} = \alpha^2 + \beta^2 - i\hat{\omega} \frac{R}{\mu_s T_s}; \\
H_{22} &= -\frac{D\mu_s}{\mu_s}; \quad H_{23} = -i\alpha \left[(m+1) \frac{DT_s}{T_s} + \frac{D\mu_s}{\mu_s} \right] + \frac{RDU_s}{\mu_s T_s}; \\
H_{24} &= i \frac{\alpha R}{\mu_s} + (m+1) \gamma M^2 \alpha \hat{\omega}; \quad H_{25} = -(m+1) \frac{\alpha \hat{\omega}}{T_s} - \frac{D(\mu'_s DU_s)}{\mu_s}; \\
H_{26} &= -\frac{\mu'_s DU_s}{\mu_s}; \quad H_{31} = -i\alpha; \quad H_{33} = \frac{DT_s}{T_s}; \quad H_{34} = i\gamma M^2 \hat{\omega}; \quad H_{35} = -\frac{i\hat{\omega}}{T_s}; \\
H_{37} &= -i\beta; \quad H_{41} = -i\alpha\chi \left(r \frac{DT_s}{T_s} + 2 \frac{D\mu_s}{\mu_s} \right); \quad H_{42} = -i\alpha\chi; \\
H_{43} &= \chi \left[-\alpha^2 - \beta^2 + r \frac{D\mu_s DT_s}{\mu_s T_s} + r \frac{D^2 T_s}{T_s} + i \frac{R\hat{\omega}}{\mu_s T_s} \right]; \\
H_{44} &= -i\chi r \gamma M^2 \left[\alpha DU_s + \beta DW_s - \hat{\omega} \left(\frac{DT_s}{T_s} + \frac{D\mu_s}{\mu_s} \right) \right]; \\
H_{45} &= i\chi \left[r \frac{(\alpha DU_s + \beta DW_s)}{T_s} + \mu'_s \frac{\alpha DU_s + \beta DW_s}{\mu_s} - r \hat{\omega} \frac{D\mu_s}{\mu_s T_s} \right]; \\
H_{46} &= -i\chi \hat{\omega} \frac{r}{T_s}; \quad H_{47} = -i\beta\chi \left(r \frac{DT_s}{T_s} + 2 \frac{D\mu_s}{\mu_s} \right); \quad H_{48} = -i\beta\chi; \\
H_{62} &= -2 \text{Pr}(\gamma - 1) M^2 DU_s; \quad H_{63} = R \text{Pr} \frac{DT_s}{\mu_s T_s} - 2i(\gamma - 1) M^2 \text{Pr}(\alpha DU_s + \beta DW_s); \\
H_{64} &= iR \text{Pr} \frac{(\gamma - 1) M^2}{\mu_s} \hat{\omega}; \quad H_{65} = \alpha^2 + \beta^2 - iR \text{Pr} \frac{\hat{\omega}}{\mu_s T_s} - (\gamma - 1) M^2 \text{Pr} \frac{\mu'_s [(DU_s)^2 + (DW_s)^2]}{\mu_s} - \frac{D^2 \mu_s}{\mu_s}; \\
H_{66} &= -2 \frac{D\mu_s}{\mu_s}; \quad H_{68} = -2 \text{Pr}(\gamma - 1) M^2 DW_s; \quad H_{83} = \frac{RDW_s}{\mu_s T_s} - i\beta \left[(m+1) \frac{DT_s}{T_s} + \frac{D\mu_s}{\mu_s} \right]; \\
H_{84} &= i \frac{\beta R}{\mu_s} + (m+1) \gamma M^2 \beta \hat{\omega}; \quad H_{85} = -(m+1) \frac{\beta}{T} \hat{\omega} - \frac{D(\mu'_s DW_s)}{\mu_s}; \\
H_{86} &= -\frac{\mu'_s DW_s}{\mu_s}; \quad H_{87} = \alpha^2 + \beta^2 - i\hat{\omega} \frac{R}{\mu_s T_s}; \quad H_{88} = -\frac{D\mu_s}{\mu_s}.
\end{aligned}$$

The vector $\hat{\mathbf{G}}_\alpha^{(L)}$ in Eq. (3.9) has the components

$$\begin{aligned}
\hat{G}_{\alpha 1}^{(L)} &= \hat{G}_{\alpha 3}^{(L)} = \hat{G}_{\alpha 5}^{(L)} = \hat{G}_{\alpha 7}^{(L)} = 0 \\
\hat{G}_{\alpha 2}^{(L)} &= -\frac{R}{\mu_s} \left(i\alpha \delta \hat{\Pi}_{\alpha x x} + \frac{\partial \delta \hat{\Pi}_{\alpha x y}}{\partial y} + i\beta \delta \hat{\Pi}_{\alpha x z} \right) \\
G_{\alpha 4}^{(L)} &= i\alpha \delta \hat{\Pi}_{\alpha y x} + \frac{\partial \delta \hat{\Pi}_{\alpha y y}}{\partial y} + i\beta \delta \hat{\Pi}_{\alpha y z} + O(R^{-1}) \\
\hat{G}_{\alpha 6}^{(L)} &= \frac{R \text{Pr}}{\mu_s} \left(\frac{(\gamma - 1) M^2}{\text{Pr}} \right)^{1/2} \left(i\alpha \delta \hat{Q}_{\alpha x} + \frac{\partial \delta \hat{Q}_{\alpha y}}{\partial y} + i\beta \delta \hat{Q}_{\alpha z} \right) - \\
&\quad \frac{R \text{Pr}}{\mu_s} \frac{(\gamma - 1) M^2}{2} \left[\left(\delta \hat{\Pi}_{\alpha x y} + \delta \hat{\Pi}_{\alpha y x} \right) \frac{\partial U_s}{\partial y} + \left(\delta \hat{\Pi}_{\alpha y z} + \delta \hat{\Pi}_{\alpha z y} \right) \frac{\partial W_s}{\partial y} \right] \\
\hat{G}_{\alpha 8}^{(L)} &= -\frac{R}{\mu_s} \left(i\alpha \delta \hat{\Pi}_{\alpha z x} + \frac{\partial \delta \hat{\Pi}_{\alpha z y}}{\partial y} + i\beta \delta \hat{\Pi}_{\alpha z z} \right)
\end{aligned}$$

3.6.2

The solvability condition for the problem (3.9) reads

$$-i \frac{dc}{d\xi} \left\langle \frac{\partial \mathbf{H}_0}{\partial \alpha} \mathbf{A}, \mathbf{B} \right\rangle + c\xi \left\langle \frac{\partial \mathbf{H}_0}{\partial x_1} \mathbf{A}, \mathbf{B} \right\rangle + \left\langle \hat{\mathbf{G}}_{\alpha}^{(L)}, \mathbf{B} \right\rangle e^{i\gamma\xi} = 0 \quad (3.31)$$

Differentiating Eqs. (6.30) with respect to x_1 we obtain

$$\left(\frac{\partial}{\partial y} - \mathbf{H}_0 \right) \frac{\partial \mathbf{A}}{\partial x_1} = \frac{\partial \alpha}{\partial x_1} \frac{\partial \mathbf{H}_0}{\partial \alpha} \mathbf{A} + \frac{\partial \mathbf{H}_0}{\partial x_1} \mathbf{A} \quad (3.32)$$

$$\frac{\partial A_1}{\partial x_1} = \frac{\partial A_3}{\partial x_1} = \frac{\partial A_5}{\partial x_1} = \frac{\partial A_7}{\partial x_1} = 0 \text{ at } y = 0 \quad (3.33)$$

$$\frac{\partial A_1}{\partial x_1}, \frac{\partial A_3}{\partial x_1}, \frac{\partial A_5}{\partial x_1}, \frac{\partial A_7}{\partial x_1} \rightarrow 0 \text{ at } y \rightarrow \infty \quad (3.34)$$

Applying the solvability condition to the problem (3.32)-(3.34) we get

$$\left\langle \frac{\partial \mathbf{H}_0}{\partial \alpha} \mathbf{A}, \mathbf{B} \right\rangle \frac{\partial \alpha}{\partial x_1} = - \left\langle \frac{\partial \mathbf{H}_0}{\partial x_1} \mathbf{A}, \mathbf{B} \right\rangle \quad (3.35)$$

From (3.31) and (3.35) and using the definition $b = \partial \alpha / \partial x_1$ we obtain Eqs. (3.12)-(3.14).

3.6.3

The spectral density $(q^2)_{\alpha, \beta, \omega}$ in Eq. (6.31) is expressed as

$$(q^2)_{\alpha, \beta, \omega} = \frac{J_{22} + J_{44} + J_{66} + J_{88} + 2\text{Re}(J_{24} + J_{26} + J_{28} + J_{46} + J_{48} + J_{68})}{\left| \left\langle \frac{\partial \mathbf{H}_0}{\partial \alpha}(x_{10}, \alpha) \mathbf{A}, \mathbf{B} \right\rangle \right|^2}$$

where Re denotes the real part,

$$J_{22} = 2R^2 \int_0^\infty \frac{T_s}{\mu_s} \left[r\alpha^2 |B_2|^2 + \mu_s^2 \left| D \left(\frac{B_2}{\mu_s} \right) \right|^2 + \beta^2 |B_2|^2 \right] dy$$

$$J_{44} = 2 \int_0^\infty \mu_s T_s [\alpha^2 |B_4|^2 + r |DB_4|^2 + \beta^2 |B_4|^2] dy$$

$$J_{66} = 2(\gamma - 1) M^2 R^2 \Pr \int_0^\infty \frac{T_s^2}{\mu_s} \left[\alpha^2 |B_6|^2 + \mu_s^2 \left| D \left(\frac{B_6}{\mu_s} \right) \right|^2 + \beta^2 |B_6|^2 \right] dy +$$

$$2 [R \Pr(\gamma - 1) M^2]^2 \int_0^\infty \frac{T_s}{\mu_s} [(DU_s)^2 + (DW_s)^2] |B_6|^2 dy$$

$$J_{88} = 2R^2 \int_0^\infty \frac{T_s}{\mu_s} \left[\alpha^2 |B_8|^2 + \mu_s^2 \left| D \left(\frac{B_8}{\mu_s} \right) \right|^2 + r\beta^2 |B_8|^2 \right] dy$$

$$J_{24} = 2i\alpha R \int_0^\infty T_s \left(mB_2 DB_4^* - \left(DB_2 - B_2 \left(\frac{D\mu_s}{\mu_s} \right) \right) B_4^* \right) dy$$

$$J_{26} = -2R^2 \Pr(\gamma - 1) M^2 \int_0^\infty \frac{T_s}{\mu_s} DU_s \left(DB_2 - B_2 \left(\frac{D\mu_s}{\mu_s} \right) \right) B_6^* dy$$

$$J_{28} = 2\alpha\beta R^2 \int_0^\infty \frac{T_s}{\mu_s} (m + 1) B_2 B_8^* dy$$

$$J_{46} = -2R \Pr(\gamma - 1) M^2 \int_0^\infty T_s [i\alpha DU_s + i\beta DW_s] B_4 B_6^* dy$$

$$J_{48} = 2i\beta R \int_0^\infty T_s \left[B_4 \left(DB_8^* - B_8^* \left(\frac{D\mu_s}{\mu_s} \right) \right) - mDB_4 B_8^* \right] dy$$

$$J_{68} = -2R^2 \Pr(\gamma - 1) M^2 \int_0^\infty \frac{T_s}{\mu_s} DW_s B_6 \left(DB_8^* - B_8^* \left(\frac{D\mu_s}{\mu_s} \right) \right) dy$$

$$D = d/dy$$

Chapter 4

Real Gas Effects on Receptivity to Kinetic Fluctuations: Mean Flow Effect

4.1 Introduction

Results in Ref.[FT17] were obtained at moderate total enthalpies because the analysis was restricted by the calorically perfect gas model for the mean flow and the stability equations. In the present work, we are interested in understanding the role that real gas effects play in receptivity and stability related to kinetic fluctuations. We consider cases of higher total enthalpy flow ($5.60 - 16.53$ MJ/kg) for which temperatures remain low enough that the dissociation of air can be ignored (frozen flow). Thus, our study of real gas effects on receptivity and stability is done exclusively through the use of two different boundary layer mean flow models; a calorically perfect gas model and a real gas model (five species model for air). The specific heat C_p is assumed constant in the perfect gas model, whereas it varies as a function of temperature in the real gas model. An overview of the latter model is provided in Section II.A, and a detailed description is given in Appendix 4.5.1. The stability solver used in both cases assumes a calorically perfect gas (i.e. constant Prandtl number and specific heats ratio γ). Unless otherwise specified, the labels 'calorically perfect gas' or 'air' (real gas) are meant only to indicate the mean flow model that was used in obtaining the result.

4.2 Numerical method

4.2.1 Mean flow boundary-layer solver

The real gas boundary-layer solver used in this work utilizes the five species model of air. One can find the boundary-layer equations for a reacting gas in monographs [Dor62, AVGP76, Lun09]. The numerical method is similar to that described by Cebeci[Ceb99] for non-reacting

flows and in Refs. [Ulk10, TUK12] for five species air and binary mixtures. In the case of a sharp leading edge, the coordinate x is the distance from the leading edge, and in a case of a blunt body, it is the distance from the stagnation point. Coordinate y represents the distance from the wall. Because the solver has not yet been described elsewhere, we provide the governing equations and details of the numerical algorithm in Appendix 4.5.1. Transformation from x and y to the Levy-Lees-Dorodnitsyn variables ξ and η is utilized. The transformations are given by

$$\xi = \int_0^x \rho_e u_e \mu_e r_0^{2j} dx, \quad \eta = \frac{\rho_e u_e r_0^j}{\sqrt{2\xi}} \int_0^y \frac{\rho}{\rho_e} dy, \quad (4.1)$$

where $r_0(x)$ is the radius of the body in the axisymmetric flow case. For two dimensional flow we set $j = 0$, and $j = 1$ for axisymmetric flow.

The results presented herein were obtained using a uniform grid with $\Delta\eta = 0.015$ on the interval $0 \leq \eta \leq \eta_{max} = 10$, and 800 even intervals on $0 \leq x \leq 0.4$ m. The pressure gradient was assumed to be zero. To check convergence, the number of intervals was doubled. Also, the mean flow solutions were checked on larger/smaller distances along coordinate x .

Using the mean profiles for temperature T , viscosity μ and the derivative of the stream-wise velocity dU/dy , where y is the distance from the wall, a standard central difference scheme can be used to obtain the mean profiles for d^2U/dy^2 , dT/dy , d^2T/dy^2 , $d\mu/dT$, and $d^2\mu/dT^2$. In some cases, particularly for the latter two quantities, numerical instabilities arose due to division by very small numbers in the derivative approximation. This issue was overcome by using a smoothing splines routine in MATLAB to avoid numerical oscillations.

4.2.2 Stability and receptivity solver

The stability and receptivity results were obtained following Refs.[Tum07, FT17] The linearized Navier-Stokes equations were reduced to a system of ordinary differential equations (ODEs) for normal modes. The fundamental solutions of the direct and adjoint eigenvalue problem were found numerically using the fourth order Runge-Kuta method by integration of the equations from y_{max} to the wall with the known analytical asymptotic solutions outside the boundary layer. The number of grid points was varied across the boundary layer (up to 1201) to check the accuracy of the computations. The Gram-Schmidt orthonormalization procedure was employed in the computation of the fundamental solutions during integration across the boundary layer. Finally, the eigenfunctions of the direct and adjoint problems were obtained as a sum of the fundamental solutions. To satisfy the boundary conditions using the fundamental solutions for modes of the discrete spectrum, the eigenvalue α was found using a Newton-Raphson iteration scheme. About 600 grid points equally-spaced across the boundary layer were needed to compute $\alpha_r = \text{Real}(\alpha)$ with 6 significant digits and $\alpha_i = \text{Im}(\alpha)$ with 4 significant digits.

Note that the stability solver used in this work assumes effective (constant) values of the Prandtl number Pr and specific heats ratio γ , which in general vary as a function of temperature across the boundary layer in the real gas model. We have thus explored the

effects of varying Pr_{eff} and γ_{eff} across the range of relevant values on the stability of the 2nd Mack mode. The effects are seen to be small, however, choosing Pr_{eff} and γ_{eff} equal to the corresponding values at the edge of the boundary layer yields results that are most consistent with those obtained using a real gas stability code[KT13b]. Demonstrations justifying this procedure are provided for a particular case in Section *III.A.1*.

4.3 Results

Real gas effects depend on the total enthalpy and pressure at flight conditions. An important question is how to choose flow parameters to construct examples which are meaningful from a practical perspective without introducing a constraint on the path of flight (e.g. altitude). Klentzman and Tumin[KT13a] discussed possible choices of flow parameters that would be practically relevant and suitable for academic study. They suggested considering total enthalpies 4.5 – 18 MJ/kg (flight at 3 – 6 km/s).

The nose/edge radii and the upper limit of the wall temperature dictate altitudes of flight (since speed of flight, radii and density are the parameters controlling the convective heat flux). Because prospective hypersonic vehicles will have high hypersonic lift-to-drag ratio (> 2.5), the nose and leading edge radii will be relatively small. Additionally, it is expected that future heat insulation materials will work up to 2000 K. Based on these insights, 8 cases were suggested in Ref. [KT15] for stability analysis in boundary layers. (To emphasize the real gas effects, boundary-layer flow in a binary mixture of oxygen was considered in Ref.[KT13a]). In the present work, we have chosen to consider the first three cases: ‘Case 1’, ‘Case 2’, and ‘Case 3’, each with two different wall temperatures: $T_w = 1000$ K and $T_w = 2000$ K. For each configuration, we compare receptivity and stability results based on the calorically perfect gas and the real gas mean flow models with the given edge parameters. For convenience, the details of Cases 1 - 3 are provided in Table 7.1. In the present work, thermodynamic and transport properties were calculated according to Ref.[GYTL90].

For the cases considered herein, the wave packets are dominated by the 2nd Mack modes. The steepest descent method used for the evaluation of the wave packet amplitudes [FT17] leads to analysis which takes into account an appropriately small neighborhood around $\beta = 0$. In the cases considered, dissociation is absent and the mean flow profiles are self-similar.

	T_e (K)	U_e (m/s)	I (MJ/kg)	M_e	P_e
Case 1	278	3.351×10^3	5.60	10.00	0.0433
Case 2	556	4.668×10^3	11.17	9.92	0.0433
Case 3	834	5.648×10^3	16.53	9.91	0.0433

Table 4.1: Details of the flow properties for Case 1 - 3 from Ref. [KT15]

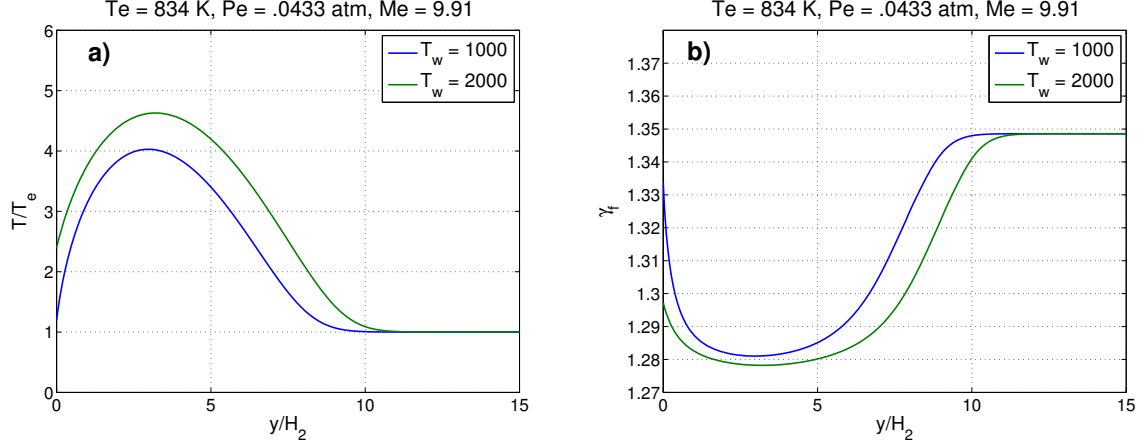


Figure 4.1: (a) Temperature profiles across boundary layers of air at $T_w = 1000$ K and $T_w = 2000$ K (b) Variation of the specific heats ratio γ_f across the boundary layers for the same two wall temperatures.

4.3.1 “Case 3”, $T_e = 834$ K, $P_e = 0.0433$ atm, $U_e = 5.648$ km/s, $M_e = 9.91$

Stability characteristics and properties of the mean flow

We begin with a discussion of ‘Case 3’, which corresponds to the boundary-layer edge parameters $T_e = 834$ K, $P_e = 0.0433$ atm, and $U_e = 5.648$ km/s. This is an example of high total enthalpy flow (16.53 MJ/kg). These flow parameters lead to frozen Mach number $M_e = 9.91$, frozen specific heats ratio $\gamma_f = 1.35$, and frozen Prandtl number $Pr_f = 0.699$, determined using the thermodynamic and transport properties of Ref.[GYTL90].

Figure 4.1 shows how the mean temperature and frozen specific heats ratio vary across the boundary layer based on the real gas model for wall temperatures $T_w = 1000$ K and $T_w = 2000$ K.

Because we are going to use effective specific heats ratio γ_{eff} and Prandtl number Pr_{eff} in the stability and receptivity calculations throughout this work, we have explored the effects of varying γ_{eff} and Prandtl Pr_{eff} across the range of relevant values. We use the present case with $T_w = 1000$ K as a demonstrative example. The variation of the frozen specific heats ratio and Prandtl number across the boundary layer for this case is shown in Figure 4.2. Figure 4.3a shows the growth rates $\sigma = -\alpha_i H_2$ for $F = 40.0e - 06$ corresponding two representative cases, $\gamma_{eff} = 1.35$, $Pr_{eff} = 0.70$ and $\gamma_{eff} = 1.28$, $Pr_{eff} = 0.68$, using the real gas mean flow model. One can see that varying γ and Pr does not have a strong effect on the growth rates.

In Figure 4.3b, we provide a comparison of the growth rates computed using the real gas mean flow model with both the calorically perfect gas stability solver ($\gamma_{eff} = 1.35$, $Pr_{eff} = 0.70$) and a real gas stability solver[KT13b]. The computations with γ_{eff} and Pr_{eff} are shown to reproduce the stability features produced by the real gas stability solver quite

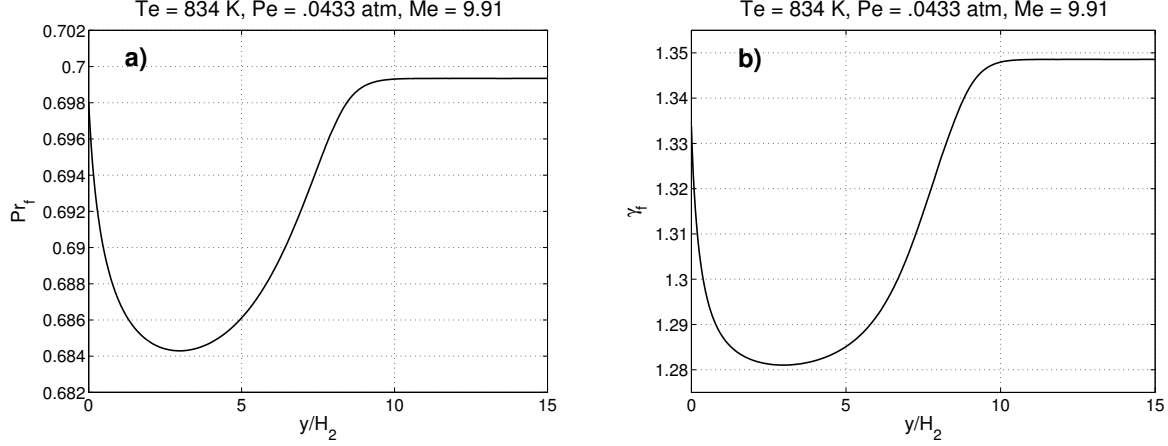


Figure 4.2: Case 3, $T_w = 1000$ K: Variation of (a) Prandtl number Pr_f and (b) the specific heats ratio γ_f across the boundary layer.

well. (Note that we have chosen γ_{eff} and Pr_{eff} to match the values of these parameters at the edge of the boundary layer. This choice is in general seen to provide results which most closely match those obtained with the full stability solver and is thus adopted for all cases discussed in this paper). The ‘tail’ seen in the stability curves computed based on the real gas mean flow model is associated with a transition of the mode from subsonic to supersonic with respect to the free-stream velocity, i.e. $c_r < 1 - 1/M_e$. (The significance of this phenomenon is discussed in more detail in Section III.A.2). The growth rates computed using the calorically perfect gas mean flow and stability solver are also shown for comparison. Note the absence of the ‘tail’ in this curve, indicating that the mode remains subsonic over the range of R_{H_2} for which it is unstable.

For this configuration, and those that follow, the edge flow parameters are made to be identical between the two mean flow models. Figure 4.4 shows comparisons of the mean velocity and temperature profiles produced using each model for ‘Case 3’, wall temperature $T_w = 1000$ K. The lower temperatures that result from the real gas model are due to the dependence of the specific heat C_p on temperature.

Supersonic modes

Figure 4.5 shows $N = -\int_{x_0}^x \alpha_i dx$ for 11 frequency parameters together with the envelope N_{max} calculated using the calorically perfect gas model. A difference in behavior is observed between N corresponding to low frequencies (lines 9 - 11) and high frequencies (lines 1 - 8). Figure 4.6 shows $\alpha_i H_2$ and $c_r = \text{Re}(\omega/\alpha)$ scaled with U_e as a function of the local Reynolds number based on the scale H_2 . Figure 4.6a shows that α_i at low frequencies has a kink close to the upper neutral point. This kink is also accompanied by the phase speed c_r crossing the line $1 - 1/M_e$ in Fig. 4.6b. Physically, this means the mode is traveling upstream supersonically relative to the free stream. Discrete modes having $c_r > 1 - 1/M_e$ are referred to as subsonic, whereas modes having $c_r < 1 - 1/M_e$ are supersonic.

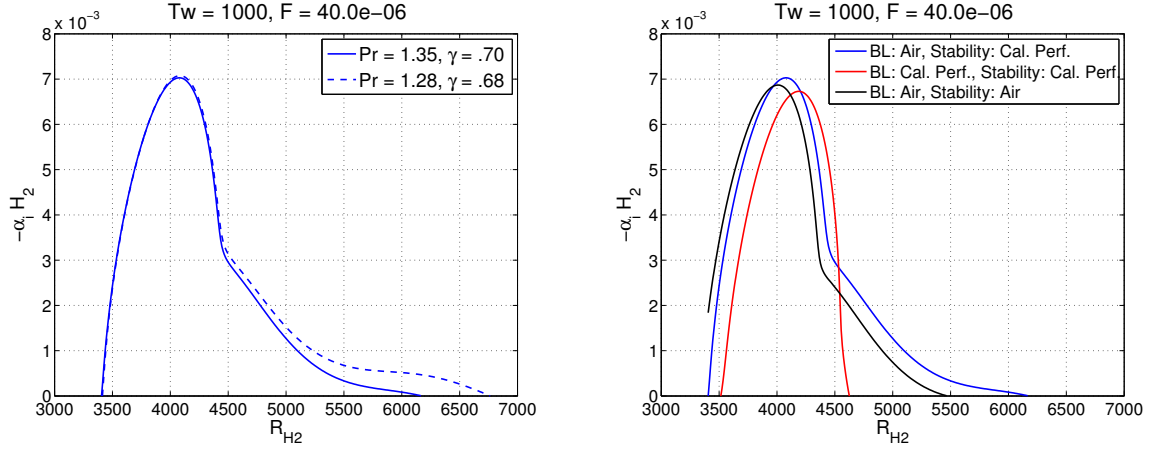


Figure 4.3: (a) The effect of variation of γ_{eff} and Pr_{eff} on growth rates for air with wall temperature $T_w = 1000$. (b) A comparison of growth rates for air using both the calorically perfect stability solver ($\gamma_{eff} = 1.35$ and $Pr_{eff} = .70$) and a real gas stability solver. Also shown is the growth rate for a calorically perfect gas using the calorically perfect stability solver.

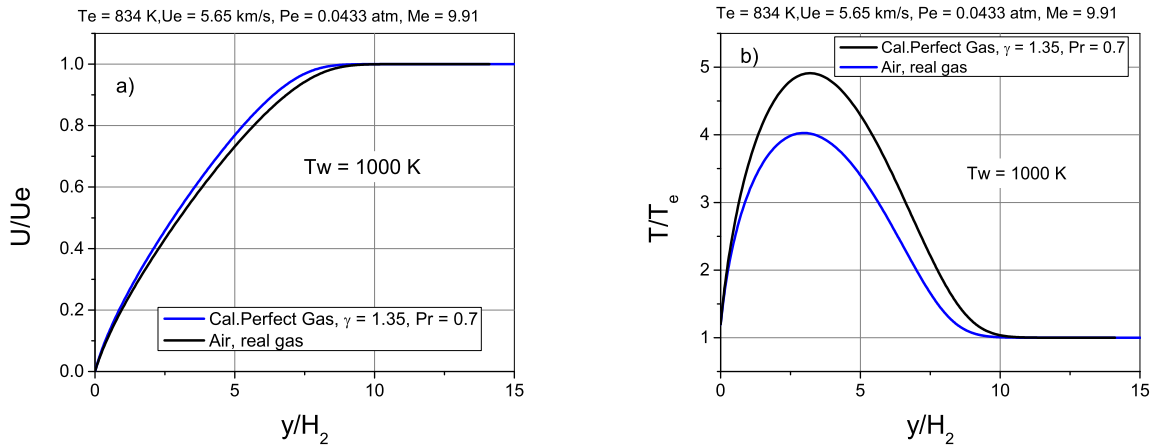


Figure 4.4: Comparisons of (a) mean velocity and (b) temperature profiles at $T_w = 1000$ K.

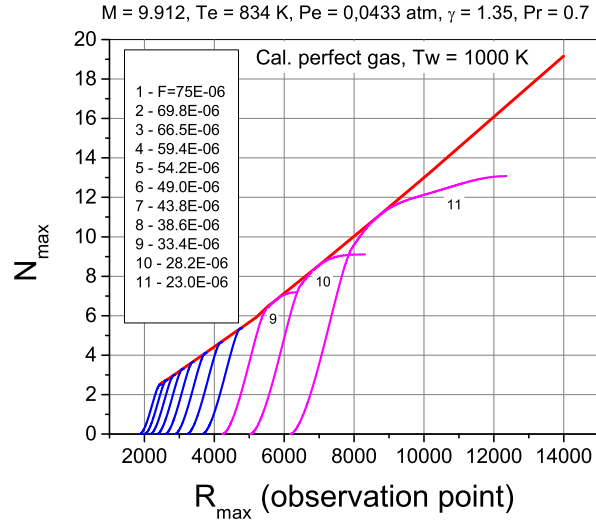


Figure 4.5: N - factors and the envelope for the calorically perfect gas mean flow model.

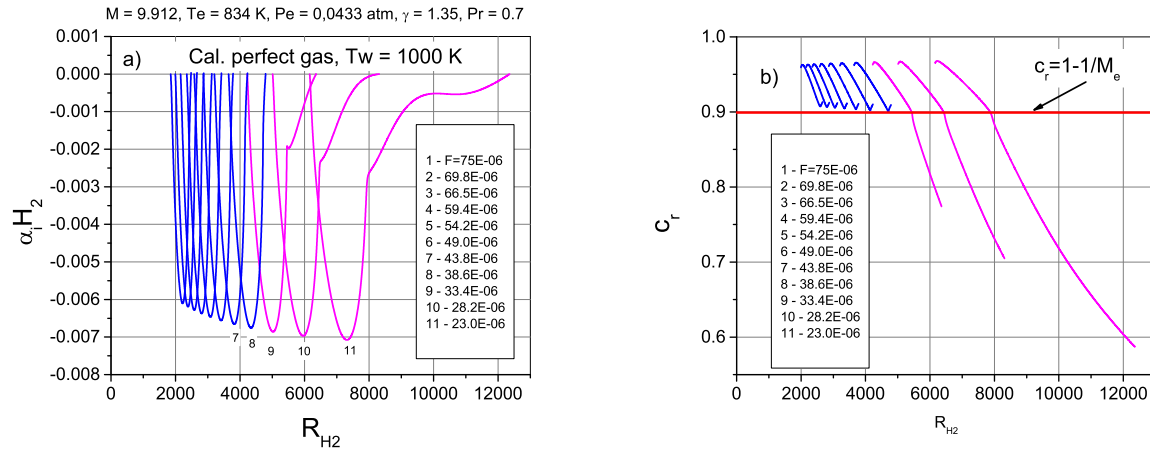


Figure 4.6: (a) $\alpha_i H_2$ and (b) phase velocities for the calorically perfect gas mean flow model.

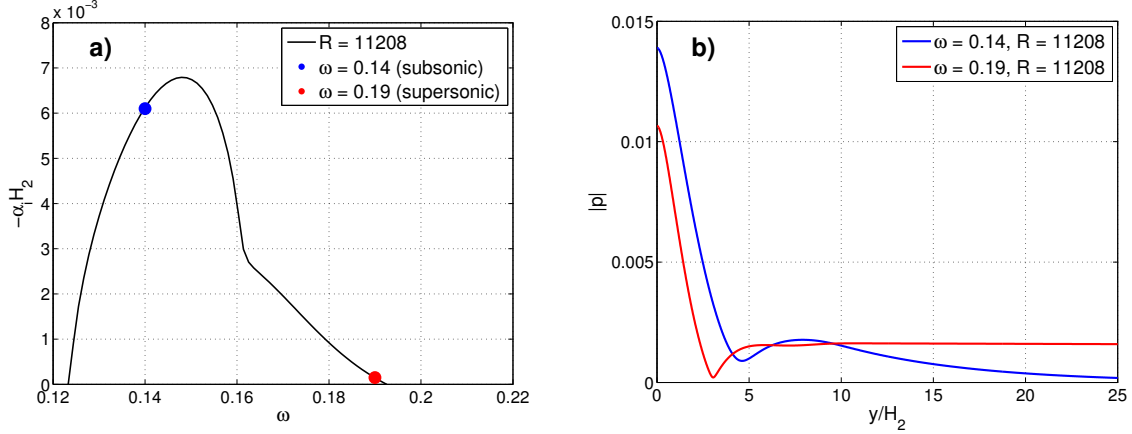


Figure 4.7: (a) Growth rates corresponding to the real gas model for Case 3, $T_w = 1000$ with $R_{H_2} = 11208$. Supersonic transition occurs at $\omega \approx 0.162$. (b) Eigenfunctions for the pressure disturbance at the locations indicated in (a).

This transition of unstable modes from subsonic to supersonic has been observed in the past. Bitter and Shepherd[BS15] recently conducted a comprehensive study of stability in high-speed boundary layers on a cold wall ($T_w/T_e = 0.2$), where they noted that the kink in the growth rate is associated with a transition of the 2nd Mack mode from subsonic to supersonic. Near the upper neutral branch, supersonic modes coalesce with the branch cut associated with slow acoustic waves, potentially leading to new physical phenomena. Figure 4.7a shows a stability plot corresponding to ‘Case 3’, $T_w = 1000$ (calorically perfect gas mean flow) in which the 2nd Mack mode transitions from subsonic to supersonic in the vicinity of $\omega = 0.162$. Figure 4.7b shows the magnitude of the pressure eigenfunctions at the two points indicated in Figure 4.7a, blue and red corresponding to subsonic and supersonic regions respectively. Note the much slower free stream decay of the eigenfunction associated with the supersonic mode. In the corresponding disturbance plots, provided in Figure 4.8, one can see that supersonic modes are associated with disturbance fields that reach far into the outer flow. In exploring this phenomenon further, Chuvakhov and Fedorov[PVC16] recently demonstrated that wave packets of supersonic 2nd Mack modes ($T_w/T_e = 0.5$) can spontaneously emit sound waves, potentially changing the dynamics of the wave packet. More investigation is needed in order to fully understand the implications of this phenomenon.

The role of the cold wall ($T_w/T_e < 1$) in the appearance of supersonic modes is emphasized in both references [BS15, PVC16]. Our result at $T_w = 1000$ K (Fig. 4.6) demonstrates that unstable supersonic modes may exist at $T_w/T_e > 1$.

The presence of supersonic unstable modes for this configuration is also observed for the real gas model. In fact, for the real gas case, the 2nd Mack mode transitions from subsonic to supersonic for the entire range of frequencies considered. In the calorically perfect gas case, the unstable modes remain entirely subsonic for frequencies higher than roughly $F = 35.0e - 06$, after which point supersonic transition occurs for sufficiently large

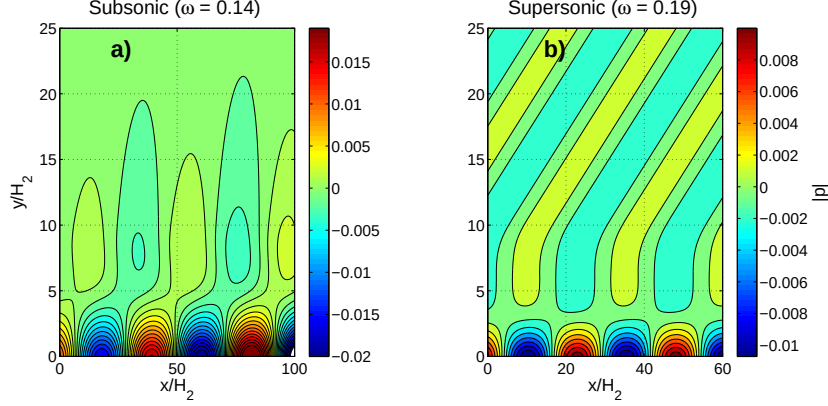


Figure 4.8: The pressure disturbance corresponding to (a) $\omega = 0.14$, $R_{H2} = 11208$ (subsonic), and (b) $\omega = 0.19$, $R_{H2} = 11208$ (supersonic).

Reynolds number. This is typical of a trend that will be observed in the examples considered in this work; that is, the real gas model results in the appearance of supersonic modes for lower Reynolds numbers than in the calorically perfect gas case.

Receptivity to KF and development of the wave packets

The primary focus of this work is on determining the maximal downstream amplitude of acoustic wave packets generated by receptivity to kinetic fluctuations. The solution to the receptivity problem accounts for the amplitude of wall pressure perturbations corresponding to the 2nd Mack mode at the lower branch of the neutral curve where the disturbance is generated. Figure 4.9a shows the root mean square of wall pressure perturbations (normalized by the edge pressure P_e) at the observation point R_{max} for both the calorically perfect gas and real gas models. Also shown is the amplitude of the wave packet rescaled to the lower neutral point by $e^{N_{max}}$, where N_{max} corresponds to the dominating frequency at the observation point R_{max} . These curves are labeled as "at the observation point" and "at the neutral point", respectively. This rescaled wave packet amplitude reflects the receptivity together with the wave packet summation downstream. A similar comparison for wall temperature $T_w = 2000$ is provided in Figure 4.9b. Both cases result in wave packets containing supersonic modes, although supersonic transition occurs for much lower Reynolds numbers in the real gas model.

Note that there is no significant difference in the receptivity results for air and calorically perfect gas. It was shown in Ref.[FT17] that the receptivity to kinetic fluctuations occurs in the vicinity of the critical layer. At high Mach numbers, the 2nd Mack modes have $c_r > 1 - 1/M_e$ at the lower branch of the neutral curve, and the critical layer is located close to the boundary layer edge. Therefore, receptivity itself is governed primarily by the edge flow parameters.

The downstream magnitude of the perturbations, however, is significantly higher for the

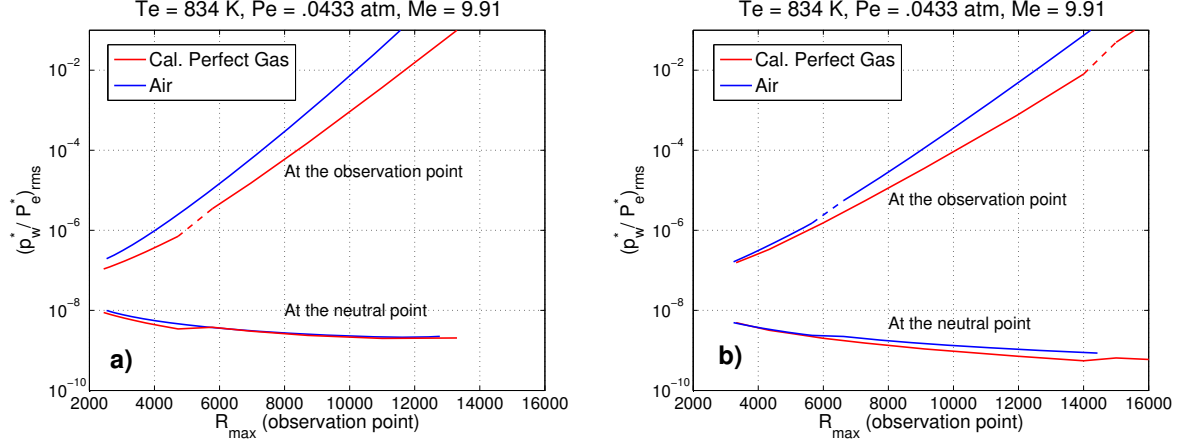


Figure 4.9: Comparison of results for air and calorically perfect gas: RMS of the wall pressure perturbation at the observation point and the “initial” amplitudes at the lower neutral curve; (a) $T_w = 1000$ K ; (b) $T_w = 2000$ K .

real gas model. This can be largely attributed to the destabilizing effect of the colder mean flow profile that results from the dependence of the specific heats on temperature in the real gas model. Indeed, Malik and Anderson[MA91] showed that real gas effects can destabilize the 2nd Mack mode and shift its unstable band to lower frequencies.

The dashed line in Fig. 4.9b corresponds to the root-mean-square of the wall pressure disturbance at the point of observation $R_{max} \approx 6500$. This Reynolds number is within the region of transition from subsonic to supersonic modes in air and the asymptotic formula used in Ref.[FT17] (and in the present work) has to be modified due to the maximum of the N -factor being achieved in the vicinity of the slow acoustic mode. Figure 4.10 shows N as a function of frequency parameter F over a range of fixed Reynolds numbers R_{H2} (in increments of $R_{H2} = 500$) for Case 3, wall temperature $T_w = 2000$. The curves corresponding to wave packets containing supersonic modes are indicated in red in the figure. The apparent truncation of these curves is due to the coalescence of the (supersonic) discrete mode with the branch cut associated with slow acoustic waves of the continuous spectrum. Observe that the maximum of N in the transitional region is located close to the end point where supersonic modes coalesce with the slow acoustic mode.

Figure 4.11 summarizes the results for the real gas model. One can see that the wave packet at $T_w = 1000$ K is made up completely of modes which transition from subsonic to supersonic at all frequency parameters of interest. On the other hand, for $T_w = 2000$ K, there is a range of Reynolds numbers for which the wave packet is comprised of only subsonic modes. It should be noted that the temperature ratio $T_w/T_e \approx 2.4 > 1$ in this case. Another observation stemming from Fig. 4.11b is that the lower wall temperature results in higher wall pressure disturbance amplitudes at the observation point.

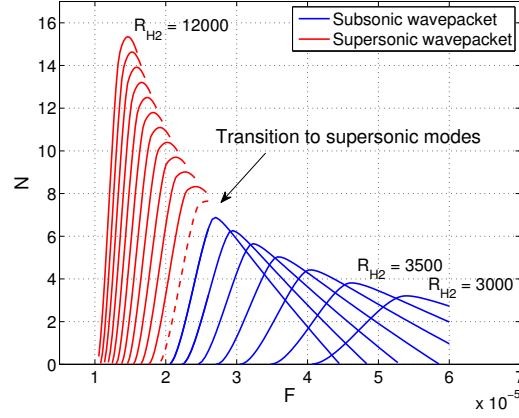


Figure 4.10: N as a function of F for a range of fixed Reynolds numbers (in increments of $R_{H2} = 500$). The red curves correspond to wave packets containing supersonic modes; the blue curves are purely subsonic. The dashed red curve corresponds to the region of transition from purely subsonic to supersonic.

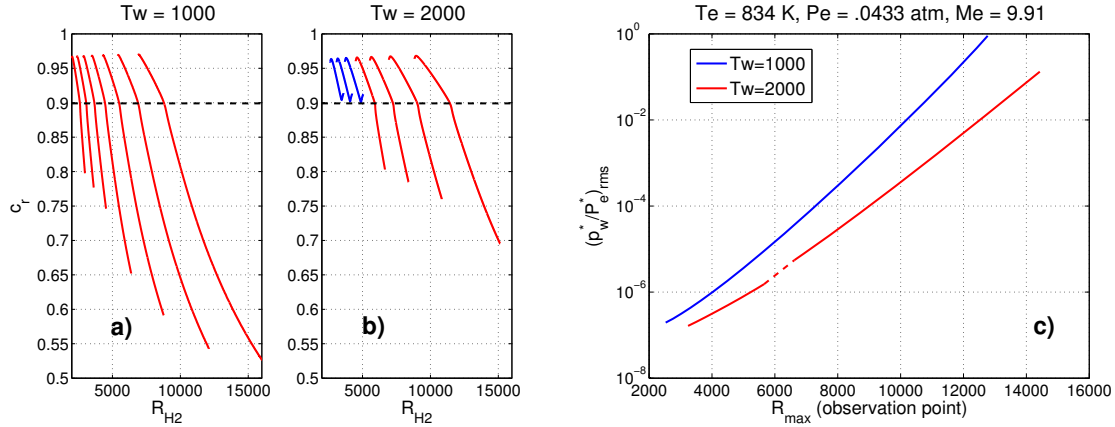


Figure 4.11: Results for ‘Case 3’: Phase speed c_r for a range of frequency parameters for a) $T_w = 1000$ K, b) $T_w = 2000$ K. For $T_w = 1000$, the curves from left to right correspond to $F \times 10^6 = (1) 67.5, (2) 57.1, (3) 47.7, (4) 39.3, (5) 31.8, (6) 25.4, (7) 19.9$. For $T_w = 2000$, the curves from left to right correspond to $F \times 10^6 = (1) 46.2, (2) 39.1, (3) 32.8, (4) 27.1, (5) 22.0, (6) 17.7, (7) 13.9$. c) RMS of the wall pressure perturbation at the observation point.

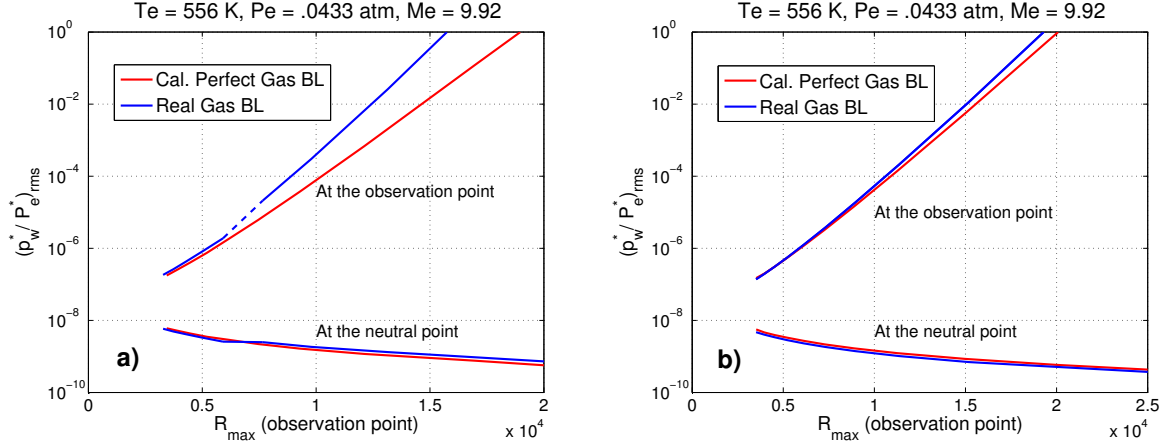


Figure 4.12: ‘Case 2’: RMS of the wall pressure perturbation at the observation point and the “initial” amplitudes at the lower neutral curve; (a) $T_w = 1000$ K ; (b) $T_w = 2000$ K.

4.3.2 “Case 2”, $T_e = 556$ K, $P_e = 0.0433$ atm, $U_e = 4.668$ km/s, $M_e = 9.92$

We now consider ‘Case 2’ which corresponds to the boundary-layer edge parameters $T_e = 556$ K, $P_e = 0.0433$ atm, and $U_e = 4.668$ km/s. The total enthalpy for this flow is (11.14 MJ/kg). These flow parameters lead to frozen Mach number $M_e = 9.92$, frozen specific heats ratio $\gamma_f = 1.38$, and frozen Prandtl number $Pr_f = 0.70$.

Figure 4.12 shows the root mean square of wall pressure perturbations (normalized by the edge pressure P_e) at the observation point R_{max} . Also shown is the amplitude of the wave packet rescaled to the neutral point, as was done for ‘Case 3’. Note that the real gas model results in wave packets containing supersonic modes, resulting in significantly higher amplification than in the calorically perfect gas case. For $T_w = 2000$ (Fig. 4.12b), no supersonic modes are present in either case. The ‘initial’ amplitudes are similar between the two cases because the mean flow profiles at the edge of the boundary layer are about the same. Despite the absence of supersonic modes, the lower temperatures that result from the inclusion of real gas effects lead to higher amplification of the pressure disturbance for air.

Figure 4.13 shows N factor plots computed using the air model for a range of frequencies with wall temperature $T_w = 1000$ K. Regions where the mode is traveling supersonically are indicated in red. This gives some sense of the influence that the transition to supersonic modes (and the tail feature that results in the spectrum) has on the amplification of the disturbance.

4.3.3 “Case 1”, $T_e = 278$ K, $P_e = 0.0433$ atm, $U_e = 3.351$ km/s, $M_e = 10$

Finally, we present results for ‘Case 1’, which corresponds to the boundary-layer edge parameters $T_e = 278$ K, $P_e = 0.0433$ atm, and $U_e = 3.351$ km/s. This case represents a flow with more moderate total enthalpy 5.6 MJ/kg. The flow parameters lead to frozen Mach number

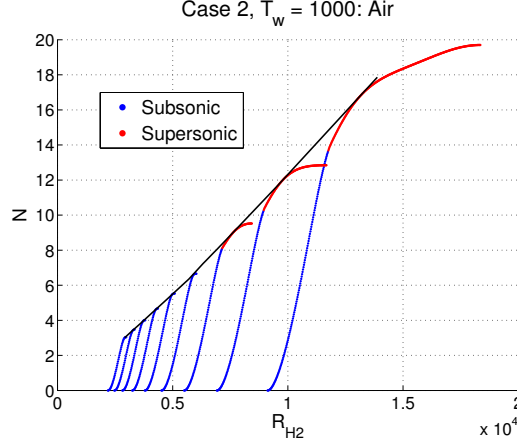


Figure 4.13: ‘Case 2’, $T_w = 2000$ K: N factor curves computed using the real gas model for a range of frequencies. Regions where the mode is supersonic are indicated in red.

$M_e = 10$, frozen specific heats ratio $\gamma_f = 1.40$, and frozen Prandtl number $Pr_f = 0.715$. For ‘Case 1’, the computations do not show the presence of supersonic modes in the case of air or calorically perfect gas for either of the two wall temperatures considered ($T_w = 1000$ and 2000 K). The amplitudes of the wall pressure perturbations are shown in Fig. 4.14. For wall temperature $T_w = 1000$ K (Fig. 4.14a), the results are in line with what we have observed in earlier purely subsonic cases. Note that the amplification is again slightly higher for air.

However, for $T_w = 2000$ K, the calorically perfect gas model leads to behavior that has not yet been considered in the previous cases. Figure 4.15a shows growth rates computed using the calorically perfect gas model for two representative frequencies. Note that for the lower frequency $F = 11.5e - 06$ (blue curve), the first Mack mode becomes unstable, albeit only slightly. Despite the elegance of the asymptotic approach utilized in this work, it is not clear how to apply the method to compute wave packet amplitudes in this case. Thus, only results corresponding to the real gas model are shown in figure 4.14b. Use of a numerical rather than asymptotic approach for evaluating wave packet integrals, such as the one employed recently by Luchini [Luc17] for incompressible gas, may mitigate this issue. Such an approach will be employed in future work in order to overcome this obstacle and to better resolve the subsonic-supersonic transition region seen in earlier cases.

4.4 Conclusion

This paper provides the first insight into real gas effects on receptivity of high-speed boundary layers to kinetic fluctuations. Because the edge temperatures $T_e = 834$ K, $T_e = 556$ K, and $T_e = 278$ K and the wall temperatures $T_w = 1000$ K and 2000 K are low enough so as not to yield very high temperatures inside the boundary layer, the high-enthalpy flows considered do not exhibit significant dissociation. The chosen pressure $P_e = 0.0433$ atm is also too low to

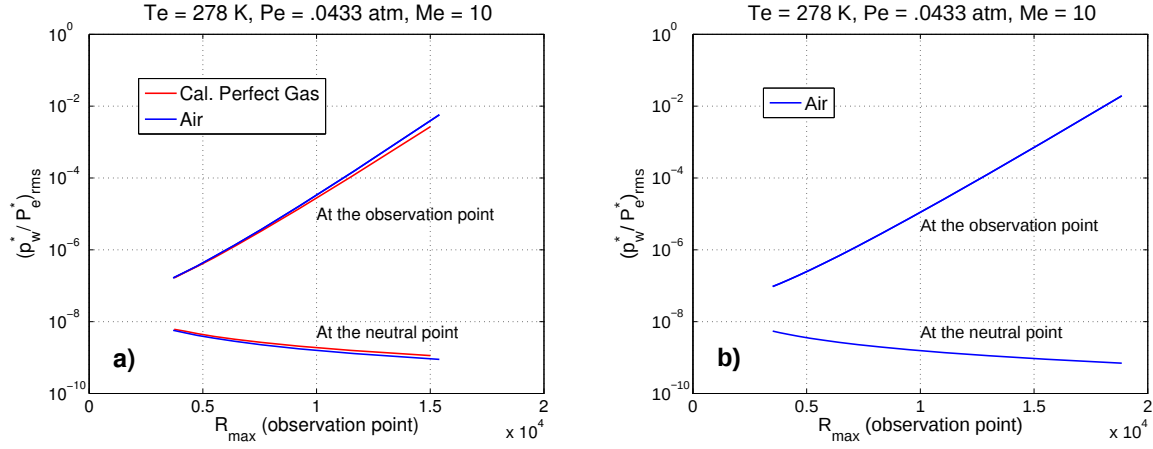


Figure 4.14: ‘Case 1’: RMS of the wall pressure perturbation at the observation point and the “initial” amplitudes at the lower neutral curve; a) $T_w = 1000$ K; b) $T_w = 2000$ K

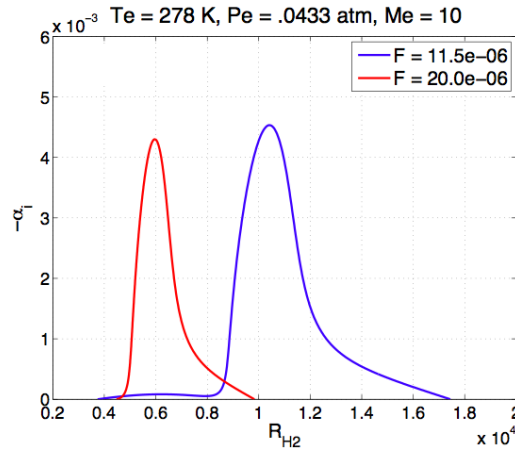


Figure 4.15: ‘Case 1’: $-\alpha_i H_2$ at $T_w = 2000$ K for two representative frequencies based on the calorically perfect gas mean flow model.

support dissociation. Therefore, the variation of the specific heats ratio and Prandtl number through the BL is not significant. We are thus able to explore real gas effects by considering two different mean flow models: a calorically perfect gas model and a 5-species model for air (referred to as real gas). The main real gas effect on the mean flow is attributed to the dependence of the specific heat on temperature which in general yields a lower temperature profile through the boundary layer. This work confirms the destabilizing effect that lower temperatures (both on the wall and through the BL) are known to have on the Mack 2nd mode. This work also suggests that real gas effects may lead to earlier transition of the 2nd mode from subsonic to supersonic with respect to the free stream.

The stability and receptivity analysis has been carried out using a solver for calorically perfect gas with effective values of the specific heats ratio and Prandtl number. The flow is most receptive to kinetic fluctuations (thermal noise) at the lower branch of the neutral curve, where the 2nd Mack modes have phase speed $c_r > 1 - 1/M_e$. At large Mach numbers the critical layers at the lower branch of the neutral curve are located close to the edge of the boundary layer. Because the receptivity mechanism of high-speed boundary layers to kinetic fluctuations is determined in the vicinity of the critical layer, the initial amplitudes of the unstable modes are about the same because the mean flow profiles are similar at the edge of the boundary layer.

It was found that there are unstable supersonic 2nd Mack modes in the high-enthalpy flow at the temperature ratio $T_w/T_e \approx 2.4 > 1$, in contrast to recent publications emphasizing the role of a very cold wall ($T_w/T_e < 1$). Actually, this case is more relevant to future hypersonic vehicles with high lift-to-drag ratio (≥ 3). To achieve high lift-to-drag ratio, the design will be constrained by low angles of attack and relatively small nose and leading edge radii. Therefore, the shock waves will be relatively weak, and the edge temperatures T_e will not be very high, leading to higher wall to edge temperature ratios, such as those considered here.

High-speed boundary layers are most receptive to kinetic fluctuations at the lower branch of the neutral curve,[FT17] and the initial amplitudes of the unstable modes are governed by the boundary layer edge parameters, whereas the wave packet amplitude downstream is dependent on the transition to unstable supersonic modes. For the cases considered, supersonic modes, when present, appeared for lower Reynolds numbers for the real gas boundary layer mean flow model than the calorically perfect gas model.

Figure 4.16 shows stability results at $T_w = 1000$ K using the real gas model for different frequency parameters in the complex phase velocity plane $c = c_r + ic_i$. In this case, an increase in the Reynolds number corresponds to a decrease in c_r . The figure also shows the branch cut for the slow acoustic mode. After transition from $c_r > 1 - 1/M_e$ to $c_r < 1 - 1/M_e$, the supersonic discrete modes remain in the vicinity of the slow acoustic modes. Chuvakhov and Fedorov[PVC16] showed that wave packets of the supersonic 2nd Mack modes may emit sound. This phenomenon may lead to a change in the dynamics of the wave packet, which suggests that the evaluation of wave packet amplitudes where supersonic modes are involved should be considered with some reservation in accordance with the results of Ref.[PVC16].

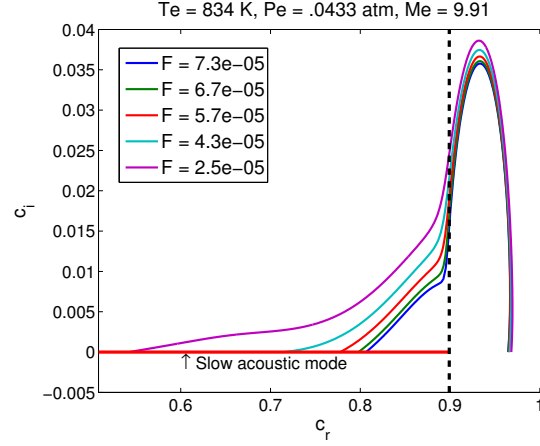


Figure 4.16: Eigenvalues in complex plane $c = c_r + ic_i$ for ‘Case 3’ at $T_w = 1000$ K using the real gas model. Increase in R_{H2} corresponds to decrease in c_r .

4.5 Appendix

4.5.1 Boundary-layer equations for a reacting gas and the numerical method

The governing equations

The underlying assumptions are the following [Dor62]:

1. “All gas species considered behave as perfect-gas species.”
2. “The flow is in a steady state.”
3. “The effects of radiation from the gas to the surface and from the surface to the gas are neglected.”
4. “The flow is two-dimensional or axially symmetric.”
5. The flow is in thermal equilibrium (internal degrees of freedom are in equilibrium with the translational and rotational degrees of freedom).

The x -momentum equation:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_e}{dx} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right), \quad (4.2)$$

where u and v are x - and y - velocity components, respectively; ρ is density of the mixture, and p is the pressure.

The mass conservation equation:

$$\frac{\partial \rho u r_0^j}{\partial x} + \frac{\partial \rho v r_0^j}{\partial y} = 0, \quad (4.3)$$

where $r_0(x)$ is the radius of the body in the axisymmetric flow case; $j = 0$ for two-dimensional flow, and $j = 1$ for axisymmetric flow.

The species conservation equation:

$$\rho u \frac{\partial c_s}{\partial x} + \rho v \frac{\partial c_s}{\partial y} = \frac{\partial}{\partial y} \left(\rho D_{sm} \frac{\partial c_s}{\partial y} \right) + \dot{W}_s, \quad (4.4)$$

where D_{sm} is the diffusion coefficient of species s in the gas mixture. $\dot{W}_s$ in Eq. (4.4) is the net mass rate of production of species s due to chemical reactions (source term). The source term can be written as follows:

$$\dot{W}_s = M_s \frac{d[s]}{dt}. \quad (4.5)$$

The energy conservation equation:

$$\begin{aligned} \rho u \frac{\partial I}{\partial x} + \rho v \frac{\partial I}{\partial y} &= \frac{\partial}{\partial y} \left[\frac{\mu}{Pr} \frac{\partial I}{\partial y} + \mu \left(1 - \frac{1}{Pr} \right) \frac{1}{2} \frac{\partial u^2}{\partial y} \right] \\ &- \frac{\partial}{\partial y} \left[\rho \sum_s \left(\frac{1}{Le_s} - 1 \right) D_{sm} h_s \frac{\partial c_s}{\partial y} \right], \end{aligned} \quad (4.6)$$

where Pr is the frozen Prandtl number and Le_s is the frozen Lewis number defined with the diffusion coefficient D_{sm} , respectively. The total enthalpy I is defined as follows:

$$I = h + \frac{u^2}{2}, \quad (4.7)$$

$$h = \sum_s c_s h_s, \quad (4.8)$$

$$h_s = \int_0^T C_{ps} dT + h_s^0,$$

where C_{ps} is the specific heat of species s at constant pressure, and h_s^0 is the heat formation of species s .

It is convenient to analyze the boundary-layer equations using the Levy-Lees-Dorodnitsyn transformation,

$$\xi = \int_0^x \rho_e u_e \mu_e r_0^{2j} dx, \quad \eta = \frac{\rho_e u_e r_0^j}{\sqrt{2\xi}} \int_0^y \frac{\rho}{\rho_e} dy. \quad (4.9)$$

The mass conservation equation (4.3) is excluded by using the stream function:

$$\begin{aligned}\psi(\xi, \eta) &= \frac{\sqrt{2\xi}}{p_e r_0^j} f(\xi, \eta), \\ \frac{u}{u_e} &= f'(\xi, \eta),\end{aligned}\tag{4.10}$$

where prime indicates the partial derivative with respect to η .

The momentum equation:

$$\begin{aligned}(Cf'')' + ff'' + \frac{2\xi}{u_e} \frac{du_e}{d\xi} \left[\frac{\rho_e}{\rho} - (f')^2 \right] &= 2\xi \left(f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right), \\ C &= \frac{\rho\mu}{\rho_e\mu_e}.\end{aligned}\tag{4.11}$$

The species conservation equation:

We introduce the species mass fraction distribution:

$$q_s(\xi, \eta) = \frac{c_s}{c_{se}}.\tag{4.12}$$

The transformed Eq. (4.4) is recast in the following form:

$$\begin{aligned}2\xi \left(\frac{\partial q_s}{\partial \xi} f' - q'_s \frac{\partial f}{\partial \xi} \right) &= \left(\frac{C}{Sm_s} q'_s \right)' + q'_s f \\ &- \frac{2\xi}{c_{se}} \frac{dc_{se}}{d\xi} f' + \frac{2\xi \dot{W}_s}{\rho \rho_e u_e^2 \mu_e r_0^{2k} c_{se}}, \\ Sm_s &= \frac{\mu}{\rho D_{sm}} = \frac{Pr}{Le_s} \quad \text{--the Schmidt number.}\end{aligned}\tag{4.13}$$

The energy equation

We introduce dimensionless total enthalpy

$$g(\xi, \eta) = \frac{I}{I_e},\tag{4.14}$$

and recast Eq. (4.6) using the new dependent and independent variables as follows:

$$\begin{aligned}\left(\frac{C}{Pr} g' \right)' + fg' &= \frac{2\xi}{I_e} \frac{dI_e}{d\xi} g f' + \left[\sum_s \frac{C}{Sm_s} \left(\frac{1}{Le_s} - 1 \right) \frac{h_s c_{se}}{I_e} q'_s \right]' \\ &+ \frac{u_e^2}{I_e} \left[\left(\frac{1}{Pr} - 1 \right) C f' f'' \right]' + 2\xi \left(f' \frac{\partial g}{\partial \xi} - g' \frac{\partial f}{\partial \xi} \right).\end{aligned}\tag{4.15}$$

We have to discuss the boundary conditions for Eqs. (4.11), (4.13), and (4.15). At the edge of the boundary layer, we have:

$$\begin{aligned}\eta \rightarrow \infty : \quad f' &\rightarrow 1, \\ g &\rightarrow 1, \\ q &\rightarrow 1.\end{aligned}\tag{4.16}$$

At the wall, the no-slip condition is recast as

$$\eta = 0 : \quad f = f' = 0. \quad (4.17)$$

For the energy equation, the wall boundary conditions can have different forms. For example, the wall temperature, T_w , is prescribed

$$\eta = 0 : \quad T = T_w. \quad (4.18)$$

Another option is to define the heat flux at the wall:

$$\eta = 0 : \quad q_{flux} = q_{fluxwall}. \quad (4.19)$$

In the case of a reacting gas, the heat flux at the wall depends on thermal conductivity and diffusion of species with their recombination on the wall.

$$q_{flux} = -k \frac{\partial T}{\partial y} - \rho \sum_s D_{sm} h_s \frac{\partial c_s}{\partial y}. \quad (4.20)$$

We need the temperature derivative at the wall in order to find the heat flux (4.20). Using the total enthalpy definition,

$$I = \sum_s c_s h_s + \frac{u^2}{2}, \quad (4.21)$$

one can find

$$\left(\frac{\partial I}{\partial y} \right)_w = \sum_s \left[\left(\frac{\partial c_s}{\partial y} \right)_w h_{sw} + c_{sw} \left(\frac{\partial h_s}{\partial y} \right)_w \right]. \quad (4.22)$$

Using new variables (5.1), we recast Eq. (4.22):

$$\frac{\rho_w u_e r_0^j}{\sqrt{2\xi}} \left(\frac{\partial I}{\partial \eta} \right)_w = \frac{\rho_w u_e r_0^j}{\sqrt{2\xi}} \sum_s \left[\left(\frac{\partial c_1}{\partial \eta} \right)_w h_{1w} + c_{1w} \left(\frac{\partial h_1}{\partial \eta} \right)_w \right].$$

Derivatives of the species enthalpies in Eq. (4.23) can be found as

$$\left(\frac{\partial h_s}{\partial y} \right)_w = C_{psw} \left(\frac{\partial T}{\partial y} \right)_w. \quad (4.23)$$

Taking into account that the frozen specific heat of the mixture is $C_{pf} = \sum_s c_s C_{ps}$, we arrive at the result:

$$\left(\frac{\partial T}{\partial y} \right)_w = \frac{\rho_w u_e r_0^j}{C_{pfw} \sqrt{2\xi}} \left[\left(\frac{\partial I}{\partial \eta} \right)_w - \sum_s \left(\frac{\partial c_s}{\partial \eta} \right)_w h_{sw} \right]. \quad (4.24)$$

Using the temperature derivative, one can find the heat flux (4.20).

$$q_{flux} = -\frac{\mu_w \rho_w u_e r_0^j}{Pr_w \sqrt{2\xi}} \left[\left(\frac{\partial I}{\partial \eta} \right)_w + \sum_s (Le_s - 1) \left(\frac{\partial c_s}{\partial \eta} \right)_w h_{sw} \right]. \quad (4.25)$$

In addition, we need a wall boundary condition for mass fraction of species s . The boundary condition also can be formulated in different ways, depending on the wall properties. For example, one can consider an *equilibrium wall condition*. In this case, the mass fractions at the wall are to be determined according to the wall temperature and the pressure. Another option is to consider a *fully catalytic wall*. This boundary condition means that all atoms recombine at the wall, and the mass fraction of atoms at the wall is equal to zero. In practical applications, the wall temperature is low, and there is not a significant difference between the equilibrium wall condition and the fully catalytic wall condition because the mass fraction of atoms is close to zero when temperature is low.

It is beneficial to have a *non-catalytic wall*. In this case, atoms do not recombine at the wall and there is no diffusion toward the wall. This boundary condition corresponds to

$$\eta = 0 : \quad \left(\frac{\partial c_s}{\partial y} \right)_w = 0, \quad (4.26)$$

and there is no additional input into the heat flux (4.20) due to the diffusion.

However, real walls are not ideal, and there is a finite recombination rate of atoms at the wall (*partially catalytic wall*). The boundary condition is formulated for mass fractions of atoms c_2 and c_4 (nitrogen and oxygen, respectively) as follows:

$$\eta = 0 : \quad D_{2mw} \left(\frac{\partial c_2}{\partial y} \right)_w = K_{RN} c_{2w}, \quad (4.27)$$

$$\eta = 0 : \quad D_{4mw} \left(\frac{\partial c_4}{\partial y} \right)_w = K_{RO} c_{4w}, \quad (4.28)$$

where K_{RN} and K_{RO} are the catalytic recombination rates (velocity units) of nitrogen and oxygen, respectively. In the limit $K_{RS} \rightarrow 0$, we have a non-catalytic wall. In the limit $K_{RS} \rightarrow \infty$, we have the fully catalytic wall boundary condition ($c_{2w} \rightarrow 0, c_{4w} \rightarrow 0$).

Usually it is assumed that there is no catalytic effect on molecules NO ($s = 5$):

$$\eta = 0 : \quad \left(\frac{\partial c_5}{\partial y} \right)_w = 0. \quad (4.29)$$

Therefore, conservation of oxygen and nitrogen atoms on the wall requires:

$$\eta = 0 : \quad \left(\frac{\partial c_1}{\partial y} \right)_w + \left(\frac{\partial c_2}{\partial y} \right)_w = 0, \quad (4.30)$$

$$\eta = 0 : \quad \left(\frac{\partial c_3}{\partial y} \right)_w + \left(\frac{\partial c_4}{\partial y} \right)_w = 0, \quad (4.31)$$

the equations (4.29), (4.30), and (4.31) are used to formulate boundary conditions for molecules NO , N_2 , and O_2 .

Using variables (ξ, η) , we arrive at

$$\begin{aligned}
\frac{\rho_w u_e r_0^j}{\sqrt{2\xi}} (D_{2m})_w c_{2e} Q d_{2w} &= K_{RN} c_{2e} Q s_{2w}, \\
\frac{\rho_w u_e r_0^j}{K_{RN} \sqrt{2\xi}} (D_{2m})_w Q d_{2w} &= Q s_{2w}, \\
\frac{\mu_w u_e r_0^j}{Sm_{2w} K_{RN} \sqrt{2\xi}} Q d_{2w} &= Q s_{2w}, \\
\gamma_{3N} Q d_{2w} &= Q s_{2w}, \\
\gamma_{3N} &= \frac{\mu_w u_e r_0^j}{Sm_{2w} K_{RN} \sqrt{2\xi}},
\end{aligned} \tag{4.32}$$

where

$$\begin{aligned}
Q d_{2w} &= \left(\frac{\partial q_2}{\partial \eta} \right)_w, \\
Q s_{2w} &= (q_2)_w.
\end{aligned}$$

$$\begin{aligned}
\frac{\rho_w u_e r_0^j}{\sqrt{2\xi}} (D_{4m})_w c_{4e} Q d_{4w} &= K_{RO} c_{4e} Q s_{4w}, \\
\frac{\rho_w u_e r_0^j}{K_{RO} \sqrt{2\xi}} (D_{4m})_w Q d_{4w} &= Q s_{4w}, \\
\frac{\mu_w u_e r_0^j}{Sm_{4w} K_{RO} \sqrt{2\xi}} Q d_{4w} &= Q s_{4w}, \\
\gamma_{3O} Q d_{4w} &= Q s_{4w}, \\
\gamma_{3O} &= \frac{\mu_w u_e r_0^j}{Sm_{4w} K_{RO} \sqrt{2\xi}},
\end{aligned} \tag{4.33}$$

where

$$\begin{aligned}
Q d_{4w} &= \left(\frac{\partial q_4}{\partial \eta} \right)_w, \\
Q s_{4w} &= (q_4)_w.
\end{aligned}$$

In the case of a sharp leading edge, $\gamma_{3s} \rightarrow \infty$ at $\xi \rightarrow 0$, and the boundary conditions (4.32) and (4.33) become equivalent to (4.26), i.e. to a noncatalytic wall. At $\xi \rightarrow \infty$, $\gamma_{3s} \rightarrow 0$. In other words, at the finite catalytic recombination rate, the solution is similar to a non-catalytic wall ($K_{Rs} \rightarrow 0$) and far downstream it is similar to a fully catalytic wall ($K_{Rs} \rightarrow \infty$).

In case of a blunt body, γ_{3s} has a finite limit at $x \rightarrow 0$.

$$\begin{aligned}\xi &= \int_0^x \rho_e u_e \mu_e r_0^{2j} dx \rightarrow \int_0^x \rho_{e0} \left(\frac{du_e}{dx} \right)_0 \mu_{e0} x^{1+2j} dx \\ &= \rho_{e0} \left(\frac{du_e}{dx} \right)_0 \mu_{e0} \frac{x^{2+2j}}{2+2j},\end{aligned}\tag{4.34}$$

$$\gamma_{3s} \rightarrow \frac{\mu_w \left(\frac{du_e}{dx} \right)_0 x^{1+j}}{\text{Sm}_{sw} K_{Rs} \sqrt{2 \rho_{e0} \left(\frac{du_e}{dx} \right)_0 \mu_{e0} \frac{x^{2+2j}}{2+2j}}} = \frac{\mu_w \sqrt{\left(\frac{du_e}{dx} \right)_0} \sqrt{1+j}}{\text{Sm}_{sw} K_{Rs} \sqrt{\rho_{e0} \mu_{e0}}}.\tag{4.35}$$

Discretized equations

The computational domain in ξ and η is discretized, and a second-order finite-difference approximation for Eqs. (4.11), (4.13), and (4.15) similar to Ref. [Ceb99] is utilized. The grid points are defined as

$$\begin{aligned}\xi^0 &= 0, \xi^n = \xi^{n-1} + k_n, \quad n = 1, 2, \dots, N; \\ \eta_0 &= 0, \eta_j = \eta_{j-1} + h_j, \quad j = 1, 2, \dots, J,\end{aligned}\tag{4.36}$$

where h_j stands for the grid step size (not to be confused with enthalpy). In the present work, the coordinate η is stretched according to Ref. [Ceb99]:

$$\begin{aligned}\eta_j &= \eta_{j-1} + h_{j-1}, \\ h_j &= K h_{j-1},\end{aligned}\tag{4.37}$$

where K is the stretching parameter ($K > 1$). Thus the grid is determined by N , K , h_1 , and by the end coordinate η_e . The number of grid points J is determined according to

$$J = \frac{\ln [1 + (K-1)(\eta_e/h_1)]}{\ln K} + 1 \quad (K > 1),\tag{4.38}$$

$$J = \frac{\eta_e}{h_1} + 1 \quad (K = 1).\tag{4.39}$$

The sum of mass fractions for all species is 1. Therefore, we have:

$$c_{1e}q_1 + c_{2e}q_2 + c_{3e}q_3 + c_{4e}q_4 + c_{5e}q_5 = 1.\tag{4.40}$$

Because we use the global mass conservation law for the mixture and introduced the stream function, we can exclude one of the species. In this work, we exclude nitrogen molecules and find the mass fraction of nitrogen molecules as

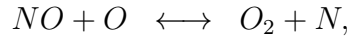
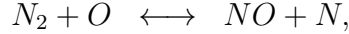
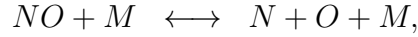
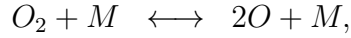
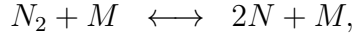
$$c_{1e}q_1 = 1 - (c_{2e}q_2 + c_{3e}q_3 + c_{4e}q_4 + c_{5e}q_5).\tag{4.41}$$

The code provides options of using two different models for each of the thermodynamic and transport properties. For each property, one option largely follows Gupta et al. [GYTL90] and the other follows the model used by Hudson [Hud96].

As in the formulation of the thermodynamic and transport properties, we consider two different models for calculating the chemical source term. The first model follows Gupta et al. [GYTL90], and the second follows Park [Par85]. The chemical source term W_s is the net rate of mass production of species s per unit volume, or the sum of the rate of production due to each reaction r :

$$W_s = M_s \sum_{r=1}^{NR} \left(\frac{d[s]}{dt} \right)_r. \quad (4.42)$$

Here, NR is the number of reactions and $[s]$ denotes the molar concentration of species s . For the five-species air model, the reactions that are considered are



where M is the third body involved in the reaction. In the following formulations for the source terms, the reactions are numbered from $r = 1$ to $r = 5$ in the order given above.

Chapter 5

Analysis of Receptivity to Kinetic Fluctuations in the Reentry-F Flight Experiment

5.1 Introduction

Bushnel[Bus90] summarized potential sources for the generation of unstable modes in boundary layers at flight conditions. They are: (a) free stream fluctuations (incoming vorticity, entropy, and acoustic disturbances), (b) particulates (ice clouds, volcanic dust, rocket exhaust particles etc.), (c) electrostatic surface charging, (d) acoustic radiation generated by the vehicle, (e) roughness, and (f) Brownian motion, which is always present in a gas at finite temperature.

In fact, ‘Brownian motion’ is not an accurate term for describing the mechanism that Bushnel was referring to. Brownian motion, first characterized in the 19th century, refers to the motion of relatively large particles due to the chaotic behavior of molecules in the surrounding gas or liquid. Brownian motion is a result of kinetic fluctuations (KF) on the molecular (microscale) level that manifest themselves on a mesoscale. These kinetic fluctuations may cause fluctuating dissipative fluxes that lead to the generation of unstable modes in the boundary layer on a hydrodynamic scale.

Recent studies[ZR84, Luc08b, FA10, Luc10, Fed12, Fed15, Luc17, FT17] have demonstrated that KF could be associated with laminar-turbulent transition in quiet wind tunnels and in flight experiments. Because KF are indispensable in virtually any environment of physical interest, analysis of the KF-induced transition provides an upper bound on laminar-turbulent transition estimates in the absence of other external disturbances (or near absence - path A in the terminology of Morkovin, Reshotko, and Herbert[MRH94]). If the analysis leads to the conclusion that the KF mechanism does not play a significant role for a particular experiment, it can be reasonably concluded that transition was caused by other sources of unstable mode generation in the boundary layer.

Wright and Zoby[WZ77] reported laminar-turbulent transition data for a flight test with

a 5-degree half-angle cone with an ablative nose section. The experimental conditions, the transition data, and their analyses are discussed by Schneider in Ref.[Sch99]. Malik[Mal03] analyzed the stability of the boundary layer flow on the cone using two gas flow models, equilibrium gas and finite rate chemistry. He performed computations corresponding to the flight conditions at altitude 30.48 km (100,000 ft) and Mach number $M = 20$, assuming that the nose radius was not distorted by ablation. The results revealed that the second Mack mode corresponding to frequencies 200 to 480 kHz could have been responsible for the transition from laminar to turbulent flow. (Note that the locus associated with the onset of transition reported in Ref. [WZ77] and used by Malik [Mal03] was based on smoothed heating rate measurements taken at 21 stations along the test body.) In particular, the mode of frequency 240 kHz could be associated with the onset of transition, which occurred at $x = 2.9$ m. The N -factor at this location was found to be about 7.9 using the quasi-parallel flow assumption (LST) and 9.8 using the parabolized stability equations (PSE). The purpose of the present work is to determine whether or not the KF-mechanism could play a role in laminar-turbulent transition at conditions corresponding to the reentry flight experiment[WZ77]. We present results for the case considered by Malik[Mal03] ($H = 30.48$ km), as well as for altitude $H = 25.0$ km.

5.2 Results

5.2.1 Mean flow

The experiment[WZ77] utilized a cone with a 5-degree half-angle and a blunted nose with a radius of 3.1 mm. At the stagnation point, the wall temperature was about 4000 K (estimated by Zoby) and it dropped down to about 400 K at $x \approx 0.4$ m. At $x \geq 0.4$ m, the measured wall temperature (reported in Ref.[How71] and reproduced in Ref.[Mal03]) was approximately constant and equal to 400 K. The results of Ref.[Mal03] using finite rate chemistry indicate that the dissociation of oxygen is negligible at $x > 1$ m.

For the purpose of the present work, we consider a sharp cone with a 5-degree half-angle and constant wall temperature $T = 400$ K. The solution for inviscid conical flow at free stream conditions corresponding to the standard atmosphere at the respective altitude leads to the boundary layer edge parameters given in the following table:

Altitude	T_e (K)	U_e (m/s)	P_e (atm)	ρ_e (kg/m)	M_f
30.48 km	420.6	6.01×10^3	0.0613	5.11×10^{-2}	14.61
25.00 km	410.7	5.94×10^3	0.1412	1.21×10^{-1}	14.61

Table 5.1: The edge flow parameters corresponding to altitudes $H = 30.48$ km and $H = 25.00$ km.

The boundary layer solver used in this work utilizes a five species model of air. One can find the boundary layer equations for a reacting gas in monographs [Dor62, AVGP76, Lun09]. The numerical method is similar to that described by Cebeci[Ceb99] for non-reacting flows

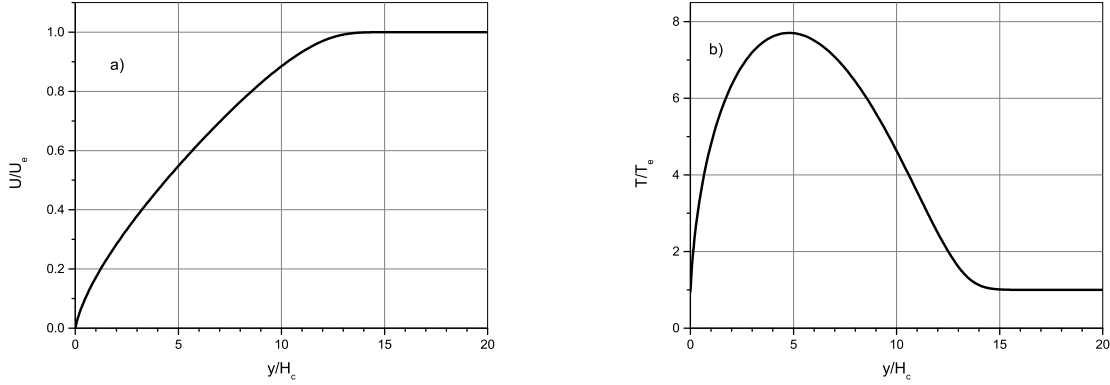


Figure 5.1: Boundary layer mean velocity (a) and temperature (b) profiles.

and in Refs. [Ulk10, TUK12] for five species air and binary mixtures. The coordinates x and y represent the distance from the cone tip and wall-normal distance, respectively. The governing equations and details of the numerical algorithm are provided in Ref. [ET17b]. Because the temperature in the boundary layer is relatively low for the cases considered here, there is essentially no dissociation, and the boundary layer flow is self-similar. In order to use the five species solver at such low temperatures, small nonzero mass fractions (10^{-6}) of atoms N , O , and of molecules NO were assigned at the boundary layer edge. The mass fractions of molecules N_2 and O_2 were close to 0.78 and 0.22, respectively. The thermodynamic and transport properties of the five species flow were determined using the models reported in Ref.[GYTL90] A transformation from x and y to the Levy-Lees-Dorodnitsyn variables ξ and η was utilized. The transformations are given by

$$\xi = \int_0^x \rho_e u_e \mu_e r_0^{2j} dx, \quad \eta = \frac{\rho_e u_e r_0^j}{\sqrt{2\xi}} \int_0^y \frac{\rho}{\rho_e} dy, \quad (5.1)$$

where ρ and u are density and stream-wise velocity, respectively, $r_0(x)$ is the radius of the body in the axisymmetric flow case, and the subscript ‘ e ’ indicates a boundary layer edge flow parameter. For two-dimensional flow we set $j = 0$, and for axisymmetric flow (as in the present case) we set $j = 1$. The results presented herein were obtained using a uniform grid with $\Delta\eta = 0.015$ on the interval $0 \leq \eta \leq \eta_{max} = 10$, and with step $\Delta x = 0.5$ mm.

Figure 1 shows the dimensionless velocity and temperature profiles with respect to dimensionless wall-normal distance y/H_c , where $H_c = (2\mu_e x / 3\rho_e U_e)^{1/2}$. At $x = 3.261$ m (10.7 ft), $H_c = 0.0414$ cm (0.001358 ft), and the dimensionless boundary layer thickness is about 14, which corresponds to a dimensional thickness of about 0.19 ft. These results are consistent with the mean velocity and temperature profiles reported in Ref.[Mal03].

Despite the absence of dissociation in the present cases, real gas effects are still manifested through the dependence of the specific heat C_p on temperature. Figure 2 shows the distribution of the frozen specific heats ratio γ_f and frozen Prandtl number Pr_f across the boundary layer.

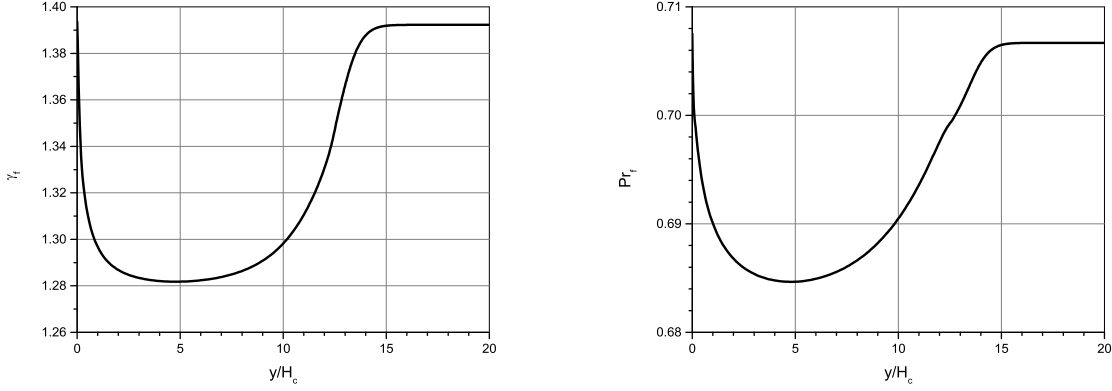


Figure 5.2: Frozen specific heats ratio (a) and Prandtl number (b) profiles in the boundary layer.

5.3 Stability analysis

5.3.1 Stability solver with effective Prandtl and specific heats ratio

In this approach, we follow the method of Ref.[ET17b] using the mean flow profiles obtained from a boundary layer solver for a 5-species air model. The stability and receptivity calculations are carried out using the stability solver described in Ref.[Tum07, FT17]. It is based on the stability equations for a calorically perfect gas with effective Prandtl number Pr_{eff} and effective specific heats ratio γ_{eff} .

In order to justify the use of γ_{eff} and Prandtl Pr_{eff} , we have explored the effects of varying these parameters across the range of relevant values. Figure 5.3 illustrates that such variations do not have a strong effect on the growth rate $\sigma = -\alpha_i H_c$ at frequency parameter $F = \omega \mu_e / (\rho_e U_e^2) = 0.30 \times 10^{-4}$. The Reynolds number R_c is defined using the length scale H_c .

5.3.2 Stability solver for 5-species air model

The stability solver for a 5-species reacting air model was developed in Ref.[KT13c]. To solve the governing equations numerically, we discretize the equations using fourth-order finite differences. A nonuniform mesh is utilized to obtain greater accuracy within the boundary layer. Following Ref. [Mal90], we let

$$y = \frac{a\eta}{b - \eta}, \quad (5.2)$$

where $a = y_{max} y_i / (y_{max} - 2y_i)$ and $b = 1 + a/y_{max}$, so that half of the grid points lie below $y = y_i$ and half above. Our grid is uniform in η , with N grid points, extending

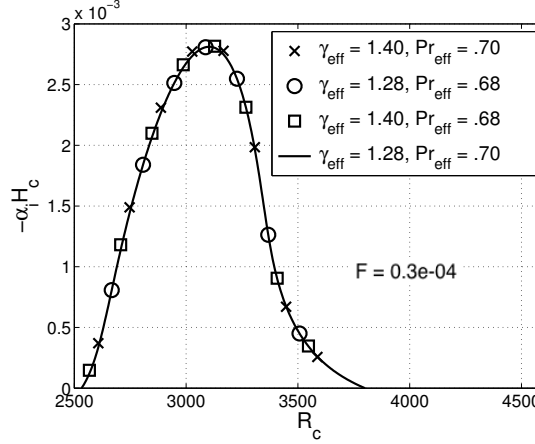


Figure 5.3: The effect of Pr_{eff} and γ_{eff} on the growth rate for $F = 0.30 \times 10^{-4}$.

from $\eta_1 = 0$ to $\eta_N = 1$. The coordinate y_{max} is chosen far away from the boundary layer edge. After experimental validation of convergence for the stability features in the range $F = 0.20 - 0.60 \times 10^{-4}$, the following grid parameters were chosen: $y_i = 6$, $y_{max} = 550$, $N = 601$.

Figure 5.4 illustrates the differences in growth rates obtained using the γ_{eff} and Pr_{eff} model with those obtained using the 5-species air model at two frequency parameters F .

Figure 5.5 illustrates the differences in N -factors obtained using the two different models. The N -factor is defined by $N = -\int_{x_0}^x \alpha_i^* dx^*$, where x_0 denotes the neutral point and '*' superscripts denote dimensional quantities. One can see that the discrepancies are not significant.

5.4 The potential effect of KF in the reentry-F flight experiment

The stability analysis presented in Sec. 5.3 indicates that the potential effect of kinetic fluctuations can be analyzed using the stability equations based on γ_{eff} and Pr_{eff} . Figure 6 shows N -factors versus dimensional distance from the cone tip for a range of dimensional frequencies. For $H = 30.48$ km, Malik[Mal03] reported that the LST approach led to $N = 7.9$ at $x = 2.9$ m, corresponding to the dominating frequency 240 kHz. These numbers agree well with our results, which show $N \approx 7.7$ at $x = 2.9$ m. The results reported by Malik in Ref.[Mal03] are also shown in Fig. 5.6 for comparison.

To estimate the potential role of thermal fluctuations in the laminar-turbulent transition for the reentry flight, we used the solver developed in Ref.[FT17] for calorically perfect gas to compute the magnitude of pressure fluctuations at the wall. Figure 5.7 shows *rms* wall pressure fluctuations scaled with the boundary layer edge pressure in the wave packet

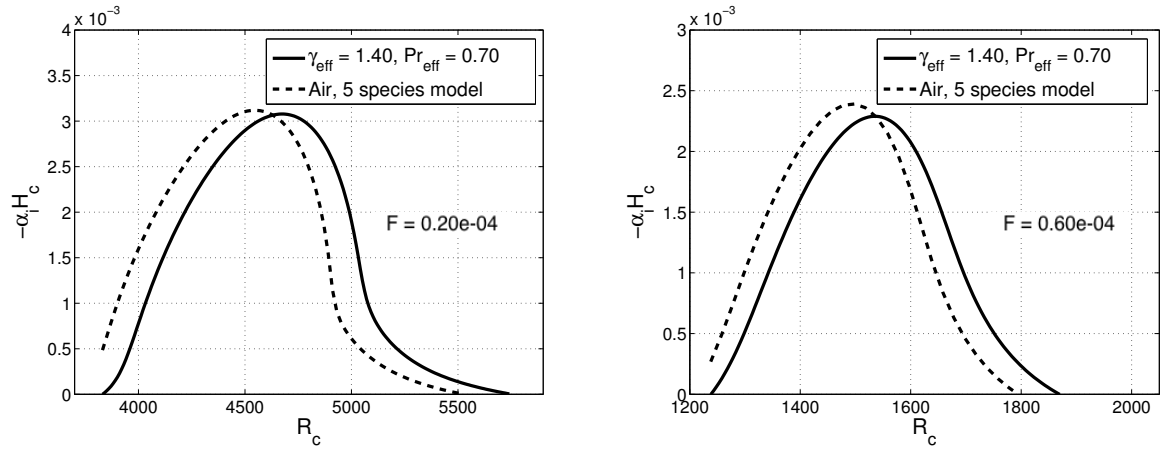


Figure 5.4: Comparisons of growth rates obtained using the γ_{eff} and Pr_{eff} model with those obtained using the 5-species air model.

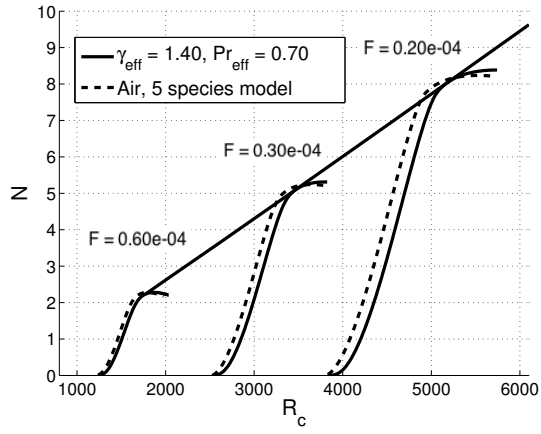


Figure 5.5: A comparison of the N -factors obtained using the γ_{eff} and Pr_{eff} model and those obtained using the 5-species air model.

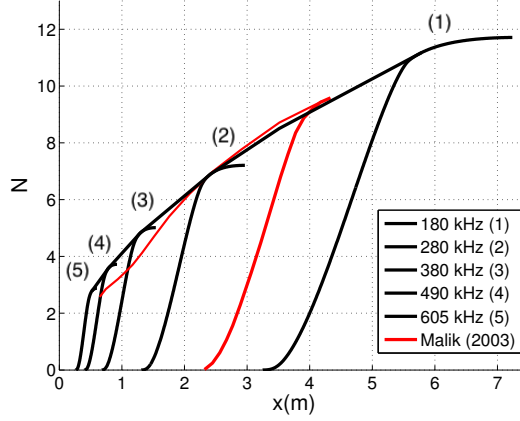


Figure 5.6: N -factors computed using γ_{eff} and Pr_{eff} using dimensional coordinate x and dimensional frequencies. There is good agreement between the present results and those reported by Malik in Ref.[Mal03].

generated by KF in the vicinity of the lower branch of the neutral curve for $H = 30.48$ km and $H = 25.00$ km. (Note that the results shown for $H = 20.00$ km are an estimate obtained by rescaling the x -coordinate of the results for $H = 25.00$ km with the appropriate free stream density. An explanation of the validity of this procedure is provided in Section IV.C of Ref. [FT17])

Typically, laminar-turbulent transition is associated with wall pressure fluctuations on the order of 10% of the pressure at the edge of the boundary layer, i.e. $(p_w^*/P_e^*)_{rms} = 10^{-1}$. One can see from Fig. 5.7 (and Table 5.2, which provides the corresponding numerical details) that, for this flight, the dimensionless wall pressure fluctuation at the observed transition locations are at least three orders of magnitude smaller than this threshold. We can thus conclude that KF could not have been the main mechanism responsible for transition.

Altitude	Observed trans. x_t (m)	$(p_w^*/P_e^*)_{rms}$ at x_t
30.48 km	2.92	1.70×10^{-5}
25.00 km	2.07	3.40×10^{-4}
20.00 km	0.87	3.00×10^{-4} (est.)

Table 5.2: The observed transition locations[WZ77] and the magnitude of the corresponding pressure fluctuations for each of three altitudes. The disturbance magnitude provided for $H = 20.0 \times 10^3$ km is an estimate obtained along the lines of Ref. [FT17]

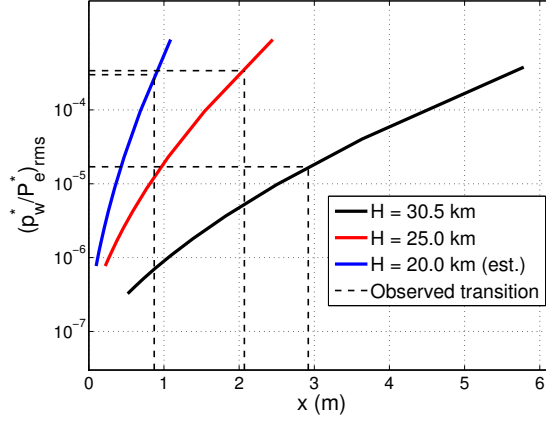


Figure 5.7: *Rms* of the wall pressure perturbation generated by kinetic fluctuations at downstream observation points for three different altitudes. The experimentally observed transition locations are indicated with dashed lines for each case. (Note that the result for $H = 20.0 \times 10^3$ km is an approximation obtained by rescaling the result at $H = 25.0 \times 10^3$ km with the appropriate density (see Ref. [FT17]).

5.5 Conclusion

Analysis of wave packets generated by kinetic fluctuations at flight conditions[WZ77] indicates that this phenomenon could not be the source of the disturbance leading to laminar-turbulent transition at $M = 20$ on a 5-deg half-angle cone. The transition could be caused by nose ablation, by a discontinuity at the graphite-beryllium junction or by other disturbances present in the atmosphere.[Bus90]

Chapter 6

Receptivity of High-Speed Boundary Layers in a Binary Mixture of Gases to Kinetic Fluctuations

6.1 Introduction

The well-known phenomenon of Brownian movement is a result of chaotic motion (kinetic fluctuations) on the molecular level. Richard Feynman in his famous Lectures on Physics[FLS77] (see also the website ¹) gave a nice visualization of the phenomenon: “...we can understand it qualitatively by thinking of a great push ball on a playing field, seen from a great distance, with a lot of people underneath, all pushing the ball in various directions. We cannot see the people because we imagine that we are too far away, but we can see the ball, and we notice that it moves around rather irregularly.”

Kinetic fluctuations were introduced into hydrodynamics by Landau and Lifshitz[LL57] for a one-component fluid. It was assumed that the kinetic fluctuations will manifest themselves through fluctuating dissipative fluxes (heat flux and shear stress tensor). The fluctuation-dissipation theorem provides the space-time correlations for fluctuating shear stresses and heat fluxes (see also Refs.[LP80, dM84, OS06]). If kinetic fluctuations can generate fluctuating dissipative fluxes on the mesoscale, they should be able to generate unstable modes in boundary layer on the hydrodynamic scale. This idea was expressed first time by R. Betchov.[Bet61] The idea had been considered unreasonable for a long time because the initial amplitudes of the unstable modes should be expected to be very small. The solution of receptivity problem was not available at this time. The first result for low speed boundary layer flow was published by Zavolskii and Reutov[ZR84] in 1984. They showed that amplification of unstable modes in boundary layers is large enough to amplify modes generated by kinetic fluctuations and to cause laminar-turbulent transition in quiet wind tunnels. Once again, the publication of the results and Betchov’s idea attracted little attention. Attention was received only in 2009 after the first publication of P. Luchini[Luc08b]

¹<http://www.feynmanlectures.caltech.edu/info/>

in 2008 where he reinforced finding in Ref. [ZR84]. At about the same time, A. Fedorov considered receptivity of high-speed boundary layers to kinetic fluctuations and showed that they could have a role in laminar-turbulent transition in quiet wind tunnels and at flight conditions [FA10]. Further studies of the phenomenon in low- and high- boundary-layer flows demonstrated that kinetic fluctuations may cause transition and should be taken into account [Luc10, Fed12, Fed15, Luc17, FT17]. Analysis of receptivity to kinetic fluctuations provides an estimate of the upper bound for the transition Reynolds number.

Previous studies of the receptivity problem did not include real gas effects that may be significant for high-speed flows. Edwards and Tumin included real gas effects into consideration using effective specific heats ratio and Prandtl number. This approach allowed for the use of the solver of Ref.[FT17] where the mean flow was solved using a boundary-layer solver for chemically non-equilibrium flow[ET17b, ET17a, ET18]. However, besides fluctuating shear stress and heat flux considered in Refs.[ET17b, ET17a, ET18], there should be fluctuating species production and diffusion flux in chemically reacting mixtures [OS06, BPdZ⁺10, BdZP⁺11]. It was shown in Ref.[FT17] that the main effect is associated with fluctuating shear stress. In the present work we are evaluating relative role of chemical reactions and diffusion in receptivity of high-speed boundary layers to kinetic fluctuations. Motivation to consider binary mixture of gases is that oxygen should demonstrate an enhanced effect of dissociation, whereas nitrogen will show the other limit when dissociation is suppressed.

6.2 Governing Equations

6.2.1 Dimensional form of the governing equations for binary mixture of gases

The governing equations in dimensional form include the continuity equation for the gas mixture, momentum equations, energy equations, and equation of conservation of species s . Because the global continuity equations is used, the one equation for species conservation is redundant.

$$\frac{\partial \rho^*}{\partial t^*} + \nabla \cdot (\rho^* \mathbf{u}^*) = 0, \quad (6.1)$$

$$\rho^* \left[\frac{\partial u_i^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) u_i^* \right] = -\frac{\partial p^*}{\partial r_i^*} + \quad (6.2)$$

$$\begin{aligned} & \frac{\partial}{\partial r_k^*} \left[\mu^* \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) + \left(\mu_2^* - \frac{2}{3} \mu^* \right) \delta_{ik} \nabla \cdot \mathbf{u}^* \right] + \frac{\partial \delta \Pi_{ik}^*}{\partial r_k^*}, \\ \rho^* \left(\frac{\partial h^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) h^* \right) &= \left(\frac{\partial p^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) p^* \right) + \frac{\partial}{\partial r_i^*} \left(\kappa^* \frac{\partial T^*}{\partial r_i^*} \right) + \Phi^* + \\ & \frac{\delta \Pi_{ik}^*}{2} \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) - \frac{\partial \delta Q_i}{\partial r_i^*}, \end{aligned} \quad (6.3)$$

$$\rho^* \frac{\partial c_s}{\partial t^*} + \rho^* u_k^* \frac{\partial c_s}{\partial r_k^*} = W_s^* + \delta W_s^* + \frac{\partial J_{si}^*}{\partial r_i^*} + \frac{\partial \delta J_{si}^*}{\partial r_i^*}, \quad (6.4)$$

where Φ^* is the dissipative function

$$\Phi^* = \frac{1}{2} \left[\mu^* \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) + \left(\mu_2^* - \frac{2}{3} \mu^* \right) \delta_{ik} \nabla \cdot \mathbf{u}^* \right] \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right). \quad (6.5)$$

Equation (6.4) includes the source term W_s^* that in case of binary mixture oxygen² is

$$W_1 = M_1 \frac{d[O_2]}{dt} = -M_1[O_2] (k_f^{13}[O_2] + k_f^{14}[O]) + M_1[O]^2 (k_b^{13}[O_2] + k_b^{14}[O]), \quad (6.6)$$

$$W_2 = M_2 \frac{d[O]}{dt} = 2M_2[O_2] (k_f^{13}[O_2] + k_f^{14}[O]) - 2M_2[O]^2 (k_b^{13}[O_2] + k_b^{14}[O]). \quad (6.7)$$

The forward and backward reaction rate coefficients in Eqs. (6.6) and (6.7) are chosen in accordance with Ref.[GYTL90] and they are given in Appendix 6.6.1. Because one equation for species conservation is redundant, one has to chose in Eq. (6.4) $s = 1$ (molecules) or $s = 2$ (atoms). Enthalpy $h(p, T)$ of the mixture is determined according approximations from Ref.[GYTL90], and pressure for the binary mixture is determined as

$$p^* = c_1 R_1 \rho^* T^* + c_2 R_2 \rho^* T^*, \quad (6.8)$$

where R_1 and R_2 are gas constants for molecules and atoms, respectively. Following the concept of fluctuating hydrodynamics, equations (6.1) - (6.4) include the fluctuating dissipative fluxes $\delta \Pi_{ik}^*$, δQ^* , δJ_i^* , and δW_s^* .

The space-time correlations of the dissipative fluxes that are required to find spectral density of amplitude in unstable mode generated by kinetic fluctuations have the same form

²The equations for nitrogen are the same, only by replacing O with N, and the reaction rate coefficients appropriately

as for one-component gas [LL57, BdZP⁺11]:

$$\overline{\delta Q_i^*(\mathbf{r}^*, t^*) \delta Q_j^*(\mathbf{r}^{*'}, t^{*'})} = 2k_B \kappa^* T^{*2} \delta_{ij} \delta(\mathbf{r}^* - \mathbf{r}^{*'}) \delta(t^* - t^{*'}), \quad (6.9)$$

$$\begin{aligned} \overline{\delta \Pi_{ik}^*(\mathbf{r}^*, t^*) \delta \Pi_{lm}^*(\mathbf{r}^{*'}, t^{*'})} = \\ 2k_B T^* \left[\mu^* (\delta_{il} \delta_{km} + \delta_{im} \delta_{kl}) + \left(\mu_2^* - \frac{2}{3} \mu^* \right) \delta_{ik} \delta_{lm} \right] \delta(\mathbf{r}^* - \mathbf{r}^{*'}) \delta(t^* - t^{*'}), \end{aligned} \quad (6.10)$$

where k_B is the Boltzman constant. Equations (6.9) - (6.10) were derived in Ref.[LL57] using the fluctuation-dissipation theorem. In case of binary mixture, space-time correlation of fluctuating diffusion dissipative fluxes is (see Ref.[OS06], thermo- and baro- diffusions are neglected):

$$\overline{\delta J_{si}^*(\mathbf{r}^*, t^*) \delta J_{sj}^*(\mathbf{r}^{*'}, t^{*'})} = 2k_B T^* \rho^* D^* \left[\left(\frac{\partial \zeta^*}{\partial c_s} \right)_{T^*, p^*} \right]^{-1} \delta_{ij} \delta^*(\mathbf{r}^* - \mathbf{r}^{*'}) \delta^*(t^* - t^{*'}). \quad (6.11)$$

Space-time correlations of the fluctuating source terms in Eq. (6.4) for binary mixture can be derived using results of Ref.[BPdZ⁺10] (the fluctuation-dissipation theorem given in Eq. (88) of Bedeaux et al. has an incorrect sign as pointed out in Ref.[BdZP⁺11])

$$\begin{aligned} \overline{\delta W_i^*(\mathbf{r}^*, t^*) \delta W_i^*(\mathbf{r}^{*'}, t^{*'})} = \\ k_B L_r^*(\mathbf{r}, t) \left[1 + \exp \left(- \frac{M_1^* \Delta g^*}{\mathcal{R}^* T^*} \right) \right] \delta(\mathbf{r}^* - \mathbf{r}^{*'}) \delta(t^* - t^{*'}), \end{aligned} \quad (6.12)$$

$$L_r^* = \frac{M_1^*}{\mathcal{R}^*} k_-^*(\mathbf{r}^*, t^*) c_1(\mathbf{r}, t) \quad (6.13)$$

where k_-^* is the backwards reaction rate coefficient as presented in Refs.[BPdZ⁺10, BdZP⁺11]. Notice that results (6.12) and (6.13) do not distinguish backward reaction rate coefficients for different catalytic particles. The issue will be discussed later when we apply the results.

6.2.2 Dimensionless equations

Coordinates, velocity, time, pressure and temperature are made nondimensional using a length-scale Δ^* which is on the order of the boundary-layer thickness, free-stream velocity U_e^* , time-scale Δ^*/U_e^* , doubled dynamic pressure $\rho_e^* U_e^{*2}$ and temperature T_e^* , respectively. Time is scaled with Δ^*/U_e^* . Viscosity and thermal conductivity are scaled with μ_e^* and κ_e^* . The species gas constants are scaled with the mixture gas constant R_e^* at the boundary-layer edge. Dimensionless enthalpy is obtained using scaling with $C_{pfe}^* T_e^*$. Molecular weights of species are scaled with the mixture molecular weight M_e^* . Finally, we arrive at the following system of governing equations.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (6.14)$$

$$\rho \left[\frac{\partial u_i}{\partial t} + (\mathbf{u} \cdot \nabla) u_i \right] = -\frac{\partial p}{\partial r_i} + \frac{1}{Re} \frac{\partial}{\partial r_k} \mu \left[\left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right) + \left(\frac{\mu_2}{\mu} - \frac{2}{3} \right) \delta_{ik} \nabla \cdot \mathbf{u} \right] + \varepsilon_L \frac{\partial \sigma_{ik}^{(L)}}{\partial r_k}, \quad (6.15)$$

$$\begin{aligned} \rho \left(\frac{\partial h}{\partial t} + (\mathbf{u} \cdot \nabla) h \right) &= (\gamma_{fe} - 1) M_{fe}^2 \left(\frac{\partial p}{\partial t} + (\mathbf{u} \cdot \nabla) p \right) + \frac{1}{Re Pr_{fe}} \frac{\partial}{\partial r_i} \left(\kappa \frac{\partial T}{\partial r_i} \right) + \\ &+ \frac{1}{Re} (\gamma_{fe} - 1) M_{fe}^2 \Phi + (\gamma_{fe} - 1) M_{fe}^2 \varepsilon_L \frac{1}{2} \sigma_{ik}^{(L)} \left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right) - \varepsilon_L \left[\frac{M_{fe}^2 (\gamma_{fe} - 1)}{Pr_{fe}} \right]^{\frac{1}{2}} \frac{\partial q_i^{(L)}}{\partial r_i}, \end{aligned} \quad (6.16)$$

$$\rho \frac{\partial c_s}{\partial t} + \rho u_k \frac{\partial c_s}{\partial r_k} = W_i + \frac{\partial J_{si}}{\partial x_i} + [\gamma_{fe} Re M_{fe}^2]^{\frac{1}{2}} \varepsilon_L \delta W_s^{(L)} + [(\gamma_{fe} - 1) Re M_{fe}^2]^{\frac{1}{2}} \varepsilon_L \frac{\partial \delta J_{si}^{(L)}}{\partial x_i}, \quad (6.17)$$

$$p = \frac{1}{\gamma_{fe} M_{fe}^2} (c_1 R_1 \rho T + c_2 R_2 \rho T), \quad (6.18)$$

Here $Re = U_e^* \rho_e^* \Delta^* / \mu_e^*$ is Reynolds number, M_{fe} is frozen Mach number at the boundary-layer edge, and

$$\varepsilon_L = \frac{1}{Re^2} \left(\frac{k_B T_e^* U_e^* \rho_e^{*2}}{\mu_e^{*3}} \right)^{1/2} \quad (6.19)$$

is a small parameter related to kinetic fluctuations introduced in Ref.[FT17]. For the high-speed flows considered in Ref.[FT17], $\varepsilon_L \sim 10^{-10}$. This parameter can be written in the form separating the medium properties from the flow properties:

$$\varepsilon_L = \frac{M_{fe}^{1/2}}{Re^2} \varepsilon_{kin}, \quad \varepsilon_{kin} = \left(\frac{k_B T_e^* a_{ef}^* \rho_e^{*2}}{\mu_e^{*3}} \right)^{1/2}. \quad (6.20)$$

6.2.3 Equations for disturbances

The unsteady flow field is expressed as a superposition of a steady basic flow and an unsteady disturbance induced by kinetic fluctuations

$$\Psi(x, y, z, t) = \Psi_s(x, y, z) + \varepsilon_L \psi'(x, y, z, t), \quad \varepsilon_L \ll 1. \quad (6.21)$$

It is further assumed that the basic flow is a laminar boundary layer, which is weakly non-parallel in x -direction and it does not depend on the spanwise coordinate z . The equations for disturbances are linearized, and in absence of the forcing kinetic fluctuations the normal mode concept leads to solution for perturbations having the following form:

$$\psi'(x, y, z, t) = \hat{\psi}(y) \exp(i\alpha x + i\beta z - i\omega t). \quad (6.22)$$

Equations for the amplitudes functions $\hat{\psi}(y)$ comprise a system of ODEs. Writing the equations as a system of first order ODEs, we arrive at 10 equations. It was shown in Ref.[KT13a] that the source term perturbation at free-stream parameters of interest is insignificant. In addition, perturbation of the diffusion flux $\partial J'_{si}/\partial x_i$ was also insignificant because the eigenfunctions of the inviscid problem and the eigenfunctions obtained using the complete set of the stability equations were close. Therefore, the equation for the mass fraction amplitude (for example, $\hat{c}_1(y)$) becomes an algebraic equation. Due to the global mass conservation law ($c'_1 + c'_2 = 0$), we can exclude one $\hat{c}_s(y)$ component from the other equations and arrive at a system of eight ODEs similar to the calorically perfect gas case [Mac65, Mac69, Nay80, FT17]. The difference is that coefficients of the ODEs and the equation of state are to be calculated for real gas. We shall call the system of equations together with the conventional boundary conditions as “Reduced LST problem.” This approach retained the core of the stability and receptivity solver used in Ref.[FT17]. However, this approach suffers singularity within the critical layer, and it will be addressed in what follows.

6.2.4 Receptivity Formulation and Statistical Analysis

The receptivity analysis follows that approach used in Ref.[FT17] since the approach of the “Reduced LST problem” allows us to use the core of the stability and receptivity solver developed in Ref.[FT17]. Here, we summarize the approach in Ref.[FT17]. The disturbance vector we consider has the form:

$$F(x, y, x, y) = \left(u, \frac{\partial u}{\partial y}, v, p, \theta, \frac{\partial \theta}{\partial y}, w, \frac{\partial w}{\partial y} \right) \quad (6.23)$$

with Fourier transform

$$\hat{\mathbf{F}}(x, y; \beta, \omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{F}(x, y, z, t) \exp(i\omega t - i\beta z) dt dz. \quad (6.24)$$

The amplitude function $\hat{F}$ satisfies a system of partial differential equations resulting from the Fourier transform of the disturbance equations. This system can be written in the matrix-operator form 8×8

$$\mathbf{H}(y, \partial_y, x_1, \varepsilon \partial_{x_1}, \omega, \beta) \hat{\mathbf{F}} = \hat{\mathbf{G}}^{(L)}. \quad (6.25)$$

The second equation in Eq. (6.25) is the x -momentum equation, the third equation is the continuity equation, the fourth equation is the y -momentum equation, the sixth equation is the energy equation, and the eighth equation is the z -momentum equation. The other equations have the role of definitions of the components in vector $\mathbf{F}$. The vector $\hat{\mathbf{G}}^{(L)}$ represent Fourier components of the Langevin terms.

The standard boundary conditions on the wall $(u, v, w, \theta) = 0$ and the boundedness condition in the outer flow give

$$\hat{F}_1 = \hat{F}_3 = \hat{F}_5 = \hat{F}_7 = 0 \text{ at } y = 0, \quad (6.26)$$

$$\left| \hat{F}_1, \hat{F}_3, \hat{F}_5, \hat{F}_7 \right| < \infty \text{ at } y \rightarrow \infty. \quad (6.27)$$

We consider a partial solution of the problem (6.25)-(6.27), which comprises a normal mode $\hat{\mathbf{F}}_m(x_1, y)$ of the discrete spectrum (for example, the Mack second mode) with the eigenvalue $\alpha_m(x_1, \beta, \omega)$ and a forced disturbance $\hat{\mathbf{F}}^{(L)}(x, y)$. It is assumed that the mode m is unstable. At the neutral point $x_{10}(\beta, \omega)$, its eigenvalue $\alpha_m(x_{10}, \beta, \omega) = \alpha_0$ is real while its growth rate $\sigma = -\text{Im}(\alpha_m)$ satisfies the condition $\partial\sigma/\partial x_1(x_{10}) > 0$. The dominant excitation of this mode occurs in the region $|x_1 - x_{10}| = O(\varepsilon^{1/2})$, where the forcing waves have wavenumbers $\alpha = \alpha_0 + \varepsilon^{1/2}\gamma$, and $\gamma = O(1)$ is real parameter characterizing detuning of the resonance condition $\alpha = \alpha_0$. We introduce the inner variable $\xi = \varepsilon^{-1/2}(x_1 - x_{10})$ and approximate the forced disturbance by the local Fourier integral

$$\hat{\mathbf{F}}^{(L)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathbf{F}}_{\alpha}^{(L)}(y) e^{i\alpha(x-x_0)} d\alpha = \frac{\varepsilon^{1/2}}{2\pi} e^{i\varepsilon^{-1/2}\alpha_0\xi} \int_{-\infty}^{+\infty} \hat{\mathbf{F}}_{\alpha}^{(L)}(y) e^{i\gamma\xi} d\gamma. \quad (6.28)$$

The local solution can be expressed in the form

$$\hat{\mathbf{F}}(\xi, y) = \frac{1}{2\pi} e^{i\varepsilon^{-1/2}\alpha_0\xi} \int_{-\infty}^{+\infty} [c(\xi, \gamma) \mathbf{A}_0(y) + \varepsilon^{1/2} \mathbf{A}^{(L)}(y, \gamma) e^{i\gamma\xi}] d\gamma. \quad (6.29)$$

Here $\mathbf{A}_0(y) = \mathbf{A}_m(x_{10}, y, \alpha_0)$ is the eigenfunction of mode m , which is a solution of the local-parallel stability problem

$$\begin{aligned} \left(\frac{\partial}{\partial y} - \mathbf{H}_0 \right) \mathbf{A} &= 0, \\ A_1 = A_3 = A_5 = A_7 &= 0 \quad \text{at } y = 0, \\ A_1, A_3, A_5, A_7 &\rightarrow 0 \text{ at } y \rightarrow \infty. \end{aligned} \quad (6.30)$$

The matrix $\mathbf{H}_0 = \mathbf{H}_0(x_1, \alpha, \beta, \omega)$ of dimension 8×8 is calculated at $x_1 = x_{10}$ and $\alpha = \alpha_0$; its explicit form $\mathbf{H}_0$ is given in Appendix 6.6.2. The main difference from Ref.[FT17] is in the matrix $\mathbf{H}_0$. Elements of the matrix are different because of the mean flow profiles depending on the real gas effects, and due to new additional terms associated with mass fractions perturbations.

Following lines of Ref.[FT17], we arrive at the correlation spectral density of the amplitude coefficient as

$$K_c = \sqrt{\frac{\pi}{\sigma_x(x_{10})}} (2\pi)^2 (q^2)_{\alpha, \beta, \omega} \delta(\beta - \beta') \delta(\omega - \omega'), \quad (6.31)$$

where $\sigma_x(x_{10}) = d\sigma/dx$ is the growth-rate gradient at the resonance point x_{10} , and the spectral density $(q^2)_{\alpha, \beta, \omega}$ is expressed in the analytical form given in Appendix 6.6.3. The space-time correlations of the fluctuating dissipative fluxes are used to find the spectral density $(q^2)_{\alpha, \beta, \omega}$. With the exception of the correlation for the fluctuating source term for a binary mixture, their application is straightforward. With regards to the fluctuating chemical source term, it is important to recognize that in Ref.[BPdZ⁺10], the space-time correlation

results were found for a single binary mixture under a single dissociation/association reaction. This is reflected by the single backward reaction rate coefficient that appears in Eq. (6.12). Here we consider a binary mixture under two dissociation/association reactions corresponding to the presence of the two possible catalytic particles in a binary mixture. Thus, we work with two forward and backward reaction rate coefficients. Here it is important to notice that despite considering two different catalytic particles, we always consider the same type of binary association/dissociation reaction. We can then consider the effect of the different catalytic particles simultaneously by considering a single forward reaction rate coefficient and a single backward reaction rate coefficient for the source term by noting the following:

$$W_1 = -M_1[O_2] (k_f^{13}[O_2] + k_f^{14}[O]) + M_1[O]^2 (k_b^{13}[O_2] + k_b^{14}[O]) = -M_1[O_2]k_-^* + M_1[O]^2k_+^* \quad (6.32)$$

where

$$k_-^* = k_f^{13}[O_2] + k_f^{14}[O] \quad (6.33)$$

$$k_+^* = k_b^{13}[O_2] + k_b^{14}[O]. \quad (6.34)$$

It is also important to note that our convention for the direction of the reactions is reversed from that of Ref.[BPdZ⁺10]. In particular, our forward reaction rate coefficient k_f corresponds to their backwards reaction rate coefficient k_-^* , and vice versa.

The asymptotic result (6.31) is based on the Laplace method for evaluation of integral in vicinity of the neutral point. Edwards and Tumin[ET18, ET19] analyzed the problem using a direct computation of integral and demonstrated that the receptivity integrand has an offset from the neutral point, and the neutral point is not representative for the receptivity phenomena if there is a near neutral behavior on an extended interval in vicinity of x_{10} .

In this work, we will focus on examining the integrals and integrands that appear in the spectral density $(q^2)_{\alpha,\beta,\omega}$. These terms are directly associated with particular physical phenomena and details of KF receptivity. Consequently, they are a particularly insightful quantity to study. In particular, we will be evaluating the relative role of the new fluctuating species diffusion and fluctuating species production at the flow parameters of interest.

6.3 Results

6.3.1 Free-stream flow parameters

Choice of flow parameters was discussed in Ref.[KT13a]. According to known prospective projects, flows having total enthalpy 5 - 18 MJ/kg are of interest. On the other hand, the edge Mach number should be above 4 in order for the second Mack's mode to exist. This leads to a constraint on the edge temperature. Fig. 6.1 shows the frozen Mach number versus temperature for different total enthalpies in equilibrium air, binary mixtures of oxygen and nitrogen, respectively. Also, as it was discussed in Ref.[KT13a], there is a constraint due to acceptable temperatures of materials that can be used for thermal protection. This constraint

together with reasonable required Lift-to-Drag ratio leads to an estimated altitude of flights. It was suggested to consider eight study Cases of flow at pressure 0.0433 atm. All examples in Ref.[KT13a, KT15] (and in the present work) were carried out at zero streamwise pressure gradient flow. This does not mean that the examples are associated with flow past a flat plate at zero angle of attack. The suggested cases cover flows with one and more oblique shock waves leading to temperatures and pressure different from corresponding values in the atmosphere. Figure 6.2 illustrates growth rates in Cases 1 - 3 of boundary layers in air and in

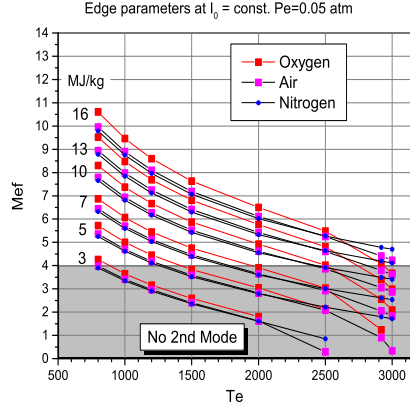


Figure 6.1: Local frozen Mach numbers and local temperatures at fixed values of total enthalpy in equilibrium state.

the binary mixtures of oxygen. Certainly, the growth rates are different but the differences are not particularly significant. More detailed flow parameters of three cases are presented in Tables 6.1 and 6.2. The wall parameters refer to boundary-layer flows over adiabatic and non-catalytic walls. Because the composition of gases in air is different than in binary mixtures of oxygen and nitrogen, using the same free stream parameters as in Table 6.1 does not allow one to maintain a constant Mach number or enthalpy as was considered in Ref.[KT13a]. However, the Mach numbers and total enthalpies are close (see Table 6.2). The main differences between the cases of air and oxygen are the temperatures of the adiabatic non-catalytic walls, T_w .

Table 6.1: Details of Cases 1, 3, and 6 in binary mixture of oxygen[KT13a].

	x (m)	T_e (K)	U_e (m/s)	Re_L	I (MJ/kg)	T_w (K)	M_{ef}	H (cm)
Case 1	0.4	278	3.351×10^3	2.988×10^6	5.597	3814	10.5	0.03273
Case 3	0.4	834	5.648×10^3	1.022×10^6	16.49	4761	10.5	0.05596
Case 6	1.199	1668	5.475×10^3	1.022×10^6	16.49	4315	7.32	0.1677

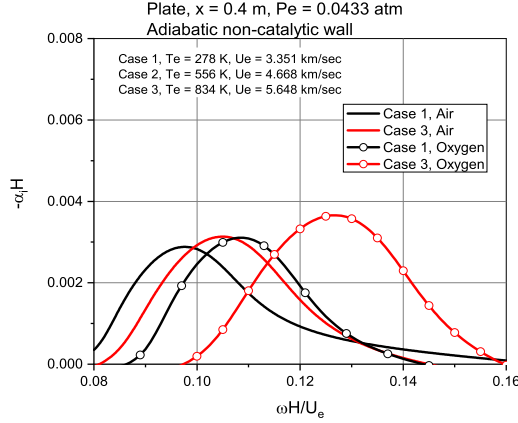


Figure 6.2: Comparison of growth rates in Oxygen and air

Table 6.2: Details of Cases 1, 3, and 6 in air[KT15].

	x (m)	T_e (K)	U_e (m/s)	Re_L	I (MJ/kg)	T_w (K)	M_f	H (cm)
Case 1	0.4	278	3.351×10^3	3.82×10^6	5.6	3950	10	0.0289
Case 3	0.4	834	5.648×10^3	1.13×10^6	16.53	7332	9.91	0.0533
Case 6	1.199	1668	5.475×10^3	1.04×10^6	16.55	7231	6.91	0.166

6.3.2 Evaluation of the reduced LST approach

For the purpose of evaluation of the proposed reduced LST model, we consider Cases 1,3,6 for binary mixtures of oxygen and nitrogen at zero pressure gradient. These choices provide a representative and meaningful range of conditions. Throughout, the perturbations are assumed two-dimensional ($\beta = 0$). Cases 1, 3, and 6 for nitrogen have the same edge flow parameters T_e, U_e as in cases for oxygen. The stability results within the reduced LST model are compared with results obtained using solver [KT13a, KT15].

Case 1

Growth rates obtained using different models are shown in Fig. 6.3. One can see that growth rates obtained with the reduced LST equations are about the same as growth rates obtained using the complete system of LST equations for Case 1. The agreement is decent near the lower frequency neutral point. Agreement in this region is important since our receptivity analysis will focus on this region in particular since kinetic fluctuations may excite unstable waves in this region [FT17].

Eigenfunctions obtained using the reduced LST model (modified equations) are shown in Fig. 6.4. Results for the reduced system of equations are close to eigenfunctions obtained using the full system of LST equations (pressure and temperature eigenfunctions). Eigenfunction of the mass fraction, c_1 , demonstrate a clear difference in the vicinity of the

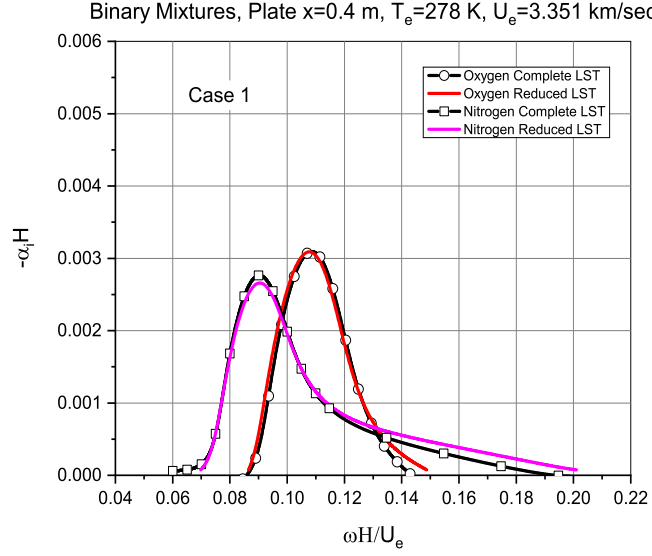


Figure 6.3: Growth rate computed using different LST models for Case 1

critical layer ($y_c \approx 13$) where the phase velocity of the second mode is equal to the local velocity $U(y_c)$. This is an expected consequence of neglecting the viscous term in the mass conservation equation.

Overall, with the exception of the mass fraction eigenfunction at the critical layer, the reduced LST showed strong agreement in Case 1 with the full model. However, Case 1 lacks significant dissociation. Case 3 on the other hand has significant dissociation. We present similar comparisons with Case 3 next, and observe a comparable level of agreement.

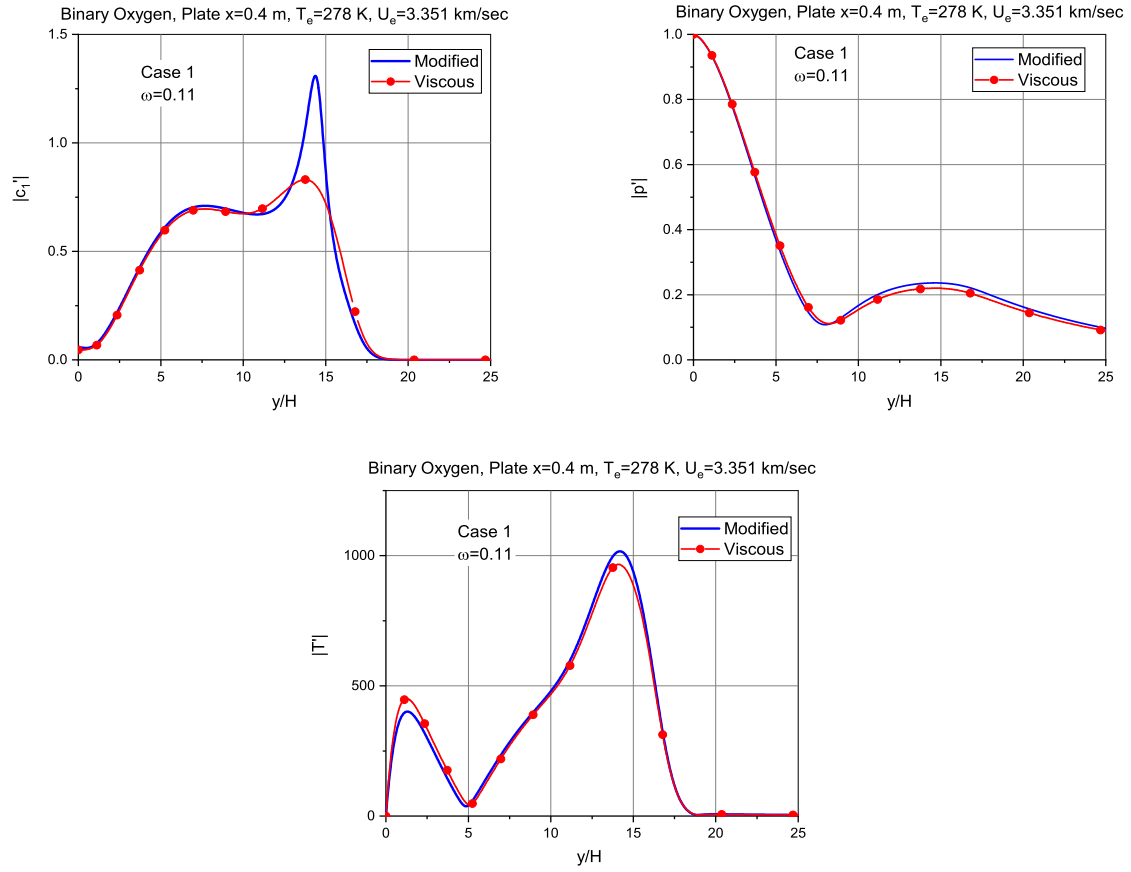


Figure 6.4: Eigenfunctions for Case 1 with angular frequency $\omega = 0.11$

Case 3

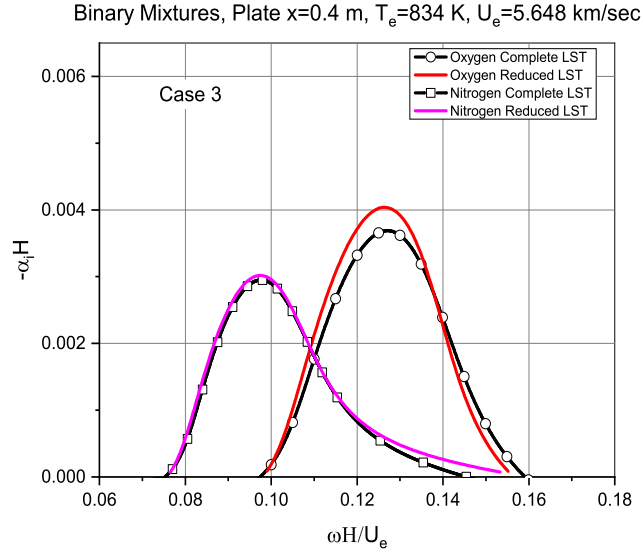


Figure 6.5: Growth rate computed using different LST models for Case 3.

Figure 6.5 compares the growth rates obtained with different models for both nitrogen and oxygen. The agreement between the full viscous model and the reduced model is not as strong as in Case 1, however it is still reasonably close. The largest disagreement occurs near the peak of the growth rate for oxygen. A comparison of the eigenfunctions in this region shows only small differences between the two models. The eigenfunctions for oxygen are plotted in Figure 6.6. The only exception to this extremely close agreement is the species mass fraction perturbation which has a peak at the critical layer due to the expected singularity. Furthermore, as discussed, our main concern lies with the lower frequency neutral point. As we can see, the agreement in this region for both oxygen and nitrogen is fairly strong. Finally, we present similar comparisons for Case 6.

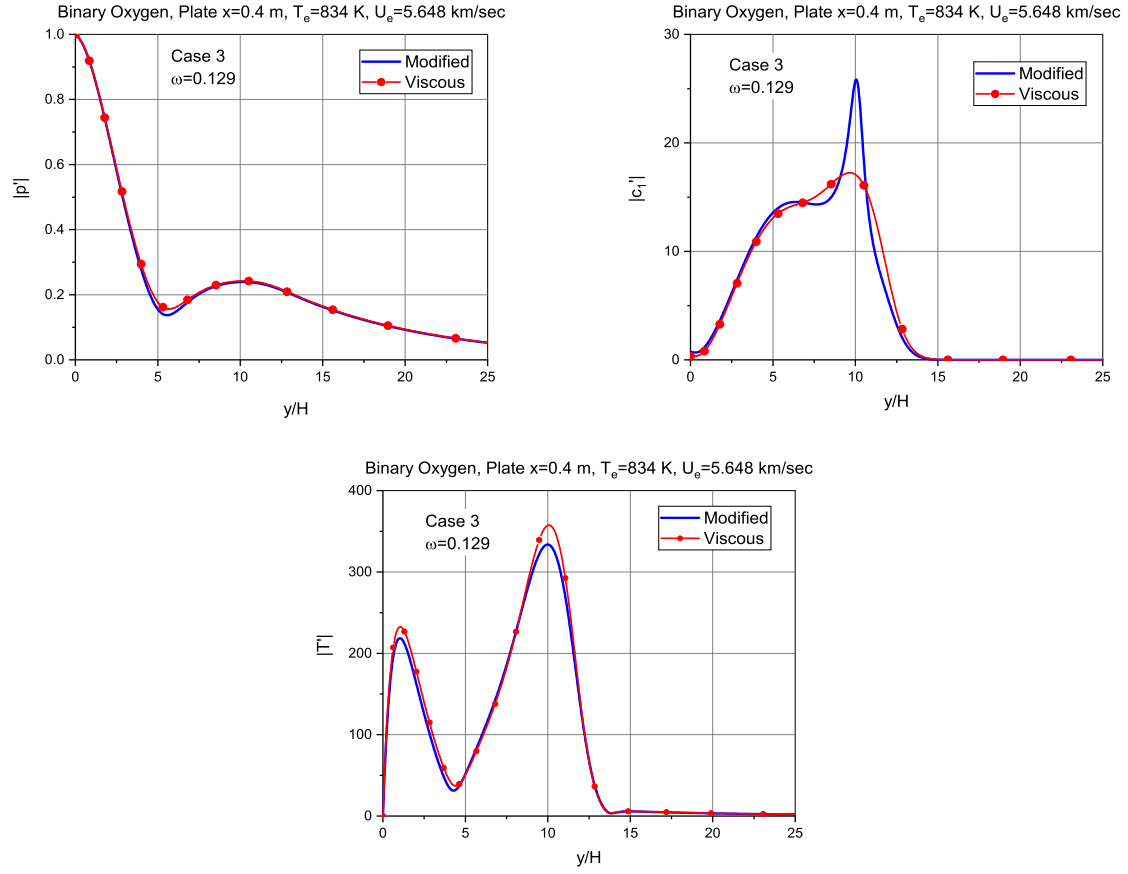


Figure 6.6: Eigenfunctions for Case 3 with angular frequency $\omega = 0.129$

Case 6

Figure 6.7 compares the growth rates obtained with different models. The agreement is strong overall, however, there is some disagreement near the peak of the growth rate curve for oxygen. This is similar to what was observed for Case 3 where the reduced set of equations seems to overestimate the values in this region in the oxygen case. However, the disagreement in this region is not particularly large, leading to a reasonable overall approximation of the complete set of equations. Comparing eigenfunctions for the oxygen case further confirms that differences are not particularly significant. Figure 6.8 compares the eigenfunctions for

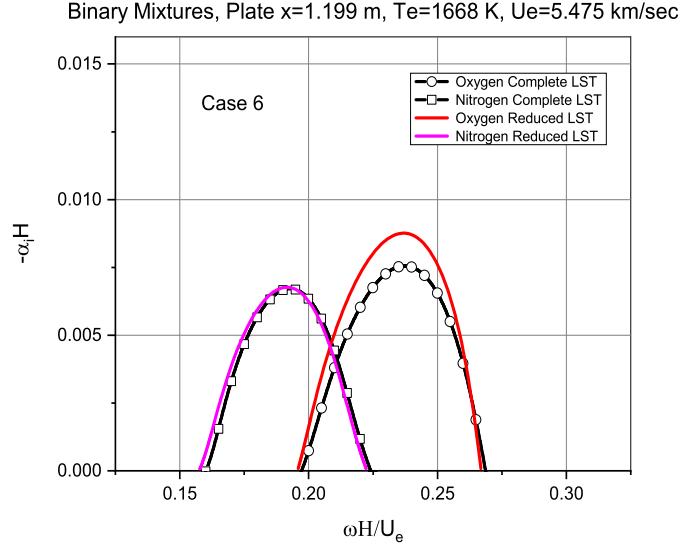


Figure 6.7: Growth rate computed using different LST models for Case 6

the different models for this case. Again, we see strong agreement between the model nearly everywhere except for the mass fraction disturbance eigenfunction in the critical layer. Again, in the critical layer we observe the expected singularity. As we will see, this singularity does not directly impact the receptivity results in a significant way.

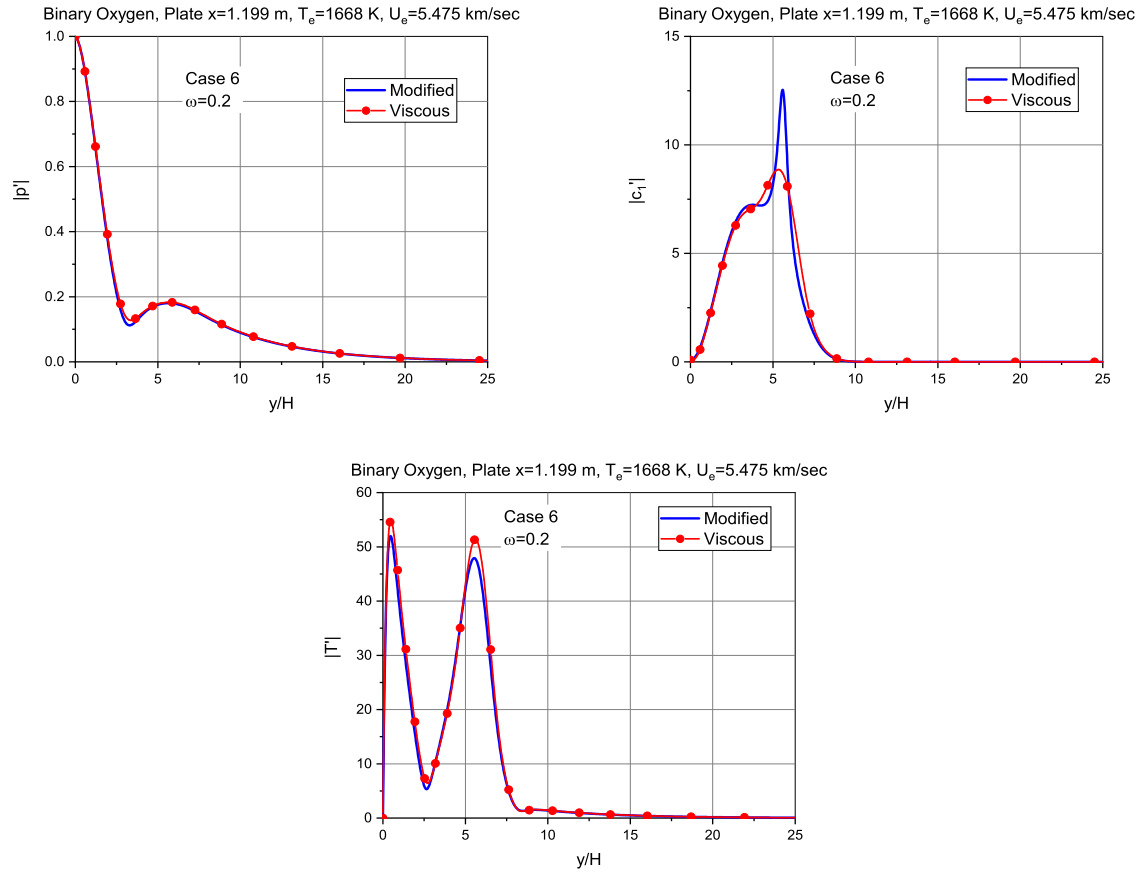


Figure 6.8: Eigenfunctions for Case 6 with angular frequency $\omega = 0.22$

6.3.3 Receptivity to Kinetic Fluctuations. Case 1. $T_e = 278 K$, $U_e = 3.35 km/s$, $I = 5.6 MJ/kg$, $M_{ef} = 10.5$.

Because the reduced LST model led to the same structure of the forced equations as in Ref.[FT17], we can use the same solver with slight modifications in equations of Appendix VIII.C of Ref.[FT17]. These modifications are presented in Appendix 6.6.2 and Appendix 6.6.3 of the present work.

We will now explore some details of the KF receptivity analysis. In Appendix 6.6.3, the spectral density $(q^2)_{\alpha,\beta,\omega}$ is explicitly given in terms of the integrals $J_{k,m}$. These $J_{k,m}$ integrals have their origin in the Langevin forcing terms introduced in the governing equations. Since these integrals are directly associated to physical phenomena of interest, we will focus on evaluating the relative role of the new considerations introduced in this paper. In particular, we will examine the role of the fluctuating species diffusion, fluctuating chemical source term (which have not been studied before), and the role of the fluctuating shear stress which was found to be the dominant fluctuation in Ref.[FT17]. We will study three integrals that appear in the spectral density. These three integrals are labeled J_{22} J_{36} J_{66} . The J_{22} term originates from the correlation of shear stress fluctuations in the x -momentum equation. The J_{66} term has its origin in the energy equation where there are contributions to the correlations from all fluctuating quantities (fluctuating shear stress, fluctuating heat flux, fluctuating diffusion flux, and fluctuating chemical source term). The J_{36} term has its origin in the correlation of the fluctuating chemical source term and fluctuating species diffusion flux terms only. The rest of the integrals in the formulation, except for one that is similar to J_{36} , have similar origins, are not particularly interesting to study since they include no new effects, and are negligible. So, we focus on J_{22} , J_{36} , and J_{66} . By examining these three integrals, and integrands of these integrals, we will be able to better understand the role the new fluctuations play.

First, we examine these integrals as functions of ω . Due to the singularity of the critical layer at the neutral point, we examine what happens to these integrals as we move toward the neutral point (the lower ω values shown) in order to ensure the singularity is not causing the integrand to blow up. We see that as we approach where $\alpha_i \sim 10^{-5}$, there are no issues for Case 1. This is demonstrated in figure 6.9. The only integrals expected to have these problems are the integrals J_{36} and J_{66} . Note that any results presenting these integrals are normalized. The normalization is discussed in Appendix 6.6.3.

An interesting and important aspect of figure 6.9 to notice is that the J_{22} is the dominant integral for both oxygen and nitrogen. In fact, it is clear from Figure 6.9 that J_{22} is over an order of magnitude larger than the second largest term J_{66} at the neutral point. This provides an interesting observation for both oxygen and nitrogen in Case 1. Namely, that the shear stress fluctuations in the x -momentum equation provide the main effect, even when we consider the new fluctuating quantities. Dominance of J_{22} term is in agreement with Ref.[FT17].

The integrands also provide insight into the KF receptivity. The integrands are shown in Fig. 6.10 as functions of y for ω such that $\alpha \approx 1.5 \times 10^{-4}$. These integrands are denoted by $j_{k,m}(y)$ and are normalized in the same way as the integrands. As expected, the $j_{22}(y)$

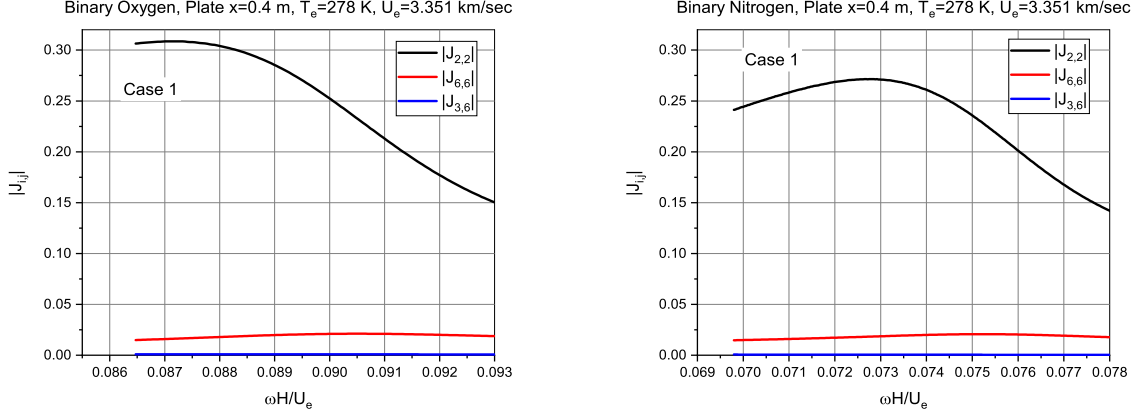


Figure 6.9: Integrals $J_{m,k}$ plotted as function of ω for Case 1. J_{22} is associated with shear stress fluctuations in stream wise momentum equation. J_{66} is associated with the fluctuations in the energy equation (all fluctuations appear in the energy equation). J_{36} is associated with the fluctuating species diffusion and fluctuating species production in the species conservation equation

integrand provides the largest contribution to the integral. Interestingly, the maximum of $j_{22}(y)$ occurs in the vicinity of the critical layer near the point $y = y_c$ (vertical line in Figure 6.10) where $U(y_c) = c_r = \omega/\alpha_r$. Another clear feature of Fig. 6.10 is that all other components are noticeable in both the critical layer and in the near-wall region. The integrals not plotted also show this exact behavior as well. However, further analysis has demonstrated that their contributions to the spectral density $(q^2)_{\alpha,\beta,\omega}$ are negligible. The interesting observation here is that the boundary layer is most sensitive to kinetic fluctuations in the critical layer. There are also spikes in some of the integrands $j_{m,k}(y)$ corresponding to the singularities introduced in the species conservation equation through neglecting the diffusion term. They are not particularly large in this case. As long as we do not get too close to the neutral point, we do not see these singularities having much influence in the final result. This is discussed in detail later on.

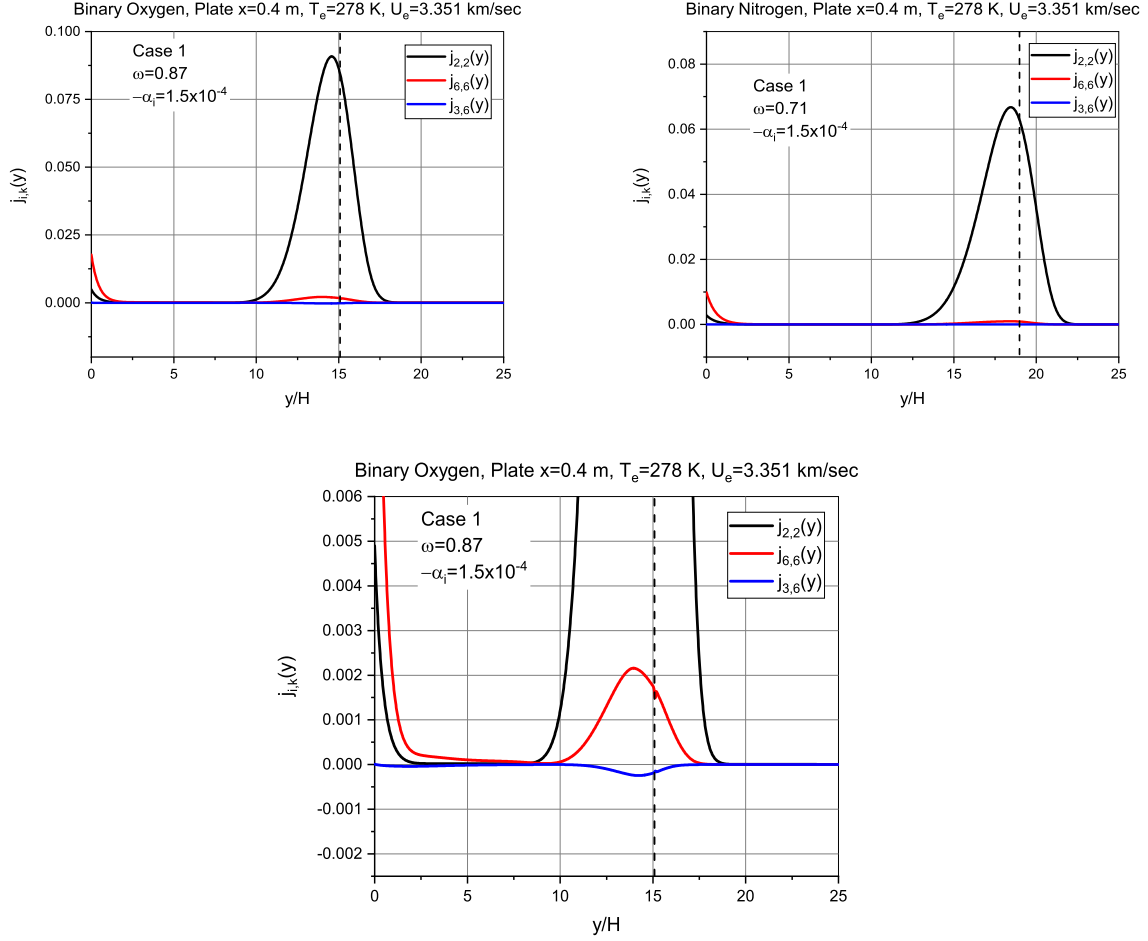


Figure 6.10: Integrands $j_{m,k}(y)$ for Case 1 oxygen and nitrogen. $j_{22}(y)$ is associated with shear stress fluctuations in stream wise momentum equation. $j_{66}(y)$ is associated with the fluctuations in the energy equation (all fluctuations appear in the energy equation). $j_{36}(y)$ is associated with the fluctuating species diffusion and fluctuating species production in the species conservation equation

6.3.4 Receptivity to Kinetic Fluctuations. Case 3. $T_e = 834 K$, $U_e = 5.65 km/s$, $I = 16.5 MJ/kg$, $M_{ef} = 10.5$.

We now consider the receptivity of binary mixtures of oxygen and nitrogen in Case 3 conditions. The results are quite similar to Case 1. First, as before, we examine the integral as a function of ω near the neutral point (lowest ω values shown). We again see that near the neutral point, the two largest contributors are J_{22} and J_{66} . There is an interesting difference between Cases 1 and 3. Unlike in Case 1, where J_{22} and J_{66} varied similarly between each other (J_{22} was an order of magnitude larger than J_{66}) for both oxygen and nitrogen near the neutral point, this is not the case here. While J_{22} is an order of magnitude larger than J_{66} for nitrogen, this is not exactly the case for oxygen. For oxygen, near the neutral point, $J_{22} \approx 0.15$, and $J_{66} \approx 0.02$. Then, we have that $|J_{22}|$ is only about 7.5 times larger than $|J_{66}|$. So, the dominance of the J_{22} term corresponding to shear stress fluctuations is not as pronounced for oxygen in Case 3. It is illuminating to also examine the integrands for the integrals near the neutral point. In Figure 6.12, the integrands are plotted for ω corresponding to $\alpha_i \approx 1.5 \times 10^{-4}$ for both oxygen and nitrogen.

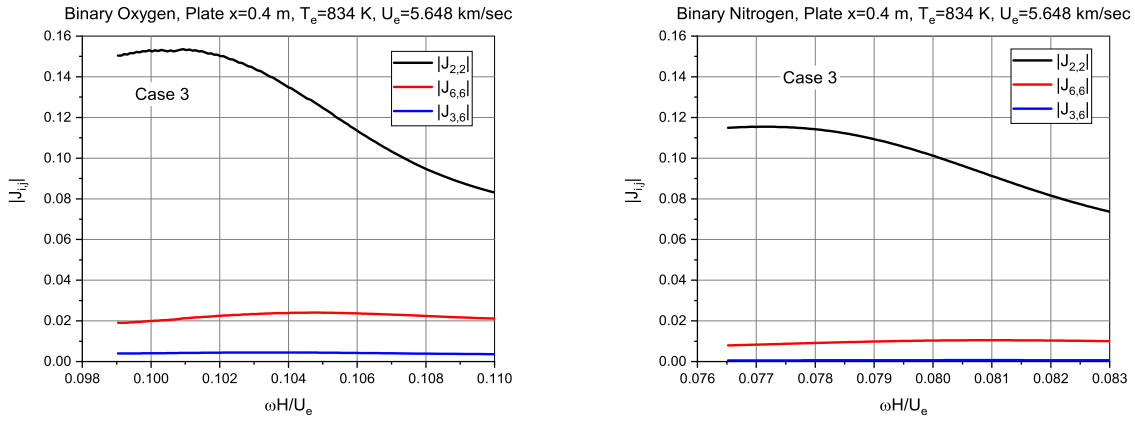


Figure 6.11: Integrals $J_{m,k}$ plotted as function of ω for Case 3 oxygen and nitrogen. J_{22} is associated with shear stress fluctuations in stream wise momentum equation. J_{66} is associated with the fluctuations in the energy equation (all fluctuations appear in the energy equation). J_{36} is associated with the fluctuating species diffusion and fluctuating species production in the species conservation equation

In Figure 6.12, we further see a number of features similar to Case 1. We see that the $j_{22}(y)$ integrand is by far the largest contributor, and reaches its maximum near the critical layer (vertical dot-dash line). We can also observe that every integrand has its largest contribution in two regions, near the wall and near the critical layer. In general, we can see that for oxygen, the integrands experience larger peaks than for the same case in nitrogen. So, it appears that for oxygen, the boundary layer is more sensitive to kinetic fluctuations in this case. One of the most pronounced differences between oxygen and nitrogen in Case 3 is in the integrands $j_{36}(y)$ and $j_{66}(y)$. Case 3 has significant dissociation, and for oxygen

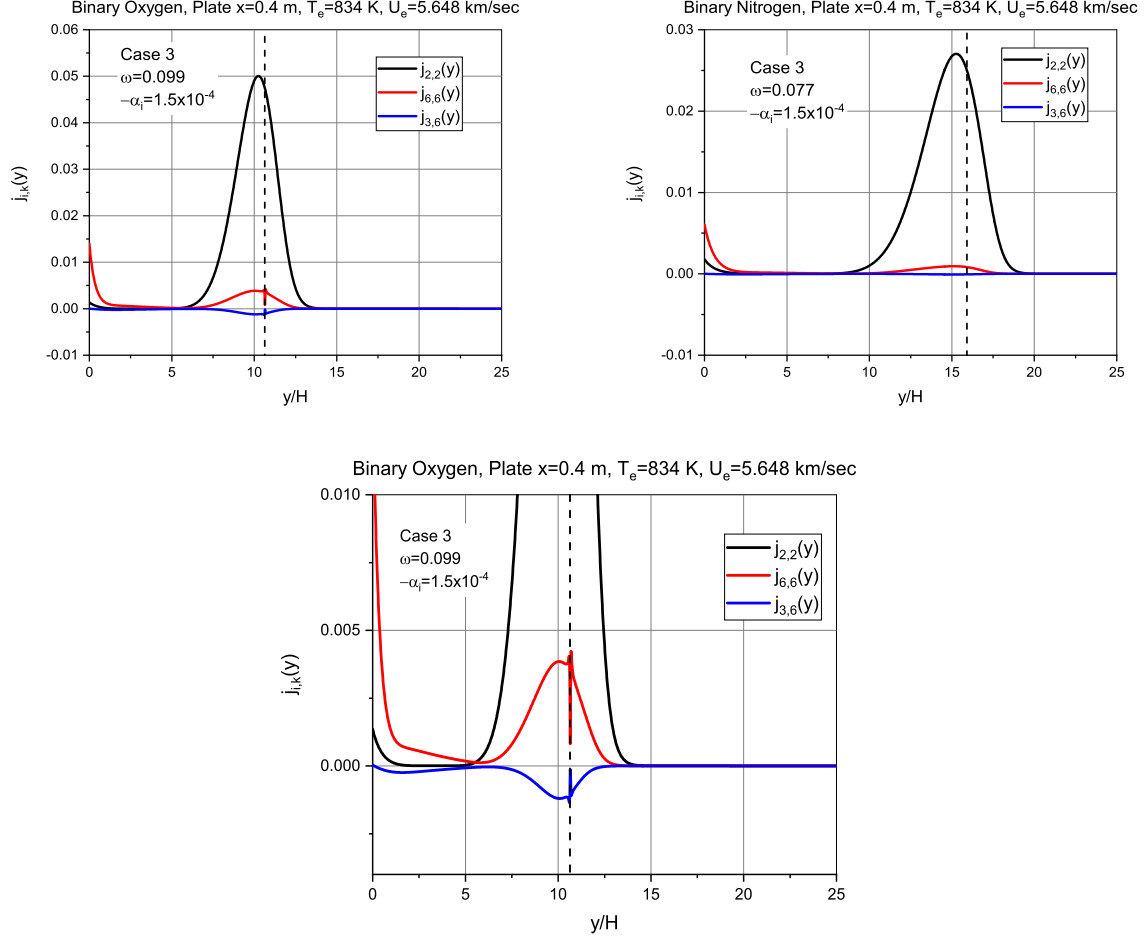


Figure 6.12: Integrands $j_{m,k}(y)$ for Case 3 oxygen and nitrogen. $j_{22}(y)$ is associated with shear stress fluctuations in stream wise momentum equation. $j_{66}(y)$ is associated with the fluctuations in the energy equation (all fluctuations appear in the energy equation). $j_{36}(y)$ is associated with the fluctuating species diffusion and fluctuating species production in the species conservation equation

in particular, a stronger effect is expected due to its lower dissociation temperature. This is reflected in the integrands of the integrals composing the spectral density term. We see that the integrands $j_{36}(y)$ and $j_{66}(y)$, both of which contain terms corresponding to the fluctuating reaction rate and fluctuating species diffusion, are considerably more pronounced than their nitrogen counterparts. There is one more clear feature that should be noticed. The singularity that was not clearly observed for Case 1 is considerably more pronounced for Case 3. It is interesting to note that the integrands shown for nitrogen for both Case 1 and Case 3 do not clearly exhibit the singularity as strongly as for oxygen. We will see that this is what occurs in Case 6. Overall, we again find agreement with Ref.[FT17] that J_{22} is the dominant term.

6.3.5 Receptivity to Kinetic Fluctuations. Case 6. $T_e = 1668\text{ K}$, $U_e = 5.48\text{ km/s}$, $I = 16.5\text{ MJ/kg}$, $M_{ef} = 7.32$.

The final case to examine is Case 6 for both oxygen and nitrogen. We again approach our analysis by examining the integral terms that contribute to the spectral density $(q^2)_{\alpha,\beta,\omega}$. When examining the integrals near the neutral point (lowest ω values shown) in Figure 6.13, just like in Cases 1 and 3 for both oxygen and nitrogen, we see that the integrands J_{22} and J_{66} are the dominant integrals. Recall J_{22} and J_{66} respectively have their origin in the shear stress fluctuations of the x -momentum equation and in the fluctuations of the energy equation. However, like Case 1, we see that the J_{22} is clearly dominant by being over an order of magnitude larger than the second largest integral J_{66} . Thus, we can again see that the dominant effect is produced by the shear stress fluctuations in the x -momentum equation. In Figure 6.14 select integrands are plotted as functions of y . We again see the

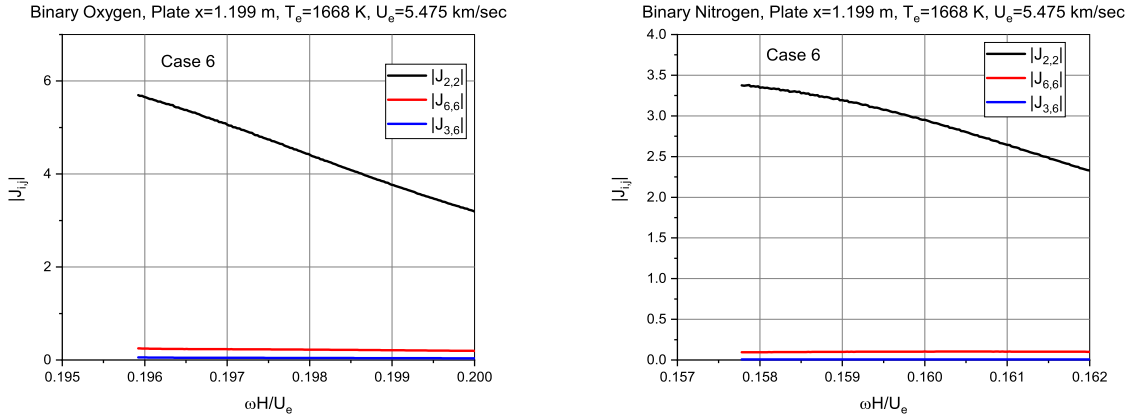


Figure 6.13: Integrals $J_{m,k}$ plotted as function of ω for Case 6 oxygen and nitrogen. J_{22} is associated with shear stress fluctuations in stream wise momentum equation. J_{66} is associated with the fluctuations in the energy equation (all fluctuations appear in the energy equation). J_{36} is associated with the fluctuating species diffusion and fluctuating species production in the species conservation equation

expected behavior that the $j_{22}(y)$ integrand is much larger than all the other integrands. We also see that all integrands mainly contribute in regions near the wall and near the critical layer. A noticeable difference between oxygen and nitrogen lies with the integrands that correspond to j_{36} , and j_{66} . Recall that these integrands contain contributions from the fluctuating chemical reaction rate and the fluctuating species diffusion flux. We see that oxygen has a significantly more noticeable presence of these terms than nitrogen for Case 6. So, the contributions of the fluctuating reactions rate and species diffusion flux are stronger for oxygen and nitrogen as seen in Case 3. This is not particularly surprising due to the fact oxygen disassociates more easily (at a lower temperature) than nitrogen. Overall we see the same pattern as observed for Cases 1 and 3. We see that that the main effect of the kinetic fluctuation occurs near the wall and the critical layer, with the strongest effect being

produced by the x -momentum shear stress fluctuations. This establishes further agreement with Ref.[FT17]. Another feature that is particularly clear in this case, are the singularities at the critical layer. However, qualitatively, we see that they are small at this ω . These will be discussed next.

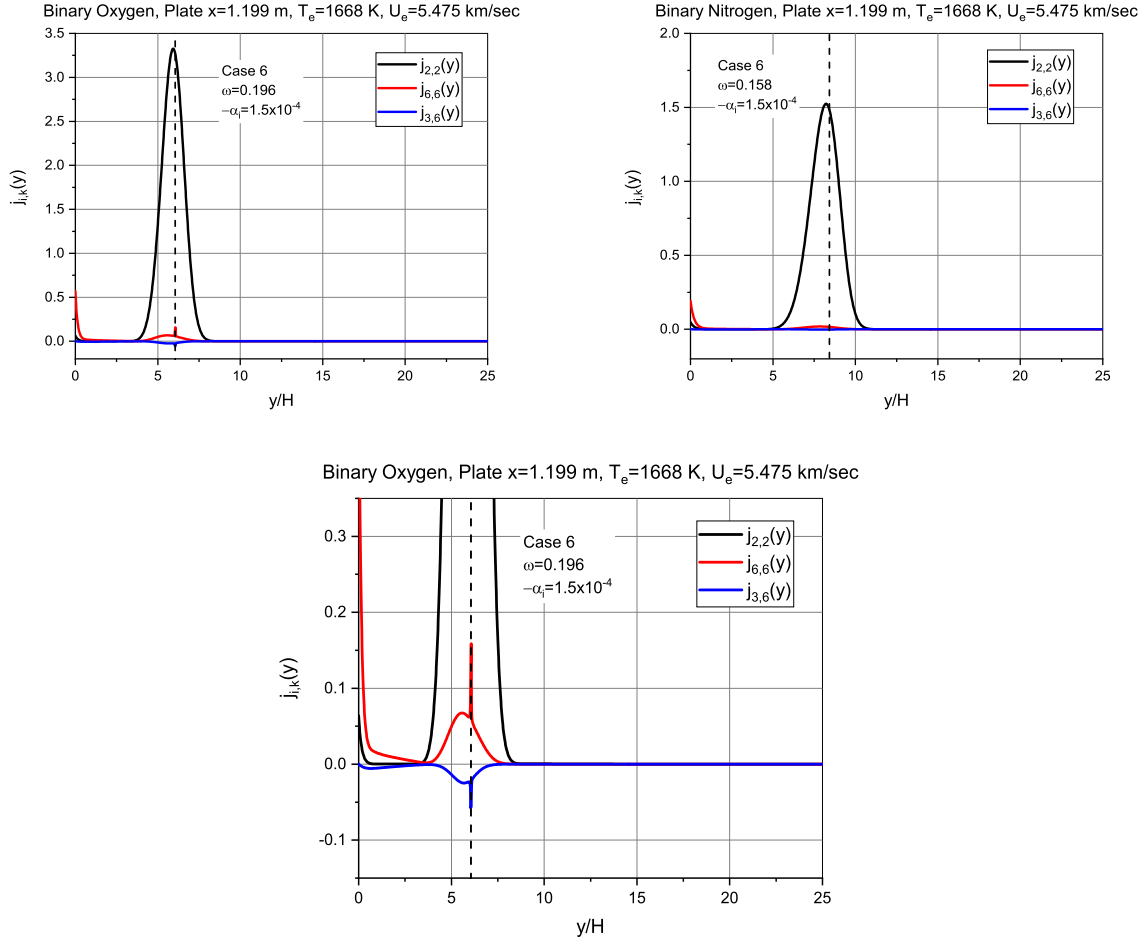


Figure 6.14: Integrands $j_{m,k}(y)$ for Case 6 oxygen and nitrogen. $j_{22}(y)$ is associated with shear stress fluctuations in stream wise momentum equation. $j_{66}(y)$ is associated with the fluctuations in the energy equation (all fluctuations appear in the energy equation). $j_{36}(y)$ is associated with the fluctuating species diffusion and fluctuating species production in the species conservation equation

6.4 Discussion

6.4.1 Discussion of Critical Layer Singularity at Neutral Point

As discussed, there is a coefficient that appears in the J_{ij} integrals that leads to the singularities at the critical layer $y = y_c$ at the neutral point. This singularity enters the formulation since we dropped the viscous diffusion term in the species conservation equation, leading to an inviscid equation. However, we can get astoundingly close to the neutral point without having the singularity throw off the results in a severe way. We can get close enough that we can consider $\alpha_i \sim 10^{-5}$. We briefly explain why this occurs. The factor that appears in the integrals is:

$$\frac{i\hat{\omega}}{W_{s\rho}\rho_s(R_1 - R_2) - i\rho_s\hat{\omega}\bar{R} - W_{sc}\bar{R}} \quad (6.35)$$

where $\hat{\omega} = \omega - \alpha U_s$, $W_{s\rho} = (\partial W_s / \partial \rho)_{T,c}$, and $W_{sc} = (\partial W_s / \partial \rho)_{T,\rho}$. The denominator term has a first order pole singularity slightly offset from the critical layer when we are at the neutral point. However, the term that appears in the numerator goes to zero exactly at the critical layer when we are at the neutral point. Consequently, the ratio of the numerator term to the denominator term "nearly cancels" out the first order pole. This can be seen in Figure 6.15 for an ω for Case 6 oxygen where $\alpha_i = 1.5 \times 10^{-4}$. However, since the zero of the numerator is offset from the pole in the denominator, this balance is not an exact cancellation of the pole in the denominator term and this becomes more apparent as we get closer to the neutral point. However, due to this near cancellation of the first order pole, we can get results that do not severely blow up close to the neutral point. In our expressions, derivatives of the coefficient term also appear. The small jump seen in the coefficient consequently leads to a larger jump in the derivative. These jumps in the derivative grow rapidly as we get closer to the neutral point, preventing us from getting precisely to the neutral point. However, since these terms are multiplied by adjoint eigenfunction components which are small, these jumps do not affect the results immediately unless we get closer to the neutral point than discussed.

6.4.2 Discussion of Asymptotic Contributions to the Integrals at

$$y \rightarrow \infty$$

The formulation of the receptivity analysis requires the integrals in the spectral density term to be evaluated out to infinity in the wall normal direction. This is typically done by numerically integrating up to some value y_{max} , then finding the asymptotic contribution to the integral. In this work, we do not treat the estimates so delicately. We only compared results for the integral evaluated up to y_{max} . As already seen, the integrands have only minute contributions outside the critical layer and the near-wall region. We also know that the adjoint eigenfunction components that appear in the integrands decay as $y \rightarrow \infty$. So, we can expect the integral contributions to diminish rather quickly. We have checked that this reasoning is correct by comparing the results presented to results where we increase y_{max}

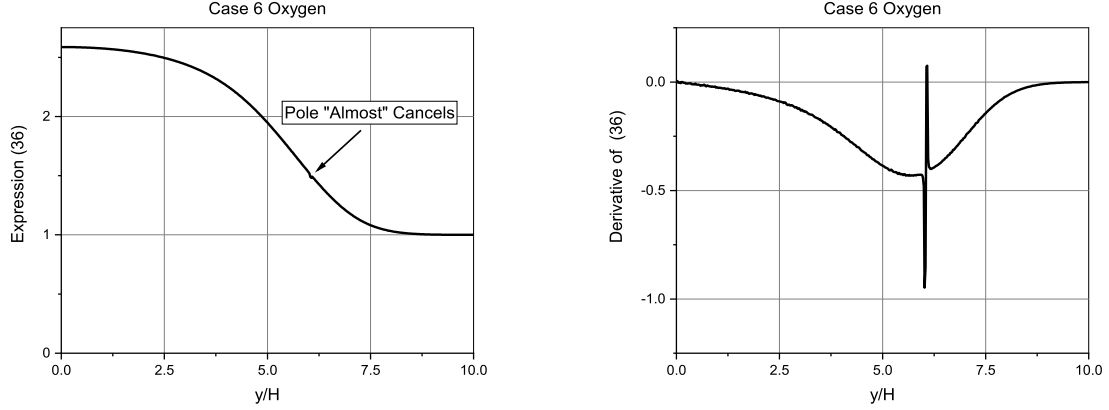


Figure 6.15: Term with singularity in J_{ij} integrals

while using only enough points so that the width of integral discretization grid does not change. As discussed, we have presented results where $y_{max} = 25$ and we use 1201 points. We compared the integrals shown earlier with integrals where we take $y_{max} = 35$ and we use 1681 points. In particular, we compared the absolute error between the two integrals for many different ω . The largest error observed was 0.5%, however, the error was often lower. Consequently, we can examine our results without being too concerned about the contributions to the integral beyond the y_{max} we examined.

6.5 Conclusion

In this paper, we have examined some details of the KF receptivity analysis for binary mixtures of oxygen and nitrogen. The approach applied relies on reducing the full LST problem to the so-called “reduced LST” problem, and applying the stability and receptivity solver developed in Ref.[FT17] with some changes. The modified terms are in Appendices 6.6.2 and 6.6.3. Though this approach does have the pitfall of introducing a singularity in our integral terms that express correlations for the random quantities of interest, we have demonstrated that this singularity happens to be “nearly” cancelled out. This allow us to get close enough to the neutral point, and gain insight into how the KF can excite the Mack second-mode instability. In general, for all cases considered for both oxygen and nitrogen, it was found that the most effective excitations occurred in the critical layer. In particular, the fluctuating shear stress in the stream-wise momentum equations was consistently the dominant contributor, often being an effect over a magnitude larger than the rest of the fluctuations. This is in direct agreement with the results of Ref.[FT17] where a one-component perfect gas was studied. Due to this strong agreement, the results obtained in Ref. [FT17, ET17b, ET17a, ET18] can be seen as representative for the the cases we examined. In particular, the estimates for the upper bound of the transition Reynolds number, and the results regarding the receptivity and development of the wave packets can be understood to be representative. Future work

will include performing the receptivity analysis using the full LST model for the binary mixture, and studying down stream amplification of wave packets. Overall, the current results here provide interesting insight into how kinetic fluctuations can excite unstable modes, and provide an interesting limiting case for studying real gas effects in an airflow.

6.6 Appendices

6.6.1 Reaction rates[GYTL90]

index	Reaction	Forward rate coefficient, k_f , cm ³ /mol – sec
13	$O_2 + O_2 \rightarrow 2O + O_2$	<u>9</u> $\times 3.61 \times 10^{18} \times T^{-1.0} \exp(-5.94 \times 10^4/T)$
14	$O_2 + O \rightarrow 2O + O$	<u>25</u> $\times 3.61 \times 10^{18} \times T^{-1.0} \exp(-5.94 \times 10^4/T)$
21	$N_2 + N_2 \rightarrow 2N + N_2$	<u>2.5</u> $\times 1.92 \times 10^{17} \times T^{-0.5} \exp(-1.131 \times 10^5/T)$
22	$N_2 + N \rightarrow 2N + N$	$4.15 \times 10^{22} \times T^{-1.5} \exp(-1.131 \times 10^5/T)$

Table 6.3: Chemical reactions and forward rate coefficients, Gupta *et al.* [GYTL90].

The tables includes indices superscripts used in Eqs. (6.6) and (6.7). The forward and backward reaction rates recommended by Gupta *et al.* [GYTL90] are given in tables 6.3 and 6.4, respectively. The reaction index indicates how the reaction is referenced in the solver[TUK12]. In the backward reaction rate (index 22), we introduce a correction coefficient (in bold) that is ≈ 1.0155 . This coefficient is introduced in order to have the same equilibrium constant for nitrogen dissociation when the collision particle is N_2 or N (reactions with indices 21 and 22, respectively). Reaction rates with indices 13, 14, and 21 include the weight coefficients (underlined in tables 6.3 and 6.4) from table I of Ref.[GYTL90].

6.6.2 $\mathbf{H}_0$ and $\hat{\mathbf{G}}_\alpha^{(L)}$ Entries

The matrix $\mathbf{H}_0$ of the local-stability problem is expressed as follows. The subscript ‘0’ is omitted for brevity. We have the following the definitions $D = d/dy$, $\mu' = d\mu_s/dT_s$, $m = 2(e-1)/3$, $r = 2(e+2)/3 = m+2$, $\mu_2/\mu = (2/3)e$; $\hat{\omega} = \omega - \alpha U_s$, $\chi = (R/\mu_s - ir\hat{\omega}/p_s)^{-1}$. The chemical source term disturbance is written in the following way for convenience: $W'_s = W_{sT}T' + W_{s\rho}\rho' + W_{sc}c'$. The terms $W_{sT}, W_{s\rho}, W_{sc}$ are partial derivatives of the source term W_s with respect to the non-s subscript. For example, $W_{sc} = (\partial W_s / \partial \rho)_{T,\rho}$. For compactness of notation, the s-subscript is dropped in terms referencing the chemical source term below, but it should be understood that all equations use only one choice of s . $s = 1$ (molecules) or $s = 2$ (atoms) are the two choices. It is also important to notice a few more remarks concerning notation. The mixture gas constant is introduced $\bar{R} = c_1 R_1 + c_2 R_2$. Furthermore,

index	Reaction	Backward rate coefficient, k_b , $\text{cm}^6/\text{mol}^2 - \text{sec}$
13	$O_2 + O_2 \leftarrow 2O + O_2$	$\underline{9} \times 3.01 \times 10^{15} T^{-0.5}$
14	$O_2 + O \leftarrow 2O + O$	$\underline{25} \times 3.01 \times 10^{15} T^{-0.5}$
21	$N_2 + N_2 \leftarrow 2N + N_2$	$\underline{2.5} \times 1.09 \times 10^{16} T^{-0.5}$
22	$N_2 + N \leftarrow 2N + N$	$2.32 \times 10^{21} \times T^{-1.5} (\mathbf{1.09} \times \mathbf{4.15}) / (\mathbf{1.92} \times \mathbf{2.32})$

Table 6.4: Chemical reactions and backward rate coefficients, Gupta *et al.* [GYTL90].

the Reynolds number is written as R rather than Re as written earlier. The non-zero elements are:

$$\begin{aligned}
H_{12} &= H_{56} = H_{78} = 1; \quad H_{21} = \alpha^2 + \beta^2 - \frac{i\rho_s \hat{\omega} R}{\mu_s}; \\
H_{22} &= -\frac{D\mu_s}{\mu_s}; \quad H_{23} = \frac{R\rho_s D U_s}{\mu_s} - i\alpha \frac{D\mu_s}{\mu_s} + [m+1] i\alpha \frac{D\rho_s}{\rho_s}; \\
H_{24} &= \frac{i\alpha R}{\mu_s} + [m+1] \frac{\alpha \hat{\omega}}{p_s}; \quad H_{25} = -[m+1] \frac{\alpha \hat{\omega}}{T_s} - \frac{D(\mu' D U_s)}{\mu_s}; \\
H_{26} &= -\frac{\mu' D U_s}{\mu_s}; \quad H_{31} = -i\alpha; \quad H_{33} = \frac{i\hat{\omega}\rho_s(R_1 - R_2)D_{C_{s,1}}}{W_{\rho}\rho_s(R_1 - R_2) - i\rho_s\hat{\omega}\bar{R} - W_c\bar{R}} - \frac{D\rho_s}{\rho_s};
\end{aligned}$$

$$H_{34} = \frac{i\hat{\omega}\gamma_{ef}M_{ef}^2}{\rho_s T_s \bar{R}} - \frac{i\hat{\omega}W_{\rho}\gamma_{ef}M_{ef}^2(R_1 - R_2)}{\bar{R}T_s [W_{\rho}\rho_s(R_1 - R_2) - i\rho_s\hat{\omega}\bar{R} - W_c\bar{R}]}$$

$$H_{35} = - \left[\frac{i\hat{\omega}(R_1 - R_2)(W_T T_s - W_{\rho}\rho_s)}{T_s(W_{\rho}\rho_s(R_1 - R_2) - i\rho_s\hat{\omega}\bar{R} - W_c\bar{R})} + \frac{i\hat{\omega}}{T_s} \right];$$

$$H_{37} = -i\beta; \quad H_{41} = -i\alpha\chi \left(2\frac{D\mu_s}{\mu_s} - r\frac{D\rho_s}{\rho_s} \right); \quad H_{42} = -i\alpha\chi;$$

$$H_{43} = \chi \left(-\alpha^2 - \beta^2 - \frac{rD\mu_s D\rho_s}{\mu_s \rho_s} - r \left(\frac{D^2 \rho_s}{\rho_s} \right) + 2r \left(\frac{D\rho_s}{\rho_s} \right)^2 + i\frac{R\hat{\omega}\rho_s}{\mu_s} \right);$$

$$\begin{aligned}
H_{44} &= -\frac{i\chi r}{p_s} \left(\alpha DU_s - \hat{\omega} \left[\frac{D\mu_s}{\mu_s} - \frac{D\rho_s}{\rho_s} \right] \right); \\
H_{45} &= i\chi \left(r \frac{\alpha DU_s}{T_s} + \mu' \frac{\alpha DU_s}{\mu_s} + r \frac{DT_s}{T_s^2} \hat{\omega} + r \hat{\omega} \frac{D\rho_s}{T_s \rho_s} - r \hat{\omega} \frac{D\mu_s}{\mu_s T_s} \right) \\
H_{46} &= -i\chi r \frac{\hat{\omega}}{T_s}; \quad H_{47} = -i\beta\chi \left(2 \frac{D\mu_s}{\mu_s} - r \frac{D\rho_s}{\rho_s} \right); \quad H_{48} = -i\beta\chi; \\
H_{62} &= -\frac{2\text{Pr}_{fe}\mu_s(\gamma_{fe}-1)M_{fe}^2 DU_s}{\kappa_s}; \\
H_{63} &= \frac{R\text{Pr}_{fe}\rho_s Dh_s}{\kappa_s} - \frac{2i\alpha\mu_s\text{Pr}_{fe}(\gamma_{fe}-1)M_{fe}^2 DU_s}{\kappa_s} + \frac{i\hat{\omega}R\text{Pr}_{fe}\rho_s[h_{s,1}-h_{s,2}]\bar{R}\rho_s Dc_{s,1}}{\kappa_s[W_\rho\rho_s(R_1-R_2)-i\rho_s\hat{\omega}\bar{R}-W_c\bar{R}]}; \\
H_{64} &= \frac{R\text{Pr}_{fe}\rho_s Dh_s}{\kappa_s} - \frac{2i\alpha\mu_s\text{Pr}_{fe}(\gamma_{fe}-1)M_{fe}^2 DU_s}{\kappa_s} + \frac{i\hat{\omega}R\text{Pr}_{fe}\rho_s[h_{s,1}-h_{s,2}]\bar{R}\rho_s Dc_{s,1}}{\kappa_s[W_\rho\rho_s(R_1-R_2)-i\rho_s\hat{\omega}\bar{R}-W_c\bar{R}]}; \\
H_{65} &= \alpha^2 + \beta^2 - \frac{i\hat{\omega}\rho_s R\text{Pr}_{fe}C_{Pf,mix}}{\kappa_s} - \frac{\text{Pr}_{fe}(\gamma_{fe}-1)M_{fe}^2(DU_s)^2\mu'}{\kappa_s} - \frac{D^2\kappa_s}{\kappa_s} \\
&\quad - \frac{i\hat{\omega}R\text{Pr}_{fe}\rho_s[h_{s,1}-h_{s,2}]\bar{R}(W_T T_s - W_\rho\rho_s)}{\kappa_s T_s[W_\rho\rho_s(R_1-R_2)-i\rho_s\hat{\omega}\bar{R}-W_c\bar{R}]}; \\
H_{65} &= \alpha^2 + \beta^2 - \frac{i\hat{\omega}\rho_s R\text{Pr}_{fe}C_{Pf,mix}}{\kappa_s} - \frac{\text{Pr}_{fe}(\gamma_{fe}-1)M_{fe}^2(DU_s)^2\mu'}{\kappa_s} \\
&\quad - \frac{D^2\kappa_s}{\kappa_s} - \frac{i\hat{\omega}R\text{Pr}_{fe}\rho_s[h_{s,1}-h_{s,2}]\bar{R}(W_T T_s - W_\rho\rho_s)}{\kappa_s T_s[W_\rho\rho_s(R_1-R_2)-i\rho_s\hat{\omega}\bar{R}-W_c\bar{R}]}; \\
H_{66} &= -2\frac{D\kappa_s}{\kappa_s}; \quad H_{83} = i\beta \left((m+1) \frac{D\rho_s}{\rho_s} - \frac{D\mu_s}{\mu_s} \right); \\
H_{84} &= i\beta \frac{R}{\mu_s} + (m+1) \frac{\beta\hat{\omega}}{p_s}; \quad H_{85} = -(m+1) \frac{\beta\hat{\omega}}{T_s}; \quad H_{87} = \alpha^2 + \beta^2 - i\hat{\omega} \frac{R\rho_s}{\mu_s}
\end{aligned}$$

The vector $\hat{\mathbf{G}}_\alpha^{(L)}$ in Eq. (6.25) has the components

$$\begin{aligned}
\hat{G}_{\alpha 1}^{(L)} &= \hat{G}_{\alpha 4}^{(L)} = \hat{G}_{\alpha 5}^{(L)} = \hat{G}_{\alpha 7}^{(L)} = 0 \\
\hat{G}_{\alpha 2}^{(L)} &= -\frac{R}{\mu_s} \left(i\alpha \delta \hat{\Pi}_{\alpha xx} + \frac{\partial \delta \hat{\Pi}_{\alpha xy}}{\partial y} + i\beta \delta \hat{\Pi}_{\alpha xz} \right) \\
\hat{G}_{\alpha 3}^{(L)} &= -\frac{i\hat{\omega}(\gamma_{fe} R M_{fe}^2)^{\frac{1}{2}}(R_1 - R_2)}{W_\rho \rho_s (R_1 - R_2) - i\rho_s \hat{\omega} \bar{R} - W_c \bar{R}} \delta \hat{W}_{\alpha, i} \\
&\quad - \frac{i\hat{\omega}[(\gamma_{fe} - 1) R M_{fe}^2]^{\frac{1}{2}}(R_1 - R_2)}{W_\rho \rho_s (R_1 - R_2) - i\rho_s \hat{\omega} \bar{R} - W_c \bar{R}} \left(i\alpha \delta \hat{J}_{\alpha, x} + \frac{\partial \delta \hat{J}_{\alpha, y}}{\partial y} + i\beta \delta \hat{J}_{\alpha, z} \right) \\
G_{\alpha 4}^{(L)} &= i\alpha \delta \hat{\Pi}_{\alpha yx} + \frac{\partial \delta \hat{\Pi}_{\alpha yy}}{\partial y} + i\beta \delta \hat{\Pi}_{\alpha yz} + O(R^{-1}) \\
\hat{G}_{\alpha 6}^{(L)} &= \frac{R \text{Pr}_{fe}}{\kappa_s} \left(\frac{(\gamma_{fe} - 1) M_{fe}^2}{\text{Pr}_{fe}} \right)^{\frac{1}{2}} \left(i\alpha \delta \hat{Q}_{\alpha, x} + \frac{\partial \delta \hat{Q}_{\alpha, y}}{\partial y} + i\beta \delta \hat{Q}_{\alpha, z} \right) \\
&\quad - \frac{R \text{Pr}_{fe} (\gamma_{fe} - 1) M_{fe}^2}{2\kappa_s} (\delta \Pi_{\alpha, xy} + \delta \Pi_{\alpha, yx}) \left(\frac{\partial U_s}{\partial y} \right) \\
&\quad - \frac{i\hat{\omega} R \text{Pr}_{fe} \rho_s [h_{s,1} - h_{s,2}] (\gamma_{fe} R M_{fe}^2)^{\frac{1}{2}} \bar{R}}{\kappa_s [W_\rho \rho_s (R_1 - R_2) - i\rho_s \hat{\omega} \bar{R} - W_c \bar{R}]} \delta \hat{W}_{\alpha, i} \\
&\quad - \frac{i\hat{\omega} R \text{Pr}_{fe} \rho_s [h_{s,1} - h_{s,2}] [(\gamma_{fe} - 1) R M_{fe}^2]^{\frac{1}{2}} \bar{R}}{\kappa_s [W_\rho \rho_s (R_1 - R_2) - i\rho_s \hat{\omega} \bar{R} - W_c \bar{R}]} \left(i\alpha \delta \hat{J}_{\alpha, x} + \frac{\partial \delta \hat{J}_{\alpha, y}}{\partial y} + i\beta \delta \hat{J}_{\alpha, z} \right)
\end{aligned}$$

6.6.3 The Spectral Density $(q^2)_{\alpha, \beta, \omega}$

As derived in Ref.[FT17], the matrix operator in equation 6.25 is expanded so that we can determine a problem for $\mathbf{A}^{(L)}$ introduced in equation 6.29. Since the new problem for $\mathbf{A}^{(L)}$ will have a non-trivial solution if its right hand side (which has $\mathbf{A}$ eigenfunction terms only) is orthogonal to eigenfunction of the corresponding the adjoint problem for $\mathbf{A}$, the adjoint eigenfunctions $\mathbf{B}$ are introduced. This solvability condition is established by taking an inner product of the RHS of the problem introduced for $A^{(L)}$ with the adjoint eigenfunctions. This then leads to an ODE equation for the amplitude coefficient c . In setting up this ODE for c , the term corresponding to the spectral density is determined at this stage. The denominator of term consists of the inner product between the direct problem eigenfunction $\mathbf{A}$ multiplied by a term in the matrix operator expansion, and the adjoint eigenfunction $\mathbf{B}$. This term can be seen below in the denominator of the spectral density. The numerator of the spectral density consists of the integrals discussed.

The spectral density $(q^2)_{\alpha, \beta, \omega}$ in Eq. (6.31) is expressed as

$$(q^2)_{\alpha, \beta, \omega} = \frac{J_{22} + J_{33} + J_{44} + J_{66} + J_{88} + 2\text{Re}(J_{24} + J_{26} + J_{28} + J_{36} + J_{46} + J_{48} + J_{68})}{\left| \left\langle \frac{\partial \mathbf{H}_0}{\partial \alpha}(x_{10}, \alpha) \mathbf{A}, \mathbf{B} \right\rangle \right|^2}$$

where Re denotes the real part. For convenience, the following coefficients are introduced to make the integrals easier to write explicitly:

$$\begin{aligned}
C_{3W} &= -\frac{i\hat{\omega}(\gamma_{fe}RM_{fe}^2)^{\frac{1}{2}}(R_1 - R_2)}{W_\rho\rho_s(R_1 - R_2) - i\rho_s\hat{\omega}\bar{R} - W_c\bar{R}} \\
C_{3J} &= -\frac{i\hat{\omega}[(\gamma_{fe} - 1)RM_{fe}^2]^{\frac{1}{2}}(R_1 - R_2)}{W_\rho\rho_s(R_1 - R_2) - i\rho_s\hat{\omega}\bar{R} - W_c\bar{R}} \\
C_{6W} &= -\frac{i\hat{\omega}R\text{Pr}_{fe}\rho_s[h_{s,1} - h_{s,2}](\gamma_{fe}RM_{fe}^2)^{\frac{1}{2}}\bar{R}}{\kappa_s[W_\rho\rho_s(R_1 - R_2) - i\rho_s\hat{\omega}\bar{R} - W_c\bar{R}]} \\
C_{6J} &= -\frac{i\hat{\omega}R\text{Pr}_{fe}\rho_s[h_{s,1} - h_{s,2}][(\gamma_{fe} - 1)RM_{fe}^2]^{\frac{1}{2}}\bar{R}}{\kappa_s[W_\rho\rho_s(R_1 - R_2) - i\rho_s\hat{\omega}\bar{R} - W_c\bar{R}]}
\end{aligned}$$

Explicitly, the integral terms are:

$$\begin{aligned}
J_{22} &= 2R^2 \int_0^\infty \frac{T_s}{\mu_s} \left[r\alpha^2 |B_2|^2 + \mu_s^2 \left| D \left(\frac{B_2}{\mu_s} \right) \right|^2 + \beta^2 |B_2|^2 \right] dy \\
J_{33} &= \int_0^\infty |B_3|^2 |C_{3W}|^2 \frac{1}{R_1} \left(k_f^{13} \frac{\rho_s^2 c_{1,s}}{M_1} + k_f^{14} \frac{\rho_s^2 c_{2,s}}{M_2} \right) c_{1,s} \left[1 + \exp \left(-\frac{M_1 \Delta g_s}{RT_s} \right) \right] \\
&\quad + 2T\rho D \left[\left(\frac{\partial \zeta}{\partial C} \right)_{T,p} \right]^{-1} (|\alpha|^2 |B_3|^2 |C_{3J}|^2 + |D(B_3 C_{3J})|^2 + |\beta|^2 |B_3|^2 |C_{3J}|^2) dy \\
J_{44} &= 2 \int_0^\infty \mu_s T_s [\alpha^2 |B_4|^2 + r |DB_4|^2 + \beta^2 |B_4|^2] dy \\
J_{66} &= 2R^2 \text{Pr}_{fe} (\gamma_{fe} - 1) M_{fe}^2 \int_0^\infty \frac{T_s^2}{\kappa_s} \left(|\alpha|^2 |B_6|^2 + \kappa_s^2 \left| D \left(\frac{B_6}{\kappa_s} \right) \right|^2 + |\beta|^2 |B_6|^2 \right) dy \\
&\quad + 2R^2 \text{Pr}_{fe}^2 (\gamma_{fe} - 1)^2 M_{fe}^4 \int_0^\infty \frac{\mu_s T_s}{\kappa_s^2} (DU_s)^2 |B_6|^2 dy \\
&\quad + \int_0^\infty |B_6|^2 |C_{6W}|^2 \frac{1}{R_1} \left(k_f^{13} \frac{\rho_s^2 c_{1,s}}{M_1} + k_f^{14} \frac{\rho_s^2 c_{2,s}}{M_2} \right) c_{1,s} \left[1 + \exp \left(-\frac{M_1 \Delta g_s}{RT_s} \right) \right] dy
\end{aligned}$$

$$\begin{aligned}
& + \int_0^\infty 2T\rho D \left[\left(\frac{\partial \zeta}{\partial C} \right)_{T,p} \right]^{-1} (|\alpha|^2 |B_6|^2 |C_{6J}|^2 + |D(B_6 C_{6J})|^2 + |\beta|^2 |B_6|^2 |C_{6J}|^2) dy \\
& J_{88} = 2R^2 \int_0^\infty \frac{T_s}{\mu_s} \left[\alpha^2 |B_8|^2 + \mu_s^2 \left| D \left(\frac{B_8}{\mu_s} \right) \right|^2 + r\beta^2 |B_8|^2 \right] dy \\
& J_{24} = 2i\alpha R \int_0^\infty T_s \left(mB_2 DB_4^* - \left(DB_2 - B_2 \left(\frac{D\mu_s}{\mu_s} \right) \right) B_4^* \right) dy \\
& J_{26} = -2R^2 \text{Pr}_{fe}(\gamma - 1) M^2 \int_0^\infty \frac{T_s}{\kappa_s} DU_s \left(DB_2 - B_2 \left(\frac{D\mu_s}{\mu_s} \right) \right) B_6^* dy \\
& J_{36} = \int_0^\infty (B_3 B_6^*) (C_{3W} C_{6W}^*) \frac{1}{R_1} \left(k_f^{13} \frac{\rho_s^2 c_{1,s}}{M_1} + k_f^{14} \frac{\rho_s^2 c_{2,s}}{M_2} \right) c_{1,s} \left[1 + \exp \left(-\frac{M_1 \Delta g_s}{RT_s} \right) \right] \\
& + 2T\rho D \left[\left(\frac{\partial \zeta}{\partial C} \right)_{T,p} \right]^{-1} (|\alpha|^2 (B_3 B_6^*) (C_{3J} C_{6J}^*) + (D(B_3 C_{3J}) D(B_6^* C_{6J}^*)) + |\beta|^2 (B_3 B_6^*) (C_{3J} C_{6J}^*)) dy \\
& J_{28} = 2\alpha\beta R^2 \int_0^\infty \frac{T_s}{\mu_s} (m+1) B_2 B_8^* dy \\
& J_{46} = -2R \text{Pr}_{fe}(\gamma - 1) M^2 \int_0^\infty \frac{T_s \mu_s}{\kappa_s} [i\alpha DU_s + i\beta DW_s] B_4 B_6^* dy \\
& J_{48} = 2i\beta R \int_0^\infty T_s \left[B_4 \left(DB_8^* - B_8^* \left(\frac{D\mu_s}{\mu_s} \right) \right) - mDB_4 B_8^* \right] dy \\
& J_{68} = -2R^2 \text{Pr}_{fe}(\gamma - 1) M^2 \int_0^\infty \frac{T_s}{\kappa_s} DW_s B_6 \left(DB_8^* - B_8^* \left(\frac{D\mu_s}{\mu_s} \right) \right) dy \\
& D = d/dy
\end{aligned}$$

Chapter 7

A Model of Distributed Receptivity to Kinetic Fluctuations in High-Speed Boundary Layers

Nomenclature

$\mathbf{A}_0$	Eigenvector of the direct stability problem
a	Speed of sound
$\mathbf{B}_0$	Eigenvector of the adjoint stability problem
c	Amplitude of the unstable mode
c_r	Real part of ω/α
c_i	Imaginary part of ω/α
c_p	Specific heat at constant pressure
F	Dimensionless frequency parameter
$\hat{\mathbf{G}}_L$	Fourier component of the Langevin vector
J	Product $S K ^{-2}$
K	$= i\epsilon^{-1} \int_{x_{10}}^{x_1} \alpha \, dx_1$
k_B	Boltzmann constant
M	Mach number
N	$= -\int_{x_0}^x \alpha_i \, dx$, N-factor
P	Pressure (p - pressure perturbation)
Pr	Prandtl number
R	Reynolds number
R_g	Gas constant
S	Receptivity spectral density
T	Temperature
t	Time
u_i	Velocity component
x	Streamwise coordinate

x_1	$= \epsilon x$
y	Distance from the wall
z	Spanwise coordinate
α	$= \alpha_r + i\alpha_i$, streamwise wave number (complex)
β	Spanwise wave number
γ	Specific heats ratio
Δ^*	Boundary layer thickness scale
$\delta\Pi_{ij}$	Fluctuating shear stress
δQ_i	Fluctuating heat flux
ϵ	$= R^{-1}$, small parameter
ϵ_L	Small parameter characterizing kinetic fluctuations
θ	Temperature perturbation
λ	Thermal conductivity
μ	Viscosity coefficient
μ_2	Bulk viscosity coefficient
ν	Kinematic viscosity coefficient
ρ	Density
σ	$= -\alpha_i$
ω	$(= FR)$ Angular frequency
<i>Superscript</i>	
(L)	Langevin forcing terms
$*$	Dimensional parameter (or complex conjugate)
<i>Subscript</i>	
e	Boundary layer upper edge condition
i	Imaginary part
rms	Root-mean-square
max	downstream location of the wave packet observation
r	Real part
s	Mean flow parameter
w	Parameter at the wall condition
β	Derivative of N with respect to β
ω	Derivative of N with respect to ω
0	Neutral point
∞	Free stream flow parameter

7.1 Introduction

The laminar-turbulent transition of hypersonic boundary layer flows is a complex phenomenon for which no universal characterization exists. The modern view of the transition process in low disturbance environments consists of three main stages: excitation of unstable modes by external forcing (receptivity), linear convective amplification, and finally,

non-linear breakdown and transition.[Mor69, Res76] In recent decades, much attention has been given to the incorporation of receptivity analysis into the standard linear stability theory (LST) framework. This is due to the potential for amplitude based methods [Mac77, Fed15] to yield improved transition onset locus prediction compared to methods based on LST alone, i.e. the conventional e^N method. The bulk of recent receptivity studies have focused on the excitation of unstable modes by free stream disturbances such as acoustic waves or vorticity and entropy perturbations, wall-induced disturbances including roughness, vibration and blowing suction, or interaction with other external elements such as atmospheric particulates.

In the present work, we are interested in the case where sources of external forcing of the type just described are absent or negligible. In this case, there is evidence that kinetic fluctuations (KF), also referred to as thermal noise, may be the source of transition onset through the excitation of convectively amplified unstable modes. This idea was first suggested by Betchov[Bet61] in the 1960's but was not explored quantitatively until asymptotic analysis of KF receptivity in low-speed boundary layers was conducted by Zavolskii and Reutov in 1984 [ZR84]. More recently, the problem has been studied by Luchini [Luc08b, Luc08a, Luc10, Luc17] in the context of low-speed incompressible boundary layers, and by Fedorov and Averkin [FA10], Fedorov [Fed15], and Fedorov and Tumin [FT17], in connection to high speed transition. Analysis of experimental data in the Langley Mach 3.5 quiet tunnel indicates that KF could play a role in triggering laminar-turbulent transition at quiet flow conditions[Fed12]. Ref. [Fed15] showed that vorticity and entropy perturbations as well as atmospheric particulates could both be ruled out as likely causes of transition for the HIFiRE-1 flight test [LCC⁺11]. KF-induced transition was proposed as an alternative explanation, and indeed, analysis of the flight configuration showed that the theoretically predicted KF-induced transition onset locus was not far from the experimentally observed location. The KF-receptivity mechanism has also been shown to be relevant at other flight conditions [FT17]. Importantly, due to the indispensable nature of KF in any environment of practical interest, the theoretically predicted KF-induced transition location can be usefully thought of as an upper bound beyond which transition cannot be avoided. Thus, there is value in developing robust methodologies for computing receptivity to KF even for cases in which transition is potentially induced by some other source of external forcing.

Analyses of receptivity to KF, including the present work, are based on the theory of *fluctuating hydrodynamics* formulated by Landau and Lifshitz[LL57]. The theory leads to the inclusion of fluctuating dissipative fluxes into the Navier-Stokes equations in the form of Langevin source terms. These additional terms correspond to fluctuating shear stress and fluctuating heat flux. Expressions for the space-time correlations between these random fluxes can be obtained through the fluctuation-dissipation theorem. This formulation allows one to solve the receptivity problem and to find the downstream amplitude of the disturbance generated by thermal noise.

Studies of receptivity to KF have also relied on the method of multiple scales (MMS), introduced through a Wentzel-Kramers-Brillouin (WKB) type ansatz. This approach is motivated by the disparity between the stream-wise scales at which the boundary layer mean

profiles vary, and the comparatively small scales at which the disturbance quantities vary. In the context of LST of boundary layer flows, the method of multiple scales was first introduced by Bouthier[Bou72], Gaster [Gas74], Nayfeh et al. [AHNM74] to study the influence of non-parallel effects on the stability of low speed boundary layers. Saric and Nayfeh [SN75, SN77] showed that by accounting for non-parallel flow effects, one could attain better theoretical agreement with experimental neutral curve data, particularly in the presence of an adverse pressure gradient. Later on, the method was used to study the stability of weakly non-parallel compressible boundary-layer flows.[PN79, Gap80, Nay80, EH80, TF82] The method of multiple scales has also proven useful in the study of distributed receptivity. Examples include an acoustic wave interacting with a boundary layer on a smooth wall [Fed82, Fed85] or a wall with distributed roughness elements [ZF87, Cho94]. One can find the mathematical details and relevant bibliography in Refs. [Cho98, Tum11, FK11, LB14, ZL14]. Kinetic fluctuations are an example of distributed forcing in a boundary-layer flow.

Studies of distributed and localized forcing in boundary layers support the idea that the most efficient coupling with boundary layer modes occurs in the vicinity of the lower branch of the neutral curve. This is because the detuning of the coupling waves is minimized when the unstable discrete mode is neutral. In certain cases, the resonance phenomenon leads to singular behavior in the mathematical formulation at the neutral point. This occurs for a vibrating wall with wave length equal to that of the Tollmien-Schlichting wave in a low-speed boundary layer [TF83] and may also be realized in the scattering of acoustic and vortical disturbances in a boundary layer on a wavy surface.[ZRR83] Refs. [TF83, ZRR83] demonstrated that dominant generation of Tollmien-Schlichting waves occurs in a small interval of width $\epsilon^{1/2}L$ around the neutral point, where L is the distance from the leading edge, and $\epsilon = R^{-1}$. Here, the Reynolds number R is based on a boundary-layer scale.¹ The steepest descent method has been commonly used to analyze the excitation that occurs in this region (see, for example, Ref. [ZF87]). In Ref. [FK02], this approach was formally incorporated into an asymptotic matching procedure in which a non-resonant outer solution is matched to a resonant inner solution near the lower neutral point. This approach is similar to the asymptotic analysis of Wu[Wu11] who studied convecting gusts interacting with sound and a boundary layer. The matched asymptotic (MA) approach is the one adopted recently in Ref. [FT17] to study KF-receptivity. In Ref. [Luc17], the KF-receptivity problem was solved through a direct numerical solution of the amplitude equation in the excitation region. For a Mach 0.04 test case, the results obtained using the MA procedure were in agreement with those obtained with the direct numerical approach.

It was pointed out in Ref. [ET17b] that asymptotic treatment of the excitation region is sometimes questionable when the discrete mode exhibits long-scale near-neutral behavior upstream of the neutral point. This is particularly the case where a near-neutral first Mack mode instability exists immediately upstream of the dominant second Mack mode.² Such behavior may occur across a range of Mach numbers. Here, we describe an alternative

¹The results of Refs. [ZRR83] and [TF83] are seen to be consistent when it is taken into account that the former uses a boundary layer scale $\sim \epsilon L$ to non-dimensionalize the stream-wise coordinate x , while the latter utilizes the body length scale L .

²See clarification of terminology in Ref. [FT11].

approach, similar to the one recently employed by Luchini.[Luc17] The method is again based on the MMS. The outcome is an expression for disturbance amplitudes which is in some cases asymptotically equivalent to that developed in Ref. [FT17] but which does not explicitly depend on the neutral point location and the behavior of the growth rate at that point. We also explore the validity of the asymptotic approach for the cases presented herein and show that it can be applied across a range of configurations.

Briefly, the structure of the paper is as follows. The governing equations and their linearized form are introduced in §7.2. Section 7.3 provides the formulation and solution of the receptivity problem. Some discussion of the receptivity solution is provided in §7.5, followed by a validation of the code developed based on the present formulation for a Mach 0.04 and a Mach 6 configuration. In §7.5.7.5.3 and §7.5.7.5.4, we consider a Mach 10 case for which the growth rates exhibit long-scale near neutral behavior upstream of the lower neutral branch and a Mach 2 case where the dominant instability is the first Mack mode. We conclude with a recap of the key results of this work and their implications.

7.2 Equations

7.2.1 Governing Equations

For the reader's convenience, the relevant hydrodynamic equations have been included here, reproduced from Ref. [FT17]. We consider three-dimensional unsteady compressible flow with velocity $\mathbf{u} = (u_1, u_2, u_3)$, density ρ , pressure p , and temperature T which are functions of time t and coordinates r_1, r_2 , and r_3 . The flow is governed by the Navier-Stokes equations written in dimensional form using the index notation as follows:

$$\frac{\partial \rho^*}{\partial t^*} + \text{div} \rho^* \mathbf{u}^* = 0, \quad (7.1)$$

$$\rho^* \left[\frac{\partial u_i^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) u_i^* \right] = -\frac{\partial p^*}{\partial r_i^*} + \frac{\partial \Pi_{ik}^*}{\partial r_k^*}, \quad (7.2)$$

$$\rho^* T^* \left(\frac{\partial s^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) s^* \right) = \frac{1}{2} \Pi_{ik}^* \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) - \frac{\partial Q_i^*}{\partial r_i^*}, \quad (7.3)$$

where the stress tensor Π_{ij}^* and the heat flux vector Q_i^* are given by

$$\Pi_{ik}^* = \mu^* \left(\frac{\partial u_i^*}{\partial r_k^*} + \frac{\partial u_k^*}{\partial r_i^*} \right) + \left(\mu_2^* - \frac{2}{3} \mu^* \right) \delta_{ik} \text{div} \mathbf{u}^* + \delta \Pi_{ik}^*, \quad (7.4)$$

$$Q_i^* = -\lambda^* \frac{\partial T^*}{\partial r_i^*} + \delta Q_i^*, \quad (7.5)$$

and s^* is entropy per unit mass, $\delta \Pi_{ik}^*$ and δQ_i^* are mean zero Langevin source terms corresponding to fluctuating shear stress and fluctuating heat flux respectively. We consider a

calorically perfect gas with equation of state

$$p^* = \rho^* R_g^* T^*, \quad (7.6)$$

where R_g^* is the gas constant per unit mass. The spatial-time correlations between these terms were derived by Landau and Lifshitz[LL57] using the fluctuation-dissipation theorem:

$$\overline{\delta \Pi_{ij}^*(\mathbf{r}^*, t^*) \delta \Pi_{kl}^*(\mathbf{r}'^*, t'^*)} = 2k_B T^* \left[\mu^* (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \left(\mu_2^* - \frac{2}{3} \mu^* \right) \delta_{ij} \delta_{kl} \right] \delta_{\mathbf{r}, t}, \quad (7.7)$$

$$\overline{\delta Q_i^*(\mathbf{r}^*, t^*) \delta Q_j^*(\mathbf{r}'^*, t'^*)} = 2k_B \lambda^* T^{*2} \delta_{ij} \delta_{\mathbf{r}, t}, \quad (7.8)$$

$$\delta_{\mathbf{r}, t} = \delta(\mathbf{r}^* - \mathbf{r}'^*) \delta(t^* - t'^*), \quad (7.9)$$

where the over-bar represents the ensemble averaging; δ_{ij} is the Kronecker symbol, and $\delta(\cdot)$ is the Dirac delta function.

We continue along the lines of Ref. [FT17] and non-dimensionalize velocity, time, pressure and temperature using boundary-layer thickness length-scale Δ^* , free-stream velocity U_∞^* , time-scale Δ^*/U_∞^* , doubled dynamic pressure $\rho_\infty^* U_\infty^{*2}$ and temperature T_∞^* . This yields the nondimensional governing equations

$$\frac{\partial \rho}{\partial t} + \text{div} \rho \mathbf{u} = 0, \quad (7.10)$$

$$\rho \left[\frac{\partial u_i}{\partial t} + (\mathbf{u} \cdot \nabla) u_i \right] = - \frac{\partial p}{\partial r_i} + \frac{1}{R} \frac{\partial}{\partial r_k} \left[\mu \left[\left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right) + \left(\frac{\mu_2}{\mu} - \frac{2}{3} \right) \delta_{ik} \text{div} \mathbf{u} \right] \right] + \epsilon_L \frac{\partial \delta \Pi_{ik}}{\partial r_k}, \quad (7.11)$$

$$\rho \left(\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T \right) = (\gamma - 1) M^2 \left(\frac{\partial p}{\partial t} + (\mathbf{u} \cdot \nabla) p \right) + \frac{1}{R \text{Pr}} \frac{\partial}{\partial r_i} \left(\mu \frac{\partial T}{\partial r_i} \right) + \frac{(\gamma - 1) M^2}{R} \Phi + \epsilon_L (\gamma - 1) M^2 \frac{1}{2} \delta \Pi_{ik} \left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right) - \epsilon_L \left[\frac{(\gamma - 1) M^2}{\text{Pr}} \right]^{1/2} \frac{\partial \delta Q_i}{\partial r_i}, \quad (7.12)$$

$$p = \frac{1}{\gamma M^2} \rho T, \quad (7.13)$$

$$\Phi = \frac{\mu}{2} \left[\left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right) + \left(\frac{\mu_2}{\mu} - \frac{2}{3} \right) \delta_{ik} \text{div} \mathbf{u} \right] \left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} \right). \quad (7.14)$$

Here $R = U_\infty^* \rho_\infty^* \Delta^* / \mu_\infty^*$ is the Reynolds number, $\text{Pr} = \mu^* c_p^* / \lambda^*$ is the Prandtl number, $M = U_\infty^* / a_\infty^*$ is the Mach number, and

$$\epsilon_L = \frac{1}{R^2} \left(\frac{k_B T_\infty^* U_\infty^{*2} \rho_\infty^{*2}}{\mu_\infty^{*3}} \right)^{1/2} \quad (7.15)$$

is a small parameter characterizing the magnitude of kinetic fluctuations. Ultimately, the derivations in the following section will lead to the observation that the downstream root-mean-square (*rms*) magnitude of the KF-induced disturbance is proportional to this parameter.

7.2.2 Disturbance Equations

We utilize the notations: $(r_1, r_2, r_3) = (x, y, z)$, $(U_1, U_2, U_3) = (U_s, V_s, W_s)$, $(u'_1, u'_2, u'_3) = (u, v, w)$, $T' = \theta$, $p' = p$. The flow field is decomposed into a steady two-dimensional base flow and an unsteady disturbance as

$$\Psi(x, y, z, t) = \Psi_s(x, y) + \epsilon_L \psi'(x, y, z, t). \quad (7.16)$$

The steady base flow is a laminar boundary layer which is weakly nonparallel in the x -direction and is independent of the span-wise coordinate z . This suggestion approximation of the base flow by

$$U_s = U_0(x_1, y) + O(\epsilon), \quad V_s = \epsilon V_0(x_1, y) + O(\epsilon^2), \quad P_s = P_0(x_1) + O(\epsilon^2), \quad (7.17)$$

where $\epsilon = R^{-1}$ is small parameter characterizing the nonparallel flow effect and $x_1 = \epsilon x$ is a slow coordinate associated with the scales on which the mean flow quantities vary in the stream-wise direction. Here $R = U_\infty^* \Delta^* / \nu_\infty^*$ is the Reynold's number based on the boundary-layer length scale Δ^* . Linearizing Eqs. (7.10)-(7.14) about the base flow and maintaining the first and second order terms in ϵ yields the following system of equations, in which non-parallel terms appear at $O(\epsilon)$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho_s u + U_0 \rho) + \frac{\partial}{\partial y} (\rho_s v + \epsilon \rho V_0) + \rho_s \frac{\partial w}{\partial z} = 0, \quad (7.18)$$

$$\begin{aligned} \rho_s \left(\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (u U_0) + v \frac{\partial U_0}{\partial y} + V_0 \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \\ \frac{1}{R} \left\{ \frac{\partial}{\partial x} \left[\mu_s \left(r \frac{\partial u}{\partial x} + m \frac{\partial v}{\partial y} + m \frac{\partial w}{\partial z} \right) + \mu \left(r \frac{\partial U_0}{\partial x} + \epsilon m \frac{\partial V_0}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[\mu_s \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \mu \left(\frac{\partial U_0}{\partial y} \right) \right] + \right. \\ \left. \frac{\partial}{\partial z} \left[\mu_s \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \right\} + \left(\frac{\partial \delta \Pi_{xx}}{\partial x} + \frac{\partial \delta \Pi_{xy}}{\partial y} + \frac{\partial \delta \Pi_{xz}}{\partial z} \right), \quad (7.19) \end{aligned}$$

$$\begin{aligned} \rho_s \left(\frac{\partial v}{\partial t} + U_0 \frac{\partial v}{\partial x} + \epsilon \frac{\partial}{\partial y} (v V_0) \right) = - \frac{\partial p}{\partial y} + \\ \frac{1}{R} \left\{ \frac{\partial}{\partial x} \left[\mu_s \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial \mu}{\partial x} \frac{\partial U_0}{\partial y} + \frac{\partial}{\partial y} \left[\mu_s \left(m \frac{\partial u}{\partial x} + r \frac{\partial v}{\partial y} + m \frac{\partial w}{\partial z} \right) + \mu \left(m \frac{\partial U_0}{\partial x} + \epsilon r \frac{\partial V_0}{\partial y} \right) \right] + \right. \\ \left. \frac{\partial}{\partial z} \left[\mu_s \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \right\} + \left(\frac{\partial \delta \Pi_{yx}}{\partial x} + \frac{\partial \delta \Pi_{yy}}{\partial y} + \frac{\partial \delta \Pi_{yz}}{\partial z} \right), \quad (7.20) \end{aligned}$$

$$\begin{aligned} \rho_s \left(\frac{\partial w}{\partial t} + U_0 \frac{\partial w}{\partial x} + \epsilon V_0 \frac{\partial w}{\partial y} \right) = - \frac{\partial p}{\partial z} + \\ \frac{1}{R} \left\{ \frac{\partial}{\partial x} \left[\mu_s \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] + \frac{\partial}{\partial y} \left[\mu_s \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[\mu_s \left(m \frac{\partial u}{\partial x} + m \frac{\partial v}{\partial y} + r \frac{\partial w}{\partial z} \right) \right] \right\} + \\ \left(\frac{\partial \delta \Pi_{zx}}{\partial x} + \frac{\partial \delta \Pi_{zy}}{\partial y} + \frac{\partial \delta \Pi_{zz}}{\partial z} \right), \quad (7.21) \end{aligned}$$

$$\begin{aligned}
& \rho_s \left(\frac{\partial \theta}{\partial t} + u \frac{\partial T_s}{\partial x} + U_0 \frac{\partial \theta}{\partial x} + v \frac{\partial T_s}{\partial y} + \epsilon V_0 \frac{\partial \theta}{\partial y} + w \frac{\partial T_s}{\partial z} \right) + \rho \left(U_0 \frac{\partial T_s}{\partial x} + \epsilon V_0 \frac{\partial T_s}{\partial y} \right) \\
& = (\gamma - 1) M^2 \left[\frac{\partial p}{\partial t} + u \frac{\partial P_0}{\partial x} + U_0 \frac{\partial p}{\partial x} + V_0 \frac{\partial p}{\partial y} + \frac{1}{R} \Phi' \right] + \\
& \frac{1}{R \text{Pr}} \left[\frac{\partial}{\partial x} \left(\mu_s \frac{\partial \theta}{\partial x} \right) + \frac{\partial \mu}{\partial x} \frac{\partial T_s}{\partial x} + \frac{\partial}{\partial y} \left(\mu_s \frac{\partial \theta}{\partial y} + \mu \frac{\partial T_s}{\partial y} \right) + \frac{\partial}{\partial z} \left(\mu_s \frac{\partial \theta}{\partial z} \right) \right] - \\
& \left(\frac{(\gamma - 1) M^2}{\text{Pr}} \right)^{1/2} \left(\frac{\partial \delta Q_x}{\partial x} + \frac{\partial \delta Q_y}{\partial y} + \frac{\partial \delta Q_z}{\partial z} \right) + \\
& \frac{(\gamma - 1) M^2}{2} \left[(\delta \Pi_{xy} + \delta \Pi_{yx}) \frac{\partial U_0}{\partial y} \right], \quad (7.22)
\end{aligned}$$

$$\frac{p}{P_0} = \frac{\theta}{T_s} + \frac{\rho}{\rho_s}, \quad (7.23)$$

where Φ' is the linearized dissipation function (one can find its explicit form, for example, in Ref. [Nay80]). Note that stream-wise derivatives $\partial/\partial x$ of mean flow quantities in the above equations, are of order ϵ . The coefficients r and m in Eqs. (7.19) - (7.21) depend on the bulk viscosity ratio $\mu_2/\mu = 2e/3$ as $m = 2(e - 1)/3$, $r = 2(e + 2)/3$. In the present work, we take $e = 0$ in accordance with the Stokes hypothesis. The disturbance of viscosity in Eqs. (7.18 - 7.23) is related to the disturbance of temperature as $\mu = (d\mu_s/dT)\theta$.

7.3 Receptivity Analysis

In this section, we derive an expression for the *rms* amplitude of KF-induced disturbances for the compressible flat plate boundary layer. We begin by introducing the disturbance vector

$$\mathbf{F}(x, y, z, t) = (u, \frac{\partial u}{\partial y}, v, p, \theta, \frac{\partial \theta}{\partial y}, w, \frac{\partial w}{\partial y}). \quad (7.24)$$

Let $\hat{\mathbf{F}}$ be the Fourier transform of $\mathbf{F}$ with respect to time t and the span-wise coordinate z , where our convention is given by

$$\hat{s}(\omega, \beta) = \int_{-\infty}^{\infty} s(t, z) e^{(i\omega z - i\beta t)} dt dz, \quad s(t, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{s}(\omega, \beta) e^{-(i\omega z - i\beta t)} d\omega d\beta. \quad (7.25)$$

Similarly, $\hat{\mathbf{G}}^{(L)}$ will denote the transformed Langevin source terms corresponding to the fluctuating shear stress and heat flux components induced by kinetic fluctuations¹. The

¹The Fourier transform of a random process such as $\mathbf{G}^{(L)}$ is not, in general, well defined because integrals of form (7.25) are not guaranteed to converge. There is no issue with our formal use of the quantity $\hat{\mathbf{G}}^{(L)}$, however, since our interest ultimately lies in the Fourier transform of the autocorrelation of $\mathbf{G}^{(L)}$ for which no such convergence issues exist. This issue was recognized by Landau and Lifshitz[LL57] who referred to the work of Rytov[Ryt87] for a discussion of *probabilistic convergence*.

elements of the random forcing vector $\hat{\mathbf{G}}^{(L)}$ are given in Appendix A of Ref. [FT17]. Note that spectral correlation formulae for the components of $\hat{\mathbf{G}}^{(L)}$ may be obtained by making the transformation:

$$\delta(z^* - z'^*) \delta(t^* - t'^*) \rightarrow (2\pi)^2 \delta(\omega - \omega') \delta(\beta - \beta'). \quad (7.26)$$

in expressions (7.7) and (7.8).

The transformed boundary layer disturbance equations can be expressed in terms of the slow coordinate x_1 as a first order linear system with the 8×8 matrix operators H_1, H_2 , and H_3 as

$$\begin{aligned} H(y, \partial_y, x_1, \epsilon \partial_{x_1}, \omega, \beta) \hat{\mathbf{F}} &= \hat{\mathbf{G}}^{(L)}, \\ y = 0 : \hat{F}_i &= 0, \quad i = 1, 3, 5, 7, \\ y \rightarrow \infty : |\hat{F}_i| &< \infty, \quad i = 1, 3, 5, 7. \end{aligned} \quad (7.27)$$

The second, fourth and eighth equations in (7.27) represent the stream-wise, wall-normal and span-wise momentum equations respectively, the third equation is the continuity equation, and the remaining three arise from reducing the momentum equations to differential equations which are first order in y . As is common in receptivity studies, we consider a partial solution to the problem which takes into account only the dominant unstable mode of the discrete spectrum. The corresponding eigenvalue is denoted by α . In this approximation, the solution to the distributed receptivity problem is expressed in terms of the generation of this mode by kinetic fluctuations.

A multiple scales expansion is introduced through the WKB type ansatz

$$\hat{\mathbf{F}} = [c(x_1) \mathbf{A}_0(x_1, y) + \epsilon \mathbf{A}_1(x_1, y) + O(\epsilon^2)] K(x_1), \quad K(x_1) = \exp \left[i\epsilon^{-1} \int_{x_{10}}^{x_1} \alpha(\eta; \omega, \beta) d\eta \right]. \quad (7.28)$$

The zeroth order amplitude vector-function $\mathbf{A}_0(x_1, y)$ in Eq. (7.28) is an eigenfunction of the conventional LST problem

$$\begin{aligned} \mathcal{L}_0 \mathbf{A}_0 &= 0, \quad \mathcal{L}_0 = \left[\frac{\partial}{\partial y} - H_0 \right], \\ y = 0 : A_{0i} &= 0, \quad i = 1, 3, 5, 7, \\ y \rightarrow \infty : |A_{0i}| &\rightarrow 0, \quad i = 1, 3, 5, 7. \end{aligned} \quad (7.29)$$

For convenience, the lower integration bound x_{10} in Eq. (7.28) is taken in this work to be the neutral point corresponding to a specified frequency ω and spanwise wave number β . We note, however, that the product $c(x_1)K(x_1)$ (and, thus, the final solution) is independent of this choice.

The coefficients of Eq. (7.29) depend on the mean flow profiles and are slow functions of the variable x_1 , as are the eigenfunction $\mathbf{A}_0 = \mathbf{A}_0(x_1, y)$ and the eigenvalue $\alpha = \alpha(x_1)$. The normalization of $\mathbf{A}_0$ is chosen based on the desired interpretation of the final result. For

example, if the eigenfunction is normalized so that the quantity of interest (wall pressure, streamwise velocity perturbation, etc.) is equal to one, the product $g(x_1) = \epsilon_L c(x_1) K(x_1)$ corresponds to the physical amplitude of that quantity as a function of the downstream coordinate x_1 .

At this stage it is necessary to point out a subtlety in the introduction of the amplitude function $c(x_1)$. This amplitude, expressed as a slow function of x_1 , represents the excitation of the discrete mode by kinetic fluctuations, here modeled as white noise. Since the spectrum of the forcing contains all wavenumbers (and frequencies), it would be reasonable to suspect that c in fact exhibits variation on all length scales. This difficulty is resolved by the fact that our interest lies not in c but instead in the physically meaningful mean-square quantity $\overline{|cK|^2}$ which is statistically well-behaved.

Inserting (7.28) into the forced disturbance equations yields the following linear equation for the amplitude correction term $\mathbf{A}_1$:

$$\mathcal{L}_0 \mathbf{A}_1 = \mathbf{H}_2 \left(\frac{dc}{dx_1} \mathbf{A}_0 + c \frac{\partial \mathbf{A}_0}{\partial x_1} \right) + c \mathbf{H}_3 \mathbf{A}_0 + \epsilon^{-1} \hat{\mathbf{G}}^{(L)} K^{-1} + O(\epsilon). \quad (7.30)$$

where $\mathbf{H}_2 = -i\partial \mathbf{H}_0 / \partial \alpha$ and $\mathbf{H}_3$ contains terms related to the non-parallel nature of the mean flow. A solvability condition exists which requires that the right-hand side of Eq. (7.30) be orthogonal to the vector-function $\mathbf{B}_0$, defined as a solution to the adjoint problem

$$\begin{aligned} \mathcal{L}_0^\dagger \mathbf{B}_0 &= 0, \quad \mathcal{L}_0^\dagger = \left[\frac{\partial}{\partial y} + \mathbf{H}_0^H \right], \\ y = 0 : B_{0i} &= 0, \quad i = 2, 4, 6, 8, \\ y \rightarrow \infty : |B_{0i}| &\rightarrow 0, \quad i = 2, 4, 6, 8. \end{aligned} \quad (7.31)$$

Here superscript H denotes the conjugate transpose (Hermitian) of the corresponding matrix and $\cdot^*$ indicates complex conjugation. The adjoint operator in (7.31) is introduced based on the inner product defined by:

$$\langle \mathbf{f}, \mathbf{h} \rangle = \int_0^\infty \sum_{i=1}^8 f_i h_i^* dy. \quad (7.32)$$

Here we are interested in the generation of a single unstable discrete mode by kinetic fluctuations. One can find a complete discussion of the full spectrum in Ref. [Tum07].

Applying the solvability condition yields a first order ODE for the amplitude coefficient $c(x_1)$:

$$\frac{dc}{dx_1} + c W(x_1; \alpha, \beta, \omega) + \epsilon^{-1} q(x_1; \alpha, \beta, \omega) \exp \left[-i\epsilon^{-1} \int_{x_{10}}^{x_1} \alpha(\eta; \omega, \beta) d\eta \right] = 0, \quad (7.33)$$

where

$$q(x_1, \alpha; \beta, \omega) = \Gamma^{-1} \langle \hat{\mathbf{G}}^{(L)}, \mathbf{B}_0 \rangle, \quad W(x_1; \alpha, \beta, \omega) = \Gamma^{-1} \left\langle \mathbf{H}_2 \frac{\partial \mathbf{A}_0}{\partial x_1} + \mathbf{H}_3 \mathbf{A}_0, \mathbf{B}_0 \right\rangle \quad (7.34)$$

$$\Gamma = \langle \mathbf{H}_2 \mathbf{A}_0, \mathbf{B}_0 \rangle. \quad (7.35)$$

The second term in Eq. (7.33) is associated with the conventional nonparallel flow effect on the downstream development of the discrete mode. The results of Refs. [FA10, Luc17] indicate that the KF-receptivity mechanism is relevant for laminar-turbulent transition only for low frequency disturbances where these effects are known to be negligible, and thus, we proceed by setting $W \equiv 0$ in (7.33).

As alluded to previously, the presence of the highly oscillatory term qK^{-1} in (7.33) seems to conflict with our assumption that $c = c(x_1)$ is a slowly varying function of the streamwise coordinate. This issue has been addressed to varying degrees in previous works. Zavol'skii and Reutov[ZRR83] offer no discussion of the issue but formally arrive at an expression for the mean square disturbance amplitude which is indeed a slow function of the coordinate we call x_1 . (No slow scale is explicitly introduced in their formalism). Luchini[Luc17] offers an intuitive explanation which resembles the argument for why a stationary phase approximation is valid despite its failure in regions of rapid oscillation. This reasoning is substantiated by the asymptotic analysis of Ref. [FT17] which shows that the forcing wavenumber contributing to excitation is determined by the degree of resonance with the discrete mode in question. This analysis shows that, indeed, forcing wavenumbers outside a resonance interval of width $O(\epsilon^{1/2}/\Delta^*)$ (where qK^{-1} will be rapidly varying) represent an exponentially small contribution to the overall excitation of the mode.

The model adopted here is similar to that of Ref. [Luc17], which is formulated for incompressible flow, and the two formulations are equivalent in the low-speed limit. No scaling is explicitly introduced in Ref. [Luc17], but comparison of the final numerical results leads to the conclusion that, in the low speed limit, our $\epsilon^{-1}\hat{\mathbf{G}}^{(L)}$ corresponds to $\mathbf{g}$ in Eq. (22) of that work. Note also that the spectral intensity q in our Eq. (7.33) is equivalent to $F^{(1)}$ in Eq. (1.4) of Ref. [ZRR83] (in the low speed limit) with opposite sign. Importantly, the final result derived herein is consistent with the asymptotic analysis of Ref. [FT17].

Equation (7.33) may be solved explicitly to obtain a formal expression for the amplitude function $c(x_1)$:

$$c(x_1) = \epsilon^{-1} \int_{x_{1u}}^{x_1} q(\eta; \alpha, \beta, \omega) \exp \left[-i\epsilon^{-1} \int_{x_{10}}^{\eta} \alpha(\xi; \omega\beta) d\xi \right] d\eta, \quad (7.36)$$

where the lower bound x_{1u} is some yet to be determined upstream location where $c \approx 0$. Note that c is a random quantity due to the dependence of q on the stochastic forcing terms $\hat{\mathbf{G}}^{(L)}$. To arrive at a physically meaningful result we consider the root-mean-square (*rms*) of the disturbance $\hat{\mathbf{F}}_m$. In this way, we render the details of a single (highly-oscillatory) realization of the forcing term irrelevant in favor of its known statistical properties.

Using the solution (7.36), and the space-time correlations (7.7) and (7.8), together with (7.26), we arrive at an expression for the mean square disturbance associated with the physical parameter of interest

$$\overline{g^2}(x_1) = \frac{\epsilon_L^2}{2\pi^2} \int_0^\infty \int_{-\infty}^\infty \left[\int_{x_{1u}}^{x_1} (q^2)_{\alpha,\beta,\omega}(\eta) \left| \frac{K(x_1)}{K(\eta)} \right|^2 d\eta \right] d\omega d\beta, \quad (7.37)$$

where $|K|$, with the neutral point x_{10} chosen as the lower integration bound, is equivalent

to the standard N -factor defined by

$$N(x_1; \omega, \beta) = \epsilon^{-1} \int_{x_{10}}^{x_1} \sigma(\xi; \omega, \beta) d\xi, \quad \sigma = -\text{Im}[\alpha]. \quad (7.38)$$

The individual components of the receptivity spectral density $(q^2)_{\alpha, \beta, \omega}$ are given explicitly in Appendix C of Ref. [FT17]. Note that expression (7.37) accounts for the wave packet comprised of all modes excited by KF, not just the one that is most amplified. On the other hand, the fact that there is a single maximally amplified frequency and span-wise wavenumber corresponding to each observation point suggests asymptotic approximation of the integrals in ω and β by Laplace's method. This yields the more computationally efficient expression

$$\overline{g^2}(x_1) = \frac{\epsilon_L^2}{\pi \sqrt{N_{\omega\omega} N_{\beta\beta} - N_{\omega\beta}^2}} \left[\int_{x_{1u}}^{x_1} (q^2)_{\alpha, \beta, \omega}(\eta) \left| \frac{K(x_1)}{K(\eta)} \right|^2 d\eta \right]_{\max}. \quad (7.39)$$

where $\cdot_{\max}$ indicates evaluation at $(\omega, \beta) = (\omega_{\max}, \beta_{\max})$, corresponding to the maximally amplified frequency and span-wise wavenumber at the observation point x_1 . $N_{\omega\omega}$, $N_{\beta\beta}$, and $N_{\omega\beta}$ represent derivatives of N with respect to the indicated variable at the maximally amplified frequency $\omega_{\max}$ and wave-number $\beta_{\max}$ at the observation point. For high Mach number cases, $\beta_{\max} = 0$ for all frequencies, and thus $N_{\omega\beta} = 0$. In cases for which $\beta_{\max} \neq 0$, e.g. the Mach 2 case discussed in §7.5.7.5.4, the integrand of (7.37) consists of two peaks, one at $(\omega_{\max}, \beta_{\max})$ and one at $(\omega_{\max}, -\beta_{\max})$. If these two peaks are essentially separate, as is the case for the aforementioned Mach 2 configuration, formula (7.39) is a valid approximation if multiplied by 2. If the support of the peaks are substantially overlapping, (7.39) will no longer be valid. Of course, (7.37) is valid regardless of the structure of its integrand.

In Section 7.5, we compare the results obtained from formula (7.39) with those from (7.37) where the integrals in ω and β are computed numerically. For the cases considered in this work, the discrepancies between the results are not significant, except in certain cases where weakly unstable first Mack modes are present. This point is discussed in detail in §7.5.7.5.3.

7.4 Numerical Method

The stability code used in this work is the same as that developed and described in Ref. [Tum07]. The direct and adjoint eigenvalue problems (7.29) and (7.31) are solved by integrating the equations from the known (analytic) asymptotic solutions outside the boundary layer down to the wall, $y = 0$. The integration was performed using a fourth order Runge-Kutta scheme with a Gramm-Schmidt orthonormalization procedure meant to minimize the accrual of numerical errors. In this work a 1201 point stencil was used to perform the integration. The direct and adjoint eigenfunctions were obtained through a linear combination of the fundamental solutions. This is accomplished through a Newton-Raphson scheme which iterates on the complex eigenvalue α until the wall condition $(u, v, w, \theta)|_{y=0} = (0, 0, 0, 0,)$ is

satisfied to within a specified tolerance. The final result (7.37) is independent of the normalization of the adjoint eigenfunction (7.31). The direct eigenfunction $\mathbf{A}_0$ was normalized according to the desired physical interpretation of the amplitude $c(x_1)$.

When evaluating (7.37) directly, the integrals over ω and β were computed numerically using a 20×20 grid in the (ω, β) -plane. The integration domain varied widely between cases and even for different observation points within a single configuration. However, because the integrand decays exponentially away from its peak in all directions, it is easy to verify on a case by case basis that the support of the peak has been adequately captured. Note that for a Gaussian function, the trapezoidal rule is known to provide better accuracy than a Simpson's rule on the same evenly spaced grid. Since, qualitatively, the integrand of (7.37) resembles a Gaussian in two dimensions (see Figure 7.13), we chose to utilize a simple trapezoidal quadrature rule. Note also that for cases where $\beta_{\max} = 0$, the integrand need only be evaluated at half of the points in the stencil due to symmetry across the β -axis. In cases where $\beta_{\max} \neq 0$, the search for $\beta_{\max}$ and the evaluation of the integrand of (7.37) may be combined into one procedure in order to avoid repeating the calculation of $N(\omega, \beta)$ across the entire grid.

7.5 Results & Discussion

In this section, we utilize the direct MMS approach just described to study KF-receptivity for configurations in which the dominant unstable mode exhibits nuanced behavior in the vicinity of the onset of instability.

In previous works, asymptotic analysis has been employed (see, for example, Ref. [ZF87]) to demonstrate that excitation of the dominant unstable mode by distributed forcing occurs in a neighborhood $|x_1 - x_{10}| = O(\epsilon^{1/2})$ of the lower neutral branch. This result has motivated a MA solution to the receptivity problem which relies on a resonant inner solution to account for generation of the dominant mode. This approach has been used to study receptivity to distributed forcing such as wall vibrations[FK02] and kinetic fluctuations.[FA10, FT17]. Implicit in those analyses is the assumption that the discrete mode eigenvalue exhibits $O(1)$ variation in the neighborhood of the neutral point. This is indeed the behavior that is observed across a broad range of configurations, particularly (but not exclusively) in low-speed cases. The motivation for the present work was to provide a framework for studying cases where such assumptions about the growth rate may not be valid. In particular, our interest lies in cases for which the dominant unstable mode exhibits long scale near-neutral behavior and/or first Mack mode instabilities.

In what follows, particular attention will be given to the integral

$$I(x_1; \alpha, \beta, \omega) = \int_{x_{1u}}^{x_1} S(\eta; \alpha, \beta, \omega) |K(\eta; \alpha, \beta, \omega)|^{-2} d\eta, \quad S(\eta; \alpha, \beta, \omega) = (q^2)_{\alpha, \beta, \omega}(\eta) \quad (7.40)$$

which appears in the final result (7.37) of the previous section. Functions S , $|K|^{-2}$, and their product $J = S|K|^{-2}$ may be used to examine the features of the excitation region. Note that we consider the integral I without the factor $|K(x_1)|^2$, which is constant for

a given observation point x_1 chosen downstream of the excitation region. The quotient $|K(x_1)/K(\eta)|$, which may be expressed as

$$\left| \frac{K(x_1)}{K(\eta)} \right| = \exp \left(\epsilon^{-1} \int_{\eta}^{x_1} \sigma(\xi; \omega, \beta) d\xi \right). \quad (7.41)$$

represents the downstream amplification (or decay) at the observation point x_1 of disturbances excited at the stream-wise location η . Expression (7.41) makes it clear that the choice of lower bound x_{10} in the definition of K was indeed arbitrary with respect to the final solution. Considered by itself, the numerator $|K|^{-2}$ of the amplification coefficient is again a function of the lower integration bound x_{10} . However, in what follows, our interest is in the qualitative features of $|K|^{-2}$ which are immune to the rescaling that occurs with a change in the value of x_{10} . The exponential factor $|K|^{-2}$ may be interpreted as accounting for the following physical effect: disturbances excited upstream of the neutral point are damped prior to being amplified, and those excited downstream are less than maximally amplified relative to those excited immediately at the onset of instability. The factor peaks at the neutral point, i.e. $|K(x_{10})|^{-2} = 1$, and decays away from this peak in both directions. Note that the quantities of interest will be presented as functions of the local Reynolds number, $R = (U_e x / \nu_e)^{1/2}$ rather than the stream-wise coordinate.

The discrepancies that arise between the present solution and that of the MA approach amount to differences in the evaluation of the quantity represented by I . The solution derived from the MA approach is equivalent to the one developed here if the integral I is approximated by Laplace's method. Thus, although such an approximation is not formally how the MA solution is obtained, it is useful to compare the full integrand I with the Gaussian that is assumed in the application of Laplace's method.

In what follows, we study the excitation region for a variety of configurations by exploring the integral I for a fixed frequency and spanwise wavenumber. In each case, we consider the frequency and wavenumber corresponding to the maximally amplified mode for a given observation point. It is worth emphasizing, however, that in computing the physical disturbance magnitude at each observation point, we ultimately account for the entire wavepacket of modes excited by KF, not just the one that is most amplified. This fact is reflected in formula (7.37) by the integrals in ω and β .

In this section, we also explore the validity of formula (7.39) in which the integrals over ω and β in (7.37) have been approximated asymptotically. This approach is generally preferred as it represents an appreciable decrease in computational cost, particularly in hypersonic cases where the maximally amplified modes are two-dimensional, i.e. $\beta_{\max} = 0$. If $\beta_{\max} \neq 0$, a numerical exploration of the space is required anyway in order to determine the maximally amplified wavenumber.

We begin with a validation of the code developed based on the present direct MMS approach for a low speed ($M = 0.04$) incompressible test case.

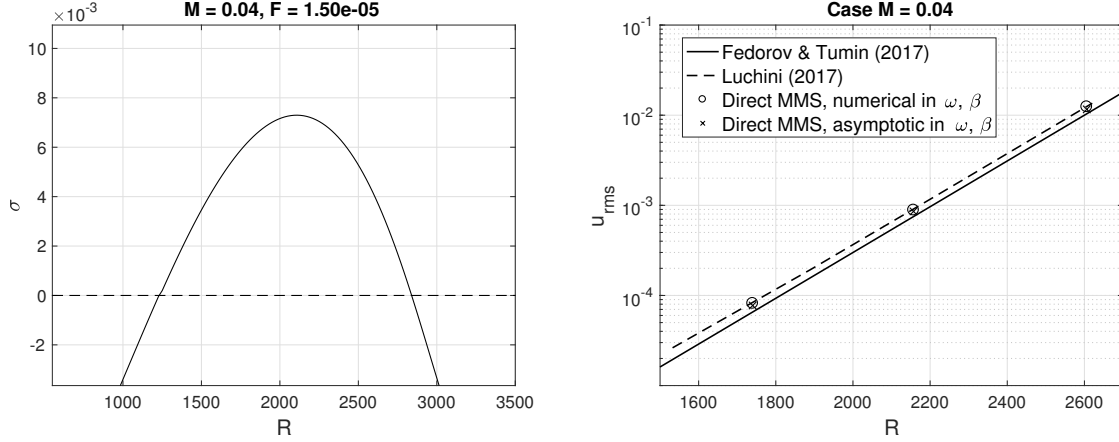


Figure 7.1: Left: The growth rate $\sigma = -\text{Im}[\alpha]$ corresponding to frequency parameter $F = 2.55 \times 10^{-5}$ for the Mach 0.04 test case. Right: A comparison of u_{rms} between the present work and results from Refs.[Luc17, FT17] for the low Mach number case ($M = 0.04$).

7.5.1 Code Validation: Low-speed Test Case, $M = 0.04$

As a verification of the code used in the present work for low speed configurations, we consider a $M = 0.04$ test case, the details of which are provided in Appendix B. The dimensionless growth rate $\sigma(R) = -\text{Im}[\alpha(R)]$ for the representative dimensionless frequency parameter $F = \omega \nu_\infty / U_\infty^2 = 1.5 \times 10^{-5}$ is shown in Figure 7.1a. We compare with results obtained by Fedorov and Tumin[FT17] using the MA approach and those obtained by Luchini [Luc17] using an incompressible solver based on an approach similar to the one described here. The *rms* maximal stream-wise velocity fluctuation u_{rms} of the wave packet for this low Mach number case computed based on the present method agrees well with the results of previous studies. The results are in particularly good agreement with those of Luchini, as expected. This comparison is provided in Figure 7.1b. In this case, the negligible discrepancy between the direct and asymptotic evaluation of the integrals in ω and β suggests that use of the dramatically more computationally efficient asymptotic evaluation is justified.

The agreement between the MA and direct MMS approaches for this case is a result of the relatively consistent evaluation of the spectral intensity integral $I(x_1; \alpha, \beta, \omega)$. The full dimensionless integrand $J = J(R; \alpha, \beta = 0, \omega)$ and its individual components, as computed based on the present approach, are shown in Figure 7.2. For this case, $\beta = 0$ is the maximally amplified wave-number. The integrand as assumed by Laplace's approximation is shown for comparison. The approximated integrand is seen to exhibit strong qualitative agreement and only minor quantitative disagreement with the behavior of J as computed directly, a trend that is observed across all frequencies and wave-numbers for this case. This contributes to the similarly minor discrepancy in the computation of u_{rms} shown in Figure 7.1. In this case, effective excitation indeed peaks in the vicinity of the neutral point and decays rapidly outside a neighborhood of that peak. This is attributed to the $O(1)$ deviation of the growth

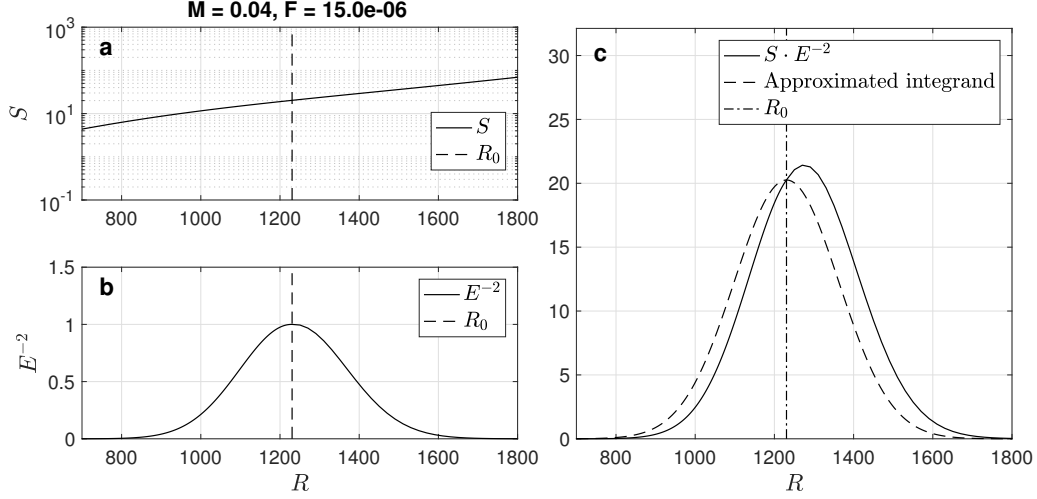


Figure 7.2: Components S (a) and $|K|^{-2}$ (b) and comparison of the full integrand J with the Gaussian assumed by the MA approach (c) for $F = 1.5 \times 10^{-5}$ and $\beta = 0$ for the $M = 0.04$ configuration.

rate from neutral on either side of the lower neutral branch and the relatively slow variation of receptivity over the excitation region.

7.5.2 Code Validation: Hypersonic Test Case, $M = 6.0$

We now study receptivity to KF for the Mach 6 test case considered in Ref. [FT17] (see Appendix A for details). In that work, the mean flow solver was based on a calorically perfect gas model. Here, we consider the same case using a 5-species model for air for the mean flow computations. From here on out, we will distinguish between use of the calorically perfect gas and air mean flow models using the labels ‘CP’ and ‘Air’. Regardless of the mean flow model used, the stability computations are done using a calorically perfect gas stability solver. For the Air model, this requires specification of a constant effective specific heats ratio and a constant effective Prandtl number. For this Mach 6 case, we utilize $\gamma_{\text{eff}} = 1.40$ and $Pr_{\text{eff}} = 0.72$. The validity of this procedure is discussed in Ref. [ET17b], along with details of the mean flow models.

Figure 7.3 shows the growth rate as a function of R for the representative frequency parameter $F = 2.2 \times 10^{-5}$ and $\beta = 0$. In this case, the mode is nearly neutral upstream of the onset of instability. This is in contrast to the previously considered low-speed $M = 0.04$ case, where the unstable mode is heavily damped upstream of the lower neutral branch. This near-neutral evolution is typical for hypersonic configurations and becomes more pronounced with increasing Mach number, as will be seen in the analysis of two Mach 10 configurations in Section 7.5.7.5.3.

It is interesting to compare the details of the excitation region for this hypersonic case

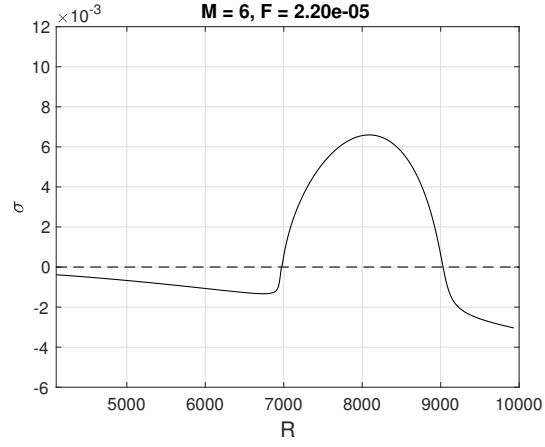


Figure 7.3: The growth rate $\sigma = -\text{Im}[\alpha]$ corresponding to frequency parameter $F = 2.20 \times 10^{-5}$, $\beta = 0$, for the Mach 6 test case.

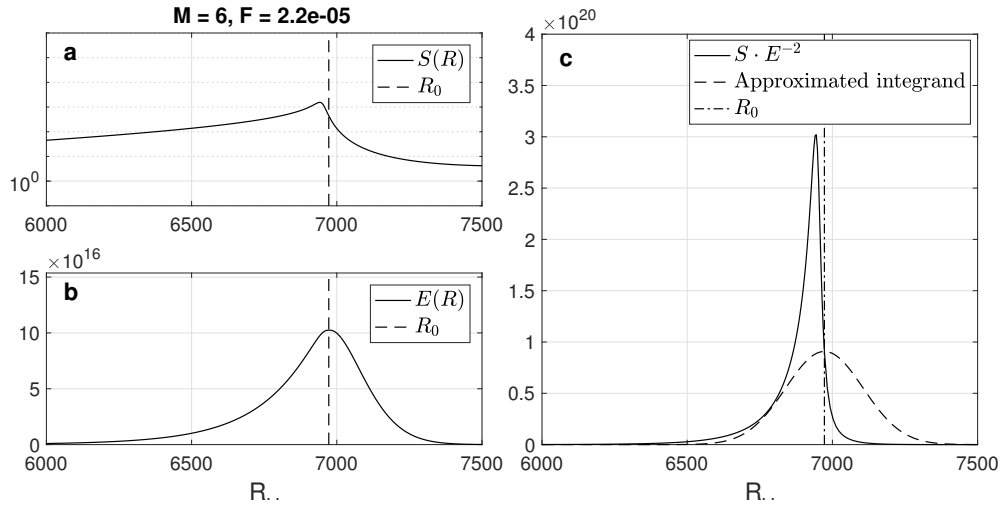


Figure 7.4: Components S (a) and $|K|^{-2}$ (b) and comparison of the full integrand J with the Gaussian assumed by the MA approach (c) for $F = 2.2 \times 10^{-5}$ and $\beta = 0$ for the $M = 6$ configuration.

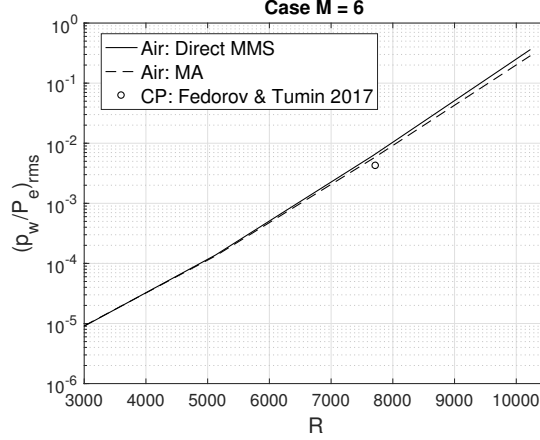


Figure 7.5: Downstream magnitude of wall pressure perturbations $(p_w/P_e)_{\text{rms}}$ for the Mach 6 test case. The published result from Ref. [FT17] for the CP mean flow model is shown for comparison.

with those of the $M = 0.04$ case. The integrand J and its components are plotted in Figure 7.4 for $F = 2.2 \times 10^{-5}$ and $\beta = 0$. The receptivity factor, shown in Figure 7.4a, exhibits a strong peak just upstream of the neutral point (note the logarithmic scale of the vertical axis). Additionally, the exponential factor $|K|^{-2}$ (Figure 7.4b) exhibits an elongated upstream ‘tail’ that is attributed to the near-neutral behavior of the growth rate in that region. Physically, this suggests a lengthening of the region upstream of the neutral point where the excitation of disturbances may be relevant. However, the effect of this elongation on actual excitation of the mode (represented by the full integrand J , Figure 7.4c) is muted. This is due to the rapid decay of receptivity upstream of the neutral point region. Since receptivity also exhibits strong decay downstream of its peak, effective excitation is restricted to a much shorter region downstream of the onset of instability than implied by the approximated integrand.

Figure 7.5 shows the normalized *rms* wall pressure disturbance $(p_w/P_e)_{\text{rms}}$ as computed based on formula (7.37), formula (7.39), and the MA solution. The good agreement between these calculations may be attributed to two factors. First, the integrals over ω and β which account for the induced wave-packet are well approximated by Laplace’s method. Second, despite substantial qualitative and quantitative discrepancies between the directly computed integrand J and its approximation by a Gaussian, there is ultimately good agreement in the evaluation of the quantity I . In this way, the agreement between the direct MMS and the MA solution is somewhat misleading. The MA solution fails to account for the details of the excitation region that are revealed through direct integration of the amplitude equation (7.33) in the direct MMS approach.

Also shown in Figure 7.5 is a value reported in Ref. [ET17b] of $(p_w/P_e)_{\text{rms}}$ for CP mean flow computed using the MA approach. This point is not far from the values corresponding to the Air mean flow model and the small discrepancy is due to the slightly lower amplification

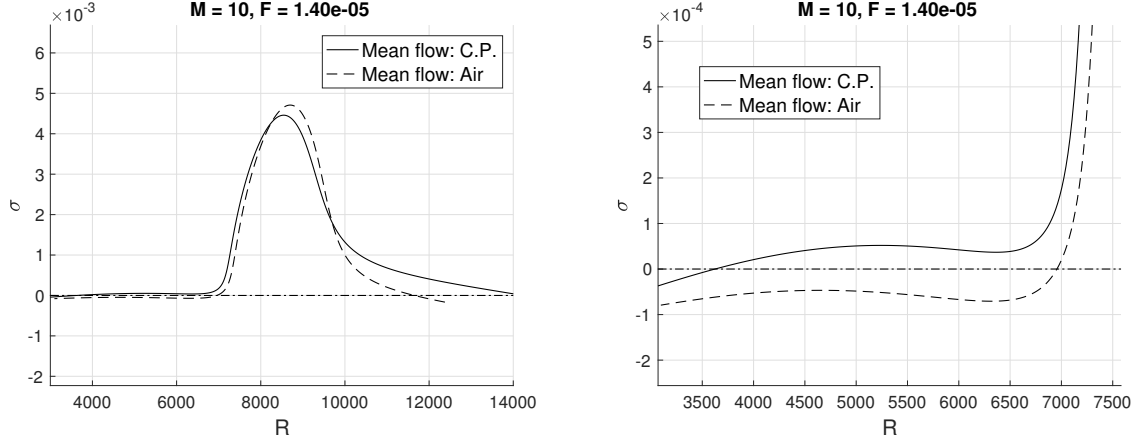


Figure 7.6: (a) Dimensionless growth rates corresponding to $F = 1.40 \times 10^{-5}$ and $\beta = 0$ for the Mach 10 Air and CP configurations. (b) Zoomed in view of the region indicated in (a) showing (stable) near-neutral behavior (Air) and a weakly unstable first Mack mode (CP).

rates that result for the CP model. The qualitative features of the excitation region for the CP case are similar to those shown in Figure 7.4.

7.5.3 Hypersonic cases, $M = 10$

In this section, we consider two high speed (Mach 10) configurations that were previously considered in Ref. [ET17b]. Results were obtained based on the MA approach of Ref. [FT17]. The cases considered therein were chosen to reflect conditions relevant for future hypersonic airplanes with lift-to-drag ratio about 3 and higher [KT13a]. Here, we consider Case 1 (in the language of that paper), wall temperature $T_w = 2000$ K, for both of the aforementioned mean flow models (CP and Air). For the Air model, the stability computations are done using $\gamma_{\text{eff}} = 1.40$ and $Pr_{\text{eff}} = 0.715$. It was pointed out in Ref. [ET17b] that use of the CP model for this configuration resulted in growth rates exhibiting a weakly unstable first Mack mode immediately preceding the dominant second Mack mode instability for all physically relevant span-wise wave-number. For this reason, the MA approach could not be used to accurately compute initial mode excitation by KF. Use of the Air model did not result in unstable first Mack modes at the maximally amplified wavenumber $\beta = 0$, however, the growth rates did exhibit near-neutral behavior over relatively long length scales upstream of the second Mack mode instability. For this case, unstable (but nearly neutral) First Mack modes do appear for $\beta > 0$. Growth rates for each case (CP and Air) corresponding to the representative frequency parameter $F = 1.4 \times 10^{-5}$ and $\beta = 0$ are shown in Figure 7.6.

Based on the discussion of §7.3, the extended near-neutral behavior exhibited in both cases suggests that identification of the neutral point with the locus of excitation may not be valid. As was done in the previous section, we explore the excitation phenomenon in both cases through examination of the integrand J .

Mean Flow: Air

Using the Air mean flow model, the growth rate of the unstable mode generically exhibits (stable) near-neutral behavior over a substantial region preceding the neutral point (Figure 7.6). Similar behavior is observed across all physically relevant frequencies for this case. The qualitative features of the excitation region are similar to those observed for the previously considered Mach 6 case. The support of the exponential factor $|K|^{-2}$, Figure 7.7b, is significantly extended upstream of its peak at the neutral point, representing a dramatic elongation of the stream-wise region where effective excitation may occur. However, as with the Mach 6 case considered previously, the effect of this tail on actual excitation is muted due to the exponential decay of the factor S upstream and downstream of the neutral point region.

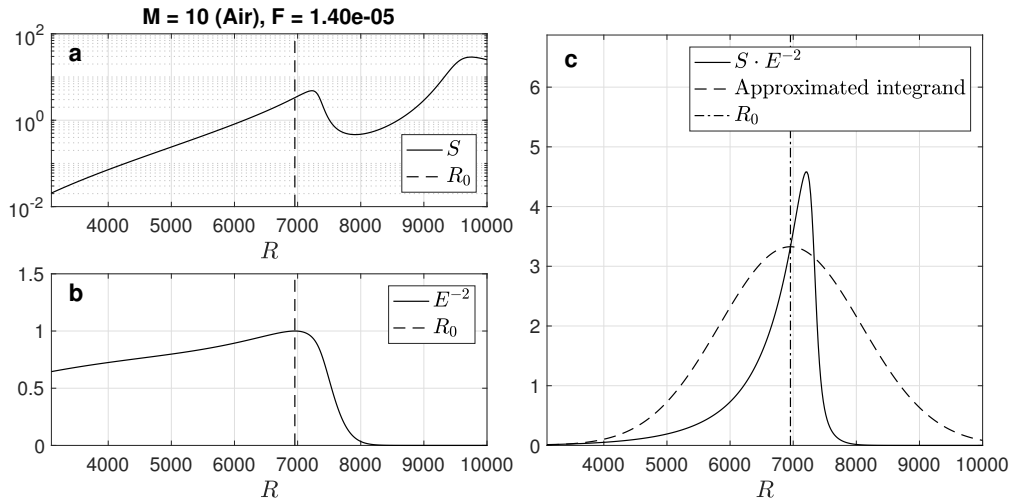


Figure 7.7: Components S (a) and $|K|^{-2}$ (b) and comparison of the full integrand J with the Gaussian assumed by the MA approach (c) for $F = 1.4 \times 10^{-5}$ and $\beta = 0$ for the Mach 10 Air configuration.

After taking the square root to compute the magnitude of the induced disturbance in accordance with expression (7.37), the discrepancy in the results obtained between the MMS and MA methods is not very significant. This can be seen in Figure 7.8, which shows the *rms* amplitude of the downstream wall pressure disturbance $(p_w/P_e)_{\text{rms}}$ for this case. Results based on the MMS approach where the integrals in ω and β have been treated either numerically or asymptotically based on Laplace's method are seen to be quite consistent. Comparison with results from Edwards and Tumin [ET17b], based on the MA approach and asymptotic treatment of the wave packet integrals, shows disagreement to the extent expected based on the disagreement in the evaluation of the integrals I .

It should be emphasized again that the fairly close agreement of the results in Figure 7.8 is not evidence for the validity of applying the MA approach for this configuration. The elongation of the effective excitation region represented by the long upstream tail of the

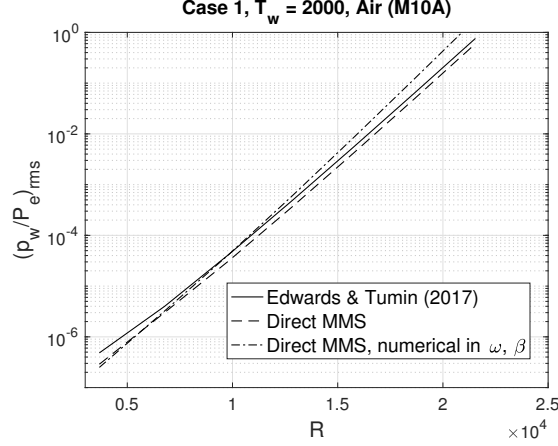


Figure 7.8: Downstream magnitude of wall pressure perturbations $(p_w/Pe)_{\text{rms}}$ for the Mach 10 Air configuration. The results shown are based on the MA approach and the present MMS approach where both numerical and asymptotic treatments of the integrals over ω and β have been considered.

resonance coefficient suggests that strong receptivity anywhere in the region upstream of the neutral point could contribute substantially to the overall excitation of the mode. The exponential decay of the factor S upstream results in a restriction of dominant excitation to the immediate vicinity of the neutral point, but this would not necessarily be the case for other configurations or other sources of external forcing.

Mean Flow: Calorically Perfect Gas

For the CP mean flow model, a weakly unstable first Mack mode immediately precedes the dominant second Mack mode instability for all β and all frequencies with the potential to be relevant for transition. No intermediate region of stability separates the first and second Mack modes, as can be seen in Figure 7.6. As a result, the N factor remains small over the long region of near neutral behavior prior to substantial amplification downstream. Plots of S and $|K|^{-2}$ for $\beta = 0$ and $F = 1.4 \times 10^{-5}$ are provided in Figures 7.9a and b.

The factor $|K|^{-2}$ has broad support, similar to the previously considered case. However, due to the downstream growth of the receptivity coefficient S over the region surrounding $x_1 = x_{10}$, the peak in effective excitation occurs well downstream of the neutral point.

This case highlights the idea that the locus of excitation is not fundamentally tied to the neutral point in the way that a superficial interpretation of the MA approach of Ref. [FT17] might suggest. Rather, the neutral point serves as a convenient locus for computing receptivity for certain configurations in an efficient asymptotic way. On the other hand, the direct MMS approach outlined herein reveals that in the presence of long scale near neutral behavior, the excitation phenomenon is relatively insensitive to the particular location where $\text{Im}(\alpha) = 0$. This can be seen in Figure 7.10 which compares the integrands J resulting from

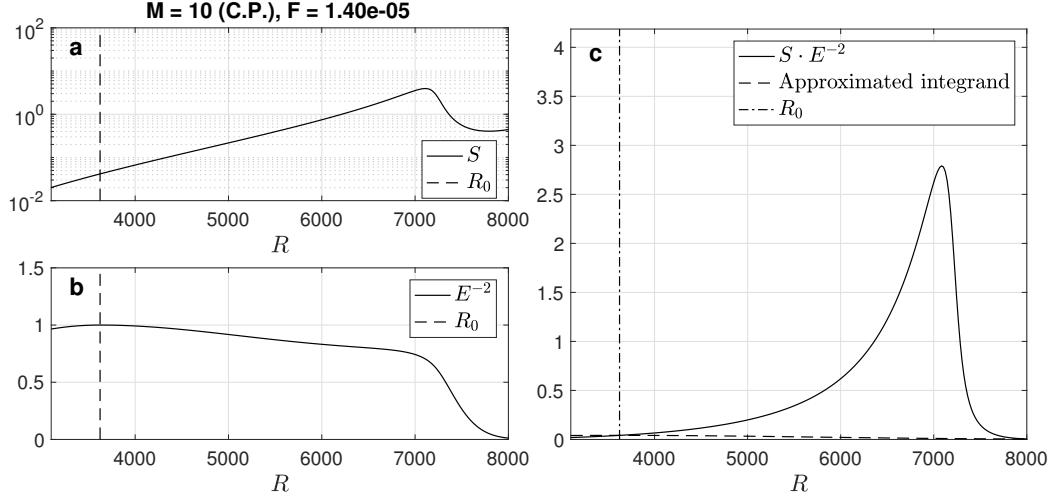


Figure 7.9: Components S (a) and $|K|^{-2}$ (b) and comparison of the full integrand J with the Gaussian assumed by the MA approach (c) for $F = 1.4 \times 10^{-5}$ and $\beta = 0$ for the Mach 10 CP configuration.

the CP and Air mean flow models. Despite a large disparity in neutral point location for the mode under consideration (e.g. $R_0 \approx 3620$ and $R_0 \approx 6955$ for CP and Air, respectively), the excitation phenomenon is largely consistent between the two cases. This is in line with physical intuition since the location of the neutral point can change dramatically with small changes in the mean flow for cases of this type. This observation suggests that the MA method could be altered in cases of this kind by viewing the onset of the dominant instability as the locus of excitation, rather than the neutral point. Such an approach, however, would necessarily be ad hoc.

Figure 7.12a shows the downstream wall pressure disturbance computed using the direct MMS approach with the wave packet integrals computed numerically, formula (7.37), and asymptotically, formula (7.39). The upswing in the curve corresponding to formula (7.39) is attributed to the fact that, while the first mode is unstable beyond some $\beta > 0$ for the frequencies that dominate in this region, it is stable at $\beta = 0$ where the approximation occurs. The curve is smooth in this region when the wave packet integrals are evaluated numerically because the unstable first modes that do appear at $\beta > 0$ are fully accounted for. The two calculations agree well at higher observation Reynold's numbers (wave packets dominated by lower frequencies) where the first mode is unstable at $\beta = 0$.

Importantly, configurations such as this one where a weakly unstable first Mack mode contributes to amplification, may be candidates for particularly high receptivity to broad spectrum forcing such as that induced by kinetic fluctuations. This is due to the destabilizing effect that an increase in span-wise wavenumber β has on the first Mack mode. This known phenomenon is demonstrated in Figure 7.11 for Air and representative frequency parameter $F = 1.4 \times 10^{-5}$.

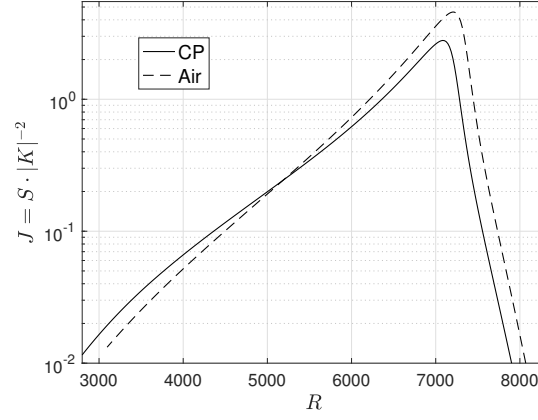


Figure 7.10: Comparison of the integrand $J = S|K|^{-2}$ for CP and Air for $\beta = 0$ and $F = 1.40 \times 10^{-5}$.

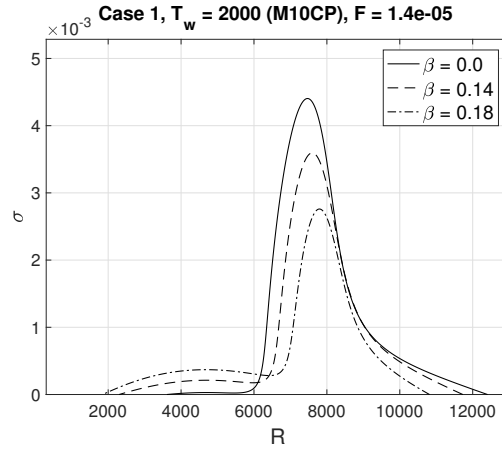


Figure 7.11: Growth rates for the Mach 10 CP configuration corresponding to $F = 1.4 \times 10^{-5}$ for $\beta = 0.0, 0.14$ and 0.18 , highlighting the destabilization of the first Mack mode and simultaneous stabilization of the second Mack mode with increasing β .

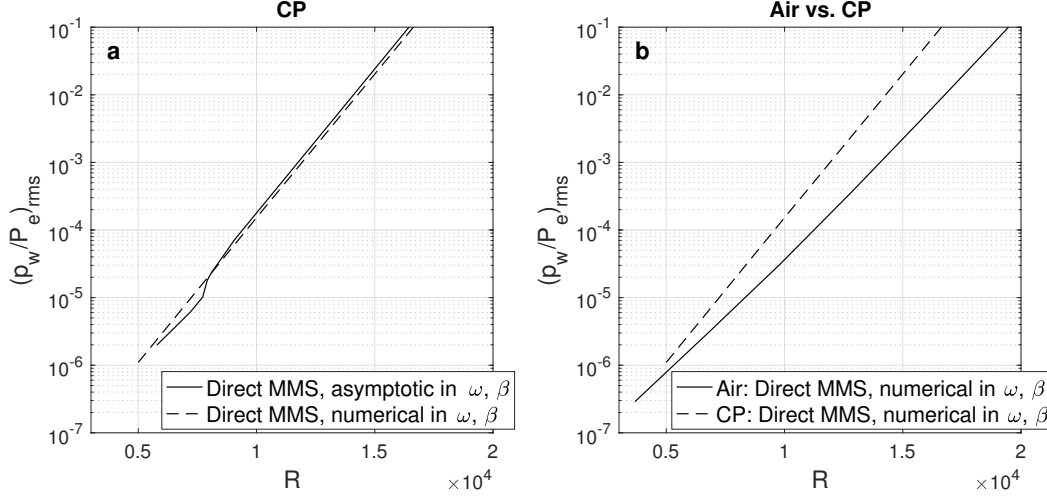


Figure 7.12: (a) Comparison of the downstream magnitude of wall pressure perturbations $(p_w/P_e)_{rms}$ for the Mach 10 CP configuration computed using formulas (7.37) and (7.39). (b) Comparison of $(p_w/P_e)_{rms}$ for the Mach 10 CP and Air configurations computed using formula (7.37).

The decrease in mode amplification resulting from the progressive stabilization of the second mode instability with increasing β is partially counteracted by the destabilization of the first mode. As a result, significant amplification is observed across a broader range of span-wise wavenumber, leading to an effective broadening of the induced disturbance wave packet. This in turn leads to greater disturbance amplitudes than would be observed in the absence of the first mode instability feature. This effect is demonstrated by comparison of KF-induced disturbance magnitudes with the CP case, which has a similar stability spectrum, but does not exhibit significant first Mack mode instabilities. (We should note that first Mack mode instabilities do begin to arise for the CP mean flow for low F and high β . However, the resulting increase in amplification never becomes significant). A comparison of the downstream wall pressure disturbance $(p_w/P_e)_{rms}$ between the Air and CP cases is given in Figure 7.12b.

7.5.4 Supersonic case ($M = 2$): First mode dominated transition

As a test of the direct MMS approach for configurations where the maximally amplified modes are not two-dimensional, we consider a Mach 2 test case for which the dominant unstable modes have non-zero span-wise wavenumber β . For this supersonic case, the dominant instability is the first Mack mode. Figure 7.13 shows the amplification $|K|^2(R_{obs}, F, \bar{\beta})$ of the unstable mode in the $(F, \bar{\beta})$ -plane for the observation Reynold's number $R_{obs} = 3000$. Here $\bar{\beta}$ is the span-wise wavenumber non-dimensionalized by the boundary layer length scale at the neutral point x_0 . The maximally amplified mode at the observation Reynold's number

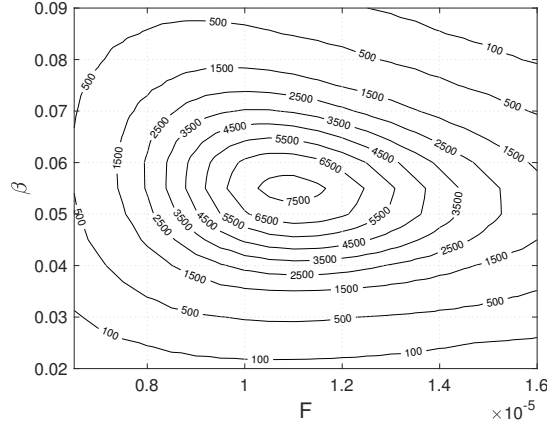


Figure 7.13: The downstream amplification spectrum $|K|^2(R_{\text{obs}}, F, \bar{\beta})$ for the unstable mode at observation Reynolds number $R_{\text{obs}} = 3000$. The first Mack mode is the dominant instability in this Mach 2 case and the maximally amplified waves are oblique.

$R_{\text{obs}} = 3000$ corresponds to $F = 1.10 \times 10^{-5}$ and $\bar{\beta} = 0.055$.

The components of the spectral excitation integral I are shown in Figure 7.14 for $F = 1.10 \times 10^{-5}$ and $\bar{\beta} = 0.055$. The qualitative behavior in this case is largely similar to what was observed for the low speed $M = 0.04$ case studied earlier. The discrepancy between the direct MMS and MA approaches arises from the steady exponential growth of the receptivity coefficient, which shifts the peak in effective excitation slightly downstream of the neutral point.

Notably, at $R \approx 440$ there is a sharp peak in the receptivity function S . This peak is due to the proximity of the discrete mode in question to another mode of the discrete spectrum in this vicinity. Figures 7.15a and 7.15b show the trajectories of discrete modes in the α -plane and complex phase speed plane for $F = 0.850$ and 0.875×10^{-5} and $\bar{\beta} = 0.055$. The arrows in Figure 7.15a indicate the direction of increasing Reynolds's number. The slow acoustic branch of the continuous spectrum (bold line) is also shown for $R = 499$. Note that the qualitative behavior of the unstable mode changes abruptly for some frequency parameter F between the two shown in Figure 7.15.

Two distinct discrete modes (thin black lines) are shown for the lower frequency. The point where these two modes are closest corresponds to $R \approx 499$ and is indicated by dots in the figure. At this location, a sharp (but not quite singular) peak is observed in the function S for $F = 0.850 \times 10^{-5}$, similar to the one seen in Figure 7.14 at $R \approx 440$ for $F = 1.10 \times 10^{-5}$. This is due to the fact that $|\Gamma|^2$, which appears in the numerator of S , tends towards zero as the eigenvalues approach one another.

This phenomenon has been addressed previously in the context of hypersonic boundary layers where a branching of two discrete modes occurs near the neutral point of the unstable mode [FK01]. To the authors' knowledge, the phenomenon has not been previously addressed in the context of supersonic boundary layers. The appearance of supersonic second Mack

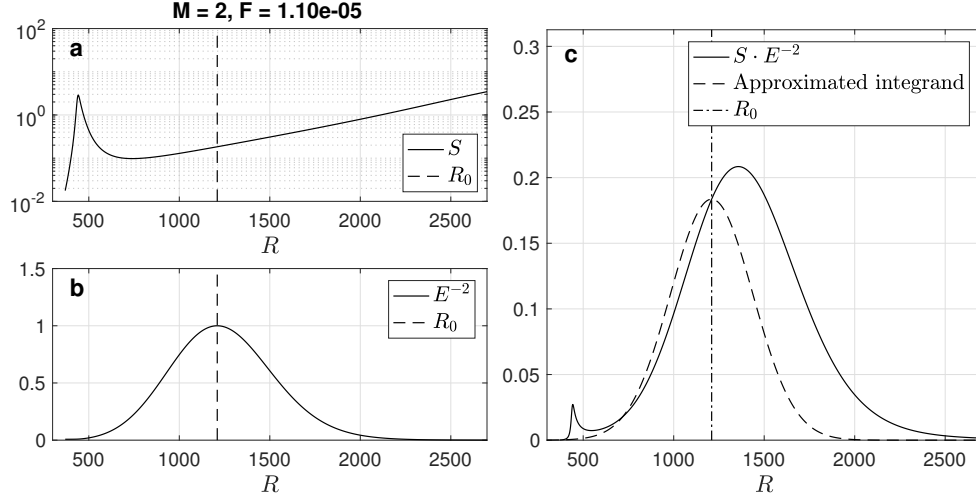


Figure 7.14: Components S (a) and $|K|^{-2}$ (b) and comparison of the full integrand J with the Gaussian assumed by the MA approach (c) for $F = 1.10 \times 10^{-5}$ and maximally amplified dimensionless span-wise wavenumber $\bar{\beta} = 0.055$ for the Mach 2 case.

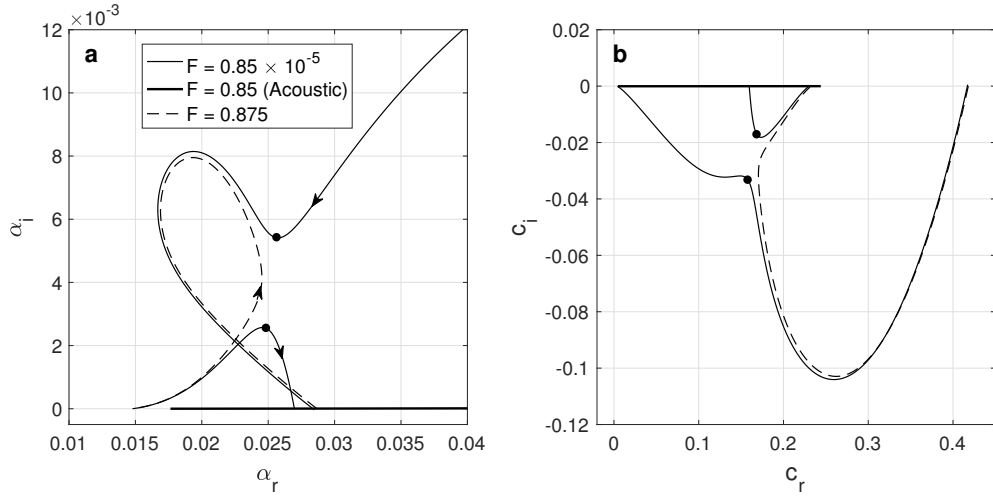


Figure 7.15: The trajectory of discrete modes in the α -plane (a) and the complex phase speed plane (b) for $F = 0.850$ and 0.875×10^{-5} and $\bar{\beta} = 0.055$. The arrows indicate the direction of increasing Reynold's number and the dots indicate $R = 499$. The bold line corresponds to the slow acoustic branch of the continuous spectrum, evaluated at $R = 499$.

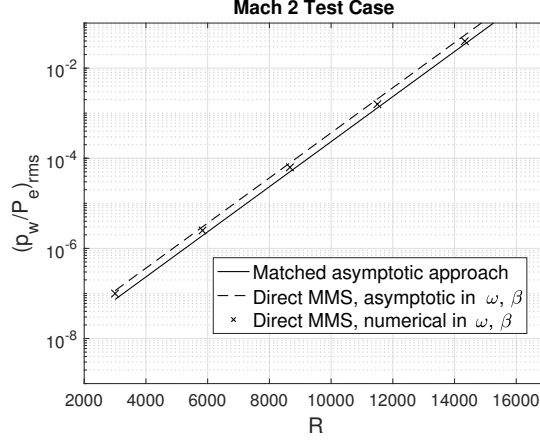


Figure 7.16: The downstream magnitude of wall pressure perturbations $(p_w/P_e)_{\text{rms}}$ for the Mach 2 test case. Results are shown for the MA approach as well as the direct MMS approach where the integrals in ω and β have been treated either asymptotically or numerically.

modes in hypersonic boundary layers is also associated with a discrete mode branching. Indeed, the qualitative features of the phase speed trajectory corresponding to $F = 0.850 \times 10^{-5}$ bear a resemblance to those of the supersonic mode shown in Figure 4a of Ref. [PVC16]. An important difference, however, is that the latter phenomenon occurs while the mode is unstable, whereas the present behavior occurs while the mode has not yet become unstable. As a result, the impact of this peak on the overall integral I is negligible, as is evident in Figure 7.14c. For the purposes of the present work, there is thus no need to address this singular behavior further, although such an approach may not be acceptable across all similar configurations.

As was the case for the low speed $M = 0.04$ case, the present direct MMS approach and the MA approach yield results that are relatively consistent both qualitatively and quantitatively. The magnitude of wall pressure disturbances $(p_w/P_e)_{\text{rms}}$ induced by kinetic fluctuations is shown in Figure 7.16, as computed based on the direct MMS approach as well as the MA approach. Again the asymptotic evaluation of the wave packet integrals in accordance with (7.39) yields results consistent with direct numerical integration. However, as mentioned previously, for cases of this kind where $\beta_{\text{max}} \neq 0$, an exploration of the (ω, β) -plane is necessary anyway, making the computational advantage of asymptotic evaluation less significant.

7.6 Conclusions

Receptivity of flat plate boundary layers to distributed forcing by kinetic fluctuations has been studied using a method of multiple scales (MMS) approach. Results have been presented for a range of configurations, from low speed to hypersonic.

The present approach may be compared to the MA approach (both of which utilize a multiple scales formulation) utilized in Ref. [FT17]. The essential difference between the two amounts to the treatment of the excitation region. In the direct MMS approach adopted here, the ODE governing mode excitation is integrated numerically, whereas the MA approach relies on an asymptotic approximation.

Through studying the components of the integrand of I in formula (7.37) for the *rms* amplitude of disturbances excited by KF, we can gain insight into the excitation phenomenon. We find that in hypersonic cases, the near-neutral evolution of the growth rate prior to the onset of stability results in a broadening of the region where effective excitation may occur. Furthermore, for second mode dominated configurations, the function S representing receptivity to KF typically exhibits a peak, away from which there is exponential decay. Although this peak typically occurs in the vicinity of the lower neutral branch of the second mode, it may be more generally associated with the onset of dominant amplification. For example, certain high speed configurations may exhibit weakly unstable first Mack modes prior to the dominant second Mack mode, with no intermediate region of stability. In such cases, the lower neutral branch may be pushed far upstream, yet the peak in receptivity is still observed near the onset of second mode amplification. The MA approach is not applicable in such cases.

We have also observed that the receptivity function for a Mach 2 test case exhibits a nearly singular peak upstream of the neutral point. We expect that this is related to a discrete mode branching that occurs in the region of the peak. This singular behavior, which does not reflect a physical spike in receptivity, has been observed before in the context of discrete mode branching in hypersonic boundary layers. In the Mach 2 case presented here, this artifact of the mode branching has a negligible effect on the overall excitation calculation and thus no attempt is made to address it further.

Finally, the results presented herein suggest that asymptotic evaluation of the wave packet integrals in ω and β via formula (7.39) is justified across a broad range of configurations. One possible exception has been identified. For hypersonic cases where unstable first modes are not present at the maximally amplified wavenumber $\beta = 0$, but do appear for $\beta > 0$, formula (7.39) may result in an underestimation of disturbances excited by KF. This is due to an underestimation of the band of span-wise wavenumber contributing to the disturbance wave packet.

7.7 Appendix

7.7.1 Flow parameters

	T_e^* (K)	U_e^* (m/s)	ρ_e^* (kg/m ³)	ν_e^* (m/s ²)	Pr	γ
Mach 0.04	300	15	1.176	1.458×10^{-5}	0.70	1.40
Mach 10.0	278	3.351×10^3	5.47×10^{-2}	3.505×10^{-4}	0.715	1.40
Mach 2.0	62	3.16×10^2	8.46×10^{-2}	4.788×10^{-5}	0.72	1.40
Mach 6.0	300	2.084×10^3	4.13×10^{-1}	4.473×10^{-5}	0.72	1.40

Table 7.1: Details of the flow parameters for each of the cases considered in this work.

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