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1 Major goals and achievements

Nowadays when modeling real-life phenomena, in many areas stochastic methods have become more and more popular in addition to traditional methods based on deterministic ordinary differential equations and partial differential equations. Here modeling means to propose a mathematical model that could reproduce complex behavior reflexed by empirical data. Broadly speaking, stochastic modeling usually consists of two steps. The first is to construct a probability model that represent certain important features of the phenomena of interest. In this step, the features of interest, or more essentially the underlying dynamics, are determined by a set of parameters of the model constructed. The second step is then to develop statistical methods to inference the parameters from a given data set; other applications shall follow once a good estimate of parameters is obtained. The construction of the model in the first step is crucial and depends on both a deep understanding of the science behind the data and the mastery of probability tools. At the same time, it also depends on the applications to be carried out, as often the number of parameters is chosen appropriately with respect to the size of the data and also the available computing powers.

The PI's research has been focused on probabilitistic aspect of modeling, more precisely on developing limit theorems for discrete models. Here, a discrete model refers to a collection of random variables over discrete temporal/spatial lattice, modeling for example the trajectory of a particle moving around in space, the fluctuations of stock prices and internet traffic rates, or the landscape of random surface representing temperatures and/or rainfall amounts in environmental studies. A ubiquitous phenomenon is that when the system under investigation becomes large, or equivalently the number of random variables becomes large, after appropriate scaling and normalization, different models may behave in very similar manners. Nowadays, it is well understood that such phenomena can be rigorously and elegantly described as limit theorems in probability theory, using stochastic processes to characterize the limiting behavior of discrete models.

In view of applications, limit theorems first provide quantitative information on how to scale the model appropriately as the dimension of the data grows, and on how at macroscopic level the dependence is represented by the stochastic process in the scaling limit. Moreover, the limit stochastic process can be viewed as a qualitative description of a given discrete model: some models may have the same scaling limit, and hence can be viewed as *very similar*, or even *indistinguishable*, at macroscopic level; while those do not have the same scaling limit should be viewed as *fundamentally different*. Both quantitative and qualitative information from limit theorems provide guidelines when building and choosing models for practical purposes. On the other hand from a purely theoretical point of view, the developments of probability theory have been motivated by many examples from real-life phenomena. Models with simple-to-describe dynamics may present formidable challenges for mathematicians, and sometimes lead to new directions of theoretical research. This is in particular the case for the theoretical investigation of *long-range dependence* [48, 56], a main theme of this research program. The field of long-range dependence was originally motivated by a problem in hydrology in the 1950s (on modeling the water level of Nile river [34, 35]), and started by a series of investigations on the issues by Benoit Mandelbrot in the 60s [42–44].

This research project aims at establishing limit theorems for various stochastic models that exhibit long-range dependence. Establishing central limit theorems and extremal limit theorems for various models will be the main focus of the current research activities of the PI. In the presence of long-range dependence, the scaling limits may differ model by model, and may require different tools accordingly. This is in stark contrast to limit theorems for models with weak dependence, where roughly speaking the scaling limits are expected to be the same as if the model consists of i.i.d. random variables. Overall many of our results have the following two features.

- A new stochastic process arises in the limit theorem for the proposed discrete model. This implies that the proposed model characterizes a different type of dynamics from other known ones.
- A phase-transition phenomenon occurs. For models with long-range dependence, also known as *with long memory*, there is often a *memory parameter* that interacts with other parameters. Drastically different scaling limits may arise depending on the relations between memory and other parameters.

The current research follows PI's previous research project on *Limit theorems for random fields*, partially supported by NSA grant H98230-14-1-0318 between 2014 and 2016. Here, the starting point is the so-called Karlin model [38] that have been shown to exhibit long-range dependence. The Karlin model (essentially the paintbox partition) has been known to be fundamental in combinatorial stochastic processes [50], and with many applications in Bayesian inference and machine learning recently. The PI's approach via limit theorems, however, is new to Karlin model. The Karlin model and its intrinsic dependence structure determined by random partitions are a main motivation for this research program (whence the 'random partition' in the title). Eventually some other closely related problems but not necessarily related to random partitions have been also addressed.

Overall, the research project had originally three major goals, all successfully achieved by the end of the grant period. In this report, we explain in more details the following four sub-topics below. A few ongoing projects here are beyond the original proposal plan, but serve as the motivating problems for PI's next ARO proposal submitted at the beginning of 2019, titled *Advances in extreme-value theory with long-range dependence* (see Section 2.5).

- **Section 2.1: Karlin random fields: limit theorems, representations, and simulations.** This part is on variations of the Karlin model [25, 38]. We first established a limit theorem for a Karlin model with heavy-tailed randomization, and then revealed that the so-obtained Karlin stable processes can be actually defined in a more general framework. In particular, we obtained an extension of the classical Lévy–Chentsov stable fields [57] to the so-called fractional Lévy–Chentsov stable fields, the law of which is uniquely determined by the geodesic metric on the manifold index set and a parameter $\beta \in (0,1)$. New stochastic-integral representations have been developed. Simulation methods are currently being investigated.

  Outcomes (published papers, or in preparation): [24, 30, 31]

- **Section 2.2: Limit theorems for random fields with long-range dependence.** A few other random-field models indexed by $\mathbb{N}^d$ or $\mathbb{R}^d$ with long-range dependence are considered, all are shown to have different scaling limits. Two models [6, 59] exhibit a phase-transition phenomenon.

  Outcomes: [6, 7, 27, 59]

- **Section 2.3: Extremes with long-range dependence.** In this part, the main goal is investigate extremes of stationary sequences with long-range dependence. The characterization of asymptotic behavior of extremes is by limit theorems on empirical random sup-measures. For some examples, the limit random sup-measures are not even of Fréchet type. In particular, in [58] the PI and coauthor solved an open problem [55] in 2004 for the extremes of a class of stable processes first introduced in [54]. We are also in the progress of developing a new aggregation framework for extreme-value theory that shall include all the random sup-measures as special cases and many new ones [64].

  Outcomes (published papers, or in preparation): [26, 28, 58, 60, 64]

- **Section 2.4: Limit theorems for interacting particle systems.** The PI and coauthor also investigated a specific example of interacting particle systems, namely the asymmetric simple exclusion processes with open boundary [21]. This process is an extensively studied model in mathematical physics modeling non-equilibrium systems, and in biology modeling protein synthesis. Here, we provided a complete picture of the phase transition of the density fluctuation [14], significantly expanding earlier results which are valid only for a small range of parameter sets, and revealed a new type of fluctuations at the critical parameters. Our method is a hybrid of the well known matrix ansatz method in combinatorics [19, 22] and recently developed tools on Askey–Wilson Markov processes [15, 16], and a few devices of our own invention [11, 12, 62]. We also demonstrated our method applies to other families of combinatorial structures [13].

  Outcomes: [12–14]

1.1 A complete list of publications, preprints, and preprints in preparation

(The following are listed in chronological order of arxiv submissions.)

1. G. Samorodnitsky and Y. Wang. Extremal theory for long range dependent infinitely divisible processes. *Annals of Probability*, 47(4):2529–2562, 2019.

2. W. Bryc and Y. Wang. Dual representations of Laplace transforms of Brownian excursion and generalized meanders. *Statistics and Probability Letters*, 140:77–83, 2018.

3. H. Biermé, O. Durieu and Y. Wang. Generalized random fields and Lévy's continuity theorem on the space of tempered distributions. *Communications on Stochastic Analysis*, 12(4): Article 4, 2018.

4. W. Bryc and Y. Wang. Limit fluctuations for density of asymmetric simple exclusion processes with open boundaries. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 55(4):2169–2194, 2019.

5. O. Durieu and Y. Wang. From random partitions to fractional Brownian sheets. *Bernoulli*, 25(2):1412–1450, 2019.

6. O. Durieu, G. Samorodnitsky and Y. Wang. From infinite urn schemes to self-similar stable processes. *Stochastic Processes and their Applications*, to appear.

7. Y. Shen and Y. Wang. Operator-scaling Gaussian random fields via aggregation. *Bernoulli*, 26(1):500–530, 2020.

8. O. Durieu and Y. Wang. A new family of random sup-measures with long-range dependence. *Electronic Journal of Probability*, 23(107):1–24, 2018.

9. S. Stoev and Y. Wang. Exchangeable random partitions from max-infinitely-divisible distributions. *Statistics and Probability Letters*, 146:50–56, 2019.

10. H. Biermé, O. Durieu and Y. Wang. Generalized operator-scaling random ball model. *Latin American Journal of Probability and Mathematical Statistics*, 15:1401–1429, 2018.

11. W. Bryc and Y. Wang. Fluctuations of random Motzkin paths. *Advances in Applied Mathematics*, 106:96-116, 2019.

12. Z. Fu and Y. Wang. Stable processes with stationary increments parameterized by metric spaces. *Journal of Theoretical Probability*, to appear.

13. S. Bai, T. Owada and Y. Wang. A functional non-central limit theorem for multi-stable processes with long-range dependence. https://arxiv/org/abs/1902.00628

14. Y. Wang and N. Zhang. *An aggregation framework for extreme-value theory with long-range dependence.* In preparation.

15. Z. Fu and Y. Wang. *Simulations for a family of stable random fields.* In preparation.

16. O. Durieu and Y. Wang. *A phase transition in a stochastic volatility model with long-range dependence.* In preparation.

17. L. Wang and Y. Chen. *Confidence Graphs for Graphical Model Selection*. In preparation.

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2 Summary of main results

2.1 Karlin random fields: limit theorems, representations, and simulations

The first main theme of the results in the past few years of the PI is on the Karlin model. The original model was an infinite urn model investigated first by Karlin [38] in the 1960s, which has turned out to play a fundamental role in combinatorial stochastic processes [50] (essentially a paintbox random partition that scales to the Poisson–Dirichlet $(\alpha, 0)$ distribution, $\alpha \in (0, 1)$). Here, in a series of recent developments from a completely different aspect, PI and coauthors revealed several interesting features of the model. In PI's first work on the Karlin model [25], it was shown that with a simple randomization of the original model, the Karlin model should be viewed as a (negatively-) correlated random walk with ± 1 steps (denoted by a stationary sequence $\{X_n\}_{n\in\mathbb{N}}$) that scales to a fractional Brownian motion with Hurst index $H < 1/2$ [25]. In other words, the Karlin model plays the role to fractional Brownian motion with $H < 1/2$ as the simple random walk to the standard Brownian motion. The correspondence is more than formal: the Karlin model provides us a new combinatorial approach to the fractional Brownian motion, as many statistics of the Karlin model are straightforward to compute based on the underlying random combinatorial structures (partitions). Following [25], after a series of results by the PI and coauthors we now understand that the dependence in the Karlin model is much more general.

- The limit theorems for Karlin model can be extended to models with both light and heavy-tailed randomization and to random-field models [24, 25, 31]. The scaling limits, referred to as *Karlin random fields*, are stable (Gaussian and non-Gaussian) random fields indexed manifolds $\mathbb{R}^d$ (the Euclidean space), $\mathbb{S}^d$ (the Euclidean sphere) and $\mathbb{H}^d$ (the hyperbolic space).

- The Karlin random fields that arise from the limit theorems include in particular a family of so-called *fractional Lévy–Chentsov stable fields*, which can be viewed as a natural extension of *Lévy–Chentsov stable fields* [31]. These are manifold-indexed stable fields with the law uniquely determined by the geodesic metric d, essentially due to a new identity regarding the measure definite kernels in integral geometry. Our models for non-Gaussian stable random fields are new, and for Gaussian stable fields we have developed new integral representations.

- We have developed tools for efficient simulations for Karlin random fields [30].

Now we provide more details on our achievements in the aforementioned three directions. For limit theorems, with $\{X_n\}_{n\in\mathbb{N}}$ a sequence of stationary random variables, and $S_n := X_1 + \cdots + X_n$, we are interested in results of the form

$$\left\{\frac{S_{\lfloor nt\rfloor}}{n^\beta}\right\}_{t\in[0,1]} \Rightarrow \{Z_t\}_{t\in[0,1]},$$

for some parameter $\beta > 0$, and we are in particular interested in what types of stochastic processes arise on the right-hand side. As mentioned earlier, the Karlin model in one dimension can be viewed as a correlated random walk, with steps $\{X_n\}_{n\in\mathbb{N}}$ a sequence of stationary random variables either taking values from $\{\pm 1\}$ [25] or being heavy-tailed ones [24]. Throughout by heavy tails we mean tails of random variables that decay asymptotically as $x^{-\alpha}$ up to a multiplicative slowly varying function. In the case that X_n has finite second moments, the limit process Z is a fractional Brownian motion with Hurst index $\beta/2 \in (0, 1/2)$. When the random variables are heavy-tailed, the limit process is expected to be a self-similar α-stable process with stationary increments, and such processes form a too large class to be indexed by a single parameter (as the fractional Brownian motion), and an extensive literature existed already.

What is surprising was in [24] we proved that the so-obtained stable process is of a new type, and later on in [31], we showed that the limit theorem can be generalized to random-field setup and obtained manifold-indexed stable random fields at the end. (For this purpose we did not work with integer lattice in general, but consider a closely related Poissonized model for the discrete model [31].) We refer to the limit random fields as Karlin random fields.

After having obtained the Karlin random fields via limit theorems, we examined their properties and representations. We focus on a sub-family of Karlin random fields that extend the classical Lévy–Chenstov stable fields here (another family included in our framework is the multiparameter fractional Brownian/stable motions recently introduced by [33]). It is known that the classical Lévy Brownian fields [41] are centered Gaussian processes $\mathbb{G}$ indexed by $\mathsf{M} = \mathbb{R}^n, \mathbb{S}^n$ or $\mathbb{H}^n$ with covariance determined by the geodesic metric

$$\mathrm{Cov}(\mathbb{G}(x), \mathbb{G}(y)) = \mathsf{d}(x, y), x, y \in \mathsf{M},$$

and their stable-process versions can be obtained by replacing the Gaussian random measure by a stable one in the stochastic-integral representation of these processes. It is known that for the geodesic metric d, d^β with $\beta \in (0, 1)$ is still a valid metric on M of negative type, which in turn determines a centered Gaussian process with

$$\mathrm{Cov}(\mathbb{G}_\beta(x), \mathbb{G}_\beta(y)) = \mathsf{d}^\beta(x, y), x, y \in \mathsf{M}.$$

The existence of fractional Lévy Brownian fields is known [18, 36]. Our contribution here is to provide a new stochastic-integral representation for these fields: the new representation does not only deepen our understanding of these Gaussian fields, but also provides a canonical extension to a family of stable random fields which we termed *fractional Lévy–Chentsov stable fields* $\{Z_{\alpha,\beta}(x)\}_{x\in\mathsf{M}}$. The law of this family is intrinsically related to the geodesic metric d on the manifold. While the stochastic-integral representation is a little technical for review purpose, it is more intuitive to think of the following property of $Z_{\alpha,\beta}$:

$$\frac{Z_{\alpha,\beta}(x) - Z_{\alpha,\beta}(y)}{\mathsf{d}^{\beta/\alpha}(x,y)} \stackrel{d}{=} c \cdot \mathrm{S}\alpha\mathrm{S}, x, y \in \mathsf{M}, \tag{1}$$

where the right-hand side stands for a constant multiplicative of a standard symmetric α-stable law (note that the above only determines the Gaussian field $\alpha = 2$ though).

In view of (1) the extension of fractional Lévy–Chentsov stable fields from Lévy–Chentsov stable field has a strong geometric flavor, and we believe the former may serve as a canonical model for manifold modeling in future. For example, the derivative of a Lévy Brownian field is often referred to as the white noise, and the derivative of a fractional Lévy–Chentsov Brownian field is the *colored noise*. Nowadays, many studies on stochastic (partial) differential equations (S(P)DEs) are with respect colored noise, and may studies have been known for S(P)DEs driven by stable Lévy processes. The fractional Lévy–Chenstov stable fields can be viewed as a colored version for Lévy–Chenstov stable fields, and we believe that relevant numerical methods should be of potential applications.

The simulation methods for Gaussian random fields have a huge literature, and, although in theory numerical methods that provide exact simulations exist, they are mostly not applicable in the manifold setup because of the huge computational cost. So for Gaussian fields we already need to develop new methods that only simulate good approximations of the fields in an efficient way. Furthermore, for stable fields it is well known that the best way to go is to divide the random field of interest into two parts, one determined by large jumps from the Poisson-point-process representation and the other by small jumps. This seminal idea was introduced by Asmussen and Rosiński [1], and has been applied to certain stable random fields indexed by Euclidean space by Cohen et al. [17]. In the latest on-going project [30], we followed the framework in the latter and extended it to general manifold-indexed stable fields, and in addition we developed a new stochastic-integral representation for the Karlin random fields to significantly improve the efficiency of the algorithm. Some simulation results are shown in Figure 1.

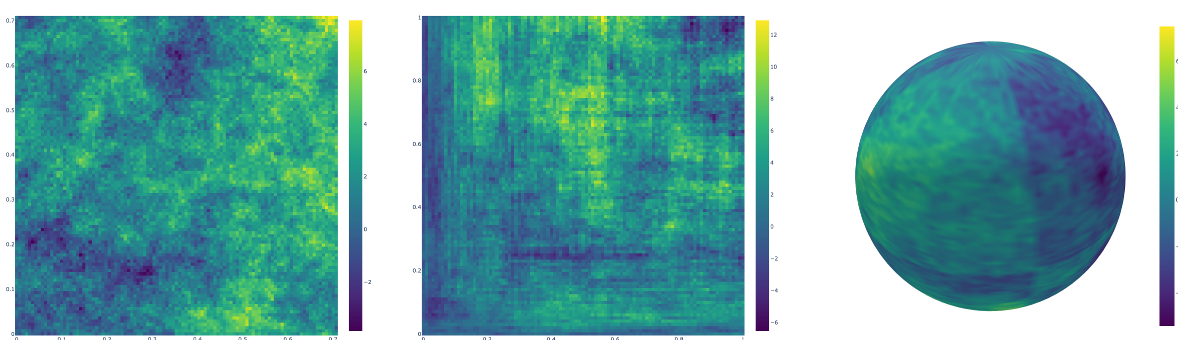

Figure 1: Simulations for (from left to right) $\mathbb{R}^2$-indexed fractional Lévy–Chentsov stable fields, multiparameter fractional stable fields, $\mathbb{S}^2$-indexed fractional Lévy–Chentsov stable fields.

2.2 Limit theorems for random fields with long-range dependence

Besides limit theorems for Karlin random fields, the PI and coauthors have also investigated a few other random-field models. Here, by random fields we mean a stochastic process indexed by $\mathbb{R}^2$ or $\mathbb{N}^2$. In the simplest setup, we are interested in the partial-sum limit of a stationary random field. Namely, let $\{X_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathbb{N}^2}$ be a stationary random field, we are interested in the asymptotic behavior of

$$S_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{i}\in[\boldsymbol{1},\lfloor \boldsymbol{n}\cdot\boldsymbol{t}\rfloor]} X_{\boldsymbol{i}},$$

where $\boldsymbol{n}\cdot\boldsymbol{t} = (n_1t_1, n_2t_2) \in \mathbb{R}^2$, as $\boldsymbol{n} = (n_1, n_2) \to \infty$ in an appropriate sense. Again, here we take the limit behavior of $S_{\boldsymbol{n}}$ as a qualitative classification of the underlying stationary random field. This task turned out to have a couple new features unseen in one dimension.

- First, even when the limit random field is Gaussian, unlike in one dimension where the scaling limit is always a fractional Brownian motion, a one-parameter family of Gaussian processes, here the limit Gaussian random fields consist of a much larger family, including for example fractional Brownian sheets but also other self-similar Gaussian random fields with stationary increments. There are much less examples for such limit theorems in the literature than the corresponding one-dimensional ones.

- Second, the choice of $\boldsymbol{n} \to \infty$ is delicate in certain situations. In some cases it suffices to assume $\min(n_1, n_2) \to \infty$ as a simple extension of the one-dimensional limit theorems. In words, this means the rectangles over which the summation is taken grow to infinity in both directions. However, in a few recent results [5, 51, 52] it was shown that for certain models with long-range dependence, the relative rates of the growth of each direction matter. More precisely, the asymptotic ratio of n_1^β/n_2 being either $0, c \in (0,\infty)$ or ∞, for some β depending on the dependence structure of the model, shall determine a distinct scaling limit. This is referred to as a *scaling-transition* phenomenon.

Besides [31] which includes a limit theorem for $\mathbb{R}^d$-indexed random field, a few other random-field models have been addressed by the PI and coauthors recently.

- Two-layer models with random partitions at the bottom were introduced [27]. In this work, models based on random partitions of $\mathbb{N}^2$ are introduced, and it is proved that their corresponding scaling limits are fractional Brownian sheets with all possible legitimate parameters, a canonical Gaussian random field with long-range dependence. The models can be roughly viewed as hybrids of the Karlin [25, 38] and Hammond–Sheffield [32] ones in one dimension.

- New anisotropic random fields with long-range dependence are shown to arise from an *aggregation* scheme [59]. A similar phenomena has been well known in the time-series setup and with many applications in financial and econometric data analysis. In words, the data from each individual may behave as with short-range dependence, while the overall accumulative data may exhibit certain qualitatively different features. The PI and coauthor characterized a similar phenomena in the spatial setup [59], by generalizing the model of Enriquez [29] in one dimension. In the spatial setup, however, the analysis becomes much more demanding, and in particular the aforementioned *scaling-transition* phenomenon occurs.

- In [6], the PI and coauthors provided a unified framework for the so-called *random-ball model*, summarizing and extending earlier results including in particular [8–10]. Most importantly, the introduced framework allows in particular to deal with the tightness of the model under scaling, a fundamental technical issue in limit theorem. In particular, the limit theorems are established as random elements in the space of *tempered distributions* [7].

2.3 Extremal limit theorems with long-range dependence

The second main theme of PI's results in the past few years is on random sup-measures with long-range dependence. Given a sequence of stationary random variables $\{X_n\}_{n\in\mathbb{N}}$, instead of the cumulative behavior often one would like to characterize the asymptotic behavior of the extremes (e.g. the top few order stastistics) in various applications. The studies of extremes of stationary sequences have a very long history. It is well known that if for example X_i is symmetric with power-law tails and the dependence is weak enough, one can establish the point-process convergence in the form of

$$\sum_{i=1}^{n} \delta_{(X_i/a_n, i/n)} \Rightarrow \sum_{i=1}^{\infty} \delta_{\left(\varepsilon_i/\Gamma_i^{1/\alpha}, U_i\right)}, \tag{2}$$

where a_n is a slowly varying function with index $1/\alpha$, and $\{(\Gamma_i, U_i)\}_{i\in\mathbb{N}}$ are points from a Poisson point process on $\mathbb{R}_+ \times (0,1)$ with intensity measure $dxdu$ and $\{\varepsilon_n\}_{n\in\mathbb{N}}$ are i.i.d. Rademacher random variables. The Poisson point process in the limit encodes the *magnitudes* of the limit order statistics and their corresponding *locations*. Under very mild assumption on the weak dependence, each order statistic occurs at a unique location, which is easily seen to be uniform (strictly speaking at a single location there might be clustered extremes). Standard framework and many tools had been developed for such convergence by the 1980s [40, 53].

We are interested in extremes for models with long-range dependence. The seminal works of O'Brien et al. [46], Vervaat [61] showed that the limit object for extremes of long-range dependence should be encoded by a random sup-measure instead of the Poisson point process on the right-hand side of (2). As a large family of random sup-measures, the so-called *Choquet random sup-measures* [45], they are essentially determined by a different type of Poisson point processes, in the form of

$$\sum_{i=1}^{\infty} \delta_{\left(\varepsilon_i/\Gamma_i^{1/\alpha}, \mathcal{R}_i\right)},$$

where this time Γ_i is as before but $\{\mathcal{R}_i\}_{i\in\mathbb{N}}$ are i.i.d. random closed sets in $[0,1]$. The fundamental change here is that asymptotically each order statistic may occur at multiple (or even uncountably infinite) locations, represented by the random closed set $\mathcal{R}_i$. Equivalently, the Poisson-point-process representation above leads to a random sup-measure, up to a multiplicative constant due to thinning,

$$\mathcal{M}(\cdot) \stackrel{\mathrm{d}}{=} \sup_{t\in\cdot} \frac{1}{\Gamma_i^{1/\alpha}} \mathbf{1}_{\{t\in\mathcal{R}_i\}}. \tag{3}$$

Surprisingly, there were very few examples of limit theorems for Choquet random sup-measures in the literature, until the recent one by Lacaux and Samorodnitsky [39]. PI's motivation in this line of research is to understand better how Choquet random sup-measures may arise from stochastic models when investigating extremes. The models are necessarily of long-range dependence, and the feature that the same order statistic may occur at multiple locations is referred as *long-range clustering*. The main achievements in this direction are as follows.

- The PI and coauthor showed that the heavy-tailed Karlin model leads to a new family of random sup-measures which we termed as the Karlin random sup-measure [26], which has the form (3) with, when restricted to $(0,1)$,

$$\mathcal{R} \stackrel{\mathrm{d}}{=} \bigcup_{i=1}^{Q_\beta} \{U_i\}$$

  where Q_β is the Sibuya distribution ($\mathbb{E}z^{Q_\beta} = 1-(1-z)^\beta, |z|<1$) and $\{U_i\}_{i\in\mathbb{N}}$ are i.i.d. uniform random variables independent from Q_β.

- The PI and coauthor solved an open question in 2004 [55] on the asymptotic extremes of a family of stable processes with long-range dependence (technically speaking, driven by a conservative and null flow) proposed by Rosiński and Samorodnitsky [54]. We showed that for certain range of parameters there the limit random sup-measure is *beyond the family of Choquet random sup-measures* [58], taking the form of

$$\mathcal{M}(\cdot) \stackrel{\rm d}{=} \sup_{t\in\cdot} \sum_{i\in\mathbb{N}} \frac{1}{\Gamma_i^{1/\alpha}} \mathbf{1}_{\{t\in\mathcal{R}_j\}}. \tag{4}$$

Here, when restricted to $[0,1]$ each $\mathcal{R}_j$ is a shifted β-stable regenerative set [3]. The structure of the random sup-measure depends on the intersection of independent stable regenerative sets, and it is known that if $\beta \in ((p-1)/p, p/(p+1)]$ for some $p \in \mathbb{N}$, then almost surely p independent stable regenerative sets intersect but $p+1$ ones do not. Furthermore, when the intersection is non-empty, it is a β_p-stable regenerative set with $\beta_p = p\beta - p + 1 \in (0,1)$. The intersections of stable regenerative sets bring several interesting properties to the random sup-measure, denoted by $\mathcal{M}_{\alpha,\beta}$. For example, as a consequence we have that $\mathcal{M}_{\alpha,\beta}([0,1])$ as a random variable converges in distribution to a one-sided α-stable distribution as $\beta \uparrow 1$ for $\alpha \in (0,1)$, and to an α-Fréchet distribution as $\beta \downarrow 0$ for all $\alpha > 0$.

- The PI and coauthor provided a general framework to show how random sup-measures in the form of (3) may arise from aggregated models, and how the aggregation may cause the limit random sup-measures to go beyond the Choquet family [64] to have the form (4) for other choices of $\mathcal{R}_j$ that have non-trivial intersections. All the random sup-measures that recently appeared in the literature [26, 58] may arise from this framework, as well as a few new ones. Note that when $\{\mathcal{R}_j\}_{j\in\mathbb{N}}$ do not intersect with probability one, the above is the same as (3).

2.4 Limit theorems for interacting particle systems

The third main achievement by the PI during the review period is on the so-called asymmetric simple exclusion process (ASEP), an extensively investigated model in interacting particle systems for non-equilibrium systems [20, 21]. The main result is a complete characterization for the asymptotic density fluctuation of ASEP in steady state for the full range of parameters. Not only we extended previous results which were known only for a very limited range of parameters [23], but also we revealed a phase transition in terms of the limit fluctuations and proved that a new type of fluctuations arise at the critical values of the parameters. To achieve this result, the PI and coauthor proceeded in several steps.

- The starting point is a recent result by Bryc and Wesołowski [16] which relates the well-known *matrix ansatz* due to Derrida et al. [22] to *Askey–Wilson processes*, a family of Markov processes recently introduced in Bryc and Wesołowski [15]. Formally, for each $n \in \mathbb{N}$ we let $\tau_i \in \{0,1\}$ indicates the occupation (or not) of location $i \in \{1,\ldots,n\}$ by a particle, and we are interested in the fluctuations of the centered height function $h_n(t) = \sum_{i=1}^{\lfloor nt\rfloor}(\tau_i - \mathbb{E}\tau_i)$. Then, the law of $\{h_n(t)\}_{t\in[0,1]}$ is essentially determined by the generating function of $\tau_1,\ldots,\tau_n$ by

$$\mathbb{E}\left(\prod_{i=1}^n t_i^{\tau_i}\right) = \frac{\langle W \mid (E+t_1D)\times\cdots\times(E+t_nD) \mid V\rangle}{\langle W \mid (E+D)^n \mid V\rangle} = \frac{\mathbb{E}[\prod_{j=1}^n(1+t_j+2\sqrt{t_j}Y_{t_j})]}{2^n\mathbb{E}(1+Y_1)^n}.$$

(Note that on the left-hand side the expectation depends on $n \in \mathbb{N}$.) The first equality above is the well known matrix ansatz, where E, D are infinite matrices and $\langle W|$, $|V\rangle$ are infinite vectors, and the second equality is the identity relating matrix ansatz to Askey–Wilson processes Y, which we know the explicit transition probability density function, determined by Askey–Wilson polynomials. Therefore, the above identity provides a new tool to compute the Laplace transform of the height function of the ASEP

$$\mathbb{E}\exp\left(-\sum_{k=1}^d c_k h_n(x_k)\right) = \mathbb{E}\exp\left(-\sum_{k=1}^d \sum_{j=\lfloor nx_{k}-1\rfloor+1}^{\lfloor nx_k\rfloor} (\tau_j - \mathbb{E}\tau_j)\,(c_k+\cdots+c_d)\right), \tag{5}$$

and we can express the right-hand side above as a functional of the Askey–Wilson process Y.

- Next, in the process of computing the asymptotic Laplace function, we discovered an intriguing connection to the so-called *tangent process* of Askey–Wilson processes recently studied by the PI and coauthor [11]. The same notion of tangent processes has been well known to be useful in extreme-value theory for Gaussian processes [49], although very few other examples have been known beyond the Gaussian case. Here the tangent process is computable thanks to the explicit formula for the probability density function of Askey–Wilson processes. The computation of the limit Laplace function in (5) is then essentially based on the tangent process, and at the end we obtained a closed form expression in terms of integrals, which should be the Laplace transform of the limit fluctuation as a stochastic process.

  However, it is not straightforward to read the limit expression and identify the corresponding stochastic process, and our result [12] was devoted to this purpose. In fact, we revealed the following identity which has its own interest in the study of stochastic processes: for all $d \in \mathbb{N}$, $s_0 = 0 < s_1 < \cdots < s_d$ and $t_0 = 0 < t_1 < \cdots < t_d \leq 1 = t_{d+1}$,

$$\mathbb{E} \exp\left(-\sum_{k=1}^{d}(s_k - s_{k-1})\mathbb{B}^{ex}_{t_k}\right) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathbb{E}_x \exp\left(-\frac{1}{2}\sum_{k=0}^{d}(t_{k+1} - t_k)X_{s_k}\right)\sqrt{x}dx, \quad (6)$$

  where $\{\mathbb{B}^{ex}_t\}_{t\geq 0}$ is a Brownian excursion and the process $\{X_s\}_{s\geq 0}$ is the so-called $1/2$-stable Biane process we characterized in [11]. This is another Markov process with explicit transition density function, first introduced by Biane [4]. A similar identity was shown to hold if we replace the measure $\sqrt{x}dx$ on the right-hand side by a different one in the form of $x^\beta dx$, and on this case in the left-hand side the Brownian excursion is replaced by a generalized Brownian meander, whence the limit fluctuation becomes the sum of a Brownian motion and a Brownian meander, the two processes being independent.

  Now back to our investigation on the fluctuations of ASEP. In one regime of the parameters, the right-hand side of (5) is asymptotically equal to, after an appropriate normalizing, the right-hand side of (6) multiplied by another term which is immediately recognized as the Laplace transform of Brownian motion. So the limit fluctuation is the sum of a Brownian motion and a Brownian excursion, the two processes being independent. At the critical regime of parameters, we obtain the right-hand side of (6) with $\sqrt{x}dx$ replaced by $x^{-1/2}dx$, corresponding to the Laplace transform of a Brownian meander.

- Last but not least, we showed that the same strategy in [14] applies to a more general framework by working out a simplified example of fluctuations of random Motzkin paths [13]. It is known that the Motzkin paths model is closely related to ASEP [19]. This result indicates that our method can apply to a large family of models from the literature of combinatorics.

2.5 Results relevant to the next proposal

The PI has submitted another proposal to ARO, titled *Advances in extreme-value theory with long-range dependence*, at the beginning of 2019. The new proposal is motivated by several results established in the past three years and a few others initiated during the same period. Below are brief descriptions on how the recent results (in progress) are related to the next proposal.

- Multiple-stable processes with long-range dependence.

  In a recent work [2], the PI and coauthors investigated limit theorems for a family of multiple-stable processes with long-range dependence. This work follows the recent breakthroughs in [47, 58], and investigates the *multi-stable processes*, stochastic processes represented by multiple stable integrals, instead of stable processes. The generalization from stable processes to multi-stable processes presents a few new challenges, and hence tools are developed accordingly. In particular, our solution is based on a thorough investigation of intersections of independent stable regenerative sets [3], and a new description of their joint local-time processes by formula of their joint moments.

As follow-ups, several open questions remain for the multiple-stable processes considered by the PI. The question on the extremes is the first one to be addressed. Some primitive investigations suggest that there exists another phase transition in terms of the relation between the memory parameter β, the tail index of the stable process α, and the degree of multiple integrals p.

- Stochastic volatility models and hierarchical models.

  Hierarchical models have received significant attentions in the literature recently. The idea here is to build up new hierarchical models using the well investigated one with long-range dependence, and then to understand the macroscopic dependence caused by interactions of (memory) parameters from different layers. Since the Karlin model has been shown to be of fundamental feature in recent studies, we expect it a fruitful direction to incorporate it into a hierarchical one. Other models with long-range dependence will also be considered.

  The first result in this direction is a two-layer model, which can be interpreted as a stochastic volatility model. As a continuation of [26], the PI is currently investigating a hybrid stochastic volatility model using the Karlin model to represent the volatility, and i.i.d. random variables with power-law tails to represent the noise signals, namely

$$X_n = Y_n Z_n$$

  where $\{Y_n\}_{n\in\mathbb{N}}$ is the Karlin model in [24] with parameters α, β, and $\{Z_n\}_{n\in\mathbb{N}}$ are i.i.d. with power-law tails indexed by α', independent from $\{Y_n\}_{n\in\mathbb{N}}$. There are three parameters in this framework, the tail index of the Karlin model $\alpha > 0$, the memory parameter $\beta \in (0,1)$ of the Karlin model, and the tail index of the noise $\alpha' > 0$. It turns out that there is a phase transition in terms of scaling limits, reflecting roughly the relation between α and $\alpha'\beta$ (=,<, or >). In words, when $\alpha < \alpha'\beta$, the tails of the Karlin model $\{Y_n\}_{n\in\mathbb{N}}$ dominate, and the noise $\{Z_n\}_{n\in\mathbb{N}}$ shows up in the limit as marks. The limit Poisson point process in this case is

$$\sum_{i=1}^{\infty}\sum_{\ell=1}^{Q_{i,\beta}} \delta_{\left(Z_{i,\ell}/\Gamma_i^{1/\alpha}, U_{i,\ell}\right)},$$

  where $\{Z_{i,\ell}\}_{i\in\mathbb{N},\ell\in\mathbb{N}}$ are i.i.d. copies of Z_1 and $\{U_{i,\ell}\}_{i\in\mathbb{N},\ell\in\mathbb{N}}$ are i.i.d. copies of uniform random variables. In the case $\alpha > \alpha'\beta$, the tails of noise dominate and the limit Poisson point process takes a different form. The critical case $\alpha = \alpha'\beta$ is the most interesting and one recovers the logistic random sup-measures [45, 60].

- Aggregated models.

  This part is based on the work in progress [64], described in Section 2.3.

- Simulations.

  This part is based on the on-going work on simulation for Karlin random fields [30], described in Section 2.1. Note that the Karlin random fields considered therein are stable random fields (instead of max-stable ones, their counterparts in extreme-value theory). However, due to the association of stable and max-stable processes [37, 63], several developed ideas in [30] could be borrowed here. Moreover in contrast to the case of stable processes where only simulations for approximated processes can be achieved, we expect exact simulation methods in general for max-stable random fields.

3 Training

The PI has provided individual supervision to the following students at the University of Cincinnati.

- Christopher Koloseike, Fall semester 2019, capstone project on *Simulations for continuum Brownian trees*.
- Danyang Long, Summer semester 2019, capstone project on *The Boolean model in stochastic geometry*.
- Prince Abunkun, Spring semester 2017, capstone project on *Algorithms on graphs*.
- Michael Chambers, Spring semester 2017, capstone project on *Binomial model in mathematical finance*.
- Zuopeng Fu. Mr. Fu is currently a Ph.D. student (expected to graduate in 2020) in the Department of Mathematical Sciences at University of Cincinnati under the supervision of the PI. He is working on stable processes and random fields with long-range dependence for his dissertation. Mr. Fu has been supported as a Research Assistant in fall semester 2017, spring semester 2018, and summer 2018, 2019.

  Mr. Fu received Taft Dissertation Fellowship in the academic year 2019–2020 from Taft Research Center.
- Linna Wang, Ms. Wang is currently a Ph.D. student (expected to graduate 2021) in the Department of Mathematical Sciences at University of Cincinnati, under the joint supervision of the PI and Prof. Yichen Qin from Department of Operations, Business Analytics, and Information Systems in Lindner College of Business at University of Cincinnati. She has been working on network analysis for her dissertation. She has been a Research Assistant in summer 2017, 2018 and spring and fall semesters 2019. She obtained a masters degree in statistics from the department in August 2017.

  Ms. Wang received Levine Summer Fellowship in summer 2019 from the Department of Mathematical Sciences at University of Cincinnati.

  The outcome of Ms. Wang's work sponsored by the ARO grant was on a topic a little distant from the current research program. The original plan was to establish connections between the PI's research agenda and Professor Qin's expertise in network analysis. It turned out that the project has become a joint work between Professor Qin and Ms. Wang only. Below is the abstract for the paper (to be submitted) *Confidence graphs for graphical model selection*.

  In this article, we introduce the concept of confidence graphs (CG) for graphical model selection. Similar to the endpoints in the familiar confidence interval for parameter estimation, CG identities two nested graphical models (small and large confidence graphs) trapping the true graphical model in between at a given level of confidence. Rather than relying on a single selected model, CG consists of a group of nested graphical models as the candidates. The proposed method can be coupled with many popular model selection methods, making it an ideal tool for evaluating the selection uncertainty of various model selection methods. To obtain CG, we proposed a new residual-based bootstrap procedure to approximate the sampling distribution of the selected models. To visualize the distribution of selected models and its associated uncertainty, we further developed new graphical tools, such as grouped model selection distribution plot (GMSD) and model uncertainty curve (MUC). Numerical studies further illustrate the advantages of the proposed method.

4 Presentations

Below is a list of invited talks of the PI in the past three years.

- University of Kansas, Probability Seminar, Lawrence, November 28, 2018.
- Indiana University, Bloomington, Probability Seminar, Bloomington, October 29, 2018.
- AMS Fall Central Sectional Meeting, Special Session on Self-similarity and Long-range Dependence in Stochastic Processes, Ann Arbor, October 20–21, 2018.
- Cornell University, Probability Seminar, Ithaca, September 11, 2018.
- University of Illinois, Urbana Champaign, Probability Seminar, Urbana, April 25, 2017.
- Purdue University, Probability Seminar, West Lafayette, March 7, 2017.
- University of Tennessee, Knoxville, Stochastics Seminar, Knoxville, February 14, 2017.

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