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The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma: 4. One-Component Charged Gas in an External Electric Field

by Michael Grinfeld and Pavel Grinfeld

NOTICES

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1. Introduction

In Part 1 of this series of reports,[1–3] we formulated a novel approach to thermodynamics of heterogeneous systems completely or partially filled with a liquid substance in a plasma state. The approach is based on the use of the Gibbs variational principles, and it enables efforts to address a variety of problems relating to equilibrium and stability of such systems.

The general motivation for this series of reports is discussed in Grinfeld and Grinfeld,[1] in which we also demonstrated how the Gibbs approach can be applied to heterogeneous systems with charged gases. We also established the general equations of equilibrium and the general conditions of stability. The adjective "general" means that the results of this part are applicable to any charged gases without special assumptions about the equations of state (EOS) of the gases.

At the same time, the general approach does not permit further mathematically exact equilibrium configurations or their ultimate numerical analysis. The next step was made in Grinfeld and Grinfeld,[2] in which we specified the EOS following the footsteps of Grinfeld.[4] For that model EOS, we found the exact solution of the classical 1-D problem for the unbounded, uniformly charged plate inserted in the one-component charged gas.

In this report, we establish the exact solution for the same EOS but different geometry. That geometry is presented in Fig. 1.

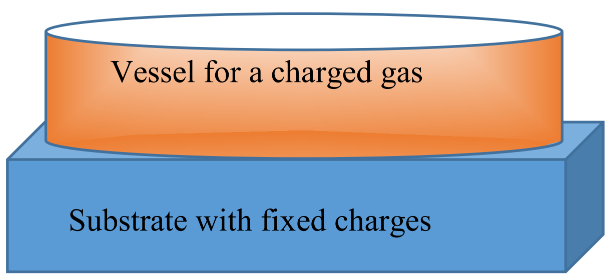

Fig. 1 Model of a heterogeneous charged system

Figure 1 shows two charged bodies. The lower (blue) body with the shape of parallelepiped contains uniformly distributed immobile electric charges. The upper (orange) body contains a charged gas; the charges inside the orange vessel are mobile; by the assumption, they will try to attain the configuration corresponding to the minimum of the total energy of the system.

The set of exact solutions for the equilibrium configurations of plasma are very limited even for the simplest EOS. The equilibrium distribution of the charged particle within the bounded 3-D container can be determined of course only with the help of numerical computation. Fortunately, when charges exist in the slabs between two parallel plates, for some EOS the exact equilibrium configurations can be calculated analytically. Such solutions are extremely valuable since they permit the deepest qualitative analysis and play an extremely important role for the validation of the developed numerical method. In the following, we establish such an exact solution for the EOS suggested in Grinfeld.[4]

2. Basic Methodology

We assume upfront that the temperature is fixed, uniformly distributed everywhere, so in the following we omit the temperature from all our functions.

The total energy E_{total} is postulated to consist of two ingredients: the electrostatic energy E_{elec} and the internal energy of the charged gas E_{int}. We chose it in the following form:

$$E_{total} = E_{elec} + E_{int} = \int_{Space} d\Omega \frac{1}{2} \nabla_i \varphi \nabla^i \varphi + \int_{Vess} d\Omega \rho e(\rho), \tag{1}$$

where φ is the electric potential, ρ is the mass density of the charged gas, and $e(\rho)$ is the internal energy density per unit mass.

The first integral in Eq. 1 is extended over the whole space, and the second only over the container with charged gas.

The total mass M_0 in the vessel is specified

$$\int_{Vess} d\Omega \rho = M_0 . \tag{2}$$

When using the variational method, we should deal with the unconstraint minimization of the functional

$$\Phi = \int_{Space} d\Omega \frac{1}{8\pi} \nabla_i \varphi \nabla^i \varphi + \int_{Vess} d\Omega \left(\rho e(\rho) - \lambda \rho \right), \tag{3}$$

where λ is the indefinite Lagrange multiplier.

Calculating the first variation of Φ, we get

$$\delta\Phi = \int_{Vess} d\Omega \left((\rho e)_\rho - \lambda + q\varphi \right) \delta\rho \tag{4}$$

as implied by the following chain:

$$\delta\Phi = \int_{Space} d\Omega \frac{1}{4\pi} \nabla^i \varphi \nabla_i \delta\varphi + \int_{Vess} d\Omega \left((\rho e)_\rho - \lambda \right) \delta\rho =$$

$$-\int_{Vess} d\Omega \frac{1}{4\pi} \varphi \nabla_i \nabla^i \delta\varphi + \int_{Vess} d\Omega \left((\rho e)_\rho - \lambda \right) \delta\rho =$$

$$\int_{Vess} d\Omega \left((\rho e)_\rho - \lambda + q\varphi \right) \delta\rho ,$$

where $\delta\rho$ is the variation of the density.

In view of Eq. 4, we arrive at the relationship

$$(\rho e)_\rho + q_- \varphi = \lambda , \tag{5}$$

where q_- is the specific charge of the gas per unit mass.

We call Eq. 5 the chemical equilibrium equation. It can be traced back at least to Gibbs.[5]

The chemical Eq. 5 should be combined with the equation of electrostatics:

$$\nabla^i \nabla_i \varphi = -4\pi\rho q_- . \tag{6}$$

Equation 5 implies

$$\varphi = \frac{\lambda}{q_-} - \frac{1}{q_-} (\rho e)_\rho . \tag{7}$$

Inserting φ from the Eq. 7 into the Eq. 5, we get

$$\nabla^i \left((\rho e)_{\rho\rho} \nabla_i \rho \right) = 4\pi\rho q_-^2 . \tag{8}$$

3. Exact Solution without the Extinction Points for the Special EOS

Consider our favorite model, such that

$$(\rho e)_{\rho\rho} \equiv a^2 , \tag{9}$$

where a^2 is a positive constant.

For this model, Eq. 8 can be reduced to the linear partial differential equation

$$\nabla^i \nabla_i \rho = \frac{4\pi q_-^{\ 2}}{a^2} \rho \tag{10}$$

within the vessel.

In the 1-D case, we arrive at the following ordinary differential equation

$$\frac{d^2 \rho}{dz^2} = \Delta^2 \rho, \tag{11}$$

where the positive constant Δ is defined as

$$\Delta \equiv \sqrt{\frac{4\pi q_-^{\ 2}}{a^2}}. \tag{12}$$

The ordinary differential Eq. 11 allows the following general solution inside the vessel,

$$\rho(z) = C_s \sinh(\Delta z) + C_c \cosh(\Delta z), \tag{13}$$

where C_s and C_c are the constants to be determined from other data.

Let M^* be the mass of the vertical column with the unit cross section. With the help of the general Eq. 13 the mass balance Eq. 2 implies

$$M^* = 2C_c \Delta^{-1} \sinh(\Delta H). \tag{14}$$

Similarly, let m^* be the mass of the vertical column with the unit cross section within the vessel with the immobile charges, and let q_+ be the charge per unit mass of the immobile charges. Then the total charge Q of the vertical column with the unit cross section within the vessel with the immobile charges is equal to

$$Q = q_+ m^*. \tag{15}$$

Applying the Gauss theorem to column comprising both charged layers we get

$$E_i(H_+) N^i_{\ +} + E_i(H_-) N^i_{\ i-} = 4\pi(qM^* + Q), \tag{16}$$

where $E_i(z)$ is the electrostatic field and N^i is the outwardly directed normal to the boundary.

Taking into account the equality of the absolute values of the electric field at infinity, we can rewrite Eq. 16 as follows:

$$-\frac{d\varphi(H)}{dz} = 2\pi\left(q_- M^* + Q\right). \tag{17}$$

Equation 7 implies

$$\frac{d\varphi}{dz} = -\frac{a^2}{q_-}\frac{d\rho}{dz}. \tag{18}$$

With the help of Eqs. 13 and 18, we get

$$\frac{d\varphi}{dz} = -\frac{a^2\Delta}{q_-}\left(C_s\cosh(\Delta z) + C_c\sinh(\Delta z)\right). \tag{19}$$

Inserting Eq. 19 into Eq. 17, we get

$$\frac{a^2\Delta}{q_-}\left(C_s\cosh(\Delta H) + C_c\sinh(\Delta H)\right) = 2\pi\left(q_- M^* + Q\right) \tag{20}$$

or

$$C_s\cosh(\Delta H) + C_c\sinh(\Delta H) = \frac{2\pi q_-}{a^2\Delta}\left(q_- M^* + Q\right). \tag{21}$$

The relationship Eq. 14 implies

$$C_c = \frac{\Delta M^*}{2\sinh(\Delta H)}. \tag{22}$$

Inserting the result, Eq. 22, into Eq. 21, we get

$$C_s = \frac{1}{\cosh(\Delta H)}\frac{2\pi q_- Q}{a^2\Delta}, \tag{23}$$

as implied by the following chain:

$$C_s = \frac{1}{\cosh(\Delta H)}\left(\frac{2\pi q\left(M^* q_- + Q\right)}{a^2\Delta} - \frac{\Delta M^*}{2}\right) = \frac{1}{\cosh(\Delta H)}\left(\frac{4\pi q_-{}^2 - a^2\Delta^2}{2a^2\Delta} M^* + \frac{2\pi q Q}{a^2\Delta}\right) = \frac{1}{\cosh(\Delta H)}\frac{2\pi q_- Q}{a^2\Delta} \quad (24)$$

Inserting the results Eqs. 22 and 23 into Eq. 13, we arrive at the following ultimate relationship for the equilibrium density distribution:

$$\rho(z) = \frac{2\pi q_- Q}{a^2\Delta}\frac{\sinh(\Delta z)}{\cosh(\Delta H)} + \frac{\Delta M^*}{2}\frac{\cosh(\Delta z)}{\sinh(\Delta H)}. \quad (25)$$

Presenting Q in the form Eq. 15 we can rewrite Eq. 25 as

$$\rho(z) = \frac{2\pi q_- q_+ m^*}{a^2\Delta}\frac{\sinh(\Delta z)}{\cosh(\Delta H)} + \frac{\Delta M^*}{2}\frac{\cosh(\Delta z)}{\sinh(\Delta H)}. \quad (26)$$

When $q_+ q_- < 0$, the two terms in Eq. 26 have opposite signs, and the solution, Eq. 27, can lead to the conclusion that the density $\rho(z)$ can assume negative values. Then this solution becomes unacceptable from the standpoint of physics, and should be replaced with another one.

The value of M^*, at which the density $\rho(z)$ vanishes at $z = H$ is called critical, and marked as M_{crit}. Equation 26 leads to the following value of M_{crit}:

$$\frac{M_{crit}}{m^*} = -\frac{4\pi q_- q_+}{a^2\Delta^2}\tanh^2(\Delta H) \quad (27)$$

or

$$\frac{M_{crit}}{m^*} = -\frac{q_+}{q_-}\tanh^2(\Delta H) \quad (28)$$

or else

$$\left|\frac{Q_{crit}}{Q_+}\right| = \tanh^2(\Delta H). \quad (29)$$

4. A Solution with Extinction Point at the Top of the Vessel

Let the extinction of the density happen at $z = H_{ext} < H$. Then with the help of Eq. 23 we get the following mass balance equation:

$$C_s\left(\cosh\left(\Delta H_{ext}\right) - \cosh\left(\Delta H\right)\right) + C_c\left(\sinh\left(\Delta H_{ext}\right) + \sinh\left(\Delta H\right)\right) = \Delta M^*. \tag{30}$$

Also, the extinction condition is

$$C_s \sinh\left(\Delta H_{ext}\right) + C_c \cosh\left(\Delta H_{ext}\right) = 0\,. \tag{31}$$

Equation 19 implies

$$\frac{d\varphi\left(H_{ext}\right)}{dz} = -\frac{a^2\Delta}{q_-}\left(C_s \cosh\left(\Delta H_{ext}\right) + C_c \sinh\left(\Delta H_{ext}\right)\right) \tag{32}$$

and

$$\frac{d\varphi\left(-H\right)}{dz} = -\frac{a^2\Delta}{q_-}\left(C_s \cosh\left(\Delta H\right) - C_c \sinh\left(\Delta H\right)\right). \tag{33}$$

In our case, Eq. 16 should be replaced with

$$E_i\left(+\infty\right) N^i{}_+ + E_i\left(-\infty\right) N^i{}_- = 4\pi\left(q_- M^* + q_+ m^*\right) \tag{34}$$

or

$$E_i\left(+\infty\right) N^i{}_+ = 2\pi\left(q_- M^* + q_+ m^*\right) \tag{35}$$

or else

$$E_i\left(H\right) N^i{}_+ = 2\pi\left(q_- M^* + q_+ m^*\right). \tag{36}$$

With the help of Eq. 32, we can rewrite Eq. 36 as

$$\frac{a^2\Delta}{q_-}\left(C_s \cosh\left(\Delta H_{ext}\right) + C_c \sinh\left(\Delta H_{ext}\right)\right) = 2\pi\left(q_- M^* + q_+ m^*\right), \tag{37}$$

which, in turn, can be rewritten as

$$C_s \cosh\left(\Delta H_{ext}\right) + C_c \sinh\left(\Delta H_{ext}\right) = \frac{2\pi q_-}{a^2 \Delta}\left(q_- M^* + q_+ m^*\right) \tag{38}$$

or else

$$C_s \cosh\left(\Delta H_{ext}\right) + C_c \sinh\left(\Delta H_{ext}\right) = \frac{\Delta}{2q_-}\left(q_- M^* + q_+ m^*\right). \tag{39}$$

Thus, for three unknowns, C_s, C_c, and H_{crit} we arrive at the algebraic Eqs. 30, 31, and 39.

Equation 31 implies

$$C_c = -C_s \tanh\left(\Delta H_{ext}\right). \tag{40}$$

Using Eq. 37, we can rewrite Eq. 30 as

$$C_s\left(\cosh\left(\Delta H_{ext}\right) - \cosh\left(\Delta H\right)\right) - C_s \tanh\left(\Delta H_{ext}\right)\left(\sinh\left(\Delta H_{ext}\right) + \sinh\left(\Delta H\right)\right) = \Delta M^* \tag{41}$$

or

$$C_s = \Delta M^* \frac{\cosh\left(\Delta H_{ext}\right)}{1 - \cosh\left(\Delta H + \Delta H_{ext}\right)}, \tag{42}$$

as implied by the following chain:

$$C_s = \Delta M^* \frac{1}{\left(\begin{array}{l}\cosh\left(\Delta H_{ext}\right) - \tanh\left(\Delta H_{ext}\right)\sinh\left(\Delta H_{ext}\right) - \\ \cosh\left(\Delta H\right) - \tanh\left(\Delta H_{ext}\right)\sinh\left(\Delta H\right)\end{array}\right)} =$$

$$\Delta M^* \frac{1}{\dfrac{1}{\cosh\left(\Delta H_{ext}\right)} - \dfrac{\cosh\left(\Delta H\right)\cosh\left(\Delta H_{ext}\right) + \sinh\left(\Delta H\right)\sinh\left(\Delta H_{ext}\right)}{\cosh\left(\Delta H_{ext}\right)}} =$$

$$\Delta M^* \frac{1}{\dfrac{1}{\cosh\left(\Delta H_{ext}\right)} - \dfrac{\cosh\left(\Delta H + \Delta H_{ext}\right)}{\cosh\left(\Delta H_{ext}\right)}} = \Delta M^* \frac{\cosh\left(\Delta H_{ext}\right)}{1 - \cosh\left(\Delta H + \Delta H_{ext}\right)}$$

Using Eq. 41, we can rewrite Eq. 40 as

$$C_c = -\Delta M^* \frac{\sinh\left(\Delta H_{ext}\right)}{1 - \cosh\left(\Delta H + \Delta H_{ext}\right)}. \tag{43}$$

Using Eqs. 41 and, 42, we can rewrite Eq. 39 as

$$\Delta M^{*}\frac{\cosh\left(\Delta H_{ext}\right)}{1-\cosh\left(\Delta H+\Delta H_{ext}\right)}\cosh\left(\Delta H_{ext}\right)- \\ \Delta M^{*}\frac{\sinh\left(\Delta H_{ext}\right)}{1-\cosh\left(\Delta H+\Delta H_{ext}\right)}\sinh\left(\Delta H_{ext}\right)=\frac{\Delta}{2q_{-}}\left(q_{-}M^{*}+q_{+}m^{*}\right) \tag{44}$$

or

$$\cosh\left(\Delta H+\Delta H_{ext}\right)=\frac{q_{+}m^{*}-q_{-}M^{*}}{q_{+}m^{*}+q_{-}M^{*}}, \tag{45}$$

and then

$$\cosh\left(\Delta H+\Delta H_{ext}\right)=\frac{1+R_Q}{1-R_Q}. \tag{46}$$

In Eq. 46 we use the following notation

$$R_Q\equiv-\frac{q_{-}M^{*}}{q_{+}m^{*}}. \tag{46}$$

Equation 46 has a real solution if

$$0<R_Q<1. \tag{47}$$

Indeed, we get the following sequence of inequalities

$$\cosh\left(\Delta H+\Delta H_{ext}\right)=\aleph, \tag{48}$$

where

$$\aleph\equiv\frac{1+R_Q}{1-R_Q}=\frac{q_{+}m^{*}-q_{-}M^{*}}{q_{+}m^{*}+q_{-}M^{*}}. \tag{49}$$

Equation 48 has the following solution:

$$\frac{H_{ext}}{H}=\frac{1}{\Delta H}\ln\left(\aleph+\sqrt{\aleph^{2}-1}\right)-1. \tag{50}$$

Needless to say that the solution (Eq. 50) is physically meaningful if

$$-1 \le \frac{1}{\Delta H} \ln\left(\aleph + \sqrt{\aleph^2 - 1}\right) - 1 \le 1 \tag{51}$$

or

$$0 \le \frac{1}{\Delta H} \ln\left(\aleph + \sqrt{\aleph^2 - 1}\right) \le 2\,. \tag{52}$$

5. Extinction Domain at the Bottom

Let the extinction of the density happens at $z = H_{ext} < -H$. Then, with the help of Eq. 13, we get the following mass balance equation:

$$C_s\left(\cosh\left(\Delta H\right) - \cosh\left(\Delta H_{ext}\right)\right) + C_c\left(\sinh\left(\Delta H\right) - \sinh\left(\Delta H_{ext}\right)\right) = \Delta M^*\,. \tag{53}$$

The extinction, Eq. 31, remains the same. Hence, Eq. 39 remains valid as well. Using Eq. 39, we can rewrite Eq. 53 as

$$C_s = \Delta M \frac{\cosh\left(\Delta H_{ext}\right)}{\cosh\left(\Delta H - \Delta H_{ext}\right) - 1}, \tag{54}$$

as implied by the following chain:

$$\begin{aligned}
& C_s\left(\cosh\left(\Delta H\right) - \cosh\left(\Delta H_{ext}\right)\right) - \\
& C_s \tanh\left(\Delta H_{ext}\right)\left(\sinh\left(\Delta H\right) - \sinh\left(\Delta H_{ext}\right)\right) = \Delta M^* \to \\
& C_s\left(\begin{array}{c} \cosh\left(\Delta H\right) - \cosh\left(\Delta H_{ext}\right) - \\ \dfrac{\sinh\left(\Delta H_{ext}\right)\sinh\left(\Delta H\right) - \sinh\left(\Delta H_{ext}\right)\sinh\left(\Delta H_{ext}\right)}{\cosh\left(\Delta H_{ext}\right)} \end{array}\right) = \Delta M^* \to \\
& C_s \frac{\left(\begin{array}{c} \cosh\left(\Delta H\right)\cosh\left(\Delta H_{ext}\right) - \cosh\left(\Delta H_{ext}\right)\cosh\left(\Delta H_{ext}\right) - \\ \sinh\left(\Delta H_{ext}\right)\sinh\left(\Delta H\right) + \sinh\left(\Delta H_{ext}\right)\sinh\left(\Delta H_{ext}\right) \end{array}\right)}{\cosh\left(\Delta H_{ext}\right)} = \Delta M^* \to \\
& C_s\left(\frac{\cosh\left(\Delta H - \Delta H_{ext}\right) - 1}{\cosh\left(\Delta H_{ext}\right)}\right) = \Delta M^* \to C_s = \Delta M \frac{\cosh\left(\Delta H_{ext}\right)}{\cosh\left(\Delta H - \Delta H_{ext}\right) - 1}
\end{aligned}$$

Using Eq. 54, we can rewrite Eq. 40 as

$$C_c = -\Delta M \frac{\sinh\left(\Delta H_{ext}\right)}{\cosh\left(\Delta H - \Delta H_{ext}\right) - 1}\,. \tag{55}$$

Combining Eqs. 54 and 55 with Eq. 13, we get

$$\rho(z) = \Delta M \frac{\cosh(\Delta H_{ext})\sinh(\Delta z) - \sinh(\Delta H_{ext})\cosh(\Delta z)}{\cosh(\Delta H - \Delta H_{ext}) - 1} = \Delta M \frac{\sinh(\Delta z - \Delta H_{ext})}{\cosh(\Delta H - \Delta H_{ext}) - 1}. \tag{56}$$

Inserting the result of the Eq. 56 into Eq. 16, we arrive at the following formula for the field:

$$E(z) = \frac{a^2}{q_-}\frac{d\rho}{dz} = \frac{a^2\Delta^2 M}{q_-}\frac{\cosh(\Delta z - \Delta H_{ext})}{\cosh(\Delta H - \Delta H_{ext}) - 1}. \tag{57}$$

In particular, the relationship of Eq. 57 implies

$$E(H) = \frac{a^2\Delta^2 M}{q_-}\frac{\cosh\Delta(H - H_{ext})}{\cosh\Delta(H - H_{ext}) - 1}. \tag{58}$$

The Gauss Eq. 16 remains unchanged, and in combination with Eq. 58 it implies

$$\frac{\cosh\Delta(H - H_{ext})}{\cosh\Delta(H - H_{ext}) - 1} = \frac{2\pi q_-}{a^2\Delta^2 M}(q_- M_- + q_+ m_+) \tag{59}$$

or

$$\frac{\cosh\Delta(H - H_{ext})}{\cosh\Delta(H - H_{ext}) - 1} = \frac{q_- M_- + q_+ m_+}{2q_- M_-} \tag{60}$$

or else

$$\cosh\Delta(H - H_{ext}) = \frac{q_+ m_+ + q_- M_-}{q_+ m_+ - q_- M_-} = \aleph^{-1}. \tag{61}$$

Equation 61 permits a real solution if

$$\frac{q_+ m_+ + q_- M_-}{q_+ m_+ - q_- M_-} = \frac{1 + \frac{q_- M_-}{q_+ m_+}}{1 - \frac{q_- M_-}{q_+ m_+}} = \aleph^{-1} \geq 1. \tag{62}$$

We can rewrite Eq. 62 as

$$\frac{\dfrac{q_- M_-}{q_+ m_+}}{1-\dfrac{q_- M_-}{q_+ m_+}} \geq 0 . \tag{63}$$

The inequality of Eq. 63 has the following solution:

$$0<\frac{q_- M_-}{q_+ m_+}<1 , \tag{64}$$

as expected.

Then, Eq. 60 implies

$$-1 \leq 1-\frac{1}{\Delta H} \ln \left(\aleph^{-1}+\sqrt{\aleph^{-2}-1}\right) \leq 1 . \tag{65}$$

Physically admissible solutions must satisfy the inequalities:

$$-1 \leq \frac{H_{ext}}{H} \leq 1 . \tag{66}$$

Inserting the Eq. 65 into Eq. 66, we get

$$-1 \leq 1-\frac{1}{\Delta H} \ln \left(\aleph^{-1}+\sqrt{\aleph^{-2}-1}\right) \leq 1 \tag{67}$$

or

$$0 \leq \frac{1}{\Delta H} \ln \left(\aleph^{-1}+\sqrt{\aleph^{-2}-1}\right) \leq 2 . \tag{68}$$

6. Conclusion

This is the fourth part of the study published under the common umbrella *The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma*. In Parts 1–3, we formulated a novel approach to thermodynamics of one- and two-component heterogeneous systems completely or partially filled with a liquid substance in plasma state. The approach is based on the use of the Gibbs variational principles, and it enables efforts to address a variety of problems relating to equilibrium and stability of such systems.

Our analysis was centered on the exact analytical solution of the boundary value problems derived in Grinfeld and Grinfeld.[1–3] To make the exact solving possible, we postulated the EOS formulated earlier in Grinfeld,[4] and limited ourselves to the simplest geometry permitting the reduction of the original 3-D boundary value problem for the system of equations in partial derivatives to the 1-D problem for the system of ordinary differential equations. More specifically, we analyzed a charged gas between two parallel unbounded plates, impenetrable for the charges. The charged gas is exposed to the uniform electrostatic field, generated by the immobile charges uniformly distributed in the infinite plate with parallel edges.

In the absence of the external field, the mobile charges concentrate near the boundary walls. The charged gas occupies the whole available space between the walls. The mass, however, changes in the space where nowhere equals neither zero nor infinity (contrary to the classical distribution of charges in a conductor). In the classical electrostatic theory of conductors, the 3-D mass density of the electrons is equal to infinity (or, as is usually said, the electrons have finite 2-D density and concentrate on the boundary walls as 2-D aggregates). Nonetheless, in our model the mobile charges still try to concentrate near the walls of the vessel and create the 3-D boundary layers there. The thickness of the boundary layer is determined by the parameter Δ (Eq. 13). The parameter Δ reflects the counteraction and balance of the electrostatic forces and the traditional thermodynamic pressure in the gas.

The external electrostatic field drastically changes the equilibrium density distribution of the charged gas. We demonstrated that now lacunas free of any electric charges can appear inside the vessel. These lacunas are described by the relationships of Eqs. 50 and 65.

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List of Symbols, Abbreviations, and Acronyms

1-D	1-dimensional
2-D	2-dimensional
3-D	3-dimensional
EOS	equation of state

1 (PDF)	DEFENSE TECHNICAL INFORMATION CTR DTIC OCA
1 (PDF)	CCDC ARL FCDD RLD CL TECH LIB
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