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CALCULATING THE AHP PRIORITY VECTOR

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This memo uses the following notation conventions: Matrix variables are set as uppercase bold (e.g., $\boldsymbol{W}$). Vector variables are set as lowercase bold (e.g., $\boldsymbol{w}$). The estimate of a variable is indicated using a circumflex over the variable (e.g., $\hat{\boldsymbol{w}}$ is an estimate of $\boldsymbol{w}$). Scalar variables are set as lowercase not bold (e.g., w), and scalar constants are set as uppercase not bold (e.g., N).

This discussion draws from the discussions by Lipovetsky [2, 3] and Saaty [4].

In the Analytic Hierarchy Process (AHP), a rater makes pairwise comparisons between alternatives. We collect these comparisons into a matrix. Ideally, the ij-th element in the matrix is the ratio of the priorities of w_i and w_j of the i-th and j-th alternatives. We don't know the values of each w_i, only the ratios. If we have N alternatives, then the matrix looks like

$$\boldsymbol{W} = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ & . \; . \; . \; . \; . \; . \; . & & \\ w_n/w_1 & w_n/w_3 & \dots & w_n/w_n \end{bmatrix} \tag{1}$$

We want to transform this set of pairwise comparisons into a priority ranking of the alternatives, $\boldsymbol{w} \equiv \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}^T$, by recovering the values of all w_i from the ratios in W. The convention in AHP is to scale $\boldsymbol{w}$ to make the sum of the elements equal to 1.

The structure of the matrix $\boldsymbol{W}$ is interesting. You might immediately notice that the values below the diagonal are the reciprocals of the values above the diagonal. The AHP literature labels this structure a *reciprocal matrix*, although some linear algebra literature uses that term as a synonym for the inverse of a matrix. In any case, this structure doesn't provide any direct leverage to help us recover the priority ranking, although the structure does ensure that all $w_{ij} > 0$, which means that $\boldsymbol{W}$ is a *positive matrix*, which may be helpful. However, note that every column is a multiple of the first column (and every row is a multiple of the first row). Either of these conditions mean that the *rank* of $\boldsymbol{W}$ is 1. Again, this may be helpful.

Note that if we multiply $\boldsymbol{W}$ by $\boldsymbol{w}$, we get

$$\boldsymbol{W}\boldsymbol{w} = N\boldsymbol{w} \tag{2}$$

This is the equation that we use to compute the eigenvalues and eigenvectors of a matrix. Equation (2) has a solution if and only if N is an eigenvalue of $\boldsymbol{W}$, and then the priority ranking vector $\boldsymbol{w}$ is an eigenvector of $\boldsymbol{W}$.

A digression: When engineers hear "eigenvector" they often think about a geometric interpretation, i.e. $\boldsymbol{W}$ is a linear transformation in an N-dimensional space, and the eigenvectors are the axes around which that transformation rotates. I don't think that this geometric interpretation works here–I can't find a way to interpret $\boldsymbol{W}$ as a linear transformation. Eigenvectors are also used to solve principal component analysis, so perhaps that would provide a geometric interpretation. In any case, even though we don't have an intuitive interpretation, we will take advantage of the rich linear algebra results related to eigenanalysis. Now back to our original discussion.

Is there a solution to Equation (2)?

1. Above, we noted that the rank of $\boldsymbol{W}$ is 1, which means that there is only one non-zero eigenvalue.
2. $\boldsymbol{W}$ is a positive matrix, so from the Perron-Frobenius theorem, we know that the non-zero eigenvalue is real valued (i.e. not a complex number).
3. We see that $\text{tr}(\boldsymbol{W}) = N$ (The trace of a matrix is the sum of the diagonal elements, which in this case are all equal to 1). From linear algebra, we know that the the sum of the eigenvalues of a matrix is equal to the trace of the matrix.

$$\sum eigenvalues = \text{tr}(\boldsymbol{W}) = N \tag{3}$$

So if we have only one non-zero eigenvalue, and the sum of all eigenvalues is N, and the eigenvalue is real-valued, so then the non-zero eigenvalue must be equal to N.

The solution for the eigenvector $\boldsymbol{w}$ is any column of $\boldsymbol{W}$ – the solutions will differ by a multiplicative constant. The convention in AHP is to scale $\boldsymbol{w}$ to make the sum of the elements equal to 1, and doing this makes the solution unique, no matter which column is chosen.

This is all very nice, except that $\boldsymbol{W}$ represents an ideal set of pairwise comparison weights. In practice, we elicit from the rater a set of judgments and construct the matrix $\boldsymbol{J}$

$$\boldsymbol{J} = \begin{bmatrix} 1 & j_{12} & \dots & j_{1N} \\ j_{21} & 1 & \dots & j_{2N} \\ & . \; . & . \; . \; . & \\ j_{N1} & j_{N2} & \dots & 1 \end{bmatrix} \tag{4}$$

We only elicit half the entries (e.g., above the diagonal) and then set the remaining values to the reciprocal of the elicited values so that $j_{ij} = 1/j_{ji}$.

In general, $\boldsymbol{J}$ is not *consistent*. Despite the rater's best efforts, each $j_{ij} \neq w_i/w_j$. Note that this is not due solely to quantization of the judgment scale (i.e. AHP uses a scale of {-9..-1,1..9} for each judgment j_{ij}), but is due to variations in the human rater's relative preferences among the alternatives.

Although $\boldsymbol{J}$ is positive, it no longer has a matrix rank equal to 1 (i.e. the columns do not differ from each other by a multiplicative constant), and so $\boldsymbol{J}$ may have multiple eigenvalues. In order to recover an estimate of the priority ranking vector $\hat{\boldsymbol{w}}$, we need to solve

$$\boldsymbol{J}\hat{\boldsymbol{w}} = \lambda_{max}\hat{\boldsymbol{w}} \tag{5}$$

The difference between λ_{max} and N is a measure of the inconsistency of $\boldsymbol{J}$. Saaty defines the inconsistency metric as (See [4, bottom of p. 237])

$$\mu = \frac{\lambda_{max} - N}{N - 1} \tag{6}$$

In order to estimate μ we need to compute λ_{max}.

"It is well known" that small perturbations in the matrix coefficients produce small changes in the eigenvalues of the matrix. Defining $j_{ij} = w_{ij} + \varepsilon_{ij}$ where ε_{ij} represents a small error between the elicited judgment and the ideal weight, then if all ε_{ij} are "small", then $\lambda_{max} \rightarrow N$. Saaty states that when λ_{max} is close to N, we can approximate $\hat{\boldsymbol{w}}$ by scaling each column of $\boldsymbol{J}$ so that the sum of the column equals 1, and then averaging across the resulting rows (see [4, p. 239]–there is no discussion, and I have found no derivation of this approximation in the literature).

Ishizaka and Lusti [1] call this method *mean of normalized values*, and provide a derivation for a consistent matrix. Srdjevic labels this approximation *additive normalization (AN)* and reports that "[The] popularity and wide use in practice AN owes to its extreme simplicity. Although considered inferior it significantly outperforms more sophisticated methods" [5]. My interpretation of this approximation is that if $\boldsymbol{J}$ was consistent, then any column would be an eigenvector. So, we treat each column of $\boldsymbol{J}$ as an independent perturbation of the eigenvector $\boldsymbol{w}$ of matrix $\boldsymbol{W}$. We assume that the errors have a mean of 0, and we compute the average across the rows to reduce the error in the approximation.

Having obtained $\hat{\boldsymbol{w}}$, we can estimate λ_{max} using Equation (5): Compute $\boldsymbol{J}\hat{\boldsymbol{w}}$, divide by $\hat{\boldsymbol{w}}$, and then average the coefficients of the resulting vector. We can then us this estimate of λ_{max} to compute μ using Equation (6).

Many AHP spreadsheet calculators uses these formulas to compute the priority ranking vector and the inconsistency measure. Another approach used by some spreadsheets is to use the power method to find an approximation for the dominant eigenvector (i.e. the eigenvector corresponding to λ_{max}) of $\boldsymbol{J}$, using a fixed number of iterations.

References

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