



Shaping Modal Dynamics for Drag Reduction: A Study of Transition Suppression in Shear Flows

Maziar Hemati
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Shaping Modal Dynamics for Drag Reduction: A Study of Transition Suppression in Shear Flows

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PRINCIPAL INVESTIGATOR:
Prof. Maziar S. Hemati
Aerospace Engineering and Mechanics
University of Minnesota

Email: mhemati@umn.edu
Phone: 612-625-6857
Fax: 612-626-1558

Submitted to: Dr. Gregg Abate
AFOSR/RTA1
Unsteady Aerodynamics and Turbulent Flows
Air Force Office of Scientific Research
875 N. Randolph St, Suite 325, Room 3112
Arlington, VA 22203

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1 | Executive Summary

This report documents outcomes of research supported under AFOSR Grant No. FA9550-17-1-0252: “Shaping Modal Dynamics for Drag Reduction: A Study of Transition Suppression in Shear Flows” (June 1, 2017–May 31, 2019) monitored by Dr. Douglas R. Smith and Dr. Gregg Abate. The long-term objective of this work is to use feedback control to suppress sub-critical transition to turbulence, and thus reduce drag in systems of interest to the Air Force and DoD. The focus of this effort was to develop and use dynamic mode shaping control strategies to reduce transient energy growth (TEG) of flow perturbations, thus facilitating transition suppression. Specifically, three objectives related to controller synthesis for dynamic mode shaping were investigated:

Objective #1: Eliminate TEG via dynamic mode orthogonalization;

Objective #2: Reduce TEG via dynamic mode matching;

Objective #3: Investigate data-driven and reduced-order dynamic mode shaping algorithms.

Technical outcomes for each of these objectives include theory and algorithms for realizing dynamic mode shaping control, as described in Chapter 3.

These investigations on dynamic mode shaping have also resulted in related developments on TEG control. Contributions include control-oriented models that enable the design of feedback controllers that minimize the maximum TEG (Chapter 4); discovery of performance limitations of observer-based feedback control strategies for TEG suppression (Chapter 5); and alternative static output feedback strategies that overcome these limitations for sensor-based feedback control (Chapter 6). Further research activities based on these results are being conducted under AFOSR Grant No. FA9550-19-1-0034. In addition, the research effort on dynamic mode shaping control have also contributed to data-driven methods for optimal actuator selection, described within the context of airfoil separation control in Chapter 7.

Publications supported in whole or in part by AFOSR Grant No. FA9550-17-1-0252 include 4 journal publications, 11 conference papers, 12 conference presentations, and 1 masters thesis. Personnel supported in whole or in part by this grant include the PI, 1 postdoctoral researcher, 3 PhD students, and 1 masters student.

2 | Introduction

An ability to suppress transition to turbulence would enable dramatic reductions in skin-friction drag in a variety of engineering systems, inevitably leading to efficiency and performance enhancements across a broad range of Air Force and DoD systems. In the context of wall-bounded flows, turbulent transition is often initiated by a *linear* transient growth mechanism [1–13]. The linearity of this transition mechanism has motivated numerous investigations on *linear* optimal and robust control strategies aimed at suppressing and understanding transition [14–24]. Optimal and robust controller synthesis strategies simplify the controller synthesis task by masking internal system design details from the user. Instead, the control engineer can simply focus on a set of “design knobs” that weigh intuitive measures of performance (e.g., balancing input energy with state-regulation error). In the context of fluid flow control, this simplification is greatly welcome, owing to the high-dimensionality of the state-space. We do note, however, that this simplification comes at a cost. By masking the internal system details from the user, optimal and robust controller synthesis frameworks do not easily lend themselves to leveraging (nor elucidating) physical insights. To this end, even with the simplifications that come with optimal and robust controller synthesis techniques, fluid flow controller design can still be quite daunting. Appropriately tuning the controller to realize an adequate control strategy remains something of an art.

Here, we propose an alternative perspective for flow control design—one that focuses on shaping the spectral properties of the linearized Navier-Stokes operator directly. It is the non-normality of the linearized Navier-Stokes operator that is responsible for transient energy growth (TEG) [25]. While non-normality can be defined in various equivalent ways [26–28], a convenient definition is based on whether or not the modes of the operator are normal (orthogonal) to one another—i.e., if the modes are not orthogonal, then the operator is deemed to be non-normal. The possibility of TEG due to non-normality of eigenmodes can be understood from a graphical comparison of the normal and non-normal systems shown in Figures 2.1(A) and 2.1(B), respectively. While both linear systems are stable and begin with the same initial state $\vec{x}(t_1)$, the non-normality of modal directions in Figure 2.1(B) gives rise to a future state $\vec{x}(t_2)$ that has a larger magnitude than it had at the initial time t_1 .

Of course, in the context of linear systems, stability implies that the state will ultimately decay to the origin; however, in the context of *linearized* fluid flows, TEG allows disturbances to be amplified in the short-term, which can trigger nonlinear instabilities and transition—i.e., once the state moves beyond the basin of attraction, turbulent dynamics ensue [29]. It is also worth noting that the resulting TEG becomes more pronounced as the non-normality between modes increases, less pronounced as non-normality decreases, and disappears completely in the case of normal modes. For instance, linearization of the Navier-Stokes equations for three-dimensional channel

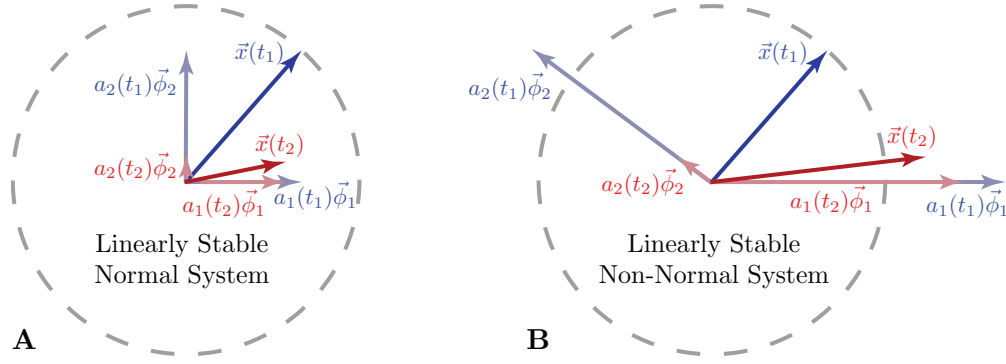


Figure 2.1: Non-normality between modes can lead to transient energy growth (TEG), even when individual modal components $a_i(t)\vec{\phi}_i$ are decaying. The normal system in (A) and the non-normal system in (B) both begin with the same initial condition $\vec{x}(t_1)$ and have the same stable eigenvalues; however, the state in (A) decays for all time, whereas the state in (B) initially grows before it decays to the origin. The dashed line in the figure corresponds to the magnitude of the state at the initial time t_1 .

flow yields the Orr-Sommerfeld-Squire (OSS) operator. As Reynolds number (Re) increases, the modes of OSS become closer to parallel and TEG becomes more pronounced. As a result, at higher Re , it becomes more likely that TEG due to linear mechanisms will trigger the nonlinear processes that govern turbulent dynamics.

From this standpoint, it becomes apparent that a feedback control strategy aimed at mitigating transition to turbulence could be aimed directly at inhibiting transient growth due to non-normality—an objective that can be achieved by *shaping* the modes of the closed-loop system to be orthogonal to one another. One can understand this notion in the context of the simple system of Figure 2.1: can feedback control be leveraged to shape the non-normal system (B) into the normal system (A)? Indeed, addressing this general question will be the focus of the present study. In particular, we will view the control problem from the standpoint of a novel perspective for fluid flow control, which we call *dynamic mode shaping*. While some of the theoretical work concerning dynamic mode shaping has been considered previously within the controls community—under the name of *eigenstructure assignment* (ESA) [30, 31]—the present work addresses a number of issues that currently stand in the way of leveraging the dynamic mode shaping perspective for *fluid flow control*. In addition to our contributions toward a synthesis of dynamic mode shaping controllers, we also address challenges of high-dimensionality, nonlinearity, and uncertainty that will be essential for relevance to practical fluid flow configurations.

As noted in [30], the solution to the pole placement problem of modern control theory is not unique, which offers additional flexibility to the control designer to specify the shape of the closed-loop modes as well. Although ESA is a familiar technique of modern control theory, the challenges associated with specifying desirable spectral characteristics for a closed-loop system (i.e., closed-loop eigenvalues and eigenvectors) has led control theorists to favor optimal and robust control methodologies, even with the demands of subsequent controller tuning. We note here, however, that the challenge of spectral specification is less daunting in the context of transient growth suppression—at the very least we know that we should target an orthogonal set of closed-loop modes. For a stable linear system, a set of orthogonal modes is a *sufficient* condition to guarantee monotonic decay of trajectories. Further, although normality is not a *necessary* condition

for monotonic decay, examination of the pseudospectrum for a normal operator reveals desirable robustness properties that make closed-loop orthogonalization an attractive choice for control in the context of uncertain systems. Perturbations to a normal operator yield well-behaved perturbations in eigenvalues; in contrast, non-normal operators are prone to undesirable sensitivities to such perturbations [26]. Hence, although orthogonality of eigenmodes is not a *necessary* condition for monotonic decay, normal operators have desirable robustness properties and provide a desirable target for closed-loop dynamics.

Here, we formulate dynamic mode shaping controllers and investigate their utility for transient growth suppression. Objective #1 of the research investigates control strategies aimed at eliminating transient growth by shaping the closed-loop modes into an orthogonal set. As part of this study, conditions for feasibility of the control objective have been established. Of course, an orthogonal set of closed-loop modes will not be admissible in all flow control configurations. Objective #2 of the proposed work formulates an alternative dynamic mode matching strategy aimed at shaping the spectral properties of a high-Reynolds number flow to match those of a lower Reynolds number flow. This strategy makes use of the fact that non-normality—and therefore transient growth—decreases with decreasing Reynolds number. Both Objectives #1 and #2 focus on developing control strategies and algorithms for suppressing transition. A third objective of the effort addresses fundamental barriers that need to be overcome for dynamic mode shaping to be a viable control strategy in more complex flow configurations in the future. Objective #3 investigates data-driven and reduced-order dynamic mode shaping algorithms to accommodate common flow control challenges, including high-dimensionality, nonlinearity, and model uncertainty. Technical outcomes for each of these objectives are described in Chapter 3.

In addition to research contributions addressing Objectives #1–#3, the research effort also resulted in a number of related outcomes for TEG control. Chapter 4 describes contributions toward control-oriented reduced-order modeling, which enables the design of feedback controllers that minimize the maximum transient energy growth (MTEG) in a channel flow. Chapter 5 describes the discovery of a fundamental performance limitation associated with TEG suppression using observer-based feedback laws based on the separation principle. Chapter 6 describes contributions toward overcoming these performance limitations by leveraging static sensor-based feedback control structures.

In addition to these contributions on dynamic mode shaping and TEG control, the modal perspective of this research effort has also contributed to data-driven techniques for selecting actuators for optimal flow control, which is discussed within the context of airfoil separation control in Chapter 7.

3 | Dynamic Mode Shaping

Here, we formulate and propose a number of controller synthesis strategies aimed at suppressing transient energy growth (TEG). The strategies proposed here will build off one another—each addressing a different practical challenge that establishes the necessary groundwork for ultimately realizing feedback fluid flow control. In Section 3.1, we present the foundations of controller synthesis for dynamic mode shaping. In Section 3.2, we show that the dynamic mode shaping framework can be used in a *dynamic mode matching* capacity. We apply the dynamic mode matching controller synthesis method to *match* the spectral properties of a non-normal system with the spectral properties of a system with a lesser degree of non-normality. In Section 3.3, we propose and formulate a *dynamic mode orthogonalization* technique to eliminate TEG, when such a solution is admissible. We demonstrate the technique on a simple linear system and consider the robustness properties of the control method to modeling uncertainties. In Section 3.4, we extend the dynamic mode orthogonalization technique for controlling low-rank large-scale systems. We show that controller synthesis can be performed efficiently by working with a low-order representation of the system dynamics. Lastly, in Section 3.5, we introduce a nonlinear controller synthesis approach based on the notion of Koopman invariant subspaces and Carleman linearization. This final approach enables TEG suppression in particular classes of nonlinear systems by means of dynamic mode shaping control. Materials in this chapter are drawn heavily from [32].

Shaping the modes of an operator by feedback control is possible and amounts to a problem in eigenstructure assignment [30, 31, 33, 34]. As noted in [30], the solution to the pole placement problem is not unique, which offers additional flexibility to the control designer to specify the shape of the closed-loop modes as well. Although eigenstructure assignment (ESA) is a familiar technique of modern control theory, the challenges associated with specifying desirable spectral characteristics for a closed-loop system (i.e., closed-loop eigenvalues and eigenvectors) has led control theorists to favor optimal and robust control methodologies, even with the demands of subsequent controller tuning. Much of the recent research on ESA has focused on alleviating this burden of choice, especially to a control engineer who may have limited physical insight about the particular system to be controlled [35, 36].

It is interesting to note that although pole placement has been considered in previous investigations of flow control [37], the notion of dynamic mode shaping by eigenstructure assignment has *never* formally been investigated in the context of fluid flow control. This last point comes as a big surprise given the predominance of modal decomposition techniques for fluid flow analysis: for instance, beyond transition control, the dynamic mode shaping framework to be developed and studied here is a natural one to consider in the context of models based on dynamic mode decomposition (DMD), which considers the dynamics of fluid flows in terms of the spectral properties

of an approximate linear dynamical system that governs the flow evolution [38–45]. The ubiquity of modal decomposition techniques for fluid flow analysis suggests that dynamic mode shaping methodologies will offer a welcome perspective for flow control; practitioners can apply the same familiar principles they already use to *interpret* fluid dynamic behaviors to the task of *designing* reliable strategies for flow control and manipulation. The methods introduced here will inevitably enable modal decomposition techniques, such as DMD, to be developed beyond flow analysis and diagnostic techniques.

3.1 Foundations of Dynamic Mode Shaping

Consider the finite-dimensional state-space representation of the linearized Navier-Stokes equations,

$$\begin{aligned}\dot{\vec{x}} &= A\vec{x} + B\vec{u} \\ \vec{y} &= C\vec{x},\end{aligned}\tag{3.1}$$

where $\vec{x} \in \mathbb{R}^n$ is the vector of flow state variables, $\vec{u} \in \mathbb{R}^m$ the vector of actuator inputs, and $\vec{y} \in \mathbb{R}^p$ the vector of measured outputs. Here, $A \in \mathbb{R}^{n \times n}$ is determined by the dynamical characteristics of the fluid flow, $B \in \mathbb{R}^{n \times m}$ is determined by the specific arrangement of actuators and their influence on the flow dynamics, and $C \in \mathbb{R}^{p \times n}$ is determined by the specific arrangement and type of sensors. In essence, (A, B, C) define the given flow control configuration for a particular operating point.

The proposed dynamic mode shaping strategy for TEG suppression is best understood from the standpoint of a modal representation of the flow response. Assuming A has n distinct eigenvalues* μ_i with associated eigenvectors $\vec{\xi}_i$ and reciprocal eigenvectors $\vec{\rho}_i$, the sum of the individual modal dynamics yields a convenient expression for the system response,

$$\begin{aligned}\vec{x}(t) &= \sum_{i=1}^n \langle \vec{\rho}_i, \vec{x}(0) \rangle e^{\mu_i t} \vec{\xi}_i + \int_0^t \sum_{i=1}^n \langle \vec{\rho}_i, B\vec{u}(\tau) \rangle e^{\mu_i(t-\tau)} \vec{\xi}_i d\tau \\ \vec{y}(t) &= C\vec{x}(t),\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes inner-product. The aim of dynamic mode shaping is to alter the spectral properties of the linearized flow, such that the individual modal contributions result in desirable spatiotemporal response characteristics for the full system. Moving the system eigenvalues from μ_i to λ_i will alter the temporal nature of the flow; shaping the system modes from $\vec{\xi}_i$ to $\vec{\phi}_i$ will alter the spatial response characteristics of the flow; and shaping the reciprocal eigenvectors from $\vec{\rho}_i$ to $\vec{\zeta}_i$ will modify the contribution of each mode to the overall flow response. As we will show next, desired closed-loop spectral properties can be achieved through a linear output feedback law of the form $\vec{u} = K\vec{y} = KC\vec{x}$, with the static controller gain matrix $K \in \mathbb{R}^{m \times p}$.

To see how we can shape the spectral properties of the closed-loop system via the feedback law $\vec{u} = K\vec{y}$, we begin by considering the eigendecomposition of the controlled (closed-loop) dynamics: $(A + BKC)\vec{\phi}_i = \lambda_i\vec{\phi}_i$, for $i = 1, \dots, n$. By re-writing this eigendecomposition as,

$$\underbrace{\begin{bmatrix} A - \lambda_i I & B \end{bmatrix}}_{G_i} \begin{bmatrix} \vec{\phi}_i \\ KC\vec{\phi}_i \end{bmatrix} = 0,$$

*Modal decompositions and dynamic mode shaping can be employed in the case of confluent eigenvalues as well, though additional considerations need to be made.

we immediately see that not all choices of closed-loop modes are admissible. The closed-loop mode $\vec{\phi}_i$ corresponding to the specified closed-loop eigenvalue λ_i will only be attainable if the vector $\begin{bmatrix} \vec{\phi}_i \\ KC\vec{\phi}_i \end{bmatrix}$ is in the nullspace of $G_i := \begin{bmatrix} A - \lambda_i I & B \end{bmatrix}$: i.e.,

$$\begin{bmatrix} \vec{\phi}_i \\ KC\vec{\phi}_i \end{bmatrix} \in \mathcal{N}(G_i). \quad (3.2)$$

The relationship in Eq. (3.2) highlights a key condition for modal admissibility and controller gain determination, which will be central to our approach here. In fact, given a flow control configuration (A, B, C) and a self-conjugate set of allowable closed-loop eigenvalues* $\{\lambda_i\}$, the relationship in Eq. (3.2) can be used to yield a complete spectral characterization of *all* closed-loop system realizations. In the event that a desired mode is not admissible, the “best” approximation of this mode can be achieved via projection onto the admissible modal subspace, as depicted in Figure 3.1. The admissibility constraint in Eq. (3.2) can be leveraged in determining admissibility conditions[†], both for dynamic mode matching and for dynamic mode orthogonalization.

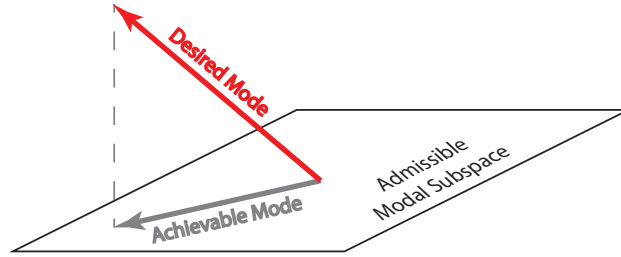


Figure 3.1: Given a flow control configuration and desired set of closed-loop eigenvalues, not all closed-loop modes are admissible; orthogonality and arbitrary shaping of modes may not be achieved exactly. A projection of the desired mode onto the admissible subspace can be performed to determine the “closest” achievable mode.

In practice, one method for computing the admissible subspace of closed-loop modes and the necessary controller gains for achieving these desired modes is based on the singular value decomposition (SVD) of G_i ,

$$G_i = U\Sigma \begin{bmatrix} \tilde{V}_i & V_i \\ \tilde{W}_i & W_i \end{bmatrix}^T,$$

where the last q_i right-singular vectors form an orthonormal basis for $\mathcal{N}(G_i)$, such that

$$\mathcal{N}(G_i) = \text{span}\left(\begin{bmatrix} V_i \\ W_i \end{bmatrix}\right).$$

Hence, any admissible mode $\vec{\phi}_i$ can be specified as a linear combination of columns of V_i —i.e., $\vec{\phi}_i = V_i \vec{\alpha}_i$, where $\vec{\alpha}_i \in \mathbb{R}^{q_i}$ is a vector of coefficients and q_i is the dimension of the associated admissible modal subspace. Further, in order to satisfy the constraint specified in Eq. (3.2), the

*Only controllable eigenvalues can be placed arbitrarily.

[†]The closed-loop modes must form a self-conjugate set to ensure the system remains real-valued. The self-conjugacy requirements can be relaxed for complex-valued systems.

following relation must also hold, $\vec{\psi}_i = KC\vec{\phi}_i = W_i\vec{\alpha}_i$. We can exploit this final expression to compute the controller gain required to achieve a *specified* set of admissible closed-loop modes. In particular, define the matrix of admissible closed-loop modes Φ and the associated auxiliary matrix Ψ needed to satisfy the nullspace constraint in Eq. (3.2),

$$\Phi = \begin{bmatrix} \vec{\phi}_1 & \vec{\phi}_2 & \cdots & \vec{\phi}_n \end{bmatrix} = \begin{bmatrix} V_1\vec{\alpha}_1 & V_2\vec{\alpha}_2 & \cdots & V_n\vec{\alpha}_n \end{bmatrix}, \quad (3.3)$$

$$\Psi = \begin{bmatrix} \vec{\psi}_1 & \vec{\psi}_2 & \cdots & \vec{\psi}_n \end{bmatrix} = \begin{bmatrix} W_1\vec{\alpha}_1 & W_2\vec{\alpha}_2 & \cdots & W_n\vec{\alpha}_n \end{bmatrix}, \quad (3.4)$$

then the control gain K can be determined from the relation

$$KC\Phi = \Psi. \quad (3.5)$$

In the remainder of the manuscript, we consider the case of full-state feedback (i.e., $C=I$). We introduce alternative strategies for TEG suppression based on the dynamic mode shaping framework outlined here. In Section 3.2, we make direct use of the dynamic mode shaping perspective for reducing the degree of TEG by means of *dynamic mode matching*. In Section 3.3, we build upon the dynamic mode shaping perspective to formulate a *dynamic mode orthogonalization* control strategy, which is aimed at eliminating TEG. We also demonstrate robustness properties of the resulting dynamic mode orthogonalization controller when applied to uncertain systems. Subsequently, in Section 3.4, we show that dynamic mode orthogonalization can be formulated using model-reduction techniques in the context of low-rank large-scale systems. In Section 3.5, we extend these dynamic mode shaping approaches to controlling nonlinear systems by appealing to the notion of Koopman invariant subspaces. To this end, we introduce a new perspective for nonlinear feedback control based on our proposed linear controller synthesis techniques.

3.2 Reducing Transient Energy Growth via Dynamic Mode Matching

In principle, TEG can be *eliminated* by achieving an orthogonal set of closed-loop modes via feedback control, as will be discussed in Section 3.3; however, such outcomes are not necessarily admissible for a given flow control configuration. Here, we propose a dynamic mode matching strategy that aims to *reduce* the degree of TEG. (We describe the dynamic mode matching strategy first because it follows from a direct application of dynamic mode shaping, introduced in Section 3.1.) By noting that non-normality decreases with decreasing Re , then it seems reasonable to investigate under what conditions it is possible to reduce non-normality by controlling a high Re flow to “behave like” a lower Re flow. Indeed, if feedback control can be used to shape the spectral properties of a high Re flow to match the spectral properties of a lower Re flow, then the degree of non-normality and associated will be reduced.

To build intuition for this strategy, consider a simple non-normal system that has been used previously to explain the role of non-normality in TEG [5, 25] and to assess TEG control strategies [21]:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/R & 0 \\ 1 & -2/R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{u}, \quad (3.6)$$

where R is a parameter that acts like Re and $\vec{u} \in \mathbb{R}^2$ is a vector of control inputs. As seen in the phase portraits for this system (see Figure 3.2), non-normality increases as the parameter R increases. All three tiles in Figure 3.2 are associated with non-normal systems; however, only trajectories in tile (c) can exhibit TEG, thus indicating that non-normality is not a sufficient condition for TEG. Indeed,

the possibility of TEG arises only beyond some threshold in the alignment of modal directions and in the value of the parameter R . Based on these observations, the task of TEG suppression for the system shown in Figure 3.2(c) does not require modal orthogonalization; rather, the objective can be achieved by shaping the closed-loop spectral properties of the system with $R = 10$ to match the spectral properties of the system with $R = 0.01$ or $R = 1$, as in Figure 3.2(a) or (b), respectively. *Dynamic mode matching* controllers for this purpose can be formulated through a direct application of the dynamic mode shaping framework introduced in Section 3.1: (i) determine the spectral properties of a desired low- R system, (ii) check for admissibility within the context of the control configuration, then (iii) design a feedback controller to match these spectral properties via dynamic mode shaping.

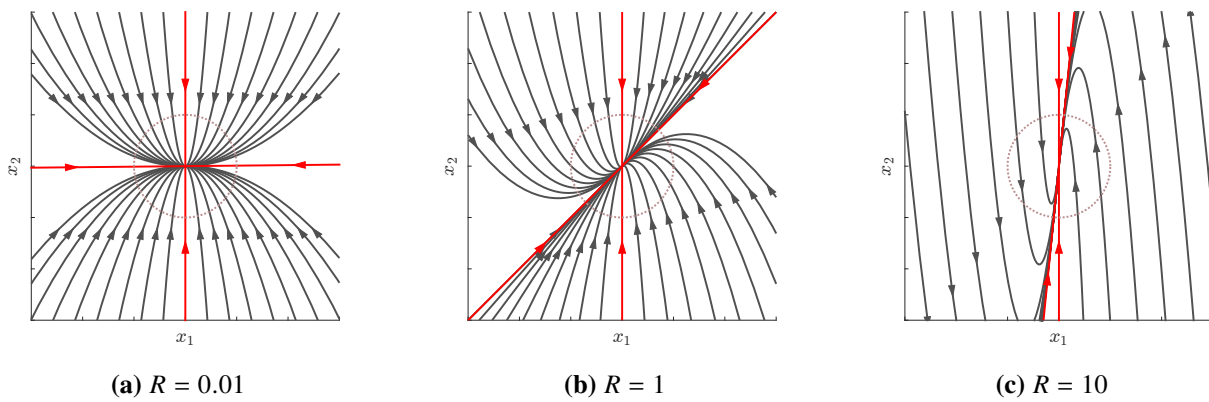


Figure 3.2: Phase portraits for the non-normal system in (3.6) show that as the parameter R increases, the modal directions (highlighted in red) approach one another, thus giving rise to the possibility of TEG away from the origin. Tiles (a) and (b) show that non-normality is not a sufficient condition for TEG—states beginning in the dashed circle remain in the dashed circle in both cases; however, beyond some threshold of non-normality, trajectories can exit the dashed circle prior to decaying to the origin, as in tile (c).

As an example of dynamic mode matching, again consider the model system in Eq. (3.6), now with $R = 500$. The goal is to *reduce* the degree of TEG by matching the spectral properties of the uncontrolled system with $R = 100$. To do so, we simply compute the target spectral properties by setting $R = 100$, then perform dynamic mode shaping with these spectral properties as the target. For this example, the spectral properties for the open-loop system with $R = 100$ are admissible, so the controller is able to successfully match this exactly, which leads to a reduction in TEG, as seen in the center tile in Figure 3.3.

Dynamic mode matching can be taken a step further: If desirable closed-loop spectral properties have already been achieved for a system at a particular parameter value, then these same spectral properties can be targeted for a different parameter value of interest with dynamic mode matching. As a simple demonstration of this notion, consider again the simple system in (3.6). Further, suppose that our objective is to design a control law that yields a set of orthogonal closed-loop modes without altering the system eigenvalues. In Section 3.3, we formulate a general approach for synthesizing such a feedback control law. For the simple system under consideration here, one simple strategy for synthesizing a control law that guarantees closed-loop orthogonalization and

unaltered eigenvalues, for any R , consists of two steps: (i) retain the mode associated with the eigenvalue $-1/R$, then (ii) shape the remaining mode to yield an orthogonal set. Following this approach yields a control law $\vec{u} = \begin{bmatrix} 0 & -x_1 \end{bmatrix}^T$, which will yield orthogonal modes with unaltered eigenvalues for all values of R . The open- and closed-loop responses for $R = 100$ are presented in Figure 3.3. Although the resulting controller here may seem like an obvious choice for this simple system, the systematic approach to computing this controller by the dynamic mode shaping framework enables a means of realizing an orthogonalizing controller in more complex scenarios for which a clear path to removal of non-normality is not obvious.

For the present discussion, we assume that a dynamic mode orthogonalization control law has already been realized for $R = 100$; then, the spectral properties of this orthogonalized closed-loop system can also be targeted via dynamic mode matching. Indeed, the spectral properties for the orthogonalized $R = 100$ system are admissible, and dynamic mode matching control successfully shapes the $R = 500$ system to match this system structure exactly (see bottom tiles in Figure 3.3).

This last example highlights an important point: while the pole placement problem can be incorporated to match the temporal character of a low- Re flow (e.g., controlling a flow at $Re=500$ to match eigenvalues of a flow at $Re=100$), doing so does not necessarily remove TEG. However, by noting that the solution to the pole placement problem is not unique, additional considerations with regards to associated mode shapes can be made to target the transient response characteristics directly. Dynamic mode matching also creates a means of reducing computational demands in controller realization: once an orthogonalizing controller is realized by computationally demanding methods for one Re , the less computationally demanding dynamic mode matching approach can be used to inform the design of a controller for a different Re .

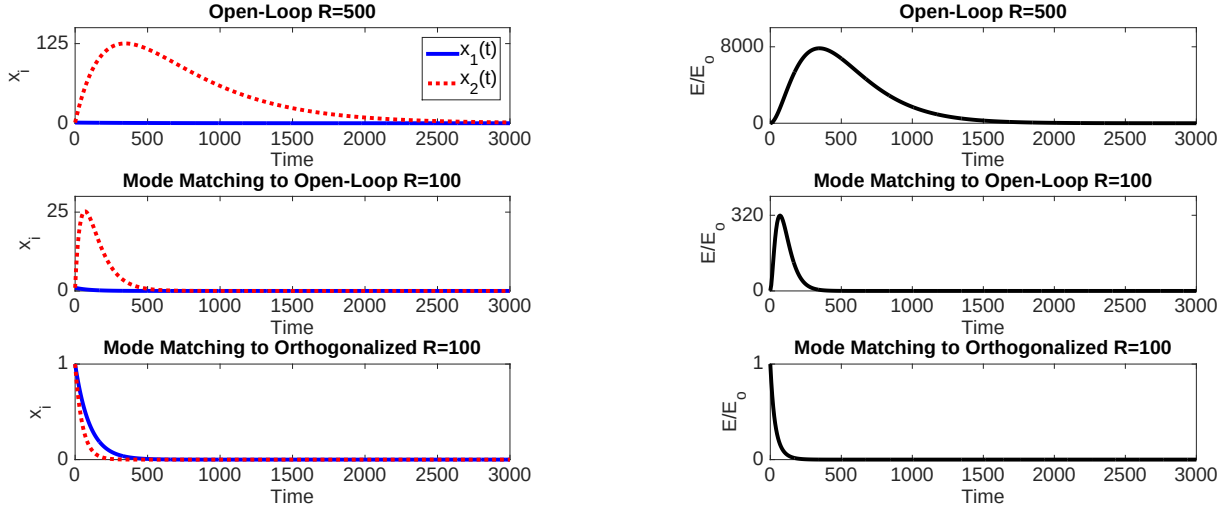


Figure 3.3: Dynamic mode shaping can be used in a “model matching” capacity by shaping the spectral properties of the closed-loop system to match the spectral properties of a different system. Here, we show three variations of the response of the model system in Eq. (3.6) with $R = 500$. Individual state variable responses are plotted on the left and associated energies on the right. **(Top)** The open-loop response exhibits relatively large TEG. **(Center)** Dynamic mode matching control is used to achieve closed-loop dynamics that match the open-loop dynamics for $R = 100$, which exhibits a lesser degree of TEG compared to the open-loop $R = 500$ case. **(Bottom)** Dynamic mode matching control is used to suppress TEG completely, by matching the closed-loop dynamics to those achieved via dynamic mode orthogonalization for $R = 100$ (to be discussed in Section 3.3). The initial condition for all state variables was set to unity here. Energy $E(t) = \vec{x}(t)^\top \vec{x}(t)$ is normalized by the initial energy $E_o = E(t_o)$ in the plots.

3.3 Eliminating Transient Energy Growth via Dynamic Mode Orthogonalization

Non-normality of the linearized Navier-Stokes operator has been shown to be a *necessary condition* for sub-critical transition to turbulence [7]. Dynamic mode shaping offers a foundation for designing controllers that eliminate this non-normality and yield orthogonal closed-loop modes, thus offering a means of inhibiting TEG and suppressing the linear mechanism for transition. Further, owing to the nature of the pseudospectra associated with normal operators [26], the resulting controllers are expected to possess an inherent robustness to modeling errors. Here, we formulate a feedback orthogonalization strategy based on mode shaping, then demonstrate the technique on a simple example.

3.3.1 A Framework for Dynamic Mode Orthogonalization

Here, we extend the notion of dynamic mode shaping and present a framework for realizing an orthogonalizing controller (or family of controllers), if one exists. The same framework can be used to realize an approximately orthogonalizing controller as well, in the event that exact orthogonalization is not admissible for the particular flow control configuration.

The condition for closed-loop orthogonalization can be asserted in terms of the expression for the admissible set of closed-loop modes in (3.3). In particular, we constrain the Gram matrix of inner-products associated with Φ to be diagonal—i.e., the closed-loop modes must be chosen from

the admissible set such that they are orthogonal. We will make use of the Euclidean inner-product here. Without loss of generality, we assume that the state has been transformed such that a weighted inner-product can be expressed with unit weighting. Then, the requirement for an *orthonormal* set of closed-loop modes amounts to $\Phi^T \Phi = I$, or

$$\begin{bmatrix} \tilde{\alpha}_1^T V_1^T V_1 \tilde{\alpha}_1 & \tilde{\alpha}_1^T V_1^T V_2 \tilde{\alpha}_2 & \cdots & \tilde{\alpha}_1^T V_1^T V_n \tilde{\alpha}_n \\ \tilde{\alpha}_2^T V_2^T V_1 \tilde{\alpha}_1 & \tilde{\alpha}_2^T V_2^T V_2 \tilde{\alpha}_2 & \cdots & \tilde{\alpha}_2^T V_2^T V_n \tilde{\alpha}_n \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\alpha}_n^T V_n^T V_1 \tilde{\alpha}_1 & \tilde{\alpha}_n^T V_n^T V_2 \tilde{\alpha}_2 & \cdots & \tilde{\alpha}_n^T V_n^T V_n \tilde{\alpha}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \quad (3.7)$$

The matrix equation in (3.7) represents a system of n^2 bilinear vector equations in the n unknown vectors $\tilde{\alpha}_i$; however, owing to symmetry of the Gram matrix, many of these equations will be redundant. In general, the vectors $\tilde{\alpha}_i$ will have different dimensions q_i , determined by the dimension of the corresponding admissible subspace. Thus, letting $m = \sum_{i=1}^n q_i$, then (3.7) corresponds to a system of at most $m(m+1)/2$ independent bilinear scalar equations in m scalar unknowns. A closer examination of (3.7) provides a means of characterizing the existence and uniqueness of solutions. Such notions are essential for determining whether an orthogonal set of closed-loop modes is admissible. Dynamic mode orthogonalization will be exactly achievable if (3.7) admits at least one solution. In the event that (3.7) does not admit a solution, the condition can be recast as a minimization problem that achieves an approximate solution corresponding to a set of closed-loop modes that are “as close as possible to being orthogonal.”

In the present study, we compute a least-squares solution to (3.7) iteratively using the Levenberg-Marquardt method [46]. In particular, we compute a locally minimizing solution for the coefficients vectors $\{\tilde{\alpha}_i\}$,

$$\{\tilde{\alpha}_i\}^{\text{opt}} = \arg \min_{\tilde{\alpha}_i} \sum_{i=1}^n \sum_{j=1}^i (\delta_{ij} - \tilde{\alpha}_i^T V_i^T V_j \tilde{\alpha}_j)^2, \quad (3.8)$$

where δ_{ij} denotes the Kronecker delta function. Subsequently, the solution to (3.8) can be substituted into (3.5) to compute the controller gain matrix K that achieves the associated set of orthogonal closed-loop modes. Indeed, the resulting set of closed-loop modes will be exactly orthogonal when feedback orthogonalization is admissible; otherwise, the solution will achieve modal orthogonality in a least-squares sense.

Next, we demonstrate the dynamic mode orthogonalization control strategy through a simple example.

3.3.2 An example of dynamic mode orthogonalization

To validate the controller synthesis techniques described above for dynamic mode orthogonalization, we consider a non-normal system that admits an orthogonal set of closed-loop modes. We construct B such that $\text{rank}(B) = n$, which guarantees that the system is orthogonalizable. To see this, consider that the non-normal operator A can be decomposed into a normal component A_n and a non-normal component A_{nn} as $A = A_n + A_{nn}$. If $\text{rank}(B) = n$, then there will always exist a controller $K = -B^{-1}A_{nn}$ that makes the closed-loop system matrix $(A + BK)$ orthogonal. In fact, as is made clear here, an orthogonalizing controller can be computed by means of alternative methods as well; however, we continue with this example simply as a validation of the dynamic mode orthogonalization approach formulated here, which is more generally applicable and useful for synthesizing orthogonalizing controllers in more complex scenarios.

In this example, We construct an arbitrary non-normal system, then compute a control law to orthogonalize the closed-loop modes and to retain the open-loop eigenvalues in closed-loop. Specifically, the system considered has eigenvalues $\lambda_i \in \{-0.1, -0.2, -0.3, -0.4, -0.5\}$ with,

$$A = \begin{bmatrix} -20.34 & -5.22 & 1.63 & 4.25 & 13.57 \\ 15.42 & 3.72 & -1.20 & -3.31 & -10.46 \\ -0.78 & -0.17 & -0.31 & 0.20 & 0.57 \\ 19.45 & 5.06 & -1.54 & -4.34 & -13.12 \\ -29.74 & -7.90 & 2.44 & 6.32 & 19.77 \end{bmatrix} \quad (3.9)$$

$$B = \begin{bmatrix} 0.10 & -1.02 & 1.74 & -1.92 & -0.04 \\ -0.52 & -0.75 & -0.57 & 0.42 & -0.92 \\ -0.12 & 1.37 & -0.72 & 0.93 & -1.39 \\ -0.72 & -0.51 & 0.67 & -1.47 & -0.92 \\ -0.97 & 0.71 & 0.96 & 1.70 & 0.87 \end{bmatrix} \quad (3.10)$$

Applying the dynamic mode orthogonalization strategy described above can be used to yield an orthogonalizing control law. We note that an infinity of controllers exist here because $\text{rank}(B) = n$; here, we report the controller gain. We report a locally minimizing solution to (3.8),

$$K = \begin{bmatrix} 25.64 & 6.64 & -1.89 & -5.21 & -17.35 \\ -1.14 & -0.33 & 0.15 & 0.09 & 0.79 \\ 30.18 & 7.88 & -2.34 & -6.35 & -20.38 \\ 19.05 & 4.99 & -1.46 & -3.90 & -12.84 \\ -6.79 & -1.76 & 0.44 & 1.41 & 4.63 \end{bmatrix}. \quad (3.11)$$

(Note: Individual elements in the controller gain K are truncated to two decimal places for reporting purposes here.) Quantifying non-normality as $\|\Phi^\top \Phi - I\|_2$, the controller reduces the level of non-normality from 3.99 in the open-loop system to 1.27×10^{-12} in closed-loop, indicating that the controller synthesis procedure successfully achieves its objective. The open- and closed-loop responses for this system from unity initial condition are shown in Figure 3.4. The time histories clearly show that TEG is fully suppressed by the orthogonalizing controller, as intended.

We next perform a brief study of the robustness properties of the orthogonalizing controller to modeling uncertainties. The simple demonstration above assumes that the orthogonalizing controller is designed with exact knowledge of the system; however, this is seldom the case in practice. For non-normal operators, eigenvalues can be highly sensitive to perturbations; however, eigenvalue sensitivity to perturbations of normal operators are “well-behaved”—as is revealed by the pseudospectra of these operators [26]. Thus, an orthogonalizing controller computed based on an inexact model $A = \bar{A} + \Delta A$ of the actual system $\bar{A}$ can be expected to perform reasonably well when applied to the actual system. (Here, ΔA represents an unknown modeling error.) Indeed, robustness to modeling uncertainties was part of the motivation for seeking an orthogonalizing controller, rather than one that only suppresses TEG.

To illustrate the robustness of dynamic mode orthogonalization controllers to modeling uncertainty, we consider the orthogonalizing controller computed for the same system above. We apply the controller to a disturbed version of the system $\bar{A} = A - \Delta A$, where each element of ΔA is drawn from $\mathcal{N}(0, \sigma^2)$, with $\sigma^2 = 0.05 \min(|\lambda_i|)$. The controlled response from unity initial condition for

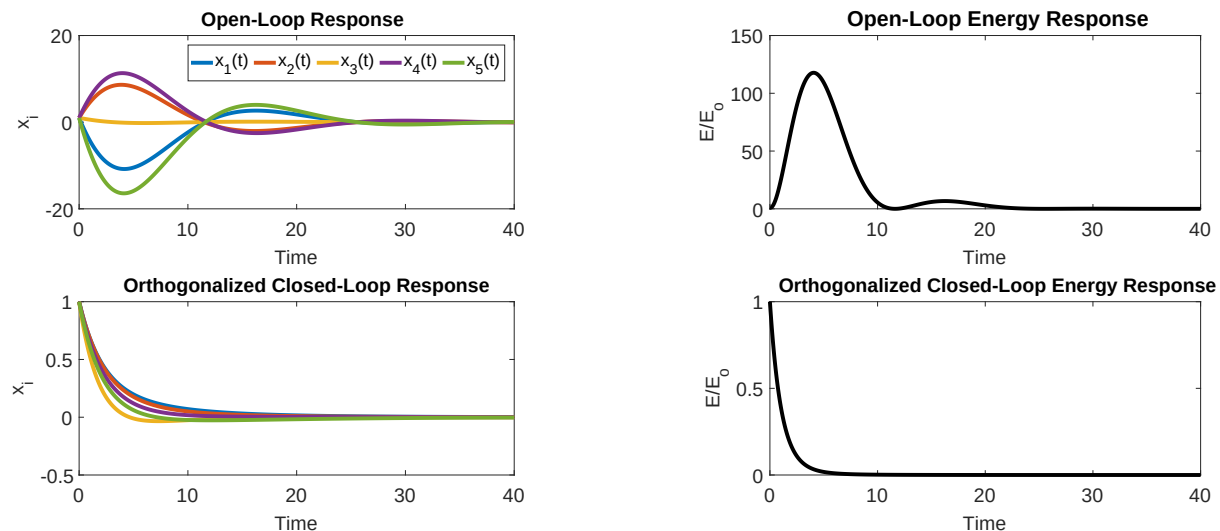


Figure 3.4: The dynamic mode orthogonalization controller successfully suppresses TEG. The plots on the left show individual state responses for the open- and closed-loop systems. Plots on the right show the transient energy responses for the open- and closed-loop systems. The initial condition for each state variable is set to unity here. Energy $E(t) = \vec{x}(t)^T \vec{x}(t)$ is normalized by the initial energy $E_0 = E(t_0)$ in the plots.

each of the 200 realizations is shown in Figure 3.5. Not only does energy decay monotonically for each realization, but the character of the controlled system response is strikingly similar between system realizations. Although this simple study does not serve as a comprehensive demonstration of robustness to modeling uncertainty, it does suggest that the control scheme does not necessarily require an exact model to be successful.

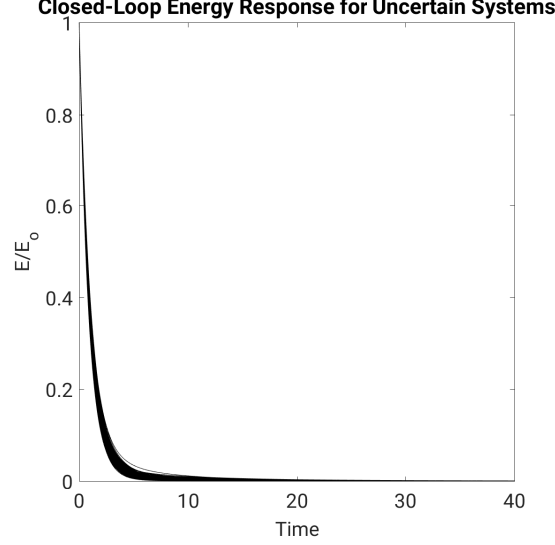


Figure 3.5: The dynamic mode orthogonalization controller possesses some robustness to modeling uncertainty. The orthogonalizing controller reported in 3.4 is applied to 200 perturbed versions of the original system, $\bar{A} = A - \Delta A$, where $\Delta A \sim \mathcal{N}(0, \sigma^2)$ and $\sigma^2 = 0.05 \min(|\lambda_i|)$. Here, we present the energy response for each of these 200 realizations, which have strikingly similar behavior. The initial condition for each state variable is set to unity for all 200 responses. Energy $E(t) = \vec{x}(t)^\top \vec{x}(t)$ is normalized by the initial energy $E_0 = E(t_0)$ in the plots.

3.4 Dynamic mode orthogonalization of low-rank large-scale systems

Upon discretization, the state dimension associated with a fluid flow tends to be quite large, which can make controller synthesis by direct computation impractical (or, at least, undesirable) in many situations. Dynamic mode orthogonalization can be applied in a more computationally tractable manner for certain classes of high-dimensional systems. In particular, consider a large-scale discrete-time system,

$$\vec{x}_{k+1} = A\vec{x}_k + B\vec{u}_k, \quad (3.12)$$

whose dynamics evolve on an r -dimensional subspace of the full n -dimensional state-space, where $r < n$. Suppose that we have determined an orthogonal basis for this r -dimensional subspace. Then, a reduced-order representation of the system dynamics can be obtained via the change of variables $\vec{x} = T\vec{z}$, where $\vec{z} \in \mathbb{R}^r$ and the r orthogonal columns of $T \in \mathbb{R}^{n \times r}$ span the associated r -dimensional subspace:

$$\begin{aligned} \vec{z}_{k+1} &= T^\top A T \vec{z}_k + T^\top B \vec{u}_k \\ &= \tilde{A} \vec{z}_k + \tilde{B} \vec{u}_k. \end{aligned} \quad (3.13)$$

We would now like to show that a dynamic mode orthogonalization controller synthesized on the low-dimensional representation in (3.13) can be used to yield a dynamic mode orthogonalization controller for the original large-scale system in (3.12). To do so, we will consider designing a controller such that the dynamics of the closed-loop system,

$$\vec{x}_{k+1} = \underbrace{(A + BK)}_{A_{cl}} \vec{x}_k, \quad (3.14)$$

evolve on the same r -dimensional subspace as the open-loop system,

$$\vec{z}_{k+1} = \underbrace{(\tilde{A} + \tilde{B}\tilde{K})}_{\tilde{A}_{\text{cl}}} \vec{z}_k, \quad (3.15)$$

where $\tilde{K} := KT$. Note that the reduced closed-loop dynamics are related to the full dynamics as $\tilde{A}_{\text{cl}} = T^\top A_{\text{cl}} T$. It is straightforward to show that the full-order and the reduced-order closed-loop systems will share non-zero eigenvalues, and that the associated modes of these systems are related as $\vec{\phi} = T\vec{\theta}$. Then, it follows, that an orthogonalizing controller on the reduced-order representation—if one exists—can be used to yield an orthogonalizing controller for the full-order system. Indeed, if $\vec{\theta}_i^\top \vec{\theta}_j = \delta_{ij}$, then

$$\vec{\phi}_i^\top \vec{\phi}_j = \vec{\theta}_i^\top T^\top T \vec{\theta}_j = \vec{\theta}_i^\top \vec{\theta}_j = \delta_{ij}. \quad (3.16)$$

Hence, the low-dimensional system representation in (3.13) can be used to synthesize a dynamic mode orthogonalization control strategy for the large-scale system. The resulting control law for the input actuation can be expressed equivalently in terms of the reduced-order state $\vec{z}$ or the full-order state $\vec{x}$, as $\vec{u} = \tilde{K}\vec{z} = \tilde{K}T^\top \vec{x}$.

As a simple demonstration, consider the low-rank large-scale system,

$$\vec{x}_{k+1} = T \begin{bmatrix} -1.1718 & -1.1384 \\ 3.8616 & 3.0218 \end{bmatrix} T^\top \vec{x}_k + T \begin{bmatrix} -0.9835 & 0.3949 \\ 0.6006 & 1.3580 \end{bmatrix} \vec{u}_k, \quad (3.17)$$

where $\vec{x} \in \mathbb{R}^{250}$, $\vec{u} \in \mathbb{R}^2$, and $T \in \mathbb{R}^{250 \times 2}$ is a randomly chosen matrix with orthonormal columns. Using the methods described above, we find that this system admits an orthogonal set of modes for all non-zero eigenvalues. An orthogonalizing gain,

$$K = \begin{bmatrix} -2.7577 & -1.5115 \\ -1.6172 & -0.8584 \end{bmatrix} T^\top, \quad (3.18)$$

leads to full suppression of TEG in closed-loop, as shown in Figure 3.6. It is important to note that the above approach can also be applied to continuous-time systems as in (5.1) as well; however, in the context of continuous-time systems, neither the open-loop nor the closed-loop response will decay to zero due to the “constant forcing” effect of zero-eigenvalues—an artifact of the “low-rank” nature of the A -matrix and its role in the continuous-time dynamics.

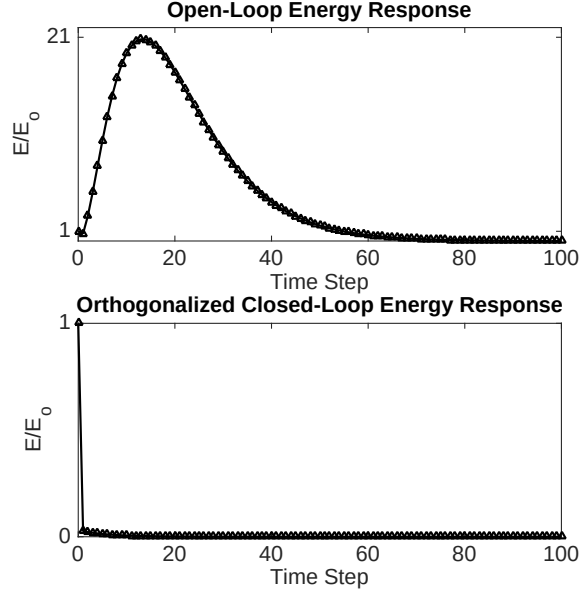


Figure 3.6: Dynamic mode orthogonalization suppresses TEG in a low-rank large-scale system with $n = 250$. **(Top)** The open-loop dynamics exhibit TEG, whereas **(Bottom)** dynamic mode orthogonalization completely suppresses TEG. The underlying dynamics of the open-loop system are governed by (3.17). The initial condition associated with these results was selected randomly. Energy $E_k = \vec{x}_k^T \vec{x}_k$ is normalized by the initial energy $E_o = E_{k=0}$ in the plots.

3.5 Dynamic Mode Shaping in Nonlinear Systems

The dynamic mode shaping and dynamic mode orthogonalization perspectives offer a potential path for suppressing TEG in nonlinear systems without resorting to linearization. By choosing an appropriate change of coordinates, a nonlinear model may admit a finite-dimensional linear representation [47, 48]. Such ideas are the subject of research on the Koopman operator and data-driven Koopman spectral analysis [39, 42, 49–53]. Here, we demonstrate the utility of dynamic mode shaping for nonlinear controller synthesis by way of a simple example.

Consider the nonlinear system,

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 + a x_2^2 \\ \lambda_2 x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad (3.19)$$

which was first used to demonstrate Koopman-based optimal control in [51]. This nonlinear system admits a finite-dimensional linear representation in terms of the extended set of variables $\{y_i\}$ for $i = 1, 2, 3$, where $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_2^2$:

$$\frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & a \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 2\lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u. \quad (3.20)$$

The linear representation admits an eigendecomposition, with eigenvalues $(\lambda_1, \lambda_2, 2\lambda_2)$ and corre-

sponding modes,

$$\phi_{\lambda_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \phi_{\lambda_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \phi_{2\lambda_2} = \begin{bmatrix} -a \\ 0 \\ \lambda_1 - 2\lambda_2 \end{bmatrix}.$$

Modes ϕ_{λ_1} and ϕ_{λ_2} always constitute an orthogonal pair; however, depending on the specific values of a , λ_1 , and λ_2 , the third mode $\phi_{2\lambda_2}$ can lead to non-normality, indicating that the uncontrolled nonlinear system can exhibit TEG.

Interestingly, the same dynamic mode shaping perspectives introduced previous can also be applied here; by leveraging the linear representation of this nonlinear system, the dynamic mode shaping approach is valid in this context. For example, TEG exhibited by this *nonlinear* system can be eliminated by applying dynamic mode orthogonalization in the space of extended variables $\{y_i\}$. Of course, the linear control law determined in this linear setting of Koopman observables can be transformed back into the nonlinear setting, yielding a nonlinear control law that can be implemented on the original system.

Here, we synthesize a nonlinear control law—by means of dynamic mode orthogonalization—to suppress TEG. A comparison of open- and closed-loop responses for this system are shown in Figure 3.7. Here, the open-loop system has $\lambda_1 = -1/2$, $\lambda_2 = -3/4$, and $a = 2$; the resulting control law effectively acts to eliminate the term a from the closed-loop system. Although this choice is, perhaps, obvious from the analytical expressions for the modes of this system, the dynamic mode orthogonalization perspective provides a means of computing such solutions in more complicated systems for which an orthogonalizing control law may not be as obvious.

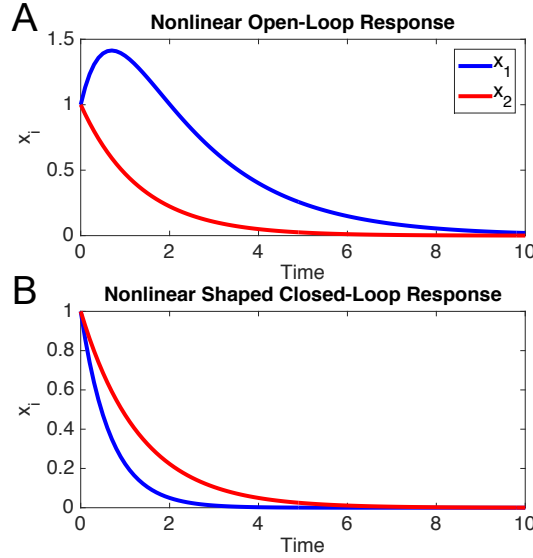


Figure 3.7: Dynamic mode shaping can be used to inhibit TEG arising due to nonlinear mechanisms in special classes of nonlinear systems. Here, we consider the exact finite-dimensional linear representation in Eq. (3.20) of the nonlinear system in Eq. (3.19). (A) The open-loop response exhibits TEG. (B) Dynamic mode shaping control is used to suppress TEG by orthogonalizing the closed-loop modes in an extended space of observables. For this demonstration, the control law is designed to keep the eigenvalues in this extended space unaltered.

3.6 Conclusions

We have introduced dynamic mode shaping as a promising feedback control technique for use in fluid flow control applications. Indeed, dynamic mode shaping offers additional flexibility beyond alternative linear control techniques for fluid flow control, allowing the user to prescribe *both* the temporal response characteristics (associated with system eigenvalues) *as well as* spatial response characteristics (associated with system modes). Further, the dynamic mode shaping framework can be used to shape the reciprocal modes (i.e., left-eigenvectors) of the closed-loop system as well, thus offering a means of altering the contributions of individual modal dynamics to the overall fluid response.

The particular investigation here focused on developing dynamic mode shaping strategies for controlling TEG in linear systems—a phenomenon that arises due to system non-normality and that is commonly associated with sub-critical transition to turbulence in various contexts. In particular, we proposed two controller synthesis strategies: (1) a *dynamic mode matching* strategy aimed at reducing TEG by shaping the spectral properties of a system to match the spectral properties of a “more desirable” system, and (2) a *dynamic mode orthogonalization* strategy aimed at suppressing TEG by shaping a non-normal system to attain orthogonal closed-loop modes. Additionally, we presented various techniques for addressing the usual challenges that arise in the context of fluid flow control: high-dimensionality, nonlinearity, and system uncertainty. The proposed strategies

were demonstrated on simple examples to illustrate the utility and promise of the dynamic mode shaping perspective.

Owing to the predominance of modal analysis techniques in fluid flow diagnostics, we believe that the dynamic mode shaping perspective will be a welcome tool to add to the arsenal of fluid flow control techniques. The perspectives and methods introduced here promise to serve as a foundation upon which such techniques can be further developed and refined. However, much remains to be done to make dynamic mode shaping suitable for practical fluid flow control. Determining a target eigenstructure remains an open challenge. While dynamic mode orthogonalization overcomes part of this challenge, the strategy can be overly conservative. An orthogonal set of modes is a sufficient condition for monotonic decay, not a necessary one. Even in systems for which an orthogonal set of dynamic modes cannot be achieved, it may still be possible to eliminate or reduce TEG. As such, Chapters 4–6 present results based on investigations of alternative control strategies for TEG reduction.

4 | Control-Oriented Model Reduction and Minimizing the Maximum Transient Energy Growth

4.1 Introduction

Numerous investigations have considered the possibility of reducing TEG and delaying transition by employing feedback control techniques. Excellent reviews of past works can be found in [54–56]. Linear quadratic optimal control has been a common approach for such applications. The full-information linear quadratic regulator (LQR) has been investigated in various capacities, and has been shown to increase transition thresholds within the channel flow system [15, 22, 57]. Observer-based output-feedback controllers have also been invoked with some success [15, 58, 59]. In [59], the authors successfully design a linear quadratic Gaussian (LQG) controller on a reduced-order Poiseuille flow system—combining the optimal LQR law with optimal state estimates from a Kalman filter. The study in [15] investigates LQG/ $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers, with particular emphasis on the use of appropriate transfer function norms to minimize the energy of flow perturbations in a channel flow system. The study in [58] further investigates LQG control of the channel flow system, giving careful attention to the design of the Kalman filter for optimal state estimation for use in observer-based feedback control. As we will show in Chapter 5, it has been shown that observer-based feedback strategies may exacerbate TEG in the controlled system, due to an adverse coupling between the fluid dynamics and the control system dynamics [60]. As a potential alternative, static output feedback formulations of the LQR problem have shown some promise in overcoming the limitations of LQG strategies under certain flow conditions within the channel flow system [61, 62] (see Chapter 6). Interestingly, although all of these investigations have shown the promise of feedback control for reducing TEG and delaying transition, it stands that linear quadratic optimal control techniques and related synthesis approaches do not necessarily minimize—nor even reduce—TEG. Indeed, in the case of LQR, the objective function to be minimized is the balance of integrated perturbation energy and input energy—not explicitly the TEG itself. Thus, the most commonly employed controller synthesis approaches aim to achieve an objective that does not necessarily address the TEG problem directly.

Given the central role of TEG and non-modal instabilities in the transition process, it seems that a more appropriate objective function for transition control would be to minimize the maximum transient energy growth (MTEG), as proposed in [63]. This objective has direct connections with notions of *worst-case* or *optimal perturbations*, which correspond to disturbances that result in the maximum TEG [2]. The objective also has connections with optimal forcing functions

determined from input-output analysis [18], though these types of “persistent disturbances” will not be considered in the present work. The minimum-MTEG optimal control problem—which can be specified for either full-state or output feedback [63]—can be posed as a linear matrix inequality (LMI). The LMI constitutes a convex optimization problem that can be solved using standard methods, such as interior-point methods [64]. However, the specific LMI problem that arises for MTEG minimization is computationally intractable for high-order systems, such as fluid flows; the memory requirements associated with existing solution methods scale as system order to the sixth power [57].

Despite the computational challenge, control laws that minimize the MTEG can be desirable over linear quadratic optimal control techniques. MTEG-minimizing controllers have been found to outperform LQR controllers in reducing TEG within a channel flow configuration [57]. To achieve this, a modal truncation was performed to obtain a reduced-order model (ROM) that would make controller synthesis tractable for the linearized channel flow system [57]. In spite of the noteworthy performance reported in their study, it is well-established that modal truncation methods tend to yield ROMs that are poorly suited for controller synthesis [65, 66]. (A demonstration of this point is given in Appendix A.) Thus, it may be possible to synthesize MTEG-minimizing controllers with even better performance than those reported in [57] by exploiting an appropriately tailored *control-oriented* model reduction strategy.

Various ROM approaches have been studied in the literature (see [67, 68] for excellent reviews of such techniques within the context of fluid dynamics). However, developing ROMs that facilitate the design of MTEG-minimizing controllers requires a tailored approach. Control-oriented ROMs within this context must address a dual need: (i) approximate the perturbation energy to concisely and adequately describe the energy-based control objective; and (ii) reduce dimensionality to faithfully represent the input-output dynamics, as needed for computationally tractable controller synthesis. Identifying states that contribute substantially to both the perturbation energy and a system’s input-output properties (e.g., controllability and observability) is a non-trivial task [69]. A projection onto proper orthogonal decomposition (POD) modes is known to be optimal for capturing the energy of a given signal; however, projection-based model reduction based on POD modes often fails to capture a system’s input-output dynamics, making such models poor candidates for controller synthesis [68, 70]. In contrast, balanced truncation can be performed to obtain ROMs that retain a system’s input-output properties [71, 72]. The balanced truncation procedure can be shown to be equivalent to a Petrov-Galerkin ROM based on a projection onto a subspace spanned by a reduced set of balanced modes. For high-dimensional fluid flow systems, the balanced POD (BPOD) and related methods provide efficient tools for computing these balanced modes [69, 72, 73]. Alternative control-oriented model reduction techniques can also be devised to retain a system’s input-output properties, e.g., using ideas from robust control [66, 72].

The contribution of the present study is to investigate the utility of control-oriented model reduction for designing MTEG-minimizing controllers. The study focuses on full-information control of a linearized channel flow system, but the lessons learned can be generalized to other flows and control architectures as well. The ROMs in this work use POD and balanced truncation in conjunction. First, we perform an output-projection of the full-state onto a set of dominant POD modes. Subsequently, a balanced truncation is performed to reduce the state-dimension, while retaining the most controllable and observable modes that contribute to the input-output dynamics.

As we will see, this dual approach results in control-oriented ROMs that can yield effective MTEG-minimizing controllers. The resulting models are able to represent the perturbation energy in terms of a small number of POD modes, thus providing a convenient approximation of the objective function. Further, state-dimension is reduced to a computationally tractable level, while retaining the most controllable and observable states that are critical for effective controller design.

The organization of the chapter is as follows: Section 4.2 summarizes preliminaries on TEG and the relevant controller synthesis strategies, including the synthesis of LQR controllers and the LMI-based synthesis of MTEG-minimizing controllers. The control-oriented model reduction approach is then introduced in Section 4.2.1. Section 4.3 presents all of the results, which are based on the linearized channel flow system described in Section 4.3.1. ROMs are analyzed in an open-loop context in Section 4.3.2. Controller performance of ROM-based controller designs is evaluated in Section 4.3.3, which also includes a comparative analysis of the flow response to LQR and MTEG-minimizing control laws. Finally, Section 7.5 summarizes the findings and contributions of this study. Materials in this chapter are drawn heavily from [74] and [75].

4.2 Reduced-order models and controller synthesis

Consider the linearized dynamics of flow perturbations $\mathbf{x}(t)$ about a steady laminar base-flow,

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \quad (4.1)$$

where $\mathbf{x}$ is a vector of n state variables, $\mathbf{u}$ is a vector of p control inputs, and t is time. Specific details on arriving at such a representation for a channel flow setup will be described in Section 4.3.1. The free-response of the system to an initial perturbation $\mathbf{x}(t_o) = \mathbf{x}_o$ at time t_o is given in terms of the matrix exponential $\mathbf{x}(t) = e^{A(t-t_o)}\mathbf{x}_o$. The perturbation energy for this system is defined as $E(t) := \mathbf{x}^T(t)Q\mathbf{x}(t)$, where $Q = Q^T > 0$. In this study, we are primarily interested in the maximum transient energy growth (MTEG),

$$\Theta_{\max} = \max_{t \geq t_o} \max_{E(t_o) \neq 0} \frac{E(t)}{E(t_o)}, \quad (4.2)$$

which corresponds to the peak energy over all disturbances and all time. The flow perturbation $\mathbf{x}_o^{\text{opt}}$ that results in this MTEG is called the *worst-case disturbance* or the *optimal perturbation* [2].

Owing to the role of transient energy growth in sub-critical transition, our aim here will be to reduce the MTEG using full-information feedback control laws of the form $\mathbf{u}(t) = -K\mathbf{x}(t)$, where K is a gain matrix determined by an appropriate synthesis strategy. In this study, two controller synthesis approaches will be considered: (i) linear quadratic regulation (LQR), and (ii) MTEG minimization via LMI-based synthesis. LQR is an optimal control technique that has been commonly employed in a variety of flow control applications, including transition control. LQR controllers are designed to minimize the integrated balance between perturbation energy and control effort; i.e.,

$$\min_{\mathbf{u}(t)} J = \int_0^\infty (\mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u}) dt \quad (4.3)$$

subject to the linear dynamic constraint in (4.1) and $R > 0$. The feedback control law that minimizes this cost function is given by $\mathbf{u} = -K\mathbf{x}$, where the control gain $K = R^{-1}B^T P$ and $P = P^T > 0$ is

determined from the algebraic Riccati equation,

$$A^T P + P A - P B R^{-1} B^T P + Q = 0. \quad (4.4)$$

LQR is a widely used optimal control strategy owing to properties like guaranteed stability margins and robustness to parameter variations. However, it must be noted that the LQR formulation does not necessarily guarantee reductions in MTEG, let alone its minimization. In principle, a controller that minimizes the balance of integrated energies in (4.3) could still yield large energy peaks.

A state feedback control gain for minimizing the MTEG in a closed-loop system can be found from the solution of a linear matrix inequality (LMI), as shown in [63]. The solution approach leverages the relationship between MTEG and a system's condition number. From this, a control law can be devised to minimize the condition number of the closed-loop system in order to minimize the associated $\Theta_{\max}$. In what follows, it is assumed that an appropriate transformation has been made so that $Q = I$. A feedback control law that minimizes the upper bound Θ_u of the MTEG can be determined from the solution to the LMI generalized eigenvalue problem [63]:

$$\begin{aligned} & \min \gamma \\ & \text{subject to} \quad I \leq P \leq \gamma I \\ & \quad P = P^T > 0 \\ & \quad AP + PA^T + BY + Y^T B^T < 0. \end{aligned} \quad (4.5)$$

This LMI problem can be solved using standard convex optimization methods, such as those available in the cvx software package for Matlab [76]. Here, γ upper bounds Θ_u . Thus, minimizing γ also minimizes Θ_u , which consequently minimizes the upper bound on $\Theta_{\max}$. The resulting full-state feedback control law is given by $\mathbf{u} = -Y P^{-1} \mathbf{x}$, where Y and P are determined from (4.5).

Standard solution techniques for the LMI problem in (4.5) are presented in [64]. However, available algorithms are computationally demanding, with memory requirements scaling as $O(n^6)$ [57], making controller synthesis intractable for the high-dimensional systems of interest in flow control. The aim of the present study is to investigate the role of reduced-order models (ROMs) for facilitating controller synthesis by making the solution of this LMI problem tractable. Further, constructing reliable control-oriented ROMs requires consideration of the specific control objective. Since the LQR and MTEG-minimizing controllers to be studied here are based on energy-based control objectives, it stands that approximating the energy $E(t)$ will be an important consideration for reduced-order modeling. Section 4.2.1 presents a method for obtaining control-oriented ROMs for LQR and MTEG-minimizing controller synthesis.

4.2.1 Control-oriented reduced-order models

Procedures for generating ROMs often rely upon truncating state variables from the system description, with the details of the truncation procedure varying with the given application. A ROM intended to facilitate controller synthesis should faithfully capture a system's input-output dynamics, motivating the use of techniques such as balanced truncation [71]. On the other hand, within the context of the energy-based control objectives considered in the present study, it will also be important for the ROM to faithfully reproduce the system energy $E(t)$. POD modes have been shown to optimally capture the energy of a given signal [67]. Here, we will combine these two approaches in order to realize ROMs that can facilitate the synthesis of controllers with energy-based control objectives. Similar approaches have been described in the literature [68, 70], but have

not been leveraged for LMI-based synthesis of MTEG-minimizing controllers. In Appendix A, we demonstrate the need for tailored ROMs that favor control design.

The basic idea underlying the ROMs proposed here is to first append an output equation to (4.1) to keep track of the state response in terms of $r < n$ POD modes,

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{z} &= \Phi_r^T \mathbf{x}.\end{aligned}\tag{4.6}$$

Here, $\mathbf{z} \in \mathbb{R}^r$ is a vector of POD coefficients and $\Phi_r \in \mathbb{R}^{n \times r}$ is a matrix whose columns are the r dominant POD modes, defined with respect to an appropriate inner-product, such that $E(t) \approx \tilde{E}(t) = \mathbf{z}^T(t)\mathbf{z}(t)$. Thus, rather than requiring full-state information to compute energy, only the r -dimensional output of POD coefficients is needed to approximate the energy response. This approach is sometimes described as an output projection onto POD modes, since the output could also be viewed as a projection of the full-state output onto the dominant POD modes [69]. It should be noted that output projection in (4.6) does not change the state dimension n of the model.

To find the POD modes, we use the snapshot POD approach [67]. Through heuristics, we found that POD modes obtained from snapshots of the system's impulse response matrix provided an adequate basis for the projection. Therefore, to compute the POD modes (Φ_r), we obtain the impulse response $G_i(t)$ from the i^{th} input to the full-state output for each of the p inputs. In generating this data, we first transform the state $\mathbf{x}$ into a coordinate system in which $\mathbf{Q} = \mathbf{I}$. These impulse response data are then collected and arranged within a single snapshot matrix $\mathbf{H} := \begin{bmatrix} G_1(t) & G_2(t) & \cdots & G_p(t) \end{bmatrix}$. The POD modes are then computed by taking the singular value decomposition (SVD) of $\mathbf{H} = \mathbf{U}\Sigma\mathbf{V}^T$. The dominant POD modes are then determined as the leading $r \leq n$ left singular vectors—i.e., $\Phi_r = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r]$, where $\mathbf{u}_i$ are columns $i = 1, \dots, r$ of $\mathbf{U}$. We note that the POD modes could also be computed analytically, though the data-driven method of snapshots is most commonly used in practice [67]. Thus, important parameters for generating the impulse response data will be the length of the simulation (t_f) and sampling interval (δ_t). We address the specifics of these parameters for ROM construction in Section 4.3.2.

Up to this point, we have shown that the system energy can be approximated using just r POD modes, with the associated POD coefficients tracked as the system output $\mathbf{z}$. However, in order to make the LMI problem for MTEG-minimizing controller synthesis computationally tractable, we still need to reduce the state dimension of the system. In the study here, we will perform a balanced truncation of (4.6) and retain only the dominant s balanced modes. Such a truncation will ensure that the input-output dynamics are preserved, making the resulting s -dimensional ROM suitable for controller synthesis [72]. In order to perform a balanced truncation, we first apply a balancing transformation $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$. In the *balanced coordinates* $\bar{\mathbf{x}}$, the controllability Gramian $\bar{W}_c$ and the observability Gramian $\bar{W}_o$ are equal and diagonal. To determine the balanced realization, one must first compute the controllability Gramian W_c and observability Gramian W_o from the Lyapunov equations, $\mathbf{A}W_c + W_c\mathbf{A}^T - \mathbf{B}\mathbf{B}^T = 0$ and $W_o\mathbf{A} + \mathbf{A}^T W_o + \mathbf{C}^T\mathbf{C} = 0$, respectively. Once the Gramians are computed, the balancing transformation $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$ can be found in three steps [77]: (i) compute the lower triangular Cholesky factorizations of $W_o = L_o L_o^T$ and $W_c = L_c L_c^T$; (ii) compute the SVD of the products of Cholesky factors $L_o^T L_c = \bar{U} \bar{S} \bar{V}^T$; and (iii) form the balancing transformation $\mathbf{T} = L_r \bar{V} \bar{S}^{\frac{1}{2}}$ and $\mathbf{T}^{-1} = \bar{S}^{\frac{1}{2}} \bar{U}^T L_o^T$.

In balanced coordinates, each mode's relative contribution to the input-output dynamics of

the system is clear. We have $\bar{W}_c = \bar{W}_o = \text{diag}(\bar{\sigma}_1, \dots, \bar{\sigma}_n)$, where $\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \dots \geq \bar{\sigma}_n > 0$ are the system's Hankel Singular Values (HSVs). Since the HSVs relay information about relative contributions to the input-output dynamics, truncating balanced modes with “small” HSVs provides a convenient strategy for reducing state-dimension while preserving the input-output dynamics [72]. Upon performing the balanced truncation, the dominant s balanced modes of the system are retained, and the $n - s$ modes corresponding to the lowest HSVs are truncated. The resulting state-space realization is given by,

$$\dot{\bar{\mathbf{x}}}_s = \bar{A}_s \bar{\mathbf{x}}_s + \bar{B}_s \mathbf{u} \quad (4.7)$$

$$\bar{\mathbf{z}}_s = \bar{C}_s \bar{\mathbf{x}}_s \quad (4.8)$$

where $\bar{\mathbf{x}}_s \in \mathbb{R}^s$ is the reduced state vector and the system matrices are defined as $\bar{A}_s := T_s^{-1} A T_s$, $\bar{B}_s := T_s^{-1} B$, $\bar{C}_s := \Phi_r^T T_s$. Here, $T_s \in \mathbb{R}^{n \times s}$ denotes a matrix whose s columns are the leading s columns of T , and $T_s^{-1} \in \mathbb{R}^{s \times n}$ denotes a matrix whose s rows are the leading s rows of T^{-1} . Finally, the perturbation energy $E(t)$ can now be approximated from this control-oriented ROM as,

$$\bar{E} = \bar{\mathbf{x}}_s^T \bar{Q} \bar{\mathbf{x}}_s, \quad (4.9)$$

where $\bar{Q} = T_s^T \Phi_r \Phi_r^T T_s \in \mathbb{R}^{s \times s}$. In this study, we take $s = r$, and so will solely report r as the dimension of the reduced-order model.

In order to assess feedback control performance with respect to the MTEG that can be experienced in closed-loop, additional care must be taken when computing the optimal disturbances in this study. Since MTEG analysis is to be performed on the full order system, feedback controllers designed using the ROMs described here are first “lifted” to a control gain with compatible dimensions as the full order model. In this way, the optimal disturbance for the full order closed-loop system can be computed directly. Of course, to actually implement the resulting feedback controllers in practice, one would simply use the reduced-order gain matrix to take advantage of the reduced computational complexity at run-time.

The procedure outlined here will enable a means of obtaining control-oriented reduced-order models that can be used to synthesize feedback controllers, especially those aimed at achieving certain energy-based objectives—as with MTEG-minimization. We emphasize that the control-oriented model reduction procedure described here and other similar approaches have been explored in a number of previous studies [22, 69–71, 73]. The contribution of the present study is to investigate the applicability of such model reduction techniques for the purposes of minimizing the MTEG using the LMI-based synthesis procedure proposed in [57, 63].

4.3 Results

Next, the control-oriented reduced-order modeling approach introduced in the previous section is investigated within the context of controlling TEG in a linearized channel flow. The channel flow system is presented in Section 4.3.1. In Section 4.3.2, the frequency-responses of various ROMs are analyzed to assess open-loop modeling performance. Finally, ROM-based controller performance is investigated in Section 4.3.3, by comparing the MTEG resulting in the closed-loop system response. The associated flow responses are also studied and discussed.

4.3.1 Channel Flow System

Consider the flow between two infinitely long parallel plates separated by a distance of $2h$ (see Figure 4.1). We are interested in the linear evolution of perturbations about a steady laminar base

flow $U(y) = U_{cl}(1 - \frac{y^2}{h^2})$. Thus, we linearize the Navier-Stokes equations about this base flow, and non-dimensionalize based on the centerline velocity U_{cl} and the channel half-height h . Taking the divergence of the continuity equation and using the y -momentum equation to eliminate the pressure term, we obtain an equivalent representation for the evolution equations, now in terms of the wall-normal velocity (v) and the wall-normal vorticity (η). A Fourier transform is applied in the streamwise and spanwise directions to obtain the well-known Orr-Sommerfeld and Squire equations [78],

$$\begin{aligned}\Delta \dot{\tilde{v}} &= \left[i\alpha U \Delta + i\alpha U'' + \frac{\Delta \Delta}{Re} \right] \tilde{v}, \\ \dot{\tilde{\eta}} &= \left[-i\beta U' \right] \tilde{v} + \left[-i\alpha U + \frac{\Delta}{Re} \right] \tilde{\eta},\end{aligned}\tag{4.10}$$

where $\Delta = \frac{\delta}{\delta y^2} - \kappa^2$. Here, $\kappa^2 = \alpha^2 + \beta^2$, and (α, β) denotes the pair of streamwise and spanwise wavenumbers, respectively. The no-slip boundary conditions in velocity-vorticity form are $\tilde{v}(y = \pm 1) = \frac{\partial \tilde{v}}{\partial y}|_{(y=\pm 1)} = \tilde{\eta} = 0$. The wall-normal direction is discretized using Chebyshev polynomials of the first kind [79], allowing $\tilde{v}$ and $\tilde{\eta}$ to be approximated at N discrete collocation points. At each point, the approximation uses Chebyshev basis functions Γ_i and the respective unknown coefficients a_i . The resulting equations of motion can be expressed in the form $\dot{\mathbf{x}} = \mathcal{A}\mathbf{x}$, where $\mathbf{x}$ is a vector of Chebyshev coefficients; i.e., $\mathbf{x} = (a_{v0}, \dots, a_{vN}, a_{\eta0}, \dots, a_{\eta N})^T$.

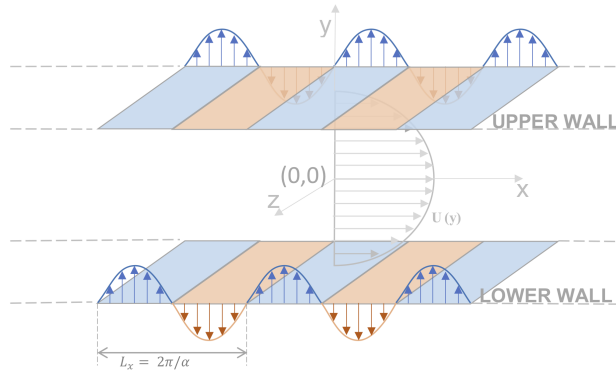


Figure 4.1: Channel flow setup.

One of the stages to bypass transition within the channel flow setting is the transient energy growth (TEG) of flow perturbations about the base-flow due to non-modal effects [78]. In accordance with previous studies, we use the kinetic energy density as a measure of this TEG,

$$E(t) = \frac{1}{V} \int_V \rho \frac{u^2 + v^2 + w^2}{2} dV, \tag{4.11}$$

where ρ is the fluid density and V is the volume of a unit streamwise length of channel. In relation to earlier discussions, the kinetic energy density can be re-expressed as $E(t) := \mathbf{x}^T \mathbf{Q} \mathbf{x}$, where $\mathbf{Q} = \mathbf{Q}^T > 0$ [80, 81].

In order to implement flow control, we introduce wall-normal blowing and suction actuation at the upper- and lower-walls—consistent with prior investigations on controlling channel flow [54,

57, 59]. Following the modeling procedure in [80], this can be modeled via the wall-transpiration boundary conditions $\tilde{v}(y = 1) = q_u$, $\tilde{v}(y = -1) = q_l$, $\frac{\partial \tilde{v}}{\partial y}|_{y=\pm 1} = 0$. Here, q_u and q_l are wall-normal velocities at the upper- and lower-wall, respectively. The final system formulation uses $\dot{q}_u$ and $\dot{q}_l$ as the control inputs, while reclassifying q_u and q_l as system states. This introduces two integrators—associated with the controls—within the system model. Finally, the actuated channel flow system can be expressed in state-space form, as in (4.1), with the state vector defined as $\mathbf{x} = (a_{v_0}, \dots, a_{v_N}, a_{\eta_0}, \dots, a_{\eta_N}, q_u, q_l)^T$ and the input vector defined as $\mathbf{u} = (\dot{q}_u, \dot{q}_l)^T$. It must be noted that the resulting system here is complex-valued, owing to the introduction of Fourier transformations in the streamwise and spanwise directions. As such, a final step is needed to transform the system into an equivalent real-valued state-space realization. Further modeling details can be found in [80].

In the remainder, we consider this linearized channel flow model at a sub-critical Reynolds number of $Re = 3000$ (unless otherwise stated) for three different wavenumber pairs (α, β) : (1, 0), (1, 1), and (0, 2). In all cases, the number of collocation points N is chosen so that the resulting state dimension is $n = 199$. The optimal spanwise disturbance to the uncontrolled system with $(\alpha, \beta) = (0, 2)$ yields the largest MTEG among all other streamwise, spanwise, and oblique wavenumber configurations. Hence, this spanwise disturbance is an important case to study for controller performance evaluation; The other two wavenumber pairs considered here have also been widely studied in the literature [57, 58].

4.3.2 Frequency analysis of control-oriented reduced-order models for the channel flow system

Here, we examine how the control-oriented ROMs introduced in Section 4.2.1 approximate the dynamics of the full-order model (FOM) for the linearized channel flow system, for which $n = 199$. Specifically, we examine whether the ROMs developed here capture the input-output behavior of the FOM accurately. In multi-input multi-output (MIMO) systems, the singular values σ of the system's transfer function over various frequencies provide a means of characterizing the input-output behavior of the system. The singular values provide information on the variation in the system's principal gains in any of the p input direction [82]. Here, we investigate these characteristics as a function of ROM order r from the input $\dot{q}_u$ to the flow-state (see Figure 4.2). In this work, the scaled frequency is given by $f^* = ftU_{cl}/h$. The frequency response data reveal that increasing the order r of the ROM decreases the approximation error with respect to the FOM response. Indeed, the trend is more clear from Table 4.1, which summarizes the root mean square error (RMSE) between the ROM and FOM frequency responses. The acceptable accuracy of a ROM in approximating a FOM response is mostly application specific. For the case here, the RMSE metric suggests that ROMs of $r \geq 20$ perform moderately well. Further, even using the higher-order $r = 50$ models—which reflect excellent agreement with the FOM frequency-response—will make LMI-based controller synthesis tractable. The transfer function from input channel $\dot{q}_l$ to the flow-state exhibits similar convergence in the frequency-response.

Finally, it should be noted that, for the ROMs presented here, the data-sampling parameters t_f and δ_t in the snapshot POD stage of the ROM-procedure were tuned to ensure convergence of the ROM response to the FOM response. In the $Re = 3000$ setting, for streamwise waves and oblique waves $\delta_t = 0.01$ and $t_f = 30$ and 50 respectively. While for spanwise waves we use $\delta_t = 1$ and $t_f = 500$ to obtain adequate models. All times reported in this work are non-dimensionalized, and

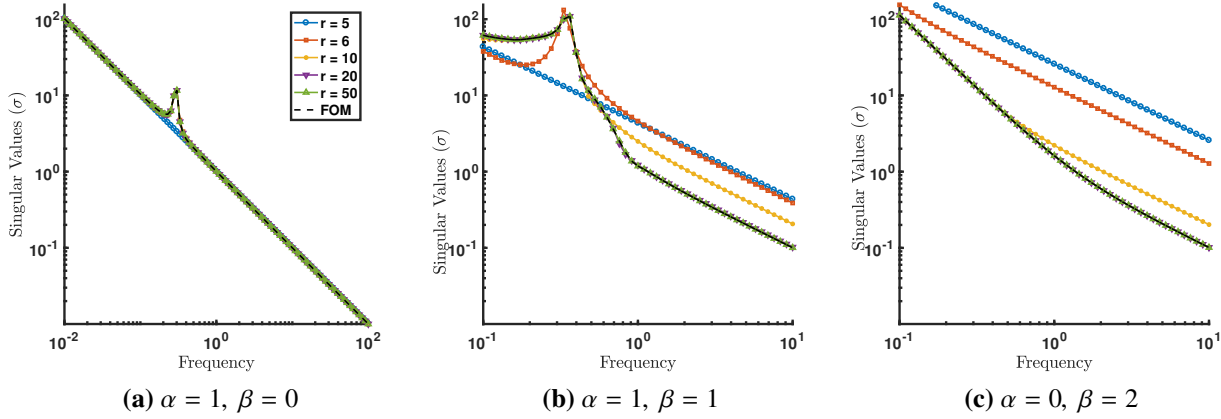


Figure 4.2: Frequency response from $\dot{q}_u$ to the flow-state at $Re = 3000$.

	$r = 5$	$r = 6$	$r = 10$	$r = 20$	$r = 50$
$\alpha = 1, \beta = 0$	1.079	0.049	0.0033	1.340×10^{-4}	5.726×10^{-6}
$\alpha = 1, \beta = 1$	18.765	14.135	2.990	0.0891	9.603×10^{-5}
$\alpha = 0, \beta = 2$	119.19	61.0560	13.332	1.705	0.0193

Table 4.1: RMSE between the frequency-response of the FOM ($n = 199$) and the ROM of order r at $Re = 3000$.

correspond to non-dimensional convective time units $\frac{tU_{cl}}{h}$.

4.3.3 Controller performance

The synthesis of MTEG-minimizing controllers via the solution of the LMI-problem in (4.5) is enabled by the control-oriented ROMs reported in Section 4.3.2 above. The resulting controllers will be referred to as *LMI-ROM controllers* here. For benchmarking, we compare MTEG performance with two sets of LQR controllers—one designed based on the FOM (“LQR-FOM”) and one designed based on the same ROM that was used for the LMI-based synthesis (“LQR-ROM”). Additionally, since the MTEG-minimizing controller in the LMI-ROM was designed without constraining the control input, we relax the penalty on the input effort within the LQR cost function in (4.3) by setting $R = 10^{-6}I$, where I denotes the identity matrix. This is done in an effort to make a fair comparison between the LMI-ROM, LQR-ROM, and LQR-FOM controllers. Note, this differs from the approach taken in [57], in which a constrained version of the controller was implemented. All controllers are applied to the full-order channel flow model described in Section 4.3.1 above, then compared based on the MTEG $\Theta_{\max}$ for each respective closed-loop system. In general, the optimal perturbation will differ between each closed-loop system, and will also differ from that of the uncontrolled system. Here we use the optimal perturbation of the respective closed-loop system to perform MTEG analysis. In this study, each optimal perturbation is calculated using the algorithm presented in [83].

As shown in Figure 4.3, the LMI-ROM controller reduces the MTEG in the system relative to each of the LQR-based controllers for each of the wavenumber pairs considered. For $(\alpha, \beta) = (1, 0)$, the LMI-ROM controller reduces $\Theta_{\max}$ by a factor of ≈ 1.88 in comparison with the LQR controllers (see Figure 4.3a). Similarly, the LMI-ROM reduces MTEG by a factor of ≈ 2.6 relative to the

LQR controllers for $(\alpha, \beta) = (1, 1)$ (see Figure 4.3b). Finally, for $(\alpha, \beta) = (0, 2)$, the difference in MTEG reduction between the LMI-ROM controller and the LQR controllers is only marginally greater (See Figure 4.3c). This result is consistent with the findings in [57], for which wall-normal blowing/suction was found to be less effective for TEG control than spanwise blowing/suction at the walls. For all controllers and all wavenumber pairs considered here, MTEG was reduced relative to the uncontrolled flow. These results are summarized in Table 4.2.

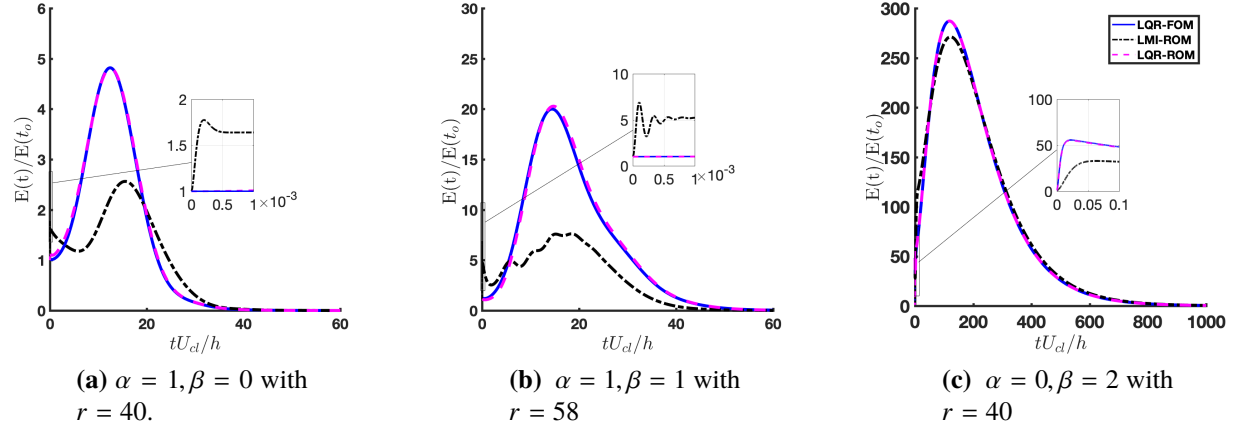


Figure 4.3: Worst-case closed-loop TEG responses at $Re = 3000$.

	r	$\Theta_{\max}$			
		Uncontrolled	LMI-ROM	LQR-FOM	LQR-ROM
$\alpha = 1, \beta = 0$	40	20.31	2.56	4.81	4.81
$\alpha = 1, \beta = 1$	58	107.00	7.58	20	20.28
$\alpha = 0, \beta = 2$	40	1762	271.2	287.1	287.1

Table 4.2: MTEG $\Theta_{\max}$ for the uncontrolled and controlled channel flow system at $Re = 3000$.

Although the LMI-ROM controller outperforms both of the LQR controllers in reducing the MTEG ($\Theta_{\max}$) in the results above, these results depend on the order r of the ROM used for controller synthesis. To investigate this influence, we vary r in the ROM design, then study controller performance on the FOM for streamwise and oblique waves— $(1, 0)$ and $(1, 1)$, respectively—still with $Re = 3000$. We only report values of r for which the open-loop ROMs successfully converge to the FOM dynamics, and so do not consider r smaller than these values here. Figure 4.4 shows that $\Theta_{\max}$ decreases with increasing ROM order r using the LMI-ROM controller. The same is true for the LQR-ROM controller, but the convergence is more rapid. MTEG for the LQR-ROM converges to that of the LQR-FOM for $r = 20$ for streamwise and spanwise waves, and for $r = 25$ for oblique waves. We found that for streamwise and oblique wavenumber pairs considered, there exists a ROM order r such that the LMI-ROM controller will outperform both of the LQR controllers for MTEG reduction. We note that although the LMI-ROM yields a reduction in MTEG relative to both LQR-based controllers in the spanwise wave case, this case was also found to be relatively insensitive to the ROM order. Again, this finding is consistent with the results reported in [57], which found that spanwise blowing/suction actuation was required to achieve meaningful reductions in MTEG. Since

the current investigation is focused on control-oriented ROM, we continue to focus on wall-normal blowing/suction actuation. Lastly, note that the fact that the LMI-ROM requires a larger r than the LQR-ROM for MTEG performance to converge is to be expected; TEG is a phenomenon intimately related to $\dot{E}(t)$, whose approximation requires higher-precision estimates of $E(t)$.

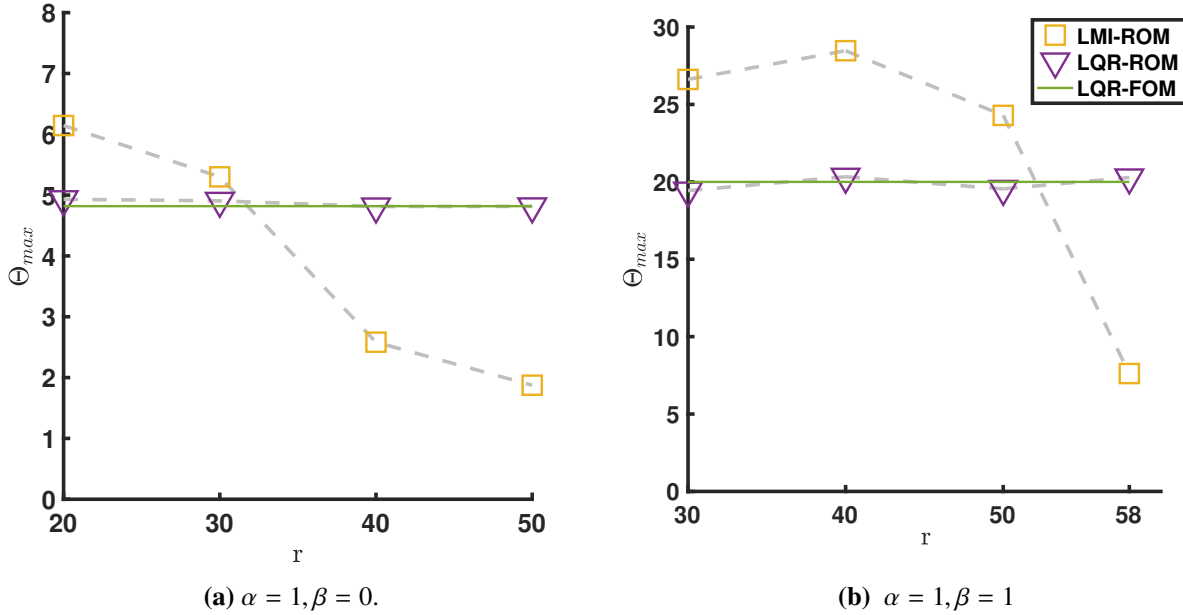


Figure 4.4: Closed-loop MTEG Θ_{max} as a function of ROM order r at $Re = 3000$.

It is clear that the LMI-ROM controller in the above cases outperforms both of the LQR controllers. In order to investigate this further, we proceed to analyze the details of the wall-normal actuation and its influence on the flow perturbations. Figure 4.5 shows the actuated wall-normal velocity at the upper- and lower-walls— q_u and q_l , respectively—for each controller. The case of $(\alpha, \beta) = (1, 0)$, shown in Figure 4.5a, reveals that LMI-ROM controller results in blowing and suction at the upper- and lower-walls that are equal in magnitude, but opposite in direction. In contrast, each of the LQR-ROM and the LQR-FOM controllers produces almost identical (both in magnitude and direction) blowing and suction at the upper- and lower-walls. Interestingly, the LQR-ROM controls differ from the LQR-FOM controls for a short period at the beginning of the response, which is an expected artifact of the modal truncation. The 5% settling time on the actuation for the LMI-ROM controllers is ≈ 43 convective time units; the LQR controllers settle in ≈ 34 convective time units, indicating a shorter duration of control.

In the oblique wave case of $(\alpha, \beta) = (1, 1)$, shown in Figure 4.6a, the LMI-ROM controller results in a maximum wall-normal blowing/suction velocity that is ≈ 1.6 times lower than either of the LQR controllers. From Figure 4.6a we find that the q_u produces a control input of larger magnitude compared to q_l , i.e., the actuator on the upper-wall is inducing a larger velocity compared to the actuator on the lower-wall. In Figures 4.6b and 4.6c the upper-wall and lower-wall actuators produce a velocity of similar magnitude, but opposite directions relative to each other. In the oblique wave case, the 5% settling time of the actuation signal for the LMI-ROM controller is ≈ 41 convective time units, while the LQR-FOM and LQR-ROM each have a settling time of ≈ 37

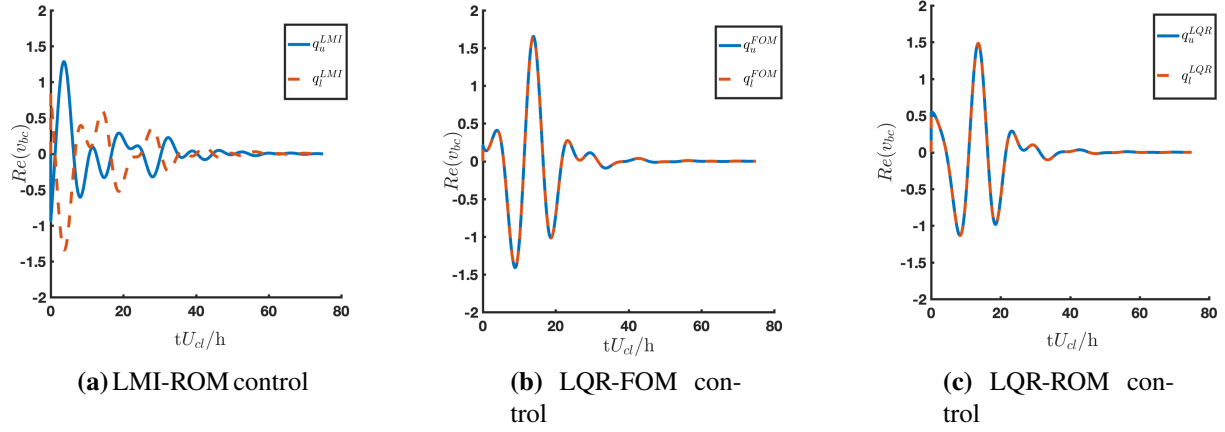


Figure 4.5: Actuated wall-normal velocity for $r = 40$, $(\alpha, \beta) = (1, 0)$, and $Re = 3000$.

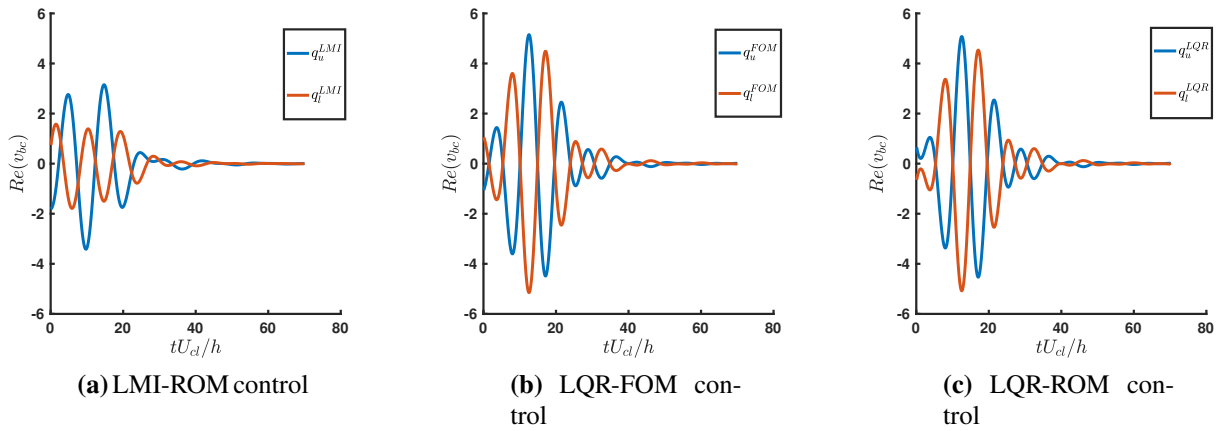


Figure 4.6: Actuated wall-normal velocity for $r = 58$, $(\alpha, \beta) = (1, 1)$, and $Re = 3000$.

convective time units. In the spanwise waves setting, shown in Figure 4.7, the LMI-ROM controller actuates the system similarly to both of the LQR controllers, but with a lower initial magnitude. For all controllers, wall-normal transpiration at the upper-wall is identical to that at the lower-wall (see Figure 4.7).

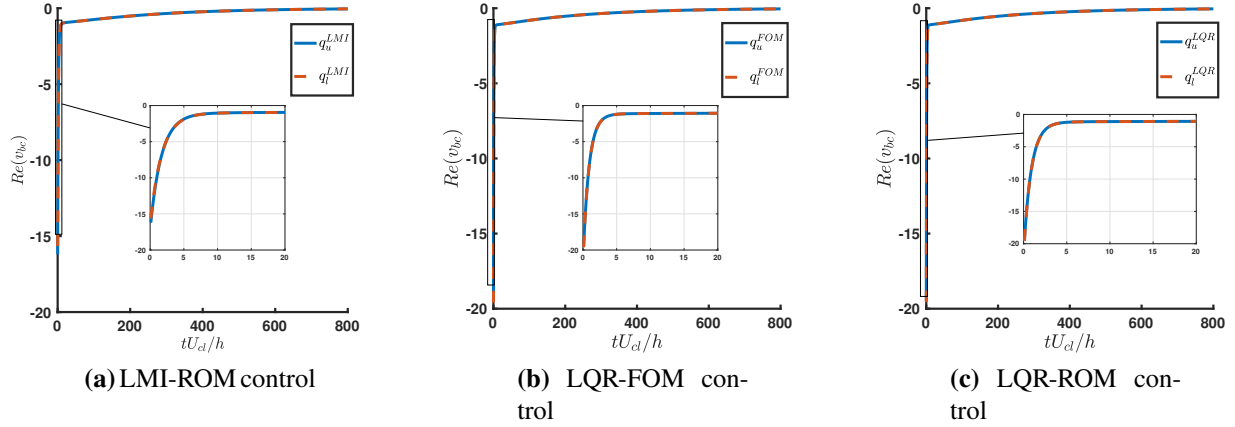


Figure 4.7: Actuated wall-normal velocity for $r = 40$, $(\alpha, \beta) = (0, 2)$, and $Re = 3000$.

Next, we examine the evolution of perturbations in streamwise velocity (u) to analyze the effect of the different controllers on the flow response. Figures 4.8 – 4.10 show the evolution of normalized u perturbations. In these figures, the initial optimal perturbation has been normalized such that $E(0) = 1$. Figure 4.8 shows the response for the case of streamwise waves. Note that the LMI-ROM controller has a streamwise perturbation profile different from that of the LQR-ROM and the LQR-FOM. The u perturbations die out faster with the LMI-ROM controller than with either of the LQR controllers. The LQR-FOM and the LQR-ROM result in a similar evolution of u perturbation, as is to be expected from the similarity in TEG profiles we noted earlier in Figure 4.3a. In the case of oblique waves (see Figure 4.9), the LQR-ROM and LQR-FOM again yield a similar response in u -perturbations. In contrast to these responses, the LMI-ROM controller results in a reduced magnitude of u at the lower walls of the channel. Again, the LMI-ROM results in a faster decay of u perturbations than the LQR-ROM and LQR-FOM controllers. Finally, in the case of spanwise waves (see Figure 4.10), all the three controllers yield similar responses in u .

Finally, we repeat our study for other Re values to ensure that the ROMs and controllers can be used in other settings as well. ROMs are developed and tuned to converge for $Re = 1000$, 5000 , and $10,000$. Then, we follow the same procedure for control synthesis as in the $Re = 3000$ cases described earlier. Note that for the case of $Re = 10,000$, the flow is linearly unstable; thus, the balanced truncation procedure for the model reduction is only performed on the stable subspace of the linearized dynamics. Figures 4.11 and 4.12 report performance results for the cases of $(\alpha, \beta) = (1, 0)$ and $(1, 1)$, respectively. The results clearly show that the LMI-ROM controllers reduce TEG to a greater extent than the LQR controllers. However, with the increase in Re , the closed-loop response to the LMI-ROM control exhibits oscillations on a transient time scale, on the order of approximately 10^{-2} to 10^{-4} convective time units. The transient oscillations appear to be a property of the closed-loop system, which has a set of eigenvalues with large imaginary parts—on the order of 10^4 —that lead to lightly damped oscillations. Further investigation is necessary to

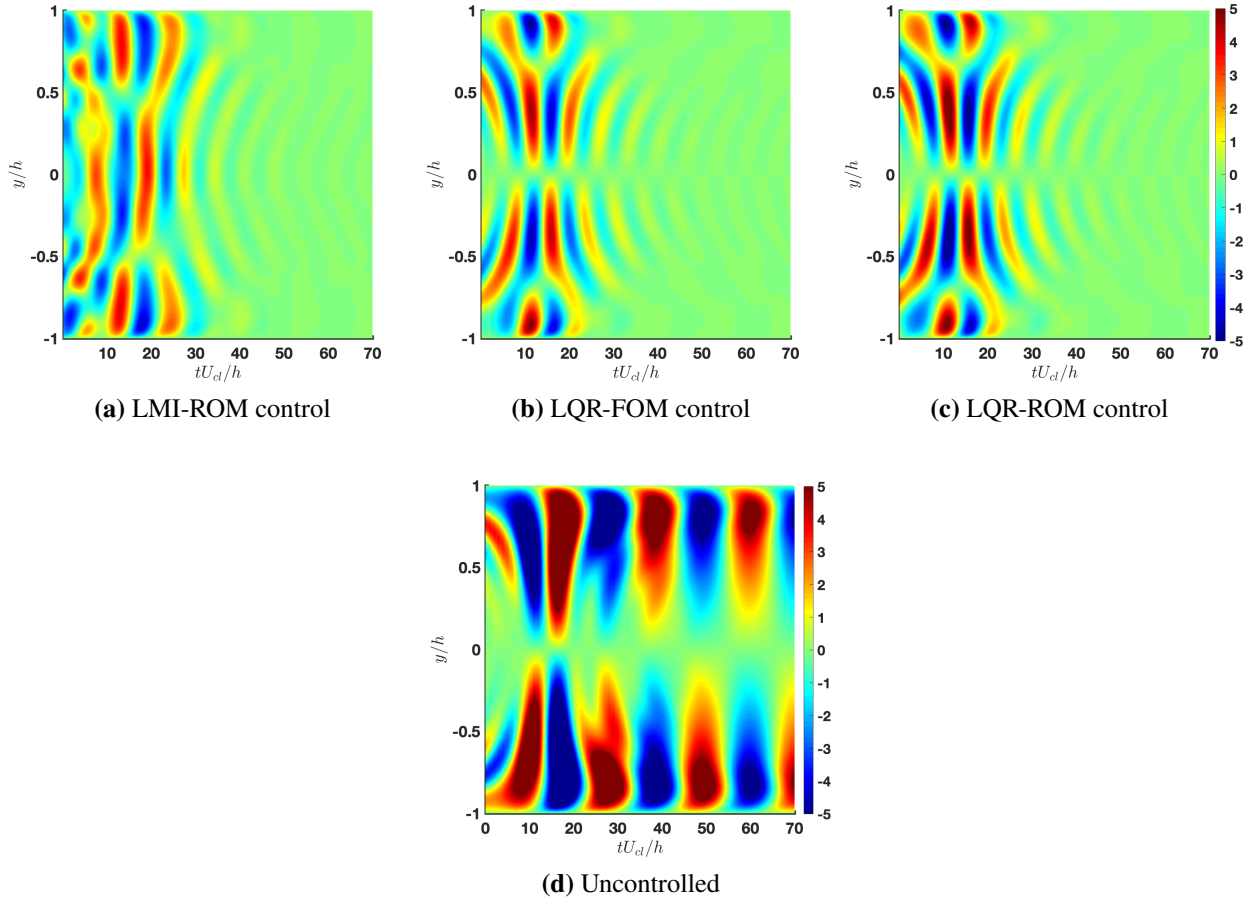


Figure 4.8: Evolution of streamwise velocity perturbations (u) for $r = 40$, $(\alpha, \beta) = (1, 0)$, and $Re = 3000$.

determine the underlying cause of these transient oscillations and the observed spectral properties of the closed-loop system.

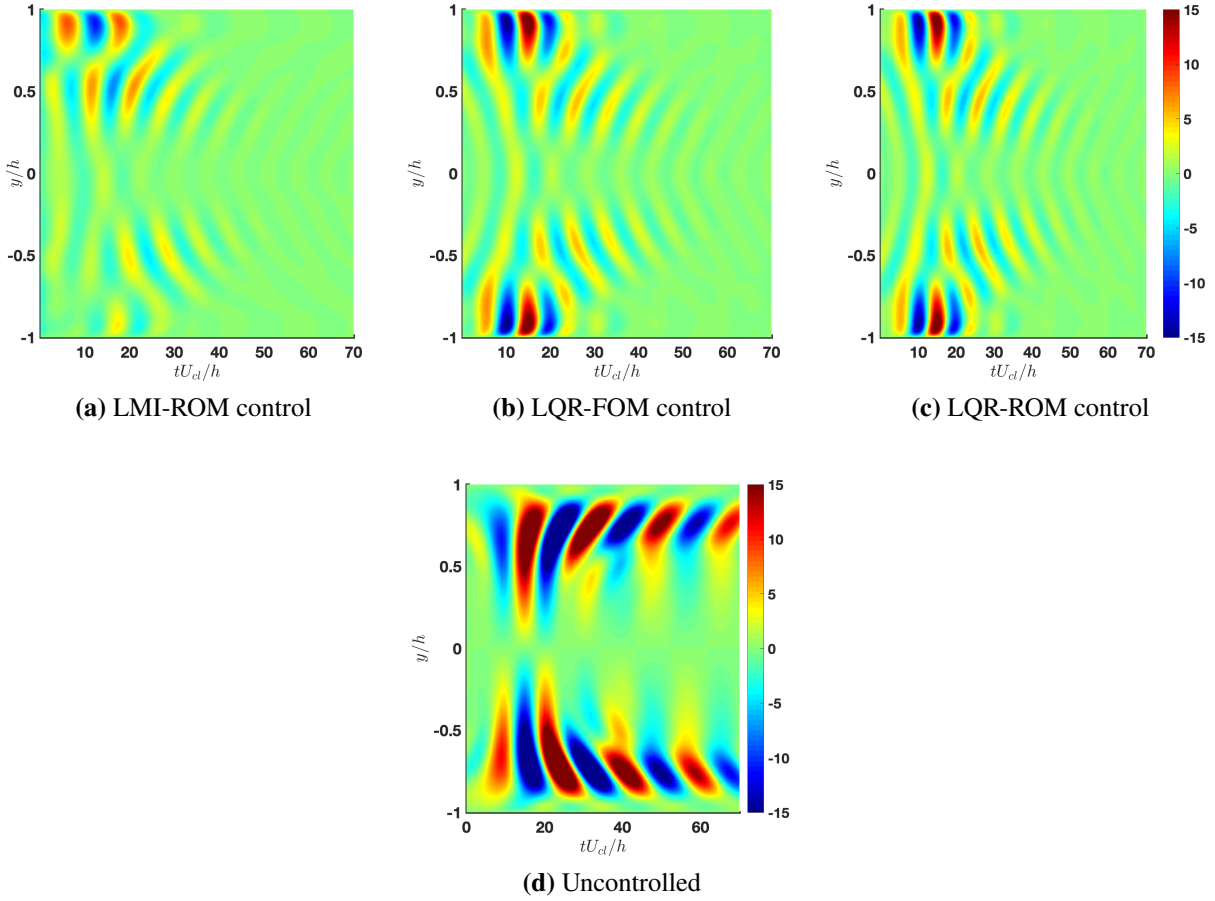


Figure 4.9: Evolution of streamwise velocity perturbations (u) for $r = 58$, $(\alpha, \beta) = (1, 1)$, and $Re = 3000$.

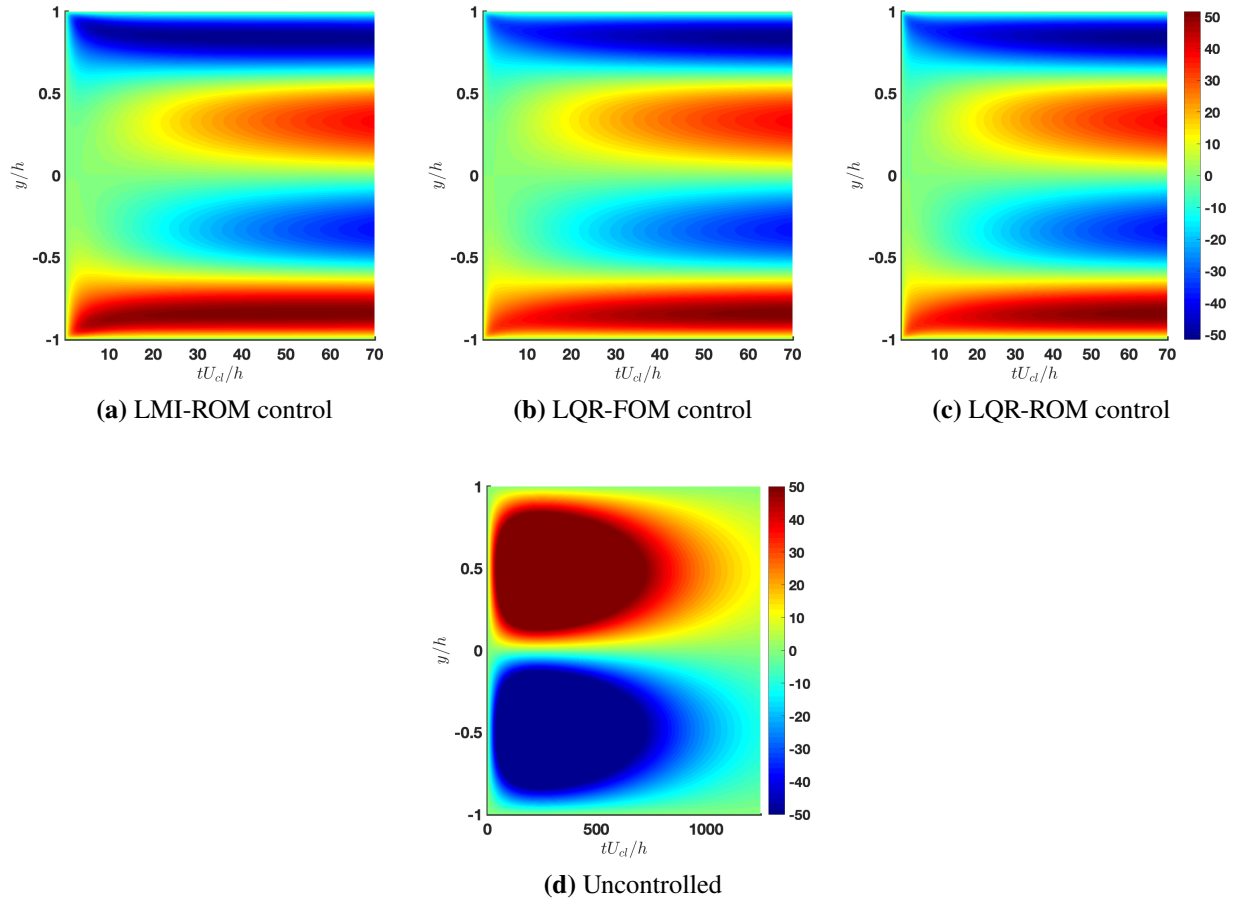


Figure 4.10: Evolution of streamwise velocity perturbations (u) for $r = 40$, $(\alpha, \beta) = (0, 2)$, and $Re = 3000$.

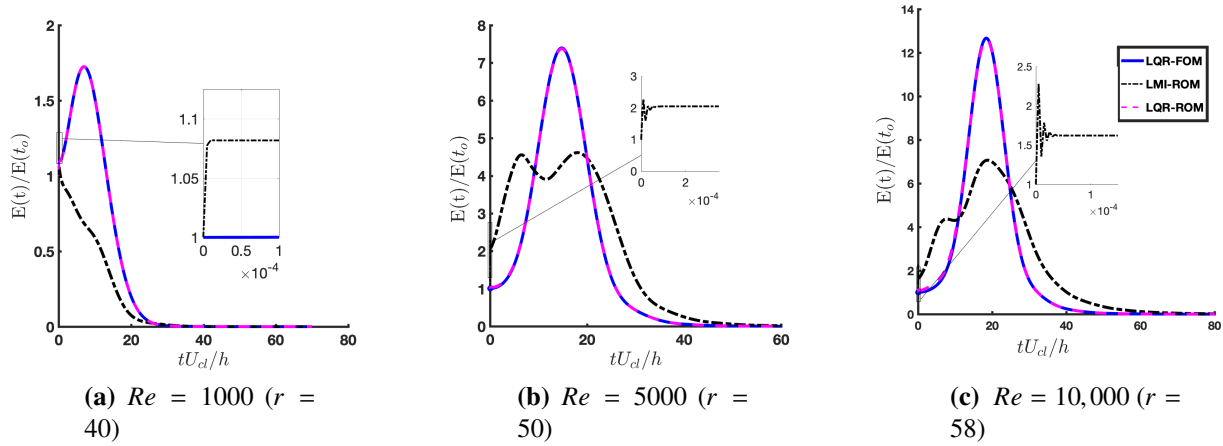


Figure 4.11: Worst-case closed-loop TEG responses for $(\alpha, \beta) = (1, 0)$.

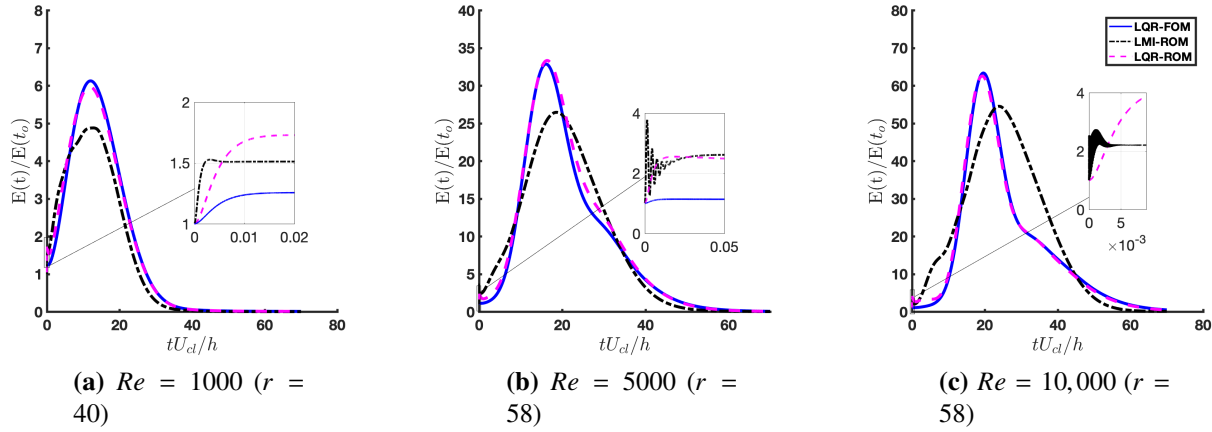


Figure 4.12: Worst-case closed-loop TEG responses for $(\alpha, \beta) = (1, 1)$.

4.4 Conclusions

In this paper, we have investigated the use of control-oriented ROMs for designing feedback controllers that minimize the MTEG of flow perturbations within a linearized channel flow. The ROMs were formed using an output projection onto POD modes, followed by a balanced truncation procedure to reduce the state dimension. POD modes were chosen in order to best approximate the perturbation energy, as needed for representing the objective function for control. Note that this output projection does not alter the state dimension of the system. As such, a balanced truncation was performed to reduce the state dimension, retaining only state variables that were essential to preserving the system's input-output dynamics. ROMs of the linearized channel flow system enabled the synthesis of MTEG-minimizing controllers through the solution of an LMI problem that would otherwise have been computationally prohibitive. Specifically, the dimension of the full-order model ($n = 199$) was reduced to yield ROMs with order $r = 40 - 60$, depending on the specific configuration. This constitutes a significant reduction in the computational demand for the subsequent LMI-based controller synthesis, since the computational requirements for LMI-based controller synthesis scale as $O(n^6)$. Further, the MTEG-minimizing controllers designed using the

proposed ROMs were found to outperform LQR controllers in suppressing TEG, even when these LQR controllers were designed based on the full-order system model. Although not explicitly reported here, the present investigation revealed that ROM-based MTEG-minimizing controllers can be sensitive to Re variations, leading to linearly unstable closed-loop systems when applied in “off-design” settings. As such, future investigations will need to focus on addressing these fragilities. It is expected that both the model reduction and controller synthesis approaches will need to be re-formulated to robustly minimize MTEG.

5 | Performance Limitations of Observer-Based Feedback Control

5.1 Introduction

Many previous studies on transition and TEG control have relied upon the well-established separation principle at some stage in the synthesis of a observer-based feedback controllers [19, 54, 55, 84, 85]—a form of sensor-based output feedback. The separation principle is commonly invoked in this design process because it considerably simplifies controller synthesis when a dynamic output feedback law is desired: a stabilizing full-state feedback controller can be synthesized independently of a stable state estimator, then combined together to yield a stabilizing dynamic output feedback compensator [86]. Despite guarantees on linear stability of the closed-loop system, invoking the separation principle can dramatically degrade closed-loop performance. Yet, the adverse consequences of controller synthesis by means of the separation principle are not fully appreciated in the context of transition delay and transient energy growth control, in which closed-loop performance is paramount. For instance, the separation principle is central to linear quadratic Gaussian (LQG) control, which remains a common controller synthesis approach for transient energy growth reduction and transition delay [19, 54, 55, 84, 85]. Although degraded closed-loop performance of observer-based feedback controllers has been reported in the literature [15], no previous studies have explicitly identified the separation principle as the source of these performance limitations. The state estimation problem is sometimes identified to be “the primary pacing item” for realizing acceptable closed-loop performance [19]; however, as we will show here, *observer peaking* and associated performance issues are more deeply rooted with the separation principle itself.

In this chapter, we will show that any linear system that exhibits transient energy growth to disturbances in open-loop will also exhibit transient energy growth to disturbances in closed-loop whenever the separation principle is invoked for controller synthesis. Indeed, even when a full-state feedback controller can fully suppress transient energy growth, a dynamic output feedback compensator designed via the separation principle will invariably lead to non-trivial transient energy growth in response to some disturbances. Further, the separation principle will guarantee that the estimator dynamics (and estimation error) will exhibit peaking in closed-loop from some initial conditions whenever the uncontrolled plant exhibits transient energy growth. The results in this chapter highlight the inherent performance limitations that arise by invoking the separation principle for observer-based feedback control in the context of transition control and transient energy growth suppression. As we will show via example, not all output feedback control approaches are restricted to the same performance limitations as observer-based feedback; indeed,

alternative output feedback approaches should be considered as candidates for controller synthesis when closed-loop performance is a primary design objective. Materials in this chapter are drawn heavily from [87].

5.2 Observer-based feedback and maximum transient energy growth

Consider the linear time-invariant system,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t),\end{aligned}\tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector, and $t \in \mathbb{R}$ is time. In considering transient energy growth in this study, we focus on the free response of the system to an initial perturbation $x(t_o) = x_o$ away from an equilibrium solution (e.g., a laminar base flow). The perturbation dynamics are given in terms of the matrix exponential, $x(t) = e^{A(t-t_o)}x_o$, and the associated perturbation energy will have a response given by,

$$E(t) = x^\top(t)Qx(t),\tag{5.2}$$

where $Q = Q^\top > 0$. Without loss of generality, we take $Q = I$, since the state can always be transformed as $\bar{x} = Q^{1/2}x$ to satisfy this definition of energy. Further, define the maximum transient energy growth G as,

$$G = \max_{t \geq t_o} \max_{E(t_o) \neq 0} \frac{E(t) - E(t_o)}{E(t_o)} = \max_{t \geq t_o} \max_{\|x_o\|=1} E(t) - 1 \geq 0.\tag{5.3}$$

When the system in (5.1) is unstable, $E(t)$ will be unbounded and G will be infinite; when the system in (5.1) is stable, G is simply the peak value in transient energy growth due to a so-called optimal disturbance [2]. The lower-bound $G = 0$ corresponds to the case of suppressed transient energy growth (i.e., monotonic stability [25]), which is the ultimate aim of controllers designed for transient energy growth suppression.

In light of these definitions, we introduce a lemma (herein, referred to as the *MTEG Lemma*) that will be central to the analysis here: $G = 0$ if and only if $A^\top + A \leq 0$ [88]. To show sufficiency, consider that if $\dot{E}(t) \leq 0$ for all $t \geq t_o$ and all initial conditions, then $G = 0$. Since $\dot{E}(t) = x^\top(t)(A^\top + A)x(t)$, it follows that $A^\top + A \leq 0$ is a sufficient condition for $G = 0$. Necessity can be shown by noting that $(A^\top + A) \not\leq 0$ implies the existence of an initial perturbation $x(t_o) = x_o$ that yields $\dot{E}(t_o) > 0$; thus, $E(t) > E(t_o)$ for some time $t > t_o$, and so $G > 0$. Therefore, $(A^\top + A) \leq 0$ is a necessary condition for $G = 0$.

The maximum transient energy growth can be reduced or suppressed by altering the system response characteristics via appropriate actuation $u(t)$. In principle, a full-state feedback control law $u(t) = -Kx(t)$ can be used to achieve various control objectives, including optimal regulation by means of a linear quadratic regulator (LQR) [86]. Here, the $K \in \mathbb{R}^{m \times n}$ is the controller feedback gain. The closed-loop dynamics of the associated stable full-state feedback system will then be

$$\dot{x}(t) = (A - BK)x(t).\tag{5.4}$$

However, in practice, the full-state $x(t)$ is typically not directly available for feedback. Instead, measurements of the system outputs $y(t)$ can be used to estimate the full-state by means of a stable state estimator of the form,

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L[y(t) - C\hat{x}(t)]\tag{5.5}$$

where $\hat{x}(t)$ is the state estimate. The observer gain $L \in \mathbb{R}^{n \times p}$ is chosen to yield desirable estimator performance, including adequate convergence rates and reliability in the face of both process and measurement uncertainties, as in the case of the optimal state estimator (i.e., the Kalman-Bucy filter).

The well-known *separation principle* establishes that a stabilizing full-state feedback controller and a stable state estimator can be designed independently of one another, then combined to yield a stabilizing dynamic compensator by means of the observer-based feedback law $u(t) = -K\hat{x}(t)$. To see this, consider the dynamics of the closed-loop observer-plant system with state $\tilde{x}(t) = (x(t), \hat{x}(t)) \in \mathbb{R}^{2n}$,

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}. \quad (5.6)$$

The observer-plant system in (5.6) can be brought into a form that replaces the estimated state $\hat{x}(t)$ by the estimation error $e(t) = x(t) - \hat{x}(t)$ via similarity transformation [86],

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \quad (5.7)$$

Since the closed-loop operator in (5.7) appears in block-triangular form, the eigenvalues of the observer-based feedback system are simply the union of the eigenvalues of the full-state feedback system ($A - BK$) and the estimator ($A - LC$); indeed, since (5.6) and (5.7) are related by similarity transformation, this establishes the well-known separation principle. In the remainder, the term *observer-based feedback* will be used to refer to a closed-loop system as in (5.6), formed by means of the separation principle.

Although the separation principle provides guarantees on closed-loop *stability*, it does not provide any guarantees on closed-loop *performance*—an important point that is often overlooked in the context of transition control and transient energy growth suppression. Consider the closed-loop energy $\tilde{E}(t) = \tilde{x}^\top(t) \tilde{Q} \tilde{x}(t)$ of the observer-plant system, with maximum transient energy growth $\tilde{G}$. Here, the first $n \times n$ sub-block of $\tilde{Q}$ will be Q , to be consistent with the original state energy defined in (5.2). However, as with (5.2) and without loss of generality, we let $\tilde{Q} = I$ in the remainder. From the MTEG Lemma, the closed-loop operator $\tilde{A}$ will exhibit $\tilde{G} = 0$ if and only if $\tilde{A}^\top + \tilde{A} \leq 0$. Considering this more closely,

$$\tilde{A}^\top + \tilde{A} = \begin{bmatrix} A^\top + A & (LC)^\top - BK \\ LC - (BK)^\top & (A - BK - LC)^\top + (A - BK - LC) \end{bmatrix}, \quad (5.8)$$

reveals that $\tilde{G}$ will depend on the plant (A, B, C) as well as the compensator gains K and L . Further, since $(A^\top + A)$ is a principal sub-matrix of $(\tilde{A}^\top + \tilde{A})$, it follows that $A^\top + A \leq 0$ is a necessary condition for $\tilde{A}^\top + \tilde{A} \leq 0$ [89]; thus, again by the MTEG Lemma, $G = 0$ is a necessary condition for $\tilde{G} = 0$. Hence, in the context of transient energy growth suppression and transition control, for which the open-loop dynamics exhibit $G > 0$, using the separation principle will limit closed-loop performance by guaranteeing that $\tilde{G} > 0$ as well. Note that $G = 0$ —or, equivalently $A^\top + A \leq 0$ —is a necessary condition for $\tilde{G} = 0$, but not a sufficient condition; *all* principal sub-matrices must

be considered to establish $\tilde{A}^\top + \tilde{A} \leq 0$ [89]. As such, even if $G = 0$, $\tilde{G}$ will also depend on the particular system (A, B, C) and the choices of K and L , in general.

Since the maximum transient energy growth $\tilde{G}$ corresponds to the “cyber-physical” state $\tilde{x} = (x, \hat{x})$ of the observer-plant system, it appears possible that only one of either $x(t)$ or $\hat{x}(t)$ is contributing to the transient energy growth. Thus, we now establish the influence of the separation principle on the maximum transient energy growth $\tilde{G}_x$ of the physical plant and $\tilde{G}_{\hat{x}}$ of the observer. The maximum transient energy growth $\tilde{G}_x$ for the physical plant under control via the separation principle can be determined by first considering a modified energy $\tilde{E}_\epsilon(t) = \tilde{x}_\epsilon^\top(t) \tilde{x}_\epsilon(t)$, where $\tilde{x}_\epsilon = W_\epsilon \tilde{x}$, $W_\epsilon = \text{diag}(I_n, \epsilon I_n)$, I_n is the $n \times n$ identity matrix, and $\epsilon > 0$ is a scalar. Then, noting that $\lim_{\epsilon \rightarrow 0} \tilde{E}_\epsilon(t) = E(t)$ and that $\tilde{A}$ and $W_\epsilon = W_\epsilon^\top$ commute, it follows from the MTEG Lemma that $\tilde{G}_x = 0$ if and only if $\lim_{\epsilon \rightarrow 0} W_\epsilon(\tilde{A}^\top + \tilde{A})W_\epsilon \leq 0$. As before, $A^\top + A \leq 0$ is a necessary condition for $W_\epsilon(\tilde{A}^\top + \tilde{A})W_\epsilon \leq 0$, so we conclude that $A^\top + A \leq 0$ —and, therefore $G = 0$ —is a necessary condition for $\tilde{G}_x = 0$. That is, if $G > 0$, then it is guaranteed that $\tilde{G}_x > 0$ as well. The same can be said about the maximum transient energy growth $\tilde{G}_{\hat{x}}$ for the estimator, which corresponds to a case of observer peaking. To show this, consider the modified energy $\tilde{E}_\epsilon(t)$ defined using the coordinate transformation $W_\epsilon = \text{diag}(\epsilon I_n, I_n)$, then proceed as before.

In summary, $A^\top + A \leq 0$ —and therefore, $G = 0$ —is a necessary condition for $\tilde{G} = 0$, $\tilde{G}_x = 0$, and $\tilde{G}_{\hat{x}} = 0$. Thus, if the uncontrolled plant exhibits non-trivial maximum transient energy growth (i.e., $G > 0$), then it is guaranteed that the separation principle will result in non-trivial maximum transient energy growth in closed-loop for the physical plant (i.e., $\tilde{G}_x > 0$), the estimator (i.e., $\tilde{G}_{\hat{x}} > 0$), and the cyber-physical observer-plant system (i.e., $\tilde{G} > 0$). The performance limitations identified here apply only to dynamic output feedback laws synthesized using the separation principle—i.e., observer-based feedback as in (5.6). Not all output feedback control laws are necessarily subject to these same limitations. As we will see in the illustrative example that follows, even when a static output feedback law can be determined to achieve zero maximum transient energy growth—a necessary condition for guaranteeing the existence of a dynamic output feedback compensator that can achieve the same [88]—the separation principle will not (and cannot) yield a closed-loop system that achieves the same if the uncontrolled plant exhibits $G > 0$.

5.3 Illustrative example: A simple non-normal system

Consider the system,

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1/R & 0 \\ 1 & -2/R \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned} \quad (5.9)$$

where $R > 0$ is a scalar parameter. Variants of this system are often used to demonstrate the role of non-normality in giving rise to transient energy growth [5, 25, 32, 88]. The origin is asymptotically stable, but the system exhibits $G > 0$ for $R > R^* = 2\sqrt{2}$. In Figure 5.1, we compare the worst-case transient energy response $E(t)$ with $R = 2 < R^*$ and $R = 3 > R^*$ for the uncontrolled plant with the corresponding worst-case closed-loop system response from each of three different controller synthesis techniques: LQR, LQG, and static output feedback (SOF). In all cases, the optimal disturbance for the worst-case response is computed by means of the iterative Algorithm 3.1 of Whidborne and Amar [83]. In the case of LQG control, the worst-case response corresponds to an

optimal disturbance on the full observer-plant state $\tilde{x} = (x, \hat{x})$. In instances for which the maximum transient energy growth is zero, there is no optimal disturbance and so the initial condition is set to either $x(t_o) = (1, 0)$ or $\tilde{x}(t_o) = (1, 0, 0, 0)$. All energy responses in Figure 5.1 correspond to the energy of the physical plant $E(t)$ normalized by the initial energy $E(t_o) = E_o$.

The LQR design here yields a full-state feedback law that minimizes the performance index,

$$J = \int_0^\infty x^\top(t)x(t) + u^\top(t)u(t)dt \quad (5.10)$$

subject to the linear dynamic constraint $\dot{x}(t) = Ax(t) + Bu(t)$. In the present study, the LQR control gain is computed directly via the Matlab command `lqr`. The response under the LQR full-state feedback control yields $G = 0$ for both $R = 2$ and $R = 3$, as seen in Figure 5.1. Note that LQR control is not guaranteed to yield zero transient energy growth in general; rather, for this example, the objective function in (5.10) was specifically selected after tuning to achieve zero maximum transient energy growth for the closed-loop response for both $R = 2$ and $R = 3$.

The separation principle is invoked for LQG optimal controller synthesis. The same LQR control gains determined by (5.10) are used in an observer-based feedback capacity (i.e., $u(t) = -K\hat{x}(t)$). The observer gain L is computed via the Matlab command `lqe`, which computes a solution to the optimal estimation problem, which is dual to the optimal control problem. Here, the observer gain L was selected by tuning the estimator objective function to reduce $\tilde{G}_x$ associated with the closed-loop energy response. Recall, that $\tilde{G}_x$ —and the associated optimal disturbance—can be computed by considering $\lim_{\epsilon \rightarrow 0} \tilde{E}_\epsilon(t) = E(t)$. As expected, in the case of $R = 3 > R^*$, the LQG controller yields $\tilde{G}_x > 0$; whereas, the same LQG controller synthesis approach applied to the case of $R = 2 < R^*$ yields $\tilde{G}_x = 0$. Further, despite tuning to reduce $\tilde{G}_x$, the physical energy $E(t)$ grows to approximately 30 times its initial value. In fact, $\tilde{G}_x \approx 30G$, meaning the LQG controller degrades the transient energy growth performance relative to the uncontrolled (open-loop) response.

Lastly, we consider the static output feedback (SOF) control law $u(t) = -y(t)$. By the MTEG Lemma, the resulting closed-loop operator $(A - BC)$ is guaranteed to exhibit zero maximum transient energy growth for all values of $R > 0$. Note that the SOF controller considered here is one of a family of SOF controllers that can achieve zero transient energy growth [88].

The results of this simple example illustrate the limitations of the separation principle that were proved in Section 5.2. These results further suggest that controllers synthesized by the separation principle—LQG or otherwise—can be unreliable for transient energy growth suppression and transition control. Although the separation principle can greatly simplify controller synthesis in many instances, the performance consequences of the separation principle must also be taken into consideration. As shown in this example, alternative output feedback approaches are not necessarily subject to the same performance limitations and should be considered as well as—if not instead of—observer-based feedback.

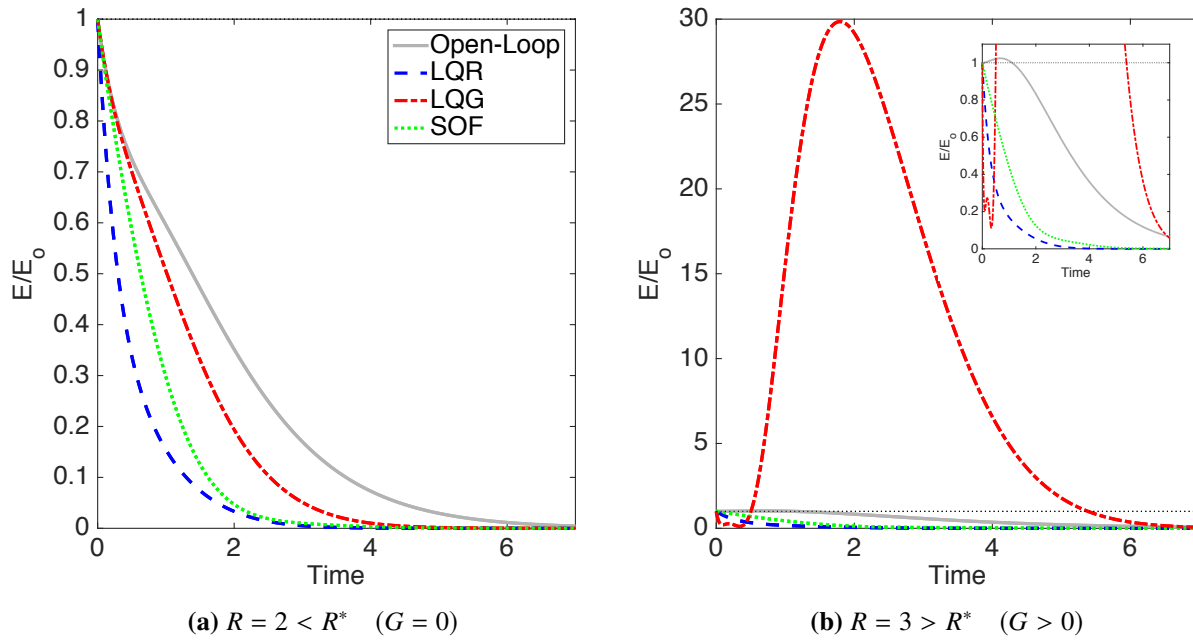


Figure 5.1: Comparison of worst-case responses for the controlled system in (5.9), for which $R^* = 2\sqrt{2}$.

5.4 Discussion and Conclusions

In this chapter, we have proven that if the maximum transient energy growth for an uncontrolled plant is non-trivial (i.e., $G > 0$), then controller synthesis by the separation principle will necessarily result in non-trivial maximum transient energy growth in closed-loop (i.e., $\tilde{G} > 0$, $\tilde{G}_x > 0$, and $\tilde{G}_{\hat{x}} > 0$). These results were established by invoking the MTEG Lemma and properties of negative semi-definite operators to show that $A^\top + A \leq 0$ —and thus $G = 0$ —is a necessary condition for $\tilde{G} = 0$, $\tilde{G}_x = 0$, and $\tilde{G}_{\hat{x}} = 0$. This result highlights a fundamental performance limitation of observer-based feedback control in the context of transient energy growth suppression and transition control. As illustrated in the example of Section 5.3, performance under full-state feedback is not necessarily a reliable indicator for closed-loop performance under observer-based feedback. Indeed, no amount of controller and estimator tuning can overcome the performance limitations of observer-based feedback. If closed-loop transient energy growth is of primary importance, then alternative approaches for output feedback control may yield better performance in this regard and should also be considered as candidate control approaches. Further, the result for $\tilde{G}_{\hat{x}}$ establishes that the separation principle will lead to observer peaking from some initial conditions in closed-loop whenever $A^\top + A \not\leq 0$ for the open-loop system. Thus, although observer peaking is commonly rooted out as a limiting factor on performance in observer-based feedback control, this result shows that observer peaking and the associated transient energy growth of the physical plant can actually be seen as direct consequences of using the separation principle to control a system for which $G > 0$, or equivalently $A^\top + A \not\leq 0$.

The results presented here can be generalized further. Note that all stabilizing controllers can be constructed by means of a Q -parameterization, in which a separation-principle-based controller/observer structure is combined with a free parameter $Q(s)$ [90]. Then, since the separation

principle yields a dynamic compensator that is strictly proper, it follows that when $Q(s)$ is strictly proper, the resulting parameterized dynamic compensator will also be strictly proper; in contrast, when $Q(s)$ is semi-proper, the resulting parameterized dynamic compensator will also be semi-proper. With this in mind, performing an analysis similar to that of Section 5.2 shows that all strictly proper stabilizing controllers will result in non-trivial maximum transient energy growth whenever $G > 0$. In contrast, it *may* be possible to achieve zero maximum transient energy growth in closed-loop when the dynamic compensator is semi-proper—indeed, this corresponds to the analogous necessary condition for the generalized result. For example, static output feedback control constitutes a semi-proper control structure, which was seen to fully suppress transient energy growth in the illustrative example of this study. Interestingly, the existence of a static output feedback controller that achieves zero maximum transient energy growth is a necessary condition for the existence of a dynamic compensator that can achieve the same [88]. Further, as shown in [88], a Q -parameterization can be used to design controllers that minimize the maximum transient energy growth.

Retrospectively, the performance limitations of observer-based feedback for transient energy growth suppression are not entirely surprising. Consider that the closed-loop operator in (5.7) is non-normal. That non-normality is a necessary condition for transient energy growth is well-established in the flow control community; it is the high degree of non-normality of the linearized Navier-Stokes operator that is commonly attributed to transient energy growth in the context of sub-critical transition in shear flows [25]. And yet, the non-normality of the closed-loop operator in (5.7) seems to have evaded attention in many flow control studies. An important ramification of this non-normality is that even if a full-state feedback law can fully remove non-normality of the physical plant (as in [32]), invoking the separation principle to synthesize an observer-based feedback law will inevitably yield a non-normal cyber-physical observer-plant system. In this case, although the modes of the physical system ($A - BK$) will still be orthogonal, the modes of the coupled observer-plant system will be oblique, since the associated operator is non-normal. Further, the degree of non-normality will depend on the specific plant (A, B, C) and the particular choice of feedback and observer gains, K and L , respectively. Thus, K and L can be tuned to improve performance for a given flow control configuration; yet, if $A^T + A \not\leq 0$, then $\tilde{G} = 0$, $\tilde{G}_x = 0$, and $\tilde{G}_{\hat{x}} = 0$ will never be achieved. Further, the performance limitations of the separation principle presented here hold true regardless of the system terms (B, C). Thus, although efforts at optimal actuator/sensor placement and selection can be useful at reducing the maximum transient energy growth, such efforts will never yield $\tilde{G} = 0$, $\tilde{G}_x = 0$, or $\tilde{G}_{\hat{x}} = 0$ when $A^T + A \not\leq 0$, if observer-based feedback is used for controller synthesis. Thus, to overcome these performance limitations, alternatives to observer-based feedback control structures need to be considered as candidates for control. Static output feedback control structures provide a convenient starting point, since they satisfy the necessary conditions for TEG elimination established here. Chapter 6 further investigates the observer-based feedback and static output feedback control approaches for TEG reduction in a linearized channel flow.

6 | Static Output Feedback Control for Reducing Transient Energy Growth

6.1 Introduction

As we have already seen, feedback controllers can be designed to reduce TEG in shear flows. Full-state feedback is usually the first choice for feasibility studies in numerical simulations. However, in practical applications, full-state feedback cannot be achieved because measurements of the full state are not directly available for feedback. Thus, in practice, only measured outputs from a limited set of sensors—typically confined to solid boundaries—can be used for output feedback [86]. One commonly used approach for output feedback design is to first estimate the full state of the flow from measured outputs using an observer, and then to use the estimated state for feedback through a full-state feedback law—i.e., observer-based feedback. The separation principle of modern control theory establishes that observer-based feedback will result in a stable closed-loop system if the full-state feedback controller is stabilizing and the observer is stable [86]. This strategy is commonly used because it greatly simplifies the design process: the controller and estimator can be designed separately, then combined to guarantee closed-loop stability.

Unfortunately, as we have shown in Chapter 5 and in [87], controller synthesis based on the separation principle can lead to adverse consequences in terms of TEG performance. Specifically, if the uncontrolled system exhibits TEG, then observer-based designs are guaranteed to exhibit TEG from some disturbances as well. This result has fundamental significance to transition delay strategies, since observer-based feedback can never fully eliminate TEG if the uncontrolled flow exhibits TEG. In this chapter, we show that observer-based feedback control of a linearized channel flow results in significantly higher levels of TEG relative to full-state feedback. In fact, we find that observer-based feedback can even degrade TEG performance relative to the worst-case TEG response of the uncontrolled flow. These findings extend our previous results regarding the performance limitations of observer-based feedback presented in Chapter 5.

The performance limitations of observer-based feedback control in the context of TEG reduction and transition delay suggest that alternative strategies for output feedback are advisable. For example, static and dynamic output feedback controllers can be designed to minimize the maximum TEG [88, 91]. These approaches have been shown to yield reduced TEG in a number of illustrative examples; however, the synthesis techniques tend to be computationally demanding, and thus difficult to use in the context of fluid flow control—especially without the availability of sufficiently reliable reduced-order models. Further, the resulting controllers can lack robustness to model uncertainties and parameter variations, making these designs less likely to perform as desired when

applied in off-design conditions on actual fluid flows.

Here, we propose to use static output feedback (SOF) controllers for optimal regulation. Specifically, we solve the linear quadratic regulation (LQR) problem with an additional constraint that the control law have a SOF structure. As we will show, the resulting SOF-LQR controllers outperform observer-based designs and reduce the worst-case TEG relative to the uncontrolled flow. Indeed, SOF controllers constitute semi-proper control laws, thus satisfying a necessary condition for overcoming the TEG performance limitations of observer-based feedback [87]. We further introduce an accelerated gradient method—based on the Anderson-Moore algorithm [92]—for determining the unique SOF-LQR law, making the synthesis task computationally tractable for many shear flows. As we will show in the context of linearized channel flow, SOF-LQR controllers exhibit robust TEG reduction in the face of model uncertainties and parameter variations. As such, SOF-LQR stands as a viable candidate for transition delay via output feedback control in future studies.

This chapter is organized as follows: In Section 6.2, we summarize the shortcomings of observer-based feedback for TEG reduction and present an accelerated Anderson-Moore algorithm for designing SOF-LQR controllers that can overcome these limitations. In Section 6.3, a linearized channel flow is used to illustrate the performance benefits of SOF-LQR control in comparison to standard observer-based designs. In Section 6.4, we discuss the results and draw conclusions. Materials here are drawn heavily from [61] and [62].

6.2 Feedback flow control synthesis

6.2.1 Shortcomings of the separation principle for transient energy growth suppression

Consider the state-space representation of the linearized Navier-Stokes equations about a laminar equilibrium solution,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{6.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector, and $t \in \mathbb{R}$ is time. For an initial flow perturbation $x(t_0) = x_o$, the system response is given in terms of the matrix exponential $x(t) = e^{A(t-t_0)}x_o$. The associated perturbation energy is given as,

$$E(t) = x^T(t)Qx(t),\tag{6.2}$$

where $Q = Q^T > 0$. Further, the maximum TEG is defined as [87],

$$G = \max_{t \geq t_0} \max_{E(t_0) \neq 0} \frac{E(t) - E(t_0)}{E(t_0)},\tag{6.3}$$

which results from a so-called *worst-case* or *optimal disturbance* [2]. The perturbation energy will never exceed its initial value when $G = 0$. In contrast, certain perturbations will result in non-trivial TEG whenever $G > 0$.

Feedback controllers have been shown to reduce TEG in various shear flows. In particular, the linear quadratic regulator (LQR) is a well-known design technique that has proven to be successful at reducing TEG in previous flow control studies [19, 54, 55]. LQR synthesis is based on solving,

$$\min_{u(t)} J = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt\tag{6.4}$$

subject to the linear dynamic constraint

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (6.5)$$

where $R > 0$. The resulting LQR controller is a full-state feedback law of the form $u(t) = -Kx(t)$, which can be determined from the solution of an algebraic Riccati equation [86]. LQR controllers are particularly appealing because they can be tuned to reduce TEG, while demonstrating robustness to parametric and modeling uncertainties. However, outside of numerical simulations, standard full-state feedback LQR controllers are typically not practically viable for flow control; standard LQR control requires knowledge of the full state of the flow, which is usually not directly available for feedback in practice.

When full-state feedback is not a viable option, an observer (i.e., state estimator) is usually designed to estimate the current state of the flow from available sensor measurements $y(t)$. A linear quadratic estimator (LQE) is often used to estimate the state of the flow. LQE provides the optimal (minimum variance) estimate of the state from noisy sensor measurements and an uncertain process model. Supplying LQE state estimates to the LQR feedback law yields a linear quadratic Gaussian (LQG) control law, which results in a stable closed-loop system. Considering the error signal $e(t) = x(t) - \hat{x}(t)$ to be the difference between the flow state and the state estimate, the closed-loop observer-based dynamics can be expressed as,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \quad (6.6)$$

where L is the observer gain. The separation principle of modern control theory is straightforward to establish from this expression, since the eigenvalues of the closed-loop system $\tilde{A}$ are simply the union of the eigenvalues of $(A - BK)$ and $(A - LC)$; thus, the stability of the closed-loop system in (6.6) is guaranteed when the controller gain K is stabilizing (i.e., $A - BK$ is stable) and the estimator is stable (i.e., $A - LC$ is stable) [86]. Note, this result is general and is not restricted to linear quadratic synthesis.

We emphasize here that the separation principle only guarantees closed-loop stability, *not* closed-loop performance. When closed-loop performance is a prime consideration, the separation principle must be invoked with caution; indeed, in the context of TEG suppression, the separation principle tends to be a poor design choice. Consider that TEG is typically attributed to being a byproduct of system non-normality [5, 25]. Then, upon inspection of $\tilde{A}$ in (6.6), it is clear that observer-based feedback yields a non-normal closed-loop system structure (i.e., $\tilde{A}\tilde{A}^T \neq \tilde{A}^T\tilde{A}$), creating the possibility of TEG. Indeed, as we proved in Chapter 5, invoking the separation principle for observer-based feedback design is guaranteed to result in TEG whenever the uncontrolled system exhibits TEG. The growth in this case can still be viewed as a byproduct of system non-normality, but now due to an adverse “cyber-physical” interaction between the control system dynamics and the fluid dynamics [87]. In fact, even if the modes of the physical system under full-state feedback $(A - BK)$ can be orthogonalized [32] (see Chapter 3), invoking the separation principle for observer-based control guarantees that the closed-loop system in (6.6) will be non-normal.

Given these observations, it is reasonable to expect that separation-principle-based designs will never outperform—in terms of TEG—the corresponding full-state feedback controllers from

which they are designed. In this chapter, we report results within the context of controlling a linearized channel flow that support this hypothesis. Indeed, observer-based feedback degrades TEG performance and results in a higher degree of TEG than the associated full-state feedback controller. Furthermore, in the channel flow study, the performance degradation is so dramatic that TEG increases even relative to the worst-case TEG associated with the uncontrolled system. While it may be possible to improve TEG performance via further tuning, such a tuning process must be conducted on both the controller and estimator simultaneously; unfortunately, the separation principle cannot be invoked to further simplify compensator design within the context of TEG control. In the next section, we introduce an alternative static output feedback strategy in an effort to overcome the performance limitations of the separation principle and associated observer-based designs.

6.2.2 Optimal static output feedback synthesis

Given the shortcomings of observer-based feedback described above, we propose an alternate strategy for TEG reduction based on the standard LQR problem introduced in Section 6.2.1, but now with an additional constraint that the resulting feedback law is to have a static output feedback (SOF) structure,

$$u(t) = Fy(t), \quad F \in \mathbb{R}^{m \times p}. \quad (6.7)$$

The benefit of the SOF control structure is that the input is determined directly from the measured output, removing the need for an observer. However, this benefit comes at the cost of a slightly more involved controller synthesis procedure, which we describe next.

The closed-loop dynamics under SOF-LQR control are of the form,

$$\dot{x} = (A + BFC)x. \quad (6.8)$$

Thus, the standard LQR objective function in (6.4) can be rewritten to conform to a SOF structure,

$$J = \int_0^\infty x^T(t)[Q + (FC)^T R(FC)]x(t)dt. \quad (6.9)$$

The solution of this SOF-LQR problem can be calculated iteratively using Anderson-Moore methods [92]. To do so, define the set of all stabilizing SOF controllers $D_s = \{F \in \mathbb{R}^{m \times p} | \text{Re}\{\lambda(A + BFC)\} < 0\}$. Then, re-write the SOF-LQR design problem as [93],

$$\text{Minimize } J(F) = \text{tr}[S(F)X] \quad \text{subject to } F \in D_s, \quad (6.10)$$

where $S(F)$ is a solution to

$$S(F)[A + BFC] + [A + BFC]^T S(F) + C^T F^T R F C + Q = 0. \quad (6.11)$$

and $X = x(0)x(0)^T$ [94]. The gradient of the cost function J with respect to the SOF control gain F can be expressed as,

$$\frac{\partial J}{\partial F} = 2[B^T S(F)L(F)C^T + R F C L(F)C^T] \quad (6.12)$$

where $L(F)$ is the solution to the Lyapunov equation,

$$L(F)[A + BFC]^T + [A + BFC]L(F) + X = 0. \quad (6.13)$$

Then, the optimal SOF gain $F^* \in D_s$ will be a minimizer of (6.10) and so must satisfy a zero gradient conditions, which reduced to

$$[B^T S(F^*)L(F^*)C^T + RF^*CL(F^*)C^T] = 0. \quad (6.14)$$

After some further manipulation, we find that a necessary condition for optimality is

$$F = -R^{-1}[B^T S(F)L(F)C^T][CL(F)C^T]^{-1}, \quad (6.15)$$

yielding a search direction for use in an Anderson-Moore method,

$$T = -F - R^{-1}[B^T S(F)L(F)C^T][CL(F)C^T]^{-1}. \quad (6.16)$$

Although the expressions above are sufficient for implementing the Anderson-Moore method, such methods tend to require a significantly large number of iterations. For high-dimensional fluid flows, each iteration can require a significant computational demand, and so it is desirable to reduce the total number of iterations through an accelerated technique. The step-size α along the gradient direction must be chosen with care, in order to balance precision with the total number of iterations. An inappropriate choice of α will either lead to slow convergence or to missing the optimal solution. In order to overcome this challenge, we formulate an accelerated Anderson-Moore algorithm that incorporates Armijo-type adaptations. Instead of using a fixed step-size α , we instead use an Armijo-rule [46] to adaptively update the step-size to achieve a better balance between precision and iteration count. A similar method to the one we propose here was originally presented in [95]; however, during our investigation on controlling linearized channel flow, we found that the specific algorithm in [95] would converge to unstable SOF control laws, thus violating the condition that $F^* \in D_s$. As such, we modified the approach to ensure convergence to a stabilizing optimal control law for the linearized channel flow configuration. Although our resulting method exhibits slower convergence than the method proposed in [95], it still outperforms the standard Anderson-Moore algorithm in terms of iteration count. Furthermore, the specific Armijo-type adaptation we propose guarantees convergence to a stabilizing SOF control law, and thus provides the optimal SOF-LQR solution for our flow control problem. The method we propose and use in this study is summarized below.

Anderson-Moore algorithm with Armijo-type adaptation

- step 0:** Initialize F_i to be any $F \in D_s$. Set $0 < \alpha < 1$, $0 < \sigma < 1/2$, and $\delta > 0$.
step 1: Solve (6.11) for $S(F_i)$.
step 2: Solve (6.13) for $L(F_i)$.
step 3: Use (6.16) to find the smallest integer $m_1 \geq 1$ such that $F_i + \alpha^{m_1} T_i \in D_s$.
step 4: Find the smallest integer $m_2 \geq m_1$ such that

$$J(F_i + \alpha^{m_2} T_i) \leq J(F_i) + \sigma \alpha^{m_2} \text{tr} \left(\frac{\partial J}{\partial F} T_i \right).$$

- step 5:** Find integer $n \in \{m_1, \dots, m_2\}$ such that

$$J(F_i + \alpha^n T_i) = \min J(F_i + \alpha^j T_i), \text{ where } j \in \{m_1, \dots, m_2\}$$

- step 6:** Set $F_{i+1} = F_i + \alpha^n T_i$, $i = i + 1$

- step 7:** Check $\|\frac{\partial J}{\partial F}\|_2 \leq \delta$, if true, stop. Otherwise, go to **step 1**.

Note that all Anderson-Moore methods require initialization with a stabilizing SOF gain $F \in D_s$ [95]. For an asymptotically stable system, setting $F_0 = 0$ is a valid choice; however, if the linearized channel flow is not asymptotically stable, then setting $F_0 = 0$ as a starting value causes the algorithm to fail. In the present study, actuator dynamics are modeled via an integral term, and so the resulting linear system model will not be asymptotically stable. Thus, we first determine a stabilizing static output feedback gain using the ILMI method proposed in [96], then proceed to compute the optimal SOF-LQR controller using the accelerated Anderson-Moore algorithm above.

6.3 Results

The controller synthesis approaches described above will be applied to a linearized channel flow. Blowing and suction actuation is applied at the upper- and lower-walls of the channel, with the blowing and suction rate at each wall serving as a system input. Shear-stress measurements at the upper- and lower-walls are used for output feedback. The channel flow dynamics are modeled by means of a Fourier-Fourier-Chebyshev spectral collocation method. In the present study, all results correspond to the streamwise and spanwise wave number pair $(\alpha, \beta) = (0, 1)$ for $Re = 3000$. Further details of this model are described in Section 4.3.1.

In studying control system performance, we will primarily consider the worst-case response of the system by determining the associated optimal disturbance x_o that leads to the maximum TEG. Here, the optimal disturbance is calculated by means of the method described in [83]. In the context of observer-based LQG control, some additional care is required in performing these calculations, since we are interested in the worst-case disturbance that maximizes the physical energy of the system. Since the observer-based feedback system has a “cyber-physical” state $\bar{x} = (x, \hat{x})$, we define the physical energy in terms of a weighting matrix $W_\epsilon = \text{diag}(Q, \epsilon Q)$, where Q is the same weighting matrix specified in equation (6.2). Then, by choosing a sufficiently small ϵ , the optimal disturbance calculated with weighting matrix W_ϵ will approximate the optimal disturbance that maximized the physical TEG of the system. Further, note that the state estimates for these optimal disturbances tend to be several orders of magnitude larger than the physical states of the system. Although it is possible to encounter such disturbances in practice, we opt to consider the response of the LQG controller to an alternative *sub-optimal disturbance* in the results presented here. After

computing the optimal disturbance for the LQG controller, we retain the physical disturbance from this solution, but initialize the estimator states to zero—as is commonly done in practice. Thus, the LQG responses presented here are not the worst-case responses; whereas, all other responses reported here do correspond to the worst-case.

Results from different control approaches are shown in Figure 6.1. Figure 6.1a shows the worst-case response of the uncontrolled channel flow (solid), which is linearly stable at $Re = 3000$ and exhibits a high degree of TEG. Also plotted in Figure 6.1a is the worst-case response of the LQR controlled system (dashed), which reduces TEG relative to the uncontrolled response. The response of twelve different LQG controlled systems subject to sub-optimal disturbances are shown in Figure 6.1b. Each LQG controller is designed using the same controller gain as is used for LQR control case, but differ from one another in that the LQE observer gains used for state estimation are designed by varying the associated weighting matrices in the objective function specification. Comparing the responses in Figure 6.1, it is clear that even the best performing LQG controller degrades TEG performance relative to the LQR controller and even relative to the uncontrolled channel flow. Finally, recall that the LQG performance reported here does not even consider the worst-case response to an optimal disturbance. In response to an optimal disturbance, LQG will result in even higher levels of TEG. Thus, in practice, even the smallest of flow disturbances can lead to undesirable TEG owing to errors in the state estimates. The results here suggest that LQG and other observer-based feedback strategies based on the separation principle can be surprisingly poor candidates for TEG reduction and transition delay.

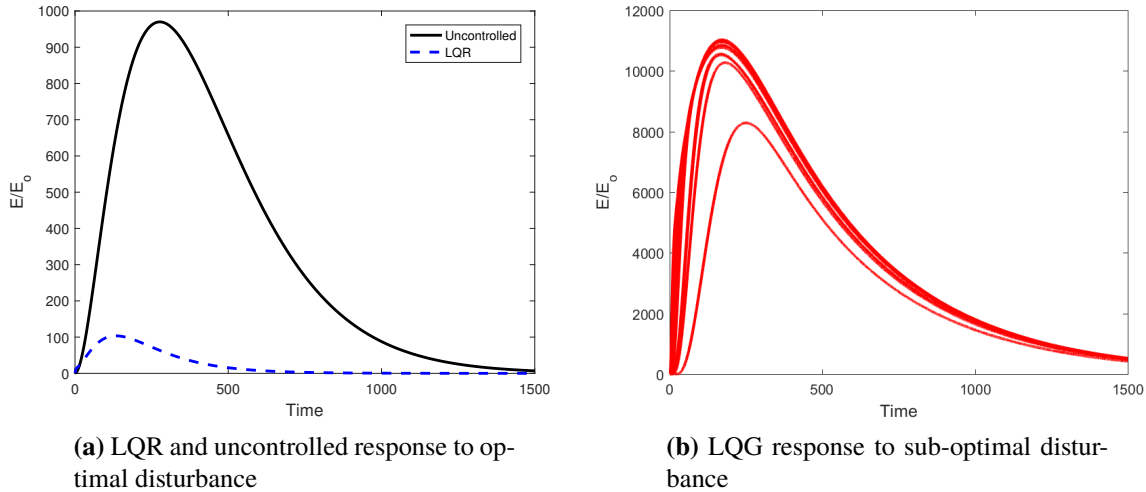


Figure 6.1: Spanwise wave case $\alpha = 0$, $\beta = 1$ at $Re = 3000$. Under critical Reynolds number, the open-loop system is linearly stable with high TEG (solid). A selected LQR controller can reduce the amount of the maximum TEG to around 103.5 (dashed).

We next investigate the worst-case response of SOF-LQR controllers. Figure 6.2 clearly shows that the worst-case SOF-LQR response reduces TEG relative to the worst-case uncontrolled response. This result alone highlights that SOF-LQR can overcome the performance limitations of observer-based feedback. Still, full-state feedback LQR control is able to reduce TEG more effectively than SOF-LQR; however, greater TEG reductions can be achieved with SOF-LQR if

additional sensors are available for feedback.

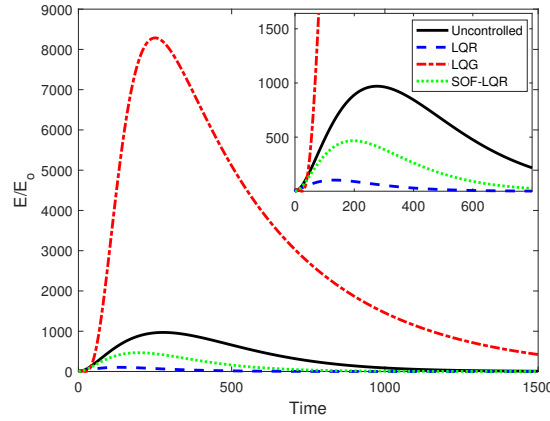


Figure 6.2: Spanwise wave case $\alpha = 0$, $\beta = 1$ at $Re = 3000$. Though cannot work as well as LQR, SOF-LQR still performs better than LQG controllers regarding to TEG suppression.

A final study that demonstrates the value of SOF-LQR control, beyond simply reducing the worst-case TEG, is to consider performance in off-design conditions. Figure 6.3 shows the worst-case response of the linearized channel flow under SOF-LQR control for $Re = 2500$ and $Re = 3500$, whereas the SOF-LQR controller was designed for $Re = 3000$. Here, the wave number pair $(\alpha, \beta) = (0, 1)$ is the same for both design and testing. The responses in these off-design conditions clearly show that SOF-LQR exhibits robustness to parameter variations and modeling uncertainty, as does the full-state feedback LQR controller. Indeed, these robustness results establish another potential advantage of SOF-LQR over LQG synthesis, which lacks robustness guarantees [97].

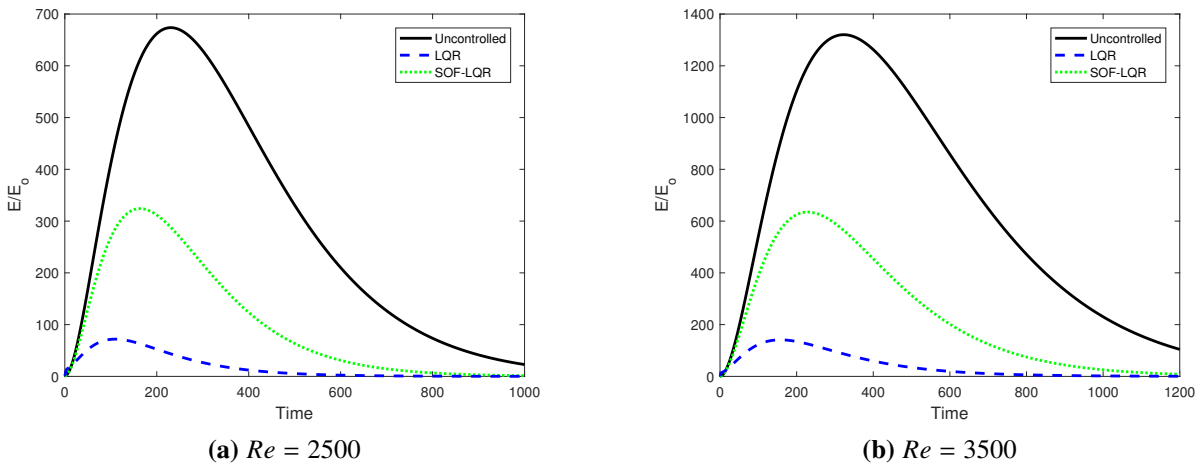


Figure 6.3: LQR and SOF-LQR both exhibit robust TEG reduction with Reynolds number variations.

6.4 Discussion and conclusion

This chapter considered the problem of TEG reduction in a linearized channel flow using output feedback control strategies. Standard observer-based feedback strategies predicated on the

separation principle of modern control were found to exhibit dramatic performance degradation with regards to TEG. Subjecting the LQG controlled system to sub-optimal disturbances led to significantly higher TEG than the worst-case response of the uncontrolled system. We proposed an alternative SOF-LQR feedback strategy and proposed an accelerated Anderson-Moore algorithm to compute the associated optimal feedback gains. SOF-LQR control of the linearized channel flow was found to reduce the worst-case TEG relative to the uncontrolled flow. Further, this method was found to exhibit robust TEG reduction in the face of modeling uncertainties and Reynolds number variations.

The present study was restricted to a limited portion of the parameter space, i.e., $(\alpha, \beta, Re) = (0, 1, 3000)$. Future investigations will focus on designing controllers for other wave number pairs and Reynolds numbers, to establish whether or not the results of this study are universal. Nevertheless, the ability of SOF-LQR control to robustly and effectively reduce TEG makes this a viable alternative to observer-based feedback strategies, which have demonstrable performance limitations. Indeed, future investigations will be conducted on devising SOF-LQR strategies for improved transition control.

7 | Data-Driven Selection of Actuators for Optimal Flow Control

7.1 Introduction

The focus of this chapter is on optimal actuator selection for flow control. This chapter deviates from the TEG control focus of previous chapters, focusing instead on airfoil separation control. However, the methods presented are general and can be applied within the context of other flow control configurations as well.

Flow separation leads to degraded performance in many engineering systems, through reduced lift, increased drag, and decreased efficiency. In an effort to alleviate the effects of flow separation on aerodynamic performance, various active flow control strategies have garnered considerable attention in the recent past. In particular, open-loop flow control strategies based on various actuator technologies—e.g., plasma actuators [98, 99] and synthetic jets [100–106]—have been shown to favorably alter separated flows, and in some cases to even cause flow reattachment.

Several studies have endeavored to understand the nonlinear dynamics associated with a fluid flow in an attempt to be able to formulate more effective control strategies [107, 108]. Further studies have focused on identifying a global description of the dynamics of the system using techniques such as Dynamic Mode Decomposition (DMD) [103–105, 109, 110]. These studies reported improved open-loop controller designs, based on zero-net-mass-flux actuation, by forcing at specific frequencies associated with dynamically important modes of the flow. While these studies provide foundations for methods that reduce flow separation, the actuator positions considered were fixed and may not necessarily translate to the optimal performance achievable in terms of separation control. In flow control applications, the placement of control devices is often guided by several factors, including separation point location and free-stream conditions. Several studies suggest placing the controller in locations that are intuitively optimal [100, 111].

For steady flows, the location of flow separation from a no-slip wall is identified exactly by Prandtl's condition for separation in the Eulerian frame through a point of zero skin friction and a negative friction gradient in the wall-tangential direction. Flow separation from a no-slip wall can also be understood by fluid tracers breaking away from a wall. It was shown in [112] that the dynamics of unsteady flow separation are better analyzed in a Lagrangian frame, wherein the Lagrangian separation point is fixed for a periodic flow. In [112], it was further shown that particles near a separation point are drawn towards an unstable manifold—i.e., an attracting material line that constitutes the separation line. Such attractors can also be identified in the flow field away from the boundary by extracting ridges in the Finite-Time Lyapunov Exponent (FTLE) field. The

FTLE field identifies the local contraction or expansion of the field over a finite time. A material line or Lagrangian Structure can then be found to exist wherever the contraction is maximum (i.e. a ridge) [113, 114].

While some of these strategies have leveraged knowledge of the unsteady flow physics to limit flow separation [98–111], the influence of actuator and sensor placement on separation control has remained relatively unexplored. In [115], it was shown that using pressure and skin-friction data, a time dependent quantity known as the *separation angle* could be computed. This was done for a set of six candidate actuator locations. A linear approximation of the separation line can then be computed by utilizing the separation angle and the separation point. It was further demonstrated that an increase in separation angle leads the material line to become concave, for any location upstream of the separation point. In turn, the concavity of the material line results in flow reattachment. In contrast, a decrease in separation angle results in an increase in the separation region. The increase in separation angle coincided with an increase in lift and a reduction in drag. Thus, a pulse location yielding a greater increase in lift corresponded to a greater degree of reattachment, providing guidance on the optimal actuator location for separation control.

The positioning of actuators forms an important basis for the evaluation of performance in a control system. In most scenarios, using all available actuators and sensors, generally, yields the highest performance for a given system. However, such an arrangement may not be feasible in practice due to physical, computational, and/or economic constraints [116]. Therefore, the selection of a subset of these actuator locations through a set of systematic criteria may assist in identifying the actuator location with the highest performance index. In [117], the effect of white noise disturbance for various actuator and sensor configurations were investigated in the context of optimal actuator and sensor placement. It was shown that the optimal actuator and sensor configuration of the Ginzburg-Landau system minimized the actuator effort and perturbation magnitude in an H_2 sense. In [118], a branch-and bound procedure was presented for optimal actuator placement with constraints on the number of actuators. The rich literature in this field provides motivation to undertake a similar study for separation control and obtain maximum achievable performance by leveraging traditional techniques in separation control. Of course, the definition of optimality would depend on the objective of the formulation and a relevant measure would have to be decided accordingly.

In this chapter, we consider the notion of optimal actuator selection for airfoil separation control. Pulse response data for lift and separation angle in response to a pulse of localized body-force actuation at six individual locations are computed from high-fidelity numerical simulations [115]. A NACA 65(1)-412 airfoil is considered for these simulations and can be seen in Figure 7.1. We obtain system models, for each of the actuator locations, by employing the Eigensystem Realization Algorithm (ERA), which can be applied on data obtained from both experiments as well as numerical simulations. The optimality of actuator locations among these six candidate locations is determined using a measure related to the minimum input energy required to drive the system to a desired output. The system models obtained by ERA, in this case, exhibit slightly unstable behavior. Therefore, the formulation that we propose takes this kind of behavior into account while determining the optimality of actuator locations.

The organization of this chapter is as follows: In Section 7.2, we present the methodology for determining optimality among candidate locations and how such strategies generalize for alternative

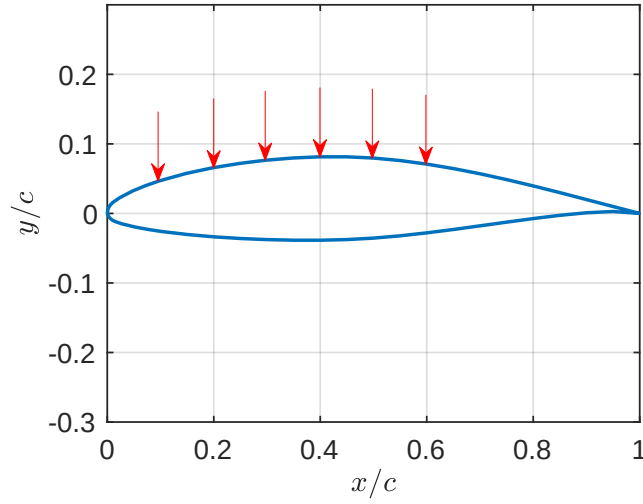


Figure 7.1: NACA 65(1)-412 airfoil. The positions $x/c = \{.1, .2, .3, .4, .5, .6\}$, on the upper surface of the airfoil correspond to the various actuator locations. These are indicated by the red arrows.

systems classes. Results are presented in Section 7.3. In Section 7.4, a brief discussion pertaining to the results is presented. Finally, conclusions are presented in Section 7.5. Materials in this chapter are heavily drawn from [119] and [120].

7.2 Methodology

A minimum control energy criterion for optimal actuator selection is introduced in Section 7.2.1. We then provide a description of how this procedure can be extended for stable/unstable systems alike in Section 7.2.2. The formulation of the procedure up to this point can be used readily to determine the optimal actuator location if all necessary system realizations are available; however, this is not the case in the present study. In Section 7.2.3, we describe how to compute the necessary system realizations from empirical data, thus enabling the method to be easily applied in the context of either numerical simulations or physical experiments to determine the optimal actuator location.

7.2.1 Minimum control energy as a measure for optimal actuator selection

Our goal here is to determine the actuator location, among a given set of locations, that minimizes the control effort needed to drive a quantity of interest (e.g., lift or separation angle) to an arbitrary value. Further, if we can determine the optimal control for each individual actuator location, then we can simply compare these minimal energies and select the least among this set. To address this larger problem, we begin by considering the minimal control energy problem for a single given flow control configuration, as described next.

For a given set of actuators and sensors, we can define the flow control configuration by a finite-dimensional state-space realization $G = (A, B, C)$:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t),\end{aligned}\tag{7.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^p$ is the input vector, and $y(t) \in \mathbb{R}^q$ is the output vector. Next, recognize that a control law that alters lift/separation angle with minimum control effort is

closely related to the problem of driving the system state to an arbitrary point in state-space using minimum control energy [121]; thus, we now seek the control input $u(t)$ that drives the system from the origin to an arbitrary point in state-space with minimal control energy over an infinite time-horizon. The optimal control problem described here can be solved by standard methods and is commonly referred to as the *minimum control energy problem* [121]:

$$\begin{aligned} \text{minimize} \quad & J = \int_0^\infty u^\top(\tau)u(\tau)d\tau \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu(t) \\ & x(0) = 0 \\ & x(\infty) = x_f, \end{aligned} \tag{7.2}$$

which admits a solution if the system is controllable (equivalently, reachable). The minimal input energy associated with the optimal control is given by,

$$J^{\text{opt}} = x_f^\top W_c^{-1} x_f, \tag{7.3}$$

where the controllability Gramian

$$W_c := \int_0^\infty e^{A\tau} B B^\top e^{A^\top \tau} d\tau \tag{7.4}$$

is the stabilizing solution to the Lyapunov equation,

$$A W_c + W_c A^\top + B B^\top = 0 \tag{7.5}$$

To determine the actuator location that yields the minimum control energy, we can simply compare the relative sizes of W_c corresponding to the dynamics of each actuator location—a larger W_c being more controllable and requiring less input energy to control (all else equal). Note that the controllability Gramian W_c is not invariant under similarity transformation; thus, care must be taken when formulating measures of optimality directly based on W_c . Some suitable choices that are invariant under similarity transformation are, e.g., $\det(W_c)$, $\text{trace}(W_c)$.

Although finite time-horizons can be considered here, we choose to focus on the infinite time-horizon case in order to maintain objectivity in the optimality measure; the solution to the finite time-horizon problem is dependent on the final time, which is undesirable because the final time can always be chosen to influence the outcome of the optimality measure.

To gain an intuition for the optimal solution, we can view the quadratic form in (7.3) as defining an ellipse that contains all points in state-space that can be *reached from the origin* using no greater than unit input energy, $X = \{x_f \in \mathbb{R}^n \mid x_f^\top W_c^{-1} x_f \leq 1\}$. The most controllable directions in state-space require the least control energy to traverse and are associated with the eigendirections associated with the largest eigenvalues of W_c ; the least controllable directions in state-space require the most control energy to traverse and are associated with the eigendirections associated with the smallest eigenvalues of W_c .

As was outlined in the beginning of this subsection, our goal here is to have maximum controllability in terms of the lift/separation angle. Although, W_c provides intuition about the most

controllable states, in practice, the quantity of interest may not directly correspond to these states; instead, it may correspond to a linear combination of these states. Hence, rather than considering the state controllability Gramian W_c directly, we can instead work with a suitably weighted version of W_c ,

$$W_{oc} := \int_0^\infty C e^{A\tau} B B^\top e^{A^\top \tau} C^\top d\tau \quad (7.6)$$

$$= C W_c C^\top, \quad (7.7)$$

which is simply the *output controllability Gramian* [122]. In some sense, output controllability is a more natural measure of optimality because it is invariant under similarity transformations, and thus constitutes a system property that is coordinate independent. This choice is particularly appealing because measures based on W_{oc} admit numerous other interpretations, beyond those afforded by the minimum control energy perspective introduced above. For instance, the output controllability Gramian is directly related to the $\mathcal{H}_2$ -norm of a stable LTI system as,

$$\|g(t)\|_2 = \sqrt{\left(\int_0^\infty \text{trace}(g^\top(t)g(t))dt \right)} = \sqrt{\text{trace}(W_{oc})} \quad (7.8)$$

where $g(t) := C e^{At} B$ is the impulse response matrix. Further, we can arrive at a frequency-domain interpretation of this measure by invoking Parseval's theorem [123],

$$\|g(t)\|_2 = \|G(s)\|_2 := \sqrt{\frac{1}{2\pi} \int_{-\infty}^\infty \text{trace}(G^\top(-j\omega)G(j\omega))d\omega} \quad (7.9)$$

$$= \sqrt{\frac{1}{2\pi} \int_{-\infty}^\infty \sum_{k\ell} |G_{k\ell}(j\omega)|^2 d\omega} \quad (7.10)$$

where $G_{k\ell}(s)$ denotes the transfer function from the ℓ -th input to the k -th output. Hence, the $\mathcal{H}_2$ -norm can be interpreted as the average system gain over all forcing frequencies. Consistent with the minimum control energy interpretation, this indicates that a system with a larger $\mathcal{H}_2$ -norm will tend to yield a larger output for the same input signal. The $\mathcal{H}_2$ -norm also admits a stochastic interpretation from the lens of Linear Quadratic Gaussian control [123]: all else equal, a system with a larger $\mathcal{H}_2$ -norm will yield a larger output power in response to unit intensity white noise inputs. Based on all these interpretations, it should be evident that output controllability and the $\mathcal{H}_2$ -norm can provide an indication of the effectiveness of an actuator location in influencing the system output with an arbitrary input.

7.2.2 Generalizability to unstable systems

At this point, it is worth discussing generalizations of the above optimality measure to unstable systems, since we are interested in comparing actuator locations for general systems, which may or may not be stable. Of course, in the context of unstable systems, neither the state controllability Gramian nor the output controllability Gramian will be bounded; thus, it seems that these optimality measures are ill-suited for comparing general flow control configurations that may exhibit unstable dynamics. However, by taking a frequency-domain perspective of the state controllability Gramian, we can arrive at a *generalized controllability Gramian* P that is bounded for unstable systems [124]:

$$P = \frac{1}{2\pi} \int_{-\infty}^\infty (j\omega I - A)^{-1} B B^\top (-j\omega I - A^\top)^{-1} d\omega \quad (7.11)$$

The generalized controllability Gramian is also related to the minimum control energy problem, as shown in Theorem 5 of [124]. Specifically, when the system under consideration is controllable, $x_o^T P^{-1} x_o = \inf \{\|u\|_2^2 \mid x(0) = x_o, x(-\infty) = 0, x(\infty) = 0\}$. As with W_c , a larger P indicates that less control energy is required to drive the state to the origin (i.e., the system is “more controllable”). In other words, the generalized controllability Gramian P has an equivalent interpretation as the conventional controllability Gramian W_c , but extends the interpretation to the context of unstable systems. Indeed, when the system under consideration is stable, the generalized controllability Gramian is equivalent to the standard controllability Gramian (i.e., $P = W_c$).

Conveniently, for a stabilizable and detectable system, the generalized controllability Gramian P can be computed directly from a state-space realization of the system. The procedure follows directly from Theorem 2 in Zhou et al. [124], which amounts to solving for the stabilizing solution X to the algebraic Riccati equation,

$$XA + A^T X - XBB^T X = 0 \quad (7.12)$$

followed by a computation of the generalized controllability Gramian P as the solution to the Lyapunov equation,

$$(A + BF)P + P(A + BF)^T + BB^T = 0, \quad (7.13)$$

where $F = -B^T X$. For stable systems, $X = 0$ and, therefore, $P = W_c$.

For the purpose of determining a measure of optimality for actuator placement, here we will define the *generalized $\mathcal{H}_2$ -norm* (denoted $\mathcal{H}_{2'}$) in analogy with Eq. (7.8), but now using the notion of generalized output controllability CPC^T instead of the conventional output controllability $CW_c C^T$:

$$\|G\|_{2'} = \sqrt{\text{trace}(CPC^T)} \quad (7.14)$$

In the remainder of this chapter, $\|G\|_{2'}$ will be used as a measure for determining the optimal actuator location among a set of candidate actuator locations. In our case, this measure is computed for all the candidate locations using the minimal realization obtained from pulse response data, as will be described in the next subsection.

7.2.3 Minimal realizations from pulse response data

An imperative step in determining optimality among the candidate set of actuator locations is obtaining mathematical models for such configurations. Once such system models are obtained, analyses corresponding to optimality can be conducted. The field of system identification deals with obtaining mathematical models for a system based on data observations obtained from the system. In general, such data is usually sampled at discrete instants of time in a large variety of applications. Hence, discrete-time system models show higher suitability for system identification methods. Identified models can subsequently be used further for analysis and control. Here, we describe one such method for determining a minimal realization of a system from empirical data.

Consider the discrete-time state-space representation of our system of interest:

$$\begin{aligned} x(k+1) &= \hat{A}x(k) + \hat{B}u(k) \\ y(k) &= \hat{C}x(k) + \hat{D}u(k) \end{aligned} \quad (7.15)$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^p$ is the input vector, and $y(k) \in \mathbb{R}^q$ is the output vector at time index k and $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are the discrete-time state-space matrices. The pulse response yields the system Markov parameters:

$$h_0 = \hat{D} \quad (7.16)$$

$$h_k = \hat{C} \hat{A}^{k-1} \hat{B} \quad (7.17)$$

where $h_k \in \mathbb{R}^{q \times p}$. Here, we assume the direct feedthrough term $\hat{D} = h_0 = 0$. For each candidate actuator location, we appeal to the eigensystem realization algorithm (ERA) [125] to compute a minimal realization of the system $\hat{G} = (\hat{A}, \hat{B}, \hat{C})$ directly from these Markov parameters. To do so, we define two Hankel matrices composed of the Markov parameters,

$$H_0 = \begin{bmatrix} h_1 & h_2 & \cdots & h_{n_o} \\ h_2 & h_3 & \cdots & h_{n_o+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n_c} & h_{n_c+1} & \cdots & h_{n_c+n_o} \end{bmatrix}, \quad H_1 = \begin{bmatrix} h_2 & h_3 & \cdots & h_{n_o+1} \\ h_3 & h_4 & \cdots & h_{n_o+2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n_c+1} & h_{n_c+2} & \cdots & h_{n_c+n_o+1} \end{bmatrix} \quad (7.18)$$

Next, compute the Singular Value Decomposition (SVD) of $H_0 = U \Sigma V^*$, then store the r largest singular values in a matrix Σ_r and the corresponding left- and right-singular vectors in the matrices U_r and V_r , respectively. Finally, a minimal realization $(\hat{A}, \hat{B}, \hat{C})$ can be computed as,

$$\hat{A} := \Sigma_r^{-\frac{1}{2}} U_r^* H_1 V_r \Sigma_r^{-\frac{1}{2}} \quad (7.19)$$

$$\hat{B} := \text{First } p \text{ columns of } \Sigma_r^{\frac{1}{2}} V_r^* \quad (7.20)$$

$$\hat{C} := \text{First } q \text{ rows of } U_r \Sigma_r^{\frac{1}{2}} \quad (7.21)$$

A complete description of ERA can be found in [125]. The resulting discrete-time state-space realizations computed by ERA may then be converted to continuous-time realizations—in the form of (7.1)—by means of Tustin's approximation [126]. As was shown in the previous subsection, the optimality measure associated with each actuator location can then be computed directly from this ERA-based minimal realization.

Our choice of utilizing pulse response data for system identification is quite natural since Markov parameters have the property of being unique for a given system and are often referred to as the “signature” of the system model [127]. In the event that other forms of input-output data are available through simulations/experiments, methods such as Observer/Kalman filter Identification (OKID) may be used to extend the applicability of ERA to general input-output response data [128]. We note that ERA introduces some elements of subjectivity to the optimal selection process, since various ERA algorithm parameters, such as r can be chosen to alter the specific realization; however, additional precautions can be taken to ensure that the realization at hand is sufficiently insensitive to these algorithmic parameters and that multiple ERA realizations based on the same pulse response data yield consistent optimal actuator rankings. Indeed, this will be the case for all of the results that are reported here, as will be described in the subsequent sections.

We further note that ERA is applicable for both stable and unstable systems [129]. For the data-sets that we will consider here, the system response appears to be mildly unstable. In principle,

it is possible to compute the output controllability Gramian by direct integration of pulse response data; however, performing a direct integration of pulse response data for unstable systems (or of unconverged responses in general) over an infinite time-horizon is not possible. (Again, we could compute measures based on finite time-horizons, but doing so would come at the undesirable cost of losing objectivity due to sensitivities associated with the choice of terminal time, as discussed earlier.) This was part of the motivation for generalizing the optimality measure in the previous subsection to accommodate both stable and unstable systems.

7.3 Results

We apply the approach described in Section 7.2 to the pulse response data from the numerical simulations described in [115] for both lift as well as for separation angle response data. The fluid flow simulations for the baseline case as well as for the case of pulse body-force actuation applied at actuator location $x/c = .4$ can be seen in Figures 7.2 and 7.3, respectively.

For each actuator location, an ERA model of order r is realized (see Figures 7.5 and 7.8). Here, r is chosen to be that order which gives the best match in terms of the original data obtained from numerical simulations. As previously indicated, a number of these realizations exhibit unstable dynamics. Indeed, in many cases, some of the discrete-time system poles are outside of the unit circle (see Figures 7.4 and 7.7). We, thus, utilize the generalization of the controllability Gramian for unstable systems as is described in Section 7.2.2. We note that the unstable realizations may be due to the nature of the response data, which seems to have not fully returned to a steady-state. Tables 7.1 and 7.2 are associated with *generalized $\mathcal{H}_2$ -norms* and indicate the degree of controllability among the various candidate locations. The ordering of locations indicates their relative rank in terms of optimality for a given output variable.

The minimality and order of ERA-based realizations were sanitized of any potential numerical artificialities by accounting for pole-zero cancellations based on a range of tolerances from $O(10^{-5})$ to $O(10^{-7})$. These tolerance values indicate the proximity of poles and zeros necessary to facilitate pole-zero cancellation. It was found that decreasing the tolerance any further resulted in realizations that were non-minimal. For the sake of representation, we use figures and tables associated with the tolerance value of $O(10^{-7})$ in the subsequent sections (Figures 7.4 through 7.9 and Tables 7.1 through 7.2).

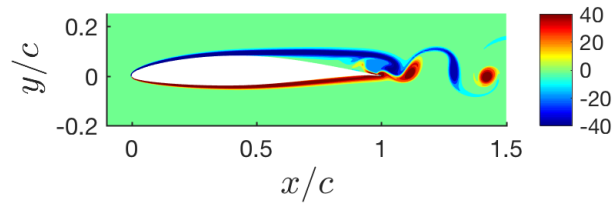


Figure 7.2: NACA 65(1)-412, $\alpha = 4^\circ$, $Re_c = 20,000$, Vorticity field at $U_\infty t/c = 0.60$ for the baseline case.

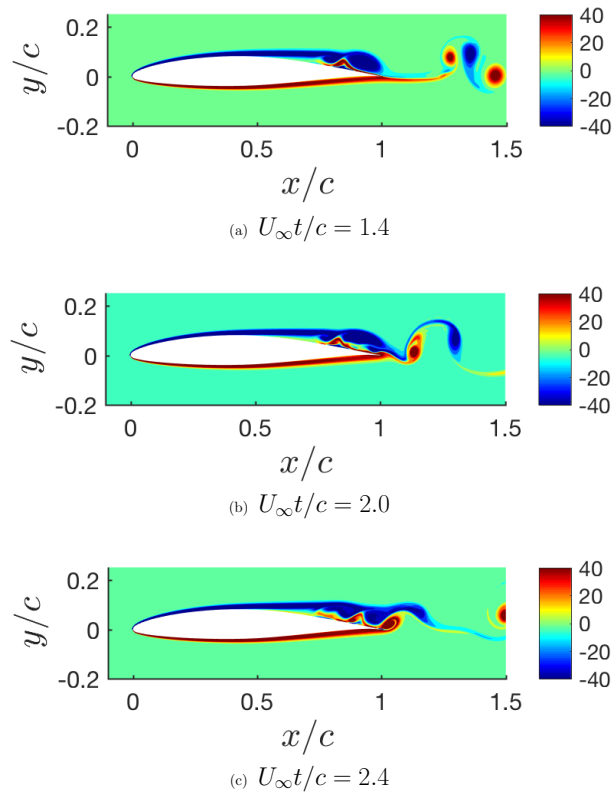


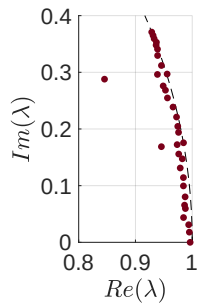
Figure 7.3: NACA 65(1)-412, $\alpha = 4^\circ$, $Re_c = 20,000$, Vorticity field at (a) $U_{\infty}t/c = 1.4$, (b) $U_{\infty}t/c = 2.0$ and (c) $U_{\infty}t/c = 2.4$ for the case with a pulse actuation at $x/c = 0.4$.

7.3.1 Optimal actuator placement for lift response data

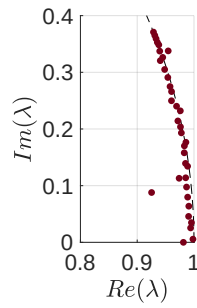
Based on the $\mathcal{H}_2$ -norm, the optimal actuator location for lift control is $x/c = 0.2$. This location has the highest controllability among all six candidate locations. The optimality study considered a variety of pole-zero overlapping tolerance values as has been described above. The optimal actuator position is largely constant although minor variations in relative ranking of other actuators is observed. Some of the eigenvalues of the discrete-time realizations obtained are outside the unit circle for all locations, thereby confirming that the identified systems are indeed unstable. This can be seen in Figure 7.4. As can be seen in Figure 7.5, the high order for the obtained minimal realizations, in all likelihood indicates that the system may have some degree of non-linearity in it, which is captured by a large number of states. The peak frequency for all actuator locations is $fc/U_\infty = 6.12$, as can be seen in Figure 7.6 and corresponds to the wake frequency. Thus, it appears that among all actuator locations, $x/c = 0.2$ is able to induce a resonance by coupling with the flow dynamics at this forcing frequency. However, some of the frequency response intuition here may have to be reconsidered in a “non-standard” manner, since the system dynamics are unstable.

x/c	$\ G\ _{2'}$
.2	51.79
.6	31.31
.1	17.41
.5	15.81
.4	15.41
.3	13.41

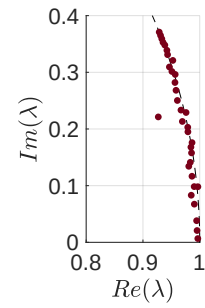
Table 7.1: Optimality of actuator locations based on “generalized $\mathcal{H}_2$ -norm”, sorted from most to least optimal for tolerance value of $O(10^{-7})$ used in minimal realization for lift response data.



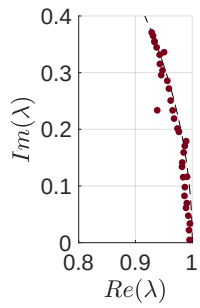
(a) Actuator at $x/c = .1$



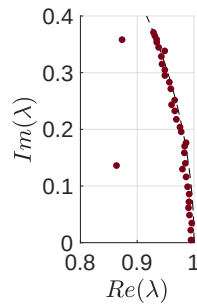
(b) Actuator at $x/c = .2$



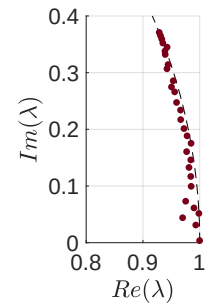
(c) Actuator at $x/c = .3$



(d) Actuator at $x/c = .4$

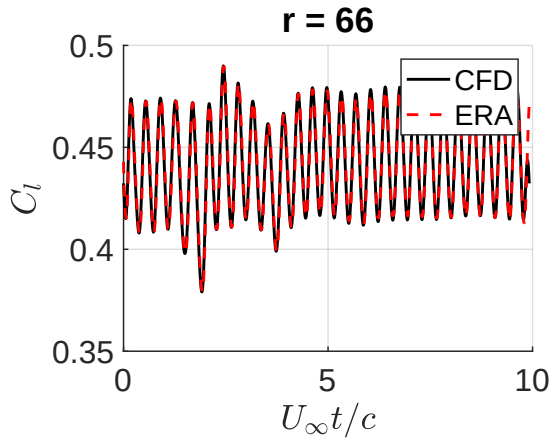


(e) Actuator at $x/c = .5$

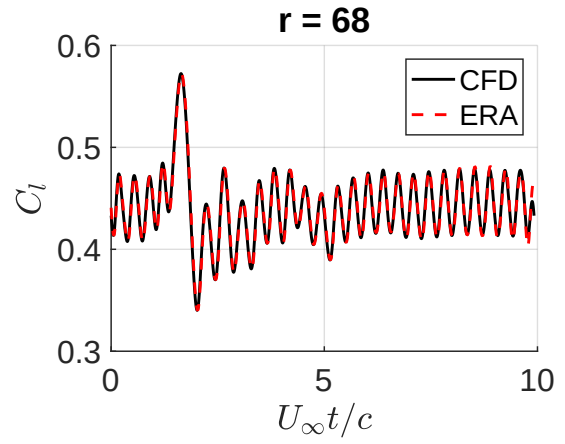


(f) Actuator at $x/c = .6$

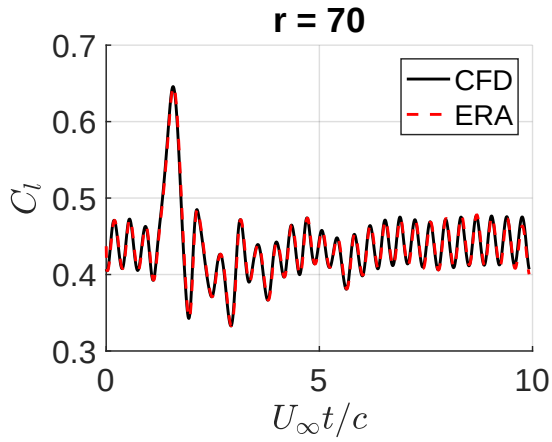
Figure 7.4: System poles of (discrete-time) minimal realization computed from lift pulse response data for each actuator location using ERA. Some poles are outside the unit circle for all locations.



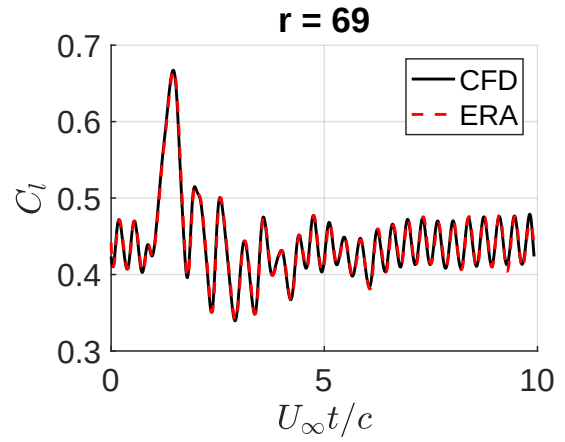
(a) Actuator at $x/c = .1$



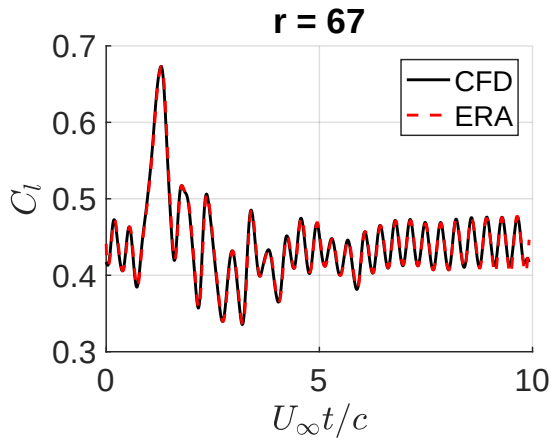
(b) Actuator at $x/c = .2$



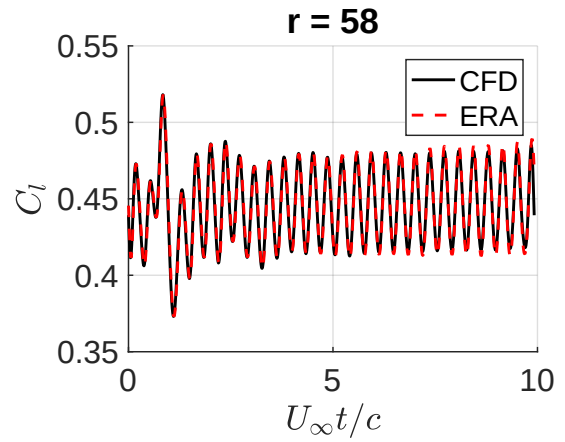
(c) Actuator at $x/c = .3$



(d) Actuator at $x/c = .4$

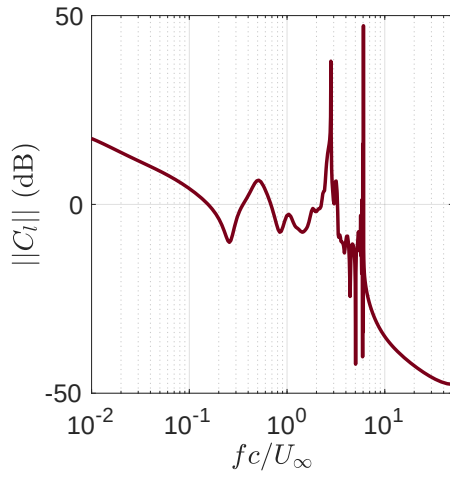


(e) Actuator at $x/c = .5$

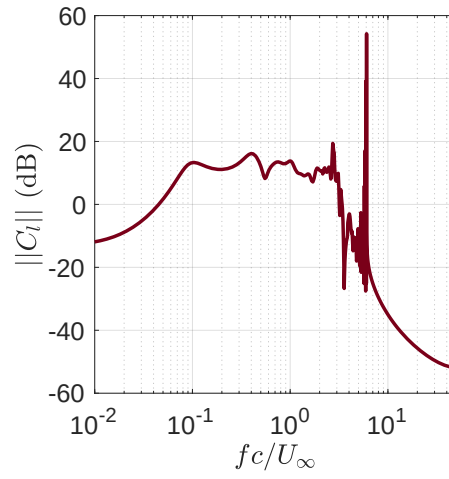


(f) Actuator at $x/c = .6$

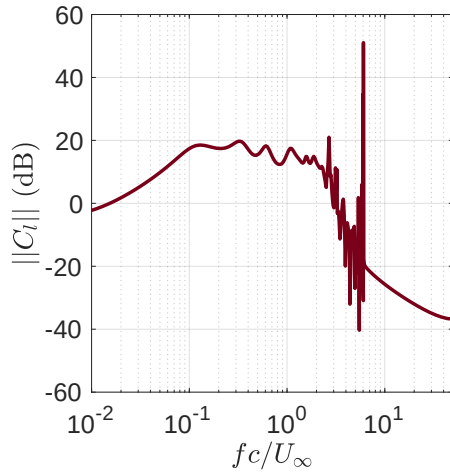
Figure 7.5: Lift coefficient pulse response data at each actuator location. Each realization is minimal with order r .



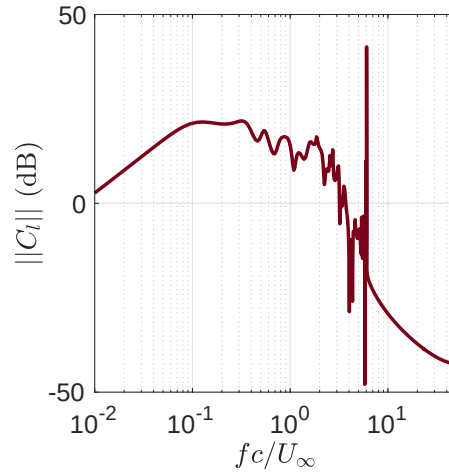
(a) Actuator at $x/c = .1$



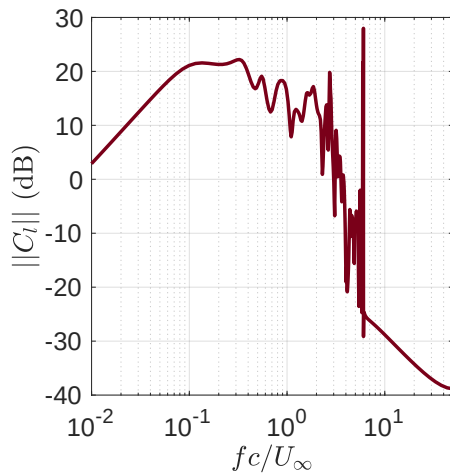
(b) Actuator at $x/c = .2$



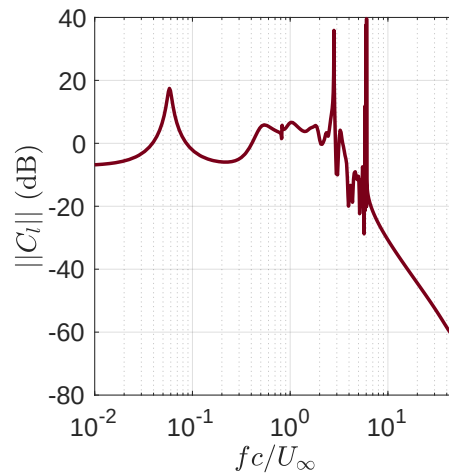
(c) Actuator at $x/c = .3$



(d) Actuator at $x/c = .4$



(e) Actuator at $x/c = .5$



(f) Actuator at $x/c = .6$

Figure 7.6: Bode magnitude plot for minimal realization at each actuator location for lift response data.

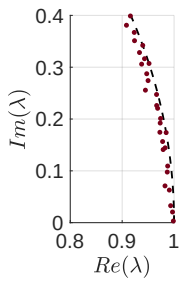
7.3.2 Optimal actuator placement for separation angle response data

A similar exercise as described above is also undertaken for the separation angle pulse response data. Based on the $\mathcal{H}_2$ -norm, the optimal actuator location for separation angle control is $x/c = 0.3$ (see Table 7.2). This location has a degree of controllability which is significantly larger than other locations. The related norm for this actuator location is an order above the next optimal location. This is also reflected in the very high peak at this location, as seen in Figure 7.8, when compared to the other candidate locations. It is worth noting here, that the next optimal location, i.e, $x/c = 0.5$, in this case is located at the separation point itself as described in [115].

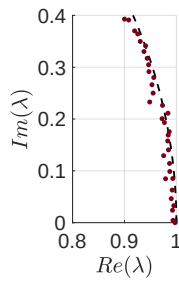
The results presented here provide two interesting observations. First, the order of the minimal realizations obtained for the separation angle response is an order above the realizations associated with lift response for all actuator locations. This probably indicates a greater degree of non-linearity associated with separation angle than with lift. Another consequence of this phenomenon can be seen in Table 7.2. The generalized $\mathcal{H}_2$ -norm associated with each of the locations is greater than their lift counterparts. This is especially prominent for the most optimal locations in the separation angle case.

x/c	$\ G\ _2$
.3	1.63×10^7
.5	3.99×10^4
.4	1059.57
.1	243.24
.6	91.95
.2	76.74

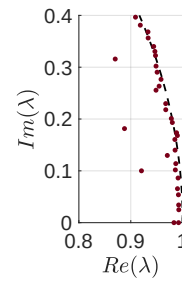
Table 7.2: Optimality of actuator locations based on “generalized $\mathcal{H}_2$ -norm”, sorted from most to least optimal for tolerance value of $O(10^{-7})$ used in minimal realization for separation angle response data.



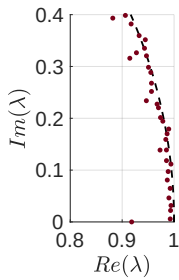
(a) Actuator at $x/c = .1$



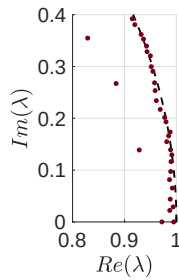
(b) Actuator at $x/c = .2$



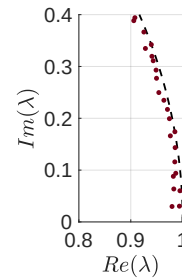
(c) Actuator at $x/c = .3$



(d) Actuator at $x/c = .4$

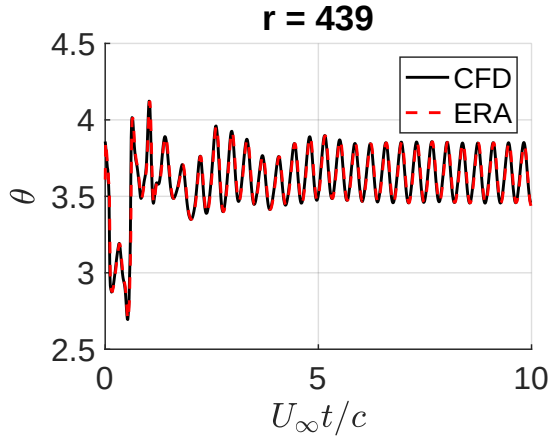


(e) Actuator at $x/c = .5$

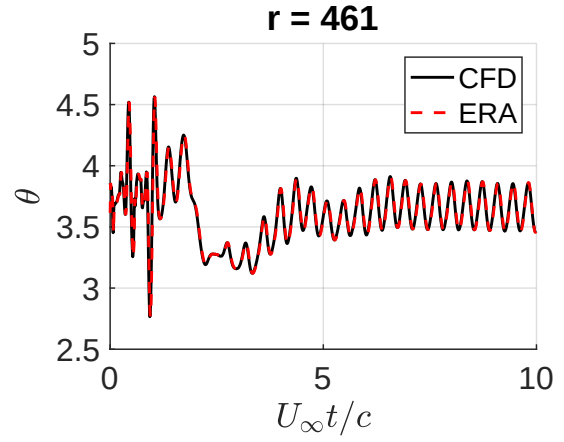


(f) Actuator at $x/c = .6$

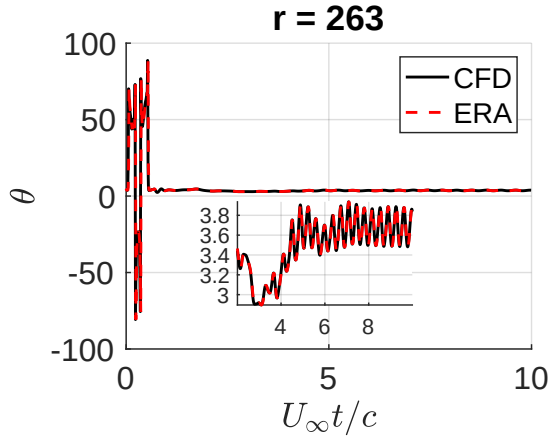
Figure 7.7: System poles of (discrete-time) minimal realization computed from separation angle pulse response data for each actuator location using ERA. Some poles are outside the unit circle for all locations.



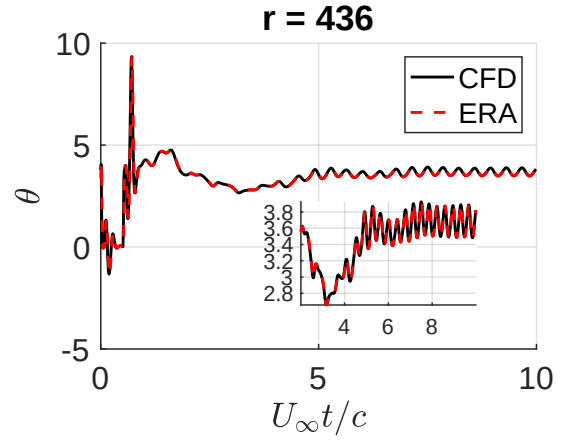
(a) Actuator at $x/c = .1$



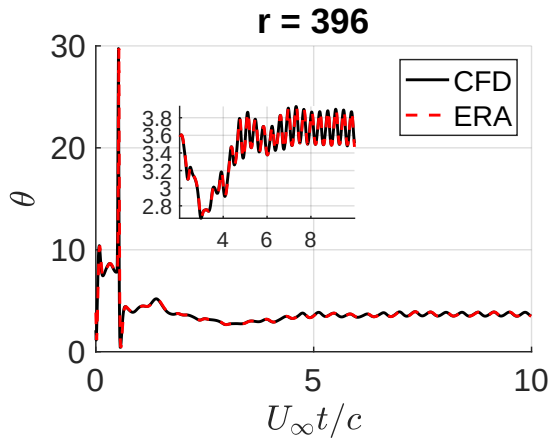
(b) Actuator at $x/c = .2$



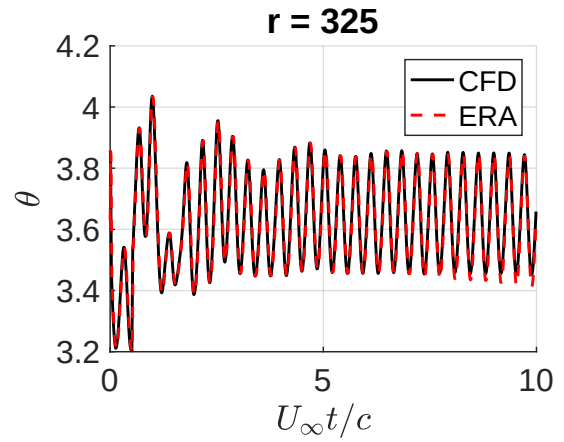
(c) Actuator at $x/c = .3$



(d) Actuator at $x/c = .4$

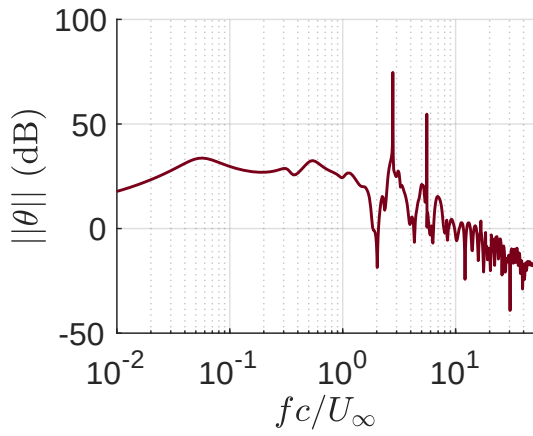


(e) Actuator at $x/c = .5$

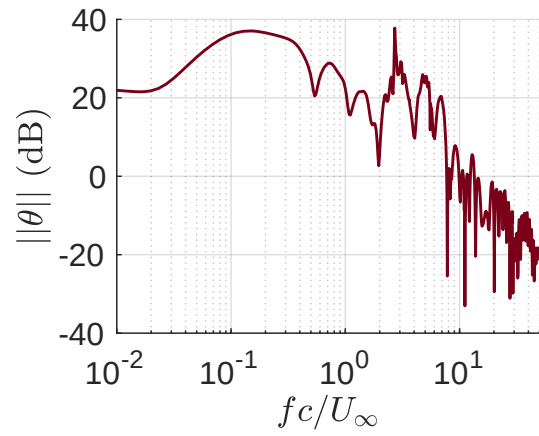


(f) Actuator at $x/c = .6$

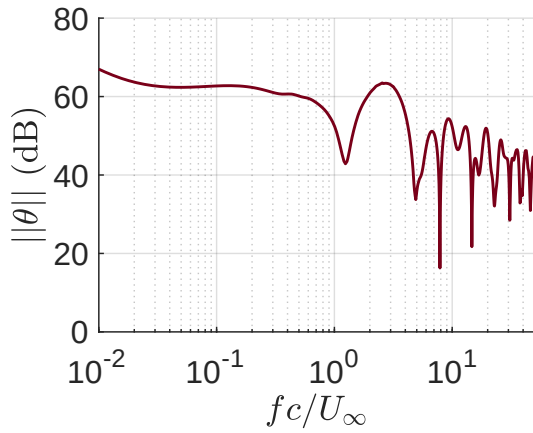
Figure 7.8: Separation angle pulse response data at each actuator location. Each realization is minimal with order r .



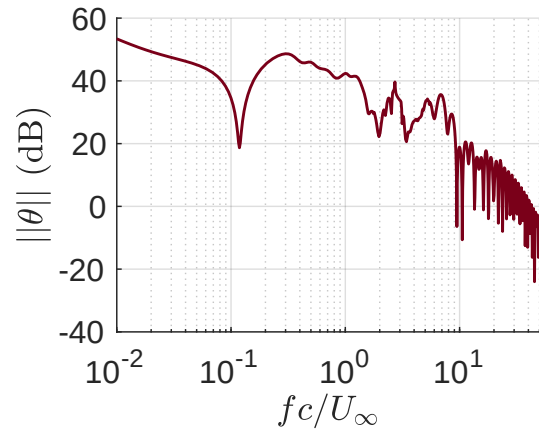
(a) Actuator at $x/c = .1$



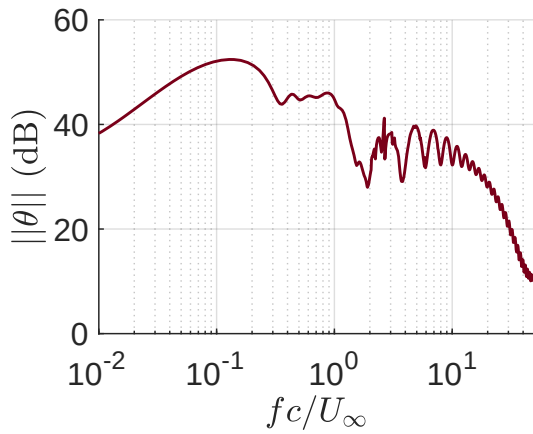
(b) Actuator at $x/c = .2$



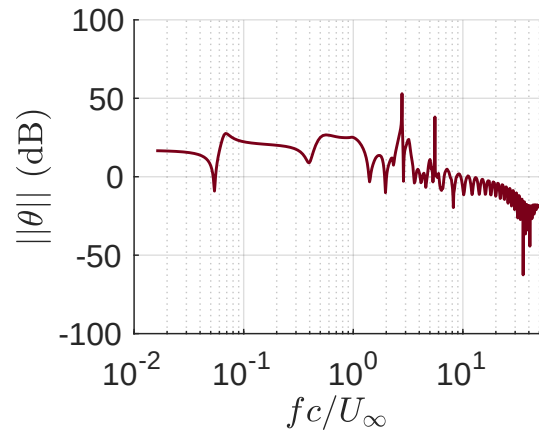
(c) Actuator at $x/c = .3$



(d) Actuator at $x/c = .4$



(e) Actuator at $x/c = .5$



(f) Actuator at $x/c = .6$

Figure 7.9: Bode magnitude plot for minimal realization at each actuator location for the separation angle response.

7.4 Discussion

The results presented here confirm that the optimality of a candidate location is dependent on the output variable being considered. The optimal location for separation angle control is found to be $x/c = 0.3$; whereas, for lift control the optimal location is found to be $x/c = 0.2$. The results from the optimality study of the separation angle response indicate that the optimal location for separation angle control would provide significantly more controllability compared to the other candidate locations; in contrast, the optimal location for lift control does not exhibit as pronounced a difference in controllability as the other locations. It was shown in [115] that an increase in the separation angle led to flow reattachment and corresponded to an increase in lift. The results presented in this chapter indicate that for separation control, the control of separation angle may be given higher priority than the control of lift.

It is well known that flow separation occurs when the boundary layer flow cannot withstand an adverse pressure gradient. The optimal control locations for the lift coefficient and separation angle at $x/c = 0.2$ and $x/c = 0.3$, respectively, are at a location just downstream of the pressure coefficient minimum at which point the laminar boundary layer becomes increasingly unstable as it works against the adverse pressure gradient. This can be seen in Figure 7.10. In this respect, the flow is most efficiently controllable at the point of initial instability in the laminar boundary layer.

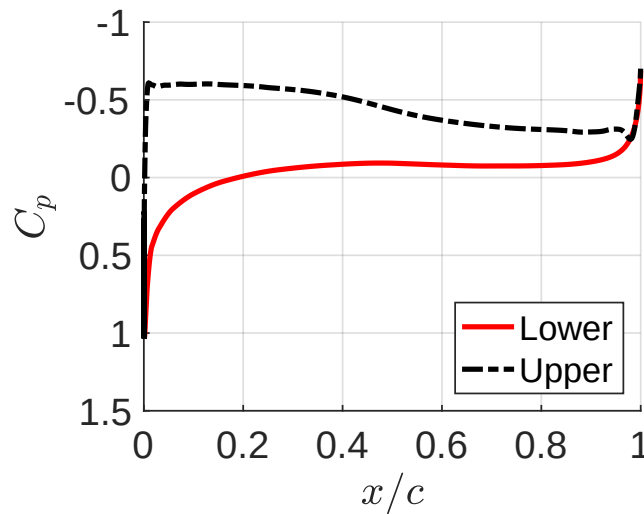


Figure 7.10: Pressure coefficients for the upper and lower surface of the airfoil for the baseline flow just before pulse actuation is applied. This corroborates the findings of the optimality study as the optimal locations are, indeed, more sensitive to the initial instability in the laminar boundary layer and are able to influence the flow more directly.

In a large variety of applications, intuition from the physical flow is used to guide active control design methods. However, there is seldom a systematic comparison of the performance attained at different actuator locations. This study extends the approach in [115] by also quantifying the degree of controllability afforded by each location. Although, most studies indicate that forcing at the separation point has the greatest impact in reducing flow separation, the results presented here provide a new perspective in determining the optimal location for separation control.

7.5 Conclusion

In this chapter, we investigated the optimal actuator selection problem for airfoil separation control. Pulse response data for lift and separation angle in response to a localized body force actuation were used to determine the optimal location among the candidate set of six locations. The optimality measure utilized here is related to the minimum energy required to drive the system output to an arbitrary value. Minimal realizations computed from pulse response data were found to exhibit unstable behavior. Therefore, we proposed a formulation of the optimal actuator selection problem to accommodate stable and unstable systems alike. For the optimality procedure to be insensitive to algorithmic parameters, several precautions were presented. This resulted in a consistent yield in terms of the rankings of the various actuator locations. For the NACA 65(1)-412 airfoil, it was found that the location $x/c = 0.2$ was optimal for the control of lift, whereas the location $x/c = 0.3$ was found to be optimal for the control of separation angle.

This study provides a quantification on the difference in performance that can be expected from actuation at different locations along the airfoil. This can be particularly useful in situations wherein actuation is difficult at the preferred location due to constraints imposed on the system. The results presented here serve as a reference for development of active control methods which would aim to reduce flow separation through optimal placement of actuators. The analysis techniques developed here are applicable more generally, and can be used to conduct optimal actuator selection studies for other flow control configurations as well.

8 | Personnel

The following is a list of individuals who have worked on research supported in whole or in part by the Air Force Office of Scientific Research under Grant FA9550-17-1-0252:

Prof. Maziar S. Hemati, Department of Aerospace Engineering & Mechanics, U. Minnesota

Dr. Yiyang Sun, postdoctoral researcher

Mr. Aniketh Kalur, graduate student (PhD in progress)

Ms. Huaijin (Grace) Yao, graduate student (MS completed, PhD in progress)

Ms. Mengying (Mia) Wang, graduate student (MS completed, PhD in progress)

Mr. Debraj Bhattacharjee, graduate student (MS completed)

9 | Publications

The publications listed below represent papers, reports, and theses supported in whole or in part by the Air Force Office of Scientific Research under Grant FA9550-17-1-0252:

Journal Papers:

1. A. Kalur and M.S. Hemati, “Control-oriented model reduction for minimizing transient energy growth in shear flows,” AIAA Journal (in press), pre-print arXiv:1907.01664.
2. K. Taira, M.S. Hemati, S.L. Brunton, Y. Sun, K., Duraisamy, S. Bagheri, S.T.M. Dawson, C.-A. Yeh, “Modal Analysis of Fluid Flows: Applications and Outlook,” AIAA Journal (in press), pre-print arXiv:1903.05750v2.
3. M.S. Hemati and H. Yao, “Performance Limitations of Observer-Based Feedback for Transient Energy Growth Suppression,” AIAA Journal, vol. 56, no. 6, 2018.
4. M.S. Hemati, C.W. Rowley, E.A. Deem, and L.N. Cattafesta, “De-Biasing the Dynamic Mode Decomposition for Applied Koopman Spectral Analysis of Noisy Datasets,” Theoretical and Computational Fluid Dynamics, vol. 31, no. 4, 2017.

Conference Papers:

1. Y. Sun and M.S. Hemati, “Suppressing subcritical transition in plane Poiseuille flow,” AIAA Fluid Dynamics Conference, Dallas, TX, 17-21 June 2019, AIAA Paper 2019-3713.
2. H. Yao and M.S. Hemati, “Advances in Output Feedback Control of Transient Energy Growth in a Linearized Channel Flow,” AIAA Aerospace Sciences Meeting, San Diego, CA, 7-11 January 2019, AIAA Paper 2019-0882.
3. M.S. Hemati and H. Yao, “Reducing Transient Energy Growth in Linearized Channel Flow with Output Feedback Control,” JSASS Fluid Dynamics Conference, Miyazaki, Japan, 4-6 July 2018, JSASS Paper 3A02.
4. D. Bhattacharjee, M.S. Hemati, B. Klose, G. Jacobs, “Optimal Actuator Selection for Airfoil Separation Control,” AIAA Flow Control Conference, Atlanta, GA, 25-29 June 2018, AIAA Paper 2018-3692.
5. A. Kalur and M.S. Hemati, “Reduced-Order Models for Feedback Control of Transient Energy Growth,” AIAA Flow Control Conference, Atlanta, GA, 25-29 June 2018, AIAA Paper 2018-3690.
6. H. Yao and M.S. Hemati, “Revisiting the separation principle for improved transition control,” AIAA Flow Control Conference, Atlanta, GA, 25-29 June 2018, AIAA Paper 2018-3693.

7. A. Medina, M.S. Hemati, and M. Rockwood, "Lift Disturbance Cancellation with Fast-Flap Actuation," AIAA Flow Control Conference, Atlanta, GA, 25-29 June 2018, AIAA Paper 2018-3090.
8. A. Medina and M.S. Hemati, "Separated Flow Response to Rapid Flap Deflection," AIAA Aerospace Sciences Meeting, Kissimmee, FL, 8-12 January 2018, AIAA Paper 2018-0574.
9. E. Deem, L.N. Cattafesta, H. Yao, M.S. Hemati, H. Zhang, and C.W. Rowley, "Experimental Implementation of Modal Approaches for Autonomous Reattachment of Separated Flows," AIAA Aerospace Sciences Meeting, Kissimmee, FL, 8-12 January 2018, AIAA Paper 2018-1052.
10. M.S. Hemati and H. Yao, "Dynamic Mode Shaping for Fluid Flow Control: New Strategies for Transient Growth Suppression," 8th AIAA Theoretical Fluid Mechanics Conference, Denver, CO, 5-9 June 2017, AIAA Paper 2017-3160.
11. E. Deem, L. Cattafesta, H. Zhang, C. Rowley, M. Hemati, F. Cadieux, and R. Mittal, "Identifying Dynamic Modes of Separated Flow Subject to ZNMF-based Control from Surface Pressure Measurements," 47th AIAA Fluid Dynamics Conference, Denver, CO, 5-9 June 2017, AIAA Paper 2017-3309.

Theses:

1. D. Bhattacharjee, "Data-driven modeling, analysis, and control of flow separation over an airfoil," Master's Thesis, August 2018, University of Minnesota Digital Conservancy, <http://hdl.handle.net/11299/201020>.

10 | Recognitions/Interactions/Transitions

In this section we summarize recognitions, interactions, and plans for transitions associated with research supported by AFOSR Grant FA9550-17-1-0252.

Presentation at Meetings

1. Y. Sun, H. Yao, A. Kalur, and M.S. Hemati (2019) “Feedback Control of Transitional Channel Flow via Reduced-Order Modeling,” AIAA Fluid Dynamics Conference, Dallas, TX, June 17–21. **(Invited)**
2. M. Budišić, J.N. Kutz, and M.S. Hemati (2019), “Dynamic Mode Decompositions and Koopman Analysis,” SIAM Conference on Applications of Dynamical Systems, Snowbird, UT, May 19–23.
3. M.S. Hemati, D. Bhattacharjee, B. Klose, and G.B. Jacobs (2019) “Optimal actuator selection for airfoil separation control using empirical data,” SIAM CSE, February 25–March 1, Spokane, WA. **(Invited)**
4. A. Kalur and M.S. Hemati, (2018) “Feedback flow control for transient energy growth minimization using convex-optimization techniques,” 71st Annual Meeting of the APS Division of Fluid Dynamics, November, Atlanta, GA.
5. H. Yao and M.S. Hemati, (2018) “Static output feedback strategies for suppressing transient energy growth in channel flow,” 71st Annual Meeting of the APS Division of Fluid Dynamics, November, Atlanta, GA.
6. M.S. Hemati, D. Bhattacharjee, B. Klose, and G.B. Jacobs (2018) “Optimal actuator selection for flow control based on empirical data,” 71st Annual Meeting of the APS Division of Fluid Dynamics, November, Atlanta, GA.
7. H. Yao and M.S. Hemati (2018) “Perspectives on Koopman Spectral Specification,” SIAM Annual Meeting, July 9–13, Portland, OR. **(Invited)**
8. E.A. Deem, L.N. Cattafesta, H. Zhang, C.W. Rowley, R. Mittal, M.S. Hemati (2018) “On Reduced Order Models for Control of a Canonical Separated Flow,” AIAA Aviation Meeting, Atlanta, GA. **(Invited)**.
9. M.S. Hemati and H. Yao, (2018) “New Strategies and Insights for Transition Delay using Dynamic Mode Shaping Control,” 89th Annual Meeting of the International Association of Applied Mathematics and Mechanics (GAMM), March 19–23, Munich, Germany. **(Invited)**.
10. M.S. Hemati and H. Yao, (2017) “Dynamic Mode Shaping for Transient Growth Suppression,” 70th Annual Meeting of the APS Division of Fluid Dynamics, November, Denver,

CO.

11. H. Yao and M.S. Hemati, (2017) “Observer-Based Feedback Control of Transient Energy Growth: Revisiting the Separation Principle,” 70th Annual Meeting of the APS Division of Fluid Dynamics, November, Denver, CO.
12. S.T.M Dawson, M.S. Hemati, and C.W. Rowley, (2017) “Practical Extensions and Applications of Dynamic Mode Decomposition” AIAA Aviation Meeting, June, Denver, CO. **(Invited)**.

Organized Symposia

1. AIAA Fluid Dynamics Conference (2019),
Reduced-Complexity Modeling for Flow Control,
co-organized w/ Karthik Duraisamy
2. SIAM Conference on Applications of Dynamical Systems (2019),
Dynamic Mode Decompositions and Koopman Analysis,
co-organized w/ Nathan Kutz and Marko Budišić.
3. AIAA Fluid Dynamics Conference (2018),
Modal Analysis of Fluid Flows,
co-organized w/ Karthik Duraisamy and Kunihiko Taira.
4. AIAA Flow Control Conference (2018),
Modal Analysis for Flow Control,
co-organized w/ Karthik Duraisamy and Kunihiko Taira.
5. APS Division of Fluid Dynamics Conference (2017),
Modal Analysis of Fluid Flows (1 Tutorial and 2 Focus Sessions)
co-organized w/ Mitul Luhar and Kunihiko Taira.
6. AIAA Fluid Dynamics Conference (2017),
Modal Analysis: Applications and New Approaches,
co-organized w/ Kunihiko Taira.

Consultative and Advisory Functions

- A number of interactions with AFRL personnel were maintained, especially with Dr. Albert Medina and Dr. Matthew Rockwood on investigations of fast-flap modeling and control.

A | Limitations of Modal Truncation for Transient Energy Growth Controller Design

Modal truncation based on the eigendecomposition of the matrix $A = V\Lambda V^{-1}$ in (4.1) is a common approach for reduced-order modeling, primarily owing to its simplicity. Since the eigenvalues λ_i in the diagonal matrix Λ provide information about the relative time-scales of the modal response, a simple strategy for modal truncation is to omit modes using time-scale arguments. For example, when long term behavior is of interest, then modes with fast decay rates and/or high oscillation frequencies can be omitted. ROMs constructed time-scale based modal truncation will not necessarily capture the input-output dynamics needed for controller synthesis. Thus, another common approach considers a measure of modal controllability/observability, given by the ratio [72]

$$\zeta_i = \frac{\|C_i\| \|B_i\|}{|\operatorname{Re}\{\lambda_i\}|}, \quad (\text{A.1})$$

where B_i and C_i correspond to the i^{th} row and column of the modal representations of the matrices B and C from (4.1), respectively. Modes with larger values of ζ_i contribute more to the input-output dynamics and should be retained, whereas those with lower values of ζ_i can be truncated.

Here, we apply both of these modal truncation approaches to arrive at ROMs with order $r < n$ of the linearized channel flow system with $(\alpha, \beta) = (1, 0)$ and $Re = 3000$. For truncation modal truncation by time-scale arguments, we truncate the “fastest” modes (i.e., stable modes with relatively large $|\operatorname{Re}\{\lambda_i\}|$). For truncation by modal controllability/observability, we truncate the least controllable/observable modes (i.e., stable modes with relatively small ζ_i). Each of these ROMs is then used to design the LMI-ROM and LQR-ROM controllers. The resulting performance of each of these controllers is compared with the LQR-FOM controller in Figure A.1. The model orders here were chosen based on convergence of the ROM open-loop frequency-response relative to the FOM. We note that for the results reported in the Appendix here, the FOM dimension is set to $n = 99$; without doing so, ROMs based on modal truncation converge with $r > 80$, which makes the LMI-based synthesis procedure intractable with the available computational resources. Figure A.2 reports the resulting MTEG $\Theta_{\max}$ as a function of r for ROMs constructed by each of the modal truncation approaches. Interestingly, the truncation of “fast modes” yields improved MTEG performance with the LMI-ROM than truncation based on ζ_i . These results indicate that modal truncation is a poor candidate for designing controllers to reduce TEG. Although in some cases these models can result in effective control laws, the model orders tend to be higher than the tailored approaches considered in this study. The short-comings of modal truncation—and other ROM methods—for TEG reduction further motivates the need for control-oriented model

reduction, like the approach presented in Section 4.2.1.

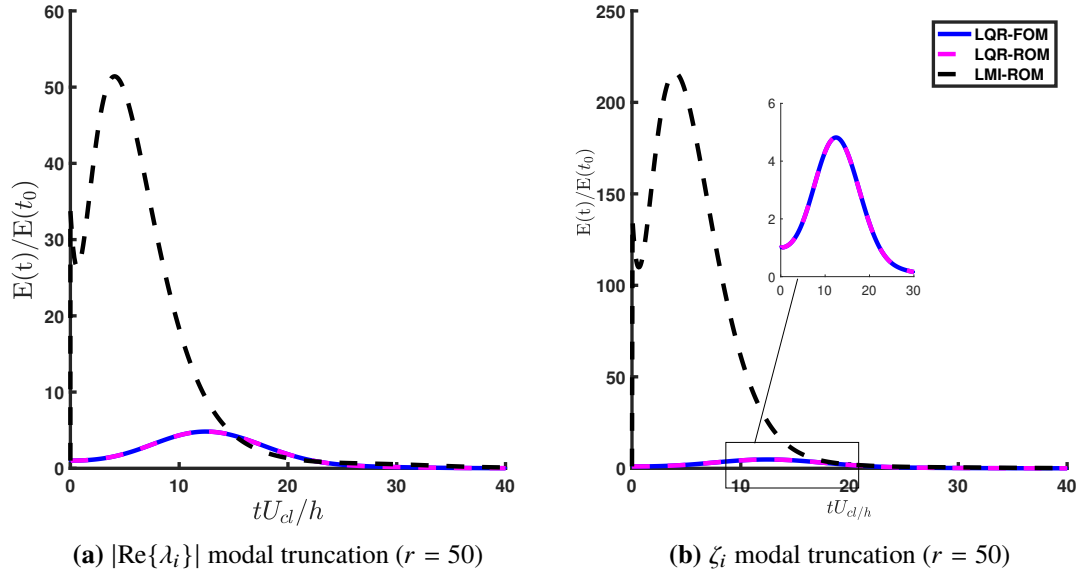


Figure A.1: Worst-case closed-loop TEG response for $(\alpha, \beta) = (1, 0)$ and $Re = 3000$.

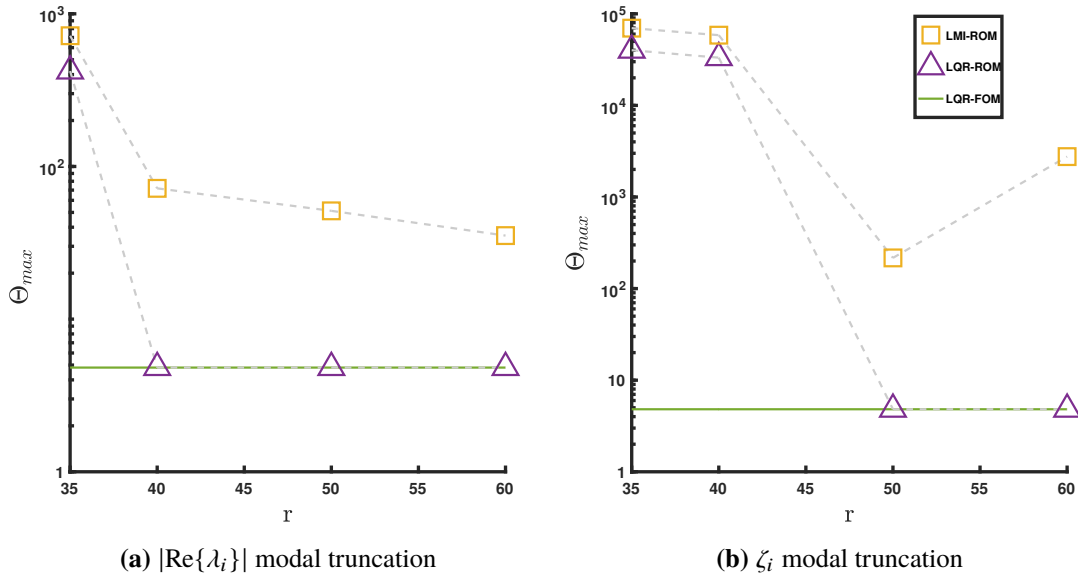


Figure A.2: Closed-loop MTEG Θ_{max} as a function of ROM order r for $(\alpha, \beta) = (1, 0)$ at $Re = 3000$.

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