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On the Systematic Construction of Discrete Filters

AYABOE EDOH (ERC, INC)

MAY 9, 2019
DUSSELDORF, GERMANY

MOTIVATION

Large Eddy Simulations (LES)

- calculate large scales
- model influence of non-resolved modes

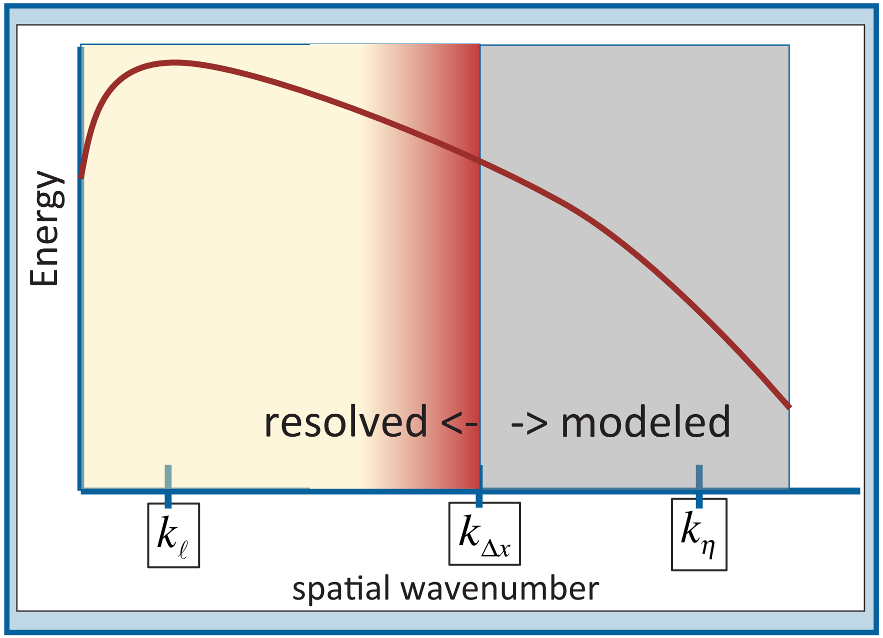
separation via low-pass filter

$$G_{\bar{\Delta}} \left\{ \partial_t u + \frac{1}{2}\partial_x u^2 = 0 \right.$$

$$\partial_t \bar{u} + \frac{1}{2}\partial_x \overline{u^2} = 0$$

$$\partial_t \bar{u} + \frac{1}{2}\partial_x \bar{u}\bar{u} + \frac{1}{2}\underbrace{\partial_x \left[\overline{uu} - \bar{u}\bar{u}\right]}_{\tau^{SFS}} = 0$$

Filter-to-grid ratio (FGR): $\Delta/\Delta x$



MOTIVATION

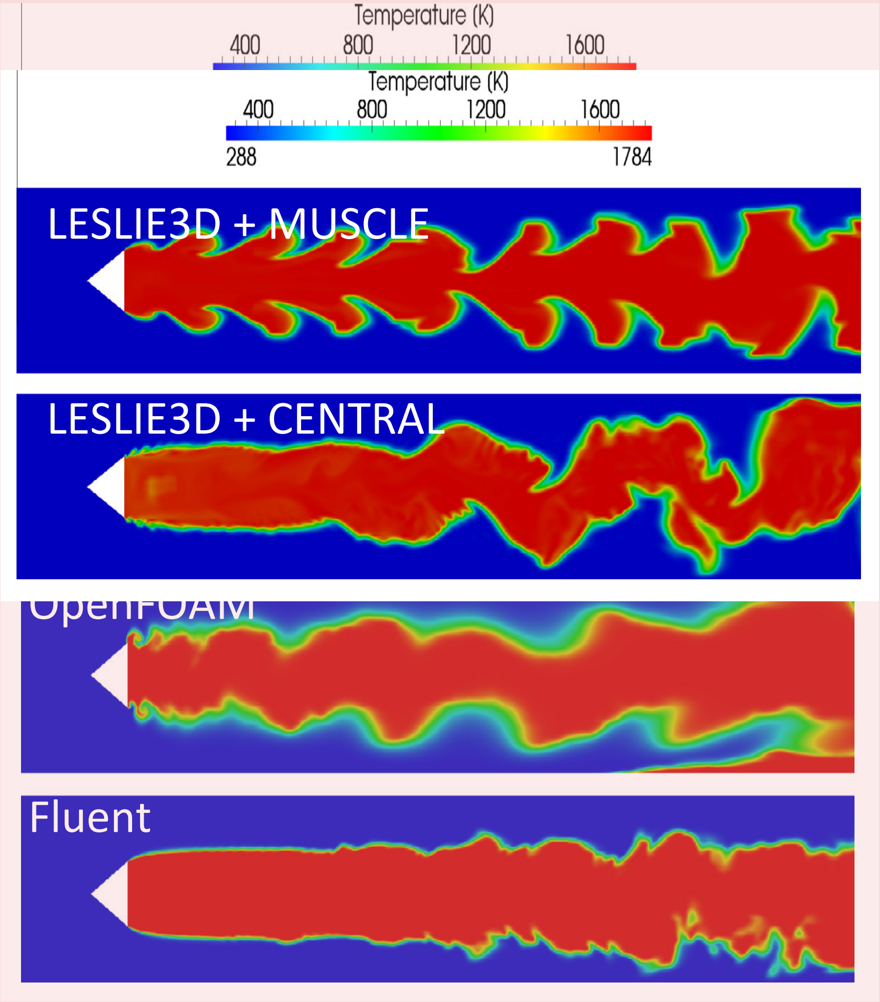


Figure 15 from [1]

Cocks *et al.* (*Combust. Flame*, 2015) [1]
- similar grid resolutions, closure models, chemistry etc.
- different codes (i.e., numerics)

Vast disagreements for reactive flow
(otherwise match w/o combustion)

Increased sensitivity wrt errors
-> discretization scheme

Mitigate numerical artifacts in order to properly evaluate/develop models

MOTIVATION

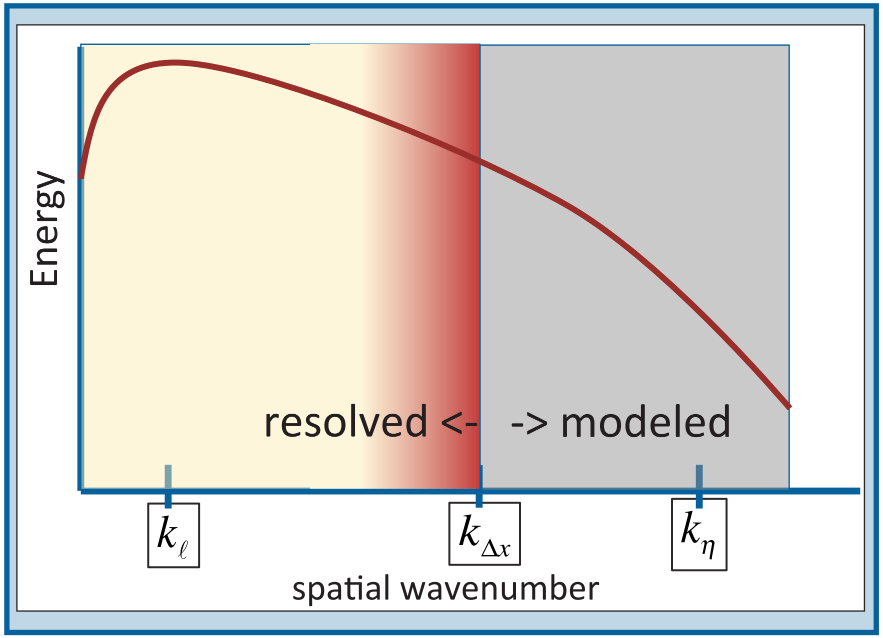
- Reduce influence of numerical error in LES calculations
  - improved solution accuracy
  - objective model assessment/development

Only a subset of grid-resolved modes are "trustable"
➔ constrain the calculation via explicit filtering
(i.e., non-unity FGR)

$$\partial_t \bar{u} + \tfrac{1}{2}\partial_x \bar{u}\bar{u} + \tfrac{1}{2}\underbrace{\partial_x[\overline{uu} - \bar{u}\bar{u}]}_{\tau^{SFS}} = 0$$

$$\partial_t \hat{\bar{u}} + \tfrac{1}{2}\partial_x \hat{\bar{u}}\hat{\bar{u}} + \tfrac{1}{2}\underbrace{\partial_x[\widehat{\overline{uu}} - \hat{\bar{u}}\hat{\bar{u}}]}_{\tau^{SFS}} = 0$$

$$\partial_t \hat{\bar{u}} + \tfrac{1}{2}\partial_x \hat{\bar{u}}\hat{\bar{u}} + \tfrac{1}{2}\underbrace{\partial_x[\widehat{\bar{u}\bar{u}} - \hat{\bar{u}}\hat{\bar{u}}]}_{\tau^{RSFS}} + \tfrac{1}{2}\underbrace{\partial_x[\widehat{\overline{uu}} - \widehat{\bar{u}\bar{u}}]}_{\widehat{\tau^{SGS}}} = 0$$



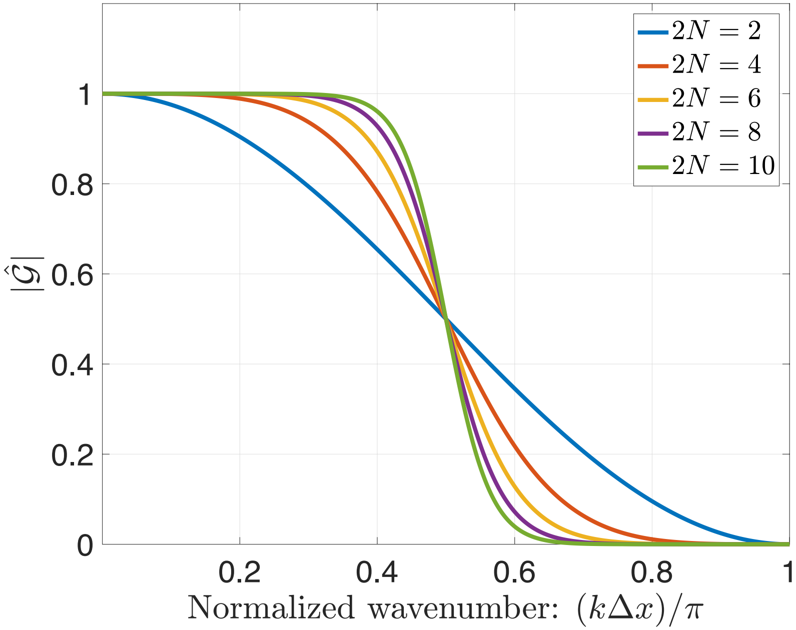
$\bar{\Delta}$: grid filter

$\hat{\Delta}$: explicit LES filter

OBJECTIVES

- Understand the systematic construction of discrete filters (and artificial dissipation)
    - scale separation, stencil-order, monotonicity, invertibility, etc…

- Formalize extensions to bounded domains



$$\bar{u}_i + \sum_{\ell\geq 1} a_\ell(\bar{u}_{i-\ell} + \bar{u}_{i+\ell}) = b_0 u_i + \sum_{r\geq 1} b_r(u_{i-r} + u_{i+r})$$

$$\left[1 + \sum_{\ell}^{R} \epsilon_{2\ell}\delta^{2\ell}\right] \bar{u}_i = \left[1 + \sum_{r}^{R} \epsilon_{2r}\delta^{2r}\right] u_i$$

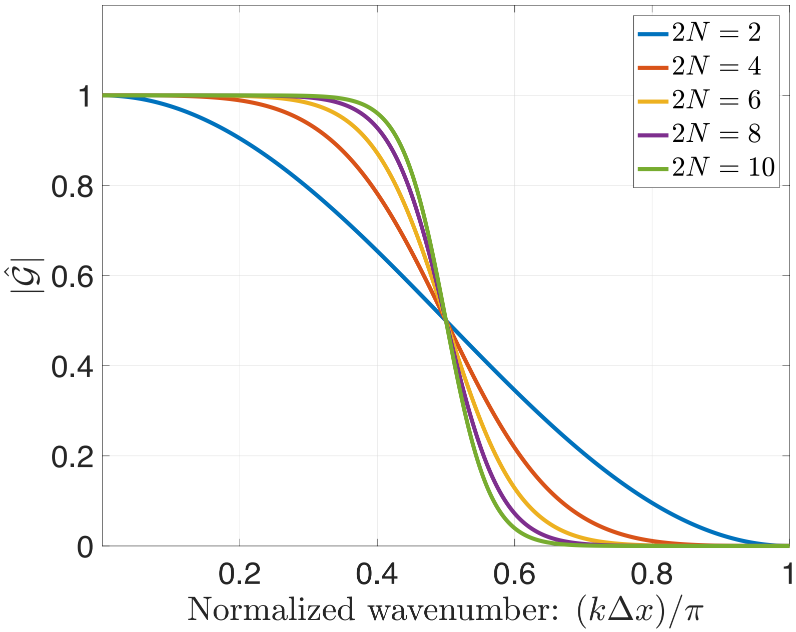
$$\begin{aligned} \delta^2 u_i &= (\Delta x^2)\partial_x^2 u_i + O(\Delta x^4) \\ &= u_{i+1} - 2u_i + u_{i+1} \\ \delta^{2n} &= (\delta^2)^n \end{aligned}$$

OUTLINE

- Discrete Purser filters
    - classic formulation
    - explicit and implicit generalizations
- Energy-stability and conservation on bounded domains
- Filter-based artificial dissipation

Purser Filters (Classic)

Purser (*J. Appl. Meteorol. Climatol.*, 1987) [2]



- Identify spectral response as the complement to a Cumulative Distribution Function (CDF)

$$C(x) = \int_0^x p(z)dz, \;\; x \in [0,1]$$

$$\widehat{\mathcal{G}}(k) = 1 - C(k) = 1 + \mathcal{D}(k)$$

- Make CDF a function of wavenumber via the response of a finite difference (FD) stencil

$$x = \hat{g}(k) = \sin^2\left(\frac{k\Delta x}{2}\right) = -\frac{1}{4}\mathcal{F}\{\delta^2\}$$

- Recover stencil from response

Beta distribution:

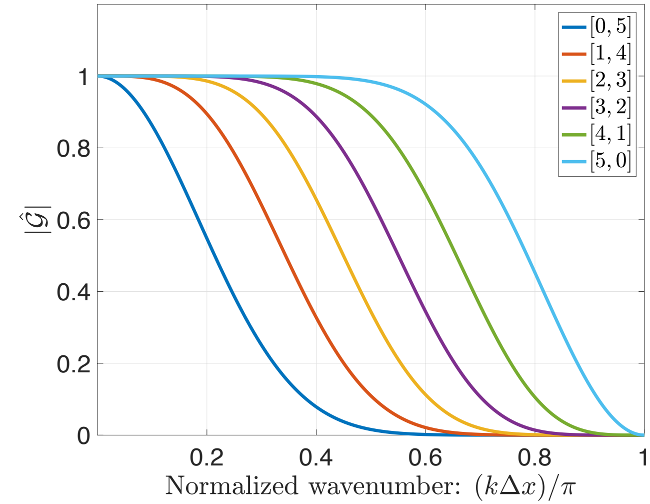
$$p(z) = \frac{(R+S+1)!}{R!S!} z^R (1-z)^S \;\text{ with }\; z \in [0,1]$$

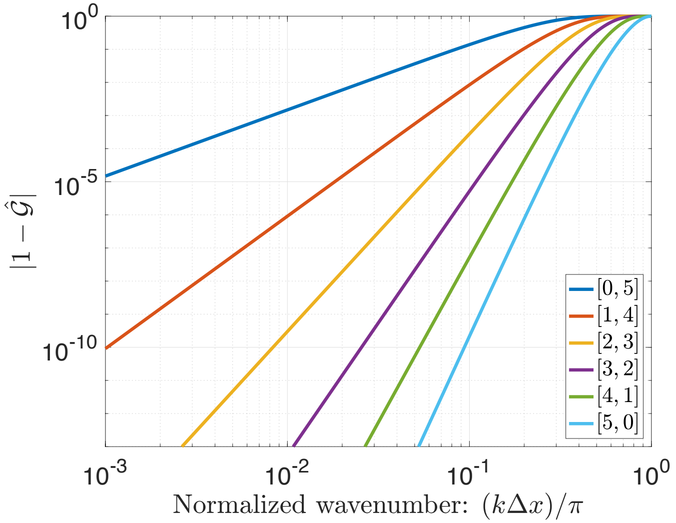
$$\downarrow$$

$$\bar{u} = \left[1 + \sum_{n=R+1}^{R+S+1} \epsilon_{2n}\delta^{2n}\right] u$$

$$\varepsilon_{EF,2n} = \frac{N!}{R!S!} \cdot \frac{S!}{(n-R-1)![S-(n-R-1)]!} \cdot \frac{(-1)^{n-R}}{n} \cdot \left(\frac{-1}{4}\right)^n$$

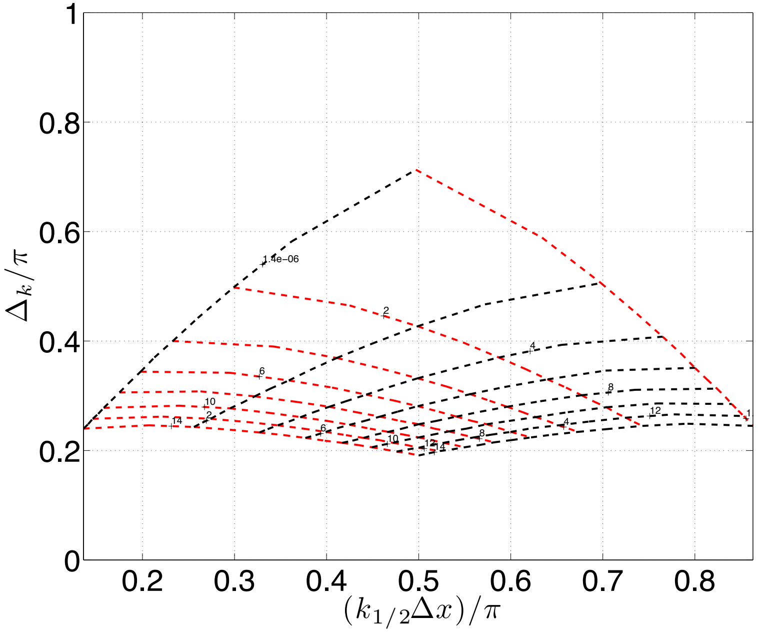
Purser Filters (Classic)





- The response shape is easily parameterized via statistical characterization of the CDF

$$z_{1/2} \approx \frac{R+2/3}{R+S+4/3}\,, \qquad \sigma^2 = \frac{(R+1)(S+1)}{(R+S+2)^2(R+S+3)}$$



filter width: $k_{1/2}$ such that $\hat{\mathcal{G}}(k_{1/2}) = \frac{1}{2}\left[\max_k\{\hat{\mathcal{G}}\} - \min_k\{\hat{\mathcal{G}}\}\right] + \min_k\{\hat{\mathcal{G}}\}$

transition width: $\Delta_k = k_{0.05} - k_{0.95}$

Purser Filters (Generalizations)

Edoh (*UCLA Thesis*, 2017) [3]

$$\bar{u}_i + \sum_{\ell\geq 1} a_\ell(\bar{u}_{i-\ell} + \bar{u}_{i+\ell}) = b_0 u_i + \sum_{r\geq 1} b_r(u_{i-r} + u_{i+r})$$

$$\left[1 + \sum_\ell^R \epsilon_{2\ell}\delta^{2\ell}\right] \bar{u}_i = \left[1 + \sum_r^R \epsilon_{2r}\delta^{2r}\right] u_i \quad \longrightarrow \quad \widehat{\mathcal{G}}(k) = 1 + \widehat{\mathcal{D}}(k) = \frac{1+\widehat{\mathcal{D}}_r}{1+\widehat{\mathcal{D}}_\ell}$$

- Consider alternate distribution functions, for example:

Beta: $p(z) = \frac{(R+S+1)!}{R!S!} z^R(1-z)^S \text{ with } z \in [0,1]$

Kumaraswamy: $p(z) = (R+1)(S+1) \cdot z^R(1-z^{R+1})^S \text{ with } z \in [0,1]$

Log-logistic: $p(z) = (\frac{R+1}{\alpha})(\frac{z}{\alpha})^R \cdot [1 + (\frac{z}{\alpha})^{R+1}]^{-2} \text{ with } z \in [0,\infty\rangle$

- Consider modifying the initial transformation: $x = -\widehat{\mathcal{D}}(k) \in [0,1]$

“embedded”: $\bar{u} = \left[1 + \sum_{n=R+1}^{R+S+1} \epsilon_{2n} \left(\sum_{\ell=r+1}^{r+s+1} \epsilon_{2\ell}\delta^{2\ell}\right)^n\right] u$

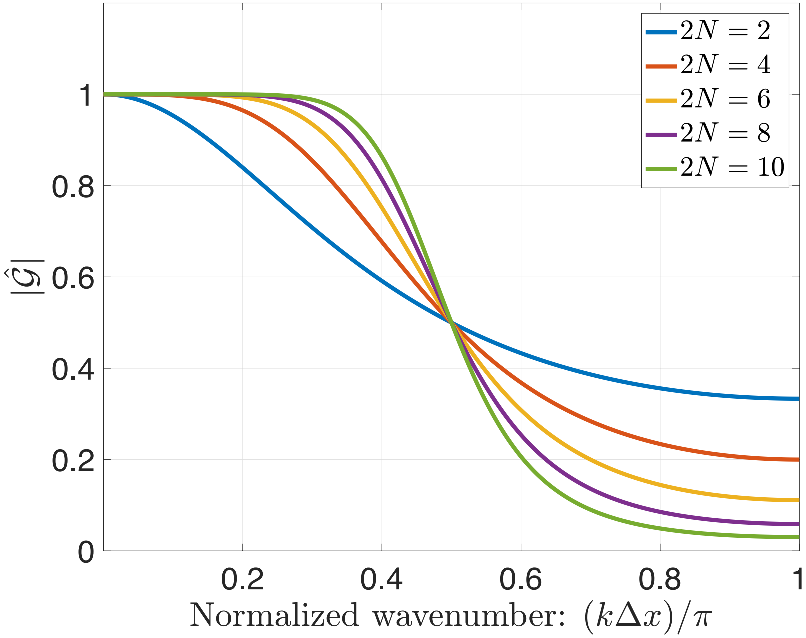
Purser Filters (Generalizations)

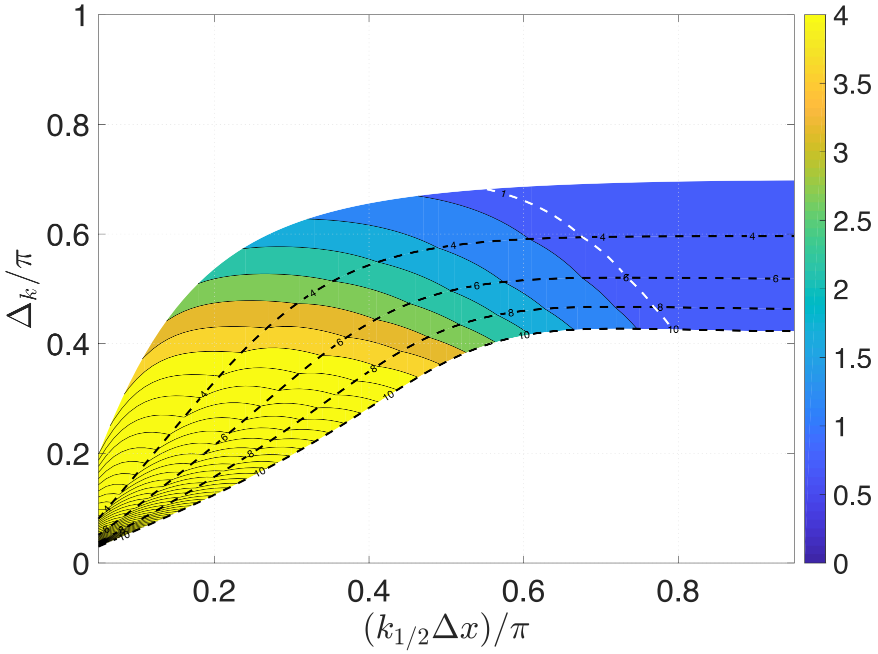
$$\bar{u}_i + \textstyle\sum_{\ell\geq 1} a_\ell(\bar{u}_{i-\ell} + \bar{u}_{i+\ell}) = b_0 u_i + \textstyle\sum_{r\geq 1} b_r(u_{i-r} + u_{i+r})$$

$$\left[1 + \textstyle\sum_{\ell}^{R} \epsilon_{2\ell}\delta^{2\ell}\right]\bar{u}_i = \left[1 + \textstyle\sum_{r}^{R} \epsilon_{2r}\delta^{2r}\right] u_i \quad \longrightarrow \quad \widehat{\mathcal{G}}(k) = 1 + \widehat{\mathcal{D}}(k) = \frac{1+\widehat{\mathcal{D}}_r}{1+\widehat{\mathcal{D}}_\ell}$$

- Consider spectral manipulations to build implicit schemes from explicit ones

Butterworth-style filters: $\dfrac{1}{1-\delta_\Delta \widehat{D}_r}$





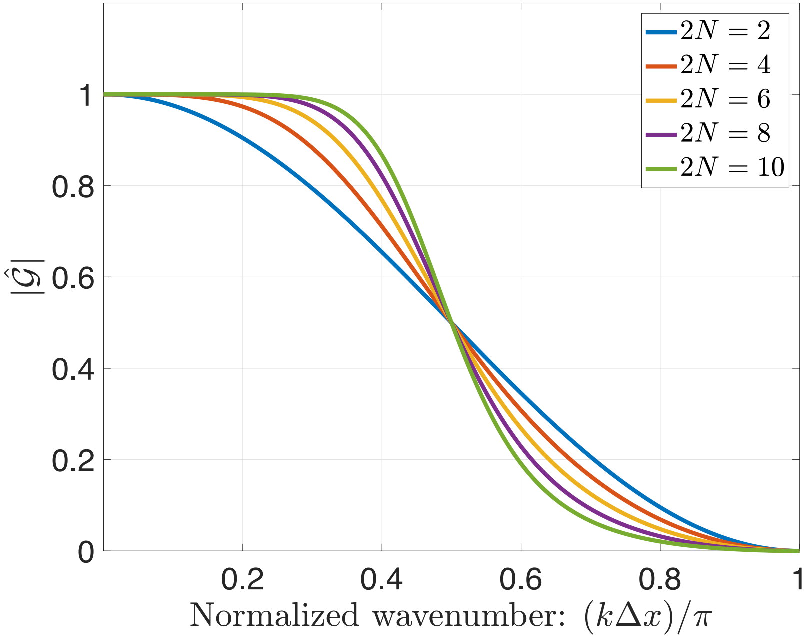
Purser Filters (Generalizations)

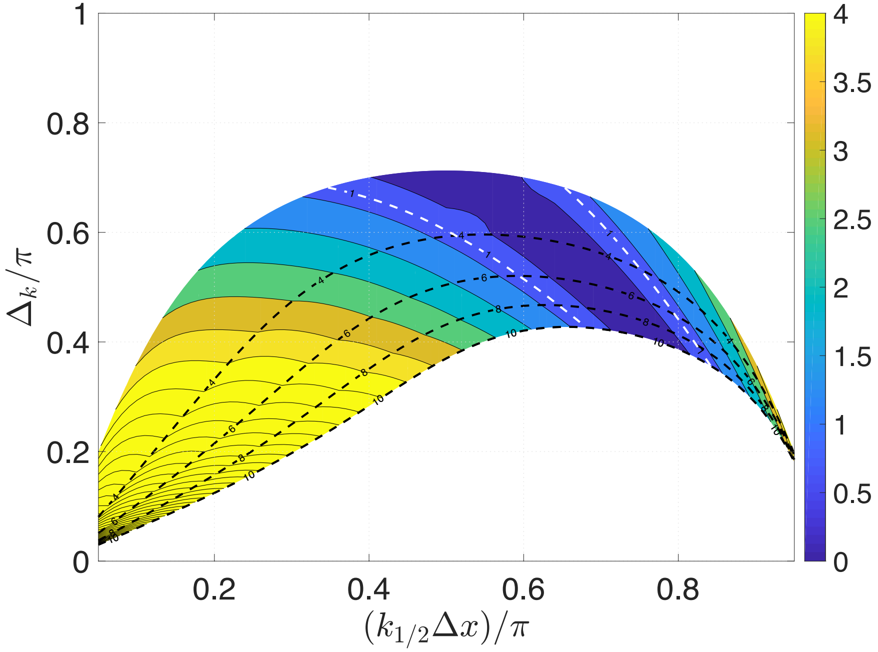
$$\bar{u}_i + \sum_{\ell\geq 1} a_\ell(\bar{u}_{i-\ell} + \bar{u}_{i+\ell}) = b_0 u_i + \sum_{r\geq 1} b_r(u_{i-r} + u_{i+r})$$

$$\left[1 + \sum_{\ell}^{R} \epsilon_{2\ell}\delta^{2\ell}\right]\bar{u}_i = \left[1 + \sum_{r}^{R} \epsilon_{2r}\delta^{2r}\right]u_i \longrightarrow \widehat{\mathcal{G}}(k) = 1 + \widehat{\mathcal{D}}(k) = \frac{1+\widehat{\mathcal{D}}_r}{1+\widehat{\mathcal{D}}_\ell}$$

- Consider spectral manipulations to build implicit schemes from explicit ones

Long-style filters: $\frac{1+\widehat{D}_r}{1+(1-\delta_\Delta)\widehat{D}_r}$





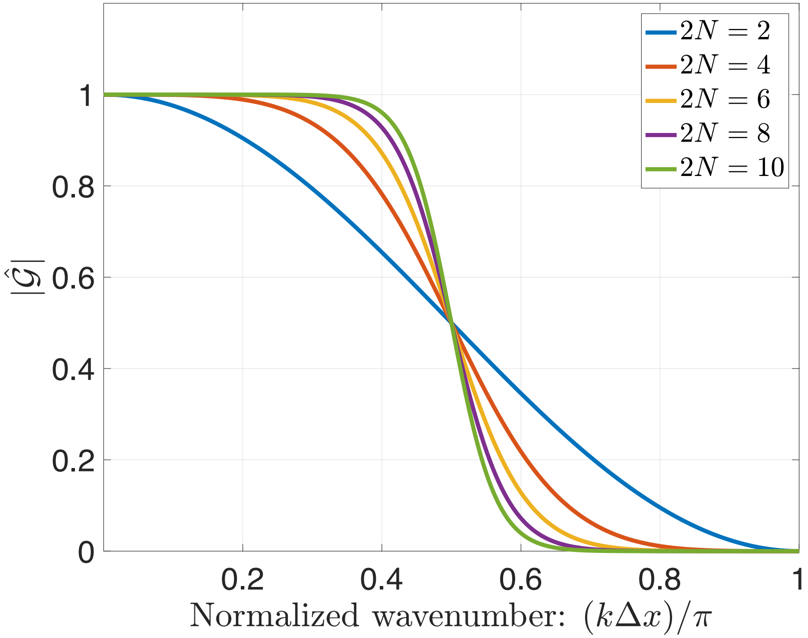
Purser Filters (Generalizations)

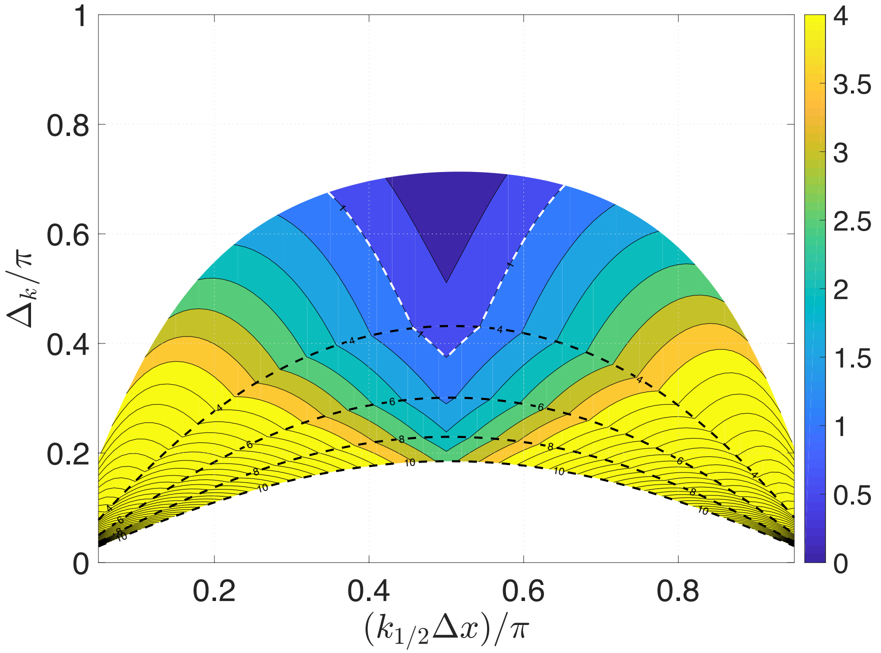
$$\bar{u}_i + \sum_{\ell\geq 1} a_\ell(\bar{u}_{i-\ell}+\bar{u}_{i+\ell}) = b_0 u_i + \sum_{r\geq 1} b_r(u_{i-r}+u_{i+r})$$

$$\left[1+\sum_\ell^R \epsilon_{2\ell}\delta^{2\ell}\right]\bar{u}_i = \left[1+\sum_r^R \epsilon_{2r}\delta^{2r}\right]u_i \quad \longrightarrow \quad \widehat{\mathcal{G}}(k) = 1+\widehat{\mathcal{D}}(k) = \frac{1+\widehat{\mathcal{D}}_r}{1+\widehat{\mathcal{D}}_\ell}$$

- Consider spectral manipulations to build implicit schemes from explicit ones

Tangent-style filters: $\frac{1+\widehat{D}_r}{1+(1-\delta_\Delta)\widehat{D}_r}$

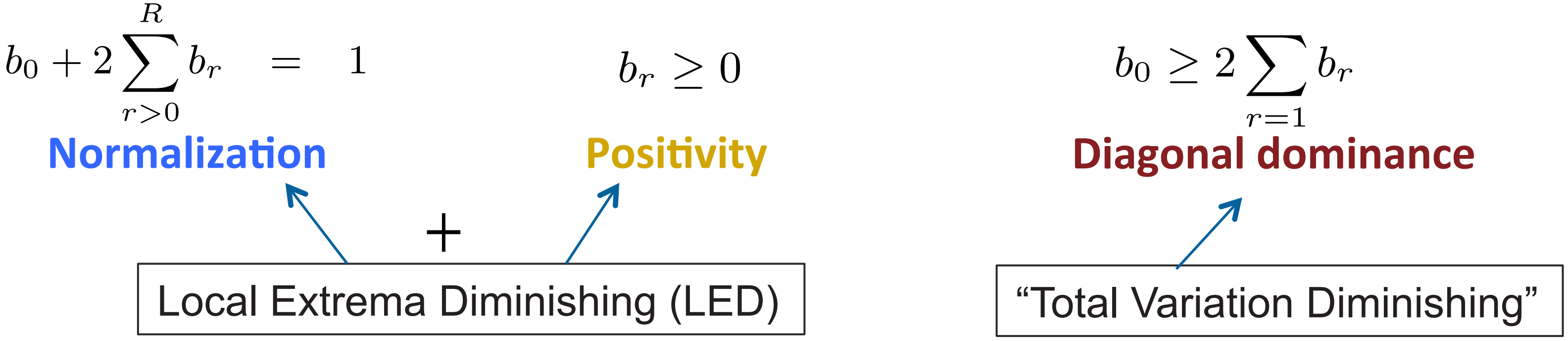




Monotonicity

$$\bar{u}_i + \textstyle\sum_{\ell\geq 1} a_\ell(\bar{u}_{i-\ell} + \bar{u}_{i+\ell}) = b_0 u_i + \textstyle\sum_{r\geq 1} b_r(u_{i-r} + u_{i+r})$$

1) stencils on RHS



$$b_0 + 2\sum_{r>0}^{R} b_r \quad = \quad 1$$

Normalization

$$b_r \geq 0$$

Positivity

$$b_0 \geq 2\sum_{r=1} b_r$$

Diagonal dominance

+

Local Extrema Diminishing (LED)

"Total Variation Diminishing"

- no new extrema are created, wrt stencil width

- strongest weighting on current node

2) An LHS stencil satisfying…

$$1 + 2\sum_{\ell>0}^{L} a_\ell \quad = \quad 1$$

Normalization

$$a_\ell \leq 0 \quad \text{for} \quad \ell \geq 1$$

Positivity-inverse matrix

$$\boxed{G_\ell \bar{\mathbf{u}} = G_r \mathbf{u} \quad \rightarrow \quad \bar{\mathbf{u}} = G\mathbf{u} = G_\ell^{-1} G_r \mathbf{u} \quad \text{with} \quad G_\ell \mathbf{1} = G_r \mathbf{1} = \mathbf{1}}$$

Monotonicity Preserving Explicit Stencils

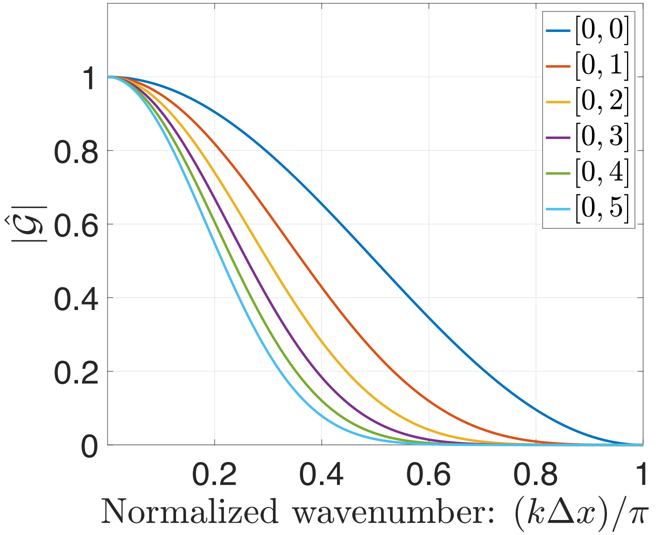
Purser Filters

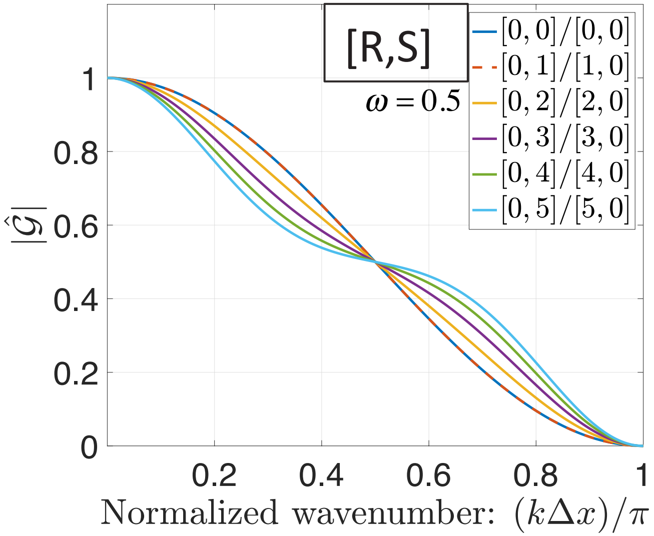
- stencil width w = 2N+1 (N = R + S + 1)
- **S = 0 (Shapiro)**
- **R = 0 (Binomial)**

R	S	$w = 4^{R+S+1}$	$w \cdot b_0$	$w \cdot b_1$	$w \cdot b_2$	$w \cdot b_3$	$w \cdot b_4$	$w \cdot b_5$
0	0	4	2	1				
0	1	16	6	4	1			
1	0	16	10	4	-1			
0	2	64	20	15	6	1		
1	1	64	32	18	0	-2		
2	0	64	44	15	-6	1		
0	3	256	70	56	28	8	1	
1	2	256	110	72	12	-8	-3	
2	2	256	146	72	-12	-8	3	
3	0	256	186	56	-28	7	-1	
0	4	1024	252	210	120	45	10	1
1	3	1024	392	280	80	-20	-20	-4
2	2	1024	512	300	0	-50	0	6
3	1	1024	632	280	-80	-20	20	-4
4	0	1024	772	210	-120	45	-10	1

Monotonicity Preserving Explicit Stencils

Binomial filters are LED:





Convex average between Binomial and Shapiro stencil is LED

$$G = \omega G^{Binom} + (1 - \omega) G^{Shap}$$

Monotonicity Preserving Implicit Filters

but why do so?
-> simple spectral tunability

Can build MP implicit schemes from MP explicit schemes

1) choose an MP explicit scheme w/ response: $\hat{\mathcal{G}}(k) = 1 + \hat{\mathcal{D}}(k)$

2) form implicit response
   - Butterworth (BW) rendition

$$\hat{\mathcal{G}}_{Bw}(k) = \frac{1}{1 - \delta_\Delta \cdot \hat{\mathcal{D}}(k)} \quad \text{with} \quad \delta_\Delta \geq 0$$

MP for $\delta_\Delta > 0$

   - Long (LG) rendition

$$\hat{\mathcal{G}}_{Lg}(k) = \frac{1 + \hat{\mathcal{D}}(k)}{1 + (1 - \delta_\Delta) \cdot \hat{\mathcal{D}}(k)} \quad \text{with} \quad \delta_\Delta > 0$$

MP for $\delta_\Delta \geq 1$

Binomial (exp.)

$$\bar{u}_i = \left[1 + \frac{1}{4}(\Delta x)^2 \delta_x^2\right] u_i$$

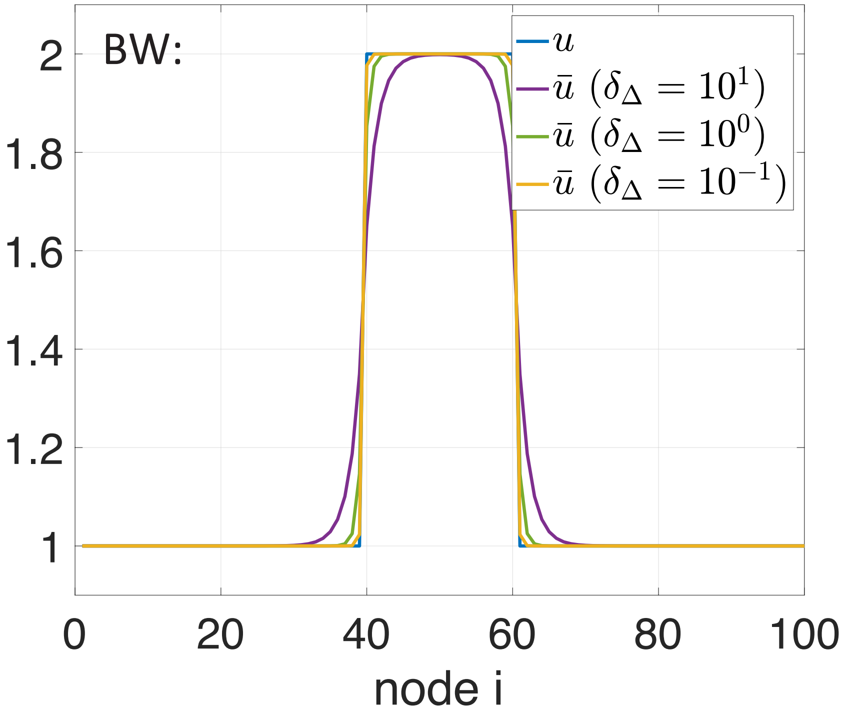
BW (imp.)

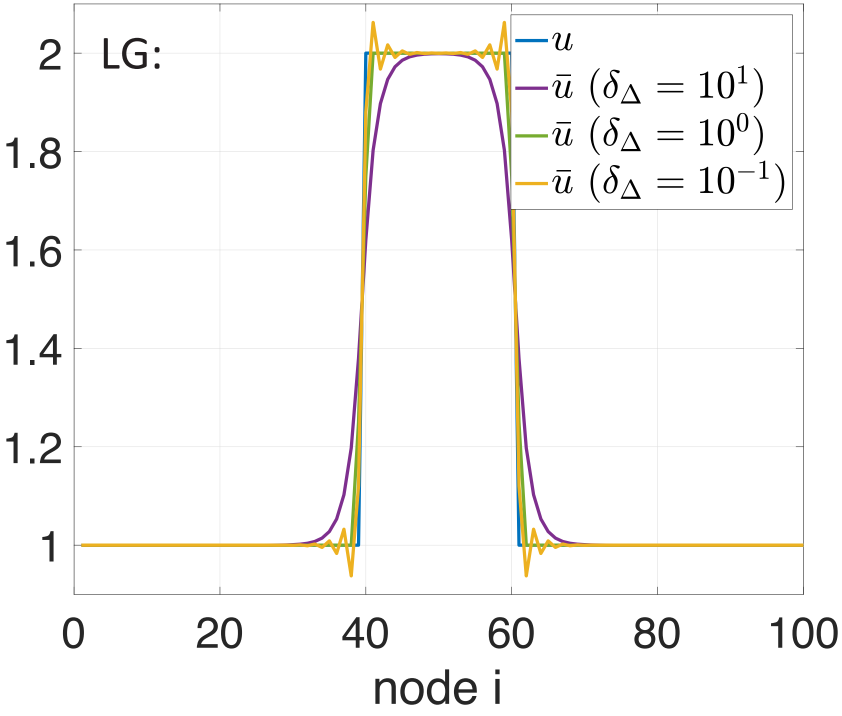
$$\left[1 - \frac{\delta_\Delta}{4}(\Delta x)^2 \delta_x^2\right] \bar{u}_i = u_i$$

LG (imp.)

$$\left[1 + \frac{(1 - \delta_\Delta)}{4}(\Delta x)^2 \delta_x^2\right] \bar{u}_i = \left[1 + \frac{1}{4}(\Delta x)^2 \delta_x^2\right] u_i$$

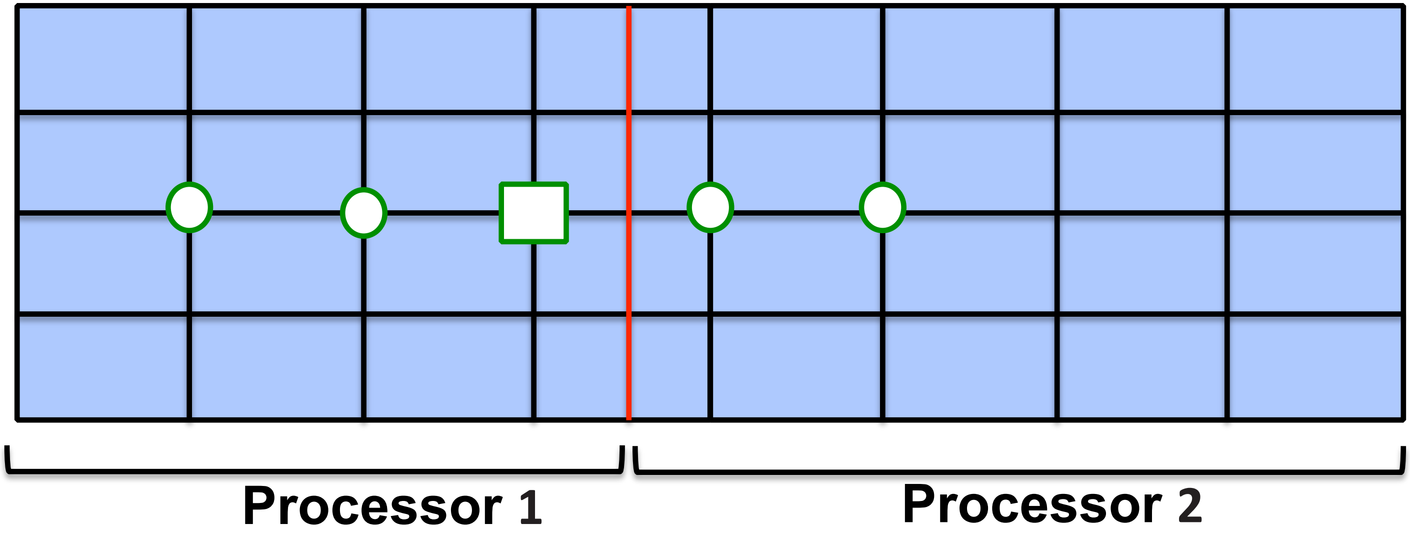
Monotonicity Preserving Implicit Filters





Extension to Bounded Domains

- Filter schemes developed and well understood on interior w/ uniform grid
- Need proper prescription of near-boundary stencils:
    - physical domain boundaries, processor blocks



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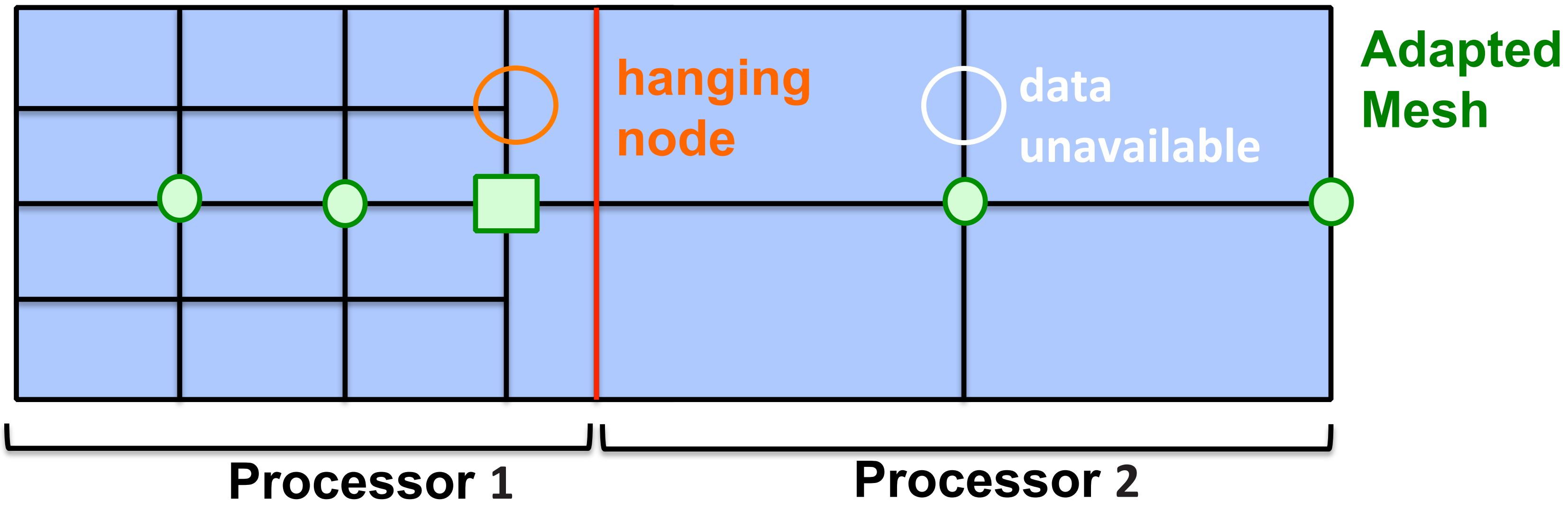

Extension to Bounded Domains

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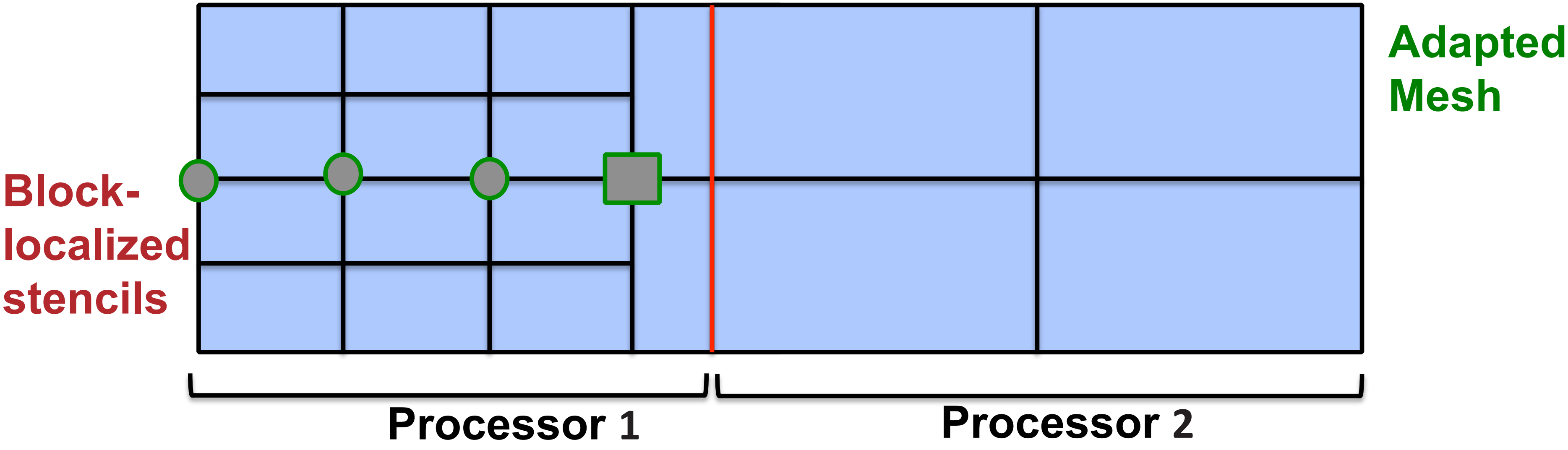
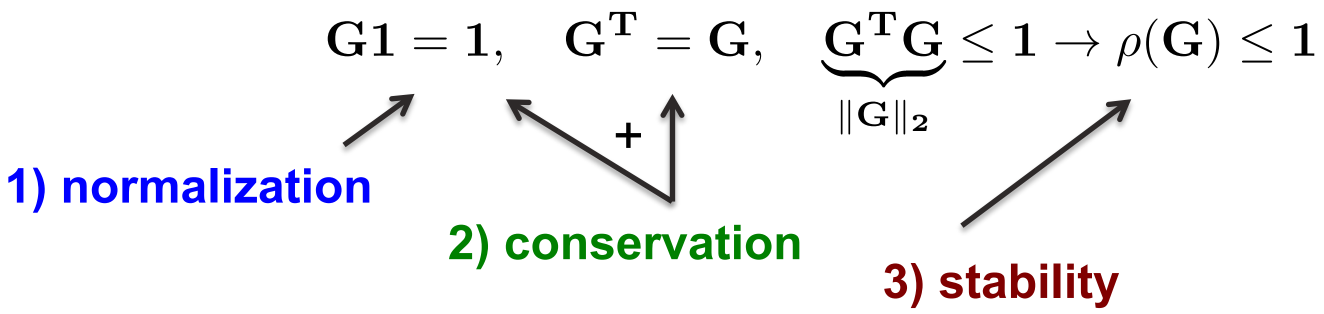


Consider construction of filter operators that are:
1) spectrally-tunable, 2) stable, 3) conservative and 4) **simple**

Required Properties of Filter Operator

$$\boxed{\bar{\mathbf{u}} = \mathbf{G}\mathbf{u}}$$



$$\mathbf{G}\mathbf{1} = \mathbf{1}, \quad \mathbf{G}^{\mathbf{T}} = \mathbf{G}, \quad \underbrace{\mathbf{G}^{\mathbf{T}}\mathbf{G}}_{\|\mathbf{G}\|_2} \le \mathbf{1} \rightarrow \rho(\mathbf{G}) \le \mathbf{1}$$

1) normalization

+

2) conservation

3) stability

1) Preservation of constant modes

$$\mathbf{G}\mathbf{1} = \mathbf{1}$$

2) Preservation of integral quantities

$$\mathbf{1}^{\mathbf{T}}\mathbf{u} \;=\; \mathbf{1}^{\mathbf{T}}\bar{\mathbf{u}} \;=\; \bar{\mathbf{u}}^{\mathbf{T}}\mathbf{1} \;=\; \mathbf{u}^{\mathbf{T}}\mathbf{G}^{\mathbf{T}}\mathbf{1}$$

3) Energy-stable wrt a general inner-norm

$$\begin{aligned} \mathbf{u}^{\mathbf{T}}\mathbf{P}\mathbf{u} \;&\ge\; \bar{\mathbf{u}}^{\mathbf{T}}\mathbf{P}\bar{\mathbf{u}} \\ &\ge\; (\mathbf{G}\mathbf{u})^{\mathbf{T}}\mathbf{P}(\mathbf{G}\mathbf{u}) \;\rightarrow\; \mathbf{P} \ge \mathbf{G}^{\mathbf{T}}\mathbf{P}\mathbf{G} \end{aligned}$$

Further Understanding the Target Interior Stencil

Consider symmetric filter stencils on a uniform grid

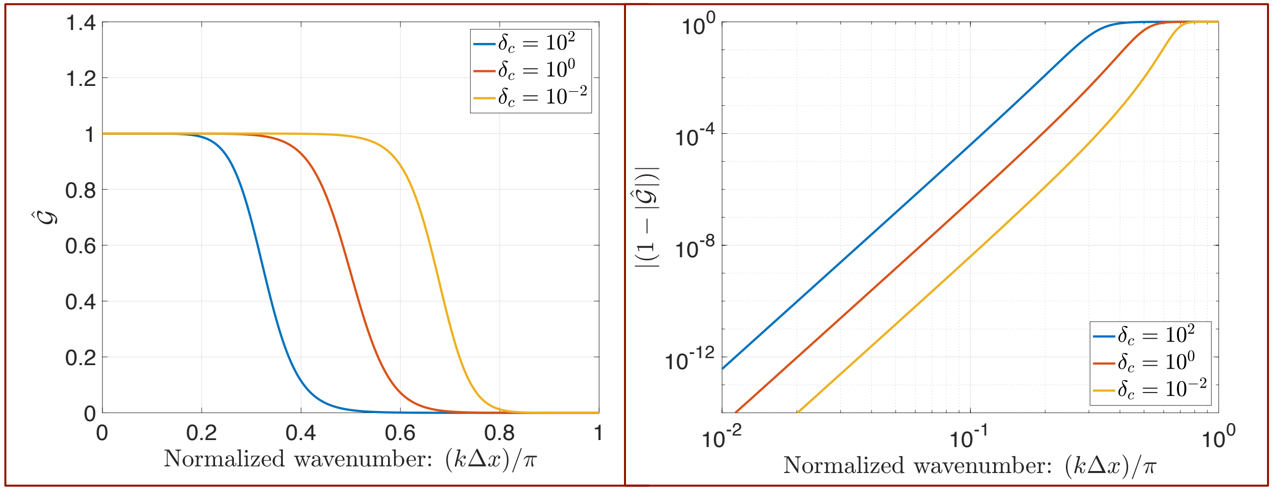
A $$\bar{u}_i + \sum_{\ell \geq 1} a_\ell (\bar{u}_{i-\ell} + \bar{u}_{i+\ell}) = b_0 u_i + \sum_{r \geq 1} b_r (u_{i-r} + u_{i+r})$$

B $$\rightarrow \left[1 + \sum_{\ell \geq 1} \epsilon_{IF,2\ell} (\Delta x)^{2\ell} \delta_x^{2\ell}\right] \bar{u}_i = \left[1 + \sum_{r \geq 1} \epsilon_{EF,2r} (\Delta x)^{2r} \delta_x^{2r}\right] u_i$$

$$\begin{aligned} \delta_x^2 u_i &= \partial_x^2 u_i + O(\Delta x^2) \\ &= \frac{u_{i+1} - 2u_i + u_{i+1}}{\Delta x^2} \end{aligned}$$

$$\delta_x^{2n} = (\delta^2)^n$$

C $$\rightarrow \hat{\mathcal{G}}(k\Delta x) = \frac{1 + \sum_{r>1} \epsilon_{EF,2r} \left[-4\sin^2\left(\frac{k\Delta x}{2}\right)\right]^r}{1 + \sum_{\ell>1} \epsilon_{IF,2\ell} \left[-4\sin^2\left(\frac{k\Delta x}{2}\right)\right]^\ell} = \frac{\overbrace{1 + \sum_{r>1} \epsilon_{EF,2r} \mathcal{X}^r}^{\mathcal{P}_{EF}(\mathcal{X})}}{\underbrace{1 + \sum_{\ell>1} \epsilon_{IF,2\ell} \mathcal{X}^\ell}_{\mathcal{P}_{IF}(\mathcal{X})}} \quad \text{with} \quad \mathcal{P}_{EF} \in [0,1], \mathcal{P}_{IF} \in \langle 0,1] \text{ for } \mathcal{X} \in [-4,0]$$



Response features polynomials that are positive and positive semi-definite over a range of input values.

Considering Base Operators

$$\tilde{D}_2 = \hat{D}_1^T C_1 \hat{D}_1 = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix},$$

Properties:
- features $(\Delta x^2)\delta_x^2$ on the interior
- is symmetric *a priori*
- has eigenvalues, $\lambda \in [-4, 0]$
  - Gerschgorin Circle Theorem

$$eig\{A\} \rightarrow |\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|$$

$$\hat{D}_1 = \begin{bmatrix} -1 & 1 & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ & & & \ddots & \\ & & & -1 & 1 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0 & & \\ & -1 & \\ & & \ddots \end{bmatrix}$$

Can recover $(\Delta x^{2n})\delta_x^{2n}$ on the interior with powers of the base operator:
“**cascading**”

$$\tilde{D}_2^2 = \begin{bmatrix} 2 & -3 & 1 & & \\ -3 & 6 & -4 & 1 & \\ 1 & -4 & 6 & -4 & 1 \\ & & \ddots & & \end{bmatrix}$$

boundary stencils generated automatically

$(\Delta x^4)\delta_x^4$

The Cascade Idea for Designing Energy-Stable and Conservative Filter Operators

Construct the filter by scaling and adding powers of a *base operator*, whose properties will yield

- the target interior stencil
- provable energy stability
- provable conservation

$$\boxed{\mathbf{G} = \mathbf{G}_\ell^{-1}\mathbf{G_r}}$$

Inspired by difference form of interior stencil

$$\mathbf{G}_\ell = \mathbf{G}_\ell^{\mathbf{T}} = \mathbf{I} + \sum_{\ell\geq\mathbf{1}} \epsilon_{\mathbf{IF},\mathbf{2}\ell}\mathbf{D}_{\mathbf{2}}^{\ell} > \mathbf{0} \qquad \mathbf{G_r} = \mathbf{G_r^T} = \mathbf{I} + \sum_{\mathbf{r}\geq\mathbf{1}} \epsilon_{\mathbf{EF},\mathbf{2r}}\mathbf{D}_{\mathbf{2}}^{\mathbf{r}} \geq \mathbf{0}$$

$$\frac{\overbrace{1+\sum_{r>1}\epsilon_{EF,2r}\mathcal{X}^r}^{\mathcal{P}_{EF}(\mathcal{X})}}{\underbrace{1+\sum_{\ell>1}\epsilon_{IF,2\ell}\mathcal{X}^\ell}_{\mathcal{P}_{IF}(\mathcal{X})}}$$

$$\mathbf{G} \quad = \quad \mathbf{M}\left[\mathbf{diag}\left\{\frac{\mathbf{1}+\sum_{\mathbf{r}\geq\mathbf{1}}\epsilon_{\mathbf{EF},\mathbf{2r}}\lambda^{\mathbf{r}}}{\mathbf{1}+\sum_{\ell\geq\mathbf{1}}\epsilon_{\mathbf{IF},\mathbf{2}\ell}\lambda^{\ell}}\right\}\right]\mathbf{M^T} \quad \text{with} \quad \lambda \in [-\mathbf{4},\mathbf{0}]$$

$$\rightarrow \quad \mathbf{G1} = \mathbf{1}, \mathbf{G} = \mathbf{G^T} \in [\mathbf{0},\mathbf{1}]$$

Identical to polynomial relations from response of interior stencil, yielding provable stability *a priori*

Edoh & Sankaran (*AIAA 2170*, 2019) [4]

Filter-based Artificial Dissipation

$$\boxed{G = I + D \longrightarrow G\{u_i\} = \left[1 + \sum_r^R \epsilon_{2r}(\Delta x)^{2r}\delta_x^{2r}\right] u_i,\ \ |\hat{\mathcal{G}}| \le 1}$$

$$\boxed{u^{n+1} = u^n + (\Delta t)\cdot \underbrace{R(u)}_{-a\delta_x u}}$$

Solution Filtering (SF)

1. $u^{n+1,*} = u^n + (\Delta t)\cdot R(u)$
2. $u^{n+1} = G\{u^{n+1,*}\}$

$$\downarrow$$

$$\frac{u^{n+1}-u^n}{\Delta t} = -a\delta_x u - a\sum_r \epsilon_{2r}(\Delta x)^{2r}\delta_x^{2r+1}u + \left(\frac{\Delta x}{\Delta t}\right)\sum_r \epsilon_{2r}(\Delta x)^{2r-1}\delta_x^{2r}u$$

- spectrally-constraining via dissipation
- preserves phase dynamics of the base scheme
- temporally inconsistent, as-is

Filter-based Artificial Dissipation (AD)

$$\frac{u^{n+1}-u^n}{\Delta t} = R(u) + \frac{1}{\tau}D\{u\}$$

$$\downarrow$$

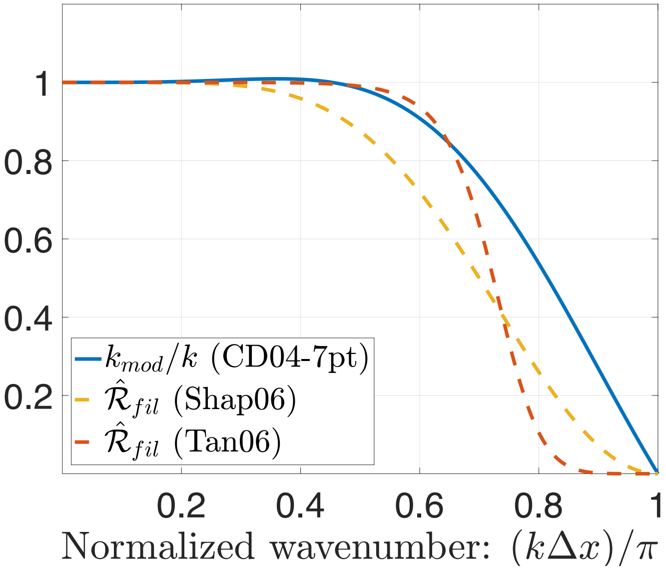
$$\frac{u^{n+1}-u^n}{\Delta t} = -a\delta_x u + |v|\sum_r \epsilon_{2r}(\Delta x)^{2r-1}\delta_x^{2r}u$$

- spectrally-constraining via dissipation
- may impact phase dynamics of base scheme
- temporally consistent
- may impact CFL limits

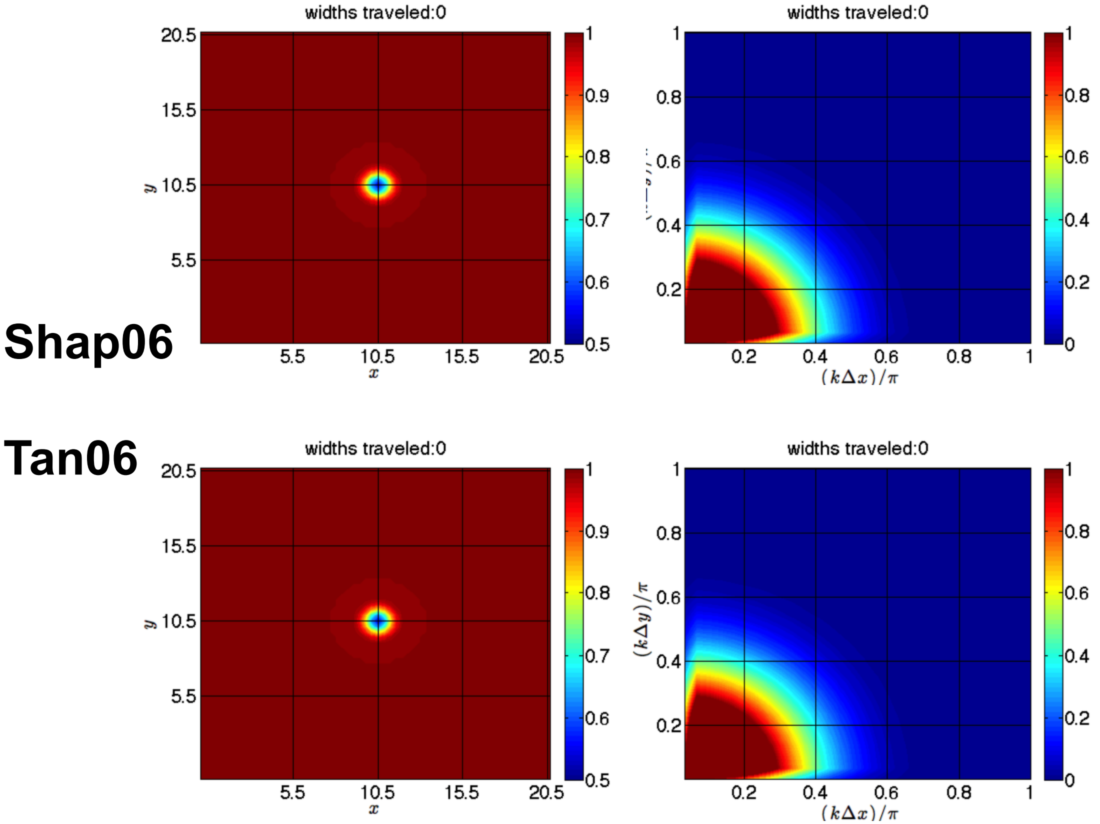
Generalized to implicit stencils:
Edoh *et al.* (JCP, 2018) [5]

Spectrally-tunable Dissipation

- match dissipation to spectral-resolvability of underlying method



Isentropic Vortex transport:

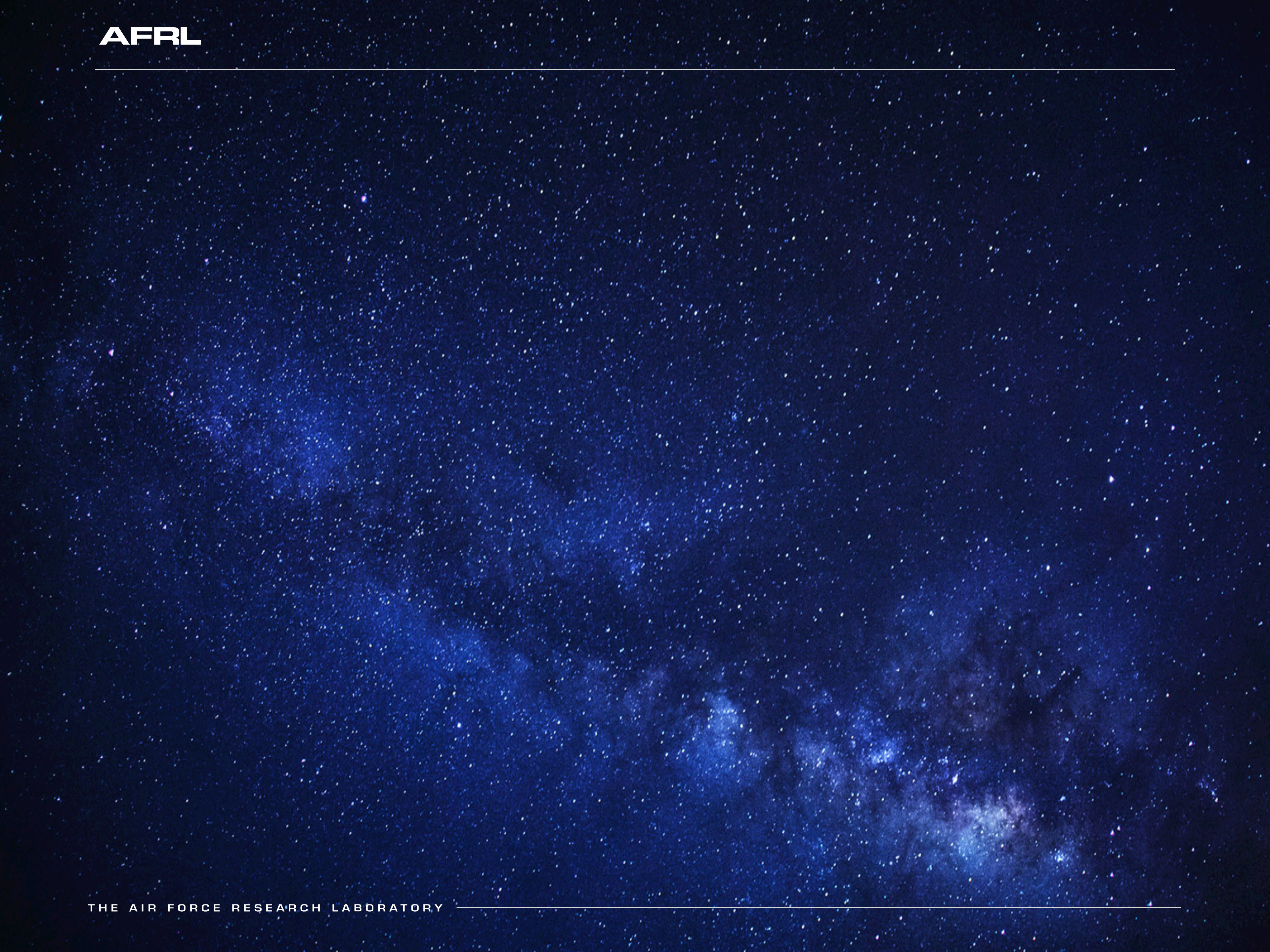


SUMMARY

- Purser filters and generalizations
  - spectral specification, high-order, monotonicity
- extension to operators on bounded domains via cascading procedure
  - energy-stability and conservation
- filter-based artificial dissipation

Future/On-going Work

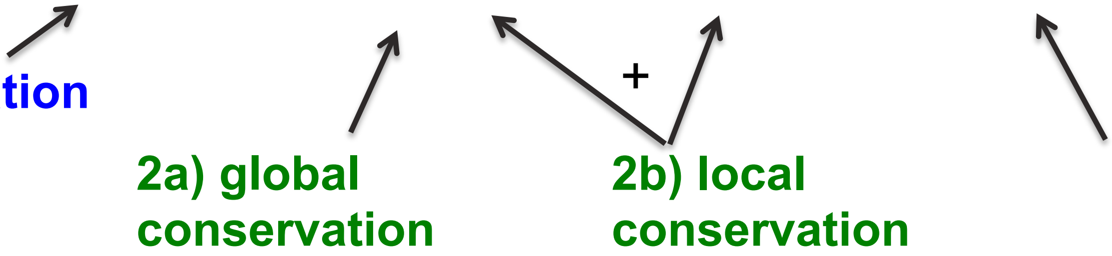
- Formulate scheme adaptivity relative to sharp and monotonic resolution of features
- Compare use of cascade-based filtering and artificial dissipation operators (NAHOM Con, May 2019)
- Construction of Pade Summation-by-Parts operators (AIAA Aviation June 2019; USNCCM, July 2019)
- Apply to explicitly-filtered LES methodology



Required Properties for AD Operator

$$\boxed{\partial_t \mathbf{u} = (\Delta x) \cdot P_x^{-1} R_{AD} \mathbf{u}}$$

$$R_{AD}\mathbf{1} = \mathbf{0}, \quad R_{AD}^T\mathbf{1} = \mathbf{0}, \quad R_{AD} = R_{AD}^T, \quad R_{AD}, R_{AD}^T \leq 0$$



1) normalization

2a) global conservation

2b) local conservation

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1) Inactive for constant modes

$$R_{AD}\mathbf{1} = \mathbf{0}$$

2a) Global conservation

$$\begin{aligned} \mathbf{1}^T P \partial_t \mathbf{u} &= \mathbf{1}^T R_{AD} \mathbf{u} \\ &= \mathbf{u}^T R_{AD}^T \mathbf{1} = \mathbf{0} \end{aligned}$$

2b) Local conservation (telescoping form)

$$\begin{aligned} R_{AD} = R_{AD}^T &= \Lambda_0 + \Delta\left(\Lambda_1[\Delta]^T + [\Delta]^T \Lambda_2 \Delta [\Delta]^T \right. \\ &\quad \left. + [\Delta]^T \Delta \Lambda_3 [\Delta]^T \Delta [\Delta]^T + \ldots\right) \end{aligned}$$

see Fisher *et al.* (JCP 2011)

3) Negative semi-definite

$$\begin{aligned} d_t \|\mathbf{u}\|_P &= \mathbf{u}^T P d_t \mathbf{u} + (d_t \mathbf{u})^T P \mathbf{u} \\ &= \mathbf{u}^T [R_{AD} + R_{AD}^T] \mathbf{u} \leq \mathbf{0} \end{aligned}$$

Interior Schemes: Filter-based Artificial Dissipation

Re-use the dissipative portion of the filter as AD (Edoh et al., JCP 2018)

manipulating spectral responses…

$$\begin{aligned} \hat{\mathcal{G}}_{fil}(k) &= 1+\boxed{\hat{\mathcal{D}}} \\ &= \frac{\hat{\mathcal{G}}_r(k)}{\hat{\mathcal{G}}_\ell(k)} \\ &= \frac{1+\hat{\mathcal{D}}_r}{1+\hat{\mathcal{D}}_\ell} \end{aligned} \quad \longrightarrow \quad \begin{aligned} \hat{\mathcal{G}}_{AD}(k) &= \hat{\mathcal{G}}_{fil} - 1 \\ &= \frac{\hat{\mathcal{D}}_r - \hat{\mathcal{D}}_\ell}{1+\hat{\mathcal{D}}_\ell} \\ &= \frac{(\mathcal{P}_{EF}-1)-(\mathcal{P}_{IF}-1)}{\mathcal{P}_{IF}} \end{aligned}$$

$$\text{with} \quad \begin{cases} \mathcal{P}_{EF}(\mathcal{X}) \leq 1 \text{ for } \mathcal{X} \leq 0 \\ \mathcal{P}_{IF}(\mathcal{X}) \in \langle 0,1] \text{ for } \mathcal{X} \in [-4,0] \end{cases}$$

interpretation as undivided differences…

$$\mathcal{R}_{AD}\{u\} = u''$$

$$\to \quad \left[1+\sum_{\ell\geq 1}\epsilon_{IF,2\ell}\delta^{2\ell}\right]u'' = \left[\sum_{r\geq 1}\epsilon_{EF,2r}\delta^{2r} - \sum_{\ell\geq 1}\epsilon_{IF,2\ell}\delta^{2\ell}\right]u_i$$

also, re-scale for dimensional consistency with gov. eq.

Cascaded Filter-based Artificial Dissipation Operators

$$\boxed{\partial_t \mathbf{u} = (\Delta x) \cdot P_x^{-1} R_{AD} \mathbf{u}}$$

$$R_{AD} = G_\ell^{-1}[\tilde{D}_r - \tilde{D}_\ell] \quad \text{with} \quad G_\ell = I + \sum_{\ell=1}^{L} \epsilon_{IF,2\ell} D_2^\ell, \quad \text{and} \quad \tilde{D}_r = \sum_{r=1}^{R} \epsilon_{EF,2r} \breve{D}_2^r, \;\; \tilde{D}_\ell = \sum_{\ell=1}^{L} \epsilon_{IF,2\ell} \breve{D}_2^\ell$$

modified kernel:

$$\breve{D}_2 = \hat{D}_1^T [\Sigma C_1] \hat{D}_1 \;\text{ with }\; \Sigma = \frac{diag\{|\sigma_i|\}}{\max\{|\sigma_i|\}}$$

operators inspired by interior schemes for filter-based AD [Edoh et al., JCP 2018]

Normalized

$$R_{AD}\mathbf{1} = G_\ell^{-1} \overbrace{[\tilde{D}_r - \tilde{D}_\ell]\mathbf{1}}^{=\mathbf{0}} = \mathbf{0}$$

Globally conservative

$$\mathbf{1}^T R_{AD}\mathbf{u} = \mathbf{u}^T [\tilde{D}_r - \tilde{D}_\ell]^T G_\ell^{-1,T} \mathbf{1} = \mathbf{0}$$

…but operator may not be symmetric for Pade schemes, thus not locally conservative!

Provably stable

$$\begin{aligned} d_t \|\mathbf{u}\|_P &= \mathbf{u}^T P d_t \mathbf{u} + (d_t \mathbf{u})^T P \mathbf{u} \\ &= \mathbf{u}^T [R_{AD} + R_{AD}^T] \mathbf{u} \\ &= \mathbf{u}^T [G_\ell(\tilde{D}_r - \tilde{D}_\ell) + (\tilde{D}_r - \tilde{D}_\ell) G_\ell] \mathbf{u} \;\leq\; 0 \end{aligned}$$