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Power-Energy/Force-Momentum Relations and Additional Boundary Conditions for Spatially Dispersive Polarized Materials

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1. Determine unambiguous macroscopic power and energy relations that hold for diamagnetic as well as paramagnetic/ferro(i) magnetic media and that elucidate the role of hidden energy;
2. Using our rigorous homogenization theory of spatial dispersion, formulate a general method for determining the additional boundary conditions (ABCs) needed for spatially dispersive dipolar materials/metamaterials and higher-order multipolar media directly from Maxwells equations and the constitutive relations;
3. Derive an exact solution for the force on Amperian (circulating current) magnetic dipoles in arbitrarily time varying fields to decide with certainty whether there exists a hidden-momentum force on Amperian magnetic dipoles, that is, on the magnetic dipoles found in materials and metamaterials. (All experiments to date have failed to find magnetic charge (monopoles) or magnetic-charge dipoles.)

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Final Performance Report for AFOSR Grant FA9550-16-C-0017

Power-Energy/Force-Momentum Relations and Additional Boundary Conditions for Spatially Dispersive Polarized Materials

Program Manager: Dr. Arje Nachman, Principal Investigator: Dr. Arthur D. Yaghjian

1 Introduction

In view of the previous advances made by the principal investigator in formulating and developing a comprehensive, rigorous homogenization theory for spatially and temporally dispersive polarized materials and metamaterials that can exhibit diamagnetic as well as paramagnetic and ferromagnetic behavior, the AFOSR three-year Grant FA9550-16-C-0017 was awarded to S4-Inc to determine robust, definitive expressions for the power-energy and force-momentum supplied by the electromagnetic fields to electric and magnetic polarized media, and to derive a deterministic set of boundary conditions with the necessary "additional boundary conditions" for interfaces between free space and spatially (as well as temporally) dispersive media. The primary tasks that were proposed and successfully completed can be briefly summarized as follows:

1. Determine unambiguous macroscopic power and energy relations that hold for diamagnetic as well as paramagnetic/ferro(i)magnetic media and that elucidate the role of "hidden energy";
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Selected highlights of these accomplishments, each of which have been published during the past three years in archival journals will be summarized in the remainder of this final report. Additional research accomplishments published during the past three years but not required by the grant proposal, namely overcoming the Chu lower bound on the quality factor of antennas with highly dispersive material, the determination of group and energy-transport velocities in spatially dispersive media, and the discovery of physical realizability limitations for the commonly used series resistance-reactance representation of antennas, are briefly described in the last section of the report.

2 Hidden Energy in the Classical Electrodynamics of Dipolar Media

As part of the completion of the first primary task, we discovered the rather remarkable result that a "hidden energy" is exhibited by permanent Amperian magnetic dipoles rotating in applied fields that makes the total energy supplied to the Amperian dipoles equal to that supplied to the equivalent hypothetical magnetic-charge dipoles. This result leads to different expressions for the time-domain energy supplied to macroscopic magnetization in diamagnetic media (zero permanent dipole moments) and paramagnetic/fero(i)magnetic media (substantial permanent dipole moments). For passive media, each of these time-domain energy expressions remains positive semi-definite (nonnegative) for all time t after an initial time t_0 at which the macroscopic fields are zero.

To determine the energy supplied by external fields to Amperian magnetic dipoles, we began by carefully applying Maxwell's equations to a perfectly electrically conducting (PEC) wire loop carrying a static magnetic dipole moment that rotates in an externally applied field. Consider the energy $P_{j_b e}(t)$ supplied to this electrically small, PEC, thin, rigid, wire loop spanning the area A and carrying an initial current I_0 such that the initial magnetic dipole moment of the loop is $\mathbf{m}_0 = I_0 A\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the normal to the loop in the direction determined by the right-hand rule applied to the circulation of I_0 (see Fig. 1). With the wire loop illuminated by external electric and magnetic fields, $\mathbf{e}_{\rm ext}(\mathbf{r},t)$ and $\mathbf{b}_{\rm ext}(\mathbf{r},t)$, we can write

$$P_{j_b e}(t) = \int_V \mathbf{j}_b(\mathbf{r},t)\cdot\mathbf{e}(\mathbf{r},t)dV = \int_V \mathbf{j}_b(\mathbf{r},t)\cdot\mathbf{e}_{\rm ext}(\mathbf{r},t)dV + \int_V \mathbf{j}_b(\mathbf{r},t)\cdot\mathbf{e}_{\rm ind}(\mathbf{r},t)dV \tag{1}$$

where $\mathbf{e}_{\rm ind}(\mathbf{r},t)$ is the induced electric field, that is, the electric field generated by the current $\mathbf{j}_b(\mathbf{r},t)$ in the loop. For an

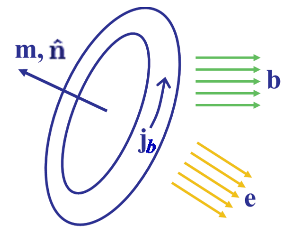

Figure 1: Electrically small, PEC, rotating, thin, rigid wire loop carrying an initial current I_0 and subject to an external field so that the total microscopic current distribution $\mathbf{j}_b$ produces a dipole moment $\mathbf{m}$.

electrically small loop carrying a current $I(t)$ at the time t, the integral in (1) involving $\mathbf{e}_{\rm ext}$ can be evaluated quasi-statically to give

$$\int_V \mathbf{j}_b(\mathbf{r},t)\cdot\mathbf{e}_{\rm ext}(\mathbf{r},t)dV = -\mathbf{m}(t)\cdot\frac{\partial\mathbf{b}_{\rm ext}(\mathbf{r}_0,t)}{\partial t}. \tag{2}$$

The result in (2) expresses the power supplied by the externally applied voltage to the current in the PEC wire loop in terms of the magnetic dipole moment and the time derivative of the external magnetic field at each instant of time t.

A relatively easy way to evaluate the integral in (1) involving $\mathbf{e}_{\rm ind}$ for an electrically small loop is to extend the integration from V to all space V_∞. Then, from Ampere's law, we eventually arrive at

$$\int_V \mathbf{j}_b(\mathbf{r},t)\cdot\mathbf{e}_{\rm ind}(\mathbf{r},t)dV = -\frac{d}{dt}\left[\frac{1}{2\mu_0}\int_{V_\infty}|\mathbf{b}_{\rm ind}(\mathbf{r},t)|^2 dV\right] = -\frac{d}{dt}\left[\frac{1}{2}LI^2(t)\right] \tag{3}$$

so that

$$P_{j_b e}(t) = \int_V \mathbf{j}_b(\mathbf{r},t)\cdot\mathbf{e}(\mathbf{r},t)dV = -\mathbf{m}(t)\cdot\frac{\partial\mathbf{b}_{\rm ext}(\mathbf{r}_0,t)}{\partial t} - \frac{d}{dt}\left[\frac{1}{2}LI^2(t)\right]. \tag{4}$$

This equation (4) says that the power $P_{j_b e}(t)$ supplied by the electric field to the current carrying PEC wire loop is equal to the total power supplied by the external fields minus the time rate of change of the energy stored in the quasi-magnetostatic field of the current loop. Moreover, it can be proven that

$$\frac{d}{dt}\left[\frac{1}{2}LI^2(t)\right] = -\frac{\partial}{\partial t}\left[\mathbf{b}_{\rm ext}(\mathbf{r}_0,t)\cdot\mathbf{m}\right] + \frac{dI}{dt}\left[\mathbf{b}_{\rm ext}(\mathbf{r}_0,t)\cdot\hat{\mathbf{n}}A\right] \tag{5}$$

so that (4) becomes

$$P_{j_b e}(t) = \mathbf{b}_{\rm ext}(\mathbf{r}_0,t)\cdot\left(\frac{d\mathbf{m}}{dt} - \hat{\mathbf{n}}\frac{dm}{dt}\right) \approx \mathbf{b}_{\rm ext}(\mathbf{r}_0,t)\cdot\frac{d\mathbf{m}(t)}{dt} \tag{6}$$

for paramagnetic materials in which the induced diamagnetic moments are negligible compared with the aligning (rotating) "permanent" magnetic dipole moments. Although the magnitude of the induced magnetic dipole moment is negligible compared to the magnitude of the total magnetic dipole moment as the dipole aligns in an external magnetic field, the change in energy produced by the induced magnetic dipole moment is not negligible.

The result in (6) is revealing: *The power supplied by the electric field applied to an inclusion or molecule with a "permanent" Amperian magnetic dipole moment (modeled by a PEC wire loop with an initial dipole moment $\mathbf{m}_0$) that predominates over any induced diamagnetic moment (as is apparently the case for most if not all known natural paramagnetic materials, which includes ferro(i)magnetic and antiferromagnetic materials) is the same as the power supplied by the magnetic field to a fixed-magnitude (m_0) hypothetical magnetic dipole moment formed by separated equal and opposite magnetic charge.*

Under the same condition that the rotating "permanent" magnetic dipole moments dominate the induced magnetic dipole moments, we have from (5) and (3) that

$$\int_V \mathbf{j}_b(\mathbf{r},t)\cdot\mathbf{e}_{\rm ind}(\mathbf{r},t)dV \approx \frac{\partial}{\partial t}\left[\mathbf{b}_{\rm ext}(\mathbf{r}_0,t)\cdot\mathbf{m}\right] \tag{7}$$

which can also be obtained by subtracting (2) from (6). This power supplied by the internal electric field to the current in the PEC wire loop is a microscopic "hidden power" drawn from the reservoir of inductive energy in the initial microscopic Amperian magnetic dipole moment. *This "hidden energy/power" caused by the induced current on the Amperian dipole has been completely overlooked in past derivations of energy and power supplied to Amperian magetic dipoles.*

3 Derivation of Additional Boundary Conditions (ABC's) Directly from Maxwell's Differential Equations

Electric quadrupolar continua satisfying a physically reasonable constitutive relation supports both an evanescent and a propagating eigenmode. Thus, three interface boundary conditions, two plus an "additional boundary condition" (ABC), are required to obtain a unique solution to a plane wave incident from free space upon an electric quadrupolar half space. By generalizing the constitutive relation to hold within the transition layer between the free space and the quadrupolar continuum, we derive these three boundary conditions directly from Maxwell's differential equations. The three boundary conditions are used to determine the unique solution to the boundary-value problem of an electric quadrupolar slab. It appears that the general method used to derive the electric quadrupolar ABC can be applied to obtain the boundary conditions for any other realizable constitutive relation in a Maxwellian multipole continuum.

Maxwell's homogeneous differential equations for the macroscopic fields in the electric quadrupolar continuum can be written as

$$\nabla \times \mathbf{E} - i\omega \mathbf{B} = 0 \tag{8}$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} + i\omega\epsilon_0 \mathbf{E} - \frac{1}{2} i\omega \nabla \cdot \overline{\mathbf{Q}} = 0 \tag{9}$$

with the following physically reasonable electric quadrupolar constitutive relation for a random distribution of spherically symmetric electric quadrupolar particles in source-free external fields

$$\overline{\mathbf{Q}} = \alpha_Q \epsilon_0 \left[\frac{1}{2}(\nabla\mathbf{E} + \mathbf{E}\nabla) - \frac{1}{3}(\nabla \cdot \mathbf{E})\bar{\mathbf{I}}\right] \tag{10}$$

where the constant α_Q is a macroscopic electric quadrupolarizability density (real-valued in a lossless continuum) and $\mathbf{E}\nabla$ denotes the transpose of the $\nabla\mathbf{E}$ dyadic. The $\overline{\mathbf{Q}}$ in (10) is a symmetric dyadic (a requirement of electric quadrupolarizability) with zero trace that can be derived from the electric quadrupole moment of an electrically small dielectric-sphere inclusion in a source-free local electric field.

To derive the three required boundary conditions from Maxwell's differential equations, we consider a planar interface between free space and the electric quadrupolar continuum. Let the interface be coincident with the xy plane of a rectangular coordinate system whose positive z axis is normal to the interface and points into the quadrupolar half space, as shown in Fig. 2. There will be an interface transition layer across which the electromagnetic properties of the medium change from those of free space to those of the electric quadrupolar continuum. We let this transition layer of thickness ℓ lie between $z = 0$ and $z = \ell$, also shown in Fig. 2. In the free space to the left of the transition layer ($z < 0$), the value of the electric quadrupolarizability

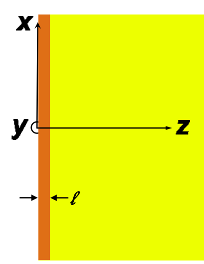

Figure 2: Geometry of interface between free space and electric quadrupolar continuum with transition layer of thickness ℓ.

density in (10) is zero, and to the right of the transition layer ($z > \ell$) its value is equal to α_Q. Since the quadrupolarizability of each of the electric quadrupoles is finite, a macroscopic electric quadrupolarizability density α_0 that holds throughout all space is finite in the transition layer and can be expressed as

$$\alpha_0 = \alpha_Q u(z) \tag{11}$$

where $u(z)$ is a continuous "unit-step" function that varies from 0 (at $z = 0$) to 1 (at $z = \ell$) across the transition layer. For an ideal continuum, $\ell \to 0$ and $u(z)$ approaches the usual unit-step distribution function. With α_0 in (11) replacing α_Q in (10), the electric quadrupolarization density $\overline{\mathbf{Q}}_0$ that holds throughout all space, including the transition layer, is given by

$$\overline{\mathbf{Q}}_0 = \alpha_Q \epsilon_0 u(z) \left[\frac{1}{2}(\nabla\mathbf{E} + \mathbf{E}\nabla) - \frac{1}{3}(\nabla \cdot \mathbf{E})\bar{\mathbf{I}}\right]. \tag{12}$$

We note that $\overline{\mathbf{Q}}_0$ equals $\overline{\mathbf{Q}}$ in (10) for $z > \ell$. With the electric quadrupolarization density in (12), Maxwell's homogeneous differential macroscopic equations that hold throughout all space are given by

$$\nabla \times \mathbf{E} - i\omega\mathbf{B} = 0 \tag{13}$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} + i\omega\epsilon_0\mathbf{E} - \frac{1}{2}i\omega\nabla \cdot \overline{\mathbf{Q}}_0 = 0. \tag{14}$$

Substitute $\mathbf{B}$ from (13) into (14) to get a single vector wave equation for $\mathbf{E}$, namely

$$\nabla \times \nabla \times \mathbf{E} - k_0^2\mathbf{E} + \frac{1}{2}\omega^2\mu_0\nabla \cdot \overline{\mathbf{Q}}_0 = 0 \tag{15}$$

with $\nabla \cdot \overline{\mathbf{Q}}_0$ obtained from (12) as

$$\nabla \cdot \overline{\mathbf{Q}}_0 = \alpha_Q\epsilon_0 \left\{ u(z)\nabla \cdot \left[\frac{1}{2}(\nabla\mathbf{E} + \mathbf{E}\nabla) - \frac{1}{3}(\nabla \cdot \mathbf{E})\overline{\mathbf{I}}\right] + \delta(z)\hat{\mathbf{z}} \cdot \left[\frac{1}{2}(\nabla\mathbf{E} + \mathbf{E}\nabla) - \frac{1}{3}(\nabla \cdot \mathbf{E})\overline{\mathbf{I}}\right]\right\} \tag{16}$$

where $\delta(z) = du(z)/dz$. For the TM solution with the magnetic field in the y direction, but no variation in the y direction, and $e^{ik_{0x}x}$ variation in the x direction, the E_y field is zero everywhere, and the evaluation of the x and z components of (15) and (16) gives the following two equations

$$\left(k_0^2 + \frac{\partial^2}{\partial z^2}\right) E_x - ik_{0x}\frac{\partial E_z}{\partial z} - \frac{1}{2}\omega^2\mu_0 \left(\nabla \cdot \overline{\mathbf{Q}}_0\right)_x = 0, \quad \left(k_0^2 - k_{0x}^2\right) E_z - ik_{0x}\frac{\partial E_x}{\partial z} - \frac{1}{2}\omega^2\mu_0 \left(\nabla \cdot \overline{\mathbf{Q}}_0\right)_z = 0 \tag{17}$$

where the x and y components of $\nabla \cdot \overline{\mathbf{Q}}_0$ are

$$\left(\nabla \cdot \overline{\mathbf{Q}}_0\right)_x = \alpha_Q\epsilon_0 \left[\frac{1}{2}\frac{\partial}{\partial z}\left(u\frac{\partial E_x}{\partial z}\right) + ik_{0x}\left(\frac{1}{6}u\frac{\partial E_z}{\partial z} + \frac{1}{2}\delta E_z\right) - \frac{2}{3}k_{0x}^2 u E_x\right] \tag{18}$$

$$\left(\nabla \cdot \overline{\mathbf{Q}}_0\right)_z = \alpha_Q\epsilon_0 \left[\frac{2}{3}\frac{\partial}{\partial z}\left(u\frac{\partial E_z}{\partial z}\right) + ik_{0x}\left(\frac{1}{6}u\frac{\partial E_x}{\partial z} - \frac{1}{3}\delta E_x\right) - \frac{1}{2}k_{0x}^2 u E_z\right] \tag{19}$$

and we have made use of the relation $\partial/\partial x = ik_{0x}$.

One cannot assume a priori that the electric field cannot contain delta functions in the transition layer. In fact, a planar multipole expansion for the continuously differentiable electric and magnetic fields outside the source plane at $z = 0$ shows that within the source plane the electric and magnetic fields can be expressed as a continuous function plus a sum of unit-step distribution functions, that is, the unit-step function and all its derivatives. Consequently, the components of the electric field everywhere, including the transition layer, can be expressed as

$$E_x(z) = E_x^r(z) + E_x^u u(z) + E_x^\delta \delta(z) + E_x^{\delta'} \delta'(z) + \cdots \tag{20}$$

$$E_z(z) = E_z^r(z) + E_z^u u(z) + E_z^\delta \delta(z) + E_z^{\delta'} \delta'(z) + \cdots \tag{21}$$

in which the $e^{ik_{0x}x}$ variation in the x direction is suppressed. The $E_x^r(z)$ and $E_z^r(z)$ are continuous ramp functions that have constant values in the transition layer, and $\partial/\partial z$ derivatives from the left and right of the transition layer that can have different values. The constant values of $E_x^r(z)$ and $E_z^r(z)$ in the transition layer are not necessarily zero. The primes on the delta functions in (20)–(21) denote differentiation with respect to z and, of course, $\delta(z) = u'(z)$. Unit-step/delta distribution theory legitimizes the use of delta functions and shows that there is no loss in generality in choosing the defining step function in (20)–(21) equal to the $u(z)$ in (11) as the thickness of the transition layer approaches zero, that is, $\ell \to 0$.

If E_x and E_z are substituted from (20) and (21) into (17), the $\left(\nabla \cdot \overline{\mathbf{Q}}_0\right)_x$ and $\left(\nabla \cdot \overline{\mathbf{Q}}_0\right)_z$ quadrupolar source terms will contain squares of δ functions and derivatives of delta functions. Judiciously utilizing this information eventually leads to the three boundary conditions

$$\mathbf{E}_s^{(2)} - \mathbf{E}_s^{(1)} = 0. \tag{22}$$

$$\hat{\mathbf{n}} \cdot \overline{\mathbf{Q}}^{(2)} \cdot \hat{\mathbf{n}} = 0. \tag{23}$$

$$\mathbf{B}_s^{(2)} - \mathbf{B}_s^{(1)} = -\frac{i\omega\mu_0}{2}\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \cdot \overline{\mathbf{Q}}^{(2)}). \tag{24}$$

In summary, the three boundary conditions in (22)–(24) at the interface between free space and an electric quadrupolar continuum satisfying the constitutive relation in (10) (with its generalization in (12) to include the transition layer) have been derived directly from Maxwell's differential equations in (13)–(14).

The three boundary conditions in (22)–(24) can be used to solve the problem of a transverse magnetic (TM) plane wave incident from free space upon an electric quadrupolar slab of thickness z_0 satisfying the constitutive relation in (10). There is a reflected wave in free space, a propagating and evanescent wave traveling in both directions in the slab, and a transmitted wave leaving the trailing interface of the slab. The boundary conditions in (22)–(24) are applied at both the leading and trailing interfaces of the slab. It is verified that the sum of the magnitudes squared of the reflection and transmission coefficients equals unity to within computer accuracy. It can be shown in that the boundary conditions in (22)–(24) ensure the continuity of the normal component of the Poynting's vector at the interfaces, and hence this conservation of energy.

Secondly, the same scattering problem is solved assuming that the evanescent modes are negligible and applying the two conventional boundary conditions. Again it is verified that the sum of the magnitudes squared of the reflection and transmission coefficients equals unity to within computer accuracy.

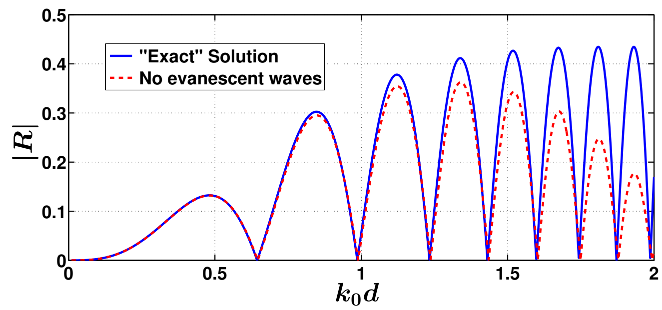

Figure 3: Magnitude of the reflection coefficient using the "exact" solution from the three boundary conditions (22)–(24) and the approximate solution from the two conventional boundary conditions omitting the evanescent waves.

In Fig. 2, the magnitude of the reflection coefficient is plotted using the "exact" solution from the three boundary conditions (22)–(24) and the approximate solution from the two conventional boundary conditions that omit the evanescent waves. The macroscopic quadrupolarizability constant α_Q and the thickness z_0 of the slab are normalized to a hypothetical average quadrupole spacing d of the material such that $\alpha_Q = .27d^2$ and the thickness of the slab is $z_0 = 20d$. The angle that the propagation vector of the incidence TM plane wave makes with the normal to the interface is equal to 80 degrees for the results shown in Fig. 2. This large angle of oblique incidence is chosen to obtain an appreciable value of the reflection coefficient. At small values of the incident angle, the scattering from the slab is so small that the reflection coefficient is $\ll 1$ until values of k_0d are reached that are much greater than 1.

An important result revealed in Fig. 2 is that the conventional electric quadrupolar boundary conditions with the evanescent waves neglected (or assumed to be part of the transition layer) produce accurate results for $k_0d \lesssim 1$. In other words, the relative magnitude of the electric quadrupolarization of both the evanescent mode and the extra delta functions in the electric quadrupolarization that exist in the transition layer become negligible as k_0d becomes less than 1.

Using the same method outlined here for electric quadrupolar continua, there appears no obstacle to deriving directly from the Maxwell differential equations additional boundary conditions (ABC's) for higher order multipole media that produce a deterministic set of boundary conditions for the extra modes that may arise depending upon the particular constitutive relations. A noteworthy result revealed by this direct method for deriving ABC's is that the boundary conditions for a spatially dispersive medium depend strongly on the constitutive relations that hold in that medium.

4 Force and Hidden Momentum for Microscopic Magnetic Dipoles

Electromagnetic force expressions are required in a variety of applications that include the determination of the radiation forces on satellites and space solar panels and other electromagnetic forces in space physics and cosmology, the prediction of the accelerations of molecules subjected to high-powered lasers, the computation of the electromagnetic levitation forces in magnetic suspension systems, the measurement of the resonant forces in image-force microscopy for nanoscale imaging, the evaluation of the safety of MRI field exposure of human tissue, and the development of electromagnetic-field therapies for improving microcirculation. Nonetheless, for arbitrarily time varying fields, there is still no consensus on the correct form of the magnetic polarization force over 100 years since Einstein and Laub published the first paper on the subject in 1908.

The "proverbial controversy" centers about the question of whether the Amperian (circulating current) magnetic dipole, which models the magnetic dipoles found in all natural materials and metamaterials, when subject to an external field, produces an internal hidden momentum and force that makes the total momentum and force on the Amperian magnetic dipole equal to that on an equivalent hypothetical magnetic-charge dipole.

After the principal investigator proved, as explained in Section 2, that permanent Amperian magnetic dipoles exhibit "hidden energy and power", it encouraged us to undertake that more difficult problem of determining unequivocally whether or not there exists an analogous hidden momentum and force within Amperian magnetic dipoles. This problem of determining the momentum/force on Amperian magnetic dipoles turns out to be much more difficult than that of determining the energy/power because in determining the momentum/force one cannot take advantage of Poynting's energy/power theorem. After many unsuccessful attempts to obtain approximate solutions for the the force exerted on Amperian dipoles by arbitrary time varying external fields, it was concluded that a more fruitful approach may be to try to find an exact solution to the force on a realistic model for an Amperian magnetic dipole, namely an electrically small perfect electric conductor (PEC).

Our work with approximate solutions indicated that an exact-solution may be possible if one could find Maxwellian equations for the quasi-static fields of PEC's that become exact as the electrical size of the PEC approaches zero. Although no such equations existed, we finally discovered a path that enabled the derivation of the required time-domain "electroquasistatic and magnetoquasistatic" equations, namely

$$\nabla \times \mathbf{E}_{es}(\mathbf{r}, t) = 0 \tag{25a}$$

$$\nabla \times \mathbf{H}_{es}(\mathbf{r}, t) - \epsilon_0 \frac{\partial \mathbf{E}_{es}(\mathbf{r}, t)}{\partial t} = \mathbf{J}_1(\mathbf{r}, t) \tag{25b}$$

$$\nabla \cdot \mathbf{E}_{es}(\mathbf{r}, t) = \frac{\rho(\mathbf{r}, t)}{\epsilon_0} \tag{25c}$$

$$\nabla \cdot \mathbf{H}_{es}(\mathbf{r}, t) = 0 \tag{25d}$$

for the electroquasistatic equations and

$$\nabla \times \mathbf{E}_{ms}(\mathbf{r}, t) + \mu_0 \frac{\partial \mathbf{H}_{ms}(\mathbf{r}, t)}{\partial t} = 0 \tag{26a}$$

$$\nabla \times \mathbf{H}_{ms}(\mathbf{r}, t) = \mathbf{J}_2(\mathbf{r}, t) \tag{26b}$$

$$\nabla \cdot \mathbf{E}_{ms}(\mathbf{r}, t) = 0 \tag{26c}$$

$$\nabla \cdot \mathbf{H}_{ms}(\mathbf{r}, t) = 0 \tag{26d}$$

for the magnetoquasistatic equations, where $\mathbf{J}_2$ is the solenoidal current and $\mathbf{J}_1$ is the remaining current. These equations predict scattered fields that possess no $1/r$ radiation far fields as $r \to \infty$. In the near fields ($r < r_0$), these approximate scattered fields become exact as the electrical size of the PEC becomes small.

With the help of these quasi-static equations, it was possible to derive the exact force on electrically small PEC's to show unequivocally for the first time that indeed a hidden-momentum force exists on an arbitrarily time varying magnetic dipole moment $\mathbf{m}(t)$ *in an external electric field* $\mathbf{E}_e(\mathbf{r}, t)$ equal to

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left[\mathbf{m}(t) \times \mathbf{E}_e(\mathbf{r}, t)\right] \tag{27}$$

where $\mathbf{r}$ is the position of the dipole and c is the speed of light. This hidden force brings the total force exerted by the external electric field on the Amperian magnetic dipole equal to

$$-\frac{1}{c^2}\frac{d\mathbf{m}(t)}{dt} \times \mathbf{E}_e(\mathbf{r}, t) \tag{28}$$

which is identical to the force that would be exerted by $\mathbf{E}_e(\mathbf{r}, t)$ on a hypothetical magnetic-charge dipole moment (this latter force on separated magnetic charge being relatively easy to derive). Moreover, it was shown that the total force in (28) is indeed transferred to the metal conductor carrying the current and does not change the kinetic momentum of the moving charges. This means that the total force in (28) is the actual force the platform holding the conductor would experience as the PEC is subjected to an external field.

The exact Mie solution to the perfectly conducting sphere under plane-wave illumination is used to prove that these expressions for the total and hidden-momentum forces on the arbitrarily shaped electrically small perfect conductors correctly predict the forces on perfectly conducting spheres. *Remarkably, it is found that the quadrupolar fields at the surface of the sphere are required to obtain the correct hidden and total forces on the sphere even though the quadrupolar moments are negligible compared to the dipole moments as the electrical size of the sphere approaches zero.*

5 Additional Published Research Acomplishments (Not Required under the Grant)

5.1 Overcoming the Chu lower bound on antenna Q with highly dispersive lossy material

With the principal investigator's determination of rigorous equations for quasi-static electromagnetic fields, it was realized that these equations could be used to evaluate the "quality-factor (Q) energy" of antennas containing highly dispersive materials. Carrying through this evaluation, it was found that electrically small antennas could be designed and tuned with highly dispersive lossy materials with twice the bandwidth predicted by the Chu lower bound as long as the antenna was not required to have high efficiency. Antenna simulations verified these conclusions, which showed for the first time since Chu's 1948 paper that Chu's bounds on the maximum bandwidth (minimum Q) of electrically small, linear, time-invariant, single-feed, single-resonant antennas could be overcome.

5.2 Group and Energy-Transport Velocities in Spatially Dispersive Media

With the help of the recently formulated electric and magnetic anisotropic representation of spatially dispersive materials or metamaterials, the electromagnetic power flow, energy density, and the energy-transport velocity that equals the group velocity of a lossless wavepacket were derived in a relatively simple straightforwardly manner for spatially dispersive media. Using the recently derived boundary conditions in spatially dispersive media, the expressions obtained for the power flow and energy density are confirmed analytically and numerically. A long-standing alternative expression for the energy-transport velocity in spatially dispersive media is shown to be invalid. These wavepackets in spatially dispersive media reduce in free space to the wavepackets of the quasi-monochromatic radiation emitted by the statistically independent sources of the sun and other stars.

5.3 Realizability Conditions for the Series Input Resistance/Reactance of the Antennas

The input impedance $Z(\omega) = V(\omega)/I(\omega)$ of a single-port, time-invariant, linear, passive antenna is often represented by a resistance in series with a reactance, $Z(\omega) = R(\omega) - iX(\omega)$. Because the input impedance is a function of frequency, the series resistance and reactance must, in general be functions of frequency to represent the antenna over a wide bandwidth. Since passivity implies causality and stability for the input impedance, it follows that $Z(\omega)$ is a causal function.

The passivity and causality of $Z(\omega)$ does not imply that $X(\omega)$ is a reactance that can be physically realized by passive causal circuit elements (capacitors, inductors or transformers) with a series resistance $R(\omega)$ that can be physically realized by a passive causal resistor. To the contrary, we prove that if and only if

$$X'(\omega) = dX(\omega)/d\omega \geq 0 \tag{29}$$

for all ω are a series $R(\omega)$ and $X(\omega)$ physically realizable by passive causal circuit elements. We prove these results by applying a time-domain current $i(t)$ and determining the energy dissipated in the resistance and the energy accumulated (stored) by the reactance. The energy accumulated by the reactance at each frequency is revealed to be proportional to $X'(\omega)$, so that the energy accumulated is negative if $X'(\omega) < 0$. As negative stored energy is impossible for a passive reactance, $X(\omega)$ and thus $R(\omega)$ cannot be physically realized by passive causal circuit elements.

These results do not imply that $X'(\omega)$, the frequency derivative of the negative imaginary part of $Z(\omega)$, cannot be negative, a condition that occurs, for example, at antiresonances of antennas. It simply means that such a reactance cannot be represented by passive causal reactive elements in series with a passive causal resistor. It is well known that many different circuit models can represent the same input impedance $Z(\omega)$. Although realizable circuits contain only passive causal resistors, inductors or transformers, and capacitors, a series R-X circuit representation for the input impedance of an antiresonant antenna must contain a noncausal resistor and active/noncausal reactance elements. We have illustrated these conclusions with simple examples and discuss their implications for determining energy expressions that can be applied to determine the quality factors of antennas containing highly lossy dispersive material, as discussed in Subsection 5.1.

Publications Resulting from the Grant

[1] A.D. Yaghjian, "Electroquasistatic and Magnetoquasistatic Equations and Fields," *Proc. EMTS*, 4 pages, 2019 (Invited).

[2] A.D. Yaghjian, "Force and Hidden Momentum for Classical Microscopic Dipoles," *PIER B*, vol. 82, pp. 165–188, November 2018.

[3] A.D. Yaghjian, "Overcoming the Chu Lower Bound on Antenna Q with Highly Dispersive Lossy Material," *IET Microwaves, Antennas & Propagation*, pp. 459–466, March 2018 (Invited).

[4] A.D. Yaghjian, "Power Flow, Energy Density, and Group/Energy-Transport Velocities in Spatially Dispersive Media," *Radio Science*, pp. 303–313, March 2018 (Invited).

[5] A.D. Yaghjian, "Physical Unrealizability of a Series Reactance and Resistance of a Passive Causal Input Impedance," *Proc. of ICEAA*, pp. 1620–1623, September 2017 (Invited).

[6] A.D. Yaghjian, "Hidden Energy in the Classical Electrodynamics of Dipolar Media," *Proc. of Metamaterials*, pp. 376–378, August-September 2017 (Invited).

[7] A.D. Yaghjian, "Classical Power and Energy Relations for Macroscopic Dipolar Continua Derived from the Microscopic Maxwell Equations," *PIER B*, vol. 71, pp. 1–37, November 2016.

[8] A.D. Yaghjian and M.G. Silveirinha, "Additional Boundary Conditions for Electric Quadrupolar Continua Derived from Maxwell's Differential Equations," *Radio Science*, vol. 51, pp. 1312–1321, September 2016 (Invited).

[9] A.D. Yaghjian, "Nonnegative Energy for Dipolar Continua," *Proc. of URSI-EMTS*, pp. 222–225, August 2016 (Invited).

[10] A.D. Yaghjian, "Maxwell's Approach to Deriving the EM Field Equations in Dipolar Continua," *Proc. of URSI-EMTS*, pp. 300–303, August 2016 (Invited).