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Advanced numerical methods for computing statistical quantities of interest from solutions of SPDEs

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The efforts undertaken and successfully completed with the aid of grant funding can be divided into the following categories. - Basic UQ algorithmic development, analysis, testing, and application. - Reduced-order algorithmic development, analysis, and testing for UQ. - Multifidelity methods for UQ. - UQ for control and optimization problems. Our efforts address both overarching approaches for improving how uncertainty quantification in the partial differential equation setting is done, that is, lowering the cost of obtaining statistical information, e.g., by developing algorithms that require a smaller number of expensive samples of outputs of interest and by lowering the costs of obtaining physical approximations, e.g., through the u

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Final Report

AFOSR Grant FA9550-15-1-0001

Advanced numerical methods for computing statistical quantities of interest from solutions of SPDEs

PI: Max Gunzburger, Florida State University

We first provide brief descriptions of efforts undertaken and successfully completed with the aid of grant funding. The product(s) emanating from these efforts are cited. The descriptions of grant-funded efforts are divided into the following categories.

1. Basic UQ algorithmic development, analysis, testing, and application
2. Reduced-order algorithmic development, analysis, and testing for UQ applications
3. Multifidelity methods for uncertainty quantification
4. Uncertainty quantification for PDE control and optimization problems

Of course there is overlap between the different categories so that some efforts listed in one category could well have been listed in another. Our efforts address both overarching approaches for improving how UQ in the PDE setting is done, that is,

- lowering the cost of obtaining statistical information
  e.g., by developing algorithms that require a smaller number of expensive samples of PDE outputs of interest
- lowering the costs of obtaining physical (space/time) approximations
  e.g., through the use of reduced-order models.

At the end of this report, we provide lists of the graduate students, postdocs, and senior collaborators whose research efforts were enabled by grant support.

Basic UQ algorithmic development, analysis, testing, and application

In this section, we describe our efforts directed at the development of new, improved algorithms for uncertainty quantification in the setting of partial differential equation and integral equation models. Almost all efforts listed involved not only algorithmic invention, but also involved the rigorous analysis and testing of the new algorithms and also their application to problems of interest.

High-dimensional discontinuity detection

We developed, analyzed, and tested a hyperspherical sparse approximation framework for detecting jump discontinuities in functions in high-dimensional spaces which improves on well-known approaches, e.g., adaptive sparse grids. A hyperspherical coordinate representation of the discontinuity surface of a function is constructed and sparse approximations of the transformed function are built in that coordinate system, with values at each point estimated by solving 1D discontinuity detection problems. Several approaches are used to approximate the transformed discontinuity surface in the hyperspherical system, including adaptive sparse grid and radial basis function interpolation, discrete least squares projection, and compressed sensing approximation. Hierarchical acceleration techniques are also incorporated to further reduce the complexity. Rigorous complexity analyses of the new methods are derived and numerical examples illustrate the effectiveness of our approach.

G. Zhang, C. Webster, and M. Gunzburger, *Hyperspherical sparse approximation techniques for high-dimensional discontinuity detection*; SIAM Review **58** 2016, 517-551.

[This paper received the distinction of being published, in substantially updated form, in the SIGEST section of SIAM Review as the best paper appearing in the SIAM Journal on Numerical Analysis over a two-year period.]

Improved discrete least-squares/reduced-basis method

We showed that the computational efficiency of the discrete least-squares (DLS) approximation of solutions of stochastic elliptic PDEs is improved by incorporating a reduced-basis method into the DLS framework. The goal is to recover the entire solution map from the the parameter space to the finite element space. To this end, first, a reduced-basis solution using the weak greedy algorithm is constructed, then a DLS approximation is determined by evaluating the reduced-basis approximation instead of the full finite element approximation. The main advantage of the new approach is that one only need apply the DLS operator to the coefficients of the reduced-basis expansion, resulting in huge savings in both the storage of the DLS coefficients and the online cost of evaluating the DLS approximation. In addition, the recently developed quasi-optimal polynomial space is also adopted in the new approach, resulting in superior convergence rates for a wider class of problems than previous analyzed. Numerical experiments are provided that illustrate the theoretical results.

M. Gunzburger, M. Schneier, C. Webster, and G. Zhang, *An improved discrete least-squares/reduced-basis method for parameterized elliptic PDEs*; to appear in Journal of Scientific Computing.

Approximation of probability density functions in the UQ-PDE setting

We studied the approximation of the probability density function of outputs of interest that depend on the solution of a stochastic partial differential equation. We first considered using, for this purpose, truncated moment and cumulant expansions of probability density functions. The SPDE is solved using two stochastic finite element (SFEM) methods, namely, Monte Carlo sampling and the stochastic Galerkin method with global polynomials. The random variable output of interest is a functional of the solution of the SPDE, such as the average over the physical domain. The truncated series are obtained considering a finite number of terms in the Gram-Charlier or Edgeworth series expansions. These expansions approximate the probability density function of a random variable in terms of another probability density function, and involve coefficients that are functions of the cumulants of the random probability density function.

We have also approximated the probability density function associated with a random output of interest approximated using a piecewise-linear polynomials. With the proposed estimate, the kernel functions are centered at the grid points rather than at the samples, as in the standard kernel density estimation approach. Unlike other strategies that place kernel functions on grid points, our method is fully algebraic and does not require the solution of a linear system. A simple rule that relates the grid size to the sample size eliminates the need for bandwidth selection procedures. Our density estimate has unitary integral, requires no constraint to enforce positivity, and is consistent. Numerical results are reported considering analytic samples, and samples associated with the solution of a stochastic partial differential equation.

G. Capodaglio, M. Gunzburger, and H. Wynn, *Approximation of probability density functions for SPDEs using truncated series expansions*; submitted.

G. Capodaglio and M. Gunzburger, *Piecewise polynomial approximation of probability density functions with application to uncertainty quantification for stochastic PDEs*; submitted.

An auxiliary equation approach for the stochastic random Navier-Stokes equations

We developed a Martingale correction regularized method for the stochastic Navier-Stokes equations with additive noise. The original system is split into two equivalent parts, a linear stochastic Stokes equations with Martingale solution and stochastic modified Navier-Stokes equations with smoother noise.

As a result, determining statistical information about outputs of interest that depend on solutions of the Navier-Stokes equation can be effected at greatly reduced cost, e.g., much fewer samples are needed when one uses Monte Carlo or sparse grid statistical estimation. In addition, an inverse fractional Laplace operator is introduced to regularize the noise term. The stability and convergence of the path-wise modified Navier-Stokes equations are proved. Numerical simulations are provided to illustrate our scheme. The comparison of non-regularized and regularized noises for the Navier-Stokes system are presented to further demonstrate the efficiency of our numerical scheme.

W. Zhao and M. Gunzburger; *Auxiliary equation approach for the stochastic unsteady-state Navier-Stokes equations with additive random noise*; submitted.

An efficient algorithm for simulating ensembles of parameterized flow problems;

In several important settings of computational fluid dynamics, a set of multiple simulations of a flow under different input conditions are required. Example of such settings include uncertainty quantification, reduced-order modeling, and optimization. The multiple approximate solutions of the Navier-Stokes equations could be generated from the use of different viscosity coefficients, initial conditions, and/or body forces. Due to the nonlinearity of the Navier-Stokes equations, straightforward linearization methods for the multiple simulations result in having not only different right-hand sides, but also different coefficient matrices. Following on work by William Layton and Nan Jiang, we have developed numerical algorithms for the efficiently determination of a set of such simulations. In the schemes, the average of the multiple solutions at any time step are used to construct multiple discrete systems all involving the same coefficient matrix but with different right-hand sides, and thus are computationally more efficient than solving for all the simulations separately. We developed two such schemes for both of which we showed that the schemes are nonlinearly and long-term stable under certain conditions on the time-step size and a parameter deviation ratio. Rigorous numerical error estimate showed that the two schemes are first-order and second-order accurate in time and optimally accurate in space. Several numerical experiments are presented to illustrate the theoretical results. Furthermore, we also developed a new numerical discretization of the variable viscosity ensemble algorithm in which the average viscosity is replaced with the maximum viscosity. It proved that this approach is significantly more stable than those in current use that use the average of the viscosity ensembles instead of the maximum value.

M. Gunzburger, N. Jiang, and Z. Wang, *An efficient algorithm for simulating ensembles of parameterized flow problems*; submitted.

M. Gunzburger, N. Jiang, and Z. Wang, *A second-order time-stepping scheme for simulating ensembles of parameterized flow problems*; to appear in Computer Methods in Applied Mathematics.

Time discretization for stochastic time-fractional PDEs with additive space-time white noise

Numerical approximation of a stochastic partial integro-differential equation driven by a space-time white noise were considered by truncating a series representation of the noise, with finite element method for spatial discretization. The stochastic time-fractional equation is discretized in time by a backward-Euler convolution quadrature for which the sharp-order error estimate are established. In particular, our results prove, for the first time, sharp optimal convergence rates of numerical solutions for both stochastic subdiffusion and diffusion-wave problems in one spatial dimension. Numerical examples were perofrmed to illustrate the theoretical analysis.

M. Gunzburger, B. Li, and J. Wang, *Sharp convergence rates of time discretization for stochastic time-fractional PDEs subject to additive space-time white noise*; Mathematics of Computation; **88** 2019, 1715-1741

M. Gunzburger, B. Li, and J. Wang, *Convergence of finite element solutions of stochastic* *partial integro-differential equations driven by the space-time white noise*; Numerische Mathematik **141** 2019, 1043-1077.

Comparisons of space/time colored and white noise forcing of the Navier-Stokes equations

We systematically constructed representations of space/time correlated and uncorrelated noises for given expectations and auto-covariances. In particular, for temporal colored noise, we used random processes to represent their properties instead of using, e.g., Karhunen-Loève expansions in the time direction. For each kind of spatial/temporal noise combination, we provided a detailed definition and discussion about finite-dimensional discretizations of the noises and their properties. The above developments are useful for general settings of partial differential equations having white/colored forcing. We apply the different types of discretized white/colored forcing to provide numerical experiments in the context of finite element discretizations of the stochastic Navier-Stokes equations and compare the performance and solution features resulting from the experiments.

W. Zhao and M. Gunzburger; *Descriptions, discretizations, and comparisons of space/time colored and white noise forcings of the Navier-Stokes equations*; submitted.

Reduced-order algorithmic development, analysis, and testing for UQ applications

Uncertainty quantification involving the solutions of partial differential equations is very costly endeavor because, in the first place, many approaches require many solutions of the spatially/temporal discretized equations and, in the second place, each of those solutions is expensive to obtain. As a result, there are two armies at work in trying to ameliorate one or the other of these sources of costs, i.e., one army tries to develop methods that reduce the number of samples needed to obtain good statistical information whereas the other army tries to reduce the cost of obtaining those samples. For the latter purpose, reduced-order modeling can play a large role, i.e., the construction of low-cost reduced-order models results in the ability to obtain many more sample solutions of the partial differential equation for a give computational budget. Here we report on our work on this type of effort.

A localized reduced-order modeling approach for PDEs with bifurcating solutions

Reduced-order modeling (ROM) commonly refers to the construction, based on a few solutions (referred to as snapshots) of an expensive discretized partial differential equation (PDE), and the subsequent application of low-dimensional discretizations of partial differential equations (PDEs) that can be used to more efficiently treat problems in control and optimization, uncertainty quantification, and other settings that require multiple approximate PDE solutions. Although ROMs have been successfully used in many settings, ROMs built specifically for the efficient treatment of PDEs having solutions that bifurcate as the values of input parameters change have not received much attention. In such cases, the parameter domain can be subdivided into subregions, each of which corresponds to a different branch of solutions. Popular ROM approaches such as proper orthogonal decomposition (POD), results in a global low-dimensional basis that does not respect the often large differences in the PDE solutions corresponding to different subregions. In this work, we develop and test a new ROM approach specifically aimed at bifurcation problems. In the new method, the k-means algorithm is used to cluster snapshots so that within cluster snapshots are similar to each other and are dissimilar to those in other clusters. This is followed by the construction of local POD bases, one for each cluster. The method also can detect which cluster a new parameter point belongs to, after which the local basis corresponding to that cluster is used to determine a ROM approximation. Numerical experiments show the effectiveness of the method both for problems for which bifurcation cause continuous and discontinuous changes in the solution of the PDE.

M. Hess, A. Alla, A. Quaini, G. Rozza, and M. Gunzburger; *A localized reduced-order modeling approach for PDEs with bifurcating solutions*; Computer Methods in Applied Mechanics and Engineering **351** 2019, 379-403.

Nonintrusive stabilization of ROMs for UQ of time-dependent convection-dominated flows

We developed a nonintrusive filter-based stabilization of reduced order models for uncertainty quantification of the time-dependent Navier-Stokes equations in convection-dominated regimes. We proposed a novel high-order ROM differential filter and use it in conjunction with an evolve-filter-relax algorithm to attenuate the numerical oscillations of standard ROMs. We also examined how stochastic collocation methods (SCMs) can be combined with the evolve-filter-relax algorithm for efficient UQ of fluid flows. We emphasize that the new stabilized SCM-ROM framework is nonintrusive and can be easily used in conjunction with legacy flow solvers. We test the new framework in the numerical simulation of a two-dimensional flow past a circular cylinder with a random viscosity.

M. Gunzburger, T. Iliescu, M. Mohebujjaman, and M. Schneier; *Nonintrusive stabilization of reduced order models for uncertainty quantification of time-dependent convection-dominated flows*; submitted.

An ensemble-proper orthogonal decomposition method for the Navier-Stokes equations

The definition of partial differential equation models usually involves a set of parameters whose values may vary over a wide range. The solution of even a single set of parameter values may be quite expensive. In many cases, e.g., optimization, control, uncertainty quantification, and other settings, solutions are needed for many sets of parameter values. We considered the case of the time-dependent Navier-Stokes equations for which a recently developed ensemble-based method allows for the efficient determination of the multiple solutions corresponding to many parameter sets. The method uses the average of the multiple solutions at any time step to define a linear set of equations that determines the solutions at the next time step. To significantly further reduce the costs of determining multiple solutions of the Navier-Stokes equations, we incorporate a proper orthogonal decomposition (POD) reduced-order model into the ensemble-based method. The stability and convergence results for the ensemble-based method are extended to the ensemble-POD approach.

M. Gunzburger, N. Jiang, and M. Schneier, *An ensemble-proper orthogonal decomposition method for the nonstationary Navier-Stokes equations*; SIAM Journal on Numerical Analysis **55** 2017, 286-304.

Exploration of efficient reduced-order modeling and a posteriori error estimation

Efficient algorithms were developed for a reduced-order model based on the proper orthogonal decomposition methodology for the solution of parameterized elliptic partial differential equations. The method relies on partitioning the parameter space into subdomains based on the properties of the solution space and then forming a reduced basis for each of the subdomains. This yields more efficient offline and online stages for the proper orthogonal decomposition method. We extended these ideas to the task of inexpensive adjoint based a posteriori error estimation of both the expensive finite element method solutions and the reduced-order model solutions, for single and multiple quantities of interest. Rigorous analyses were used to justify claims made about the new methodology and also to guide the selection of parameters that appear in the algorithms. Several numerical results illustrated the efficacy of the approach.

J. Chaudhry, D. Estep, and M. Gunzburger, *Exploration of efficient reduced-order modeling and a posteriori error estimation*; International Journal on Numerical Methods in Engineering **111** 2017, 103-122.

Reduced basis methods for nonlocal diffusion problems with random input data

In several application settings, diffusive behavior is observed to not follow the rate of spread predicted by parabolic partial differential equations (PDEs) such as the heat equation. Such behaviors, often referred to as anomalous diffusion, can be modeled using nonlocal (e.g., integral equations) in which points at a finite distance apart can interact. An example of such models is provided by fractional derivative equations. Because of the nonlocal interactions, discretized nonlocal systems have less sparsity, often significantly less, compared to corresponding discretized PDE systems. As such, the need for reduced-order surrogates which can be used to cheaply determine approximate solutions is much more acute for nonlocal models compared to that for PDEs.

We investigated the construction, application, and testing of proper orthogonal decomposition (POD) reduced-order models for an integral equation model for nonlocal diffusion. For certain modeling parameters, the model we consider allows for discontinuous solutions and includes, as a special case, fractional Laplacian kernels. Computational results illustrate the potential of using POD to obtain accurate approximations of solutions of nonlocal diffusion equations at much lower costs compared to, for example, standard finite element methods.

We also considered the construction, analysis, and application of reduced-basis methods for uncertainty quantification problems involving nonlocal diffusion problems with random input data. We developed reduced-basis approximations for nonlocal diffusion equations with affine random coefficients. Efficiency estimates of the proposed greedy reduced-basis methods were provided. Numerical examples were used to illustrate the effect that varying various model parameters have on the efficiency and accuracy of the reduced-basis method relative to sparse-grid interpolation using the full finite element method. It was shown that the proposed reduced-basis approach can indeed provide substantial savings while preserving sufficient accuracy compared to standard sparse-grid methods when combined with full finite element solvers.

Q. Guan, M. Gunzburger, C. Webster, and G. Zhang, *Reduced basis methods for nonlocal diffusion problems with random input data*; Computer Methods in Applied Mechanics and Engineering **317** 2017, 746-770.

Stochastic optimization for turbofan noise reduction using parallel reduced-order modeling

Simulation-based optimization of acoustic liner design in a turbofan engine nacelle for noise reduction purposes can dramatically reduce the cost and time needed for experimental designs. Because uncertainties are inevitable in the design process, a stochastic optimization algorithm is posed based on the conditional value-at-risk measure so that an ideal acoustic liner impedance is determined that is robust in the presence of uncertainties. A parallel reduced-order modeling framework is developed that dramatically improves the computational efficiency of the stochastic optimization solver for a realistic nacelle geometry. The reduced stochastic optimization solver takes less than 500 s to execute. In addition, well-posedness and finite element error analyses of the state system and optimization problem are provided.

H. Yang and M. Gunzburger, *Algorithms and analyses for stochastic optimization for turbofan noise reduction using parallel reduced-order modeling*; Computer Methods in Applied Mechanics and Engineering **319** 2017, 217-239.

A regularized ensemble-reduced order method for convection-dominated flows

Partial differential equations (PDEs) are often dependent on input quantities which are uncertain. To quantify this uncertainty, PDEs must be solved over a large ensemble of parameters. Even for a single realization this can a computationally intensive process. In the case of flows governed by the Navier-Stokes equations, an efficient method has been devised for computing an ensemble of solutions. To further reduce the computational cost of this method, an ensemble proper orthogonal decomposition (POD) method was recently developed. We then developed POD spatially filtered ensemble-POD methods so as to enhance the applicability of or approach to flows with higher Reynolds numbers. The POD spatial filter made it possible to construct a Leray ensemble-POD model, i.e., a regularized-reduced order model for the numerical simulation of convection-dominated flows. The Leray ensemble-POD model employs the POD spatial filter to smooth (regularize) the convection term in the Navier-Stokes equations and greatly diminishes the numerical inaccuracies produced by the ensemble-POD method in the numerical simulation of convection-dominated flows. Specifically, for the numerical simulation of a convection-dominated two-dimensional flow between two offset cylinders, we show that the Leray ensemble-POD method yields accurate results, whereas the unfiltered ensemble-POD is highly inaccurate. Error estimates for the novel Leray ensemble-POD algorithm are also proven.

M. Gunzburger, T. Iliescu, and M. Schenier; *A Leray regularized ensemble-proper orthogonal decomposition method for parameterized convection-dominated flows*

Multifidelity methods for uncertainty quantification

In many situations across computational science and engineering, multiple computational models are available that describe a system of interest. These different models have varying evaluation costs and varying fidelities. Typically, a computationally expensive high-fidelity model describes the system with the accuracy required by the current application at hand, whereas lower-fidelity models are less accurate but computationally cheaper than the high-fidelity model. Outer-loop applications, such as optimization, inference, and uncertainty quantification, require multiple model evaluations at many different inputs, which often leads to computational demands that exceed available resources if only the high-fidelity model is used. The overall premise of the multifidelity methods we have developed is that low-fidelity models are leveraged for speedup whereas the high-fidelity model is kept in the loop to establish accuracy and/or convergence guarantees.

We first developed an optimal model management strategy that exploits multifidelity surrogate models to accelerate the estimation of statistics of outputs of computationally expensive high-fidelity models. Existing acceleration methods typically exploit a multilevel hierarchy of surrogate models that follow a known rate of error decay and computational costs; however, a general collection of surrogate models, which may include projection-based reduced models, data-fit models, support vector machines, coarser grid models, and simplified-physics models, does not necessarily give rise to such an explicit hierarchy. Our multifidelity approach provides a framework to combine an arbitrary number of surrogate models of any type. Instead of relying on error and cost rates, an optimization problem balances the number of model evaluations across the high-fidelity and surrogate models with respect to error and costs. We show that a unique analytic solution of the model management optimization problem exists under mild conditions on the models. Our multifidelity method makes occasional recourse to the high-fidelity model; in doing so it provides an unbiased estimator of the statistics of the high-fidelity model, even in the absence of error bounds and error estimators for the surrogate models. Numerical experiments with linear and nonlinear examples show that speedups by orders of magnitude are obtained compared to Monte Carlo estimation that only invokes a single high-cost, high-fidelity model.

We also investigated the properties of multifidelity Monte Carlo method we developed in the setting in which a hierarchy of approximations can be constructed with known error and cost bounds. Our main result is a convergence analysis of multifidelity Monte Carlo estimation, for which we prove a bound on the costs of the multifidelity Monte Carlo estimator under assumptions on the error and cost bounds of the low-fidelity approximations. The assumptions we make are typical in the setting of similar Monte Carlo techniques. Numerical results were obtained to illustrate the derived theoretical bounds.

Our work on multifidelity methods culminated in a comprehensive survey, published in SIAM Review, of multifidelity methods that accelerate the solution of outer-loop applications by combining high-fidelity and low-fidelity model evaluations, where the low-fidelity evaluations arise from explicit low-fidelity models that approximates the same output quantity as the high-fidelity model. The review is broad in scope in that in addition to dealing with statistical estimation, it deals with applications in inference and optimization and also in that three multifidelity method strategies are considered: adaptation, fusion, and filtering.

B. Peherstorfer, K. Willcox, and M. Gunzburger, *Optimal model management for multifidelity Monte Carlo estimation*; SIAM Journal on Scientific Computing **38** 2016, A3163-3194.

B. Peherstorfer, M. Gunzburger, and K. Willcox, *Convergence analysis of multifidelity Monte Carlo estimation*; Numerische Mathematik **139** 2018, 683-707.

B. Peherstorfer, K. Willcox, and M. Gunzburger, *Survey of multifidelity methods in uncertainty propagation, inference, and optimization*; SIAM Review **60** 2018, 550-591.

UQ for PDE control and optimization problems

Comparison of different matching functionals for UQ/PDE optimization problems

We considered an optimal control problem for an elliptic partial differential equation with a random coefficient and a right-hand side control function. The goal is determine a deterministic control such that the solution of the partial differential equation matches as well as possible a prescribed random target function. To determine the deterministic control, we consider the four cases which we compare for efficiency, feasibility, and matching performance. The first case involves a random control and a matching functional defined as the expected value of a spatial matching functional; the deterministic control is defined as the expected value of the optimal random control obtained by minimizing the matching functional. The second case involves the same type of matching functional, but only considers deterministic controls. These two cases are the ones in most common use for the problem at hand. The third case also only considers deterministic control but now the matching functional involves the spatial matching of the expected values of the difference between the solution of the differential equation and the target function. The final case is fully deterministic; a deterministic PDE is determined by using the expected value of random input coefficient, the control is deterministic, and the a standard deterministic spatial matching functional is used.

We prove the existence of optimal states, adjoint states, and optimality conditions and derive the optimality systems for all four cases. The optimality system is then discretized by a standard finite element method and sparse grid collocation method for physical space and probability space, respectively. In this manner we produce four candidate deterministic controls. Illustrative numerical examples indicate that the third choice proves to be superior to the other three choices with respect to efficiency, feasibility, and matching performance. We also consider the introduction of noise when the deterministic controls are applied. In this setting, the third choice proves to be more robust as well.

M. Gunzburger and H.-C. Lee, *Comparison of approaches for random PDE optimization problems based on different matching functionals*; Computers and Mathematics with Applications **73** 2017, 1657-1672.

Optimal control of stochastic cylinder flow using data-driven compressive sensing method.

A stochastic optimal control problem for incompressible Newtonian channel flow past a circular cylinder is used as a prototype for treating, using a data-driven compressive sensing based polynomial chaos expansions, optimal control problems for the stochastic Navier-Stokes equations. The inlet flow and the rotation speed of the cylinder are allowed to have stochastic perturbations. The control acts on the cylinder via adjustment of the rotation speed. Possible objective of control could include, among others, tracking a desired (given) velocity field or minimizing the kinetic energy, enstrophy, or drag of the flow over a given body. Owing to the high computational requirements, direct application of the classical Monte Carlo methods for our problem is strictly limited. An effective gradient-based optimization method for the discrete optimality systems resulting from the data-driven compressive sensing based polynomial chaos expansion discretization is derived and can be applied broadly to stochastic flow control problems. Numerical tests are performed to illustrated the efficacy of our methodology.

H. Liang, Q. Du, M. Gunzburger, and J. Ming; *Optimal control of stochastic cylinder flow using data-driven compressive sensing method*; submitted.

Stochastic collocation method for the optimal boundary control of the Navier-Stokes equations

We considered the optimal control of a system governed by the Navier-Stokes equations with a stochastic Dirichlet boundary condition. Control conditions applied only on the boundary are associated with reduced regularity, as compared to distributed controls. To ensure the existence of solution and the efficiency of numerical simulations, the stochastic boundary conditions and controls are required to belong almost surely to the Sobolev space of functions having one weak derivative along the boundary. To simulate the system, state solutions are approximated using the stochastic collocation/finite element approach with sparse grid techniques applied to the boundary random field. An optimality system is derived for a matching-type cost functional. Error estimates are derived for the optimal state, the adjoint state, and boundary control. Numerical examples are provided that compare solutions and errors for cases in which the control is applied on part of or on the whole boundary. Comparisons are also made between using Monte Carlo and sparse grid methods for determining statistical information about the optimal solution.

W. Zhao and M. Gunzburger; *Stochastic collocation method for stochastic optimal boundary control of the Navier-Stokes equations*; submitted.

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