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COMPUTATIONAL PHASE CORRECTION OF A PARTIALLY COHERENT MULTI-APERTURE SYSTEM

**Sarah Elaine Krug
University of Dayton**

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14. ABSTRACT Multi-aperture arrays can be used to reduce the size, weight, cost, and power of an imaging system. However, all of the apertures in an array need to be properly phased in order to create a synthesized image with the maximum possible resolution gain. Phase errors that obstruct aperture phasing can be caused by hardware misalignments or atmospheric turbulence. Remapping the aperture fields between the entrance and exit pupils of an imaging system can be used for piston, tip, and tilt corrections. Remapping the pupils separates the components of the spatial frequency spectrum of an image, allowing maximum likelihood estimation to be used for piston estimation while a least squares matrix method estimates the tip and tilt errors. Images generated to simulate different amounts of atmospheric turbulence and image noise were used to test these piston, tip, and tilt correction algorithms. It was found that atmospheres with Fried parameters the size of a single aperture show the strongest correction results. Anisoplanatic images can be corrected if they are masked to the size of an isoplanatic patch and as long as there is enough spatial frequency support. The final phase corrected results are comparable with or better than the results of blind deconvolution for higher signal to noise ratios. Pupil remapping is done using blazed gratings located at an intermediate image plane to create a wavelength dependent aperture shift that results in a constant shift in the spatial iv frequencies. If the apertures in an array are placed along only one axis, the target or imaging system can be rotated and multiple images can be collected, corrected, and rotationally synthesized to create resolution gain in all directions. Future work could include expanding on isoplanatic correction techniques, improving piston, tip, and tilt estimation methods, and creating an experimental system with more than two apertures that could be tested in the field.						
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Multi-aperture arrays can be used to reduce the size, weight, cost, and power of an imaging system. However, all of the apertures in an array need to be properly phased in order to create a synthesized image with the maximum possible resolution gain. Phase errors that obstruct aperture phasing can be caused by hardware misalignments or atmospheric turbulence. Remapping the aperture fields between the entrance and exit pupils of an imaging system can be used for piston, tip, and tilt corrections. Remapping the pupils separates the components of the spatial frequency spectrum of an image, allowing maximum likelihood estimation to be used for piston estimation while a least squares matrix method estimates the tip and tilt errors. Images generated to simulate different amounts of atmospheric turbulence and image noise were used to test these piston, tip, and tilt correction algorithms. It was found that atmospheres with Fried parameters the size of a single aperture show the strongest correction results. Anisoplanatic images can be corrected if they are masked to the size of an isoplanatic patch and as long as there is enough spatial frequency support. The final phase corrected results are comparable with or better than the results of blind deconvolution for higher signal to noise ratios. Pupil remapping is done using blazed gratings located at an intermediate image plane to create a wavelength dependent aperture shift that results in a constant shift in the spatial

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LIST OF ABBREVIATIONS AND NOTATIONS

AO	Adaptive optics
DC	Center of the spatial frequency spectrum (direct current)
EP	Entrance pupil
FDC	Field dependent contrast
FOV	Field of view
FT	Fourier transform (text)
LS	Least squares
lp	line pairs
MCAO	Multi-conjugate adaptive optics
MLE	Maximum likelihood estimation
OPD	Optical path difference
OTF	Optical transfer function (text, see H)
PSF	Point spread function (text, see h)
P-V	Peak-to-valley
RMS	Root mean square
RMSE	Root mean square error
SAT	Slit aperture telescope
SNR	Signal to noise ratio
XP	Exit pupil
A	Sum of all auto-correlations
a_{xp}	Total aperture shift in exit pupil
b	Constant used to with the radius of the mask array
b_n, a_n	Aperture shift in x and y directions

ap_{new}	Aperture field after phase error is added
ap_{old}	Aperture field before phase error is added
C_n^2	Refractive index structure constant
C_{nm}	Correlation between the n th and m th apertures with not piston error
$\mathcal{C}_{nm}$	Complex field associated with the correlation between apertures n and m for higher order aberrations where the tilt has been separated
D	Aperture diameter
D_n	Diameter of n th lens
D_{pix}	Pixel pitch
D_{xp}	Array diameter at exit pupil
d_i	Distance from exit pupil to image plane
d_o	Distance from object to entrance pupil
d_{02}	Distance between Lens 1 and the grating
d_1	Distance from Lens 1 to the mirror
d_2	Distance from the mirror to the grating
d_3	Distance from the grating to Lens 2
d_4	Distance from Lens 2 to the image plane
d_n	Groove spacing for n th grating
$\mathcal{F}$	Fourier transform (equation)
f	Wiener deconvolved image
f_a	Spatial frequency shift of cross-correlations
f_{img}	Target spatial frequency adjusted by image magnification
f_n	Focal length for n th aperture
f_{targ}	Target spatial frequency
(f_x, f_y)	Spatial frequency coordinates
Δf	Illumination frequency bandwidth
$\mathcal{G}$	Frequency response of the system with an extended target

H	Optical transfer function (equation, see OTF)
h	Coherent impulse response (equation, see PSF)
h	Altitude (meters)
h_o	Constant altitude for horizontal propagation (meters)
I_{auto}	Intensity for spatial frequency in auto-correlation
I_{cross}	Intensity spatial frequency in cross-correlation
K	Power spectral density of the image noise (fld) or noise to signal ratio (scalar)
k	Wave number
L	Total propagation distance through atmosphere (meters)
L_{coh}	Coherence length
l	Grating mode
M_a	Lens 2 angular magnification
M_{img}	Image magnification
M_p	Pupil magnification
M_2	Magnitude of Lens 2
N	Number of apertures
N	Size of focal plane array in pixels
N_c	Number of correlations
N_{cc}	Number of cross-correlations
N_p	Number of overlapping correlations
N_T	Number of overlapping cross-correlations
NA	Numerical aperture of a lens
NA_{sys}	Numerical aperture of the array in the image domain
NA_{cam}	Numerical aperture of camera
NA_{cam}^*	Adjusted numerical aperture of camera
P_n	Mask for nth aperture
$\mathcal{P}$	Complex pupil mask

R^2	Sum of the squared residuals between Φ_{yn} and $\hat{\Phi}_{yn}$
$\mathbf{r}$	Polar array the same size as the image array
r_o	Fried parameter
S	Scale factor for mask
U_o	Complex field before the entrance pupil, FT of u_o
U_n	Complex field for nth aperture mask
u_i	Image field
u_o	Object field
W_n	Phase aberration field for nth aperture
$\mathcal{W}$	Wiener filter
w	Aperture radius
w	Wind speed
(x_i, y_i)	Image field coordinates
(x_o, y_o)	Object field coordinates
$\hat{y}_1$	Random variable of the relative shift between the FTs of two correlations
z	Propagation distance variable through atmosphere (meters)
Δz	Distances between discrete phase screens (meters)
z_1	Distance from the object to the phase screen (meters)
$\Gamma_{nm,kl}$	Amplitude of complex sum for the elemental multiplication between correlations nm and kl
γ_{nm}	Amplitude of the complex values in the eigenvector associated with the piston difference of correlation nm
η_0	Distance of each aperture from optical axis in entrance pupil
θ_{ap}	Beam angle from an aperture
θ_{blz}	Blaze angle
θ_i	Isoplanatic angle
θ_l	Diffraction angle for l th mode
θ_n	Angle that nth aperture shift makes within the pupil plane

$\theta_{shift,n}$	Angle the shifted aperture field makes with the image plane
θ_z	Zenith angle
θ_{10}	Angular difference between 1 st and 0 th grating modes
Λ_n	Amplitude of the complex values in the eigenvector associated with the piston of the nth aperture
λ	Wavelength
$\Delta\lambda$	Illumination wavelength bandwidth
$\bar{\lambda}$	Mean wavelength
(ξ, η)	Pupil field coordinates
φ_n	Piston of nth aperture
$\hat{\varphi}_n$	Estimated piston for nth aperture
φ_{nm}	Piston difference between the nth and mth apertures
φ_{ny}	Tilt for nth aperture
Φ_{yn}	True tilt value associated with $\hat{y}_1$
$\hat{\Phi}_{yn}$	Tilt random variable associated with $\hat{y}_1$
Ω	Steradians viewed by 1 pixel in the pupil remapped images
$\odot$	Elemental multiplication operator
$\otimes$	Convolution operator
$\star$	Autocorrelation operator

CHAPTER 1

INTRODUCTION

Multi-aperture arrays can be used to obtain higher resolution images while reducing the size, weight, power, and cost of an imaging system. The resolution of a multi-aperture array can be defined by the array's diameter if all of the apertures are phased. The phasing of optical systems is often affected by hardware misalignments and atmospheric turbulence, which can cause piston, tip, and tilt phase errors. Atmospheric turbulence can also lead to phase errors which vary across a field of view. A variety of phasing techniques have been developed for multi-aperture systems using incoherent, coherent, and partially coherent illumination. Arrays with incoherent illumination are often phased using adaptive-optics to move all apertures to within a fraction of a wavelength of each other. Coherent arrays are often phased by interfering each complex aperture field with a reference beam before computationally retrieving and correcting the phase. Recently, a method for correcting piston phase errors for partially coherent illumination has been developed. This technique can correct piston as long as the optical path difference (OPD) between apertures is shorter than the coherence length of the illumination.

The spatial frequency response function from a multi-aperture array can be described as a scaled auto-correlation of the complex pupil function, where the cross-correlations between apertures contain the phase errors and a central correlation contains the sum of the auto-correlations of all of the individual aperture fields. Phase errors result

in a loss of contrast for frequencies in the cross-correlations that overlap with the summed auto-correlations or with each other. For an array in which the entrance pupil provides a filled frequency response, correlations for apertures with similar baseline vectors should overlap. This overlap prevents the direct calculation of the phase errors. However, if these cross-correlations can be separated from the auto-correlations and each other while retaining the common spatial frequency information between them, the phase errors can be computationally estimated and corrected. This separation can be done using an anamorphic pupil relay to separate the complex aperture fields between the entrance and exit pupils. The spatial frequency content of the image collected by the system will have separated cross-correlations which can be used to computationally find and correct the phase errors between apertures [1].

Previous research on this computational phase correction method for partially coherent illumination has only used a two-aperture array as a proof-of-concept. It was assumed that each aperture was smaller than the atmospheric coherence diameter, or Fried parameter, of the atmosphere and the field of view (FOV) was less than the isoplanatic angle, leaving only one piston phase correction for the system. Further research will be done for a system with additional apertures. Here, tip and tilt corrections will be tested and the field of view will be large enough to test isoplanatic piston corrections as well. A system analysis will reveal the effectiveness of this computational phase correction method for varying atmospheric parameters, target sharpness, and signal to noise ratios (SNR).

CHAPTER 2

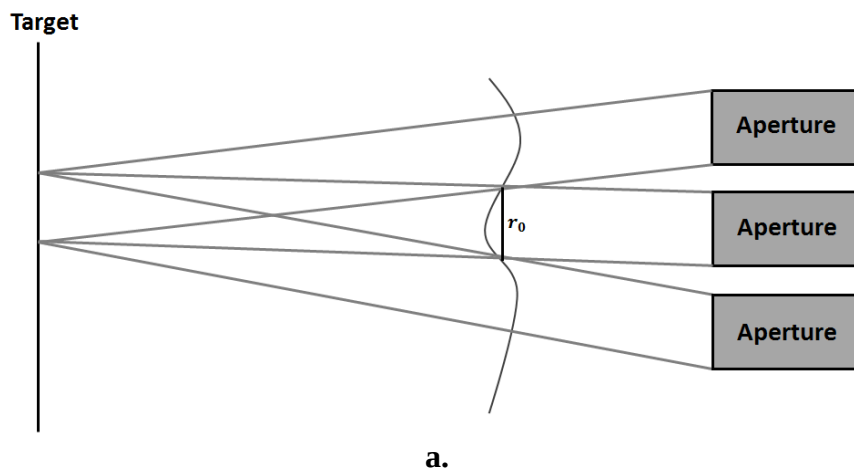
BACKGROUND

Diffraction limited resolution can be obtained from a multi-aperture array if the phase errors between and within apertures are corrected. Phase errors can come from hardware misalignments or from atmospheric turbulence. Different causes of phase errors are explained in Section 2.1. Section 2.2 discusses some of the techniques used to correct these phase errors.

2.1 Phase Errors

The causes of phase errors that are addressed are hardware misalignments, as well as the Fried parameter and isoplanatic angle measurements for atmospheric turbulence. Piston errors caused by hardware misalignments are fairly straightforward. In a multi-aperture system, light travels along different paths for each aperture. If there are differences in the optical path lengths between various apertures, the light from each path will have a different phase when it interferes at the image plane. This piston phase difference weakens the interference between the apertures. Similarly, optical components within each beam path can cause variations in the complex aperture fields which lead to higher order aberrations such as tip and tilt. These also affect interference at the image plane and decrease the spatial frequency cutoff of the image.

Atmospheric turbulence can cause similar problems before the light ever reaches the multi-aperture array. Wind, temperature, humidity, and many other factors create differences in the refractive index of the atmosphere. Light travelling from a target to each aperture in an array propagates through different parts of the atmosphere that have slightly different indices of refraction, resulting in phase differences in the light that reaches each aperture. The atmospheric coherence diameter, or Fried parameter, r_0 , is the distance along a wavefront in which the root mean square (RMS) of this phase difference is one radian [2]. If the Fried diameter is larger than an aperture, the atmospheric phase errors at the pupil plane are largely limited to piston with minimal tip and tilt [1]. Imaging systems with longer target distances require larger aperture sizes to maintain resolution but are also subject to greater impacts from atmospheric turbulence, leading to much lower Fried parameters. The combination of smaller Fried parameters and larger apertures results in pupil planes with higher order aberrations. Figure 1 displays diagrams of two Fried parameter sizes.



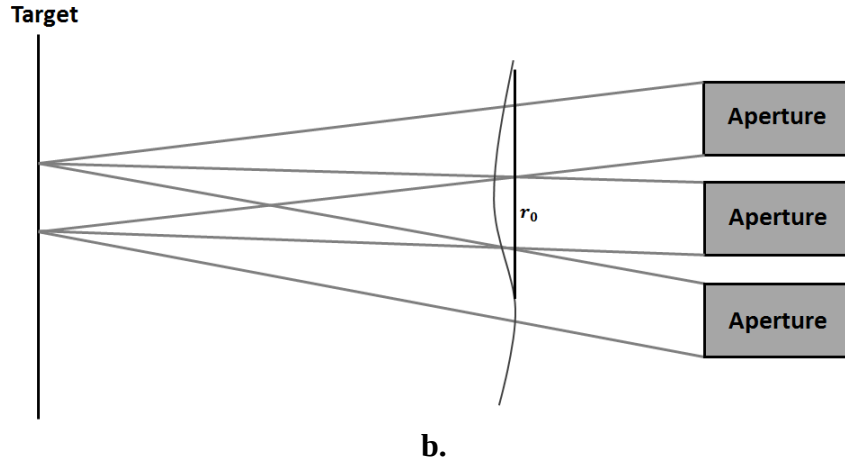


Figure 1: a. Diagram of a. small and b. large Fried parameters.

A Fried parameter measures the phase differences of light propagating from the same point on a target to different apertures in an array. In contrast, an isoplanatic angle measures the phase differences between different points on the target propagating to the same part of the array. It shows the angular diameter across the sky in which a wavefront remains approximately coherent. The size of this angle varies with the strength of the atmospheric turbulence [14]. This can be visualized by a target viewed through several layers of atmosphere, as illustrated in Figure 2. Light traveling from two different points on the target travel through different parts of the atmosphere, so their phases are different when they reach the aperture array. The closer the points on the target, the more similar the phases and the more coherent the wavefront. When the RMS phase difference between two aberrated points is one radian, the angle that the two target points make with regard to the imaging system is the isoplanatic angle and the distance between those two points on the target is the isoplanatic patch diameter.

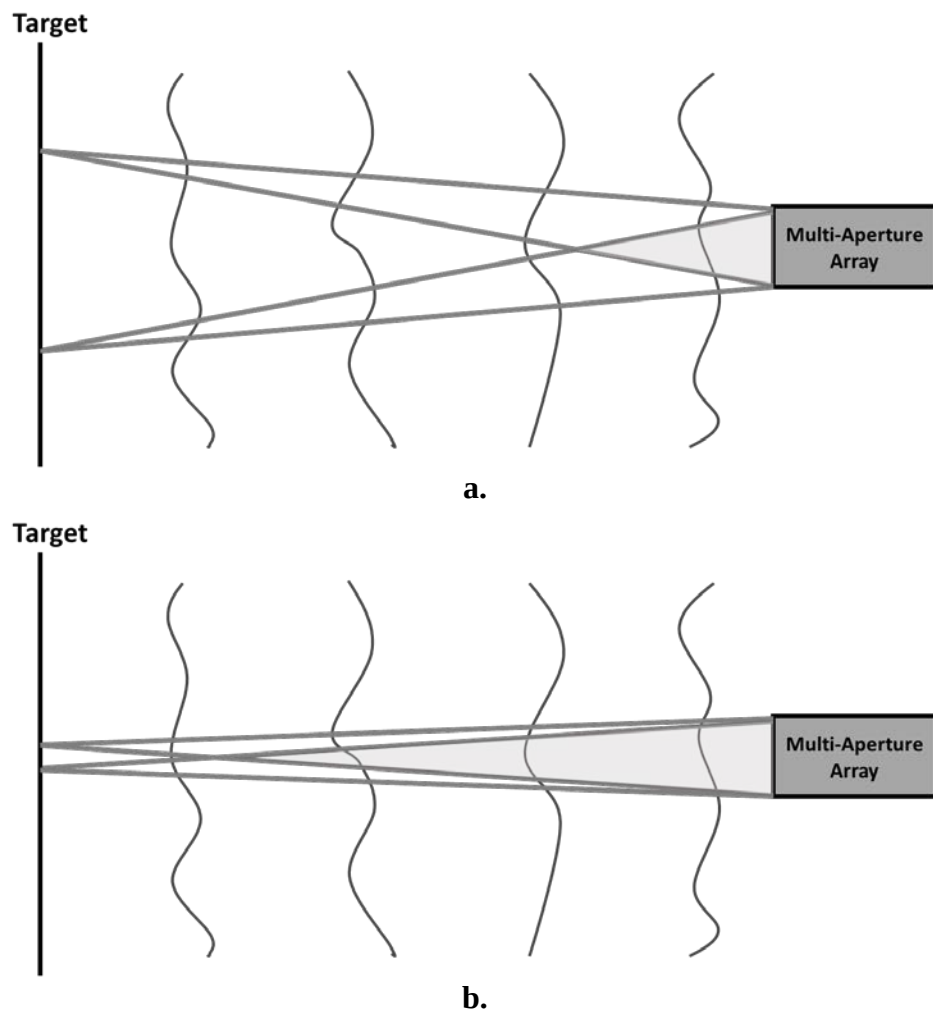


Figure 2: a. Diagram of two points separated by a large distance and b. a smaller distance.

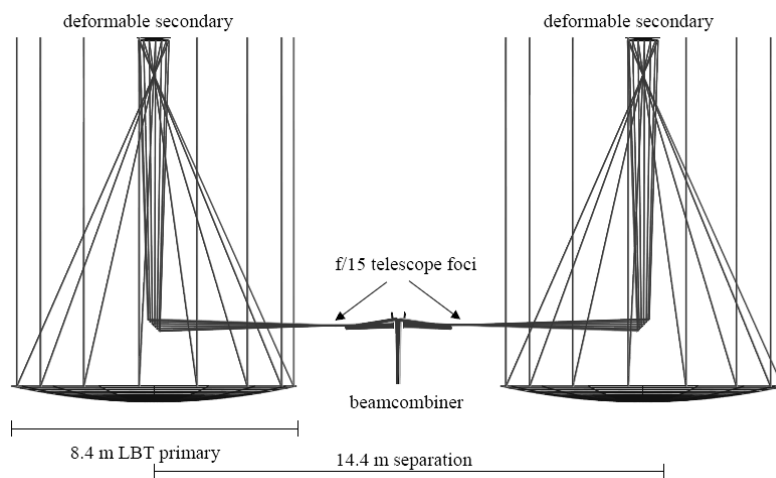
This research focuses on correcting the piston, tip, and tilt phase errors caused by hardware misalignments and the Fried parameter, with some additional analysis on how well phase corrections work for single isoplanatic patches.

2.2 Phase Correction Methods

Traditionally, phase correction methods have focused on iterative methods for correcting images collected with either incoherent or coherent illumination. Some examples of these methods are described in this section, followed by a discussion of a proposed method for phase correcting partially coherent images with a non-iterative approach.

2.2.1 Incoherent Phase Corrections

Adaptive optics (AO) is a well-known method for phase correcting incoherent systems. Incoherent imaging systems often rely on passive illumination and AO calculates phase information from the light it collects and iteratively adjusts a series of mirrors in order to match the phase errors caused by the atmosphere. Adaptive optics systems interfere the images from each aperture onto a single focal plane array using a beam combiner. Wavefront sensors are used to detect aberrations in the incoming illumination and corrections are iteratively applied to the beam-paths to correct piston, tip, and tilt. Sometimes deformable mirrors are used to correct the OPDs within a single aperture. Generally, incoherent phase error corrections are successful if the final OPDs are less than the wavelength of the illumination. The Large Binocular Telescope (LBT) and Star 9 both use adaptive optics and diagrams of each telescope array are shown in Figure 3. The LBT contains deformable secondary mirrors while Star 9 only has mirrors that adjust the piston, tip, and tilt since each aperture is small enough to neglect higher order aberrations from atmospheric turbulence [4], [5].



a.

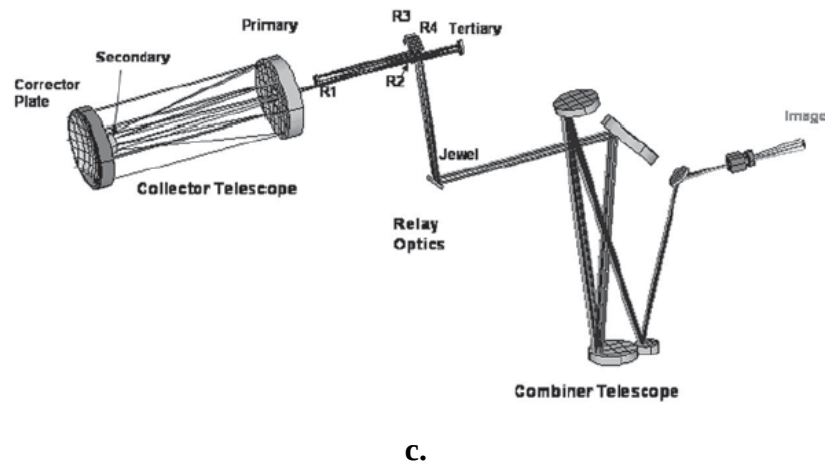
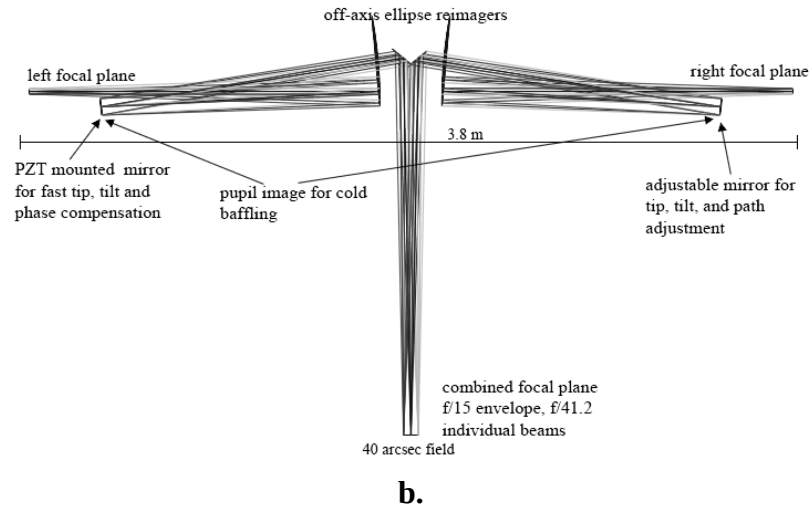


Figure 3: a. LBT setup with deformable mirrors b. LBT interferometer with tip/tilt mirrors for corrections between apertures c. A single Star 9 aperture with piston and tip/tilt mirrors. Star 9 has nine apertures [4], [5]

As with Fizeau interferometers, the beam combiner must ensure that the exit pupil of the system is proportional to the entrance pupil [6]. Otherwise, the image's spatial frequencies will be shifted incorrectly and could experience a loss of contrast [1].

Wavefront detectors in adaptive optics (AO) systems use a guide star to measure the curvature of the wavefront. The deformable mirror corrects the phase of any target within an isoplanatic angle of the guide star. However, the isoplanatic angle is generally very small, which greatly restricts the FOV that can be corrected. Any target outside the isoplanatic angle is, at best, only partially corrected. This is known as the anisoplanatic

effect. Also, the phase errors from the layer of the atmosphere conjugate to the deformable mirror are corrected most effectively. The angle of effectiveness is even more limited for any turbulence not created in that atmospheric layer [15]. These two problems are illustrated in the diagrams below.

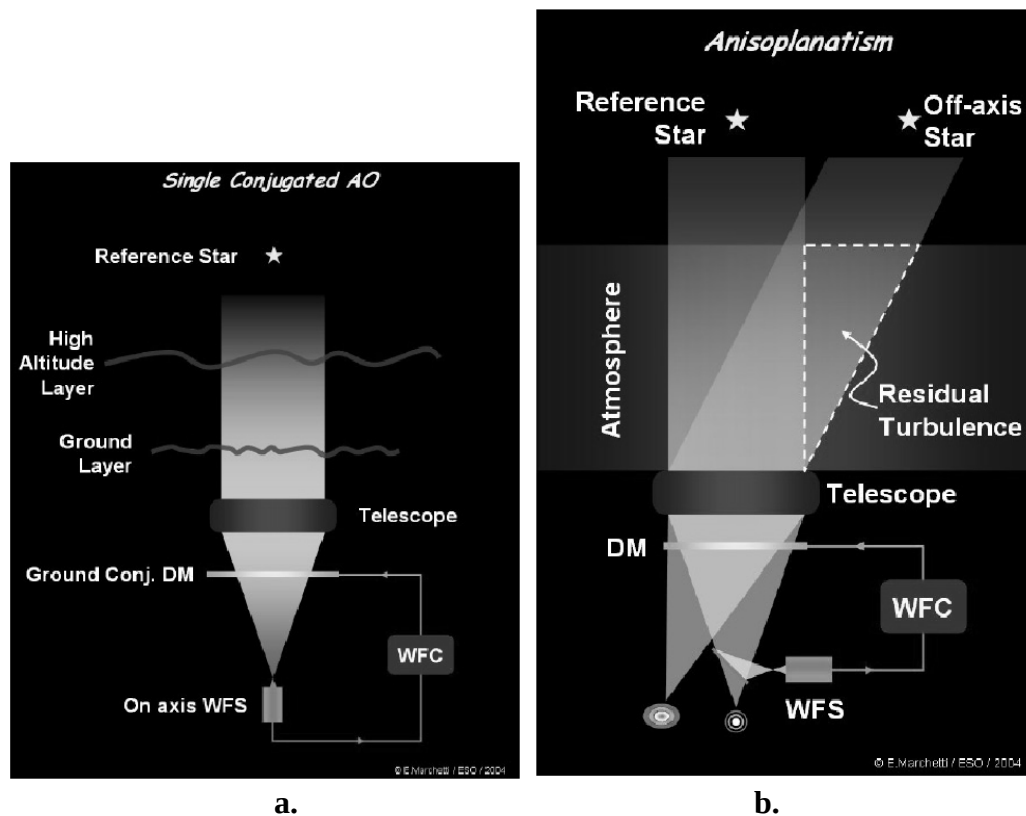


Figure 4: a. Diagram of an AO telescope where only the ground layer turbulence is corrected b. Only stars within an isoplanatic angle of the reference star can be corrected while off-axis stars have residual, uncorrected turbulence. [15]

One solution designed to detect phase errors from multiple layers of the atmosphere and increase the FOV is multi-conjugate adaptive optics (MCAO). MCAO have multiple deformable mirrors located at distances behind the entrance pupil which are conjugate to different atmospheric layers. The wavefront curvatures from several guide stars are collected, where each guide star is measured by a separate wavefront detector. As shown

in Figure 5, this increases the accuracy of the wavefront correction and increases the correctable FOV to a size larger than the isoplanatic angle. [15]

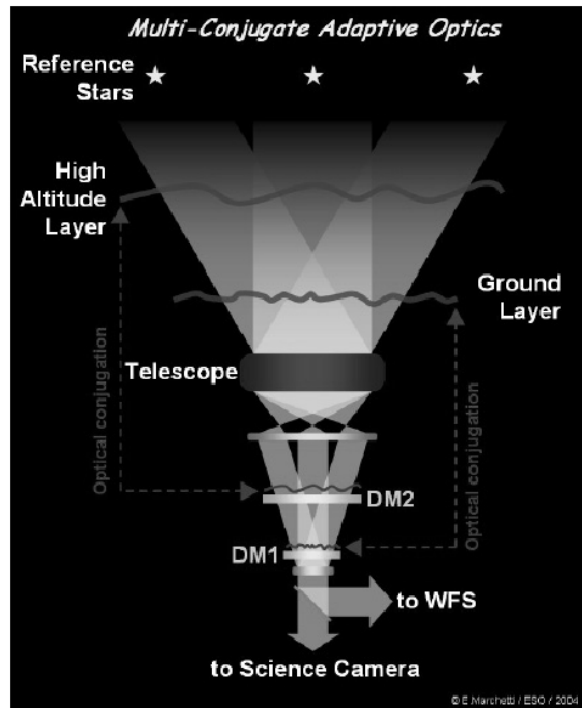


Figure 5: MCAO diagram using multiple guide stars and deformable mirrors to make corrections for field of views larger than the isoplanatic angle [15]

Unfortunately, AO systems, and MCAO telescopes in particular, are large and costly to build and maintain. Each deformable mirror is controlled by an iterative process that combines the wavefront errors collected from each wavefront detector and applies them back to the deformable mirrors. Therefore, such systems contain moving parts and require high-speed processors that allow atmospheric corrections in real time.

2.2.2 Coherent Phase Corrections

Coherent phase correction systems typically have simpler physical setups without moving parts and the phase corrections are applied digitally after the data is collected. Coherent systems are generally active systems in which a laser with a coherence length longer than the scene depth is directed toward a target and reflects back to a receiver. They

often make use of local oscillators and a variety of phase retrieval algorithms or digital holography techniques. In Figure 6, a local oscillator interferes with the complex laser field at each receiver. Digital holography techniques use the resulting intensity patterns to computationally reconstruct the complex fields at the apertures, which can then be coherently synthesized [3].

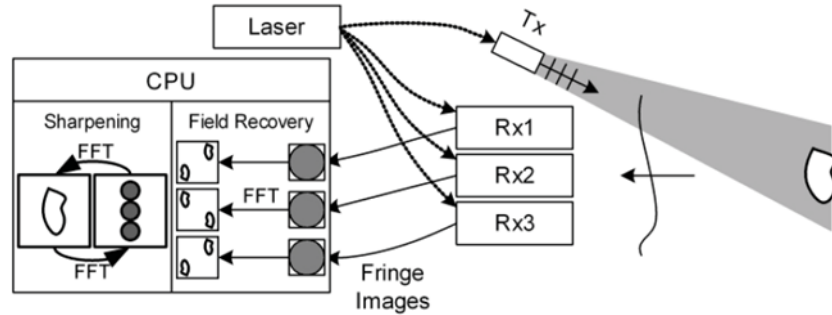


Figure 6: Diagram of coherent aperture synthesis using a local oscillator [3]

Both digital holography and phase retrieval methods collect intensity fields using multiple focal plane arrays and computationally reconstruct the amplitude and phase of each complex field before combining the fields to create a phase-corrected image. Sharpness metric maximization can be used for similar results. It uses a non-linear optimization algorithm to find the phase error estimation that best maximizes a specified sharpness metric [16]. Additionally, sharpness metric maximization is a fully computational approach that can also be applied as a stand-alone correction method for both coherent and incoherent systems, to varying effect.

2.2.3 Partially Coherent Phase Corrections

Computational phase corrections for partially coherent illumination use components from both coherent and incoherent correction techniques. Phase errors can be estimated and corrected computationally; but unlike coherent methods, local oscillators,

phase retrieval algorithms, and AO are not needed. Instead, images from each aperture are collected on a single focal plane array similar to incoherent correction methods and the interference from each of the images at the focal plane contains the phase error information necessary for phase corrections. Unfortunately, this phase information cannot be found if the beam combiner allows the entrance and exit pupils to be proportional. Instead, an anamorphic lens setup remaps the aperture fields at the entrance pupil to shifted locations in the exit pupil. The frequency response resulting from this remapping allows the phase errors to be found and corrected computationally as long as the OPD between the apertures paths is less than the coherence length of the illumination.

Greenaway discusses the possibility of isolating the phase errors between an array of pinhole apertures using pupil remapping and the equations for phase closure. He mentions the need for larger apertures, concerns related to using a larger bandwidth of light, and the possibility for correcting multiple isoplanatic patches for the same image [7].

Perrin, et. al. addresses the need for larger apertures by coupling the light from each of the entrance pupil apertures into single-mode fibers. This serves multiple purposes such as simplifying the remapping process, filtering the phase aberrations so that only first order piston errors are left for each aperture, and allowing each aperture to be represented by a single complex value. Once the light leaves each fiber at the exit pupil, the beams are focused onto a focal plane array [8]. In a related paper, Lacour, et. al., shows that the best phase error estimates between apertures are found if the entrance pupil is redundant and the exit pupil is non-redundant. Pupil remapping separates the baseline vectors between each pair of apertures in the redundant entrance pupil so that the auto-correlation of the

exit pupil results in separated spatial frequencies for each aperture pair, as seen in Figure 7.

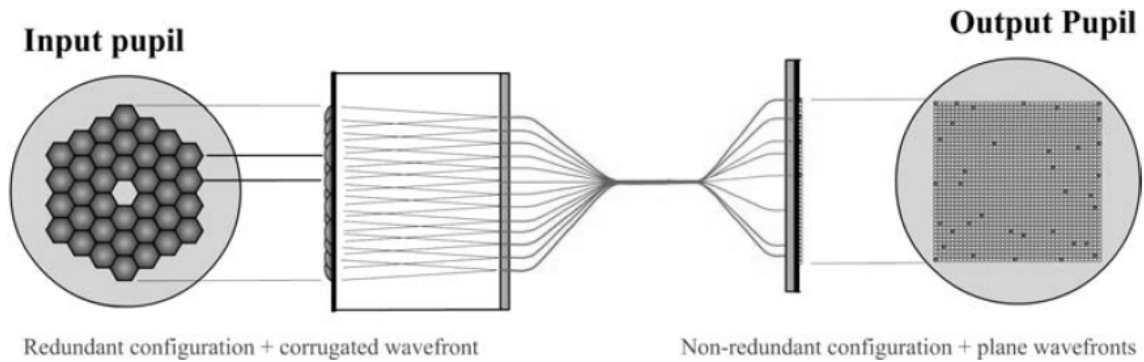


Figure 7: Diagram of pupil remapping with single-mode fibers [8]

The complex values for each spatial frequency can be used to find the relative piston errors between pairs of apertures and the least squares method can estimate the piston correction for each individual aperture [9]. However, there are a variety of drawbacks to aperture remapping with single mode fibers. For example, low coupling efficiency causes a loss of signal power; the use of one fiber per aperture averages out higher order aberrations, such as tip and tilt, so that only piston can be corrected; and the maximum FOV is limited to the diffraction limited spot size of a single aperture. The FOV can also be effected by the degree of coherence. When pupil remapping is used with a relatively wide illumination bandwidth, the degree of coherence decreases and the system suffers from a chromatic effect. The spatial frequencies collected by the aperture array are shifted as a function of wavelength, which causes a loss of contrast across the FOV of the image, which will be referred to as field dependent contrast (FDC). Lacour and Perrin generally used an illumination bandwidth low enough to ensure that FDC did not affect the image plane within the FOV restricted by the coupling angle. However, in some situations they also used hyperchromatic magnifiers, such as a Wynne lens system, to mitigate the

effects of FDC [10]. They also mentioned the possibility of using dispersive optics such as diffraction gratings [9].

In this dissertation, pupil remapping is done through an anamorphic relay system using lenses and mirrors with diffractive optics to correct for FDC caused by partially coherent illumination. One improvement that this offers over single mode fiber remapping is that less light is lost to low coupling efficiency. Also, remapping apertures with diffractive optics instead of fibers improves phase estimations since higher order phase errors are not lost and more phase content can be resolved in the apertures' correlations. In addition, the use of mirrors and diffraction gratings increases the possible FOV of the system compared to remapping systems using single mode fibers.

Without fibers or diffractive optics, the FOV of a remapped, phase corrected system is restricted by the various optical elements and FDC. Diffractive optics serve the dual purpose of correcting for the FDC and anamorphically remapping the pupil. Tai shows a similar use of diffractive optics in his work on passive synthetic aperture imaging using a grating interferometer. Gratings are placed at the two apertures of an interferometer in order to redirect the beams and recombined using another grating to counteract the dispersive effects of the first set [11]. However, instead of using gratings to combine the aperture beams, the experiment discussed in this dissertation uses them to separate the beams as a function of wavelength while allowing the spatial frequencies to shift by a constant amount.

Since phase errors in an image collected with an aperture remapping system can be corrected using the information contained within the image, multiple isoplanatic patches can be corrected without the use of guide stars or wavefront detectors. Therefore, the FOV

of the telescope is limited only by the optics of the system itself and not by the turbulence for a well-supported image. Isoplanatic corrections can be completed by masking a segment of an image that is approximately the size of an isoplanatic patch before correcting the phase. A full anisoplanatic image could be phase corrected by cropping multiple isoplanatic sections of the image and correcting each segment separately before recombining them to display the full, diffraction limited image.

The entrance pupil for the fiber remapping technique is a two dimensional lenslet array [7], [8]. After remapping, the subsequent image is oversampled so that no spatial frequency information is lost. This is fairly simple when the exit pupil consists of small point sources. However, when mirrors and reflective gratings are used in place of fiber optics the exit pupil is made up of a set of extended aperture fields. The higher spatial frequencies contained within these extended fields make it more difficult to oversample images as aperture number and resolution increases. One efficient orientation for apertures could be a one dimensional lenslet array at the entrance pupil that is remapped into a two dimensional exit pupil which utilizes both spatial frequency dimensions of the focal plane array. Taking multiple images with the one dimensional aperture array while rotating the target provides gain for all dimensions of spatial frequency information.

After correcting for phase errors in each of the rotated images, the corrected frequency responses can be synthesized to form an image with high gain in all directions. This is somewhat similar to the astronomical technique for high resolution imaging using a rotating slit aperture telescope (SAT) and the inverse radon transformation. A rotating SAT contains a long, thin pupil with a width that is small enough to be considered one dimensional. An image taken by a SAT is also one dimensional. As the slit aperture

rotates, multiple images can be taken of the target and plotted as a function of the rotation angle, this plot is a radon transform of the target. The full two dimensional image can be reconstructed by taking the Fourier transform (FT) of each line image, interpolating the data to give a full two dimensional FT, and taking the inverse FT in order to give a reconstructed and high resolution two dimensional image [12]. However, the multi-aperture array discussed here is not fully one dimensional, so a direct application of this reconstruction technique is not necessary. Instead of interpolating the FTs of each rotated image, the sum of the FTs for all the rotated images are taken in order to provide a rotationally synthesized spatial frequency spectrum.

2.3 Research Summary

A proof-of-concept for computational, or digital, partially coherent multi-aperture piston correction has been conducted using a two aperture entrance pupil and a pupil relay to separate the apertures in the exit pupil. No diffraction gratings were used to prevent FDC and partially coherent light was simulated by taking multiple images of a target illuminated by a tunable laser with a range of 50 nm. These images were summed in order to appear as a partially coherent image. [13]

In this dissertation, diffraction gratings are used to remap the pupil and an LED provides a true source of partially coherent light. An imaging system is also simulated to utilize more lenses for a higher resolution gain and test tip and tilt phase corrections as well as piston. Algorithms are described that computationally shift the resulting cross-correlations of the aperture fields back to their original places from before pupil remapping, estimate the phase errors caused by hardware misalignments and atmospheric turbulence, and correct these errors in the spatial frequency domain. Phase error corrections include

corrections of piston, tip, and tilt for multiple apertures as well as an analysis for the correction of isoplanatic patches. Images of a rotated target are taken with the one dimensional pupil relay. After phase corrections are applied, each of the rotated images are synthesized to create an image with two dimensional resolution gain.

This phase correction technique is highly dependent on the profile of the atmospheric turbulence. The algorithm's ability to correct piston, tip, and tilt phase errors depends on the Fried parameter and how it compares with the size of the apertures in the array. Its ability to correct isoplanatic patches within an image is controlled by the FOV and the diffraction limited spot size of one of the apertures. An atmospheric analysis is conducted to determine the SNR, target frequency content, and Fried parameters that are needed for successful phase corrections. Additional analysis is done to gauge the effects of anisoplanatism on this phase correction technique. The results are compared with a blind deconvolution image deblurring method.

The theory behind pupil remapping, computational phase correction, and atmospheric turbulence parameters are discussed in more detail in CHAPTER 3. This is followed in CHAPTER 4 by a computational implementation of pupil remapping and atmospheric propagation. CHAPTER 5 contains details of the phase correction algorithms used to correct piston, tip, and tilt in multi-aperture systems and CHAPTER 6 gives the results of these phase correction codes for images propagated through simulated atmospheric phase screens with different Fried and isoplanatic parameters. CHAPTER 7 compares these results with an accepted blind deconvolution image deblurring technique and discusses some real-life situations for which pupil remapping and computational phase corrections could be effective. The design of a two aperture pupil relay system is given in

CHAPTER 8. This includes a discussion of the importance of diffractive optics in correcting FDC and techniques used to set up and align the optical hardware. CHAPTER 9 provides phase corrected results from the experimental system and discusses the spatial frequency synthesis technique used to combine rotated images for rotationally symmetric gain. Concluding comments are found in CHAPTER 10.

CHAPTER 3

THEORY

Partially coherent phase corrections can be completed computationally by collecting an image using a multi-aperture array whose entrance pupil can be remapped into an exit pupil that separates the complex fields of each aperture. The separated complex fields result in non-overlapping cross-correlations that contain common spatial frequency information from the target. After taking a FT of the image resulting from the remapped exit pupil, this common spatial frequency information can be registered and compared in order to estimate and correct the piston, tip, and tilt phase errors that are not common between the apertures. The analytical equations describing the piston, tip, and tilt phase errors can be more easily explained through a mathematical description of a linear imaging system. This is discussed in Sections 3.1 and 3.2. Analytical equations describing atmospheric turbulence are described in Section 3.3 and the remapped exit pupil is described analytically in Section 3.4.

3.1 Linear Imaging System

A single lens, or an array of lenses, that focusses a target from an object plane to an image on the conjugate image plane can be described as a linear system. In a linear optical system, the final field at the image plane is a convolution of the input field from the object plane and the impulse response of the system. This convolution can be written as,

$$u_i(x_i, y_i) = \iint_{-\infty}^{\infty} u_o(x_o, y_o) h(x_i - x_o, y_i - y_o) dx_o dy_o \quad (1)$$

or,

$$u_i(x_i, y_i) = u_o(x_o, y_o) \otimes h(x_i, y_i; x_o, y_o) \quad (2)$$

where (x_o, y_o) and (x_i, y_i) are the coordinates at the object and image planes, u_o and u_i are the complex fields at those planes, and h is the coherent field impulse response. An impulse response shows the effect of the system on a point source placed on the optical axis at the object plane and is also called a point spread function (PSF) [17].

The impulse response can be found analytically by defining $u_o(x_o, y_o)$ as a Dirac delta centered on the optical axis, applying a Fresnel diffraction integral, multiplying by the aperture field and the lens, and applying another Fresnel diffraction integral. This process is shown below. The first Fresnel diffraction integral is,

$$\frac{e^{ikd_0}}{i\lambda d_0} e^{i\frac{\pi}{\lambda d_0}(\xi^2 + \eta^2)} \iint_{-\infty}^{\infty} \left[u_o(x_o, y_o) e^{i\frac{\pi}{\lambda d_0}(x_o^2 + y_o^2)} \right] e^{-i\frac{2\pi}{\lambda d_0}(x_o\xi + y_o\eta)} dx_o dy_o \quad (3)$$

Where d_0 is the distance from the object to the pupil plane, (ξ, η) are the pupil plane coordinates, and $\frac{e^{ikd_0}}{i\lambda d_0}$ is a propagation constant. In addition, $e^{i\frac{\pi}{\lambda d_0}(x_o^2 + y_o^2)}$ is an effective field curvature that focusses the field at the object plane onto the pupil plane. Therefore, the integral's solution is a field at the pupil plane with a curvature focused on the object plane. The term $e^{i\frac{\pi}{\lambda d_0}(\xi^2 + \eta^2)}$ counteracts this curvature so that the field is parallel to a flat pupil mask while $\frac{e^{ikd_0}}{i\lambda d_0}$ is a constant. For a point source, the solution to the Fresnel diffraction integral is a flat field at the entrance pupil,

$$\frac{e^{ikd_0}}{i\lambda d_0} e^{i\frac{\pi}{\lambda d_0}(\xi^2 + \eta^2)} \quad (4)$$

Multiplying this field by the pupil mask, $\mathcal{P}$, and a lens phase curvature gives,

$$\frac{e^{ikd_0}}{i\lambda d_0} e^{i\frac{\pi}{\lambda d_0}(\xi^2+\eta^2)} e^{-i\frac{\pi}{\lambda f}(\xi^2+\eta^2)} \mathcal{P}(\xi, \eta) \quad (5)$$

Where f is the focal length of the lens. This new field is again propagated a distance d_i from the pupil to the image plane through a second Fresnel diffraction integral,

$$-\frac{e^{ik(d_0+d_i)}}{\lambda^2 d_0 d_i} e^{i\frac{\pi}{\lambda d_i}(x_i^2+y_i^2)} \iint_{-\infty}^{\infty} \left[e^{i\frac{\pi}{\lambda d_0}(\xi^2+\eta^2)} e^{-i\frac{\pi}{\lambda f}(\xi^2+\eta^2)} \mathcal{P}(\xi, \eta) e^{i\frac{\pi}{\lambda d_i}(\xi^2+\eta^2)} \right] e^{-i\frac{2\pi}{\lambda d_i}(x_i\xi+y_i\eta)} d\xi d\eta \quad (6)$$

The field inside the brackets can be rewritten,

$$-\frac{e^{ik(d_0+d_i)}}{\lambda^2 d_0 d_i} e^{i\frac{\pi}{\lambda d_i}(x_i^2+y_i^2)} \iint_{-\infty}^{\infty} \mathcal{P}(\xi, \eta) e^{i\frac{\pi}{\lambda}(\xi^2+\eta^2)\left(\frac{1}{d_0}+\frac{1}{d_i}-\frac{1}{f}\right)} e^{-i\frac{2\pi}{\lambda d_i}(x_i\xi+y_i\eta)} d\xi d\eta \quad (7)$$

and the phase curvature term inside the integral becomes unity when the thin lens equation is considered,

$$\frac{1}{d_0} + \frac{1}{d_i} = \frac{1}{f} \quad (8)$$

Therefore, the final complex field incident on a focal plane array located at the image plane is,

$$-\frac{e^{ik(d_0+d_i)}}{\lambda^2 d_0 d_i} e^{i\frac{\pi}{\lambda d_i}(x_i^2+y_i^2)} \iint_{-\infty}^{\infty} \mathcal{P}(\xi, \eta) e^{-i\frac{2\pi}{\lambda d_i}(x_i\xi+y_i\eta)} d\xi d\eta \quad (9)$$

Equation (9) can be described as the impulse response of a linear imaging system multiplied by an additional curvature term to create a flat field at the image plane if the impulse response is defined relative to a curved surface focused on the pupil plane. The term $-\frac{e^{ik(d_0+d_i)}}{\lambda^2 d_0 d_i}$ is a propagation constant. The impulse response, or PSF, is,

$$h(x_i, y_i) = \iint_{-\infty}^{\infty} \mathcal{P}(\xi, \eta) e^{-i\frac{2\pi}{\lambda d_i}(x_i\xi+y_i\eta)} d\xi d\eta \quad (10)$$

This is merely a scaled inverse FT of the pupil mask as defined in section 6.2.1 of Goodman [17],

$$h(x_i, y_i) = \mathcal{F}^{-1}\{\mathcal{P}(\xi, \eta)\} \quad (11)$$

where scaling constants define the relationship between the spatial frequency and the spatial coordinates at the pupil plane,

$$f_x = \frac{\xi}{\lambda d_i} \quad (12)$$

$$f_y = \frac{\eta}{\lambda d_i} \quad (13)$$

Equation (11) is given in terms of spatial coordinates instead of spatial frequencies in order to highlight the relationship between the image and the pupil plane since phase errors will be corrected with respect to the pupil plane.

Convolving this PSF with the complex field at the object plane from equation (2), gives the complex field at the image plane,

$$u_i(x_i, y_i) = h(x_i, y_i) \otimes u_o(x_o, y_o) \quad (14)$$

where $u_o(x_o, y_o)$ has a flat phase curvature and $u_i(x_i, y_i)$ has a curvature focused on the pupil plane, similar to $h(x_i, y_i)$. The intensity of the image for a coherent spectrum is,

$$I(x_i, y_i) = |u_i(x_i, y_i)|^2 \quad (15)$$

Taking the FT of the intensity gives the frequency response of the system for an extended target, $\mathcal{G}(f_x, f_y)$,

$$\mathcal{G}(f_x, f_y) = \mathcal{F}\{I(x_i, y_i)\} \quad (16)$$

The autocorrelation theorem shows that,

$$\mathcal{F}\{I(x_i, y_i)\} = \mathcal{P}(\xi, \eta) U_o(\xi, \eta) \star \mathcal{P}(\xi, \eta) U_o(\xi, \eta) \quad (17)$$

Where U_o is the scaled FT of u_o , just as equation (11) shows that $\mathcal{P}$ is the scaled FT of h . [17] Therefore, equations (16) and (17) result in,

$$\mathcal{G}(f_x, f_y) = \mathcal{P}(\xi, \eta) U_o(\xi, \eta) \star \mathcal{P}(\xi, \eta) U_o(\xi, \eta). \quad (18)$$

Keeping equations (12) and (13) in mind, this relationship illustrates that the frequency response of the system can be defined as the scaled auto-correlation of the product of the pupil function and the propagated object field directly before the pupil mask. This is particularly convenient when calculating the frequency response of a point source by defining $U_o = 1$, since the optical transfer function is a scaled auto-correlation of the pupil function [17],

$$OTF(f_x, f_y) = \mathcal{P}(\xi, \eta) \star \mathcal{P}(\xi, \eta) \quad (19)$$

3.2 Multi-Aperture System

In a multi-aperture array, the pupil mask can be defined as the complex quantity,

$$\mathcal{P}(\xi, \eta) = \sum_{n=1}^N P_n(\xi, \eta) e^{jW_n(\xi, \eta)} \quad (20)$$

Where N is the number of apertures, P_n is the amplitude transmission and W_n specifies the phase aberrations of the n th aperture. When the only aberration is piston, W_n is a constant defined as φ_n . For example, the two aperture array depicted in Figure 8.a has a complex pupil mask of,

$$\mathcal{P}(\xi, \eta) = P_1(\xi, \eta) e^{j\varphi_1} + P_2(\xi, \eta) e^{j\varphi_2} \quad (21)$$

And its frequency response is,

$$\begin{aligned} \mathcal{G}(f_x, f_y) = & U_1(\xi, \eta) \otimes U_1^*(-\xi, -\eta) + U_2(\xi, \eta) \otimes U_2^*(-\xi, -\eta) \\ & + [U_1(\xi, \eta) \otimes U_2^*(-\xi, -\eta)] e^{-j(\varphi_2 - \varphi_1)} \\ & + [U_1^*(-\xi, -\eta) \otimes U_2(\xi, \eta)] e^{j(\varphi_2 - \varphi_1)} \end{aligned} \quad (22)$$

where U_n is the product of U_o with P_n . In the case of a point source, the first two terms of equation (22) correspond to the auto-correlation in Figure 8.b, the bottom two terms correspond to the cross-correlations, and $\varphi_2 - \varphi_1$ defines the piston difference between

the two apertures. The cross-correlations lose contrast where they overlap with the auto-correlation as the piston difference increases. [1].



Figure 8: a. Pupil function with 2 circular apertures. Each aperture has a radius, w , and centers located $\pm\eta_0$ from the origin. b. Frequency response of the pupil when illuminated by a point source.

Equations (21) and (22) can also be expanded to accommodate N apertures. An N -aperture array with piston phase errors has a pupil mask given by,

$$\mathcal{P}(\xi, \eta) = \sum_{n=1}^N P_n(\xi, \eta) e^{j\varphi_n} \quad (23)$$

a complex pupil field given by,

$$U_o(\xi, \eta) \sum_{n=1}^N P_n(\xi, \eta) e^{j\varphi_n} = \sum_{n=1}^N U_n(\xi, \eta) e^{j\varphi_n} \quad (24)$$

and a frequency response of,

$$\mathcal{G}(f_x, f_y) = \sum_{m=1}^N \sum_{n=1}^N [U_m(\xi, \eta) e^{j\varphi_m} \otimes U_n^*(-\xi, -\eta) e^{-j\varphi_n}] \quad (25)$$

For simplicity, the entrance pupil apertures in this project are only placed linearly along the ξ -axis. In an array in which the apertures are evenly spaced less than an aperture-width apart the frequency response in equation (25) is a continuous function. A scaled inverse FT of equation (25) provides the image intensity at the focal plane array.

$$I(x_i, y_i) = \sum_{m=1}^N \sum_{n=1}^N [u_m(x_i, y_i) u_n(x_i, y_i)] e^{-j\varphi_{nm}} \quad (26)$$

where $\varphi_{nm} = \varphi_n - \varphi_m$.

3.3 Atmospheric Effects

Equations (21), (22), (23), and (25) contain only piston phase errors. These errors can be caused by hardware misalignments or atmospheric turbulence. Hardware misalignments and atmospheric turbulence can also lead to tip and tilt phase errors while atmospheric turbulence can also cause even higher order aberrations. The phase errors caused by atmospheric turbulence are dependent on the Fried parameter at the entrance pupil of an imaging system. In addition, the piston examples above contain only one set of piston differences per image. The isoplanatic angle can affect the image so that different parts of the image have different sets of piston and higher order phase errors. The Fried parameter and isoplanatic angle are discussed in the next two sections.

3.3.1 Fried Parameter

Along the plane of a wavefront the Fried parameter, or atmospheric coherence diameter, defines the distance over which the phase difference remain below one radian. An analytical model for this parameter has been derived using the refractive index structure parameter, $C_n^2(h)$, which characterizes the atmosphere based on atmospheric pressure, altitude, wind speed, and temperature among other variables. A common form of the structure function is the Hufnagel-Valley model [18],

$$C_n^2(h) = C_n^2(0)e^{-h/100} + 5.94 \times 10^{-53} \left(\frac{w}{27}\right)^2 h^{10} e^{-h/1000} + 2.7 \times 10^{-16} e^{-h/1500} \quad (27)$$

where $C_n^2(0)$ is the structure parameter at sea level and h is the altitude, which is often defined through the variable propagation distance, z , and the zenith angle, θ_z .

$$h(z, \theta_z) = h_0 + z \cos \theta_z \quad (28)$$

The wind speed, w , is often given as 21 meters per second [18]. The Fried parameter can be calculated through [19],

$$r_0(\hbar(z, \theta_z), L) = \left[0.423 \left(\frac{2\pi}{\lambda} \right)^2 \int_0^L C_n^2(\hbar(z, \theta_z)) dz \right]^{-3/5} \quad (29)$$

where λ is the wavelength and L is the total propagation distance. When calculating the Fried parameter for a vertical propagation, $\hbar_0 = 0$, $\theta_z = 0$, and $\hbar = z$. For a horizontal propagation, $\theta_z = \pi/2$ and $\hbar = \hbar_0$, a constant. This simplifies equation (29) to [19],

$$r_0(\hbar_0, L) = \left[0.423 \left(\frac{2\pi}{\lambda} \right)^2 C_n^2(\hbar_0) L \right]^{-3/5} \quad (30)$$

On a clear day the structure function for the turbulence at ground level can vary from approximately $C_n^2(0) = 1 \times 10^{-13}$ for warm days with high turbulence to approximately $C_n^2(0) = 1 \times 10^{-15}$ for cool days with low turbulence, as well as at sunrise and sunset [18]. The structure function and Fried parameter are inversely proportional: a larger $C_n^2(0)$ leads to a lower r_0 and vice versa. Since $C_n^2(h)$ decreases with height in the entrainment zone where wind shear takes effect, turbulence is higher closer to sea level and a horizontal propagation at low altitudes will have a smaller Fried parameter than a vertical propagation of the same distance [20]. For example, a laser with a 1500 nm wavelength starting at sea level and traveling 50 meters will have a Fried parameter of 0.69 m traveling horizontally and 0.77 m traveling upward on a day with low turbulence. This disparity becomes more pronounced with propagation distance since the atmosphere thins significantly in the vertical direction.

Short propagation distances, low turbulence, or a combination of the two leads to large Fried parameters. A Fried parameter much larger than the diameter of the entrance pupil will create a constant phase across the whole aperture array and will have no effect on the imaging system. When r_0 becomes smaller than the entrance pupil's diameter but larger than an individual aperture, the phase across each aperture will be approximately

constant, leading to the piston phases used in equation (23). As the propagation distance or turbulence increases, the Fried parameter continues to decrease. When it is on the order of or smaller than the size of an aperture, tip, tilt, and higher order aberrations begin to occur and the phase error of each aperture field is no longer constant. In this case, the wavefront phase equations, W_n , become functions of (ξ_n, η_n) that contain higher order aberrations as well as piston. The simplest of these higher order aberrations are tip and tilt.

$$\mathcal{P}(\xi, \eta) = \sum_{n=1}^N P_n(\xi, \eta) e^{j(\varphi_n + \varphi_{ny}\eta)} \quad (31)$$

The tilt is shown in equation (31). The slope of the tilt is φ_{ny} . In this case, the frequency response becomes more complicated,

$$\mathcal{G}(f_x, f_y) = \sum_{m=1}^N \sum_{n=1}^N [U_m(\xi, \eta) e^{j(\varphi_m + \varphi_{my}\eta)} \otimes U_n^*(-\xi, -\eta) e^{-j(\varphi_n + \varphi_{ny}\eta)}] \quad (32)$$

and the number of variables that must be solved in order to correct the phase errors increases since tilt slopes must be estimated alongside pistons.

3.3.2 Isoplanatic Angle

The isoplanatic angle is defined by the angular diameter of the sky over which piston and other phase aberrations are relatively constant. If two points on a target are separated by more than the isoplanatic angle, the image will have a different piston and higher phase aberrations for each point. As with the Fried parameter, the analytical model for this parameter uses the refractive index structure parameter,

$$\theta_0(\mathcal{H}(z, \theta_z), L) = \left[2.91 \left(\frac{2\pi}{\lambda} \right)^2 \int_0^L C_n^2(\mathcal{H}(z, \theta_z)) z^{5/3} dz \right]^{-3/5} \quad (33)$$

When calculating the isoplanatic angle for a vertical propagation, $\mathcal{H} = z$. Conversely, a horizontal propagation causes $\mathcal{H} = \mathcal{H}_0$ and equation (33) becomes,

$$\theta_0(\hbar_0, L) = \left[2.91 \frac{3}{8} \left(\frac{2\pi}{\lambda} \right)^2 C_n^2(\hbar_0) L^{8/3} \right]^{-3/5} \quad (34)$$

The isoplanatic angle is also inversely proportional to the refractive index structure parameter. Utilizing the same guidelines for $C_n^2(h)$ as discussed in Section 3.3.1, the horizontal isoplanatic angle at low turbulence for 1500 nm starting at ground level and propagating 50 m is 0.0078 radians and the upward isoplanatic angle is 0.0093 radians. Short propagation distances, low turbulence, or a combination of the two leads to large isoplanatic angles. An isoplanatic angle much larger than the FOV results in a single set of phase corrections that are valid across the whole FOV. As θ_0 becomes smaller than the FOV the phase errors vary across the image and different parts of the image will need to be phase corrected separately. As the propagation distance or turbulence continues to increase, the isoplanatic angle continues to decrease and becomes much smaller than the FOV, creating anisoplanatic images. An anisoplanatic image has different sets of phase errors for each isoplanatic patch in an image but the phase correction algorithms presented in this research estimate only one set of relative pistons, tips, and tilts between apertures at a time. Phase estimates may be possible for an anisoplanatic image if the image is masked to the size of an isoplanatic patch and there is enough support in the masked region. However, when an isoplanatic patch only covers a small number of pixels, there is not enough support available in such a small section of the image and phase corrections are not possible. With proper support, multiple isoplanatic patches in an image can be corrected by masking different areas of the image. Recombining each of these corrected patches into one phase corrected image requires the masked areas to overlap with each other to allow patch registration.

3.4 Remapped Exit Pupil

Looking at equations (25) and even (32), phase corrections seem relatively simple. The pistons and tilts in the equations are clearly visible in the auto- and cross-correlations between each set of apertures. However, Figure 8.b shows that many of the auto- and cross-correlations either partially or completely overlap with one another. At first glance, this overlap creates a summation of different pistons, tips, and tilts that appear to be detrimental to phase correction. Fortunately, this overlap also provides the means to computationally correct phase errors in the spatial frequency domain of the image if the aperture fields from the entrance pupil can be remapped into a non-redundant exit pupil.

Figure 8.a shows an entrance pupil for a two-aperture system and Figure 8.b displays its OTF, which is the spatial frequency spectrum for a point source target. The spatial frequency for a diffuse target is the auto-correlation of the product between the pupil function and the complex field of the target propagated to the pupil field, as described in Sections 3.1 and 3.2. In particular, equation (22) gives the equation for the spatial frequency spectrum of an image collected from a two-aperture system with piston. An anamorphic imaging system is used to separate the aperture fields into a remapped, non-redundant exit pupil. This separation changes equation (21) to,

$$\mathcal{P}(\xi, \eta) = P_1(\xi, \eta + M_p a) e^{j\varphi_1} + P_2(\xi, \eta - M_p a) e^{j\varphi_2} \quad (35)$$

and equation (22) to,

$$\begin{aligned} \mathcal{G}(f_x, f_y) = & U_1(\xi, \eta) \otimes U_1^*(-\xi, -\eta) + U_2(\xi, \eta) \otimes U_2^*(-\xi, -\eta) \\ & + [U_1(\xi, \eta) \otimes U_2^*(-\xi, -\eta) \otimes \delta(\xi, \eta + 2M_p a)] e^{-j(\varphi_{21})} \\ & + [U_1^*(-\xi, -\eta) \otimes U_2(\xi, \eta) \otimes \delta(\xi, \eta - 2M_p a)] e^{j(\varphi_{21})} \end{aligned} \quad (36)$$

where M_p is the pupil magnification of the system and a is the aperture separation distance with regard to the entrance pupil. The pupil mask and OTF for the remapped exit pupil is shown in Figure 9.

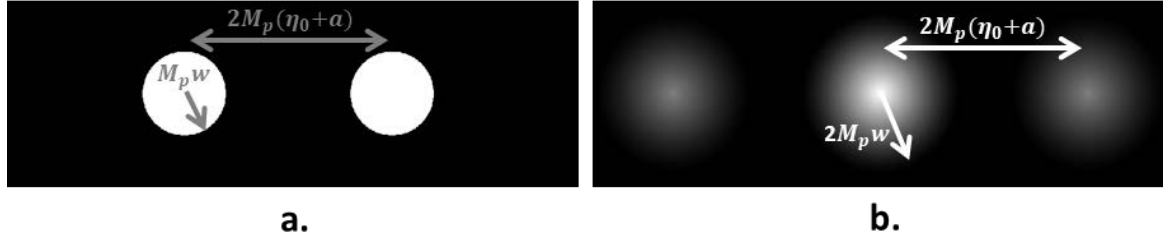


Figure 9: a. Exit pupil function with 2 anamorphically separated apertures. The radius and separation of each aperture is scaled by the pupil magnification. b. Frequency response of the non-redundant exit pupil when illuminated by a point source.

This example of a two-aperture system only shows pupil remapping in one direction. However, a greater number of apertures can be used as long as a non-redundant configuration for the apertures exists. For instance, a one dimensional six-aperture array can be reshaped into a non-redundant, separated Golay-6 array [28].

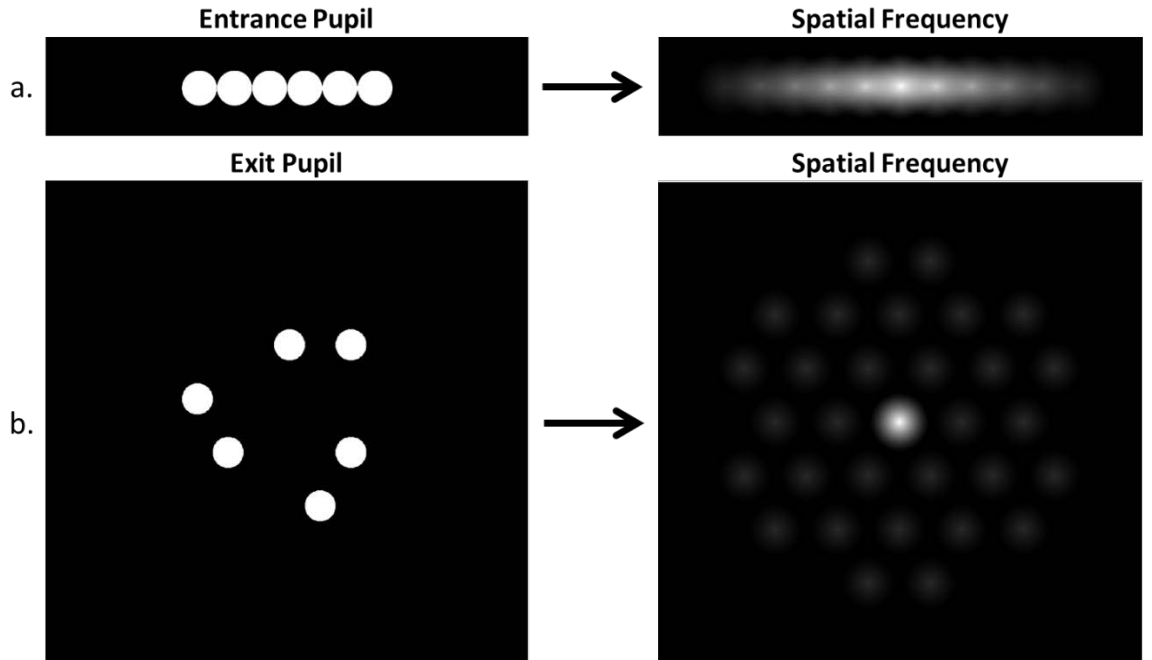


Figure 10: a. 6 aperture 1D entrance pupil and its redundant spatial frequency spectrum b. Separated Golay-6 array exit pupil and its non-redundant spatial frequency spectrum. The aperture fields and subsequent spectra assume a point source target.

A general equation for remapping an N-aperture array with piston phase errors is,

$$\mathcal{P}(\xi, \eta) = \sum_{n=1}^N P_n(\xi - M_p b_n, \eta - M_p a_n) e^{j\varphi_n} \quad (37)$$

and the spatial frequency of a corresponding image is,

$$\mathcal{G}(f_x, f_y) = \sum_{m=1}^N \sum_{n=1}^N [U_m(\xi, \eta) \otimes U_n^*(-\xi, -\eta) \otimes \delta(\xi - M_p b_{mn}, \eta - M_p a_{mn})] e^{-j\varphi_{nm}} \quad (38)$$

Where $a_{mn} = a_m - a_n$ and $b_{mn} = b_m - b_n$.

The goal of pupil remapping is to create an imaging system that collects images through the process described in Figure 11. Light from a target propagates from the object plane through the atmosphere until it reaches an aperture array that makes up the entrance pupil. The aperture fields in the entrance pupil are all remapped to new, non-redundant locations in the exit pupil before propagating on to the focal plane array at the image plane.

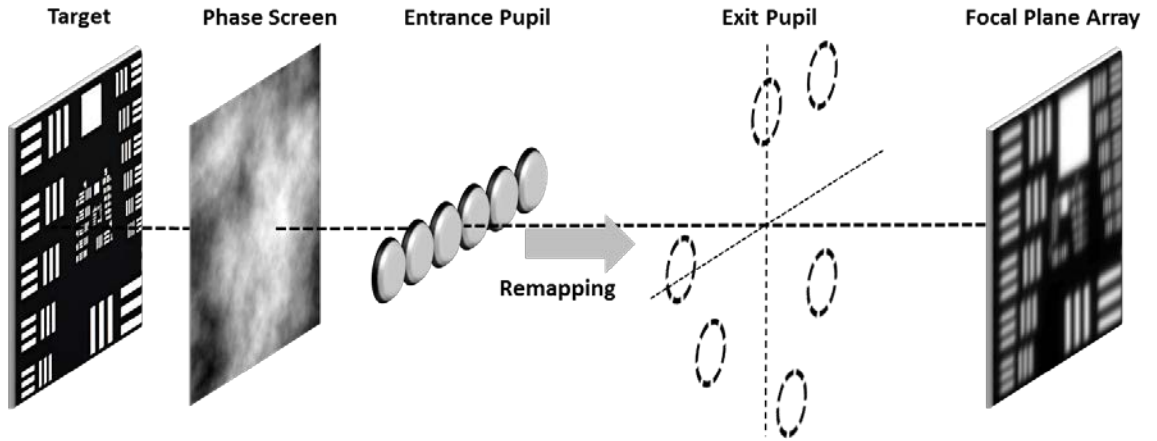


Figure 11: Diagram of the physical imaging system designed to create images with remapped pupil functions.

Physically remapping the pupils breaks the golden rule of aperture reimaging since the spatial frequencies are shifted from their original entrance pupil locations, which leads to an image with incorrect spatial frequencies [10]. However, piston, tip, and tilt from partially coherent images can be computationally estimated and corrected using remapping. The remapped pupils are corrected computationally during post-processing

along with the phase errors. This computational correction technique is discussed in detail in CHAPTER 5.

3.4.1 Field Dependent Contrast

Computationally correcting the remapped exit pupil involves shifting all of the cross-correlations in the non-redundant spatial frequency profile back to their redundant locations with regard to the entrance pupil as well as correcting piston phase. However, broader illumination bandwidths can cause a problem with the spatial frequencies of the cross-correlations which cannot be solved by a computational shift. In the case where pupil remapping is constant across an illumination bandwidth, the spatial frequencies in the cross-correlations are shifted as a function of wavelength. To make this effect more apparent, the relationship between spatial and spatial frequency coordinates given in equations (12) and (13) will be used to rewrite equation (38). First, the spatial coordinates undergo a substitution,

$$\begin{aligned} & \mathcal{G}(f_x, f_y) \\ &= \sum_{m=1}^N \sum_{n=1}^N [U_m(f_x \lambda d_i, f_y \lambda d_i) \otimes U_n^*(-f_x \lambda d_i, -f_y \lambda d_i) \otimes \delta(f_x \lambda d_i + M_p b_{mn}, f_y \lambda d_i + M_p a_{mn})] e^{-j\varphi_{nm}} \end{aligned} \quad (39)$$

Next, λd_i is divided out of every coordinate in the equation,

$$\mathcal{G}\left(\frac{f_x}{\lambda d_i}, \frac{f_y}{\lambda d_i}\right) = \sum_{m=1}^N \sum_{n=1}^N \left[U_m(f_x, f_y) \otimes U_n^*(-f_x, -f_y) \otimes \delta\left(f_x - \frac{M_p b_{mn}}{\lambda d_i}, f_y - \frac{M_p a_{mn}}{\lambda d_i}\right) \right] e^{-j\varphi_{nm}} \quad (40)$$

In this form, the shift's dependence on wavelength is visible. For an illumination bandwidth of $\Delta\lambda = \lambda_2 - \lambda_1$, the mean shift in the f_y -direction is:

$$\frac{\frac{M_p a_{mn}}{\lambda_1 d_i} + \frac{M_p a_{mn}}{\lambda_2 d_i}}{2} \approx \frac{M_p a_{mn}}{\bar{\lambda} d_i} \quad (41)$$

And the equation for the f_x -direction is similar. The mean wavelength in the bandwidth is $\bar{\lambda}$. And the spatial frequency spectrum at that wavelength is,

$$\mathcal{G}\left(\frac{f_x}{\bar{\lambda}d_i}, \frac{f_y}{\bar{\lambda}d_i}\right) = \sum_{m=1}^N \sum_{n=1}^N \left[U_m(f_x, f_y) \otimes U_n^*(-f_x, -f_y) \otimes \delta\left(f_x - \frac{M_p b_{mn}}{\bar{\lambda}d_i}, f_y - \frac{M_p a_{mn}}{\bar{\lambda}d_i}\right) \right] e^{-j\varphi_{nm}} \quad (42)$$

All of the other wavelengths around $\bar{\lambda}$ cause slightly different spatial frequency shifts away from the mean shift that was calculated in equation (41). Rotating the coordinate system in the direction of each aperture's remapping, integrating equation (40) across all of the shifts caused by the illumination bandwidth, and rotating back to the original coordinate system should result in the spatial frequency spectrum for the full illumination bandwidth,

$$\begin{aligned} \hat{\mathcal{G}}\left(\frac{f_x}{\bar{\lambda}d_i}, \frac{f_y}{\bar{\lambda}d_i}\right) &= \sum_{m=1}^N \sum_{n=1}^N \left[1 + \delta_{mn} \left(\frac{M_p \Delta \lambda}{\bar{\lambda}^2 d_i} \sqrt{a_{mn}^2 + b_{mn}^2} - 1 \right) \right] \\ &\quad \left[U_m(f_x, f_y) \otimes U_n^*(-f_x, -f_y) \otimes \delta\left(f_x - \frac{M_p b_{mn}}{\bar{\lambda}d_i}, f_y - \frac{M_p a_{mn}}{\bar{\lambda}d_i}\right) \right. \\ &\quad \left. \otimes \text{rect}\left(\frac{\left(\frac{a_{mn}}{b_{mn}} f_x + f_y\right) \bar{\lambda}^2 d_i}{M_p \Delta \lambda \sqrt{a_{mn}^2 + b_{mn}^2}}\right) \right] e^{-j\varphi_{nm}} \end{aligned} \quad (43)$$

where δ_{mn} is the Kronecker delta. The cross-correlations in equation (43) are centered around the shift caused by the mean wavelength. The spread of this shift as a function of the illumination bandwidth is defined by a rectangular function. The full working of this rotation and integration is given in APPENDIX A. The inverse FT gives the intensity field at the focal plane array.

$$I(x_i, y_i) = \sum_{m=1}^N \sum_{n=1}^N \left[\left[1 + \delta_{mn} \left(\left(\frac{M_p \Delta \lambda}{\bar{\lambda}^2 d_i} \right)^2 (a_{mn}^2 + b_{mn}^2) - 1 \right) \right] e^{-j(xb_{mn} + ya_{mn})/\bar{\lambda}d_i} \right. \\ \left. \times u_m(x, y) u_n(x, y) \text{sinc} \left(\frac{\left(\frac{a_{mn}}{b_{mn}} x + y \right) M_p \Delta \lambda \sqrt{a_{mn}^2 + b_{mn}^2}}{\bar{\lambda}^2 d_i} \right) e^{-j\varphi_{nm}} \right] \quad (44)$$

The new exponential term is the tilt caused by the mean shift of each aperture and the sinc() function is an envelope that defines the contrast across the FOV, or the FDC. The exponential can be removed by computationally cropping the separated cross-correlations and registering them with the central auto-correlation, thus correcting the shifts caused by the aperture remapping. However, the sinc() function cannot be corrected in the same way. Instead, dispersion from diffractive optics can be used.

3.4.2 Diffraction Grating

Section 3.4.1 assumes that the aperture shifts, a_n and b_n , caused by pupil remapping are constant across illumination wavelengths, leading to FDC. However, the use of diffraction gratings placed at an intermediate pupil plane to facilitate pupil remapping leads to shifts that are a function of the wavelength. Therefore, substituting $a_n(\lambda)$ and $b_n(\lambda)$ into equation (40) should cancel out the cross-correlation shifts' dependences on wavelength and prevent FDC. In order to illustrate this result, begin with the equation of a reflective grating,

$$\theta_l(\lambda) = \sin^{-1} \left[\frac{l\lambda}{\mathfrak{d}} - \sin(\theta_l) \right] \quad (45)$$

where $\mathfrak{d}$ is the distance per groove for the grating, θ_l is the angle of incidence, and θ_l is the diffraction angle for the specified mode, l . The angular difference between the diffraction angles of the zeroth and first modes is,

$$\theta_{10}(\lambda) = \theta_1(\lambda) - \theta_0(\lambda) = \sin^{-1} \left[\frac{\lambda}{\mathbb{d}} - \sin(\theta_I) \right] + \theta_I \quad (46)$$

This is the angular shift that is responsible for pupil remapping. The effect of θ_I on $\theta_{10}(\lambda)$ over the wavelength bandwidth is negligible, so an approximation to equation (46) can be used,

$$\theta_{10}(\lambda) = \sin^{-1} \left[\frac{\lambda}{\mathbb{d}} \right] \quad (47)$$

At the exit pupil, this angular shift results in a spatial shift of $\sqrt{a_n^2(\lambda) + b_n^2(\lambda)}$ in the direction defined by,

$$\theta_n = \tan^{-1} \left(\frac{a_n}{b_n} \right) \quad (48)$$

The spatial shift at the exit pupil results in an angular shift at the image plane,

$$\theta_{shift,n}(\lambda) = \tan^{-1} \left[\frac{M_p \sqrt{a_n^2(\lambda) + b_n^2(\lambda)}}{d_i} \right] \quad (49)$$

In an optical setup that includes a lens between the diffraction grating at an intermediate image plane and the final image plane, equations (47) and (49) are related by the angular magnification of that lens, M_a ,

$$M_a \sin^{-1} \left[\frac{\lambda}{\mathbb{d}_n} \right] = \tan^{-1} \left[\frac{M_p \sqrt{a_n^2(\lambda) + b_n^2(\lambda)}}{d_i} \right] \quad (50)$$

In this case, different gratings may need to be used for different separation distances, which is specified with the additional subscript in $\mathbb{d}_n$. Using the paraxial approximation and solving for the spatial shift in terms of λ results in,

$$\sqrt{a_n^2(\lambda) + b_n^2(\lambda)} = \frac{M_a \lambda d_i}{M_p \mathbb{d}_n} \quad (51)$$

After making the assumption that there is a constant ratio between $b_n(\lambda)$ and $a_n(\lambda)$ and defining the shifts at $\bar{\lambda}$ as $\bar{b}_n$ and $\bar{a}_n$, each shift can be solved for individually,

$$b_n(\lambda) = \frac{M_a \lambda d_i}{M_p \mathbb{d}_n} \sin \left[\tan^{-1} \left(\frac{\bar{a}_n}{\bar{b}_n} \right) \right] \quad (52)$$

$$a_n(\lambda) = \frac{\delta_m M_a \lambda d_i}{M_p \mathbb{d}_n} \cos \left[\tan^{-1} \left(\frac{\bar{a}_n}{\bar{b}_n} \right) \right] \quad (53)$$

These simplify to,

$$b_n(\lambda) = \frac{M_a \lambda d_i}{M_p \mathbb{d}_n} \frac{1}{\sqrt{1 + (\bar{b}_n/\bar{a}_n)^2}} \quad (54)$$

$$a_n(\lambda) = \frac{M_a \lambda d_i}{M_p \mathbb{d}_n} \frac{\bar{b}_n/\bar{a}_n}{\sqrt{1 + (\bar{b}_n/\bar{a}_n)^2}} \quad (55)$$

Plugging equations (54) and (55) into equation (40) results in,

$$\begin{aligned} \mathcal{G} \left(\frac{f_x}{\lambda d_i}, \frac{f_y}{\lambda d_i} \right) = & \sum_{m=1}^N \sum_{n=1}^N \left[U_m(f_x, f_y) \otimes U_n^*(-f_x, -f_y) \otimes \right. \\ & \delta \left(f_x + \frac{M_a}{M_p} \left(\frac{1}{\mathbb{d}_m \sqrt{1 + (\bar{b}_m/\bar{a}_m)^2}} - \frac{1}{\mathbb{d}_n \sqrt{1 + (\bar{b}_n/\bar{a}_n)^2}} \right) \right) \\ & \left. \delta \left(f_y + \frac{M_a}{M_p} \left(\frac{\bar{b}_m/\bar{a}_m}{\mathbb{d}_m \sqrt{1 + (\bar{b}_m/\bar{a}_m)^2}} - \frac{\bar{b}_n/\bar{a}_n}{\mathbb{d}_n \sqrt{1 + (\bar{b}_n/\bar{a}_n)^2}} \right) \right) e^{-j\varphi_{nm}} \right] \end{aligned} \quad (56)$$

Here, λ has been cancelled out and the shifts are no longer dependent on wavelength.

Taking the Fourier transform of this frequency response results in,

$$\hat{u}_i(x, y) = \sum_{m=1}^N \sum_{n=1}^N \left[u_m(x, y) u_n(x, y) e^{-j \frac{M_a}{M_p} \left(\frac{x+y(\bar{b}_m/\bar{a}_m)}{\mathbb{d}_m \sqrt{1 + (\bar{b}_m/\bar{a}_m)^2}} - \frac{x+y(\bar{b}_n/\bar{a}_n)}{\mathbb{d}_n \sqrt{1 + (\bar{b}_n/\bar{a}_n)^2}} \right)} e^{-j\varphi_{nm}} \right] \quad (57)$$

There is no sinc() function in this equation. Therefore, it has been shown that, in an ideal case, remapping the entrance pupil using diffraction gratings placed at an intermediate image plane avoids the problem of FDC. While there are still the exponentials

caused by aperture remapping, these can be computationally corrected, which will be discussed in CHAPTER 5. Prior to that, however, the software design used for simulating atmospheric turbulence and aperture remapping will be covered in CHAPTER 4.

CHAPTER 4

SIMULATION DESIGN

Three goals of the phase correction algorithm are to correct the relative piston differences between multiple apertures, correct the relative tip and tilt differences between that same set of apertures, and do this independently for single isoplanatic patches within an anisoplanatic image. In order to design and test this algorithm it was useful to design a simulation of an optical system similar to Figure 11 with a remapped pupil and atmospheric turbulence. To this end, a six aperture optical simulation is described below. The simulations discussed in this chapter are used to illustrate the correction algorithms described in CHAPTER 5.

4.1 Target Propagation

The three objectives of the computational phase correction algorithm are piston, tip and tilt, and isoplanatic corrections. Different simulated targets were chosen to test each correction type. Targets similar to what can be seen in the field are used when possible. Therefore, a dead leaves target is used to test piston, tip, and tilt correction performance. Dead leaves targets are statistically generated images containing multiple layers of circles with differing sizes and intensities [23]. They are designed to resemble a forest floor but can also be described as a simulation of a pile of coins. An example of the dead leaves target used for testing is shown in Figure 12.a.

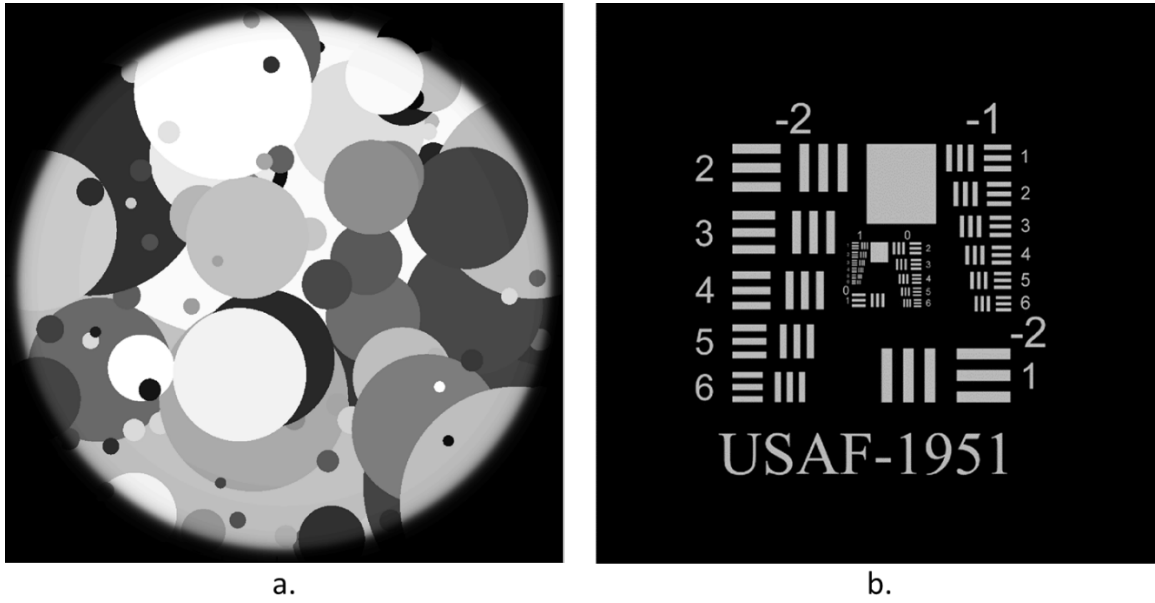


Figure 12: a. Dead leaves target, tapered to reduce vertical and horizontal spatial frequency artifacts [23] b. 1951 USAF resolution target. Elements and groups are not accurate. The target is used to demonstrate the general effectiveness of phase corrections techniques, not specific spatial frequencies.

The 1951 USAF resolution target is also used to test piston, tip, and tilt corrections. While not as realistic as the dead leaves target, it provides a discrete set of resolutions which allows an observer to more easily determine the effectiveness of the correction technique. The dead leaves and 1951 USAF targets also have varying sharpness metrics and intensity distributions. This leads to an analysis of the effectiveness of the phase correction algorithm for differing image content. An image of the 1951 USAF resolution target is shown in Figure 12.b.

In order to efficiently simulate the propagation of these targets through a six aperture optical system a point target is placed at the object plane. This point target is propagated through a vacuum to the pupil plane using a Fresnel diffraction integral like in equation (3). The propagation of the point target results in a flat field at the pupil plane that is multiplied by a circular mask with a diameter of the same length as the maximum gain of the entrance pupil. The filtered flat field is then propagated back to the object plane

using the inverse Fresnel diffraction integral. Images of the point target, masked flat field at the entrance pupil, and filtered point target are shown in Figure 13.

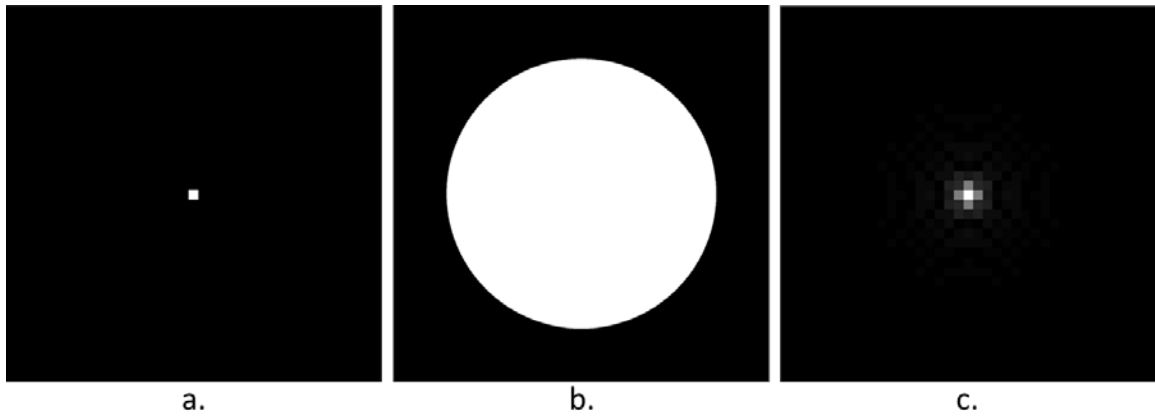


Figure 13: a. Point target b. Masked flat plane at the entrance pupil c. Filtered point target

After the initial spatial filtering, the point target is re-propagated to the pupil plane using a prewritten atmospheric propagation software [26]. The complex field at the pupil is multiplied by the pupil mask. In this case, the pupil mask is the mask of six apertures that is shown in Figure 10.a. More information about the application of atmospheric turbulence is presented in Section 4.3. However, before discussing the addition of atmosphere, a simplified test of the phase correction algorithm may be done by applying a known piston, tip, and tilt directly to the aperture mask after a propagation through a vacuum. This is discussed in Section 4.2.

After the point target is propagated pupil plane and the pupil mask is applied, each of the six apertures in the array are cropped into separate complex fields and each aperture is correlated with all of the other apertures. For six apertures there are 36 correlations: six auto-correlations, fifteen cross-correlations, and fifteen complex conjugates to those cross correlations. For N apertures, the total number of correlations can be defined as,

$$N_c = N + 2 \sum_{m=1}^{N-1} m \quad (58)$$

The spatial frequency spectra corresponding to an anamorphic optics system where the aperture fields in the entrance pupil are remapped in the exit pupil are shown in Figure 10.a and b. The correlations in the remapped spatial frequency domain are separated so that each one can be cropped separately and computationally shifted back to its location relative to the entrance pupil. In the simulation, the correlations can be computationally shifted to non-redundant locations in order to add noise, demonstrate the remapped spatial frequency spectrum, or simulate FDC. However, shifting the cross-correlation is not necessarily needed if there is no intent to add noise or model FDC from multiple wavelengths. A list of each of the aperture correlations is shown in Figure 14.a and b. Figure 14.a shows correlations located close to their original locations corresponding to the entrance pupil before remapping. Columns show correlations that fully overlap in the entrance pupil's spectrum and adjacent columns show partially overlapping correlations. Figure 14.b shows the locations of each of the correlations in the spatial frequency spectrum of the remapped exit pupil. These overlaps will be very useful in the phase correction process discussed in CHAPTER 5.

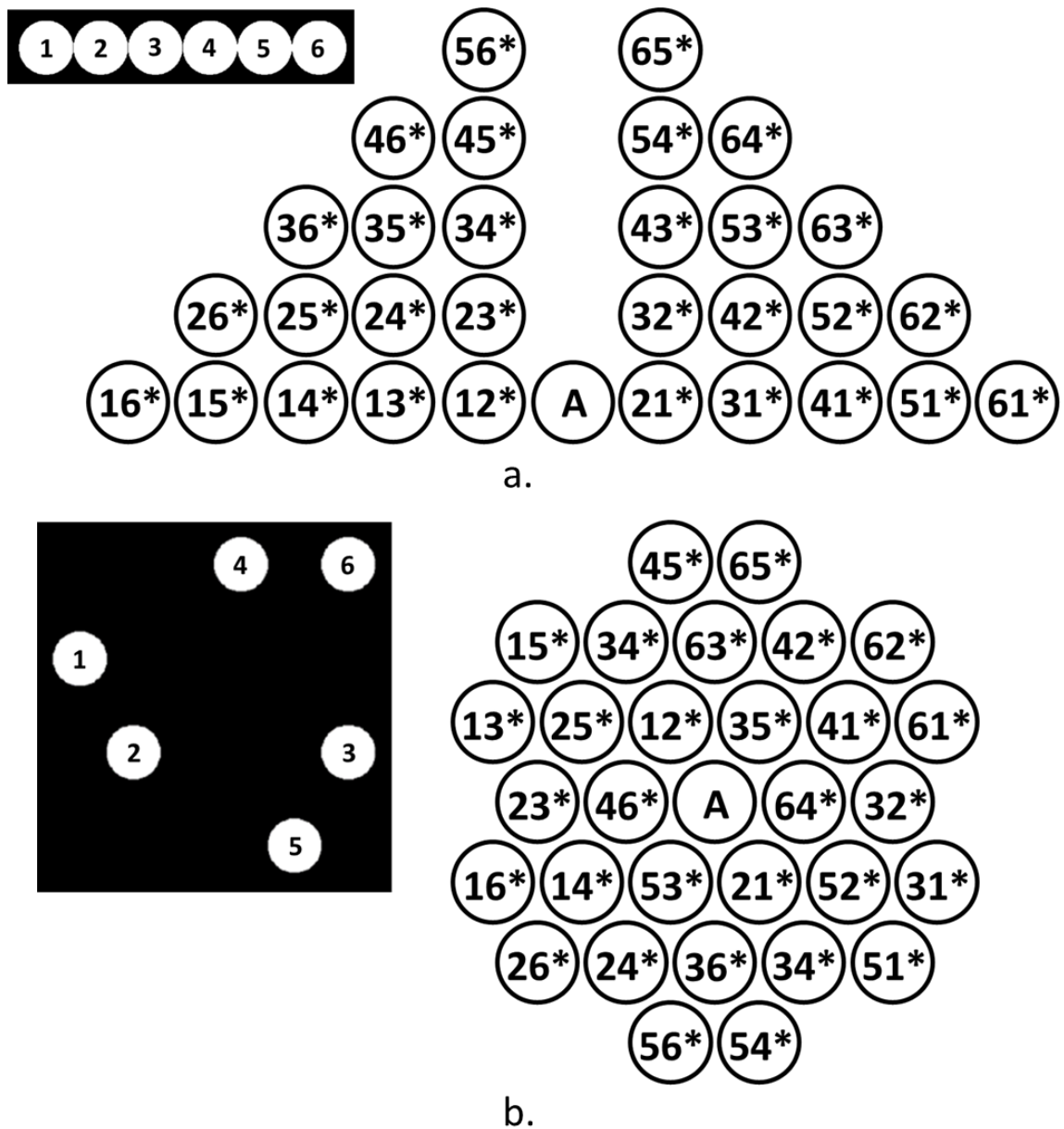


Figure 14: List of all correlations for six apertures a. near their original locations and b. in their remapped locations. In a, columns show correlations that are fully overlapping and adjacent columns show partially overlapping correlations.

The sum of all the correlations is the OTF of the six aperture system as seen from the entrance pupil and its FT is the PSF. Multiplying the FT of an extended target with the OTF gives the spatial frequency spectrum of the extended target propagated through the same system. Taking the inverse FT of this spatial frequency spectrum gives a simulated image of the extended target. This technique has the advantage of avoiding simulated

speckle noise, and therefore reducing the number of iterations necessary to generate a speckle-averaged image. This technique is applied to the 1951 USAF target and the dead leaves target described at the beginning of this section, which are used for piston, tip, and tilt correction analysis.

However, a different extended target is used for testing the effects of masking part of an image on the performance of the phase correction algorithm. To test the effects of masking, a homogeneous target is preferred that provides a similar spatial frequency spectrum regardless of mask size. For this purpose, a grid of point targets is used. The grid measures 32 by 32 points spaced so that the PSFs do not overlap after propagation. A flat intensity is added to the background to lessen the sharpness of the target and reduce the spatial frequency content at higher frequencies to make it more similar to the dead leaves target. An image of this grid target after propagation is shown in Figure 15.

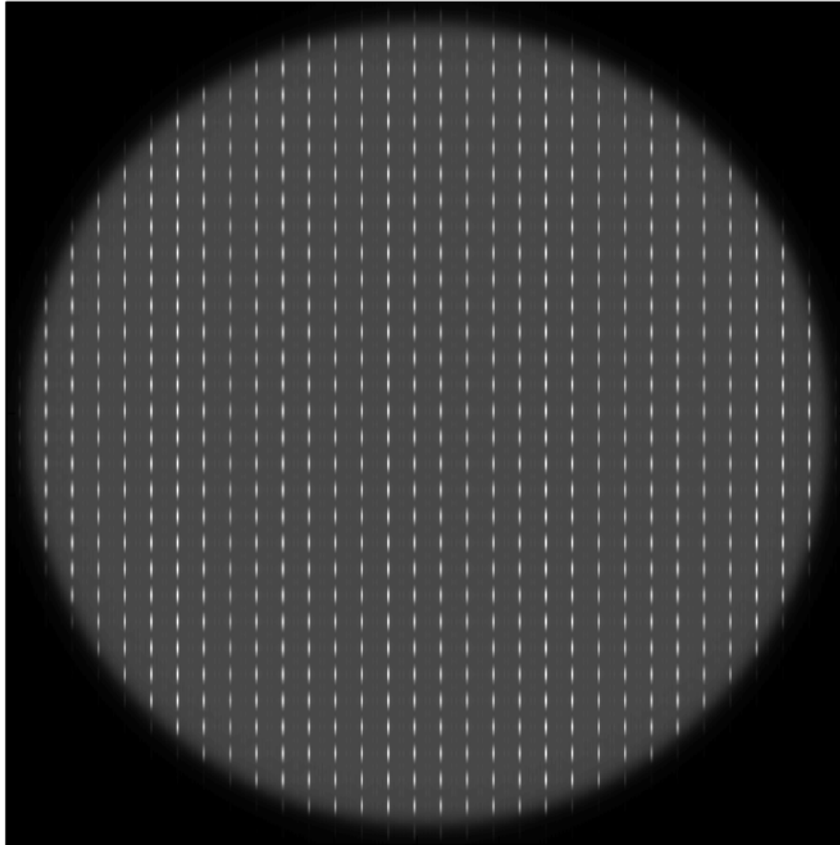


Figure 15: Propagated 32 x 32 point grid with a flat plate background

Propagating a point target and convolving the resulting point spread function with an extended target only works for simulating images with an infinite isoplanatic angle. The propagated point target only contains the phase errors from one location on the target and only that phase error gets transferred to the extended target. Smaller isoplanatic angles can be created by propagating the extended target directly. Unfortunately, that also adds speckle to the image, which requires multiple iterations to average out and greatly increases the time it takes to run the simulations. Another way to create smaller isoplanatic angles is to propagate multiple point targets at different locations in the field of view and apply them to the image separately. However, this also requires multiple iterations for one image. In order to simplify this problem, a 16 x 16 point target grid is used. Since the target is

sparse, with non-interfering point targets, it does not suffer from speckle. When propagated through atmosphere, each point on the grid has a variable PSF caused by anisoplanatism. An image of this target propagated through a vacuum is shown in Figure 16 while an anisoplanatic image is shown later.

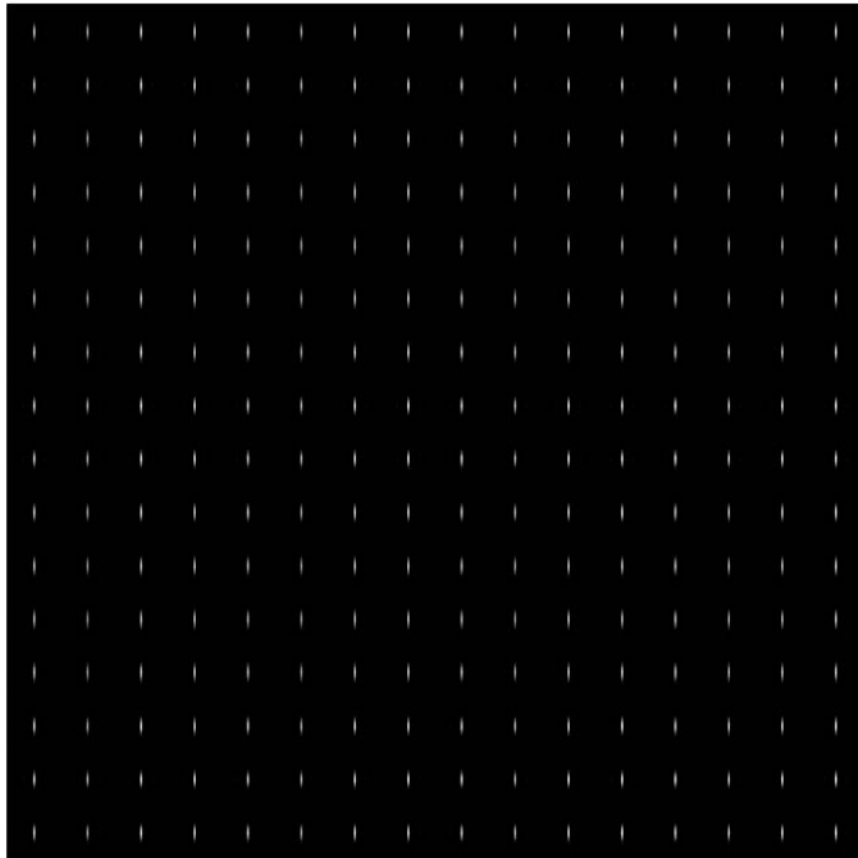


Figure 16: 16 x 16 grid target

4.2 Piston, Tip, and Tilt

The previous section gave a general outline for simulating the propagation of an image through a six-aperture entrance pupil. In this section, the specifics of simulating piston, tip, and tilt in that system are discussed. All three types of phase errors are introduced during the creation of the pupil mask shown in Figure 10.a. Each aperture

begins as circle in the middle of the field with an amplitude of one, a phase of zero, and a radius of w . Piston, tip, and tilt phases are added using the Zernike polynomial [24],

$$phase = Z_0 + Z_1\rho \cos(\theta) + Z_2\rho \sin(\theta) \quad (59)$$

where (ρ, θ) are the polar coordinates of the aperture and Z_0 , Z_1 , and Z_2 are the Zernike coefficients of the piston, tip, and tilt, respectively. All three coefficients are randomly generated for each aperture. The piston coefficient, Z_0 , is constrained between $[\pi, -\pi]$ while the tip and tilt coefficients, Z_1 and Z_2 , are kept between $\left[\frac{\pi}{2r}, -\frac{\pi}{2r}\right]$ to ensure that there is no phase wrapping for any one coefficient. The phase field is multiplied with the aperture,

$$ap_{new} = ap_{old}e^{-i*phase} \quad (60)$$

where ap_{old} is the aperture field before phase is applied and ap_{new} is the aperture with phase errors.

After the piston, tip, and tilt is applied to an aperture, that aperture is shifted to its correct location with regard to the other apertures in the entrance pupil, which corresponds with equation (20) in Section 3.2. Examples of the phase of an entrance pupil with piston, tip, tilt and all three combined is shown in Figure 17. The use of these known phase errors in the simulation allows for the direct measurements of the effectiveness of the phase correction code.

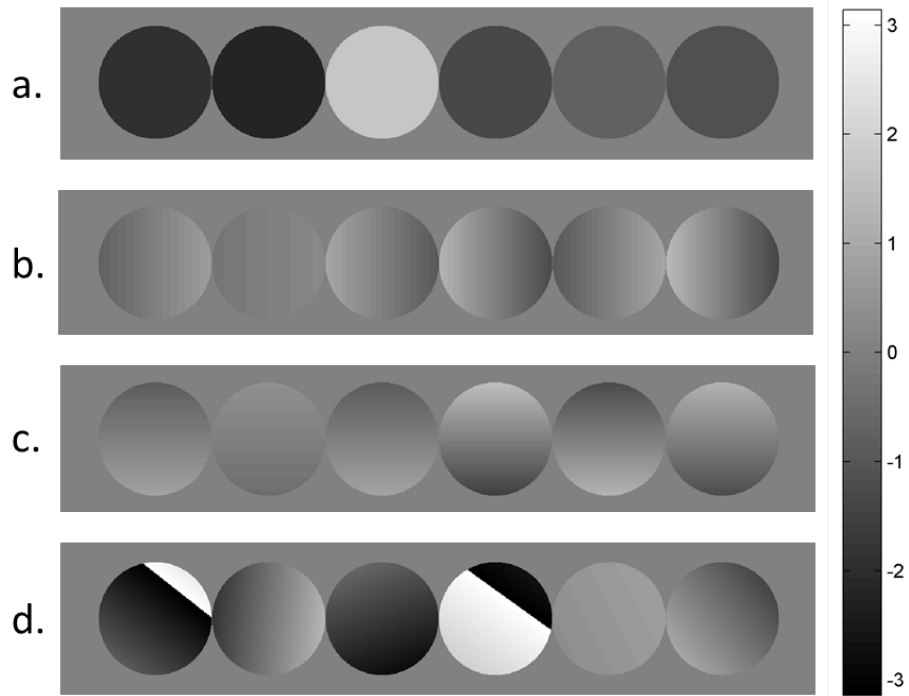


Figure 17: Examples of the phases of apertures with a. piston, b. tip, c. tilt, and d. all three

The sum of the correlations for the aperture array in Figure 17.d and the corresponding point function are shown in Figure 18.

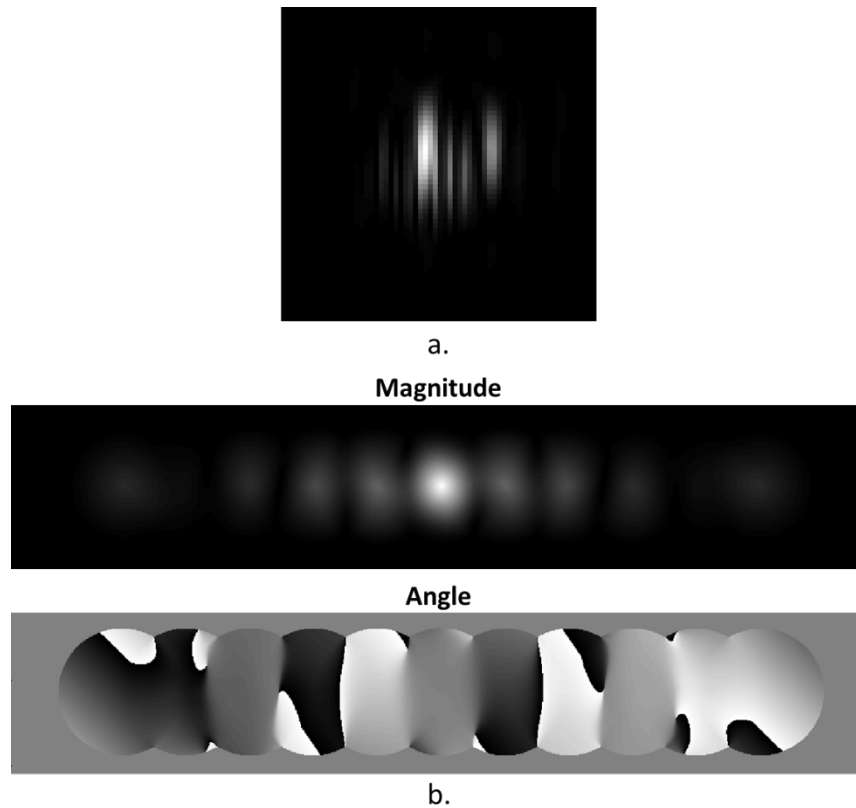


Figure 18: a. Point source from the apertures shown in Figure 17.d and b. the magnitude and phase of the FT of that point source

4.3 Atmospheric Turbulence

While a known piston, tip, and tilt are useful for initial testing of a phase correction algorithm, these phase errors do not effectively demonstrate those caused by atmospheric turbulence. As discussed in Sections 3.3.1 and 3.3.2, the Fried parameter affects how much phase error is found in each aperture and the isoplanatic angle causes different phase errors across a FOV. Both of these measures are affected by the amount of atmospheric turbulence in the atmosphere and where that turbulence is located between the target and the entrance pupil. In order to test the effectiveness of phase corrections for different Fried parameters and isoplanatic angles, these values are introduced through simulated atmospheric turbulence.

Based on the effects of Fried parameter on the phase of a multi-aperture entrance pupil, it was decided to test Fried parameters of $\frac{1}{3}$, $\frac{1}{2}$, 1, and 2 times the diameter of an aperture. Since the simulated apertures each have a diameter of 2.54 cm, the Fried parameters are 0.85 cm, 1.27 cm, 2.54 cm, and 5.08 cm respectively. This aperture diameter was chosen to match the size of the apertures in the experiment; however, the aperture diameter and corresponding Fried parameters could be scaled to any arbitrary size. Likewise, isoplanatic angles $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, 1, and 2 times the angular diameter of the FOV are used. A rough estimate of the experimental field of view is 0.0045 degrees, so the isoplanatic angles in this simulation are 0.001125, 0.0015, 0.00225, 0.0045, and 0.009 degrees respectively. Again, the FOV is dependent upon the optical setup and the values given above can be scaled to another size.

The Fried parameter equation is shown in equation (29) and the isoplanatic angle in equation (33). These equations are analytical and assume a continuous atmosphere along the propagation path. They can be used for simulations by creating a discrete version of each equation in which the atmosphere is defined by a finite number of phase screens. These new versions are,

$$r_0 = \left[0.423 \left(\frac{2\pi}{\lambda} \right)^2 \sum_{i=1}^{N_p} C_{ni}^2 \left(1 - \frac{z_i}{L} \right)^{5/3} \Delta z \right]^{-3/5} \quad (61)$$

$$\theta_0 = \left[2.91 \left(\frac{2\pi}{\lambda} \right)^2 \sum_{i=1}^{N_p} C_{ni}^2 z_i^{5/3} \Delta z \right]^{-3/5} \quad (62)$$

for the Fried parameter and isoplanatic angle respectively. In this simulation, the total propagation length is also similar to the pupil distance that could be seen in a lab, $L = 2.82$ m. The distance between each phase screen is given by Δz , N_p is the number of phase

screens, and i identifies which phase screen is currently in use. For simplicity, only one phase screen is used. In this case, $\Delta z = L$ and the equations become,

$$r_0 = \left[0.423 \left(\frac{2\pi}{\lambda} \right)^2 C_n^2 \left(1 - \frac{z_1}{L} \right)^{5/3} L \right]^{-3/5} \quad (63)$$

$$\theta_0 = \left[2.91 \left(\frac{2\pi}{\lambda} \right)^2 C_n^2 z_1^{5/3} L \right]^{-3/5} \quad (64)$$

Since L is known and there are specific values of r_0 and θ_0 for which to solve, only the C_n^2 and z_1 values for the phase screens are unknown. The ratio of equations (64) and (63) leads to a solution for z_1 and equation (63) can be used to find a relatively simple solution for C_n^2 in terms of z_1 .

$$z_1 = \left[\frac{\theta_0}{r_0} \left(\frac{2.91}{0.423} \right)^{35/3} + \frac{1}{L} \right]^{-1} \quad (65)$$

$$C_n^2 = \left[2.91 \left(\frac{2\pi}{\lambda} \right)^2 (\theta_0 z_1)^{5/3} L \right]^{-1} \quad (66)$$

A Fourier series based atmospheric phase screen generator can use these values to create multiple random phase screen iterations for each atmospheric turbulence parameter described above [25]. Examples of phase screens with Fried parameter of 0.5 and 2 times the diameter of the atmosphere are displayed in Figure 19.

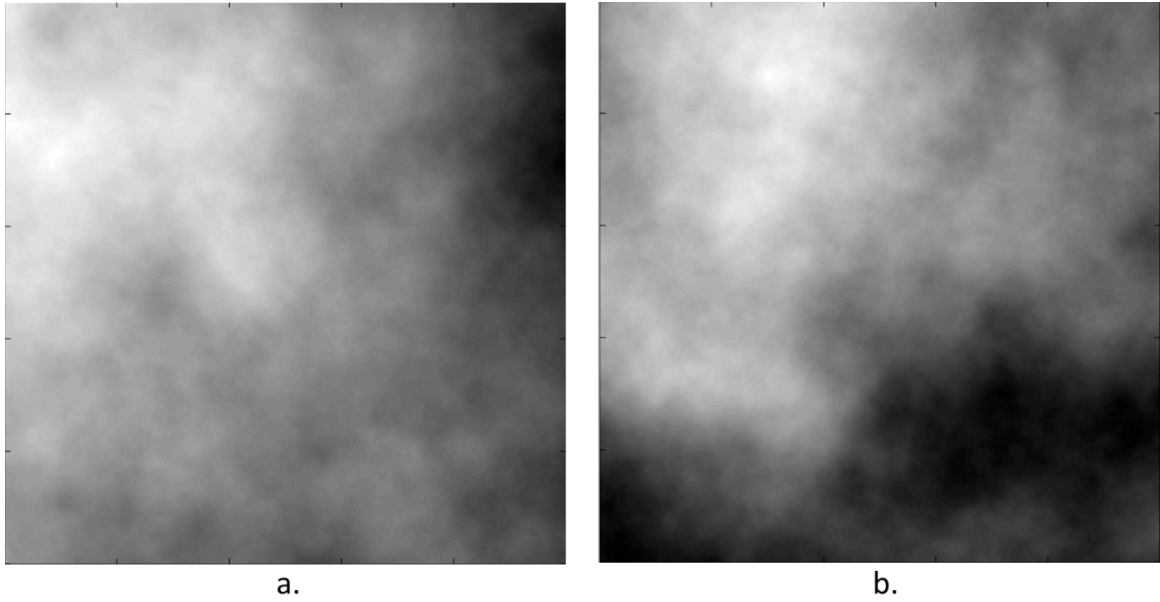


Figure 19: Phase screen for Fried parameters of a. 1.27 cm b. 5.08 cm

While Figure 19.a and b look similar, they are each located at different distances from the pupil field and therefore result in different Fried parameters. After propagating a centered point source through an atmosphere comprised of these phase screens and multiplying by the pupil displayed in Figure 13.b, the fields at the entrance pupil should take the forms shown in Figure 20 [26].

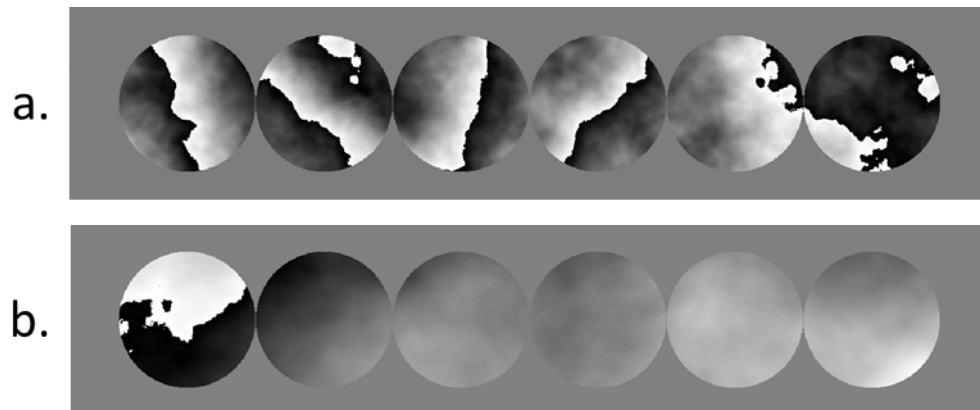


Figure 20: Masked phase fields at the entrance pupil for Fried parameters of a. 1.27 cm b. 5.08 cm

The corresponding point functions and spatial frequency spectra are shown in Figure 21.

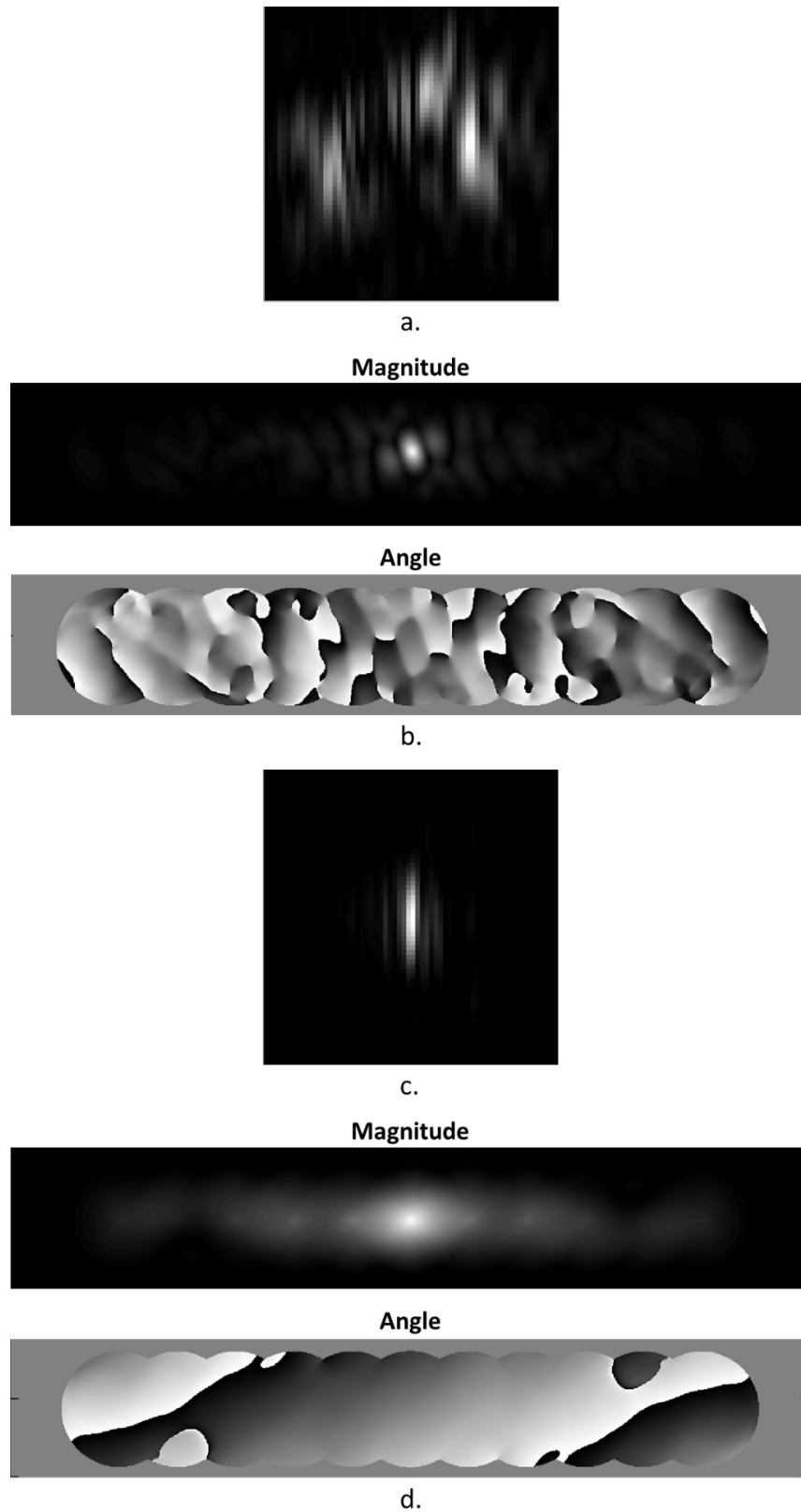


Figure 21: a. Point target propagated through an atmosphere with a Fried parameter of 1.27 cm and b. the magnitude and angle of its spatial frequency spectrum. c. Point target propagated through an atmosphere with a Fried parameter of 5.08 cm and d. the magnitude and angle of its spatial frequency spectrum.

The phase errors seen in the spatial frequency spectra display more phase wrapping for lower Fried parameter values and the magnitude of those spectra suffer as well. Convolution of the point target shown in Figure 21.a with an extended target results in a perturbed image of that target propagated through simulated turbulence. The effect of atmosphere on extended targets is shown in Figure 22.

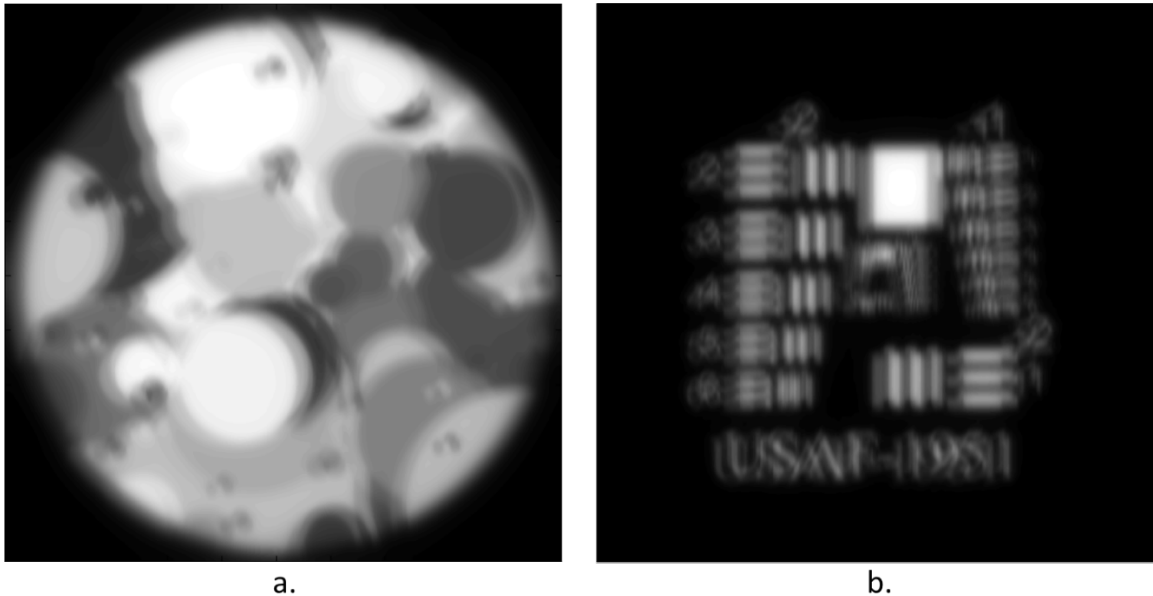


Figure 22: Atmospheric turbulence with a Fried parameter of 1.28 cm applied to the a. dead leaves target and b. 1951 USAF resolution target

The isoplanatic angle has an additional effect on extended targets. Since light from each point of a target travels through slightly different parts of the atmosphere, it interacts with the phase screen differently. As a result, different parts of an image have different phase errors. Figure 23 shows the PSF's at two different parts of an image and their corresponding spatial frequency spectra.

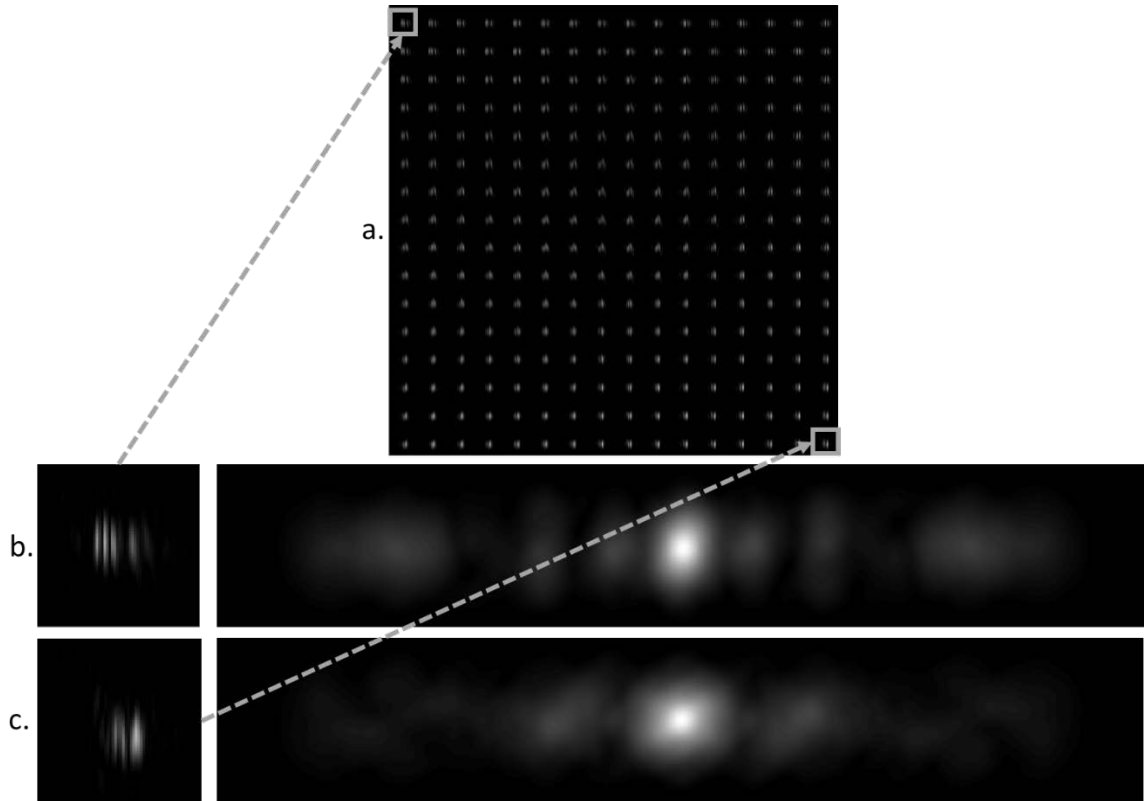


Figure 23: a. A grid target propagated through an atmosphere with a Fried parameter of 2.54 cm and isoplanatic angle of 1/4 the size of the FOV. b. The PSF at the top left of the grid and the magnitude of its spatial frequency spectrum. c. The PSF at the bottom right of the grid and the magnitude of its spatial frequency spectrum.

When a grid target is propagated through the system, the perturbation of each point in the grid target differs slightly depending on the size of the isoplanatic angle.

4.4 Signal to Noise Ratio

After simulating the propagation of a target through a six-aperture imaging system and applying phase errors, noise should also be added to the image. The two types of noise applied to this simulation are thermal noise from the camera and shot noise from the illumination intensity. Thermal noise is Gaussian and constant across the image plane regardless of intensity. It can be referred to as camera noise and the camera used in the experimental setup, a Sensors Unlimited GA1280JSX, has a camera noise of 25 electrons per pixel at a commonly used shutter speed and frame rate [27]. Shot noise is Poisson

distributed and dependent on intensity across an image. However, as the number of photons reaching the camera becomes increasingly high, the Poisson distribution approaches a normal distribution.

Since the shot noise is intensity dependent, it is also spatial frequency dependent. In order to add shot noise to the image, it should be applied while the spatial frequency spectrum is still remapped from the exit pupil as shown in Figure 10.d. Section 4.1 mentioned that this remapping is forgone in the initial simulation in order to reduce processing time; however, it is needed in order to add shot noise correctly. The locations of the remapped spatial frequencies can be calculated using the values given for a six-aperture Golay array [28]. The locations for each aperture in a non-redundant version of the Golay-6 array are shown in Table 1.

Table 1: Aperture locations for a non-redundant Golay-6 array, as shown in Figure 14.a [28]. D is the diameter of the aperture.

Aperture	x-axis	y-axis
1	-3D	$\frac{\sqrt{3}}{3}D$
2	-2D	$-\frac{2\sqrt{3}}{3}D$
3	2D	$-\frac{2\sqrt{3}}{3}D$
4	0	$\frac{4\sqrt{3}}{3}D$
5	1D	$-\frac{5\sqrt{3}}{3}D$
6	2D	$\frac{4\sqrt{3}}{3}D$

Twice the distance between two of the apertures listed above gives the distance of their cross-correlation from DC and the angle between the apertures gives the angular location with regard to DC.

After shifting each cross-correlation to the locations shown in Figure 10.d, the resulting spatial frequency spectrum is Fourier transformed to show the remapped image as seen from the exit pupil. This image is given in terms of intensity, but noise is added by measuring the photons-electrons per pixel. The intensity image, img , is changed to a photon-electron image, img_{ph} , using the equation,

$$img_{ph} = img \frac{N_{ph}}{\text{mean}(img)} \quad (67)$$

where N_{ph} refers to the mean number of photo-electrons assumed to be detected per pixel.

After converting to a photo-electron image, both thermal and shot noise are added using,

$$img_{ns} = img_{ph} + N_e \text{randn}(\mathbf{N}) + \sqrt{img_{ph}} \text{randn}(\mathbf{N}) \quad (68)$$

In which N_e is the standard deviation of the thermal noise in electrons per pixel and $\mathbf{N}$ is the size of the image array. The second term creates the thermal noise and the third term gives the intensity dependent shot noise. After applying the noise to the photo-electron image, it is converted back to intensity and the inverse FT is taken in order to give the spatial frequency spectrum again. After adding noise, each of the cross-correlations in the spatial frequency spectrum are cropped and shifted back to their original entrance pupil locations as displayed in Figure 10.b.

The desired number of photo-electrons per pixel is loosely based on the approximate number of photo-electrons at 1550 nm detected by the camera on a sunny day and is estimated to be about 5,000 photo-electrons. Different photo-electron values lead to different signal to noise ratios. In addition, the same number of desired photon-electrons leads to different SNR's for different images. In order to calculate the SNR for a specific image use,

$$SNR = \frac{mean(img_{ph})}{mean\left(\sqrt{img_{ph} + N_e^2}\right)} \quad (69)$$

The numerator gives the mean number of photo-electrons per pixel while the denominator gives the mean of the photo-electron variance per pixel. Since the images need no local oscillator, noise is dependent on intensity and the 1951 USAF target is more structured than the dead leaves target so its SNRs are higher, as shown in Table 2.

Table 2: SNRs for the 1951 USAF target and dead leaves target

Mean Photons Per Pixel	1951 USAF SNR	Dead Leaves SNR
50,000	346	249
5,000	89	72
500	16	15
50	2	2

Images of a 1951 USAF resolution target and a dead leaves target with a desired mean photo-electron count of 500 are shown in Figure 24. Different SNRs are used to analyze the capabilities of the phase correction algorithm discussed in CHAPTER 5.

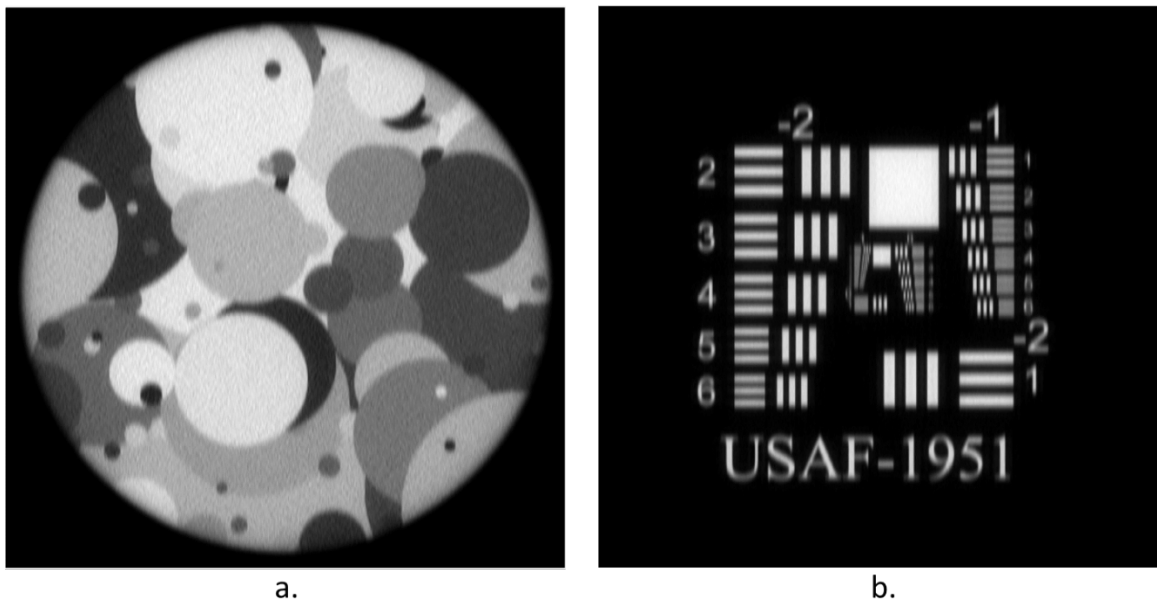


Figure 24: a. Dead leaves target and b. 1951 USAF resolution target with 500 mean photo-electrons per pixel

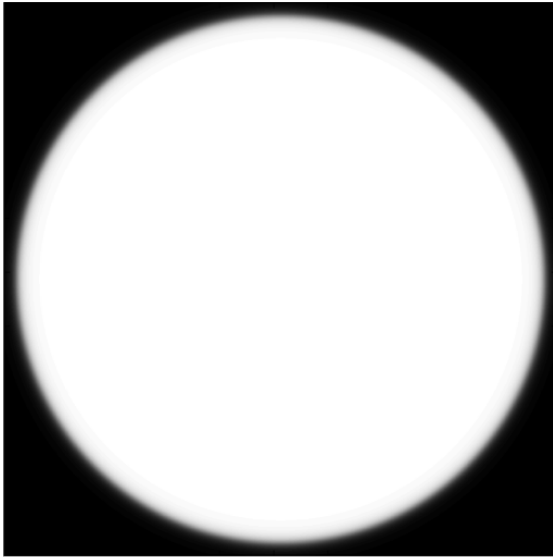
4.5 Image Masking

The phase correction algorithm's effectiveness is also evaluated for different amounts of image masking. Image masking can be used while correcting for isoplanatic patches smaller than the FOV in anisoplanatic images, as well as tapering the edges of a rectangular image so that the spatial frequencies of those edges do not interfere with the spatial frequency of the target.

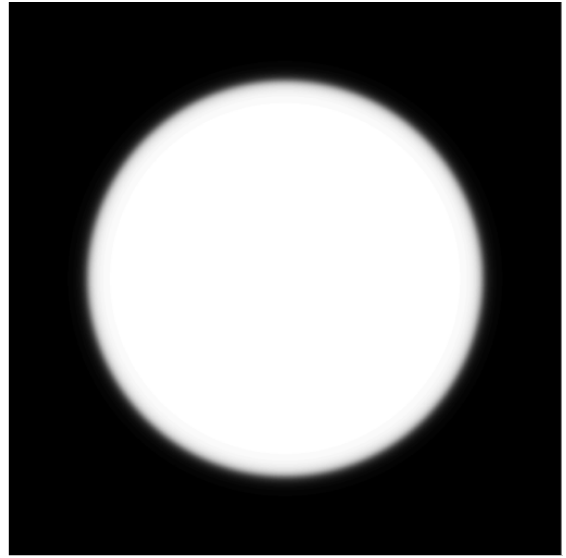
Masks are created using the equation,

$$taper = \frac{1}{2} + \frac{\sqrt{1 + (bS)^2} \cos\left(1.4 \frac{r^2}{2S}\right)}{2 \left(1 + (bS)^2 \cos^2\left(1.4 \frac{r^2}{2S}\right)\right)} \quad (70)$$

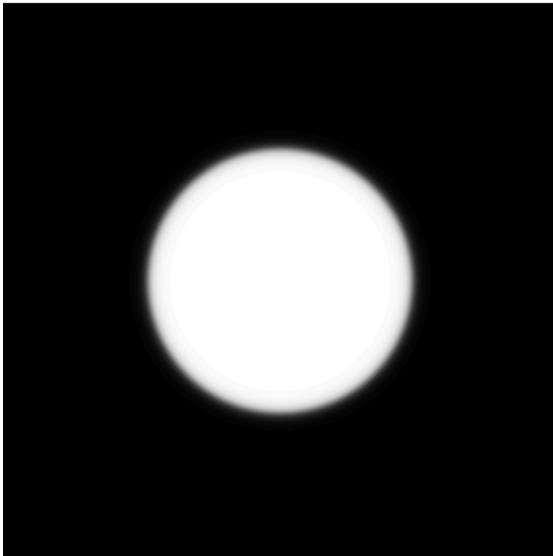
The tapering mask uses the polar array, $\mathbf{r}$, comprised of 1024 by 1024 pixels with a radius of $(\pi/2)$, the variable b controls the radius of the masking area, and S is the scale factor for the mask. When the scale factor is one, the diameter of the taper mask is equal to the length of the image. Likewise, a scale of one half gives a diameter equal to half the side length of the image plane. Figure 25 shows the different sizes of masks that are tested with the phase correction algorithm discussed in CHAPTER 5. Tapering can be seen at the edges of the masks.



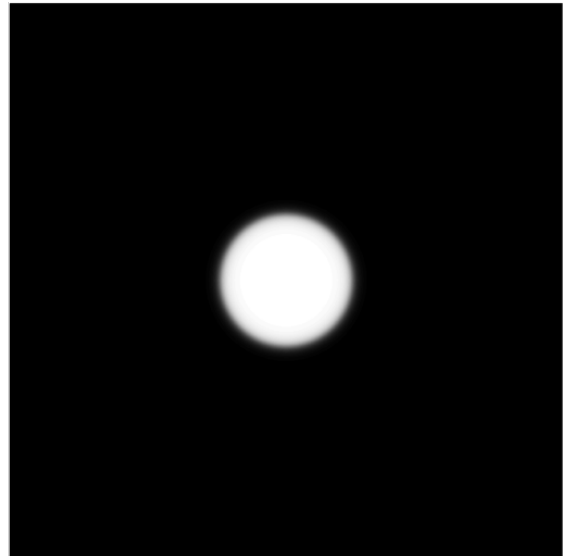
a.



b.



c.



d.

Figure 25: Taper masks with a diameter a. the length of the image, b. $\frac{3}{4}$ the length of the image, c. $\frac{1}{2}$ the length of the image, and d. $\frac{1}{4}$ the length of the image.

CHAPTER 5

PHASE CORRECTION ALGORITHMS

The piston, tip, and tilt simulations discussed in Section 4.2 were used as reference images while creating the phase correction algorithms for a six-aperture synthetic array. The atmospheric turbulence and noise added in Sections 4.3 and 4.4 were used to further test these correction algorithms. Isoplanatic simulations mentioned in Section 4.1 and image cropping included in Section 4.5 were used to further test the piston, tip, and tilt correction algorithms.

The basics of piston phase correction are discussed in detail in previous works [1], [13]. Sections 3.2 and 3.4 discuss the theory behind multi-aperture systems and what happens when the apertures are remapped between the entrance and exit pupils. The scaled auto-correlation of the entrance pupil gives the spatial frequency spectrum of the system, where the sum of the auto-correlations of individual apertures is the DC content and the distances of the cross-correlations from DC are dependent on the baseline between their two apertures. Any overlapping content between correlations in the entrance pupil contains equivalent spatial frequency information from the target, as shown previously in Figure 8.b. The two overlapping correlations also have additional, differing piston errors. When two correlations overlap, all of that spatial frequency information is mixed together and is very difficult to computationally separate.

However, physically remapping the apertures provides the opportunity to retain the matching target information from the overlaps while isolating the phase errors, as shown in Figure 9. After remapping, the correlations can be cropped into separate arrays and shifted so that they are in position to overlap again [29]. The distance each correlation is shifted is found by registering sets of two correlations with one another. In image registration, two images are correlated with one another and the distance from the center of the array to the correlation's maximum value gives the shift between the images [30]. For any given optical remapping setup, correlation shifts should remain the same between data collections so that image registration only has to be applied to each set of correlations once in order to record all of the necessary shifts. As long as the optical setup is not adjusted, these same shifts can be used for all phase corrections.

To isolate the phase errors in the overlapping regions of the correlations, the phase associated with the target's spatial frequency information can be removed by elementally multiplying one correlation by the complex conjugate of the other. The angle of the sum of the resulting complex field is the relative piston between the two correlations. An example of this multiplication between two correlations is shown in Figure 26 [1]. In this example, the relative piston error is corrected by multiplying the registered cross-correlation by the complex conjugate of the phase error so that only the angular information of the target remains.

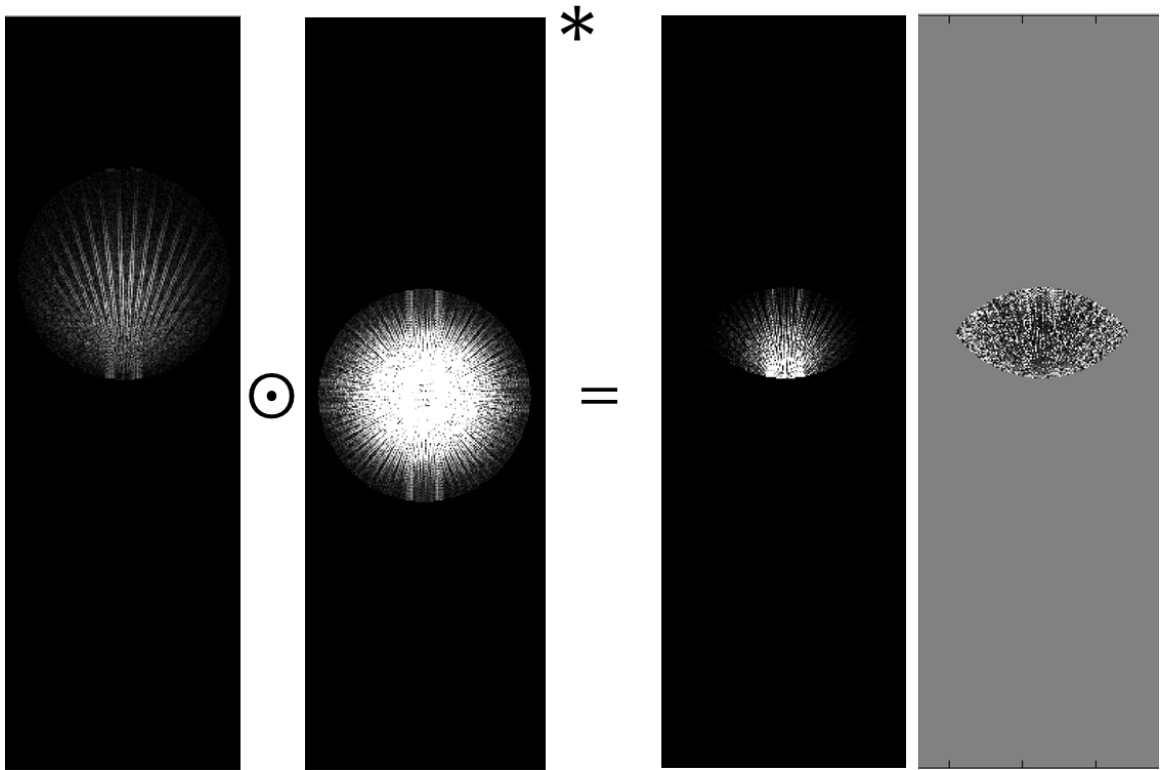


Figure 26: The element wise multiplication of a cross-correlation with the complex conjugate of the auto-correlation. The amplitude and angle of the complex product are shown to the right. The spatial frequency spectrum is from a star target. [1]

5.1 Piston Phase Corrections

The piston calculation method discussed above works very well for a two aperture system where there is only one relative piston difference between those two apertures. Multiplying one cross-correlation term with the DC content is enough to find the piston error needed for a correction. However, this technique becomes more complicated when there are more than two apertures. For a three aperture system where the apertures form a line, there are six cross-correlations and the piston errors between each of these cross-correlations are coupled. A six aperture system introduces even more variables. Multiplying one cross-correlation with the complex conjugate of another gives the relative

piston between four apertures and some of the cross-correlations do not even overlap, as shown in Figure 14.

In order to find the relative piston errors between all n apertures, a maximum-likelihood estimation (MLE) is used. It was demonstrated in 1993 that a correlation matrix can be created between the complex phases from a synthetic aperture radar system. The eigenvector associated with the highest eigenvalue of the matrix contains the phases of the synthetic apertures. [29]

A variation of this technique is used to find the relative piston errors in an optical, multi-aperture system. Figure 14.b and equation (25) are reproduced in this section for ease of reference:

$$\mathcal{G}(f_x, f_y) = \sum_{m=1}^N \sum_{n=1}^N [U_m(\xi, \eta) \otimes U_n^*(-\xi, -\eta)] e^{-j\varphi_{nm}} \quad (71)$$

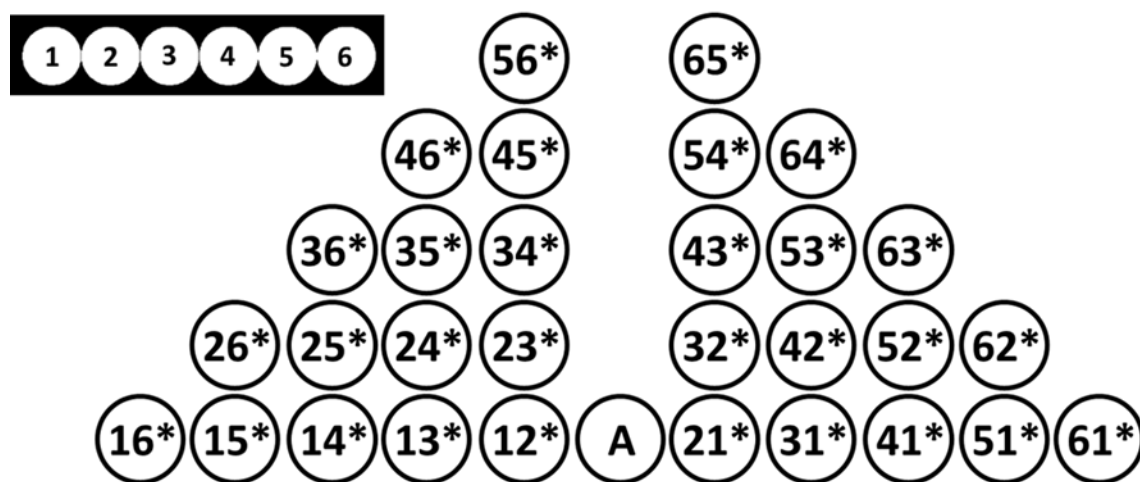


Figure 27: List of all correlations for six apertures. Columns show correlations that are fully overlapping and adjacent columns show partially overlapping correlations. Each correlation is labelled with the apertures used to create it.

Figure 27 shows all of the cross- and auto-correlations defined in equation (71) for a six aperture system, where equation (71) contains only piston errors. More generally, a linear

array with N apertures has N_{cc} cross-correlations and the same number of complex conjugates,

$$N_{cc} = \sum_{n=1}^{N-1} (n-1) \quad (72)$$

However, a six aperture array is used for all examples. The correlations in each column overlap with each other completely after registration while adjacent columns partially overlap. Each correlation is labelled with the apertures used to create it. For example, the scaled correlation between the first and second apertures is,

$$C_{12}(f_x, f_y)e^{-j\varphi_{12}} = U_2(\xi, \eta)e^{j\varphi_1} \otimes U_1^*(-\xi, -\eta)e^{-j\varphi_2} \Big|_{\substack{f_x = \xi/\lambda d_0 \\ f_y = \eta/\lambda d_0}} \quad (73)$$

where C_{12} is the correlation without the relative piston error between the two apertures. The auto-correlation in Figure 27 is the sum of the auto-correlations for all six apertures and the piston errors cancel out,

$$A(f_x, f_y) = \sum_{n=1}^N [U_n(\xi, \eta) \otimes U_n^*(-\xi, -\eta)] \quad (74)$$

The relative piston errors between all of these correlations are found before estimating the relative pistons for the apertures themselves. As shown in Figure 26, the piston error between two correlations is found by elementally multiplying one correlation by the complex conjugate of the other to cancel out the target frequencies, then taking the sum of the complex product. The angle of this complex sum reveals the relative piston phase error between the two correlations. As an example, the piston error between C_{12} and C_{23} is

$$\Gamma_{12,23}e^{-j(\varphi_1-2\varphi_2+\varphi_3)} = \sum_{f_x, f_y} [C_{12}(f_x, f_y)e^{-j\varphi_{12}} \odot C_{23}^*(f_x, f_y)e^{j\varphi_{23}}] \quad (75)$$

where $\odot$ represents the elemental multiplication and $\Gamma_{12,23}$ is the amplitude of the complex sum.

The spatial frequency spectrum of the six-aperture array in Figure 27 has the one auto-correlation term from equation (74), 15 cross-correlations and their complex conjugates, and 65 overlapping regions. Therefore, there are 65 known relative pistons between the correlations, such as the one shown in equation (75), and MLE can be used to estimate first, the 15 unknown cross-correlation pistons and second, the six unknown aperture piston errors. A 16 x 16 matrix containing these 65 complex sums is shown in equation (76).

$$\begin{bmatrix} |\Gamma_{A,A}|^2 & \Gamma_{A,12}e^{j\varphi_{12}} & \dots & \Gamma_{A,64}e^{j\varphi_{64}} & \Gamma_{A,65}e^{j\varphi_{65}} \\ \Gamma_{12,A}e^{-j\varphi_{12}} & |\Gamma_{12,12}|^2 & & \Gamma_{12,64}e^{j(\varphi_{64}-\varphi_{12})} & \Gamma_{12,65}e^{j(\varphi_{65}-\varphi_{12})} \\ \vdots & & \ddots & & \vdots \\ \Gamma_{46,A}e^{-j\varphi_{46}} & \Gamma_{46,12}e^{j(\varphi_{12}-\varphi_{46})} & & |\Gamma_{46,46}|^2 & \Gamma_{46,65}e^{j(\varphi_{65}-\varphi_{46})} \\ \Gamma_{56,A}e^{-j\varphi_{56}} & \Gamma_{56,12}e^{j(\varphi_{12}-\varphi_{56})} & \dots & \Gamma_{56,64}e^{j(\varphi_{64}-\varphi_{56})} & |\Gamma_{56,56}|^2 \end{bmatrix} \quad (76)$$

Any terms referencing cross-correlations that do not overlap are set to zero. In more general terms, an array of N apertures has a size of $(N_{cc} + 1) \times (N_{cc} + 1)$ with N_p known relative piston errors,

$$N_p = \sum_{n=1}^{N-2} \left[\sum_{m=1}^n (m) + n(n-1) \right] \quad (77)$$

The eigenvector of equation (76) corresponding to the highest eigenvalue gives a complex vector with a length of 16. The angle of each of these components is the MLE of the piston error for each cross-correlation. However, the first term represents the auto-correlation, which should not have a piston error. Multiply equation (78) by the complex conjugate of the first term's phase in order to give the piston estimations for the cross-correlations relative to the sum of the auto-correlations,

$$\begin{bmatrix} \gamma_A \\ \gamma_{12}e^{-j\hat{\varphi}_{12}} \\ \vdots \\ \gamma_{46}e^{-j\hat{\varphi}_{46}} \\ \gamma_{56}e^{-j\hat{\varphi}_{56}} \end{bmatrix} \quad (78)$$

The constants, γ_{nm} , are the amplitudes for each complex correlation term in the eigenvector. These complex terms are used to find the relative piston errors between the apertures themselves. The values from equation (78) are input into a 6×6 complex matrix since there are 6 apertures. The diagonals are zero since equation (78) does not include the auto-correlations between individual apertures.

$$\begin{bmatrix} 0 & \gamma_{12}^*e^{j\hat{\varphi}_{12}} & \gamma_{13}^*e^{j\hat{\varphi}} & \gamma_{14}^*e^{j\hat{\varphi}_{14}} & \gamma_{15}^*e^{j\hat{\varphi}_{15}} & \gamma_{16}^*e^{j\hat{\varphi}_{16}} \\ \gamma_{12}e^{-j\hat{\varphi}_{12}} & 0 & \gamma_{23}^*e^{j\hat{\varphi}_{23}} & \gamma_{24}^*e^{j\hat{\varphi}_{24}} & \gamma_{25}^*e^{j\hat{\varphi}_{25}} & \gamma_{26}^*e^{j\hat{\varphi}_{26}} \\ \gamma_{13}e^{-j\hat{\varphi}_{13}} & \gamma_{23}e^{-j\hat{\varphi}_{23}} & 0 & \gamma_{34}^*e^{j\hat{\varphi}_{34}} & \gamma_{35}^*e^{j\hat{\varphi}_{35}} & \gamma_{36}^*e^{j\hat{\varphi}_{36}} \\ \gamma_{14}e^{-j\hat{\varphi}_{14}} & \gamma_{24}e^{-j\hat{\varphi}_{24}} & \gamma_{34}e^{-j\hat{\varphi}_{34}} & 0 & \gamma_{45}^*e^{j\hat{\varphi}_{45}} & \gamma_{46}^*e^{j\hat{\varphi}_{46}} \\ \gamma_{15}e^{-j\hat{\varphi}} & \gamma_{25}e^{-j\hat{\varphi}_{25}} & \gamma_{35}e^{-j\hat{\varphi}_{35}} & \gamma_{45}e^{-j\hat{\varphi}_{45}} & 0 & \gamma_{56}^*e^{j\hat{\varphi}} \\ \gamma_{16}e^{-j\hat{\varphi}_{16}} & \gamma_{26}e^{-j\hat{\varphi}_{26}} & \gamma_{36}e^{-j\hat{\varphi}_{36}} & \gamma_{46}e^{-j\hat{\varphi}_{46}} & \gamma_{56}e^{-j\hat{\varphi}_{56}} & 0 \end{bmatrix} \quad (79)$$

The eigenvalues and eigenvectors are found again for this matrix. The eigenvector corresponding to the highest eigenvalue is,

$$\begin{bmatrix} \Lambda_1e^{-j\hat{\varphi}_1} \\ \Lambda_2e^{-j\hat{\varphi}_2} \\ \Lambda_3e^{-j\hat{\varphi}_3} \\ \Lambda_4e^{-j\hat{\varphi}_4} \\ \Lambda_5e^{-j\hat{\varphi}_5} \\ \Lambda_6e^{-j\hat{\varphi}_6} \end{bmatrix} \quad (80)$$

The constants, Λ_n , are the amplitudes for the complex values associated with each aperture and the angles of the complex vector in equation (80) are their estimated pistons,

$$\begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \\ \hat{\varphi}_3 \\ \hat{\varphi}_4 \\ \hat{\varphi}_5 \\ \hat{\varphi}_6 \end{bmatrix} \quad (81)$$

In an ideal situation, these estimated pistons should match the true values. Use these values to estimate the piston difference of each cross-correlation and correct the piston errors by multiplying each correlation by the complex conjugate of its estimated piston difference. Ideally, $\hat{\varphi}_1 - \hat{\varphi}_2 = \varphi_{12}$ so that,

$$C_{12}(f_x, f_y) = C_{12}(f_x, f_y) e^{-j\varphi_{12}} e^{j(\hat{\varphi}_1 - \hat{\varphi}_2)} \quad (82)$$

Taking the FT of the sum of all of the corrected correlations results in a piston corrected image. Tests of this piston correction method are shown in CHAPTER 6, Sections 6.1 and 6.4.

5.2 Tip and Tilt Phase Corrections

Section 5.1 discusses the MLE method for calculating the piston errors in a multi-aperture system. This section expands to include tip and tilt phase corrections using a least squares (LS) approach. In this discussion, tip runs parallel to the aperture gain while tilt runs perpendicular to the direction of the gain. Tip and tilt are estimated and corrected in parallel with one another, but this explanation is simplified by only referring to tilt. APPENDIX B shows that the correlation of two apertures with tilts has the average tilt between those two apertures. This means that one term from equation (32) has the mean tilt of the two apertures from which it is comprised as well as their piston difference,

$$\begin{aligned} & [U_m(\xi, \eta) e^{j(\varphi_m + \varphi_{my}\eta)} \otimes U_n^*(-\xi, -\eta) e^{-j(\varphi_n + \varphi_{ny}\eta)}] \\ & = C_{mn}(\xi, \eta; \varphi_{my}, \varphi_{ny}) e^{-j\left(\varphi_{nm} - \frac{\eta(\varphi_{my} + \varphi_{ny})}{2}\right)} \end{aligned} \quad (83)$$

where $\mathcal{C}_{mn}$ is a complex field. The tilt term from equation (83) and its FT is shown in equation (84).

$$\mathcal{F} \left\{ e^{-j \left(\frac{\eta(\varphi_{my} + \varphi_{ny})}{2} \right)} \right\} = \delta \left(f_y - \frac{\varphi_{my} + \varphi_{ny}}{2(2\pi)} \right) \quad (84)$$

Since the FT of a complex exponent with a tilt is a shifted Dirac delta function, the FT of a correlation is a shifted version of the image. This means that the FTs of two correlations with different tilts have different shifts. This relationship is used to calculate the relative tilts between all of the apertures.

Running an image registration algorithm on the FTs of two cross-correlations gives the relative shift between them. For example, registering the cross-correlations with $\mathcal{C}_{12}$ and $\mathcal{C}_{23}$ gives the shift for the tilt,

$$\hat{\psi}_1 = \frac{\varphi_{1y} + \varphi_{2y} - \varphi_{2y} - \varphi_{3y}}{2(2\pi)} = \frac{\varphi_{1y} - \varphi_{3y}}{2(2\pi)} \quad (85)$$

and a similar shift for the tip, where φ_{2y} cancels and $\hat{\psi}_1$ is a random variable.

The random variable, $\hat{\psi}_1$, represents the relative shift between two cross-correlations calculated by image registration. The auto-correlation is not used in this registration process since it is the summation of the auto-correlations of all six apertures. Each of these auto-correlations has the same tilt as its aperture and the sum of more than two complex fields with tilts results in a tilt that is non-linear. This is further discussed in APPENDIX C. Without the auto-correlation, the number of known cross-correlation shifts that can be used to find N aperture tilts is,

$$N_t = N_p - (N - 1) \quad (86)$$

For six apertures, $N_t = 60$. The LS method is applied as follows. First, the tilts from the overlapping cross-correlations, $\hat{\Phi}_y$, are calculated from the pixel shifts such as the one shown in equation (85).

$$\hat{\Phi}_{y1} = 2(2\pi)\hat{\varphi}_1 = \varphi_{1y} - \varphi_{3y} \quad (87)$$

Second, the sum of the squared residuals of the tilts is written as,

$$R^2 = \sum_{n=1}^{N_t} (\Phi_{yn} - \hat{\Phi}_{yn})^2 \quad (88)$$

Since each relative tilt between correlations, $\hat{\Phi}_{yn}$, is a function of some subset of the six individual aperture tilts, the derivative of R^2 can be taken with regard to φ_{1y} through φ_{6y} . To minimize R^2 , these six derivatives are set to zero and the resulting equations can be written in the form of the LS matrix equation,

$$\mathbf{A}^T \vec{\Phi}_y = (\mathbf{A}^T \mathbf{A}) \vec{\varphi}_y \quad (89)$$

Interestingly, $\mathbf{A}$ is an $N \times N_t$ matrix that can be created by listing which of the six apertures are used to create each of the 60 random variable tilts. For instance, $\hat{\Phi}_1$ is the difference between φ_{1y} and φ_{3y} , so the first value of column one is 1 and the third value is -1,

$$\mathbf{A} = \begin{pmatrix} 1 & & & & & \\ 0 & & & & & \\ -1 & & & & & \\ 0 & & \dots & & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix} \quad (90)$$

The 60 tilts from the overlapping correlations make up the vector, $\vec{\Phi}$, and the 6 estimated aperture tilts are given by the vector, $\vec{\varphi}_y$.

$$\vec{\varphi}_y = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \vec{\Phi}_y \quad (91)$$

To solve for $\vec{\varphi}_y$, find the pseudo-inverse of $\mathbf{A}$ using singular value decomposition [31],

$$(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T = \mathbf{A}^+ \quad (92)$$

and solve for the vector of the tilt estimations [31],

$$\overrightarrow{\varphi}_y = \mathbf{A}^+ \overrightarrow{\Phi}_y \quad (93)$$

The tips are estimated in parallel using the same method.

Once both the tip and tilt estimates of each aperture have been found, they are used to correct the cross-correlations. As discussed in APPENDIX B, the tilt of each cross-correlation is the mean of two apertures. Multiplying a cross-correlation by a field containing the complex conjugate of the estimate of its tilt should correct the phase error so that all correlations have the same relative tilt.

$$\mathcal{C}_{mn}(\eta) = \mathcal{C}_{mn}(\eta) e^{\frac{j\eta(\varphi_{my} + \varphi_{ny})}{2}} e^{-\frac{j\eta(\hat{\varphi}_{my} + \hat{\varphi}_{ny})}{2}} \quad (94)$$

Again, estimating and correcting the tip is done using the same method.

The auto-correlation may also be able to be phase corrected indirectly by taking all of the calculated tips and tilts, generating a simulated set of apertures with those phase errors, and then creating an auto-correlation from those apertures. Multiplying the phase error auto-correlation with the original auto-correlation could lessen or, ideally, cancel the non-linear tip and tilt effects described in APPENDIX C. Once the auto-correlation is corrected the tip and tilt algorithms could be run again, this time using the overlaps between the auto-correlation and cross-correlations, as well as just the overlaps between cross-correlations, which was discussed above.

5.3 Phase Correction Algorithm

Tip and tilt corrections are unaffected by piston phase errors. However, piston corrections assume that apertures are free of tip and tilt and is adversely affected by residual

tip and tilt errors. Therefore, tip and tilt should be corrected first while piston corrections should be corrected second. A summary of all the steps needed to correct the piston, tip, and tilts of an imaging system is given here and a flowchart is provided in Figure 28.

After an image is collected using a pupil remapping system, as described in Figure 11, its FT is taken in order to reveal the separated spatial frequency spectrum. Each of the correlations in the spectrum are masked into separate arrays before being shifted back to their original locations from before remapping. These shifted correlations are used to estimate and correct the tip and tilt phase errors.

To estimate the tip and tilt, the FTs of all the cross-correlations are found. Image registration is performed on the FTs of all of the overlapping cross-correlations in order to find their shifts. An example of this is shown in equation (85). These shifts are converted into the tip and tilt phase errors for each of the cross-correlations, like in equation (87), and used to populate the matrix, $\mathbf{A}$, described in equation (90), and find its pseudoinverse. The tip and tilt phase errors for the cross-correlations and the pseudoinverse of $\mathbf{A}$ are used to find the LS solution, as shown in equations (91) and (93). This gives the estimated tips and tilts for all of the apertures. The tips and tilts in the perturbed cross-correlations are corrected using equation (94).

The tip and tilt corrected cross-correlations are now ready for piston correction. The piston estimates are calculated by multiplying every correlation with the complex conjugates of its overlapping correlations. The complex sum of each of these products is taken and all of the complex sums are placed in a matrix such as the one shown in equation (76). The eigenvector of the matrix with the highest eigenvalue is found and multiplied by the complex conjugate of its first term, resulting in piston estimates for each of the cross-

correlations without having an arbitrary piston in the auto-correlation, as shown in equation (78). The complex estimates are placed into another matrix like the one in equation (79). The eigenvector with the highest value for this second matrix gives the piston estimates for all of the apertures. An example of this is given in equations (80) and (81). The piston errors in the cross-correlations are corrected using equation (82).

After the piston differences in the correlations have been corrected, all of the correlations are summed together and the inverse FT is taken in order to get a piston, tip, and tilt corrected image. Results from this phase correction algorithm are discussed in CHAPTER 6.

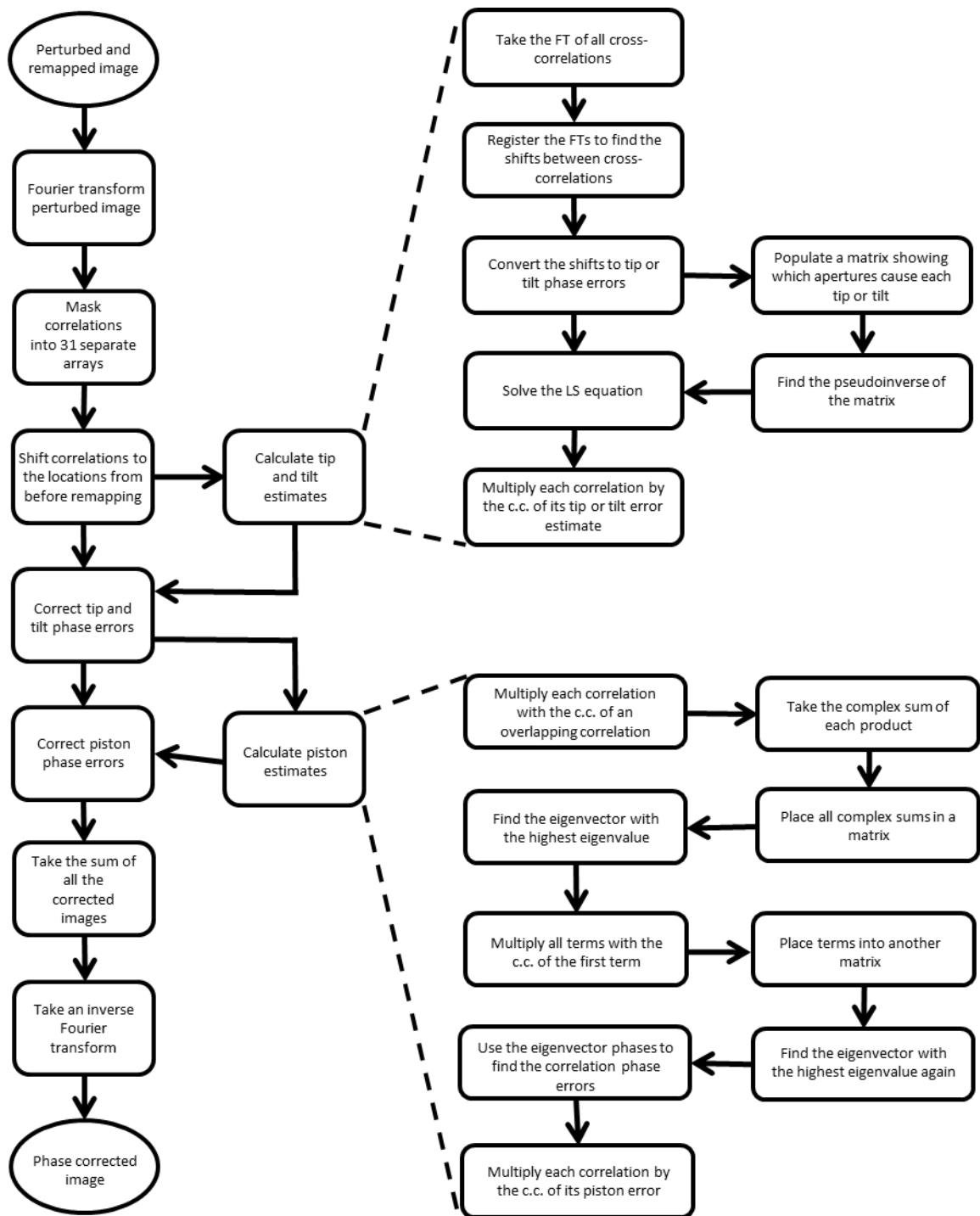


Figure 28: Flow chart showing the order of operations for the full phase correction algorithm.

CHAPTER 6

PHASE CORRECTION ALGORITHM RESULTS

The aperture simulations described in CHAPTER 4 are used to test the effectiveness of the piston, tip, and tilt correction algorithms laid out in CHAPTER 5. Simulations are used rather than experimental images in order to better control the atmospheric parameters for analysis as well as due to the complexity of hardware required for a multi-aperture design. However, these phase correction algorithms should also be effective for physical data collected under specific atmospheric conditions defined by the results in this chapter using a setup reminiscent of Figure 11.

Section 6.1 shows examples of images with known piston perturbations and the corresponding corrections while Section 6.2 does the same for images with known tip and tilt perturbations. Section 6.3 goes over the correction metric used to quantitatively measure how well the phase correction algorithm works. Sections 6.4 and 6.5 go on to analyze how well the correction algorithm works with atmospheric perturbations. Section 6.4 looks at the effects of different Fried parameter and SNR values on the phase correction algorithm. Examples of corrections using piston corrections only are shown first and then tip and tilt corrections are included in order to show how both algorithms contribute to atmospheric phase corrections. Section 6.5 demonstrates how varying degrees of

anisoplanatism affect phase corrections and Section 6.6 measures how much support is needed to phase correct one isoplanatic patch at a time.

6.1 Piston Corrections

Before presenting quantitative results for the piston correction algorithm some qualitative examples of corrected images are shown. Section 4.2 demonstrated that a different piston phase error can be added to six simulated apertures in order to generate a perturbed point spread function. An example of this is shown in Figure 29. The phase of the piston perturbed aperture array is shown in Figure 29.a, the perturbed PSF is shown in b, and c and d contain the magnitude and phase of the spatial frequency spectrum.

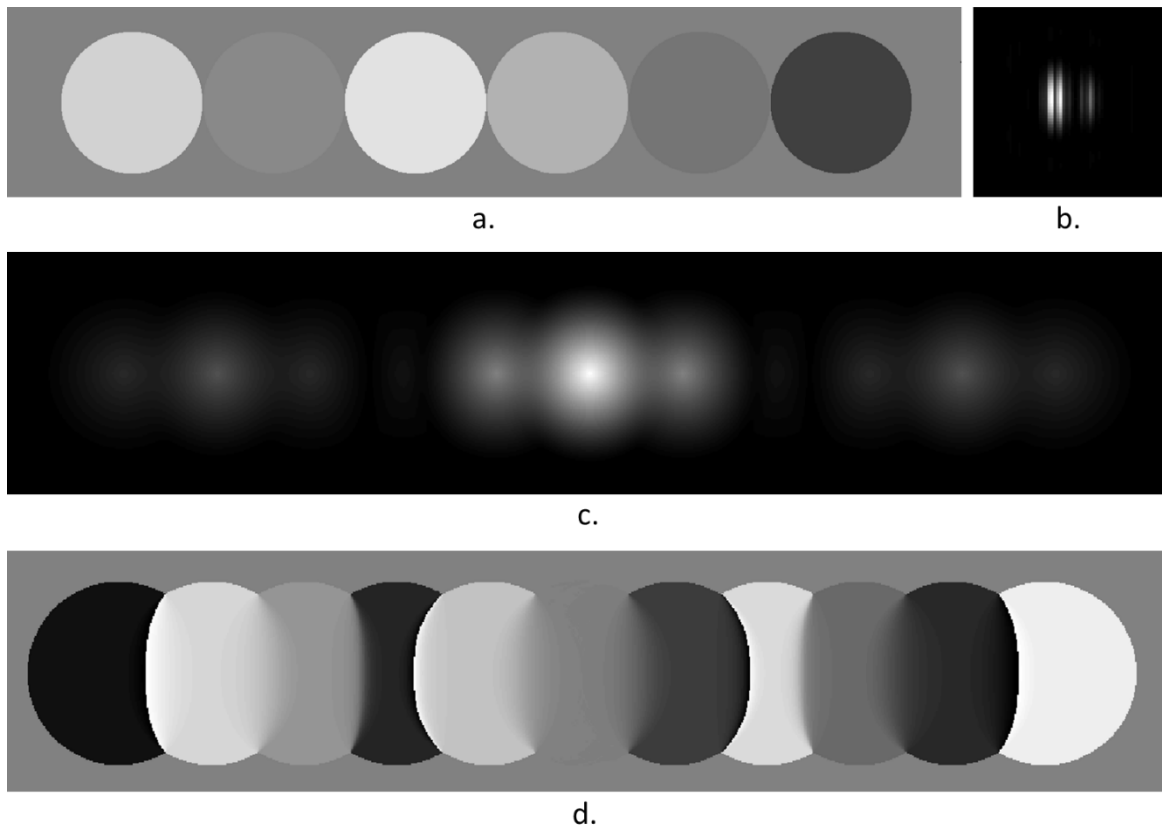
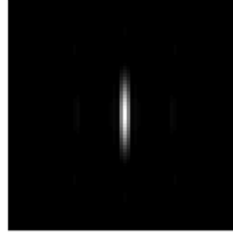
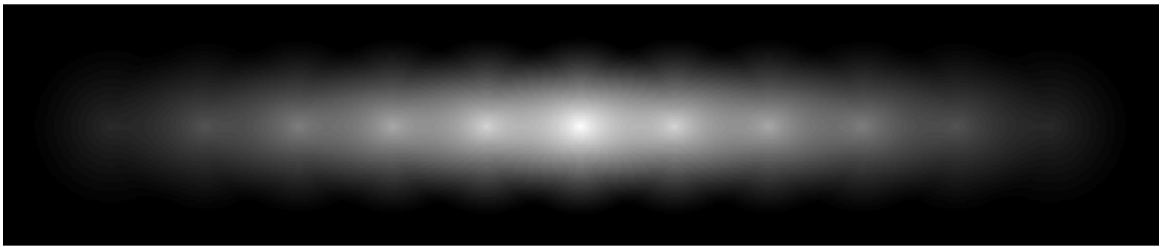


Figure 29: a. Angle of 6 piston perturbed apertures b. Piston perturbed PSF c. Squared modulus of the PSF's spatial frequency spectrum d. Angle of the PSF's spatial frequency spectrum

Applying the piston correction algorithm directly to the PSF above results in the corrected PSF and spatial frequency spectrum shown in Figure 30.



a.



b.



c.

Figure 30: a. Piston corrected PSF b. Squared modulus of the PSF's spatial frequency spectrum c. The angles of the PSF's spatial frequency spectrum are all corrected to zero.

Applying this perturbed PSF to a 1951 USAF resolution target and a dead leaves target and correcting them results in the images shown in Figure 31.a through d.

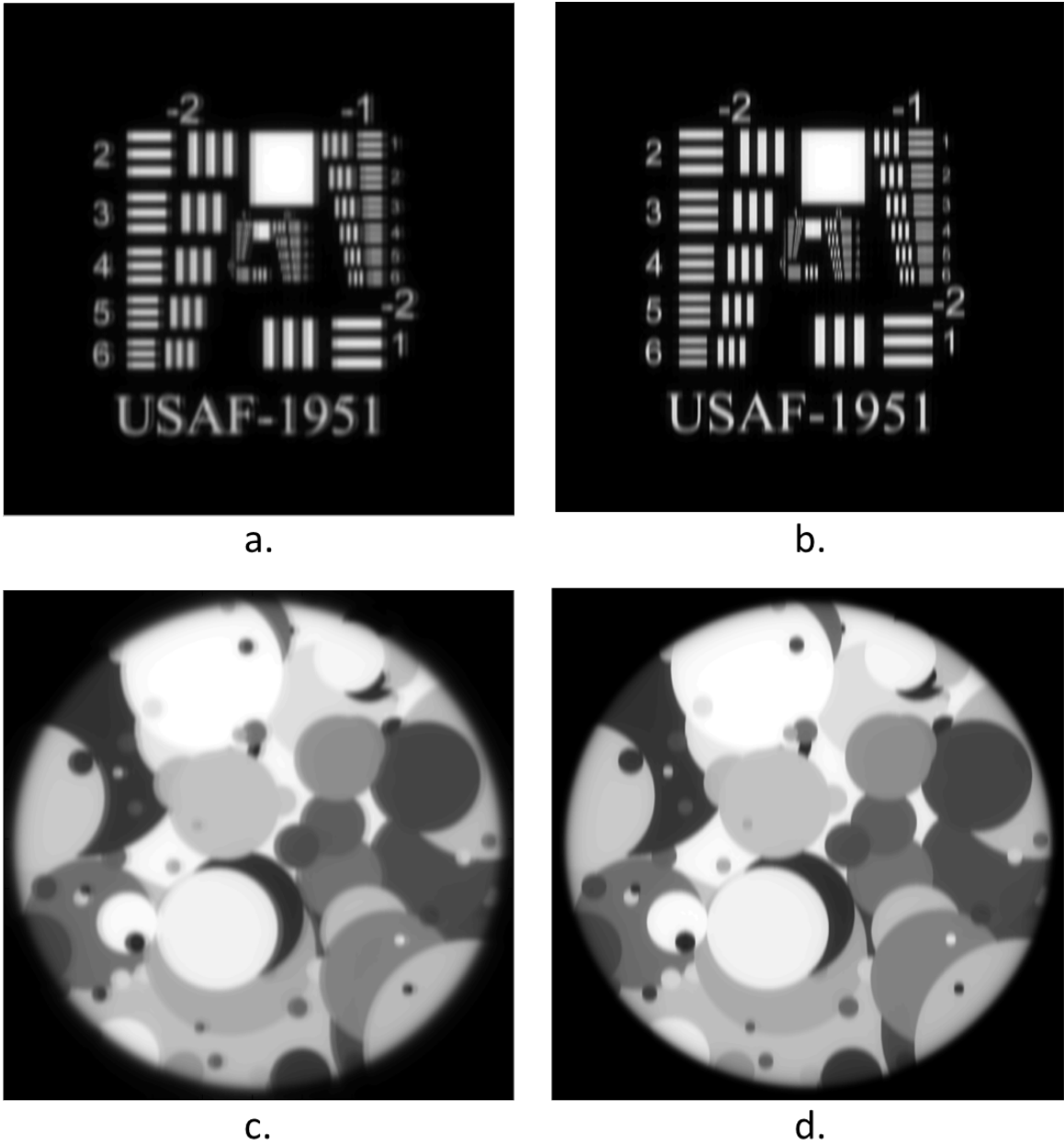


Figure 31: a. Piston perturbed 1951 USAF target b. Corrected 1951 USAF target c. Piston perturbed dead leaves target d. Corrected dead leaves target.

The piston causes a ghosting effect in the uncorrected images that is fixed in the corrected images. The original pistons added to each aperture and the pistons estimated by the correction algorithm for the PSF and both extended targets are shown in Table 3.

Table 3: Comparison of original and corrected pistons of a piston perturbed PSF in radians

Original Pistons:	2.0523	0.2860	2.4417	1.1857	-0.2622	-1.5697
--------------------------	--------	--------	--------	--------	---------	---------

Estimated Pistons: (PSF)	0.0000	-1.7663	0.3895	-0.8665	-2.3145	2.6612
Estimated Pistons: (USAF)	0.0000	-1.7663	0.3894	-0.8667	-2.3146	2.6610
Estimated Pistons: (Dead Leaves)	0.0000	-1.7720	0.3792	-0.8809	-2.3333	2.6366

A global piston can be seen in these results. However, the final correction that is applied to each correlation is the difference between the piston estimates of two apertures, so it is not effected by any global offset. These pistons are calculated with noiseless images so they are very similar to the original values. The effects of increasing image noise are shown in Section 6.4.

6.2 Tip and Tilt Corrections

In addition to piston, Section 4.2 shows that simulated tip and tilt can be added to individual apertures. An example of this, along with the resulting PSF and spatial frequency spectrum is shown in Figure 32.

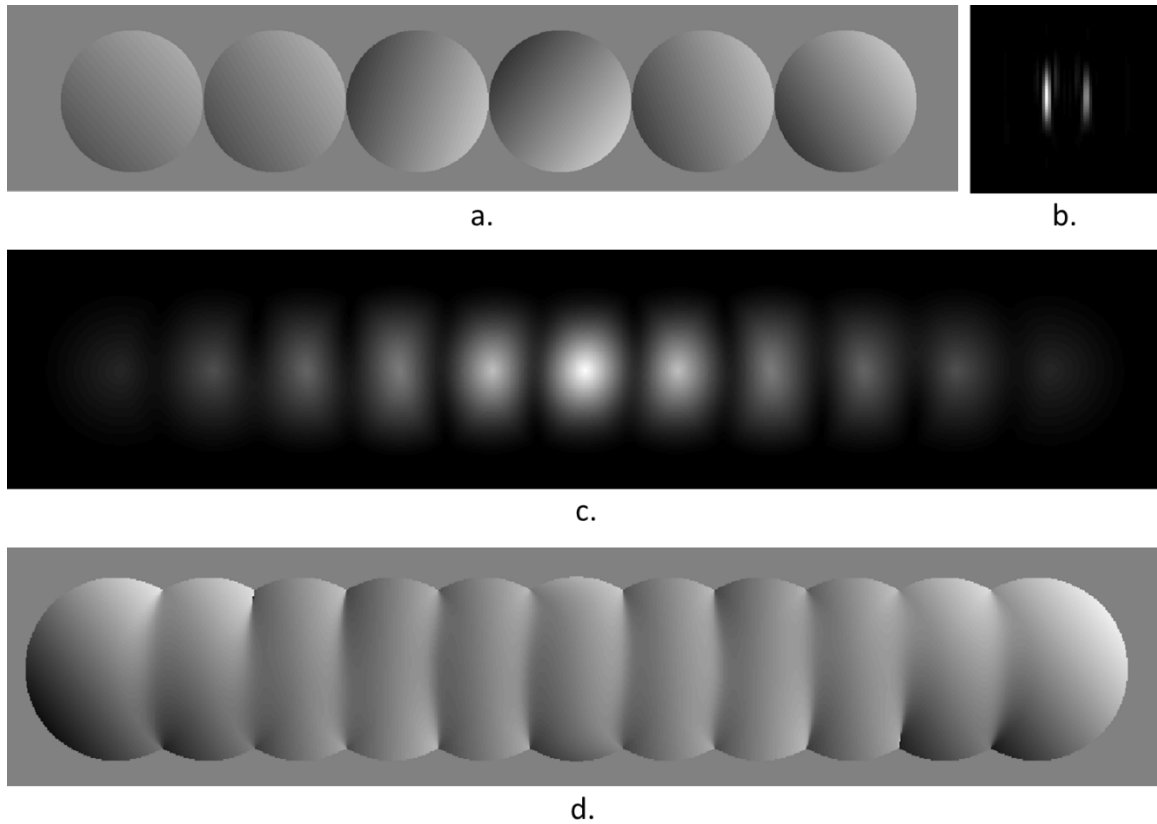


Figure 32: a. Angle of 6 tip and tilt perturbed apertures b. Tip and tilt perturbed PSF c. Squared modulus of the PSF's spatial frequency spectrum d. Angle of the PSF's spatial frequency spectrum

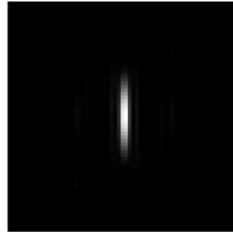
Applying the tip and tilt estimation algorithm directly to the PSF above results in the tip and tilt estimates provided in Table 4.

Table 4: Original and estimated tips and tilts from the PSF given in Figure 32 in radians per aperture diameter

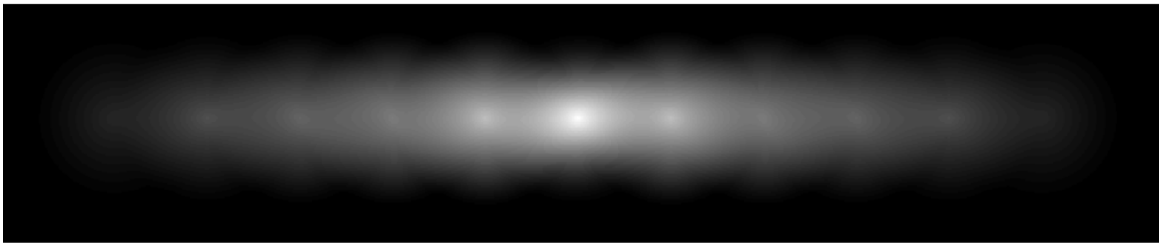
Original Tips:	1.5653	1.9902	3.0812	2.9034	2.7825	2.8928
Estimated Tips:	-0.9706	-0.5457	0.5453	0.3675	0.2466	0.3569
Original Tilts:	-1.3956	-1.1855	0.9356	2.0850	-1.0720	-1.8564
Estimated Tilts:	-0.9807	-0.7707	1.3504	2.4998	-0.6572	-1.4416

These estimates reveal a global tip and tilt that the algorithm is not able to estimate. All of the estimated tips are 2.5359 radians lower than the original tips and the estimated tilts are

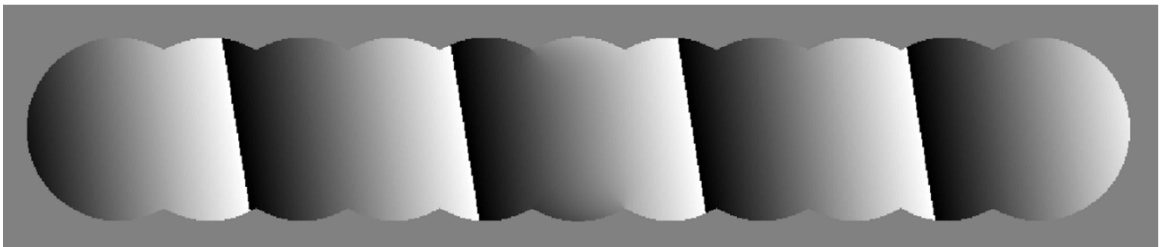
0.4148 radians higher than the original tilts. Correcting the tip and tilt in the correlations using equation (94) and the estimated tips and tilts should result in all of the correlations having matching global tip and tilt values. However, equation (94) does not account for the piston offset created from the global tip. Instead, the phase at the center of each correlation remains at 0 radians. Applying the piston correction algorithm after tip and tilt corrections adds the piston offset needed to create a single smooth global tip and tilt across the full spatial frequency spectrum. Section 5.3 explained that piston corrections should follow tip and tilt corrections because it is difficult to estimate piston while there are tip and tilt in the individual apertures. However, the current example actually shows that piston corrections should always follow the tip and tilt corrections in order to account for the piston offset caused by the global tip value. After applying the piston correction algorithm to the PSF as well as tip and tilt corrections, the corrected PSF and spatial frequency spectrum are shown in Figure 33.



a.



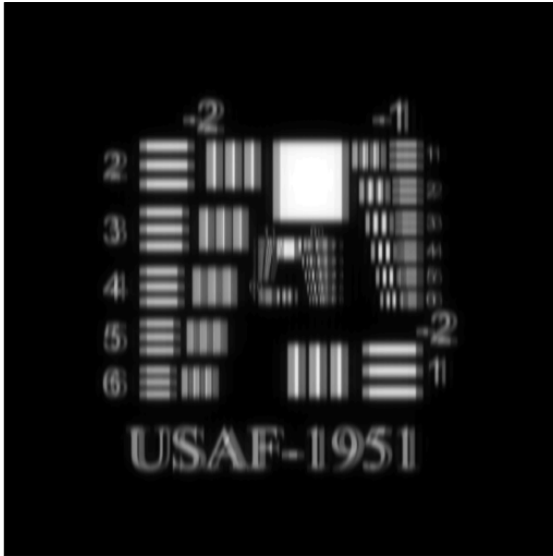
b.



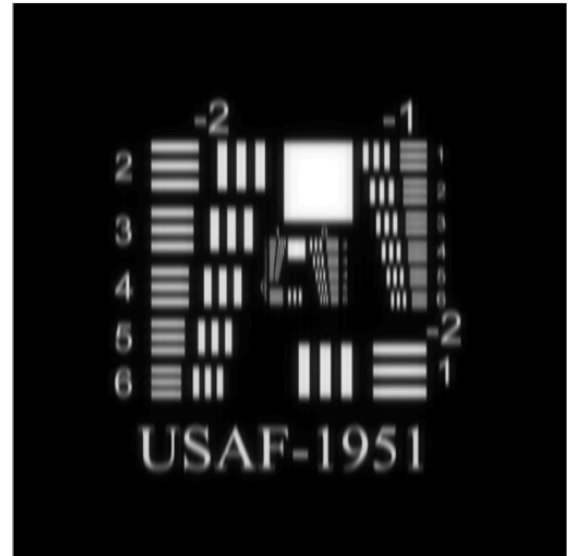
c.

Figure 33: a. Tip and tilt corrected PSF b. Squared modulus of the PSF's spatial frequency spectrum c. The angles of the PSF's spatial frequency spectrum are all corrected to zero.

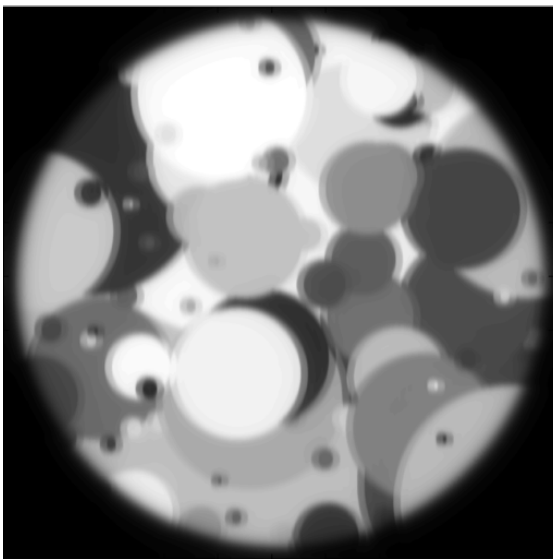
Convolving the perturbed PSF with the 1951 USAF and dead leaves targets, then correcting them, results in the images shown in Figure 34 a through d.



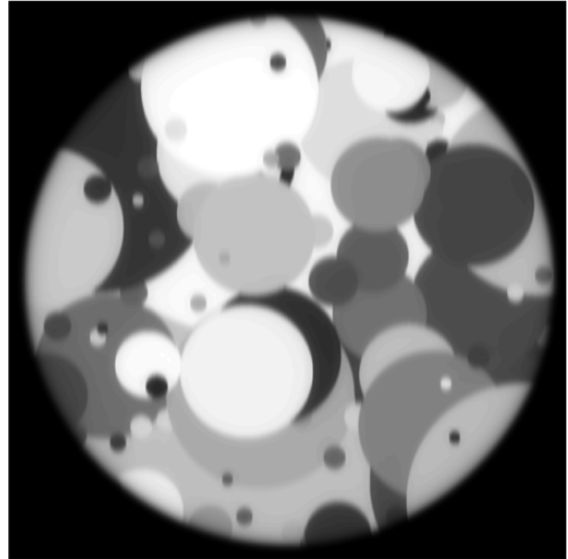
a.



b.



c.



d.

Figure 34: a. Tip and tilt perturbed 1951 USAF target b. Corrected 1951 USAF target c. Tip and tilt perturbed dead leaves target d. Corrected dead leaves target.

Similar to piston, tip and tilt phase errors cause a ghosting effect in the uncorrected image that is fixed in the corrected image. The original tilts added to each aperture and estimated by the correction algorithm for the PSF and both extended targets are shown in Table 5.

Table 5: Comparison of original and corrected tilts of a tip and tilt perturbed PSF in radians per aperture diameter

Original Tilts:	-1.3956	-1.1855	0.9356	2.0850	-1.0720	-1.8564
Estimated Tilts: (PSF)	-0.9807	-0.7707	1.3504	2.4998	-0.6572	-1.4416
Estimated Tilts: (USAF)	-0.9807	-0.7707	1.3504	2.4998	-0.6572	-1.4415
Estimated Tilts: (Dead Leaves)	-0.9815	-0.7673	1.3484	2.4956	-0.6538	-1.4413

And the results for tip correction are given in Table 6.

Table 6: Comparison of original and corrected tips of a tip and tilt perturbed PSF in radians per aperture diameter

Original Tips:	1.5653	1.9902	3.0812	2.9034	2.7825	2.8928
Estimated Tips: (PSF)	-0.9706	-0.5457	0.5453	0.3675	0.2466	0.3569
Estimated Tips: (USAF)	-0.9706	-0.5457	0.5453	0.3675	0.2466	0.3569
Estimated Tips: (Dead Leaves)	-0.9704	-0.5437	0.5453	0.3679	0.2469	0.3539

These tips and tilts are calculated with noiseless images so they are very similar to the original values. The effects of increasing image noise are shown in Section 6.4.

6.3 Correction Metric

Sections 6.1 and 6.2 show some qualitative examples of piston, tip, and tilt phase corrections. However, the original piston, tip, and tilt will not always be available for comparison with the corresponding estimated values. For example, to quantitatively measure the effectiveness of the phase correction algorithm for atmospheric turbulence, a standardized metric is used to compare corrections made on a variety of extended targets. In Section 4.1, remember that perturbations are added to extended images by multiplying the correlations in the spatial frequency profile of the extended target with the correlations of the perturbed OTF. Since the simulated extended targets are perturbed using the OTFs of point targets, a comparison of the Strehl ratios before and after phase correction is an

effective metric for comparing phase corrections between different extended targets. This is done by calculating the piston, tip, and tilt estimates for the extended target, and then applying these same estimates to correct the OTF as well as the extended image. Taking the FTs of the uncorrected and corrected OTFs gives the perturbed and corrected PSFs. Strehl ratios are calculated for these PSFs by finding their maximum values and dividing them by the maximum value of an ideal, non-aberrated PSF. An example of an ideal, a perturbed, and a corrected PSF are shown in Figure 35. In this example, the Strehl ratio before correction is 0.35 and 0.57 after correction. Ideally, the Strehl ratios for corrected PSFs should reach the Marechal criterion for diffraction limited resolution, which is 0.82 or greater [32]. However, attention is also given to noticeable resolution improvements even if the images do not always reach diffraction limited resolutions.

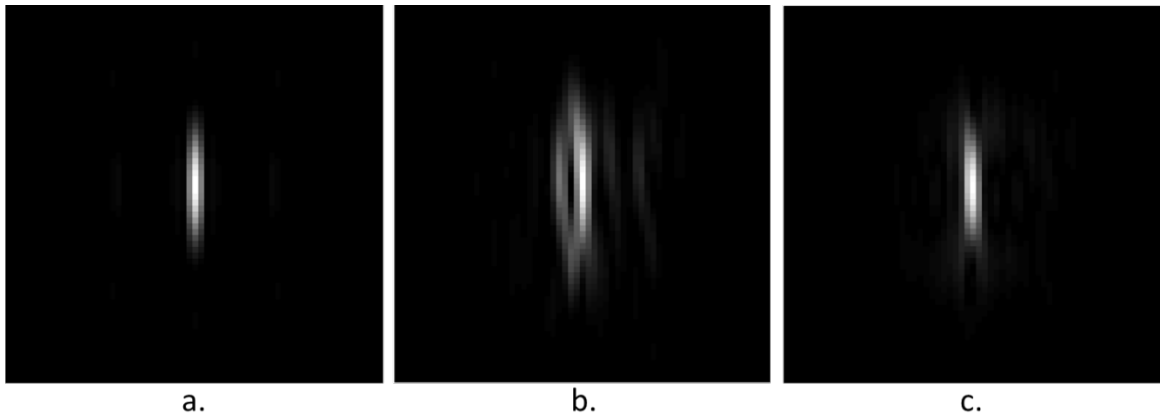


Figure 35: a. Ideal PSF with no phase errors. b. PSF perturbed with an atmosphere with a Fried of 2.54 cm and an infinite isoplanatic angle. c. Corrected PSF

6.4 Atmospheric Phase Corrections

Sections 6.1 and 6.2 demonstrate that phase corrections work in ideal cases with no noise in which each aperture has a specifically defined piston, tip, and tilt. Atmospheric turbulence, however, can contain higher order phase errors in addition to piston, tip, and tilt that are influenced by the Fried diameter. The phase correction algorithm does not fix

these higher order phase aberrations and the final image is only partially corrected. Piston correction alone may correct some phase errors caused by atmospheric turbulence, but a combination of the tip, tilt, and piston correction algorithms leads to a stronger image correction. To demonstrate this, some examples of piston only atmospheric corrections are shown first followed by corrections correcting both piston and tip and tilt phase errors.

An example of a PSF perturbed by a single atmospheric phase screen with a Fried diameter the size of an aperture is shown in Figure 36.

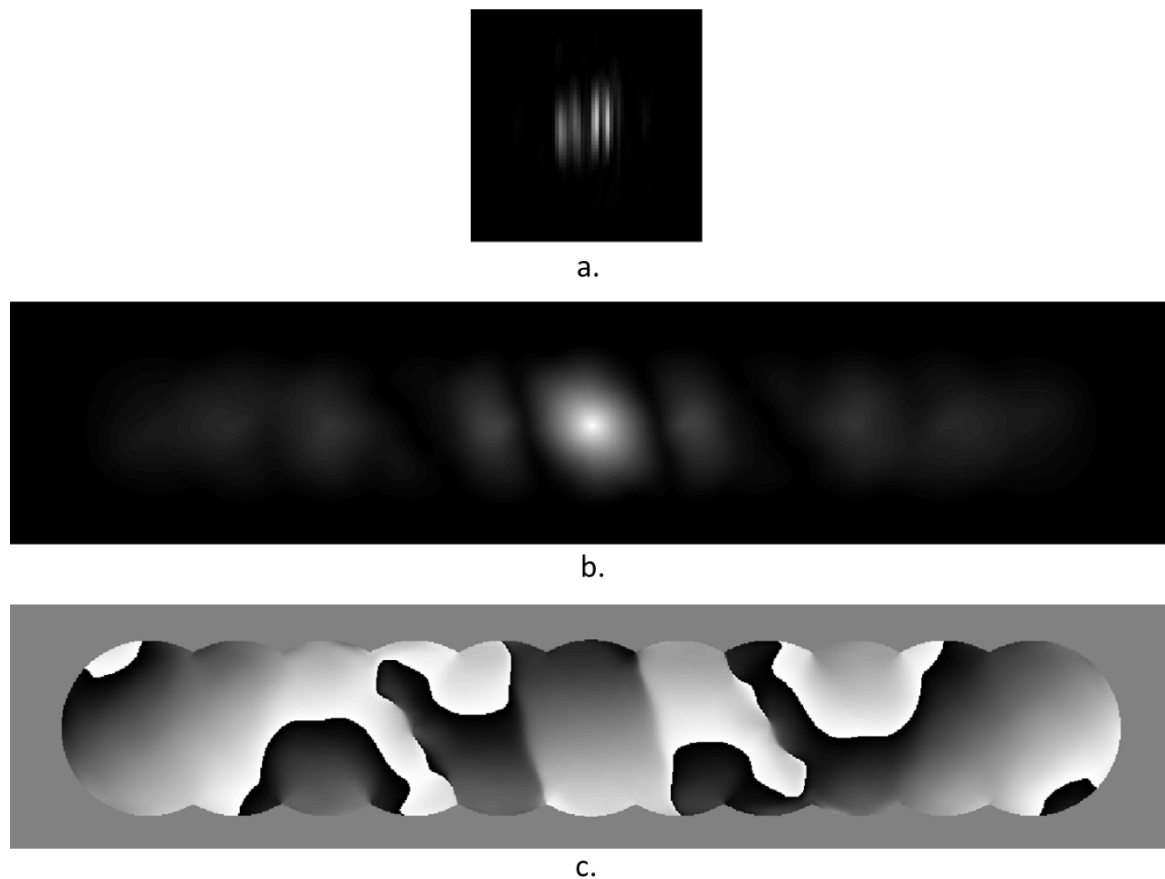


Figure 36: a. PSF perturbed by simulated atmosphere with Fried diameter the size of the aperture b. Squared modulus and c. Angle of the PSF's spatial frequency spectrum

After estimating the piston errors for the PSF and applying corrections, the corrected PSF is shown in Figure 37. The Strehl ratio before corrections was 0.29 and 0.35 after corrections and perturbations are still visible in the PSF and OTF.

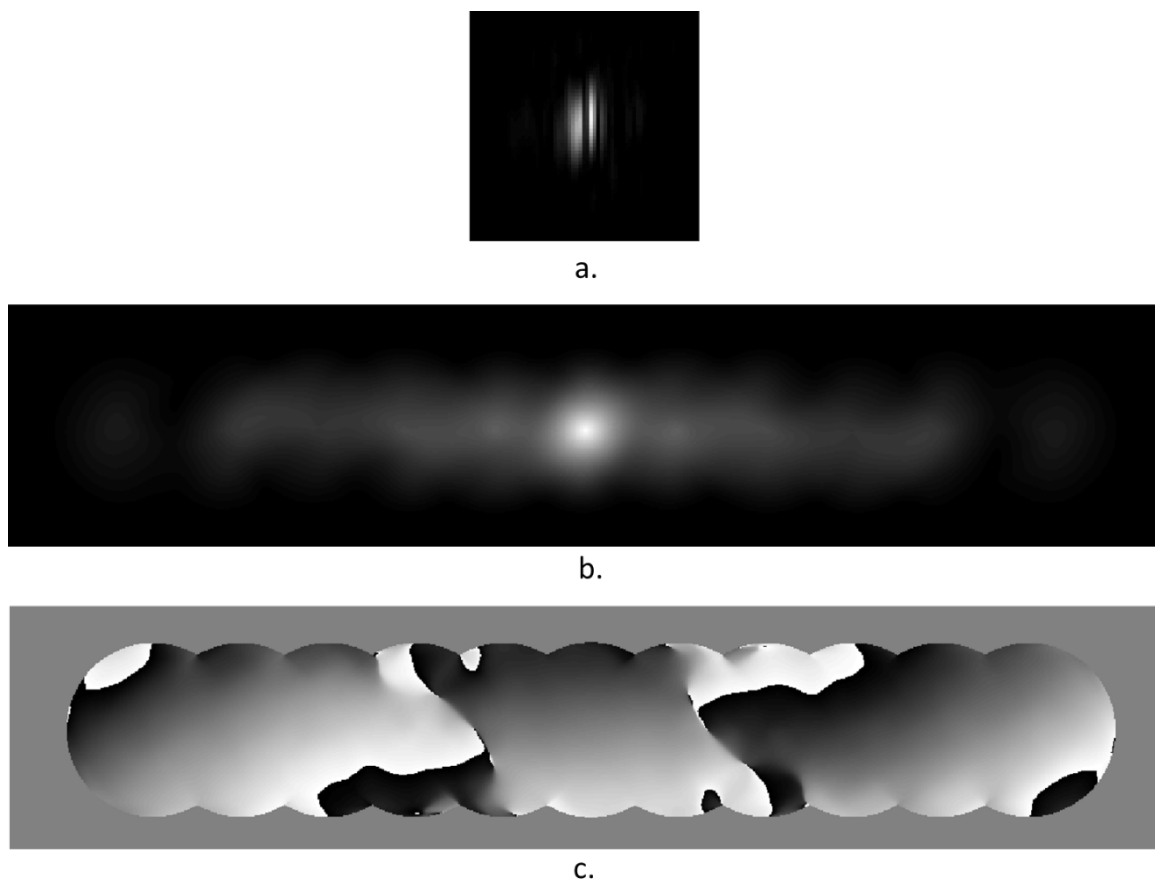
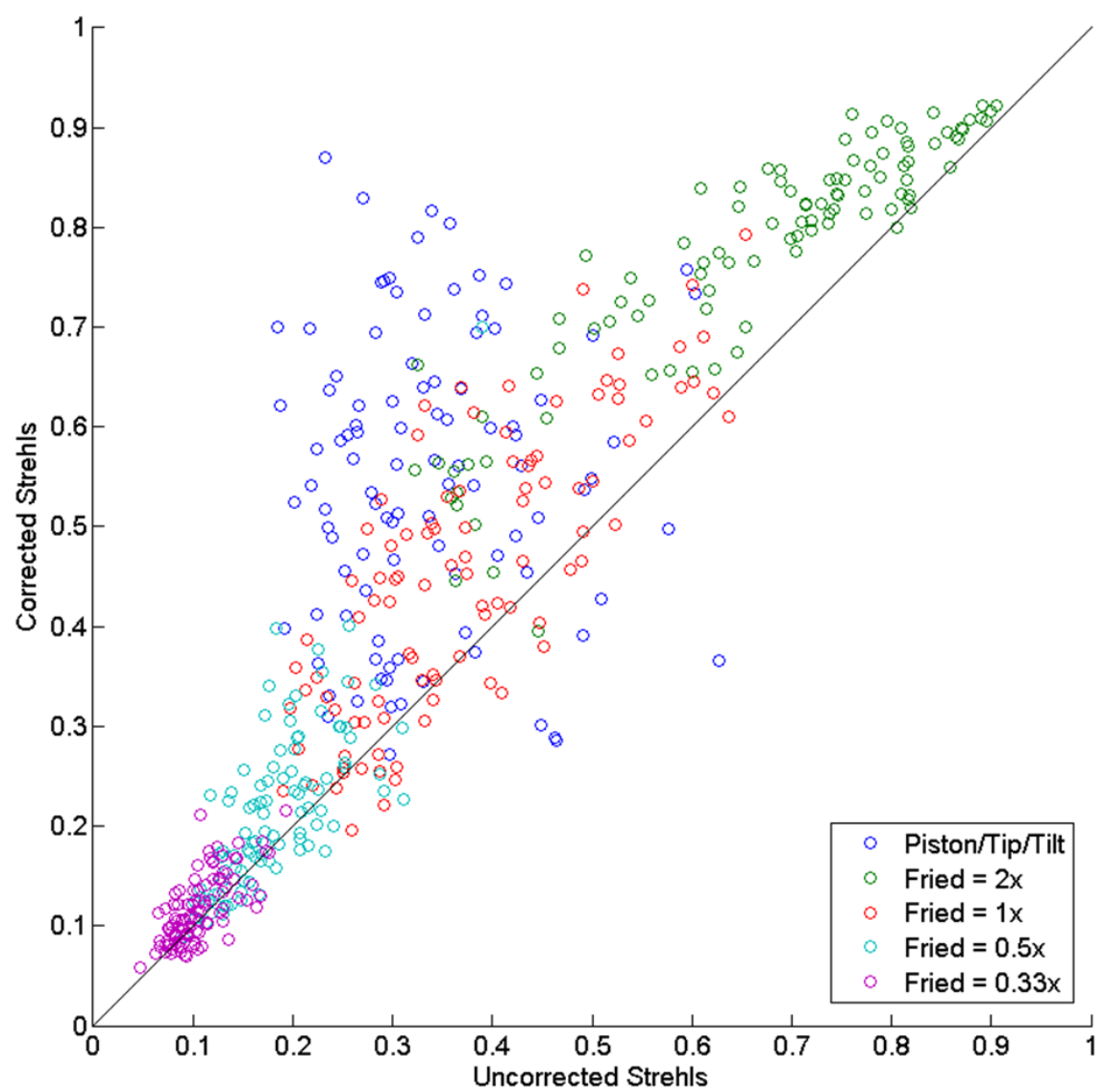


Figure 37: a. Piston corrected PSF b. Squared modulus and c. Angle of the PSF's spatial frequency spectrum

OTFs of similarly perturbed PSFs were applied to the 1951 USAF and dead leaves targets. The graphs in Figure 38 show the Strehl ratios before and after piston correction for both the 1951 USAF resolution target and the dead leaves target. One hundred atmospheric turbulence iterations were applied for Fried parameters of 2, 1, $\frac{1}{2}$, and $\frac{1}{3}$ times the size of an aperture, represented by the colors green, red, teal, and purple, respectively. One hundred iterations of known piston, tips, and tilts without atmospheric turbulence were also corrected, with the results shown in blue. The horizontal axes show the uncorrected Strehl ratios and the vertical axes show the corrected results. Points above the diagonal line are instances where the corrected images are better than the uncorrected images.



a.

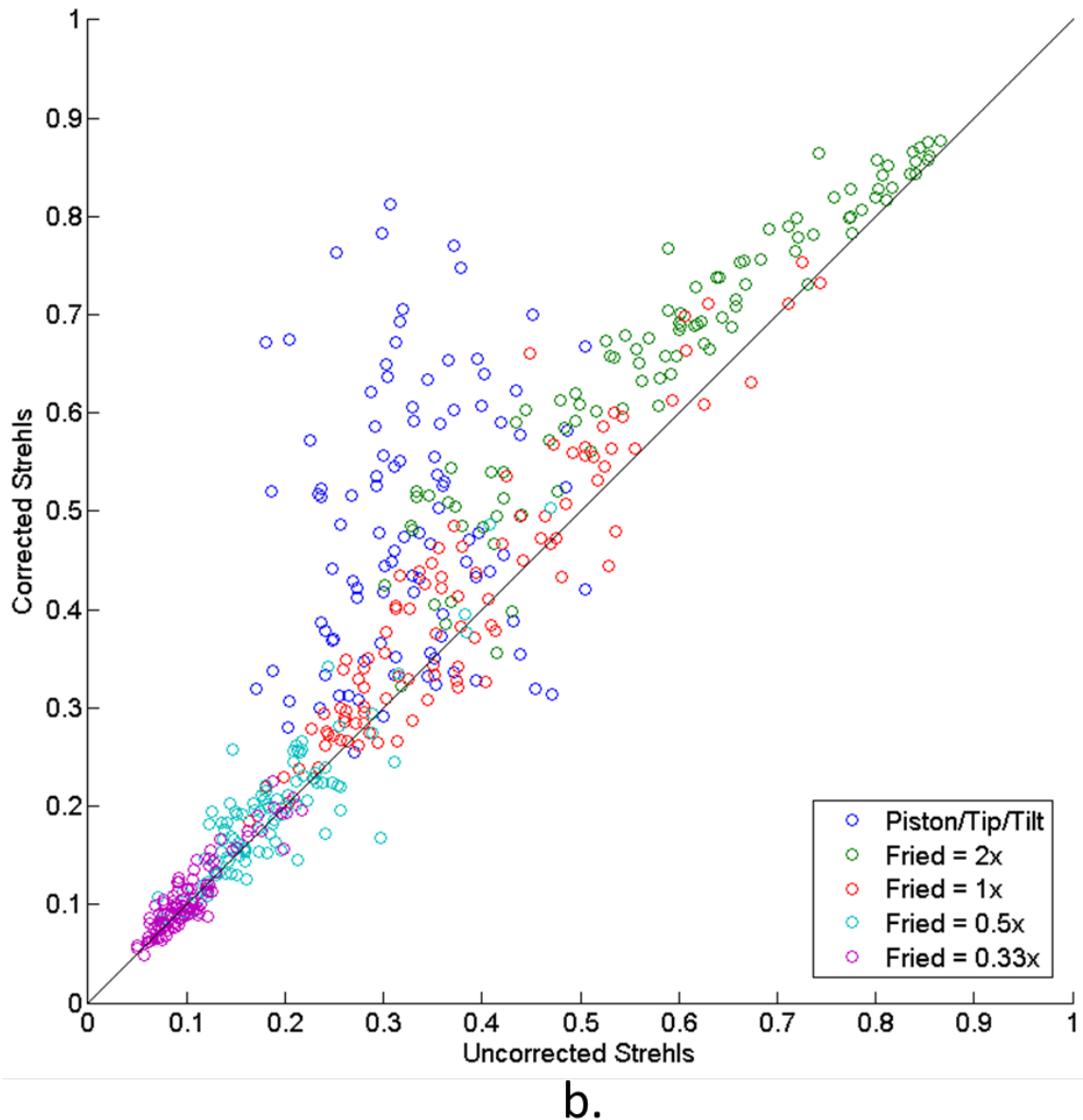


Figure 38: Scatterplot of the Strehl ratios both before and after piston correction for the a. 1951 USAF resolution target and b. dead leaves target. The graph shows 100 atmospheric iterations at Fried diameters of 2, 1, $\frac{1}{2}$, and $\frac{1}{3}$ the size of an aperture.

The information in these scatterplots can be condensed by plotting the means of the uncorrected and corrected Strehl ratios for each Fried parameter. This is done in Figure 39. The solid and hollow squares show the mean corrected and uncorrected Strehl ratios respectively for an ideal PSF with no noise.

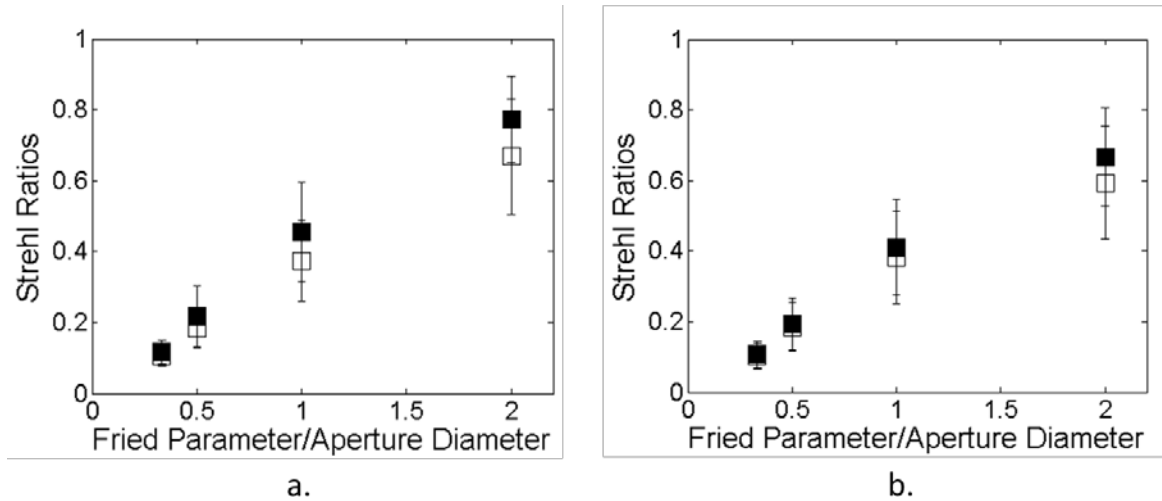


Figure 39: Mean corrected Strehl ratios for the a. 1951 USAF target and the b. dead leaves target

These graphs show that piston correction has some effect on images perturbed with atmospheric turbulence. However, the standard deviations of the corrected values overlap with those of the uncorrected Strehl ratios, showing that the piston correction algorithm does not consistently provide strong corrections.

When piston corrections are supplemented with tip and tilt corrections, the increase in the corrected Strehl ratios becomes more pronounced. More effective corrections are seen if tip and tilt phase errors are corrected first followed by piston. When Figure 36 is corrected with tip and tilt as well as piston, the resulting PSF and OTF shown in Figure 40 are much cleaner than they were in Figure 37 and the Strehl ratio is 0.70. The same is true for extended targets.

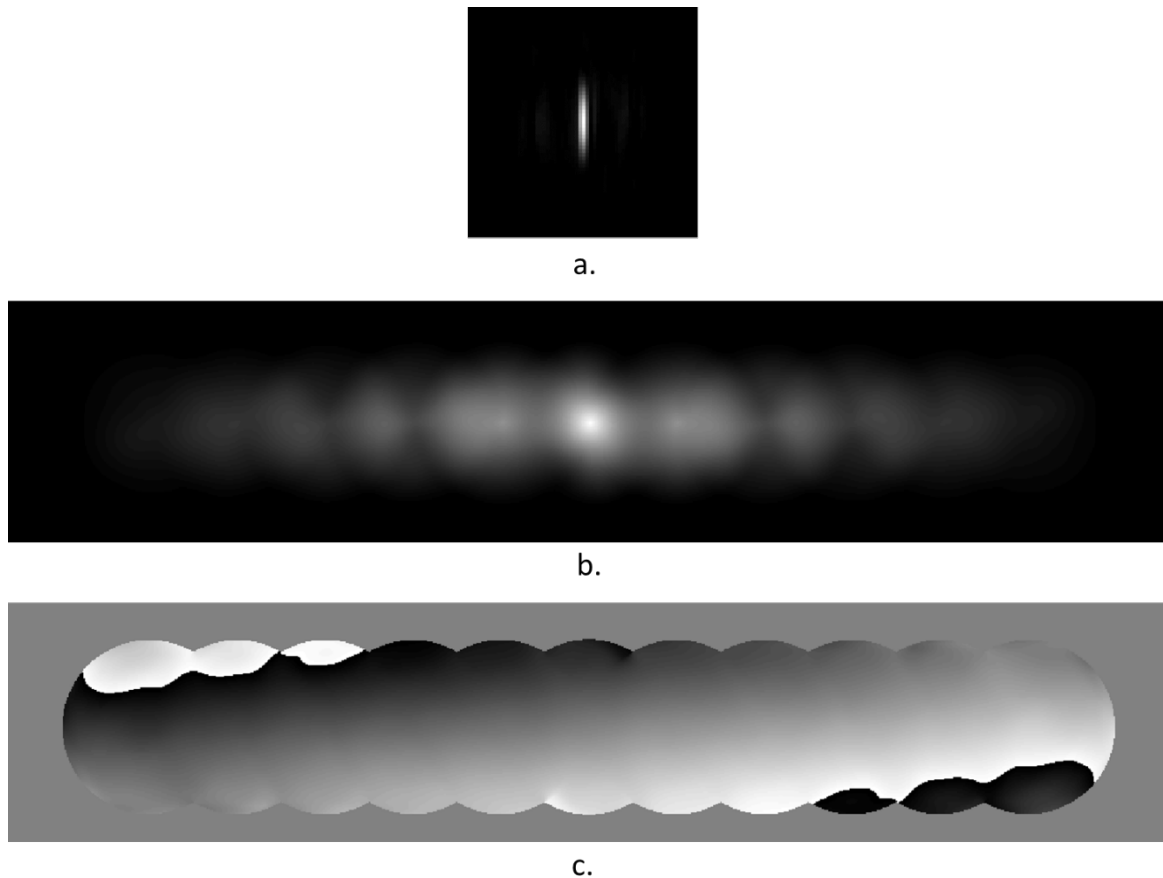
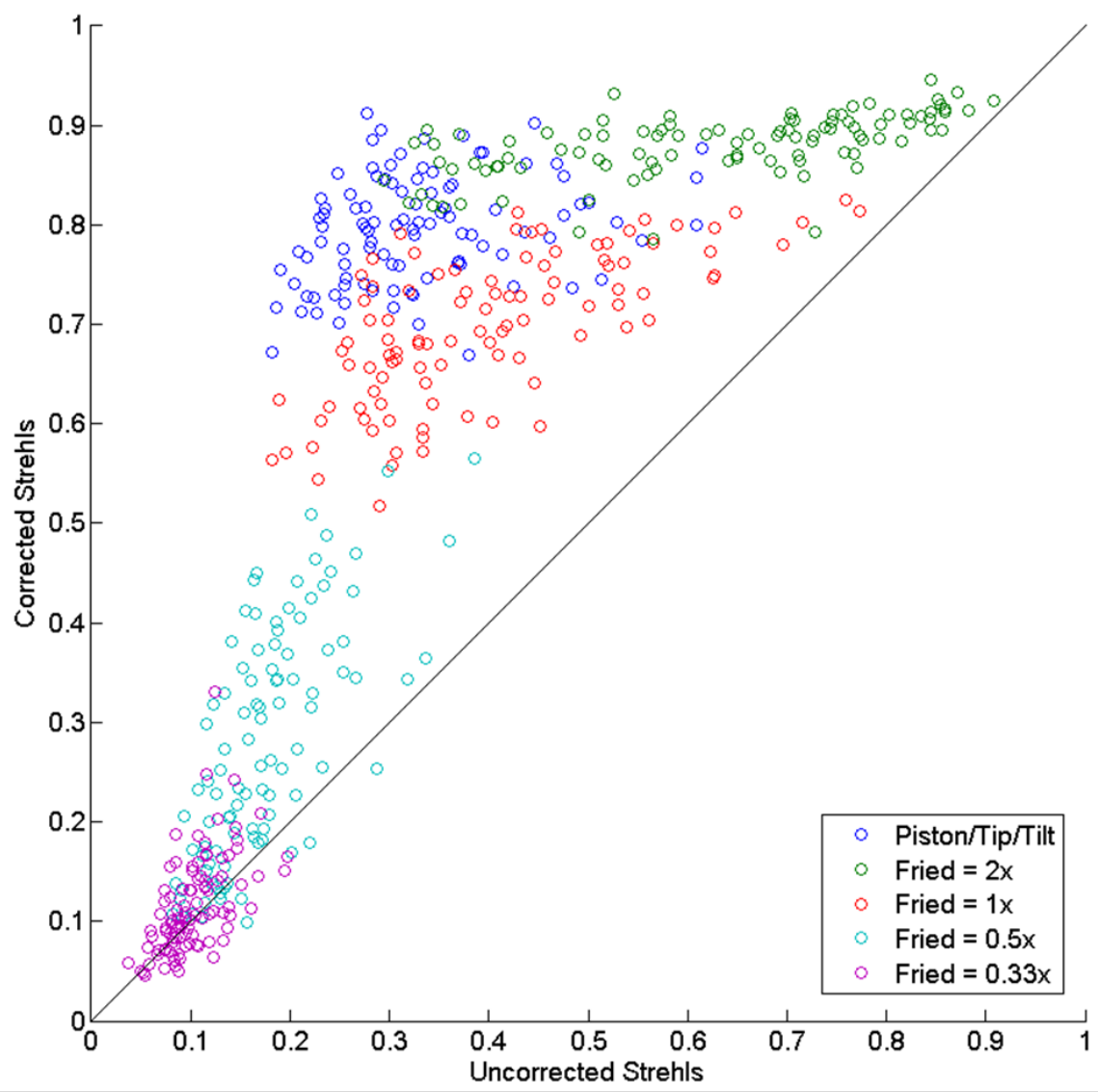


Figure 40: a. Tip, tilt, and piston corrected PSF b. Squared modulus and c. Angle of the PSF's spatial frequency spectrum

The graphs in Figure 41 show the Strehl ratios before and after phase corrections for both the 1951 USAF resolution target and the dead leaves target. These are the tip, tilt, and piston corrected counterparts to the graphs in Figure 38. Nearly all the corrections for a Fried parameter two times the size of the aperture meet the Marechal criterion, as well as about half of the piston, tip, and tilt perturbed images defined with the blue circles. In addition, corrections for a Fried parameter the size of an aperture are noticeably stronger than the previous piston only corrections. Fried parameters half the size of an aperture and lower have higher orders of aberration beyond tip and tilt, and are therefore not as well corrected.



a.

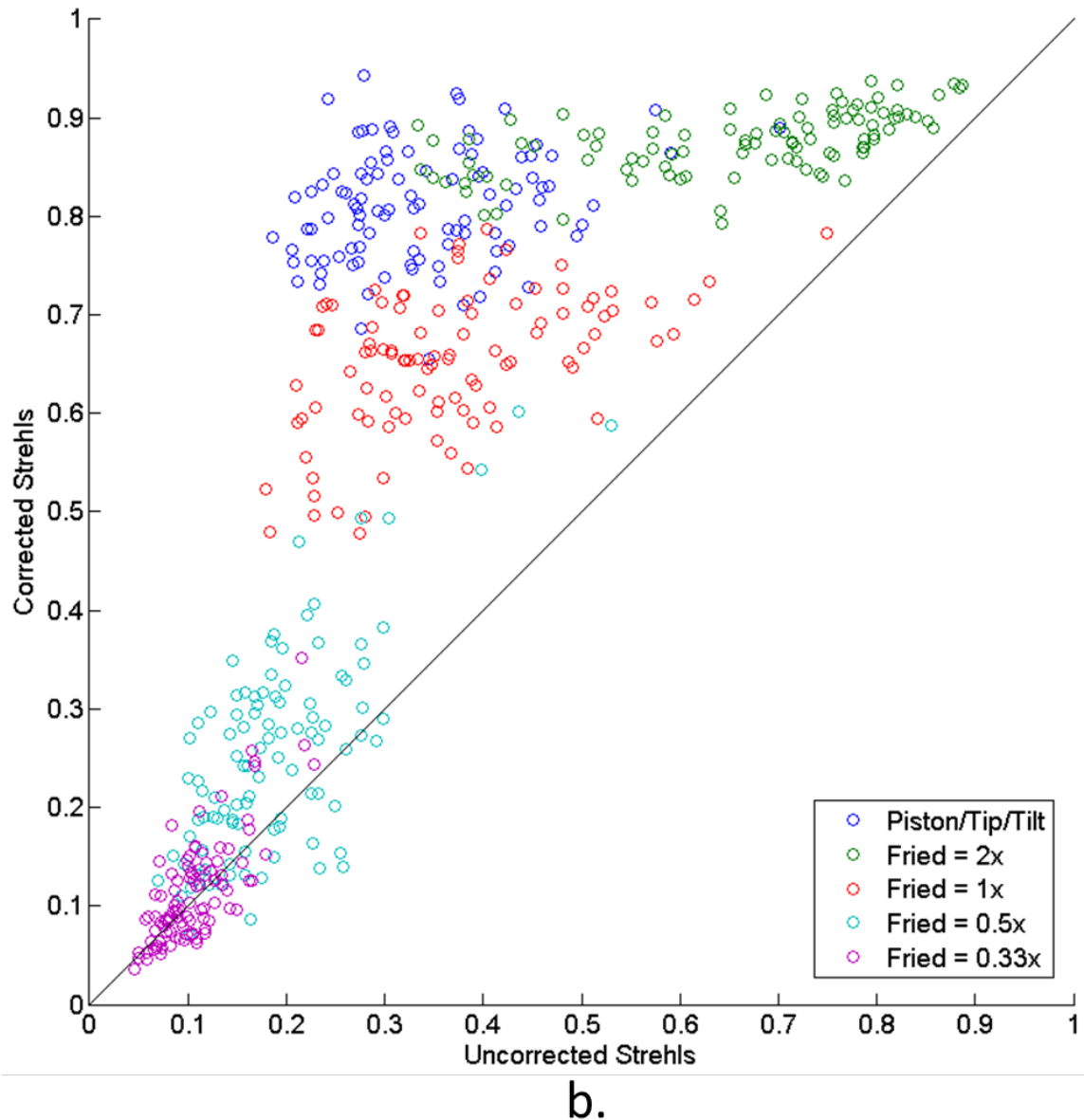


Figure 41: Scatterplot of the normalized Strehl ratios before and after tip, tilt, and piston correction for the a. 1951 USAF resolution target and b. dead leaves target. The graph shows 100 atmospheric iterations at Fried diameters of 2, 1, $\frac{1}{2}$, and $\frac{1}{3}$ the size of an aperture.

These figures display tip, tilt, and piston corrections for simulated atmospheric phase errors with no noise. As in Figure 39, the consolidated graphs of this data are shown in Figure 42 for Fried parameters two, one, one half, and one third the diameter of an aperture. Compared with the images corrected with only the piston correction algorithm, the difference between the corrected and uncorrected Strehl ratios is larger for the images that

were originally perturbed with Fried parameters at least the size of an aperture. The standard deviations for those same corrected Strehl ratios are also much smaller.

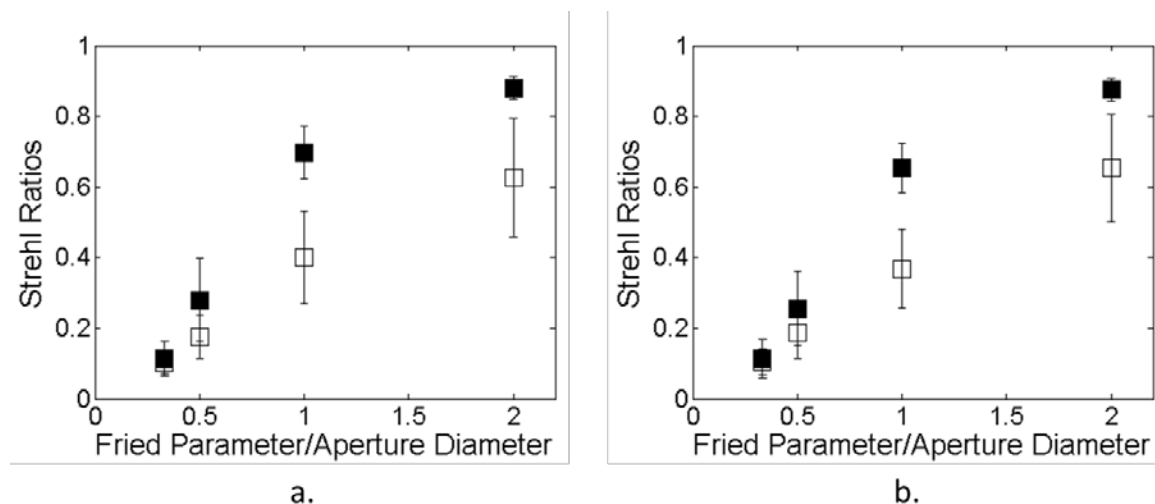


Figure 42: Mean corrected Strehl ratios for the a. 1951 USAF target and the b. dead leaves target

Adding noise to these images results in a decrease in correction effectiveness as the noise begins to dominate the signal. Equation (69) was used to calculate SNRs for the 1951 USAF target and dead leaves target simulated using various mean numbers of photoelectrons per pixel. These SNRS are shown in Table 2 of Section 4.4. Phase correction graphs for the 1951 USAF target with increasing amounts of noise are shown in Figure 43. Notice that the performance of the phase correction algorithm significantly decreases at an SNR of 2. For SNRs above this value, the image is well corrected for Fried parameters of two and one times the aperture diameter. Correction effectiveness decreases at lower Fried values. As expected, any phase estimates attempted when there is no signal and the SNR is 0 are completely random and the final Strehl ratios are worse than the uncorrected values.

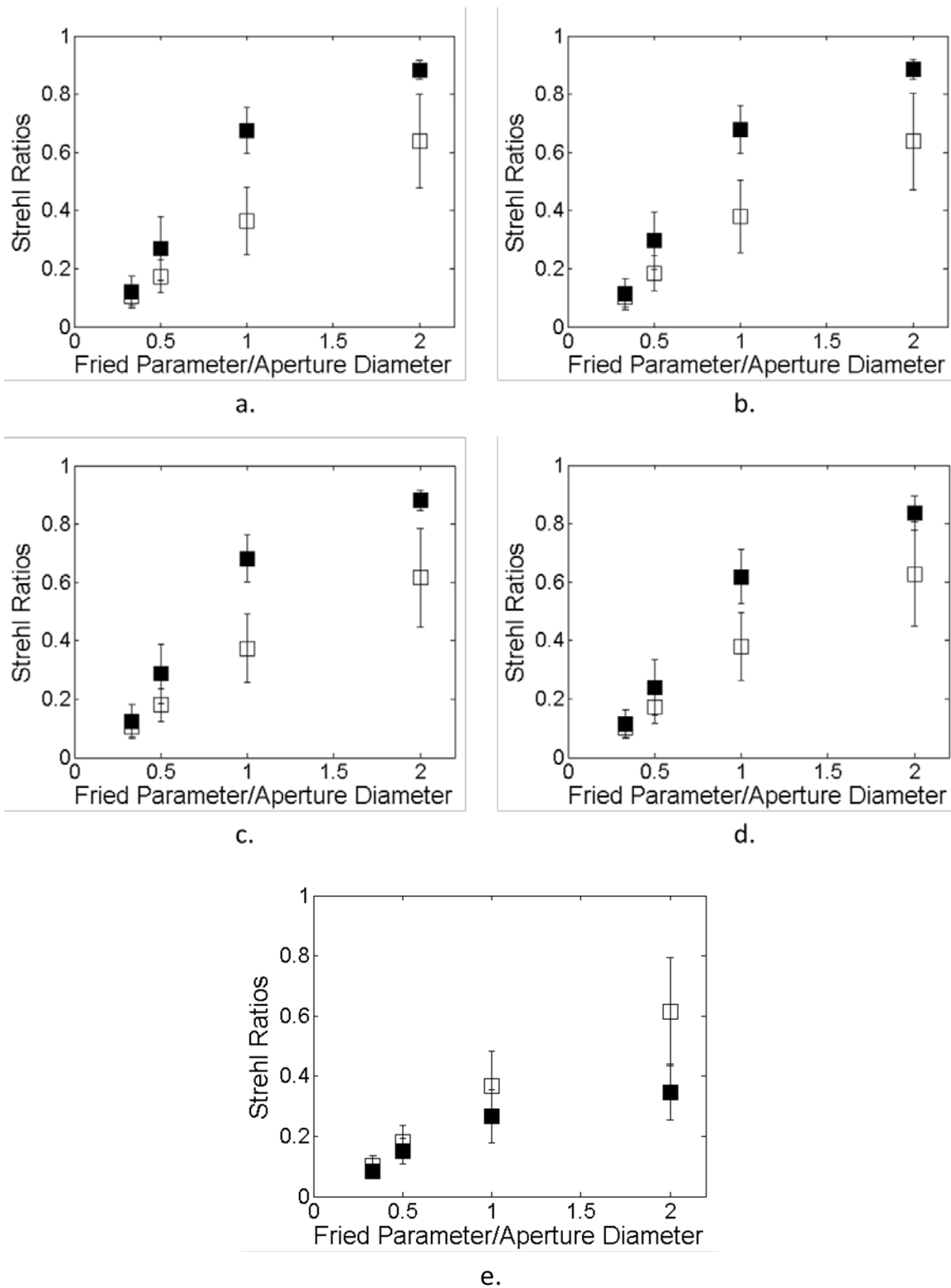


Figure 43: Phase correction results for 1951 USAF resolution target at SNRs of a. 347 b. 87 c. 16 d. 2 e. 0

Similar results are seen for the dead leaves target in Figure 44. However, in this case phase correction performance begins to drop before an SNR of 72 instead of 2.

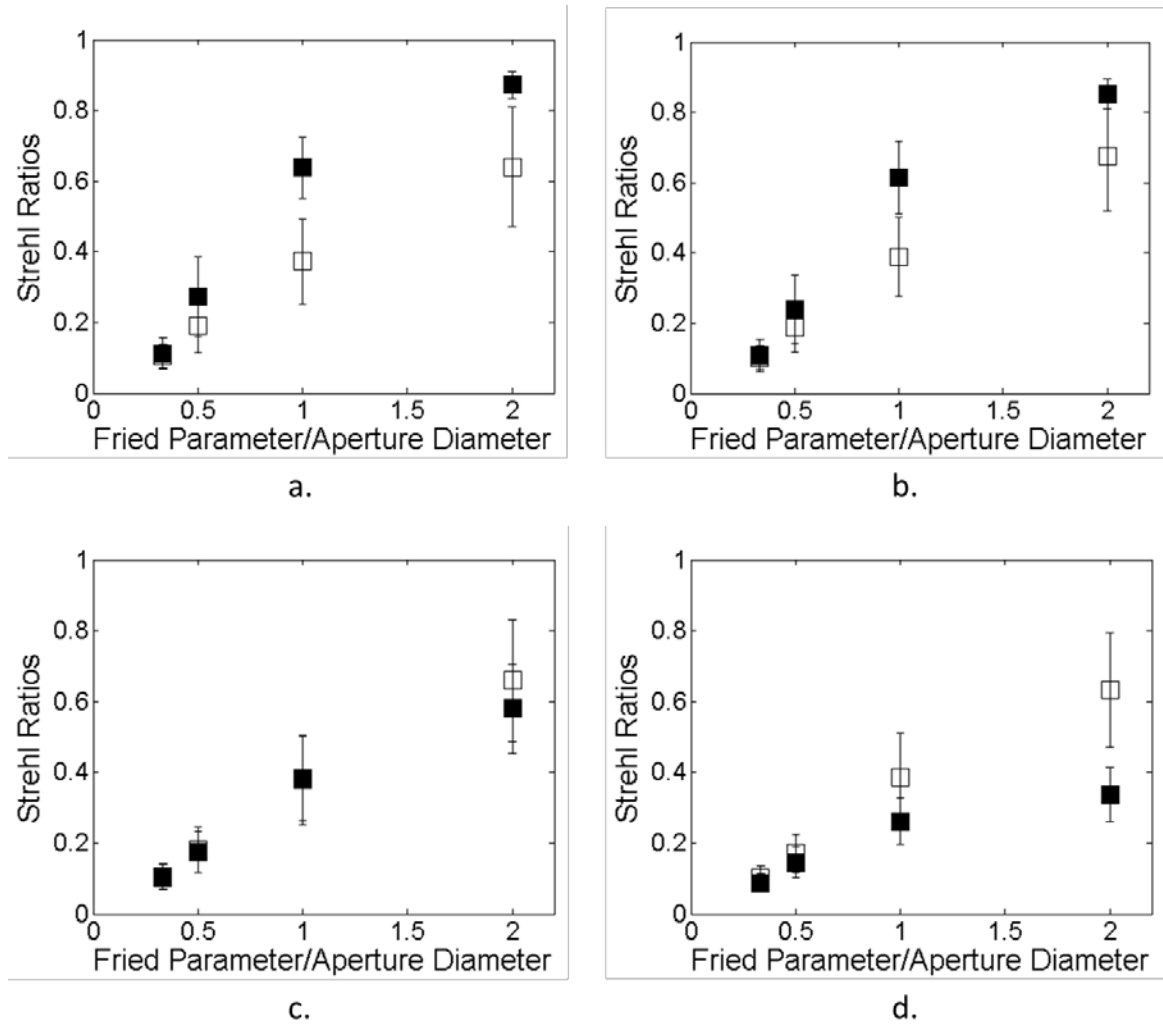


Figure 44: Phase correction results for dead leaves target at SNRs of a. 249 b. 72 c. 15 d. 0

This may point toward an inverse relationship between image sharpness and SNR for phase corrections. In particular, there is an inverse relationship with the sharpness of the higher order spatial frequencies in an image since the phase correction algorithm uses the overlapping spatial frequencies between correlations to estimate phase errors, not the central DC content. The stronger the spatial frequencies are in the overlapping regions, the better the phase estimates and the more resistant those estimates are to noise. A typical

equation for calculating image sharpness, not accounting for the difference between spatial frequencies, is,

$$sharpness = \sum |img|^2 \quad (95)$$

To more accurately measure the higher spatial frequencies found in the overlapping regions, a modified sharpness value can be used that is calculated by subtracting the sharpness of an image created with only the auto-correlation from the sharpness of the full image.

$$sharpness^* = \sum |img_{cross}|^2 - \sum |img_{auto}|^2 \quad (96)$$

The spatial frequency spectrum of the auto-correlation has the same resolution cutoffs as an image taken with a single aperture, so the modified sharpness can be seen as the difference between the sharpness of an image collected with an aperture array and the sharpness of an image collected with a single aperture. Using equation (96), the 1951 USAF target has a modified sharpness of 7.5×10^{-7} while the dead leaves target has a modified sharpness of 9.4×10^{-9} . These two values are approximately two orders of magnitude apart. As stated previously, the phase correction algorithm's results begin to drop at an SNR of around 2 for the 1951 USAF target and an SNR above 72 for the dead leaves target. These two SNR values are a little more than 1.5 orders of magnitude apart. This trend would most likely be supported if a set of targets with a greater variety of modified sharpness values were tested.

6.5 Anisoplanatic Limitations on Phase Corrections

The phase correction results displayed in Sections 6.4 all assume an infinitely large isoplanatic angle across the images' FOV. Unfortunately, this is not always the case.

Smaller isoplanatic angles can result in multiple sets of tip, tilt, and piston errors across a single image, which causes the strength of the phase corrections to decrease along with the isoplanatic angle. This affect can be mitigated by masking an anisoplanatic image to contain only one isoplanatic patch, as long as this patch still provides enough spatial frequency content and adequate SNR for the phase correction algorithm.

To analyze this anisoplanatic effect, images of a 16 by 16 point grid target were simulated with atmospheres of varying isoplanatic angles. Unlike the 1951 USAF and dead leaves targets, the grid target is directly propagated through the atmosphere in order to simulate anisoplanatism instead of convolving it with a perturbed point target. Figure 45 shows an example of a grid target propagated through an atmosphere with an isoplanatic angle of one quarter the FOV. The change in aberrations is visible across the point sources in the image.

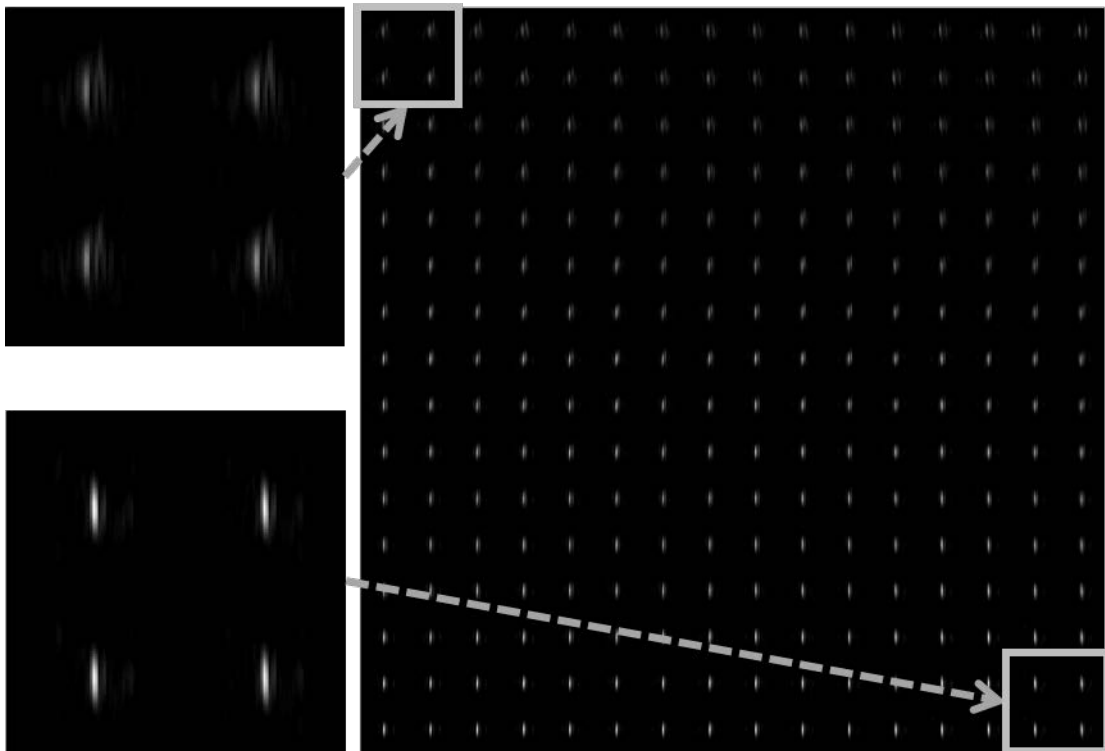


Figure 45: Grid target propagated through a six aperture pupil and a simulated aperture with an isoplanatic angle one quarter the size of the field of view

Twenty atmospheric iterations of this grid image were simulated using isoplanatic angles of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, 1, and 2 times the FOV. The FOV of these simulations is 0.0045 radians. Each set of images were phase corrected. Phase corrections were measured by taking the mean Strehl ratio of each point in the grid and then the mean Strehl ratio of all twenty images.

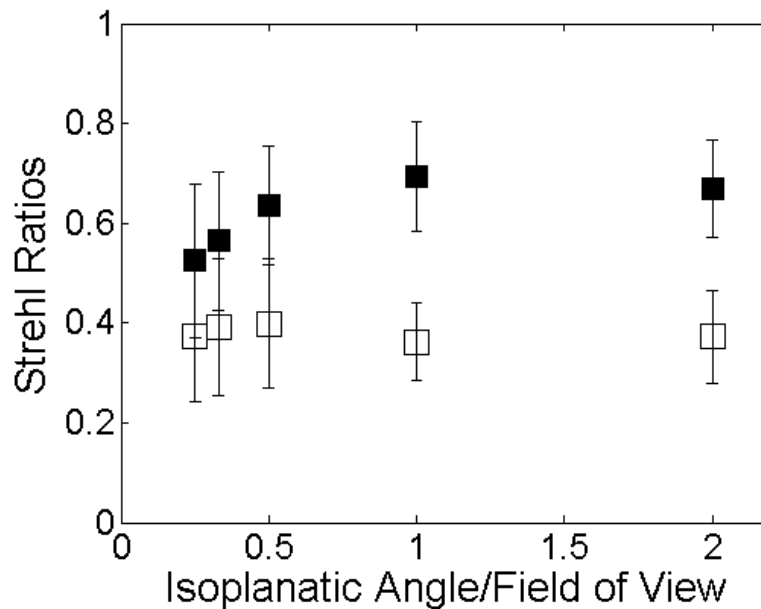


Figure 46: Graph of phase corrections with regard to the image's isoplanatic angle. The SNR is 2.

Figure 46 displays the results of this analysis. The performance of the phase correction algorithm decreases as anisoplanatism increases and the size of the isoplanatic angle gets smaller. The modified sharpness metric of the grid target is 3.2×10^{-5} , which is approximately 1.5 orders of magnitude larger than the modified sharpness of the 1951 USAF target discussed in the last section. Therefore, an SNR of 2 did not have much effect on the phase corrections. The SNR would have to be even lower in order to begin affecting the phase correction algorithm for this particular target.

The phase correction algorithm is not designed to correct for anisoplanatism. It is currently designed to estimate only one set of piston, tip, and tilt. Masking off an area of

an image approximately the size of a single isoplanatic patch should limit the correction area to one set of phase errors. However, this will only work as long as the masked area is large enough to have proper support and signal strength. Section 6.6 measures how much support is needed for phase corrections at various signal strengths.

6.6 Image Support Measurements for Phase Correction Algorithm

To measure how much support is needed for different mask sizes and the effects of SNR on that number, a 32 by 32 point grid target with a flat plate background is used. As stated previously, a grid target has an extremely high sharpness metric for which noise is negligible outside of very low SNRs. The flat plate background reduces the modified sharpness of the target to 8.9×10^{-8} , which falls between the values of the 1951 USAF target and the dead leaves target and makes it easier to measure the effects of noise on masking. Unlike the target from Section 6.5, this target was not propagated directly through the atmospheric simulations, but was instead convolved with a perturbed PSF in order to ensure a fully isoplanatic image. In this case, a dense grid target was chosen since it has relatively homogeneous spatial frequency content at all locations, not because of any ability to display anisoplanatism.

Images were masked to $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and the full diameter of the image, as shown in Figure 47. These images were given an infinite isoplanatic angle as a control and a Fried diameter the size of an aperture. One hundred images were corrected for each mask size.

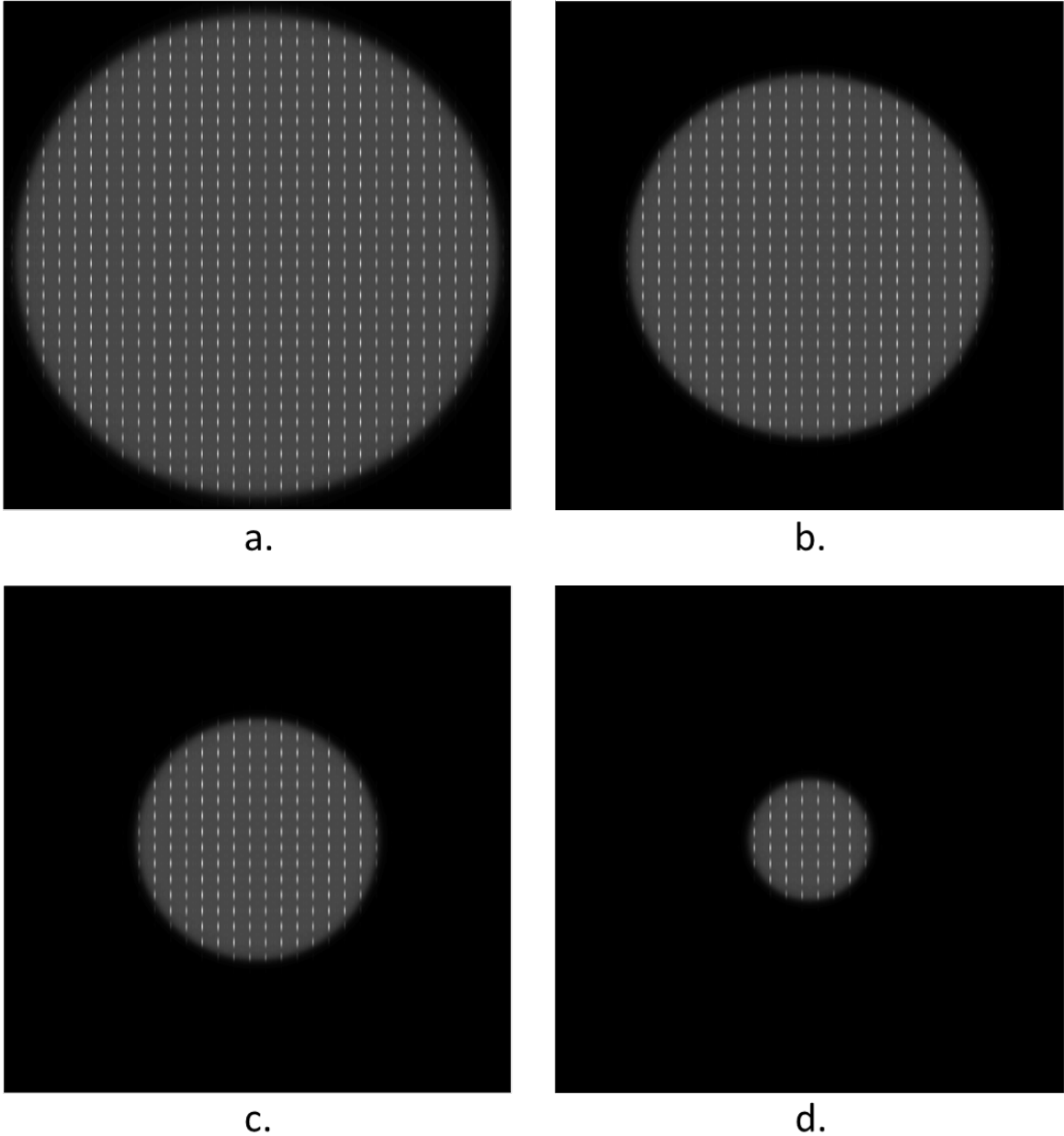


Figure 47: Grid target with a flat plate background masked to a. $\frac{1}{4}$, b. $\frac{1}{2}$, c. $\frac{3}{4}$, d. and the original diameter.

The results are shown in Figure 48. Phase corrections for smaller masking areas begins to degrade as the SNR decreases. With this particular target and lens setup, an isoplanatic angle $\frac{1}{4}$ the FOV should be correctible with masking at an SNR or 95 or higher, but the phase correction algorithm's performance grows weaker as noise increases. These results can be generalized by finding how much image support would be needed to get similar

results for a similar aperture array expanded to real world distances and wavelengths. Each of the six apertures in the imaging array have a 25.4 mm diameter, the illumination wavelength used for the simulations is 1.55 μm , and the FOV of the image is 0.0045 radians, so there are approximately 18 diffraction limited spots across $\frac{1}{4}$ of the FOV. Any similarly diffuse target with higher spatial frequency content should be correctable as long as the isoplanatic angle is larger than 18 diffraction limited spots and the SNR is sufficiently high. The SNR necessary for corrections should change based on the amount of spatial frequency information available within each isoplanatic patch, which should be measureable with the modified sharpness metric.

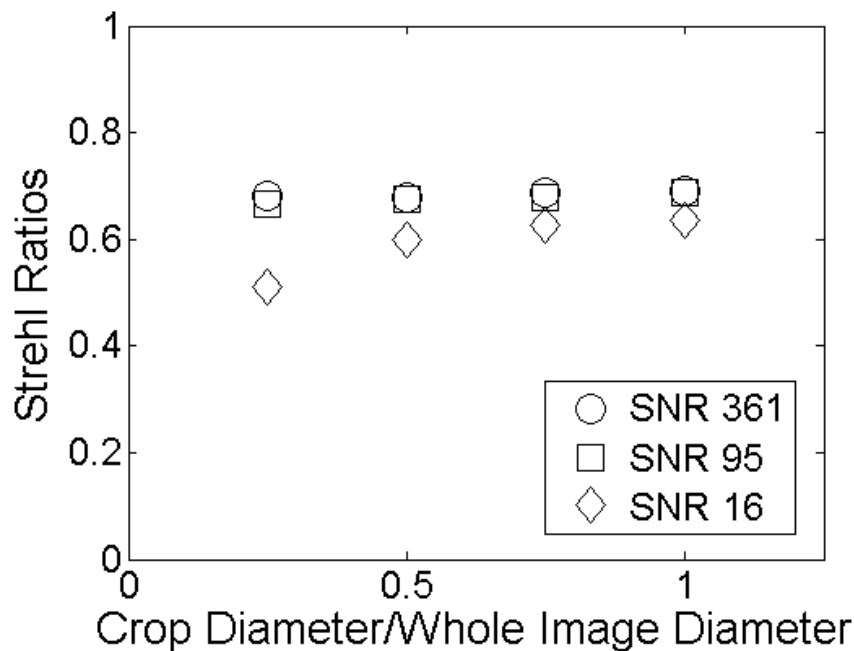


Figure 48: Masked grid target phase corrections for SNRs of 361, 95, and 16.

Using this masking technique, anisoplanatic images could potentially be corrected by masking multiple parts of an image and correcting these sections individually. After registering each masked section to account for universal tips and tilts, all of the images could be summed back together into a single image. With this technique, anisoplanatic

phase corrections could be completed without the use of an MCAO system described in Section 3.3.2.

CHAPTER 7

PHASE CORRECTION ANALYSIS

In this chapter, the results from the phase correction algorithm are compared with blind deconvolution and some use cases are considered based on the phase correction results. The analysis comparing the phase correction technique with blind deconvolution, which is a previously tested and accepted image deblurring technique, is given in Section 7.1. For the analysis, the aperture configuration and data discussed in Chapter 6 are used, where the Fried parameter is equal to the aperture diameter and the images are fully isoplanatic. Section 7.2 considers use cases for the phase correction results. Some potential examples are presented where partially coherent phase correction could be useful under field conditions.

7.1 Blind Deconvolution Comparison

A recursive gradient-projection blind deconvolution algorithm [36] was chosen to compare with the phase correction algorithm. This blind deconvolution technique starts with multiple short exposure image frames and an initial PSF estimate for those frames. No pupil remapping is needed when collecting these images. An iterative process is used to estimate the original target image and PSFs that better match the perturbations of each of the initial image frames. The number of blind deconvolution iterations needed to

produce the best possible target estimate is found when the RMSE between the estimated target and the original target used to create the simulated images reaches a minimum.

The goal is to compare this blind deconvolution technique with the pupil remapping phase correction algorithm. However, there are two major differences between these two methods that need to be addressed before their results can be directly compared. First, the phase correction algorithm corrects for piston, tip, and tilt, but does not take the PSF caused by the profile of the aperture array into account. In contrast, blind deconvolution attempts to correct the perturbations from both the atmosphere and the array profile. As a work-around for this difference, a Wiener deconvolution is applied to all images corrected by the phase correction algorithm prior to comparison with the blind deconvolution results.

For example, even after piston corrections the image in Figure 49.a still seems to have blurred, undefined edges. A Wiener filter can be used to deconvolve the original PSF from the image. The Wiener filter can be calculated through the equation,

$$\mathcal{W}(f_x, f_y) = \frac{H^*(f_x, f_y)}{|H(f_x, f_y)|^2 + K(f_x, f_y)} \quad (97)$$

where $H(f_x, f_y)$ is the FT of the PSF and $K(f_x, f_y)$ is the power spectral density of the noise in the image [34]. The noise factor can be simplified to a scalar constant equal to the noise to signal ratio (NSR) of the image [35]. Multiplying the Wiener filter with the FT of the image, $\mathcal{G}(f_x, f_y)$, and taking the inverse FT gives the deconvolved image, $f(x, y)$ [34].

$$f(x, y) = FT^{-1}\{\mathcal{W}(f_x, f_y)\mathcal{G}(f_x, f_y)\} \quad (98)$$

These images have increased edge sharpness and it is easier to see the gain from phase corrections.

The PSF used in the Wiener deconvolution is generated by creating a simulation of the original pupil function defined by the aperture array and adding the estimated pistons, tips, and tilts to the simulated apertures. All of the perturbed apertures are correlated and then those same estimated phase errors are directly corrected in the correlations. Adding all of the correlations together and taking their FT gives a PSF with an amplitude perturbation caused by the estimated phase errors. Since the phase correction algorithm corrects the phase but not the amplitude of the perturbed images, deconvolving by this PSF should increase the correction effectiveness as well as minimizing the effects of the original shape of the aperture array. After the Wiener deconvolution, the corrected images are more similar to the images processed with blind deconvolution. An example of a phase corrected dead leaves image and the same image after Wiener deconvolution is shown in Figure 49. After the deconvolution, the image appears sharper than it did before. The Wiener deconvolution was completed using $K = 92$. This value changes with Fried parameter and SNR.

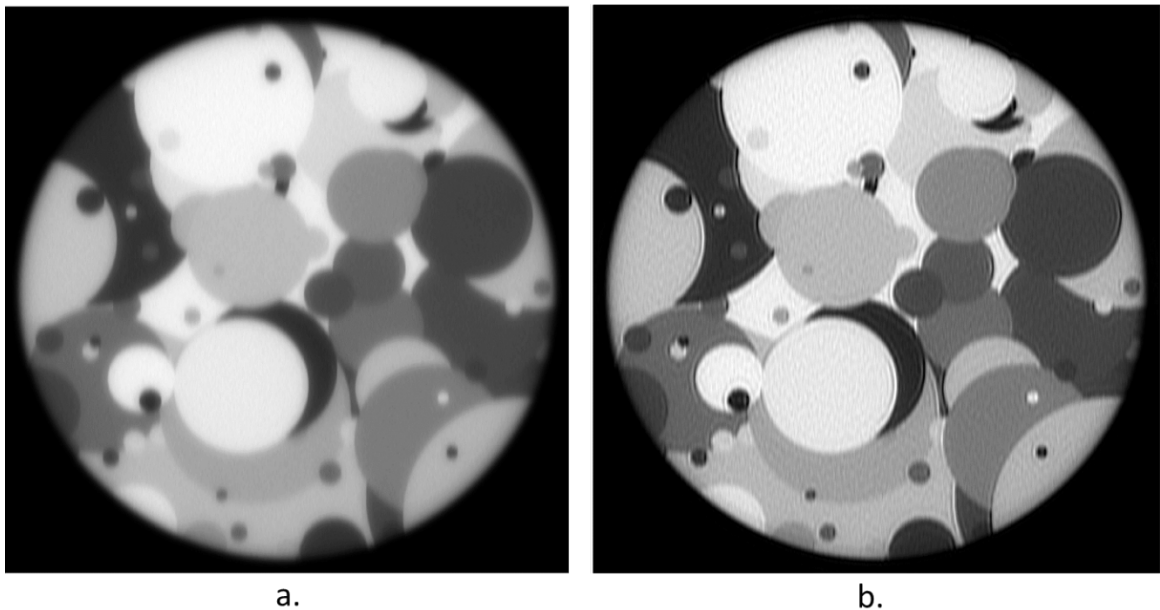


Figure 49: a. Phase corrected dead leaves image with a Fried parameter the size of an aperture and an SNR of 72 b. Same image after Wiener deconvolution.

Second, blind deconvolution needs multiple frames in order to find its estimate of the original target, while the pupil remapping phase correction algorithm needs only one frame. In order to compare an image corrected using one frame to an image corrected with multiple frames, remember that the data for the blind deconvolution does not undergo pupil remapping. Therefore, the spatial frequency spectra of the frames collected for blind deconvolution should resemble the spectrum for the array before remapping, as seen in Figure 10.b, while the frames for the phase correction algorithm have spectra resembling Figure 10.d. In order to include all of the separated spatial frequencies and consequential resolution artifacts from the remapped pupil, the frames for the phase correction algorithm need to have about six times more pixels in the non-gain direction than the blind deconvolution frames. Therefore, using a ratio of six frames for blind deconvolution for every one frame for the phase correction algorithm is an acceptable ratio of comparison. An example of an image before and after blind deconvolution is shown in Figure 50. Figure 50.a shows the original mean of all six frames while Figure 50.b shows the estimated image. The initial guess for the PSF for the frames is shown in Figure 50.c and the final PSF estimates for the six frames are shown in Figure 50.d.

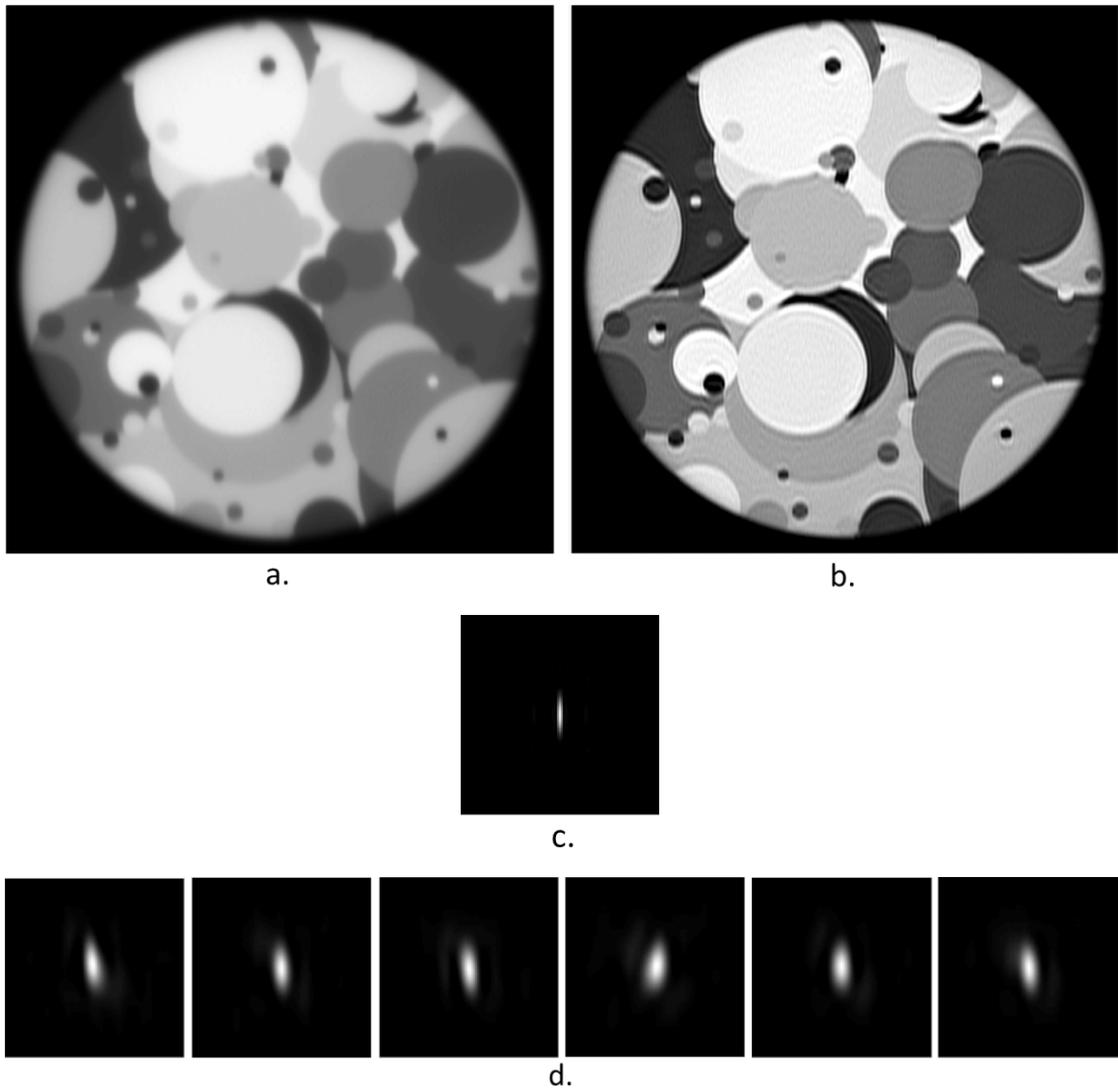


Figure 50: Results of a six frame blind deconvolution of a dead leaves target with a Fried parameter the size of an aperture and SNR of 72. a. Mean of six frames before blind deconvolution b. Image estimate after blind deconvolution c. Initial PSF given to the blind deconvolution algorithm d. PSF estimates for the six frames that give the highest RMSE.

Two different situations are studied for the comparison between blind deconvolution and the phase correction algorithm. The first is the effectiveness of both techniques under atmospheric conditions and the second is their effectiveness when there are only piston, tip, and tilt from hardware misalignments. In Section 6.4, it was found that the phase correction algorithm works best when the Fried parameter is at least 2.54 cm, or approximately the diameter of an aperture, so that is the Fried parameter chosen to test

blind deconvolution for atmospheric perturbations. While correcting the dead leaves target with the phase correction algorithm, as shown in Figure 44, an SNR of 249 worked well and there was only a small drop in effectiveness for an SNR of 72. An SNR of 15 showed a significant loss in phase correction effectiveness. Therefore, these are the three SNRs that are used in the comparison with blind deconvolution for both the atmospheric and hardware misalignment corrections.

One hundred images were used to test the phase correction algorithm for each of the SNRs in Section 6.4. The phase corrected images used in this section are deconvolved with an amplitude adjusted Wiener filter before calculating their RMSEs for comparison with the blind deconvolution images. Since the results in CHAPTER 6 were given in terms of Strehl ratios while the results in this chapter use RMSE, a comparison between some Strehl ratios from CHAPTER 6 and the corresponding RMSE values after Wiener deconvolution are shown in Table 7.

Table 7: Comparison between the mean Strehl ratios of 100 dead leaves images with a Fried parameter the size of an aperture corrected by the phase correction algorithm and the mean RMSEs of those same images after Wiener deconvolution.

SNR	Strehl Ratios	RMSE
Before Corrections	0.3813	0.0723
249	0.6373	0.0414
72	0.6147	0.0457
15	0.3782	0.0604

In this instance, an RMSE of approximately 0.072 corresponds to a completely uncorrected image without performing Wiener deconvolution, while an RMSE smaller than approximately 0.05 shows a well-corrected and deconvolved image. The piston, tip, and tilt of the dead leaves images at an SNR of 15 are not well corrected, leaving only the deconvolution of the pupil function to have any effect, so the RMSE is above 0.05 but still better than 0.072.

One hundred similarly perturbed images were generated for the blind deconvolution algorithm, but without aperture remapping. Since six images are used for each blind deconvolution, 16 images can be estimated using six frames for each estimate. The results for the blind deconvolution and phase correction algorithms are shown in Figure 51.a for images perturbed with an atmospheric turbulence with a Fried parameter the size of the aperture. Figure 51.b shows the corresponding results for images perturbed with only the piston, tip, and tilt of hardware misalignments.

In order to show a comparison between the phase corrected images and blind deconvolution images, a metric other than SNR needs to be used along the horizontal axes of the graphs since the SNRs are not the same for the remapped pupil frames used for phase corrections and the non-remapped frames used for blind deconvolution. Both methods image the same FOV at the same target plane, but blind deconvolution uses approximately six times less pixels. This means that six times more photo-electrons reach each pixel, and using these new values in equation (69) changes the SNR values. The mean number of photo-electrons per pixel and the resulting SNRs are compared for the remapped images used for phase corrections and the non-remapped images used for blind deconvolution in Table 8. Specifically, there are 6.24 times less pixels used for blind deconvolution in this example.

Table 8: SNRs for the dead leaves target with noise added with and without aperture remapping

Photo-Electrons: Remapping	SNR	Photo-Electrons: No Remapping	SNR
50,000	249	312,200	630
5,000	72	31,220	195
500	15	3,122	54
50	2	312.2	10

The FOV of a single pixel in the pupil remapping case is 1.93×10^{-11} steradians (sr) since the FOV for a simulated image is set at 0.0045 radians across, or 20.25 μ sr, and the image size is 1024 by 1024 pixels. The value 1.93×10^{-11} sr will henceforth be referred to as Ω . The number of photo-electrons per Ω is set such that it does not change with either pixel size or SNR, so mean photo-electrons per Ω is used along the horizontal axis in Figure 51 and for all future graphs comparing the phase correction algorithm and blind deconvolution.

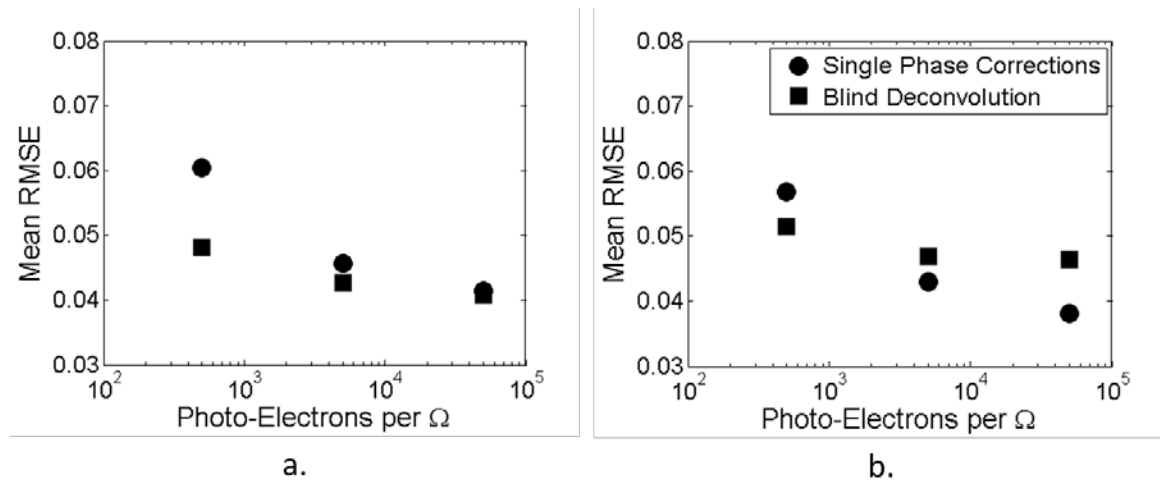


Figure 51: Graph comparing the RMSEs of the phase corrections algorithm and blind deconvolution when there is a. A Fried parameter the size of the aperture and b. Only piston, tip, and tilt perturbations from hardware misalignments. The circles are the single frame phase correction results while the squares show the blind deconvolution results.

For the tests containing atmospheric perturbations, blind deconvolution has lower RMSEs than the Wiener deconvolved images for all three photo-electron values, with a noticeable difference at 500 mean photo-electrons but much closer results for 50,000 photo-electrons. In comparison, corrections for the hardware misalignments show that the RMSEs are lower for Wiener deconvolved images when there are 5,000 or more photo-electrons detected per Ω . Atmospheric perturbations are generally more random across an aperture than hardware misalignments and have more higher order aberrations, so it makes sense that the

statistically based blind deconvolution algorithm have better results in that case. It is encouraging that the non-iterative, single frame phase correction algorithm can compete with the iterative, six-frame blind deconvolution method at high photo-electron counts. The piston, tip, and tilt correction algorithm is more well equipped to deal with the piston, tip, and tilt phase errors created by hardware misalignments and this is shown in the results, where phase corrected and Wiener deconvolved images are stronger than their blind deconvolution counterparts for mid and high phot-electron counts.

Comparing a single frame phase correction to a six-frame blind deconvolution is still not a perfect apples-to-apples comparison, though. As stated previously, the greater number of photo-electrons hitting each pixel for the blind deconvolution frames raises the SNR. In addition, using six frames for a single estimation averages out even more of the noise from those images. This naturally causes the blind deconvolution RMSEs to be lower than the single frame phase corrections. If the images used for blind deconvolution are given a shorter exposure time to compensate for the need of six frames to every one for the phase correction algorithm, the RMSE values may be more comparable. Another option is to take the mean of six images corrected with the phase correction algorithm before applying Wiener deconvolution and then calculating the RMSE. In this case, the Wiener filter only includes the PSF of the aperture array without any additional amplitude perturbations, but the noise averaging effect should still provide a better RMSE than was seen in Figure 51.

Figure 52 shows this additional set of data. Blind deconvolution and single frame phase correction results are the same, but the third set of results contains RMSE values calculated by taking 16 sets of six phase corrected images, registering them, finding their

mean, and then taking the Wiener deconvolution of that mean image. Taking the mean of 6 images before finding the RMSEs averages out some of the noise from the final images so the RMSEs are somewhat lower than they were when only single frames were Wiener filtered.

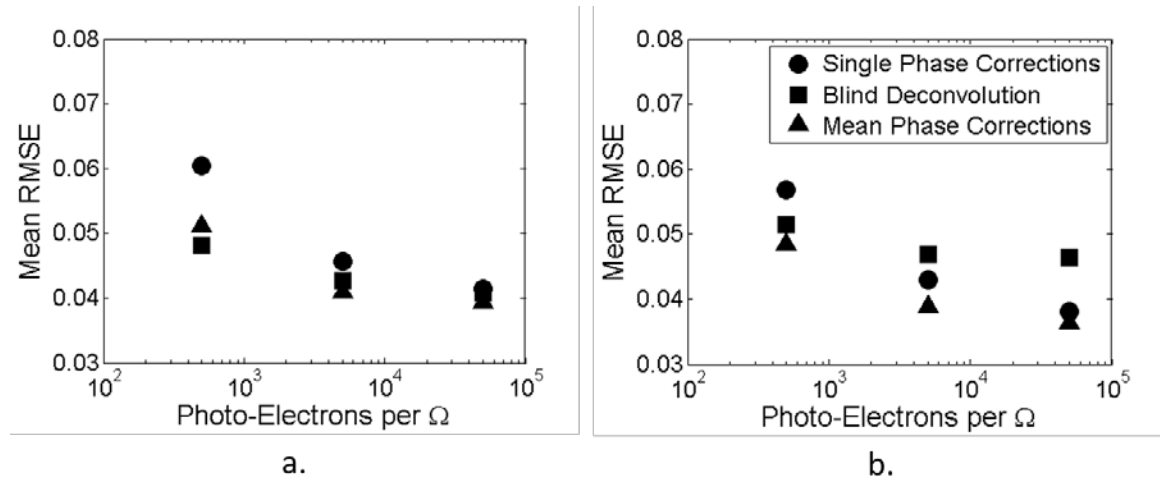


Figure 52: Graph comparing the RMSEs of the phase corrections algorithm and blind deconvolution when there is a. A Fried parameter the size of the aperture and b. Only piston, tip, and tilt perturbations from hardware misalignments. The circles are the single frame phase correction results, the squares show the blind deconvolution results, and the triangles show the results for the mean of six phase corrected images.

In this case, the RMSEs for the phase correction algorithm with atmospherically perturbed images are slightly lower than the blind deconvolution results for photo-electron values of 5,000 and above. For the case of hardware misalignments only, the phase correction algorithm does better for all three photo-electron amounts.

These results show that blind deconvolution and the phase correction algorithm are relatively well matched in the case of atmospheric perturbations with a Fried parameter the size of an aperture, while the phase correction algorithm works better in cases where the dominant source of phase errors is piston, tip, and tilt from hardware misalignments. The phase correction algorithm works better with higher photo-electron counts, which can come from stronger intensities or longer exposure times. Longer exposures will work as

long as the exposure time is still less than the Greenwood time of the atmosphere. Blind deconvolution is better at lower photo-electron counts, but multiple frames and an iterative estimation method raises processing times, which is not a problem for the phase correction algorithm.

7.2 Possible Field Examples

The original results in CHAPTER 6 and the blind deconvolution comparison in Section 7.1 both show that pupil remapping and the phase correction algorithm developed here can work well for atmospheric perturbation corrections at mid to high photo-electron counts when the Fried parameter is approximately the size of a single aperture or higher. They also work well for cases in which piston, tip, and tilt phase errors from aperture misalignments are dominant. Anisoplanatism can affect the effectiveness the phase correction algorithm, but Section 6.6 demonstrated that anisoplanatic images can still be corrected as long as each correction is limited to the size of a single isoplanatic angle. These corrections work well for an SNR of 95 as long as the masking size for the isoplanatic angle contains at least 18 diffraction limited spots across for a single aperture. After phase corrections with six apertures, the final diffraction limited spot size will be six times smaller.

Two examples are shown of pupil remapping phase correction systems that could potentially be used in the field. Both are astronomical examples. The first includes atmospheric corrections and the second contains only phase errors from hardware misalignments. They both take into account the signal power from passive illumination that is needed for the phase correction algorithm to be successful.

The first example has a design similar to other large optical telescopes that are found on Earth and is placed at a high altitude of approximately 3000 m to decrease atmospheric effects. A typical atmospheric turbulence level is $C_n^2(0) = 1.7 \times 10^{-14}$ along with a Bufton wind model and a wavelength bandwidth in the mid infrared from 3.4 to 3.6 μm . At a wavelength of 3.5 μm , the Fried parameter is 2.28 m, the isoplanatic angle covers 0.100 mrad, and the Greenwood time is 0.214 seconds. The Fried parameter allows individual apertures to be 2.28 m in diameter. At this aperture size and wavelength, the diffraction limited spot size is 1.5 μrad and 67 diffraction limited spots can fit inside the isoplanatic patch. As long as the telescope received enough light from the astronomical objects it is observing, a phase corrected image should be possible. If six apertures are used, the final gain for the array will be 13.7 m.

A large number of meteoroids strike the Moon each day and studying the effects of these lunar impacts helps to gain greater insight into the make-up of the Moon and of the meteoroid environment around the Earth [37]. To measure the power reflected from the Moon to the Earth blackbody radiation shows that, 2.98 W/m^2 should reach the Moon from the Sun in the 3.4 μm to 3.6 μm bandwidth. A single pixel looks at $1,776.5 \text{ m}^2$ of the Moon's surface. Assuming that the Moon's albedo is 12%, a total of 635 W should be reflected from that area. Six 2.28 m apertures have a combined surface area of 24.6 m^2 . Assuming a system transmission of 85%, a 70% quantum efficiency for the camera, an atmospheric transmission of 90%, and also remembering that the moon is $3.8 \times 10^8 \text{ m}$ away from the Earth, approximately $9.0 \times 10^{-15} \text{ W}$, or 160,505 photo-electrons per second, should be incident on each pixel of the camera. Using an exposure time equal to the Greenwood time results in a mean of 34,403 photo-electrons detected per pixel per

exposure. To calculate the SNR of the image of a single star, use equation (69) in Section 4.4 and assume a negligible electron read noise. The isoplanatic corrections in Section 4.4 had 14 pixels per diffraction limited spot. Use equation (69) in Section 4.4 to calculate the SNR, assuming an electron readout noise of 25, and the SNR is calculated as 184. The results for isoplanatic angles one quarter of the FOV, spanning 18 diffraction limited spots, in Figure 48, shows that corrections work for a grid with a flat plate as long as the SNR is 95 or higher, so lunar imaging should be possible with this setup. After phase corrections, the imaging resolution on the surface of the moon should be 98.3 meters per diffraction limited spot for a 6-aperture array.

The second example may work with regard to telescopes put into orbit. Such telescopes do not have to contend with atmospheric turbulence. Multi-aperture arrays, however, may have difficulties with hardware alignments. For instance, two techniques that can be used to point and track with multi-aperture arrays include moving all of the apertures monolithically or pointing each of the apertures separately to focus on the same area. The latter is especially beneficial in the case of using multiple apertures to reduce the depth of the system or when the entrance aperture itself is somewhat sparse. The monolithic technique would allow the array to maintain equal OPDs between apertures, but tracking and pointing times would be slower and require more energy. Moving each aperture individually allows faster tracking, but may cause a loss of path matching between apertures. The phase correction algorithm would allow some of these path length errors to be corrected after the image is taken. Also, without atmosphere, aperture size will not be restricted by Fried parameter and illumination by atmospheric pass-band, making it easier to find astronomical targets that offer enough signal power support for phase corrections.

Further research and testing could result in more possible uses for this form of aperture phasing.

CHAPTER 8

EXPERIMENTAL DESIGN

This chapter focuses on the physical design requirements for successful aperture remapping in an optical setup in which the target is illuminated with a partially coherent light source. Temporal coherence is inversely related to the chromatic bandwidth of a light source. A nearly monochromatic source has a much longer coherence time and length than illumination with a broader bandwidth. Partially coherent illumination is defined here as having a much broader bandwidth than monochromatic light, but not to the extent of full temporal incoherence. Previous research has shown that piston phase errors can be corrected using a two-aperture array, a small illumination bandwidth, and an anamorphic pupil relay to separate the aperture fields [1]. At larger bandwidths, piston corrected images suffer from FDC. Section 3.4.2 proposes a method to mitigate the effects of FDC in the remapping of partially coherence images. In this method, blazed diffraction gratings are placed at an intermediate image plane and redirect each aperture field to a new location in the exit pupil. A broadband LED is used as the partially coherent illumination source. Once successful partially coherent phase corrections have been demonstrated for a two-aperture array, the technique should be expandable to multi-aperture arrays such as the six-aperture simulations used in CHAPTER 4, CHAPTER 5, and CHAPTER 6.

8.1 General Setup Design

The diagram for a previous remapping setup for a two-aperture system is displayed in Figure 53 [1], [13]. Light from the target enters the entrance pupil (EP) and continues on to a second set of lenses that are offset to direct both images to the image plane. The offset causes a separation in the aperture fields that leads to a virtual remapped exit pupil (XP) located between the two sets of lenses. This particular pupil remapping example uses a setup similar to a Galilean telescope, but other setups can also be effective.

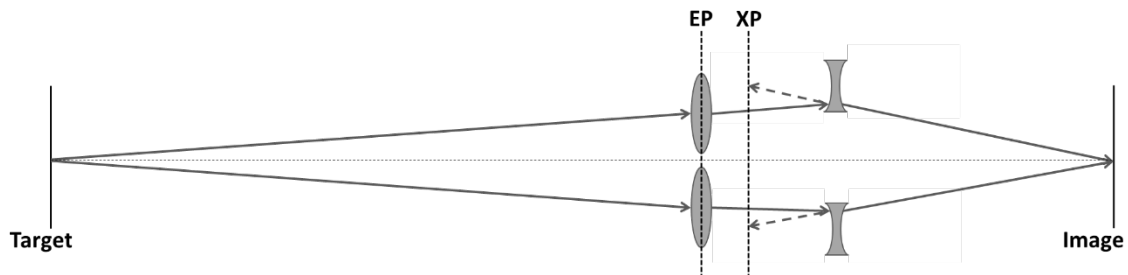


Figure 53: Diagram of anamorphic imaging system for aperture remapping [13]

The auto-correlation of the remapped exit pupil can be used to find the piston difference between the two apertures. As shown in Section 3.1, the auto-correlation of the pupil is equal to a scaled FT of the image intensity, which is also known as the frequency response of the system. In order to have access to the full frequency response the image is oversampled so that the highest frequency of the remapped optical system is less than the Nyquist frequency of the camera. While the setup shown in Figure 53 is Nyquist limited and was used for piston corrections, it still suffers from FDC as described in Section 3.4.1 since the pupil remapping is dependent upon wavelength.

A new pupil remapping design is proposed in Figure 54 that corrects FDC as discussed in Section 3.4.2. The angles in the figure are exaggerated for clarity and readability on the page. This design is similar to a Newtonian telescope in which light passes through a set of lenses at the EP, propagates to an intermediate image plane, then

continues on through a second lens that focuses it onto the final image plane. Reflective blazed diffraction gratings are located at the intermediate image plane. In order to avoid FDC, all remapping must be completed with the blazed gratings. In other words, replacing the gratings with mirrors should result in an imaging system with no pupil remapping. For this to occur, the grating should be positioned such that the blaze angle directs the light along its first order rays toward a second set of lenses, the remapped XP, and the image plane. Meanwhile, the zeroth order rays are directed so that the EP and non-remapped XP are proportional. This is represented by the dotted line in Figure 54. The blaze angle directs most of the power into the first order and the remapped XP, but replacing the gratings with mirrors would use only the zeroth order. Unlike Figure 53, the XP in Figure 54 is real. Imaging the EP through the second set of lenses flips each aperture field in the XP. The zeroth orders are directed to cross the optical axis before reaching the exit pupil in order to address this issue.

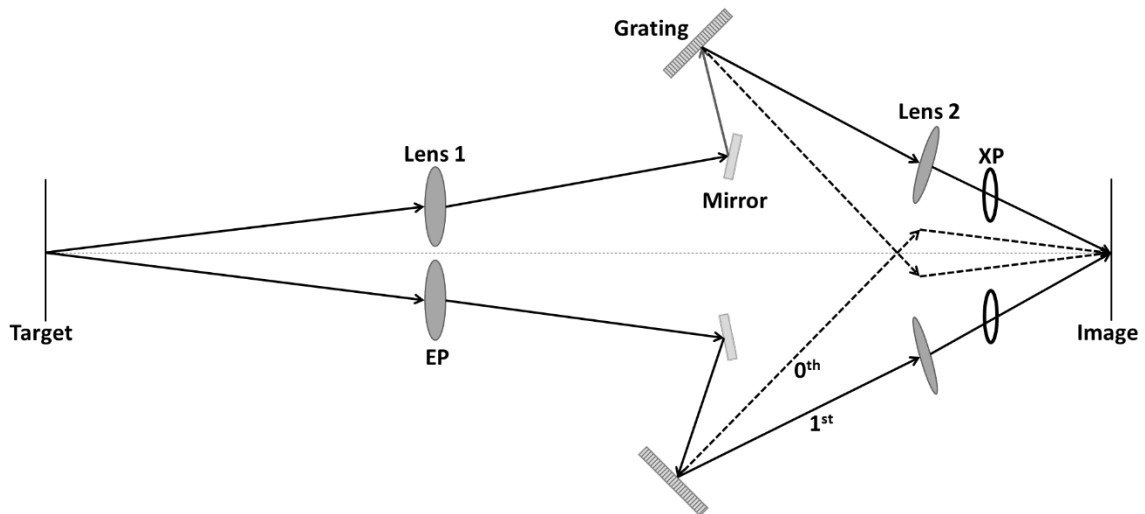


Figure 54: Pupil remapping diagram with diffraction gratings. The first order rays remap the XP while the zeroth order rays (dotted) lead to a proportional EP and XP

Placing the diffraction gratings at the intermediate image plane creates dispersion at the exit pupil but not at the final image plane, counteracting FDC. They could be placed

at points other than the intermediate image plane. However, this would require more than one grating for each aperture field in order to have the same effect. The use of only one diffractive component for each aperture beam leads to a simpler, more compact optical setup.

Ideally, the cutoff spatial frequency for all pupil remapping setups, including Figure 54, should be at or below the Nyquist frequency of the camera. However, there are limited circumstances in which this requirement can be relaxed, widening the range of optical components that will work in the lab. The FT of a PSF is the OTF of the system and the edges of the OTF are the cutoff spatial frequencies. Ideally, the remapped OTF of Figure 54 should appear as shown in Figure 55.a, where the spatial frequency cutoffs are within the Nyquist frequency of the camera. In this case, the camera has a pixel pitch of $12.5\text{ }\mu\text{m}$ and can show up to 40 line pairs per mm. This places rather stringent constraints on gratings that can be used for pupil remapping. Allowing gratings that remap to greater pupil separations increases the spatial frequency cutoffs beyond Nyquist in the direction of the gain. However, this frequency increase is only seen in one direction, so small increases can be accommodated by rotating the camera by a 45 degree angle so that the cutoff frequencies are shared equally between the x- and y- axes of the camera. This is shown in Figure 55.b. Increasing XP separation even more in Figure 55.c causes the OTF to wrap past the camera's Nyquist frequency. Fortunately, pupil sparsity now begins to play a role. A sparse aperture results in a sparse OTF. As long as the OTF wraps into locations that do not contain part of the spatial frequency spectrum, these frequencies can still be isolated, registered, and used to calculate piston, tip, and tilt phase errors.

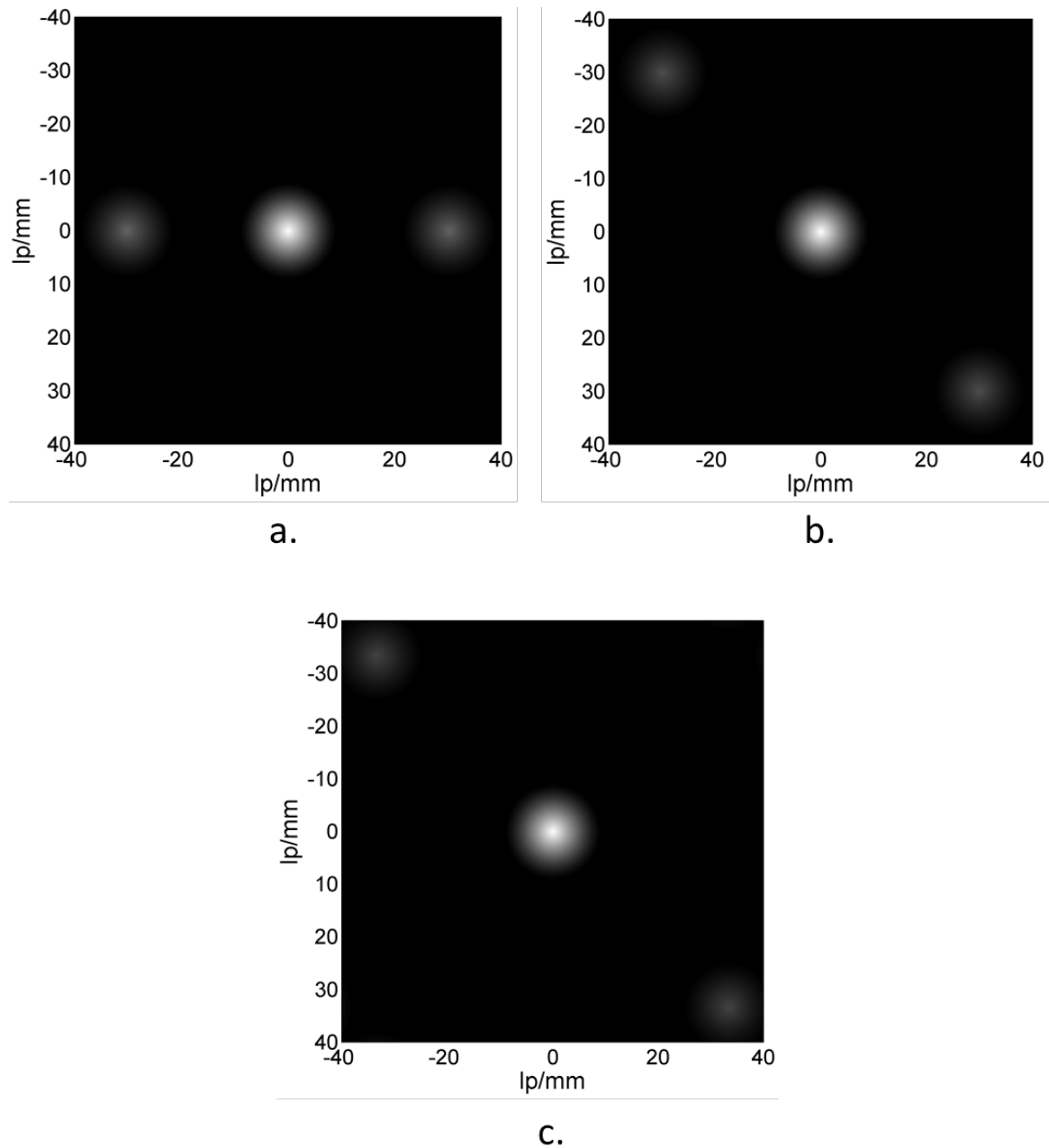


Figure 55: a. Remapped OTF of Figure 54 with spatial frequency cutoffs below Nyquist in all directions. b. Further remapped OTF where camera is rotated so that the x and y cutoff frequencies are below Nyquist. c. Further remapped OTF where cutoffs wrap beyond Nyquist, but do not overlap other frequencies

Both the frequency response of the final image with regard to the Nyquist frequency of the camera and the locations of the exit pupil caused by the zeroth and first orders from the diffraction gratings were used to design the experimental setup. Image magnification and vignetting also play a role in the design process.

8.2 Experimental Models

The setup discussed above was modelled analytically, with OSLO, and with CAD in order to find the correct measurements and components for the experimental setup. First, an analytical model was used to create a paraxial approximation of the imaging system. Second, these paraxial values were input into OSLO in order to adjust for angles wider than the paraxial approximations and optimize the wavefront errors of the system. Third, a CAD model was created in order to accurately visualize the physical setup and make sure that all optical components and mounts fit in the space provided.

8.2.1 Paraxial Models

The paraxial model primarily focuses on choosing the optical components and finding their location constraints caused by the Nyquist frequency of the camera and the blaze angles of the diffraction gratings. Ideally, the phase correction algorithms can be implemented as long as the resolution of the camera is greater than the resolution of the remapped imaging system. This can be easily measured by looking at the numerical apertures (NA) of the camera and imaging system since a higher NA corresponds to a higher resolution.

The definition for the NA of a lens is often approximated as

$$NA = \frac{nD}{2f} \quad (99)$$

where $n = 1$ is the index of refraction in air, D is the diameter, and f is the focal length. To adjust this to the NA of the imaging system the remapped pupil diameter, D_{xp} and the distance from the pupil to the image plane, d_i , are used.

$$NA_{sys} = \frac{D_{xp}}{2d_i} \quad (100)$$

This is illustrated in Figure 56.

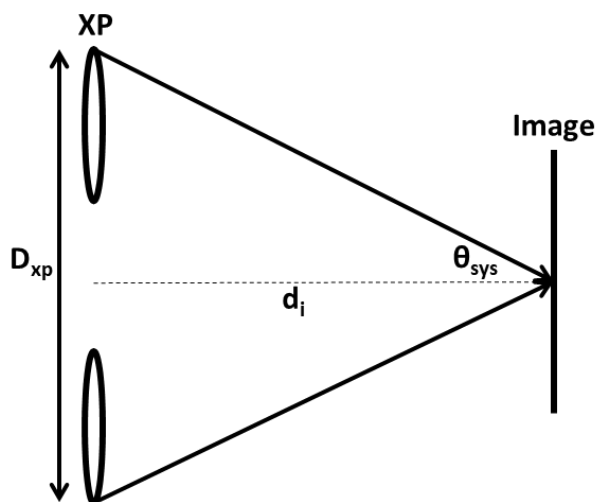


Figure 56: Numerical aperture of XP

The NA of the camera can be found by starting with the smallest resolvable distance on the focal plane array, which is twice the pixel pitch, $2D_{pix}$. Similar to the math for the diffraction of a single slit, the resolution length can be calculated with the equation,

$$2D_{pix} = \frac{f\lambda}{D} \quad (101)$$

Rearranging equation (101) to give D/f and inserting the result into equation (99) gives the NA of the camera,

$$NA_{cam} = \frac{\lambda}{4D_{pix}} \quad (102)$$

Therefore, in the ideal case the imaging system is Nyquist limited if equation (102) is greater than equation (100). Figure 55.a in Section 8.1 shows an example of a Nyquist limited OTF for a remapped system. As shown in Figure 55.b, rotating the camera or pupil by 45° allows a slightly larger frequency range. At this spatial frequency range the adjusted NA for the camera, NA^* , is,

$$NA_{cam}^* = \frac{\lambda\sqrt{2}}{4D_{pix}} \quad (103)$$

And the imaging system will have a small enough resolution cutoff as long as,

$$\frac{\lambda\sqrt{2}}{4D_{pix}} > \frac{D_{xp}}{2d_i} \quad (104)$$

Since the pupil function is sparse, the pupil frequencies can be separated even further into the corners of the frequency range as seen in Figure 55.c allowing the inequality to be relaxed slightly if necessary. In order to solve the inequality in equation (104), either D_{xp} or d_i needs to be known. Both are dependent on the blaze angle, θ_{blz} of the diffraction gratings. For instance, D_{xp} is the diameter of the remapped pupil function and remapping is caused by the gratings. As discussed in Section 3.4.2, the angle between the zeroth and first orders of a diffraction grating is given by equation (47). In Littrow configuration, the blaze angle is defined as [33],

$$\theta_{blz} = \sin^{-1} \left[\frac{\lambda}{2d} \right] \quad (105)$$

Paraxially, the angle between the zeroth and first orders is approximately $2\theta_{blz}$. The gratings are at an intermediate image plane so the angle of pupil separation before Lens 2 (see Figure 54) is related to the angle of separation after Lens 2 by the angular magnification of the lens. The angular magnification of a lens is the reciprocal of its imaging magnification.

$$\frac{1}{M_2} 2\theta_{blz} = \frac{M_p a}{d_i} \quad (106)$$

The left side of equation (106) contains the separation before the lens, and the right side contains the separation after the lens and is simplified, paraxial form of equation (50). The imaging magnification of Lens 2 is M_2 , the full separation as measured between an original

and remapped aperture in entrance pupil coordinates is a , the pupil magnification of the system that changes a to exit pupil coordinates is M_p , and the distance from the exit pupil to the final image is d_i .

When designing a pupil remapping system, all four of the previously defined variables are dependent upon various factors including the focal lengths and diameters of the chosen lenses and the distances between optical components. Unless ordering custom, lens sizes and focal lengths are restricted to the parameters of off-the-shelf products so these parameters should be predefined at the beginning of the optical design process. The target distance, size, and the magnification needed for the image to fit on the camera's focal plane array can also be predefined. Using these values and the guidelines given in equations (104) and (106), the rest of the optical component distances can be calculated. Other tolerances, such as making sure that none of the components vignette or block the beams as they pass through the system also aid these calculations.

As an example, the aperture separation, a , is calculated using the ratio of the angle made by a single aperture, θ_{ap} , and the blaze angle. The single aperture is related to the diameter of Lens 1, D_1 the same way the blaze angle is related to half the aperture separation so,

$$\frac{\theta_{ap}}{\theta_{blz}} = \frac{D_1}{a/2} \quad (107)$$

The angle produced by an aperture can be defined as,

$$\theta_{ap} = \frac{D_1}{d_{02}} \quad (108)$$

where d_{02} is the distance from Lens 1 to the grating. Combining equations (107) and (108) and solving for a gives,

$$a = 2d_{02}\theta_{blz} \quad (109)$$

and d_{02} can be found using the thin lens equation,

$$\frac{1}{d_{02}} = \frac{1}{f_1} - \frac{1}{d_0} \quad (110)$$

where f_1 is the focal length of Lens 1 and d_0 is the distance from the target to Lens 1.

All of the tolerances, guidelines, and equations discussed above were used to create a paraxial model of a pupil remapping setup in Excel. The predefined values, such as the lens parameters and target distance, were empirically adjusted until a physically feasible model was found. Since many of the guidelines discussed above were tolerances and not equalities, different solutions are available for this experimental model. All of the distances defined using this method are illustrated in Figure 57. The distance from Lens 1 to the intermediate image is the sum of the distance from Lens 1 to the mirror and from the mirror to the grating.

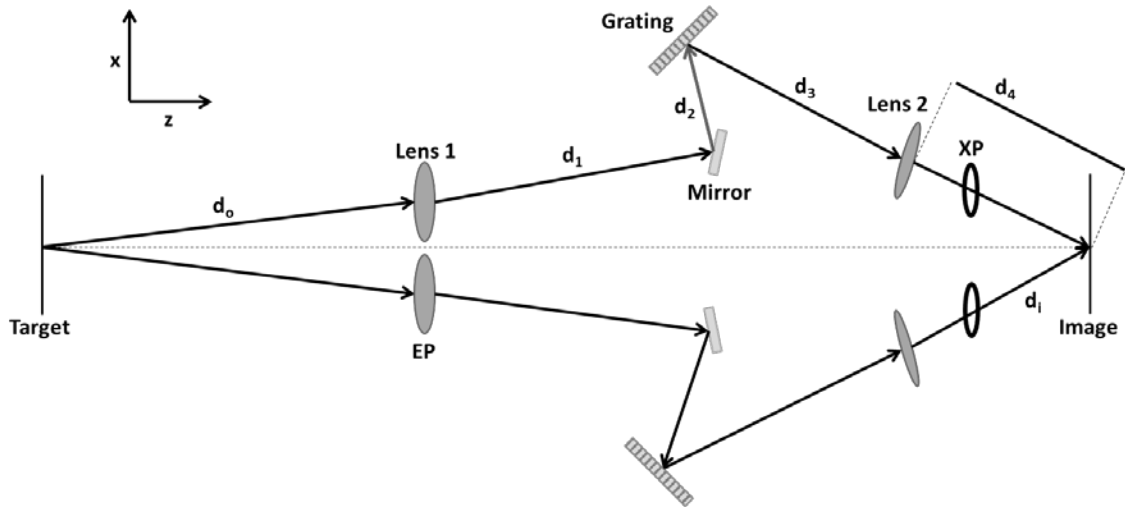


Figure 57: Pupil remapping diagram with distances between components.

8.2.2 OSLO Model

All of the values calculated using the paraxial, analytical model were input into OSLO. Before creating the entire setup, Lens 1 and Lens 2 were tested separately. The thin lens approximation was used previously, but the measurements given to OSLO included their thicknesses and wavelength adjusted focal lengths. The distance of the target from Lens 1 and of the intermediate image to Lens 2 were used. The distance from each lens to its image was optimized for the lowest wavefront error. After optimization, the two lens systems were put together and the distance from Lens 2 to the final image was optimized one more time. Figure 58 shows this optimized lens system along with the wavefront diagrams for the first lens, second lens, and full system. The peak to valley (P-V) OPD was 0.05788 waves for the first lens, 0.2749 waves for the second lens, and 0.1011 waves for the combined system.

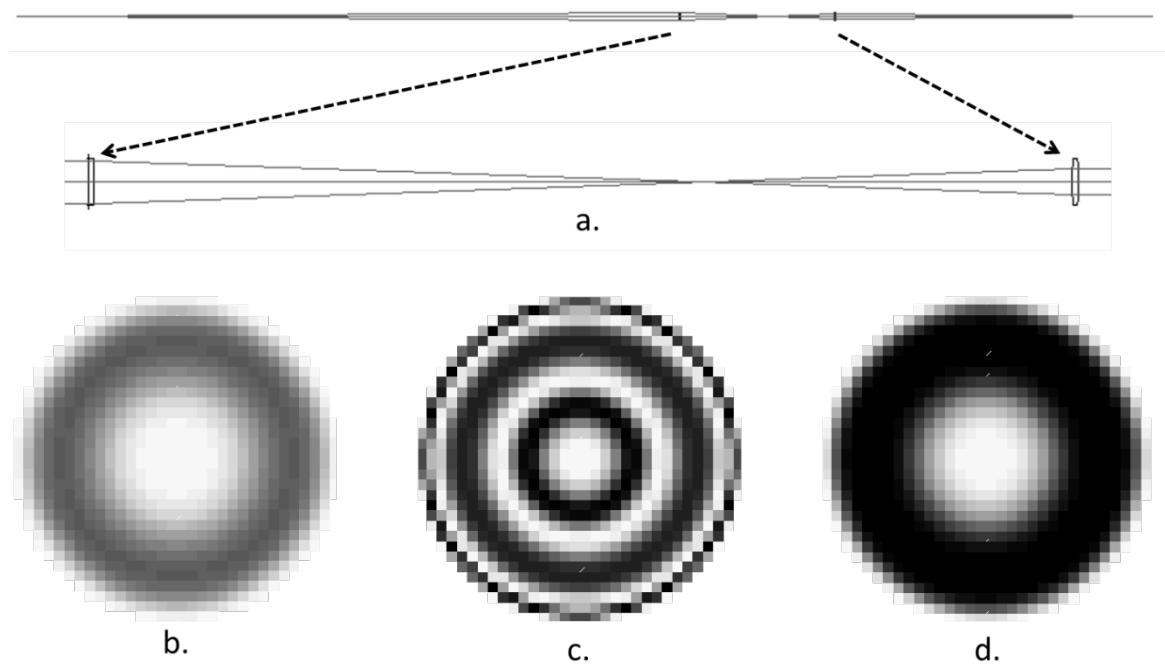


Figure 58: a. Lenses 1 and 2 with an optimized wavefront error. b. Spot diagram for first lens. c. Spot diagram for second lens. d. Spot diagram for the full system.

Once the images from Lenses 1 and 2 were optimized, the mirror and diffraction gratings were added to create one of the optical paths in the pupil remapping setup. Component angles and distances were input from the paraxial calculations using the global coordinate settings and multiple surfaces for some components. After inputting all of the paraxial measurements into OSLO, a few measurements were named as variables while the rest were set to remain constant in preparation for optimization. These variables were the angle of the mirror and the angle of the grating. In the event of errors found in the paraxial model, these variables can be changed in order to find new values using OSLO without having to completely recalculate the paraxial measurements. Next, operands, or the optimization guidelines, were defined. The central ray, the ray going from the center of the target to the center of the entrance pupil, was directed to hit the center of Lens 2 and zero at the final image plane. In addition, the angle at which the central ray hits the final image was directed to be the angle created by the height of Lens 2 and its distance from the final image. In this specific setup, that value was 0.034 radians. Optimizing the optical system caused its measurements to follow these operands as closely as possible.

After running the optimization, the locations of the exit pupil coming from the zeroth and first orders of the diffraction grating were calculated. The remapped exit pupil directed by the grating's first order was found using the current values of the system. The location of the unmoved exit pupil from the zeroth order was calculated by temporarily changing Lens 2 to an ideal lens. The difference between the locations of the remapped and non-remapped exit pupil very closely matches the analytical model. Ray fans were tested at various target heights and illumination wavelengths to check for vignetting and

the P-V OPD was found to be 0.1108 waves for one aperture path. The diagram of the wavefront that goes with this wavefront error is shown in Figure 59.

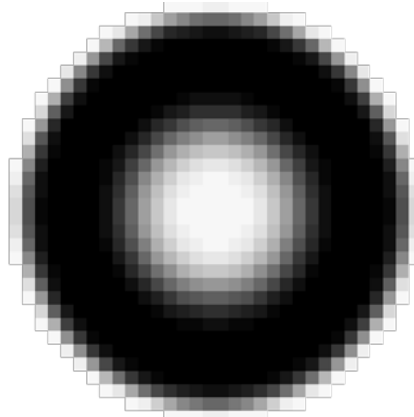


Figure 59: Wavefront diagram for optimized aperture remapping wave path.

The final component locations for one side of the lens setup are found in Table 9, where the optical axis at Lens 1 is the origin.

Table 9: OSLO values for all optical component locations.

Component	x axis (mm)	z axis (mm)	Tilt (degrees)
Target	0.00	-2039.02	0.00
Lens 1	15.25	0.00	0.43
Mirrors	16.95	227.59	-9.14
Gratings	38.76	163.22	-17.06
Lens 2	32.67	345.09	-1.92
Camera	0.00	1321.44	0.00

The specifications for the components themselves are given in Table 10.

Table 10: OSLE values for component specifications

Variable Set	Variable	Value (mm)
Lens 1	Diameter	25.4 (CLAP 23.8)
	Thickness	2.6
	Focal length	257.1
Lens 2	Diameter	25.4 (CLAP 23.8)
	Thickness	3.1

	Focal length	154.0
Mirror	Diameter	12.7
Grating	Blaze angle	6.2°
	Line spacing	150 lines/mm

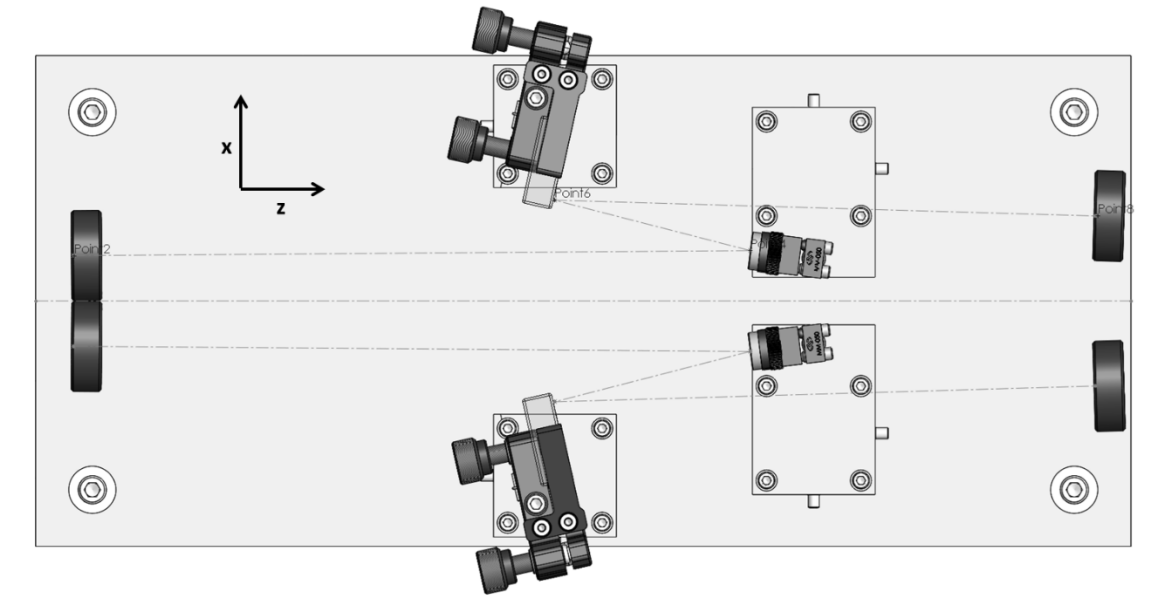
As a final test, all of these values were used to create a non-sequential OSLO model showing the paths of both of the apertures from the pupil. The images from both aperture paths were overlaid on the final image plane to compare aberrations and ensure that their images would interfere correctly. The final OSLO model is shown in Figure 60.



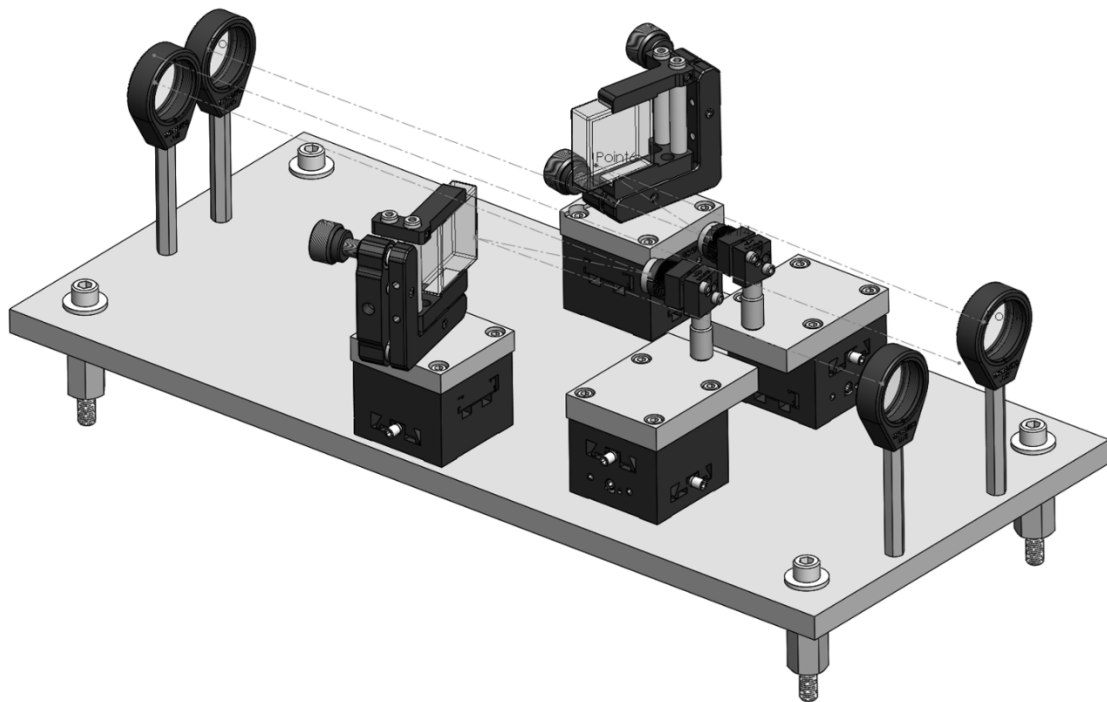
Figure 60: Final OSLO model

8.2.3 CAD Model

The information from Table 9 was given to Mr. Andrew Mason, who created a CAD model of the setup. This was used as a final check to make sure that the mounts and hardware worked with the model from OSLO without vignetting or otherwise blocking the beam path. Once these final checks were made, the necessary parts were ordered, custom base and adapter plates were fabricated, and the final setup was assembled. Top and side views of the CAD design are found in Figure 61.



a.



b.

Figure 61: CAD model a. Top b. Side

The angles and distances in this figure are slightly off due to some initial errors in the paraxial calculations. However, these values were fixed using OSLO and some small

hardware adjustments, so a new CAD design was not needed. The changes were small enough that a transparency of Figure 61.a was still used in the alignment process.

8.3 Setup and Alignment

The final optical setup after assembly is shown in Figure 62. The lines indicate the path that the central ray follows as it propagates through the system. In addition to the components pictured here, a rotating target mount for rotational image synthesis, a Sensors Unlimited GA1280JSX camera, and a baffle for the camera were also used in the setup. Baffles were also used as needed at Lens 1 and 2.

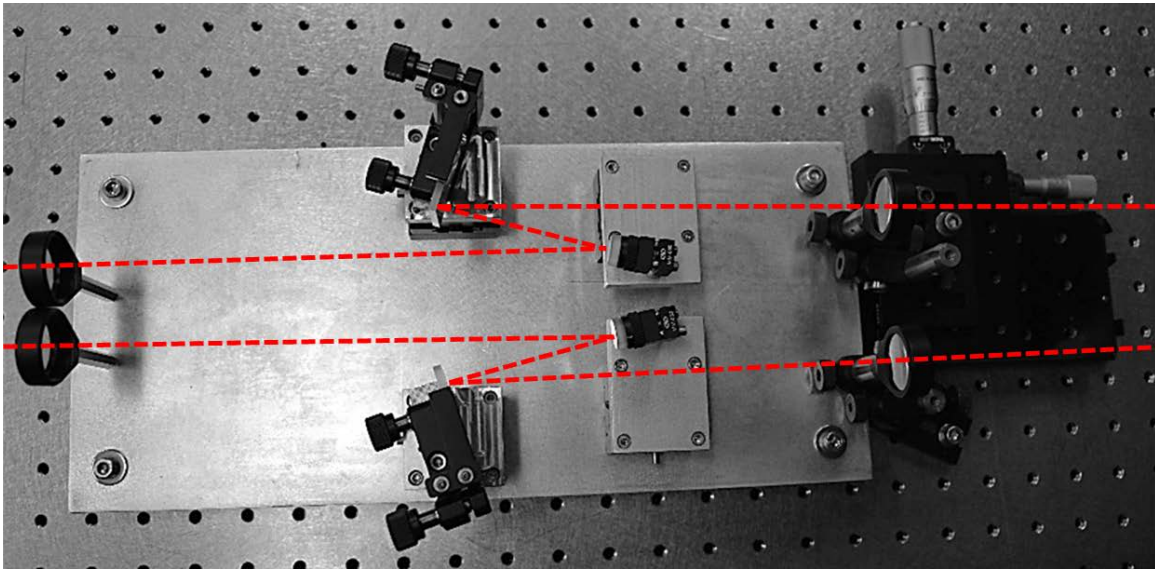


Figure 62: Optical setup for lab

8.3.1 Initial Alignment

As an initial check, a visible lamp provided illumination for the system to make sure that the center of the beam hit the correct places on the mirrors and diffraction gratings. Figure 61.a was printed as a transparency and overlaid with the lens setup, allowing for a more accurate beginning alignment. Then, the reflection angles of the blazed gratings were checked by directing a 1550 nm laser from the target plane through each of the two

apertures, ascertaining that the beams passed close to the center of Lens 2 and began to converge toward the image plane.

After this initial setup, a more accurate alignment was completed using a combination of targets, digital measurements, and observations of image sharpness and interference. While final illumination for the system came from an LED, a model 6300-LN (TLB-6328) New Focus Velocity Tunable Diode Laser was used during the alignment to provide a longer coherence length while still allowing for multiple wavelengths. The laser beam passed through a -9mm focal length lens before propagating approximately 40 cm to a rotating diffuser to eliminate speckle. The beam diameter of the laser was approximately 1 mm, so its diameter at the diffuser was 46 mm and the diffusion angle was 20 degrees. The target was placed about 18 cm from the diffuser. Based on the diameter and transparency of the target, a 50.8 mm diameter, 150mm focal length lens could be placed between the first lens and the diffuser to decrease the beam size at the diffuser and focus more light onto the target when necessary. An image of this setup is displayed in Figure 63. A mirror was used to save space along the edge of the optical table.

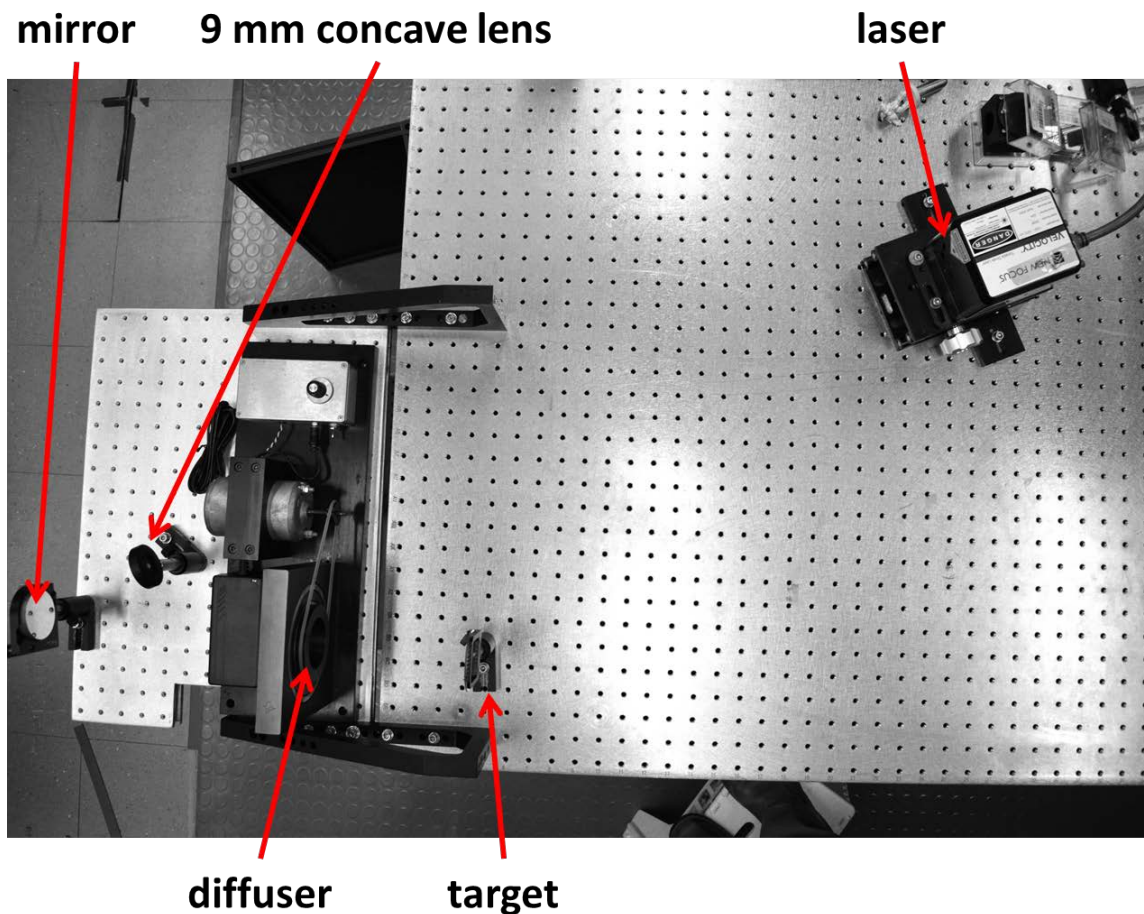


Figure 63: Laser illumination setup.

Labview was used for image acquisition. The code is designed to snap single images or display a stream of images. The FTs of the streamed images can also be displayed so that the spatial frequency spectrum of the image can be viewed in real time. The exposure time and frame rate can be specified and the histogram of the images can be controlled. Using the imaging capabilities of the Labview code, the lens setup was adjusted so that the images from the two beam paths interfered upon the focal plane array. A 1951 USAF resolution target was found to be the most useful target for image alignment.

The initial design of the experiment, as shown in Figure 61, allows for x, z, and angular adjustments of the mirrors and gratings. Only these variables were used in the

initial setup. However, further variability was useful for meeting the final alignment conditions. These conditions are the beam focus, exit pupil placement, and OPD. Ensuring that the beams pass near to the center of Lens 2 is also desirable since it helps to reduce aberrations in the system.

8.3.2 Focus Adjustments

Both aperture paths should be focused at the focal plane array. Defocus in one or both of the beam paths leads to decreased image interference and less power in the cross-correlations of the spatial frequency spectrum, which is needed for piston phase corrections. Initially, one of the two aperture paths was better focused and had greater throughput than the other. Throughput was equalized by correcting various asymmetries between the beam paths, such as the angle that the illumination from the target enters the aperture array and aberrations caused by small variations between optical component locations and orientations. The process of correcting the defocus helped to fix some of these aberrations and defocus was addressed by blocking the weaker beam and moving the camera back and forth along the beam path to find the best focus for the stronger image. Then the weaker beam path was adjusted to match the focus of the first. This was done by blocking the stronger beam path and moving Lens 2 back and forth along the optical path to change the focus distance to match the stronger beam path. Once the focuses for the two beam paths were in approximately the same plane, the mirror and lens angles were adjusted so that both images interfered on the camera. Figure 64 highlights the components that are most important for focusing the beams. A 1550nm point source placed at the target plane works best for these adjustments.

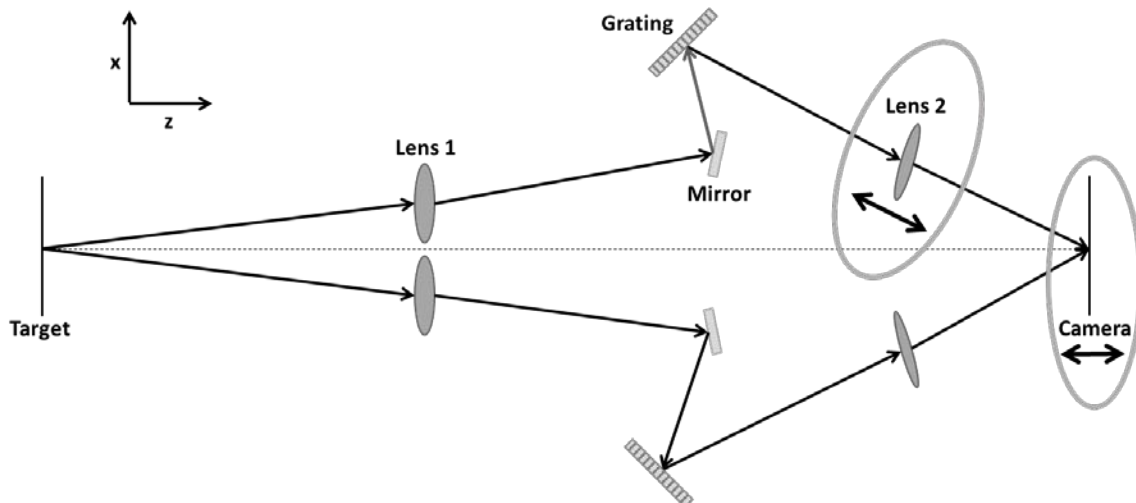


Figure 64: The camera location should be moved to find the focus for the stronger beam path. Lens 2 can be changed to adjust the focus for the weaker beam path.

8.3.3 Exit Pupil Adjustments

For the exit pupils, the blazed gratings direct most of the beam power into the first diffraction orders. The remapped exit pupil is created from the first order beams. However, the exit pupil's placement should be such that the grating's zeroth order is directed toward the unmapped aperture positions. This condition for the exit pupil ensures that all remapping is done by the gratings, allowing for a chromatically shifted exit pupil. If the angle of the zeroth order is off, the spatial frequency spectrum created by the remapped pupils will vary with wavelength and result in FDC, as described in Section 3.4.1. When using a Ronchi ruling as a target, the separated cross-correlations of the spatial frequency domain should contain discrete points at the Ronchi ruling's frequency in an ideally aligned system. If the zeroth orders are off, however, FDC will cause these points to become elongated with partially coherent illumination and the location of the Ronchi ruling's frequency will shift with the cross-correlations as the wavelength changes on a tunable laser.

The correct locations for the remapped pupils were found by adjusting the grating angles. The amount and direction that a Ronchi ruling's frequency moves with wavelength can be used to estimate how the gratings should be adjusted. However, many of the components in the optical setup are interconnected. After changing the angle of a grating, the angle of the beam with regard to Lens 2 changes and the location of the focused image is also affected. The mirror angle can be adjusted to counteract this and re-interfere the images at the focal plane array. Another way to fix this problem could be to iteratively correct both the pupil locations and the beam focus until both are correct. It was found that pupil location was relatively robust once the pupils were aligned correctly. Figure 65 illustrates the grating and mirror angles that can be used to adjust the exit pupil locations in the system.

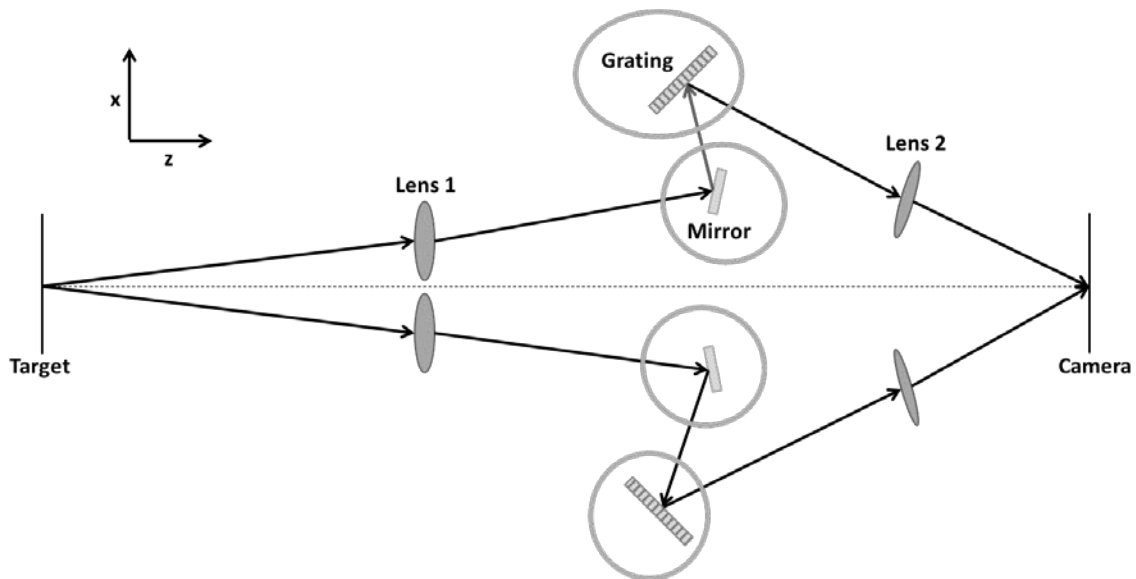


Figure 65: Adjust the grating angles to change the locations of the remapped exit pupils. Then adjust the mirror angles to counteract changes to the image location.

8.3.4 Optical Path Difference Adjustments

The OPD is the difference between the distances light travel for the two aperture beam paths. Phase errors can be calculated and corrected as long as the OPD is less than

the coherence length of the illumination. The coherence length of an LED with a 100 nm wavelength bandwidth from 1500nm to 1600nm, however, is very short. It can be calculated through the equation,

$$L_{coh} = \frac{\bar{\lambda}^2}{\pi\Delta\lambda} \quad (111)$$

Where $\Delta\lambda$ is the wavelength bandwidth and $\bar{\lambda}$ is the mean wavelength. The LED used in this experiment has a coherence length of 7.5 μm , so the OPD must be less than that value.

In order to find the current OPD of the system, images of a 4 line pairs per millimeter Ronchi ruling were taken at eleven wavelengths surrounding 1550 nm from 1545nm to 1555nm using a tunable laser. The coherence length of the laser is quite long so the phase correction algorithms are still effective when there is a large OPD. For the 4 line pairs per millimeter target, a single, discrete point is found within both the cross- and auto- correlations of the images' spatial frequency spectra. Registering these cross- and auto-correlations reveals how much the cross-correlations need to be shifted in order to correctly overlay them with the auto-correlation. Once the cross-correlations are shifted, the piston correction algorithm for two apertures can be used to find the pistons for each of the images [13]. A path length difference between apertures causes the light to interfere at slightly different phases and those phases change depending on the wavelength of the beam. Therefore, plotting the pistons as a function of wavelength reveals linear relationship with a slope that can be used to calculate the OPD. An example of this slope is shown in Figure 66.

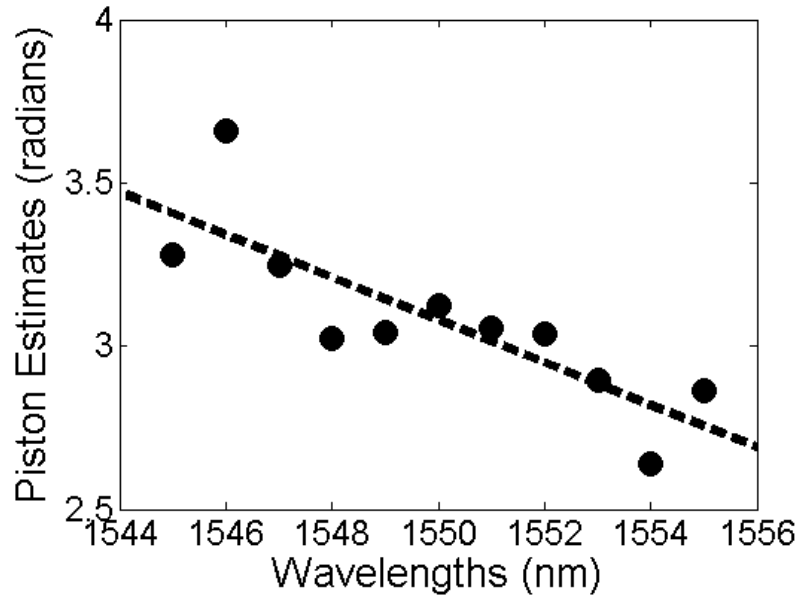


Figure 66: Piston slope as a function of wavelength

A full description of the relationship between OPD and illumination bandwidth can be found in Krug 2015 [1]. The equation relating OPD to the piston slope is,

$$\Delta\varphi = 2\pi OPD \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) \quad (112)$$

And the OPD can be found with,

$$OPD \approx \left| \frac{\bar{\lambda}^2 \Delta\varphi}{2\pi \Delta\lambda} \right| \quad (113)$$

where $\Delta\varphi$ is the piston change between wavelengths λ_1 and λ_2 . The slope from Figure 66 is -0.0653 radians per nm and the OPD is 2.513 μm .

The best way to correct OPD is to move the mirrors back and forth to change the length of each aperture path. One mirror is moved by a small amount, then the other mirror is moved to re-interfere the images on the focal plane array. Some iterative adjustments may need to be made along the way for focus and exit pupil location, but these adjustments

become negligible as the OPD decreases. Figure 67 highlights the motions of the mirrors that were used for OPD adjustments.

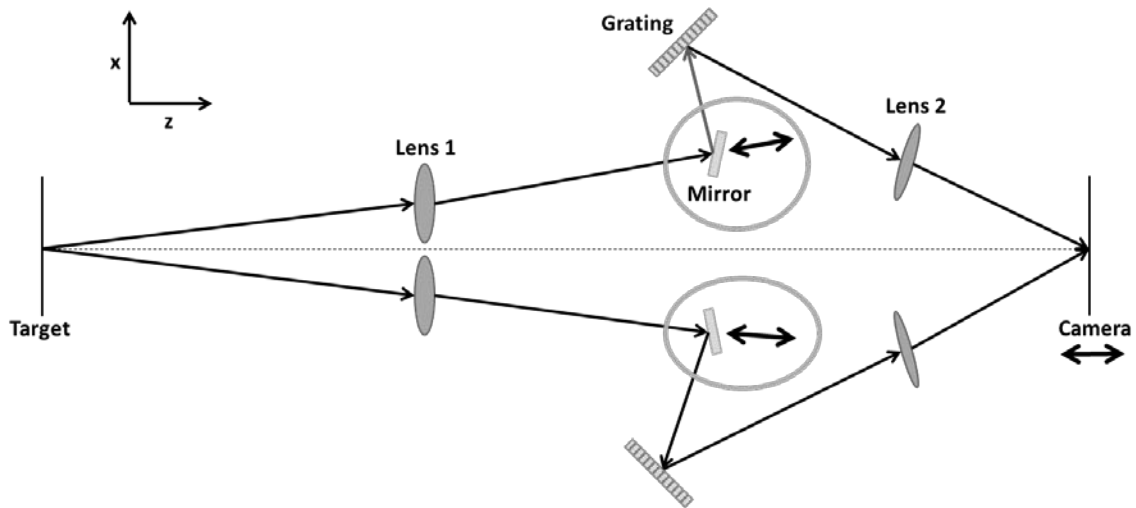


Figure 67: Mirror motions for OPD adjustments

The OPD was adjusted until the path difference was less than the coherence length of the LED. An iterative process containing the corrections for beam focus, exit pupil location, and OPD was used to align the optical system. Additional adjustments were also used to center beams through optics and mitigate vignetting. If the beam is off center at Lens 2, refraction affects the location of the exit pupil and this needs to be taken into account when adjusting beam focus and exit pupil location. Ultimately, final alignment results were a mix of trial and error and iterative adjustments.

The data from Figure 66 exhibits a fair amount of noise that can effect slope determination. The fold in the optical paths created between the reflective elements makes the system more susceptible to thermal drift, which can vary the separation distances between the mirrors and the gratings in an unequal fashion. This results in noise and a slope denoting a change in OPD that is visible over time. To display this, twelve images of the Ronchi ruling were taken at 1550nm, alternating with each of the eleven images from

1545nm to 1555nm. Each of the 1550nm images was taken about 60 seconds apart and both noise and a slope are visible in Figure 68.

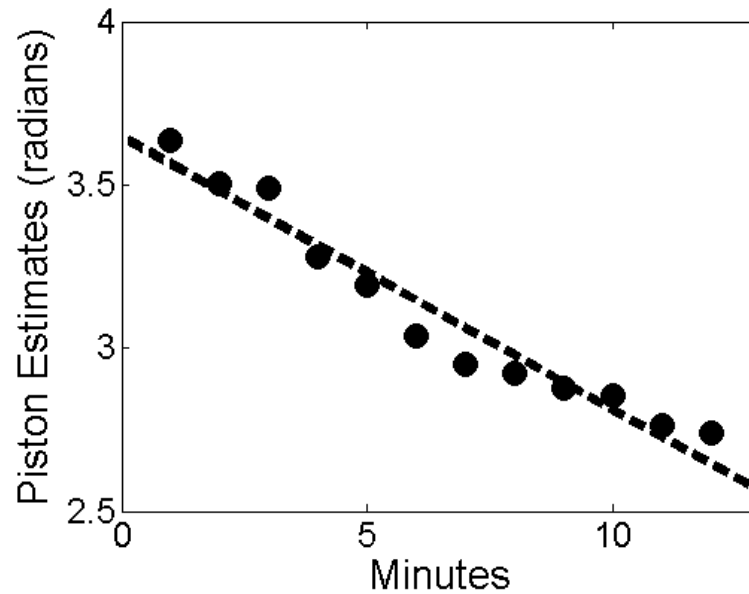


Figure 68: Thermal drift from lens system

Similar data was collected using the same laser in 2015 without so much drift or noise, so the possibility of frequency drift within the laser at these wavelengths is low [1]. To account for the slope from thermal drift, the drift slope was calculated using data like that from Figure 68 and subsequently subtracted from the OPD slope in order to find the true OPD. For example, Figure 68 shows a thermal drift of -0.0836 radians per minute during the collection of the data from Figure 66. Since the data in Figure 66 was collected at approximately one wavelength per minute, subtracting the thermal drift from the piston slope is 0.0183 radians per nm, or an OPD of 0.7 μm . Through multiple observations, it was found that for times of less than an hour while the system remained untouched, the effect that thermal drift had on OPD was generally within coherence length tolerances. However, larger effects were seen on the OPD from day to day, which were probably a

combination of thermal drift and component jitter, and led to the system being realigned each day before data collections. These re-alignments would have been much more time consuming, however, if the OPD had to be smaller than a single wavelength like the requirements for incoherent adaptive optics instead of smaller than only the bandwidth of the partially coherent LED. A final image of the aligned aperture setup is shown in Figure 69.

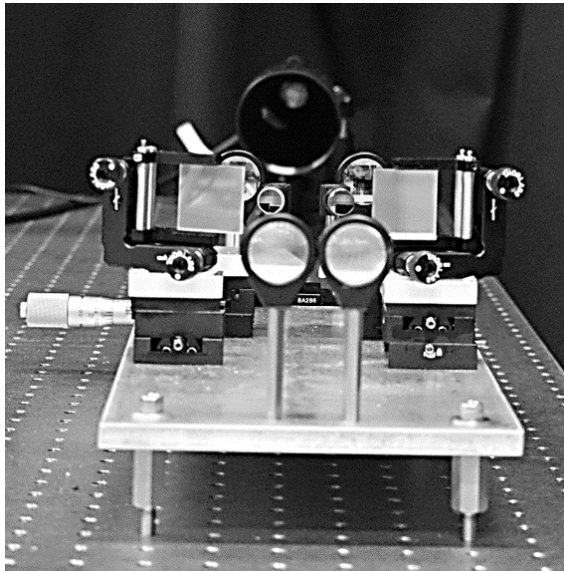


Figure 69: Imaging setup. This image was taken near the target plane, shows the two aperture remapping setup, and contains the baffle and camera in the background.

The image was taken near the target plane and the pupil remapping setup with two apertures, mirrors, blazed gratings and a second set of lenses is visible. The camera and its baffle are in the background.

8.3.5 LED Illumination and Final Data Collection

Once the pupil remapping setup was nearly aligned, a Thorlabs M1550L3 LED with power under 36mW and a FWHM bandwidth of 102 nm was used for illumination. An image of the LED illumination setup is shown in Figure 70.

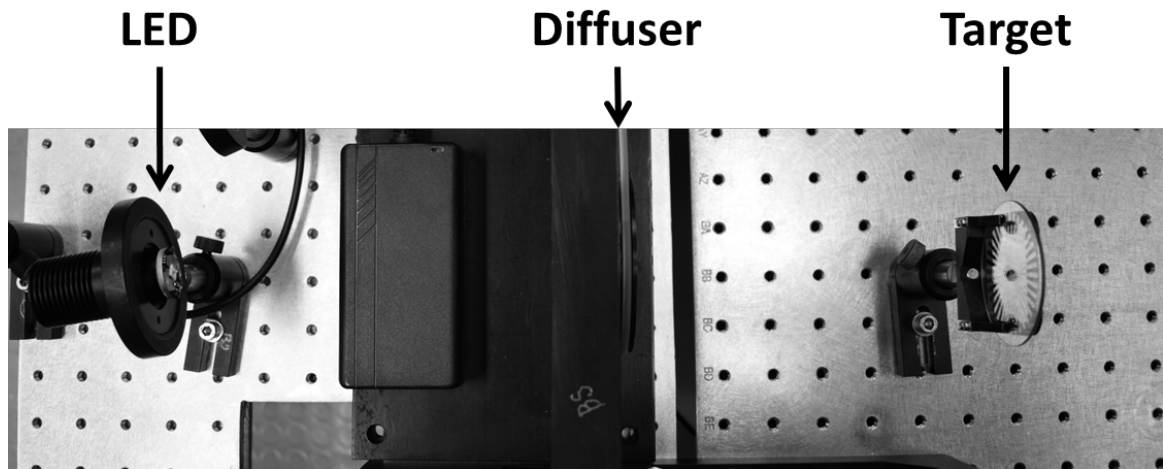


Figure 70: LED illumination setup. The light passes through the diffuser before reaching a transparent target.

Due to the spatially coherent nature of the LED, its beam diameter-divergence product at any singular point on the transparent target is too small to fully fill the aperture array at the entrance pupil. Therefore, the LED was placed behind the diffuser to greatly reduce the spatial coherence of the illumination. The diffuser did not have to rotate since the partial temporal coherence of the LED prevents speckle from being visible. Light passes through a transparent target and on through the imaging system. Non-specular reflective targets were considered, but the LED was not powerful enough to provide low noise images with this setup.

The OPD was aligned to within the LED's coherence length, $7.5 \mu\text{m}$. Figure 66 and Figure 68 show that the OPD at the beginning of the collection period was $0.7 \mu\text{m}$. Ten images were taken with LED illumination for each of the targets, which included Ronchi rulings at 4 and 6 lp/mm, a star target, and a 1951 USAF target rotated at 0 and 90 degrees for rotational image synthesis. After this data was collected, the OPD was tested again and had drifted to $5.6 \mu\text{m}$, which was still within the coherence length of the LED. Additional images were taken with the laser at 1550nm and 1521nm through 1571nm for

further testing. These included ten images of the 1951 USAF target at 0 and 90 degrees and ten images of the star target. Other images were also taken such as laser and LED flat fields, images of various targets using individual beam paths, and five images of the Ronchi rulings at 3, 4, 5, 6, 10, and 15 lp/mm with the laser set at 1550nm. Exposure time for the camera was set to 10ms, or 16.5ms for greater image brightness. A full chart of the collected images is shown in Table 11.

Table 11: List of collected images, along with their wavelengths and exposure times. The LED has a FWHM from 1500 nm to 1600 nm. Other listed wavelengths signify frames taken at regular wavelength intervals, not partially coherent illumination

Target	Wavelengths (nm)	Exposure Time	# of Frames
4 lp/mm Ronchi Ruling	LED	10 ms	10
6 lp/mm Ronchi Ruling	LED	10 ms	10
Star Target	LED	16.5 ms	20
1951 USAF. 0° rotation	LED	10, 16.5, 16.5 ms	20, 20, 20
1951 USAF, 90° rotation	LED	10, 10, 16.5 ms	20, 20, 20
1951 USAF. 0° rotation	1550, 1521-1571	16.5 ms	10,10
1951 USAF, 90° rotation	1550, 1521-1571	16.5 ms	10,10
Star Target	1550, 1521-1571	16.5 ms	10,10

Many of these images were used for piston correction in CHAPTER 9 to test partially coherent phase corrections, grating correction of FDC, and rotational image synthesis.

8.4 Design Difficulties

The experimental setup described in this chapter uses a two-aperture array. If the same pupil remapping technique is expanded to larger multi-aperture arrays, there are some potential problems that should be taken into account. Specifically, the possibilities of image rotation from the mirrors and distortion from the grating angle. These problems are relatively simple to solve in a two aperture system, but solutions become more difficult with three or more apertures.

Image rotation occurs because both the mirror and the diffraction grating in the current pupil remapping setup are reflective. If these two optical components reflect the

image outside of a single plane, the image will appear rotated. The setup displayed in Figure 54 and Figure 60 contains reflections that remain within the plane of the paper. On the other hand, if the mirror reflected that light out of the plane and the grating reflected it back into the plane of the paper, the image would be rotated by some degree. Images have to be rotated less than the resolution of the optical system at the edges of the camera for interference to be ideal. Multiple aperture paths with different rotations prevents image interference at the focal plane array. Separate aperture paths can be within separate planes, but each individual path should stay within its own plane. This restriction for image remapping increases the difficulty of designing a system in which mounts and other hardware do not block the beam paths. Custom optics and mounts can help with this, but at a higher price, more precise machining, and a potentially longer alignment time.

Image distortion is a product of the angular orientation of the diffraction grating. A pupil remapping system is designed so that replacing the grating with a mirror results in the remapped pupil (assuming that Lens 2 is large enough to intercept the zeroth order beam). For a blazed grating, the zeroth order reflects light at an angle equal and opposite to the angle of incidence, similar to a mirror, but the first order acts like a mirror tilted at a different angle. This does not impact small targets imaged on the grating or larger objects when the angle between the incident and the reflected angles is small, but it does effect extended images that are relayed by the grating at large angles.

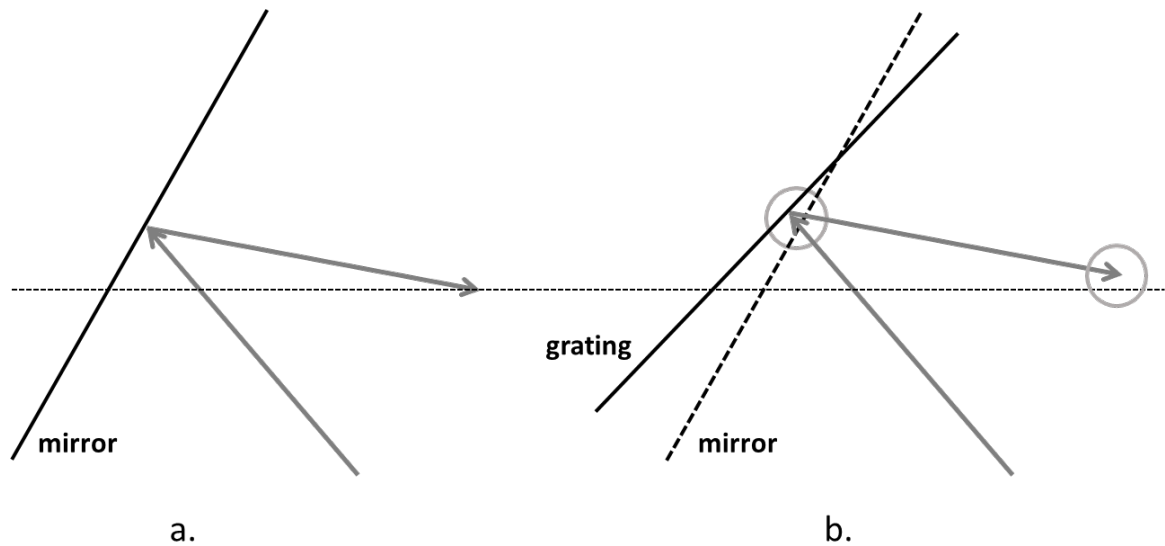


Figure 71: a. A mirror reflecting light. b. A blazed grating angled so that its first order is reflected at the same angle as the mirror.

Figure 71.a shows the path of the light if a mirror were used to direct the light toward the remapped pupil. Figure 71.b shows a blazed grating angled to direct light in the same direction as the mirror along the path of its first order. Notice that a ray that hits below the intersection of the mirror and grating travels slightly further than it would with a mirror. This small change in distance results in a small change in image height that propagates through Lens 2 to the final image. The opposite effect occurs for a ray that hits above the intersection.

This effect results in image distortion along the axis of grating separation. Two apertures remapped along the same axis by the same amount will have the same distortion and interference is not affected. Uneven remapping along one axis or remapping along multiple different axes will result in non-matching distortions that prevent ideal image interference. One way to mitigate this effect is to ensure targets are centered and small enough for the distortion to be negligible to the camera. Another option is to minimize the angle in the pupil remapping design at which the grating is used.

CHAPTER 9

EXPERIMENTAL RESULTS

Three topics are discussed in this chapter. The first is a comparison between the images collected by the optical setup that was introduced in CHAPTER 8 and the images that were expected based on simulations of the optical setup. The results of the comparison help to gauge how well the system is performing. The second shows that partially coherent phase corrections are possible with negligible FDC and the third is to demonstrate that phase corrected images can be rotationally synthesized with one another in order to create gain in more than one direction.

9.1 Two Aperture Simulation

A model of the two-aperture remapping setup was created to simulate the images collected by the system and their spatial frequency spectra in order to gauge the performance of the experimental setup. The entrance pupil in the lab consists of a two-aperture array where each aperture has a radius of 11.45mm and is located 15.25mm away from the central optical axis. Based on OSLO calculations, there should be a pupil magnification of -0.48. Multiplying the entrance pupil by this value scales the aperture fields to resemble a non-remapped version of the exit pupil, where each aperture has a radius of 5.48mm and location of 7.30mm. An image of the non-remapped pupil function is shown in Figure 72.

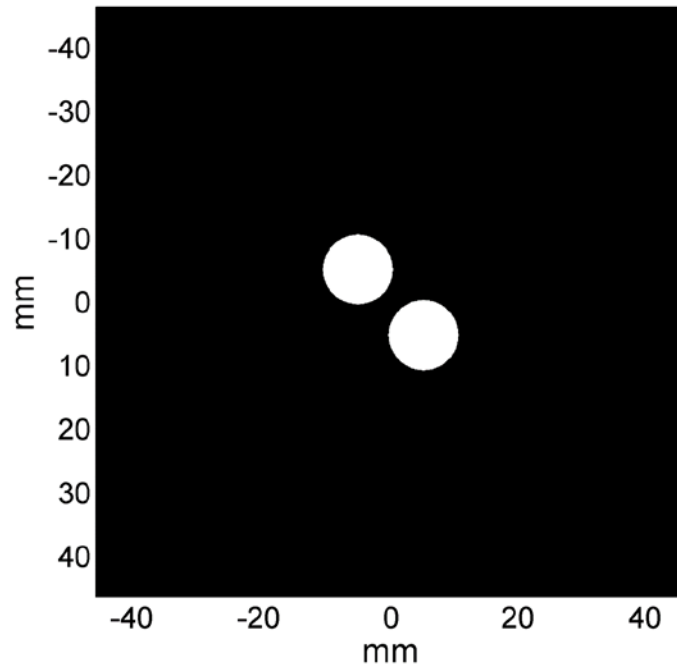


Figure 72: Original entrance pupil scaled to exit pupil coordinates.

To apply an extended target to the pupil function, multiply Figure 72 by the FT of the complex target field. As discussed in the previous chapter, the apertures are placed along the diagonal of the camera plane to permit greater spatial frequency cutoffs.

Section 3.1 showed that the frequency spectrum is a scaled auto-correlation of the complex field at the pupil. The scaled auto-correlation of Figure 72 is shown in Figure 73. It shows the continuous spatial frequency spectrum of the pupil function without remapping.

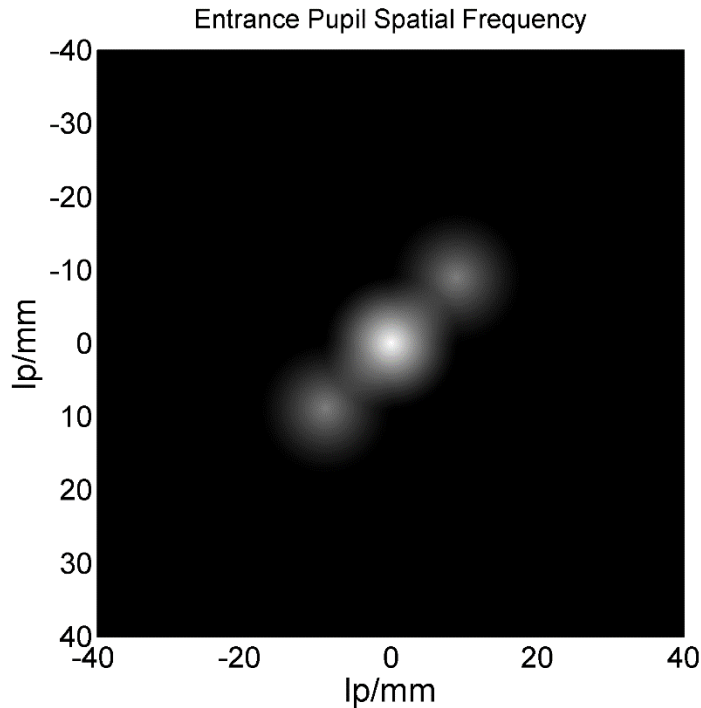


Figure 73: The auto-correlation of the original pupil function, scaled to the spatial frequency in the image domain after system magnification.

The spectrum of the remapped pupil function, however, is not continuous. To simulate this, each aperture in Figure 72 is masked and separately correlated with itself and the other aperture. This results in two aperture auto-correlations and two cross-correlations that are complex conjugates of one another. The auto-correlations are summed and the cross-correlations are able to be shifted to their remapped positions.

The amount the cross-correlations are shifted in the experimental setup can be found by observing the FT of a Ronchi ruling image taken in the lab. The spectrum of a 4 lp/mm Ronchi ruling is shown in Figure 74. Remember that the pupil magnification is negative, so the remapping shifts the apertures, and therefore the cross-correlations, across the optical axis. The Ronchi ruling's spatial frequency and its shifted counterpart in the cross-correlation are circled. The distance between these two spatial frequencies is the spatial frequency shift of the laboratory system.

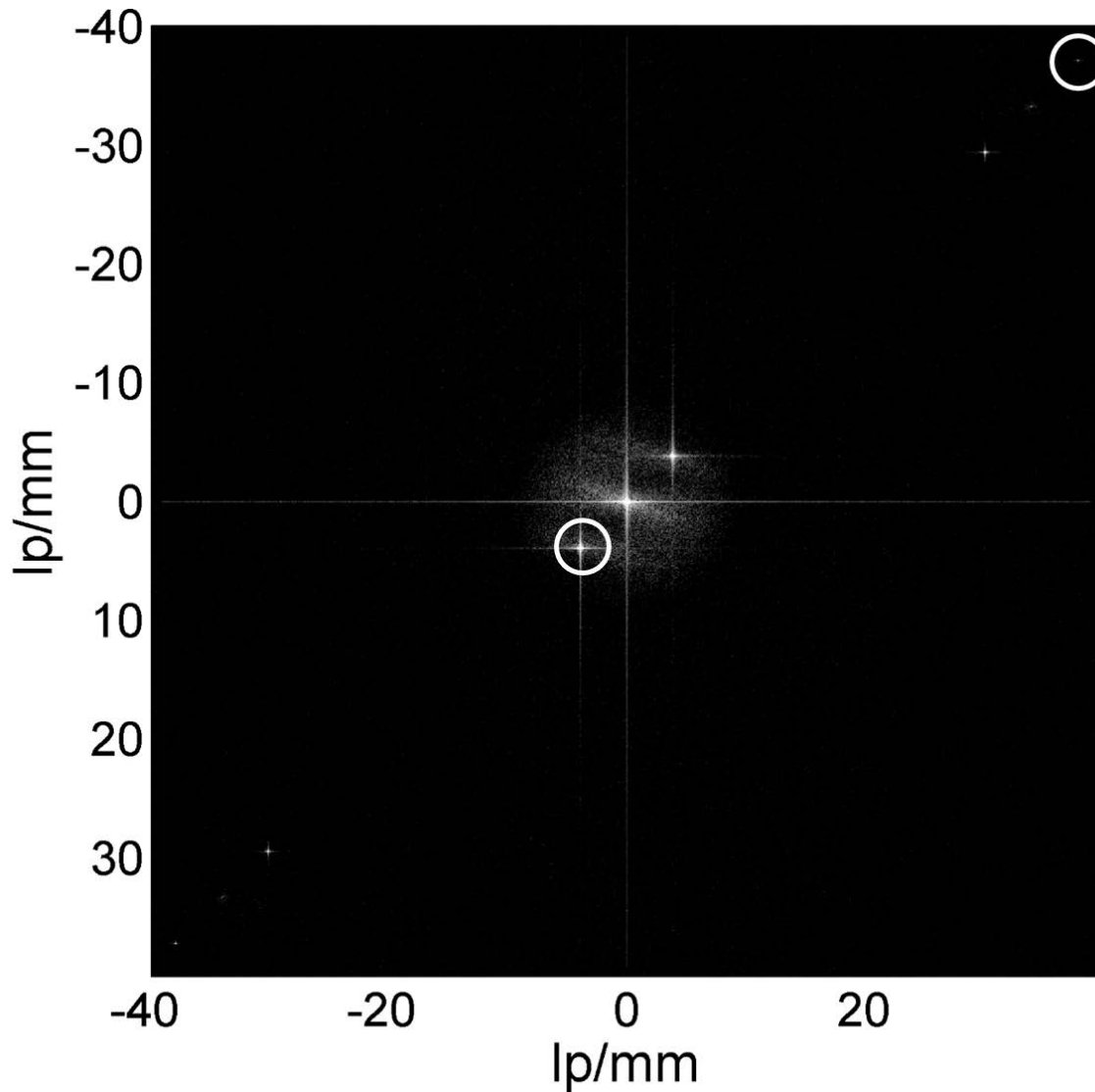


Figure 74. Spatial frequency spectrum of a 4 lp/mm Ronchi ruling image taken at a wavelength of 1550nm. The spatial frequency and its shifted counterpart are circled.

Since the maximum spatial frequency of the camera is 40 lp/mm, the step size in Figure 74, which is a 1024 by 1024 array, is,

$$\frac{2(40)}{1024} = 0.078125 \text{ lp/mm} \quad (114)$$

There is a separation of 750.98 pixels in the figure, which corresponds to a frequency shift of $f_a = 58.67$ lp/mm for the cross-correlations. From this value, the spatial separation in the exit pupil, a_{xp} , can be found through,

$$a_{xp} = \frac{z_{xp}\bar{\lambda}f_a}{2} = 34.01 \text{ mm} \quad (115)$$

After applying this shift and taking the sum of the auto- and cross-correlations, the separated spatial frequency spectrum is shown in Figure 75.

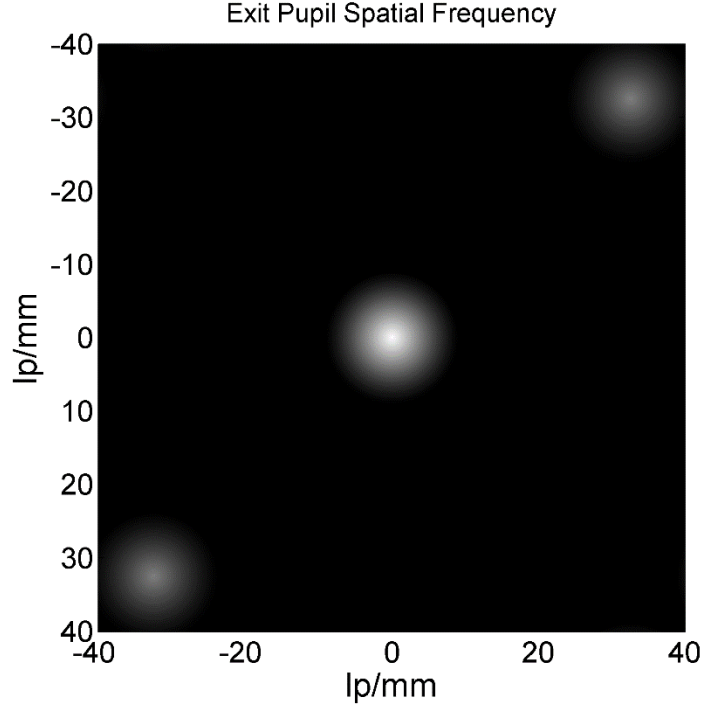


Figure 75: Spatial frequency spectrum of the system after pupil remapping.

Figure 75, however, only gives the spatial frequency spectrum of the remapping system but does not include the spectrum of the camera. The spatial frequency spectrum of the camera is controlled by its pixel depth. A single pixel in the camera is simulated with the function,

$$\text{rect}\left(\frac{x}{D_{pix}}, \frac{y}{D_{pix}}\right) \quad (116)$$

Taking the FT of equation (116) gives the spatial frequency spectrum of the pixel,

$$\text{sinc}(f_x D_{pix}, f_y D_{pix}) \quad (117)$$

This is shown in Figure 76.

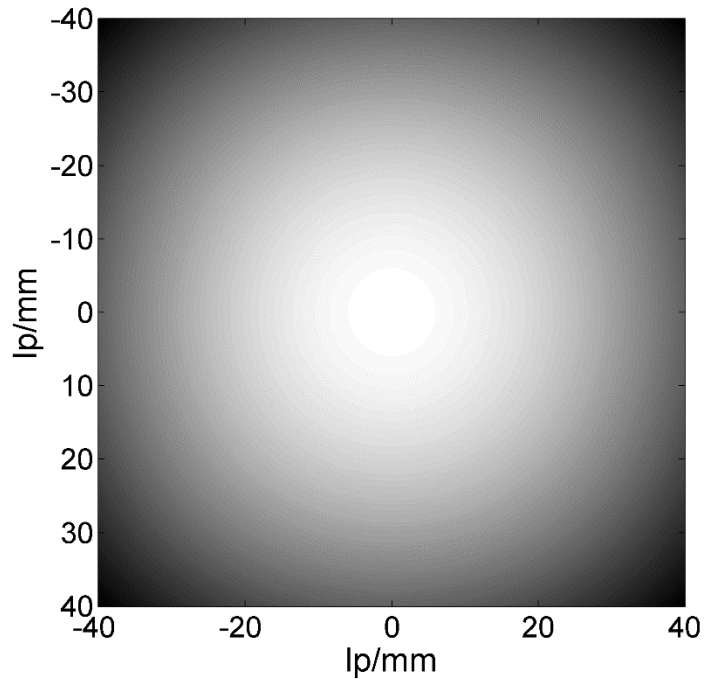


Figure 76: Spatial frequency spectrum of camera

Multiplying the frequency spectra of the system and the camera simulates what the spatial frequency spectrum should look like in the lab. Notice that the spectrum of the camera decreases toward the edges of the camera, which causes an intensity loss in the cross-correlations of the shifted frequency spectrum. The final simulated frequency response of the system is shown in Figure 77.

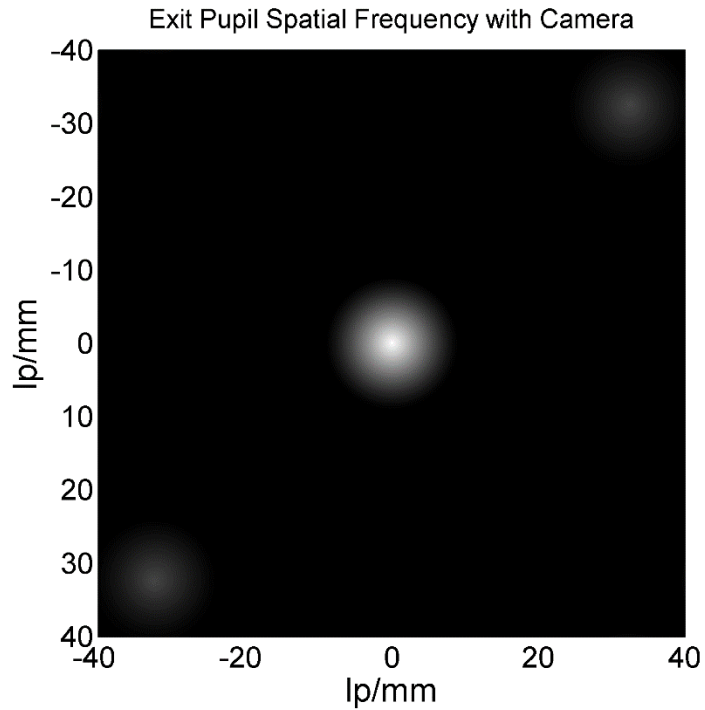


Figure 77: OTF of the system after taking the spatial frequency spectrum of the camera into account.

It is this frequency spectrum, with the decreased cross-correlation intensities, that is collected by the camera after pupil remapping. The cross-correlations are registered with the auto-correlation before the piston phase correction algorithm is applied. Figure 78 shows a spatial frequency spectrum after registration.

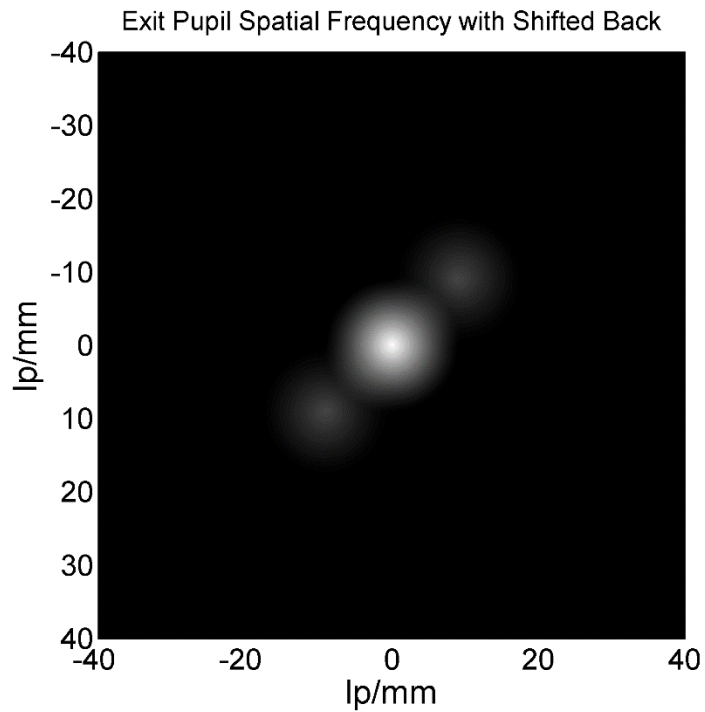
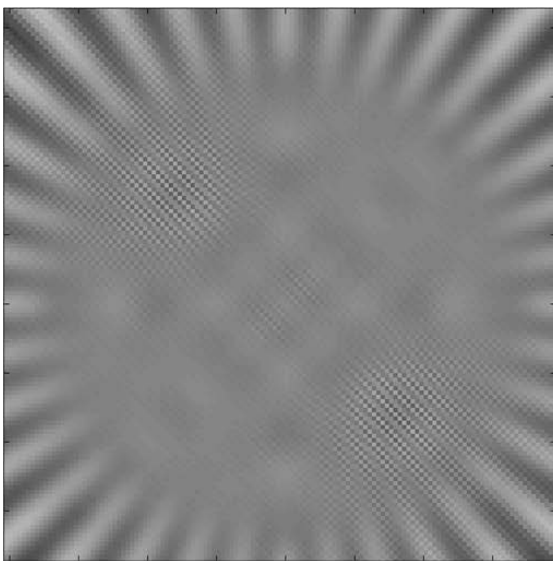
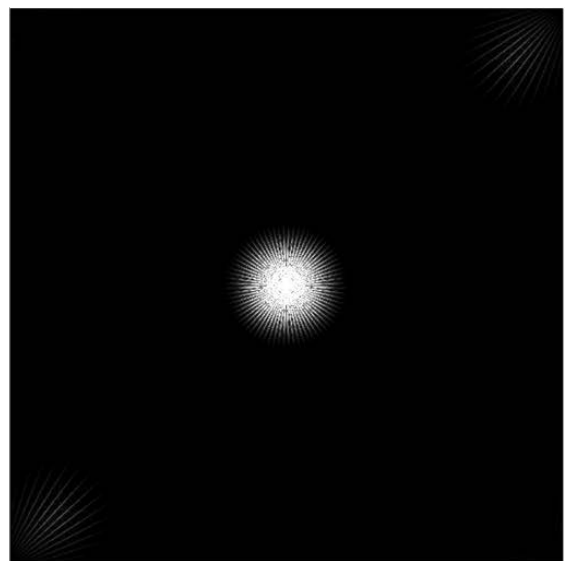


Figure 78: Product of system and camera spatial frequency spectra after registration.

A comparison of the spatial frequency spectra of simulated and lab images of a star target are shown in Figure 79. These images and spectra look relatively similar with noise causing the main difference.



a.



b.

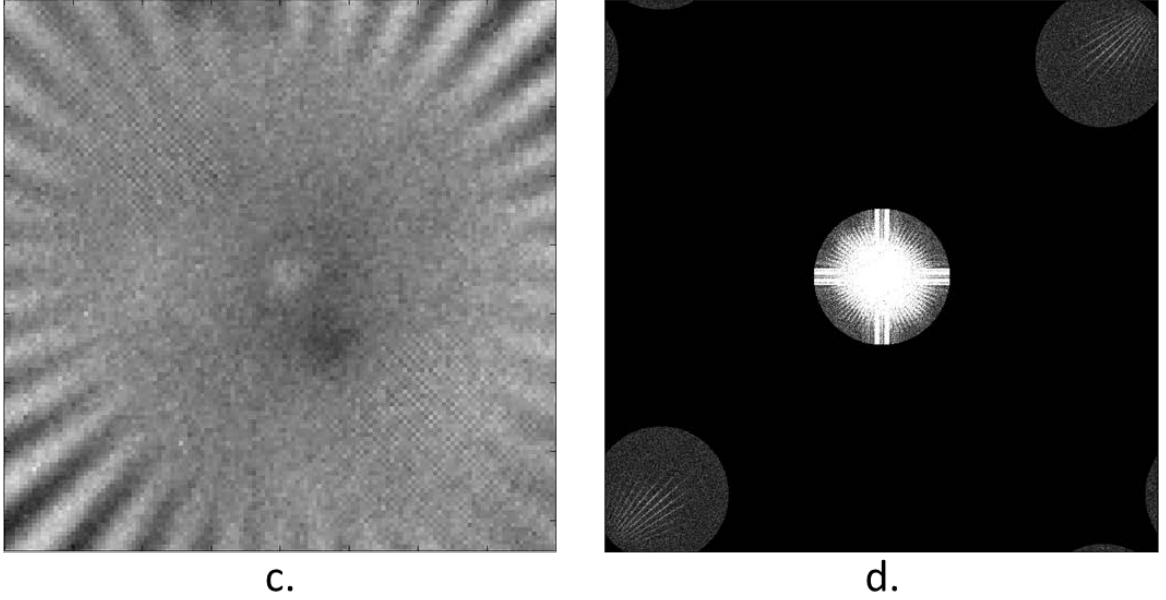


Figure 79: a. Simulated image of a star target after pupil remapping b. and its spatial frequency spectrum. c. Lab image of a star target d. and its spatial frequency spectrum. Only the centers of the star targets are shown.

As a more quantitative comparison, the ratios of specific spatial frequencies in the cross- and auto- correlations for the simulated and lab images are compared. The lab ratios were found by imaging Ronchi rulings of 3, 4, 5, and 6 lp/mm at a wavelength of 1550 nm. Their intensities in the auto- and cross-correlations were recorded and the values from the cross-correlations were divided by the values of the auto-correlations in order to find the ratios between the two for each spatial frequency value. The locations of the spatial frequencies from the Ronchi rulings in the image domain are slightly different from the target spatial frequencies due to the image magnification of the system, M_{img} . If a spatial frequency of a target is f_{targ} , then the corresponding spatial frequency in the image plane is,

$$f_{img} = \frac{f_{targ}}{M_{img}} \quad (118)$$

In the lab, a Ronchi ruling of 4 lp/mm has a spatial frequency of 5.49 lp/mm in the auto-correlation of its spatial frequency spectrum in the image domain. This means that the image magnification of the system for this data collection is $M_{img} = 0.729$. Using this value and the relationship in equation (118), the intensities at the correct locations of the simulated auto-correlation in Figure 77 can be recorded. Adding f_a to these frequencies gives their locations in the cross-correlation and these simulated intensities can be recorded as well. Figure 80 displays a graph comparing the cross- to auto-correlation intensity ratios for both the lab and simulation.

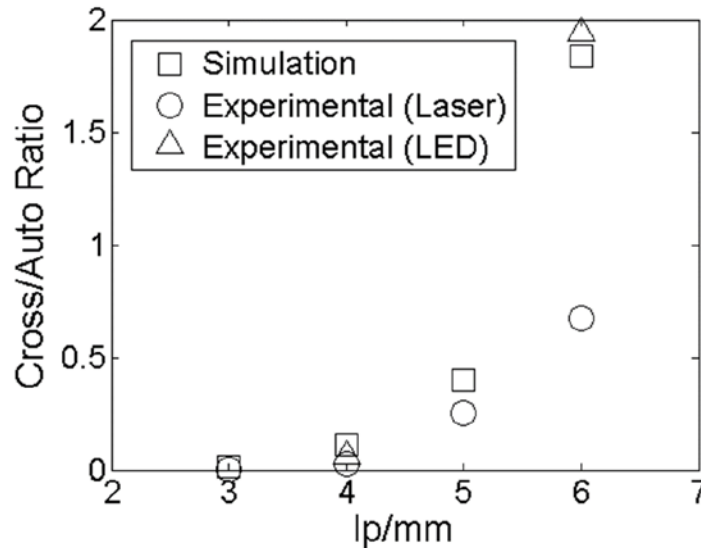


Figure 80: Graph displaying the cross- to auto-correlation intensity ratios of spatial frequencies in the auto- and cross-correlations for the lab and simulation. Images collected in the lab used both laser and LED illumination.

The values are relatively similar. However, aberrations, illumination uniformity, noise, and other differences between the simulation and the lab may explain the disparity seen in Figure 80. The system was realigned before each data collection in order to account for thermal drift. Each realignment resulted in a slightly different image magnification and even small changes in magnification resulted in significant changes in the ratio between

the cross- and auto- correlations. For instance, Ronchi rulings with 4 and 6 lp/mm were imaged using an LED on a different date and their ratios were 0.065 and 1.94, respectively. These values are much closer to the simulation values and are also given in Figure 80. The slight alignment differences between the two collections, as well as the change in light source, could be factors in these results.

9.2 Grating Correction of FDC

Section 3.4.1 explains that pupil remapping causes FDC when a constant separation of aperture fields in the pupil plane leads to a wavelength dependent shift in the spatial frequency domain. In response, Section 3.4.2 shows that using diffraction gratings for remapping creates a wavelength dependent separation of the aperture fields in the pupil plane, leaving the shift in the spatial frequency domain constant. In the experimental setup, the pupil separation at the central wavelength of 1550 nm was shown to be 34.01 mm in Section 9.1. If gratings were not used for pupil remapping, the shift of the cross-correlations in the spatial frequency domain could be calculated using,

$$\Delta f = \frac{2a_{xp}}{\lambda_2 d_i} - \frac{2a_{xp}}{\lambda_1 d_i} \quad (119)$$

where $\lambda_1 = 1600$ nm is the largest illumination wavelength and $\lambda_2 = 1500$ nm is the smallest. This is a shift of 3.79 lp/mm in the spatial frequency domain.

For a single frequency target, such as a Ronchi ruling, this would result in a rect() function with a width of 3.79 lp/mm in the direction of the shift. The FT of this function would define the intensity envelope of that frequency in the image plane, which is the FDC.

$$\text{rect}\left(\frac{f_x}{\Delta f_x}, \frac{f_y}{\Delta f_y}\right) \rightarrow \text{sinc}(x\Delta f_x, y\Delta f_y) \quad (120)$$

The subscripts of Δf represent the projections of the frequency shift along each axis. Since only the spatial frequency information in the cross-correlation is affected by wavelength, there is no shift in the auto-correlation. Therefore, the strength of the FDC envelope is also controlled by the ratio of the frequency strength in the auto- and cross-correlations. The final equation for the FDC of a single frequency is,

$$2I_{auto} + 2I_{cross}\text{sinc}(x\Delta f_x, y\Delta f_y) \quad (121)$$

where I_{auto} and I_{cross} are the intensities of a spatial frequency in the auto- and cross-correlations respectively. This is a simplification of equation (44) from Section 3.4.1. Figure 81 displays what the FDC envelope for a 6 lp/mm Ronchi ruling would look like if pupil remapping were done with a mirror instead of a diffraction grating at the intermediate pupil plane.

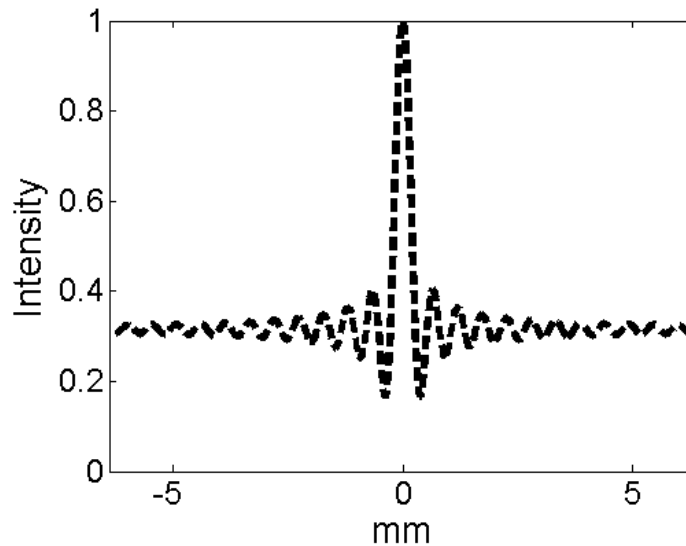


Figure 81: Simulated envelope of the FDC of a 6 lp/mm Ronchi ruling for the current lab setup with mirrors replacing diffraction gratings

In previous studies, FDC's similar to the one shown in Figure 81 were visible in images of Ronchi rulings after the cross-correlations were registered and shifted back to overlap the

auto-correlations. One such image and its simulated counterpart are shown in Figure 82. The image contains 50 incoherently summed frames with laser illuminations between 1521 and 1571 nm and is filtered to minimize artifacts from uneven illumination. The filter included two circular masks, with radii of 5 pixels, placed at the 6 lp/mm frequency locations in the spatial frequency domain along with a single pixel at the central DC value. The filter masks were large enough to accommodate the spread of spatial frequency content that causes FDC. [1]

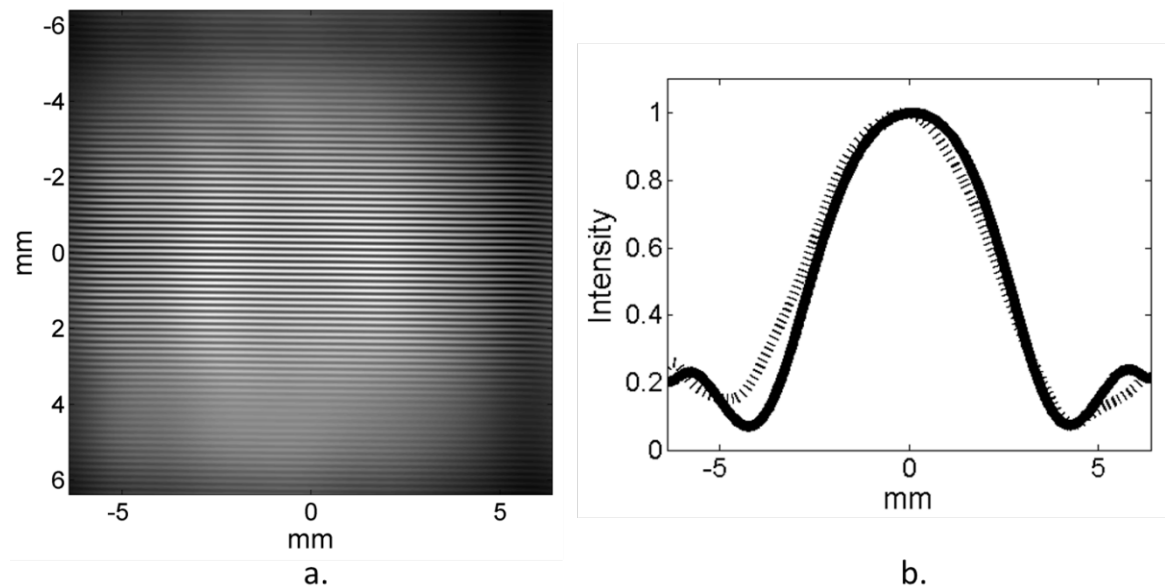


Figure 82: a. Filtered 6 lp/mm Ronchi ruling image and b. the corresponding simulated FDC. The solid line is the simulation and the dotted line is the experimental FDC. [1]

In contrast, FDC does not appear to be visible at all in the registered data from the current setup. An image with equivalent illumination filtering is shown in Figure 83.

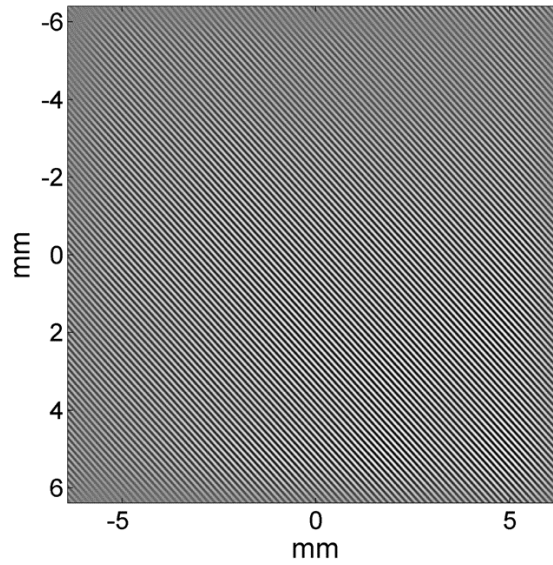


Figure 83: Filtered 6 lp/mm Ronchi ruling image from the current setup. The target was illuminated with an LED with a FWHM from 1500 to 1600 nm.

This is also visible in the spatial frequency domain. Figure 84 displays the FT of the Ronchi ruling before image registration. No obvious shifts can be seen in the cross-correlations.

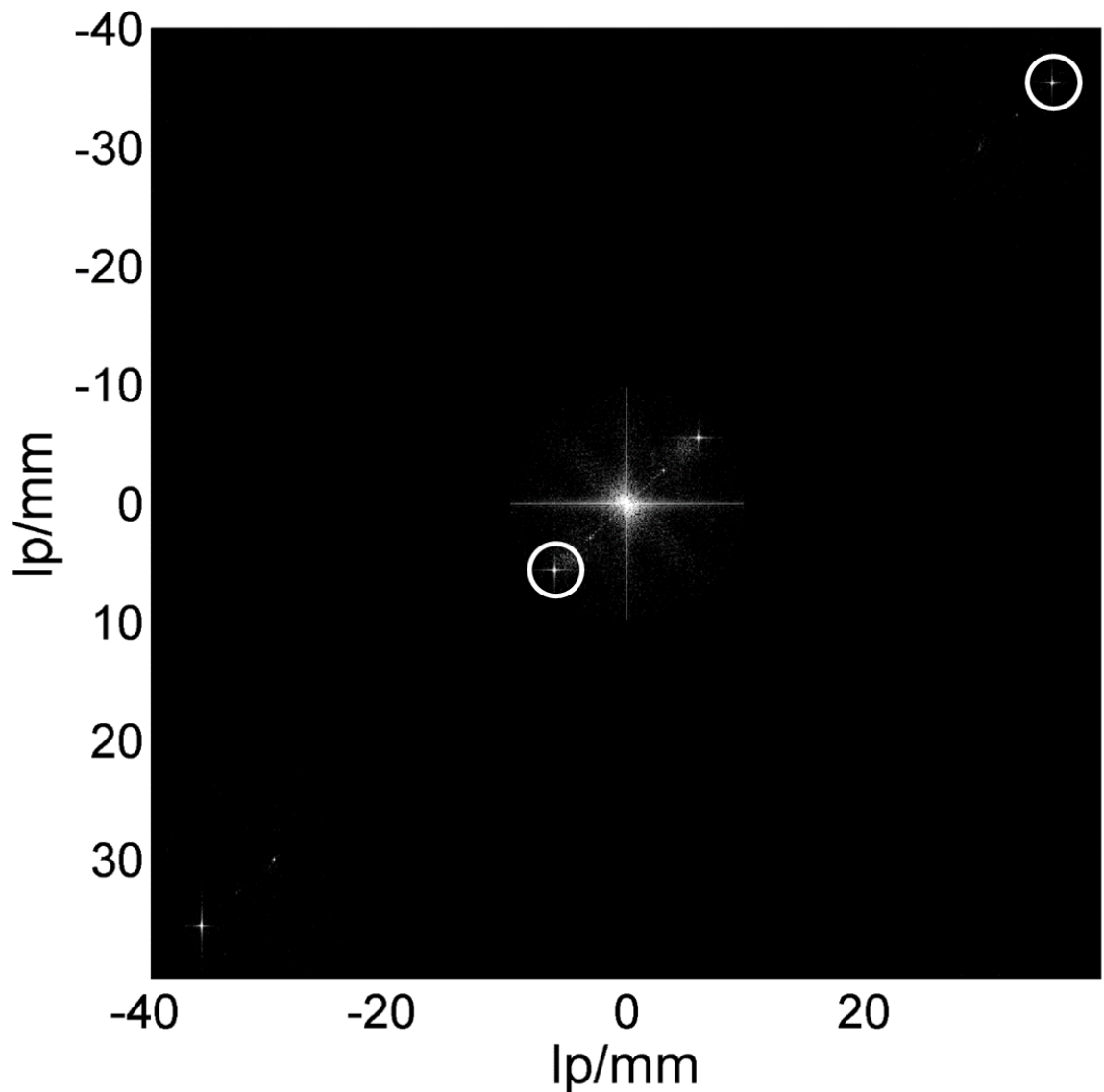


Figure 84: FT of the 6 lp/mm Ronchi ruling before registration, displaying the separated auto- and cross-correlations.

This remains the case when a closer look is taken at the 6 lp/mm point in the cross-correlation in Figure 85. Figure 85.a shows 6 lp/mm as collected with single wavelength laser illumination and Figure 85.b shows the same point imaged with the LED. The spatial frequency in the cross-correlation taken with the LED shows a minute elongation, but nowhere near the length of 48 pixels.

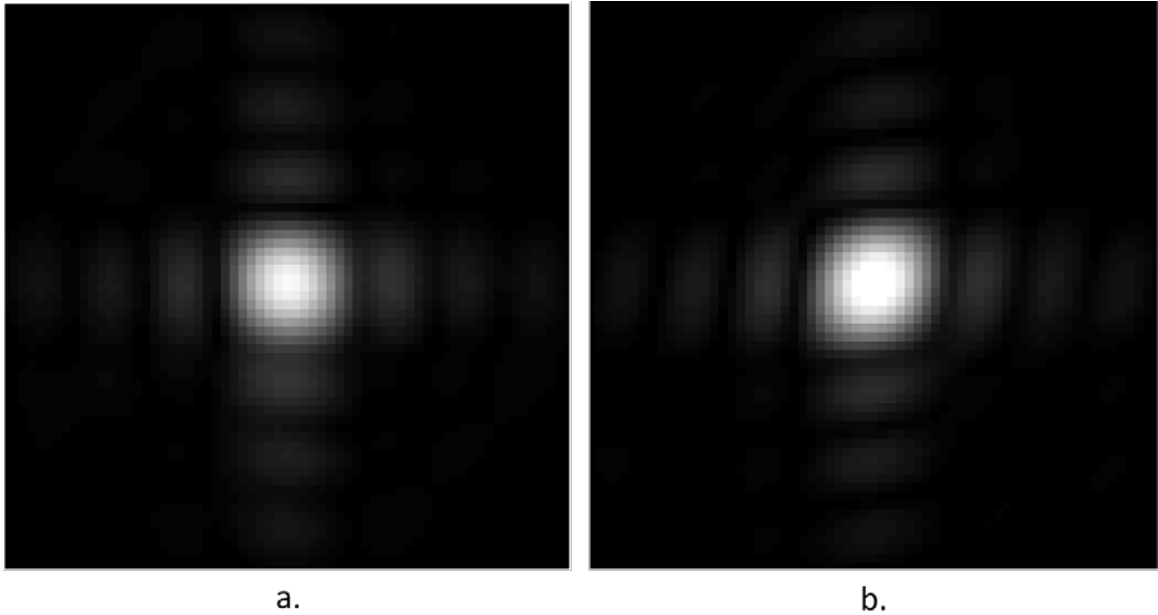


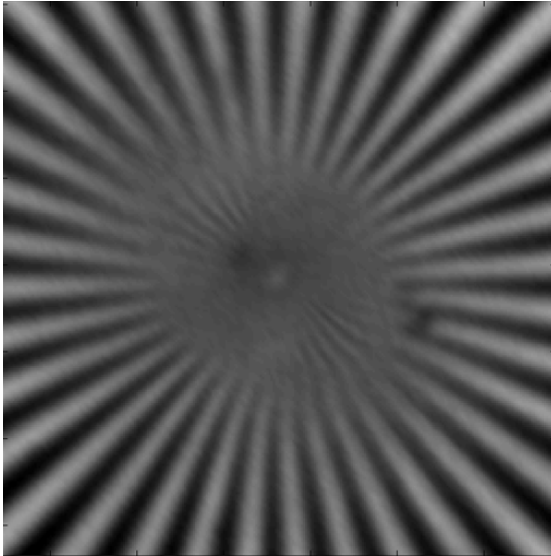
Figure 85: a. The cross-correlation spatial frequency point of a 6 lp/mm Ronchi ruling taken with an illumination of a. 1550 nm and b. 1500-1600 nm. These images have been upsampled by a factor of 8 to show greater detail.

Therefore, the blazed gratings placed at the intermediate image plane are almost exactly remapping the exit pupil to the correct locations and piston correction can be performed for an illumination bandwidth of 100nm with little degradation from FDC.

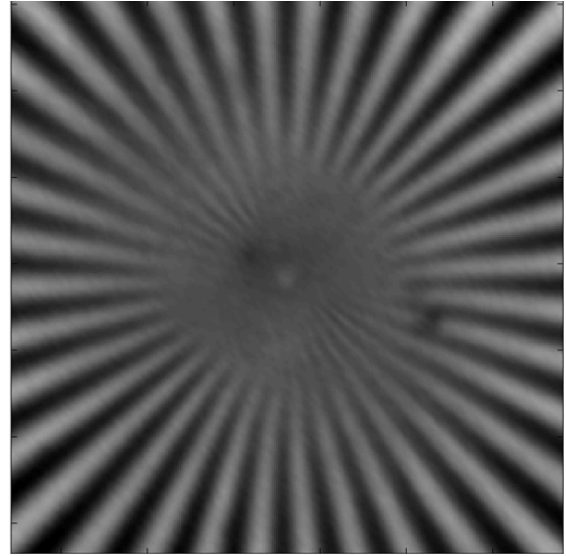
9.3 Piston Corrected Images

After showing that the experimental setup was aligned with negligible FDC using Ronchi ruling targets, two targets with multiple spatial frequencies were used to show how well the system can be piston corrected. The first was a star target and the second was a 1951 USAF resolution target. Images were taken of the 1951 USAF target rotated at 0 and 90 degrees in preparation for rotational synthesis. The procedure for the piston phase correction of two apertures was summarized at the beginning of CHAPTER 5 [1], [13]. The cross-correlations in the image's spatial frequency spectrum are registered with the auto-correlation, allowing the overlapping sections of the cross-correlation to be multiplied with the complex conjugate of the auto-correlation. The angle of the complex sum from

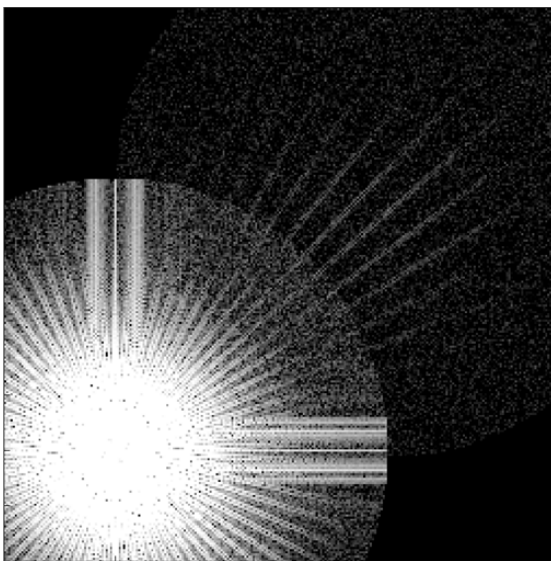
this multiplication is the relative piston between the two apertures. Images of the registered star target before and after piston phase correction are shown in Figure 86 along with their spatial frequency spectra. Only the central part of the star target is shown, containing the highest resolutions of the spokes. This data was collected using an exposure time of 16.5 ms.



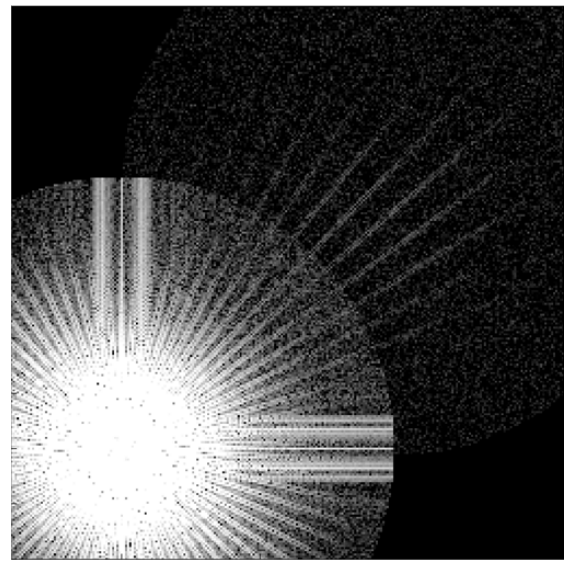
a.



b.



c.



d.

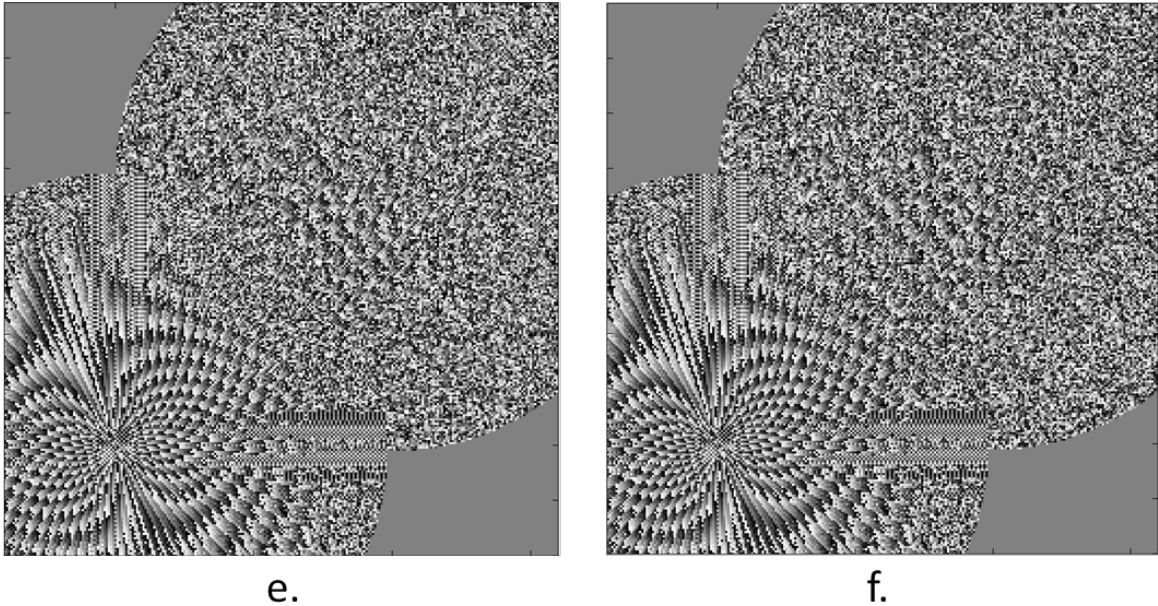


Figure 86: Registered images of a star target with LED illumination a. before and b. after piston phase correction along with the corresponding magnitudes (c and d) and angles (e and f) of their spatial frequency spectra. All images are zoomed in to show the area of gain.

Prior to corrections, the image in Figure 86.a has a loss of contrast at high spatial frequencies and a slight shift can be seen in those same frequencies. While the spokes of the star target still appear to be visible at high spatial frequencies, this shift changes with the piston error so the exact location of the inner-most spokes is not known in relation to the rest of the image. The corrected image in Figure 86.b corrects both the loss of contrast and the shifts. An increased amount of spatial frequency information is also slightly visible in the corresponding spatial frequency spectra. Figure 86.c and d show the magnitude of the spatial frequency spectrum before and after corrections respectively. There is a slight increase of intensity in the overlapping region between the auto- and cross-correlations. This is also seen in the angles of the spatial frequency content in Figure 86.e and f, where a small amount of additional content is visible in the corrected spectrum. However, the higher resolutions have low intensity and it is difficult to see where the cutoff frequency is located.

Even after piston corrections, the image still seems to lack sharpness. One of the reasons for the blurred edges in the star target is the PSF of the pupil function. While piston corrections fix some artifacts caused by atmospheric turbulence and alignment errors, it does not take the shape of the pupil function into account. After using the Wiener filter discussed in CHAPTER 7 to deconvolve the original PSF from the images, the images have increased edge sharpness and it is easier to see the gain from phase corrections. The piston error calculated for this image was 2.15 radians, $K = 1/400$ was found to give the clearest deconvolution, and the FT of image Figure 78 was used as the PSF. The deconvolved images from before and after phase corrections are shown in Figure 87.

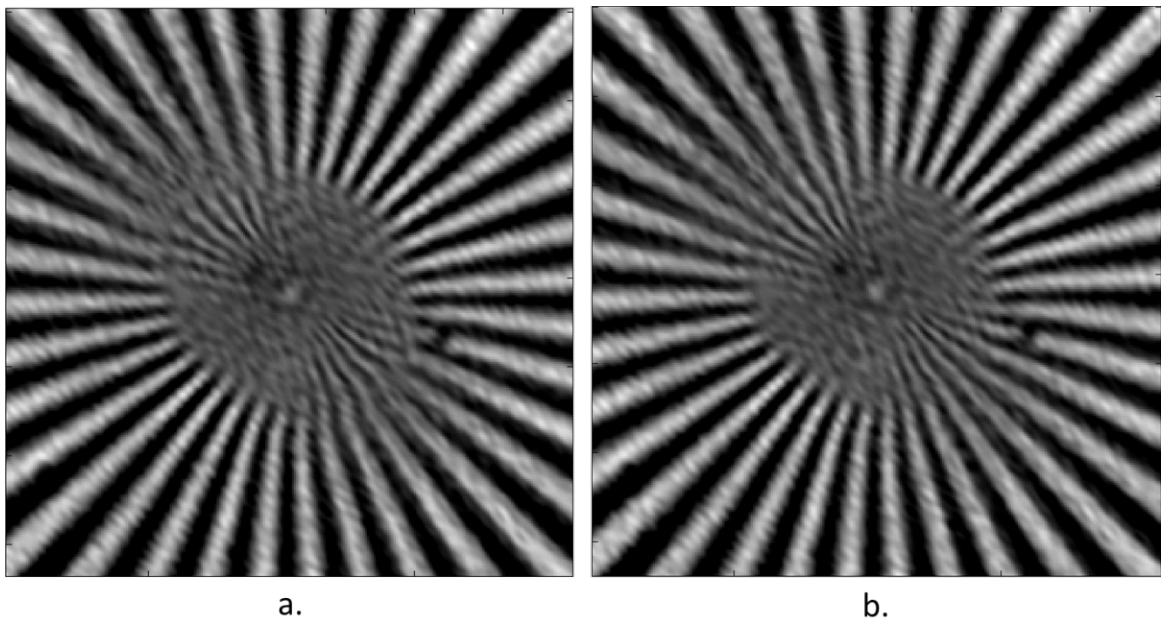
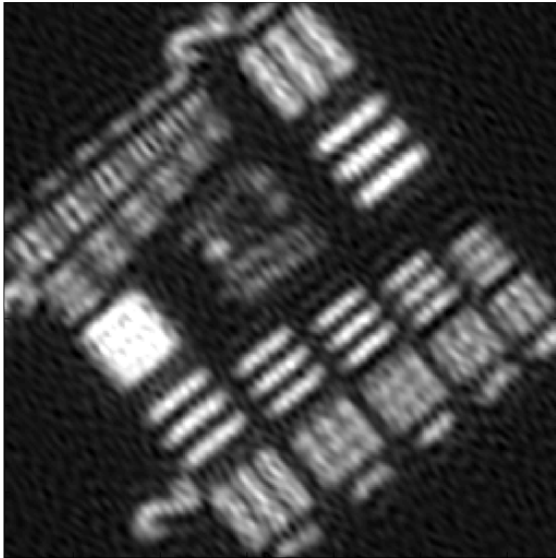
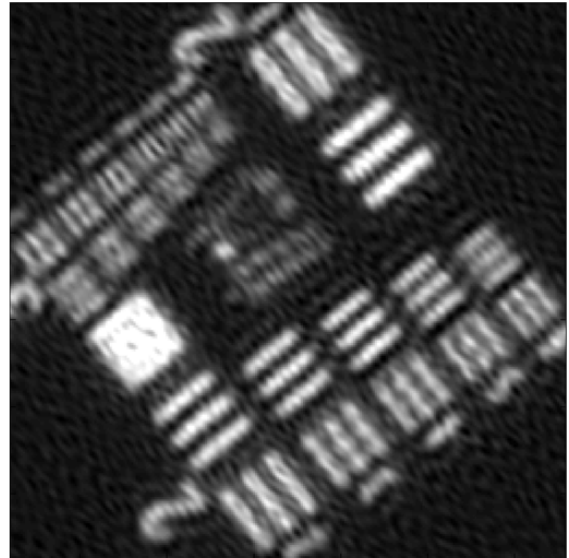


Figure 87: Deconvolved images of the star target a. before and b. after piston phase correction.

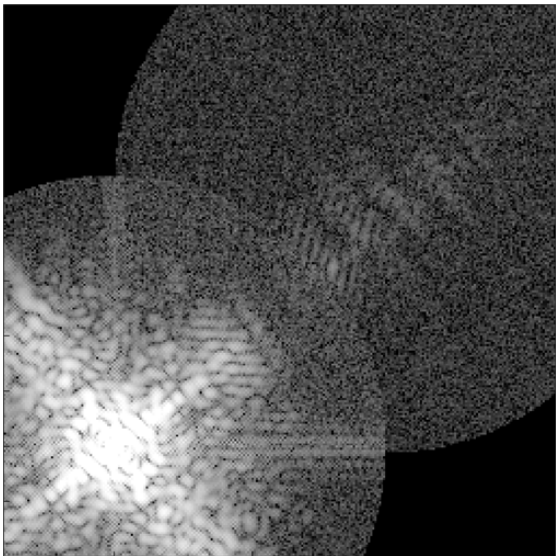
Similar images of the 1951 USAF resolution target are given in Figure 88. Figure 88.a and b have already been Wiener filtered for easier viewing.



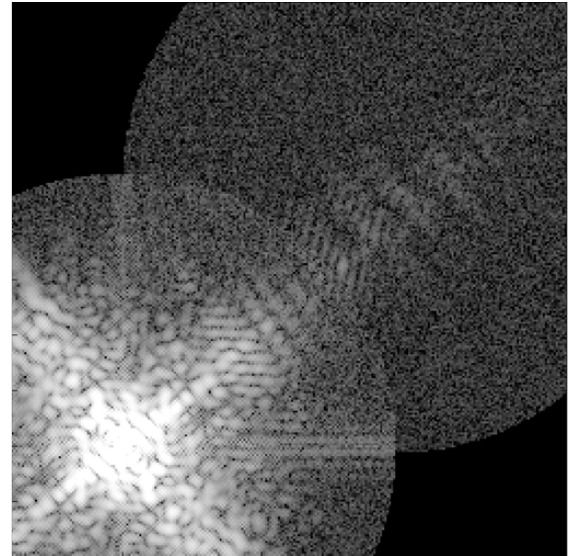
a.



b.



c.



d.

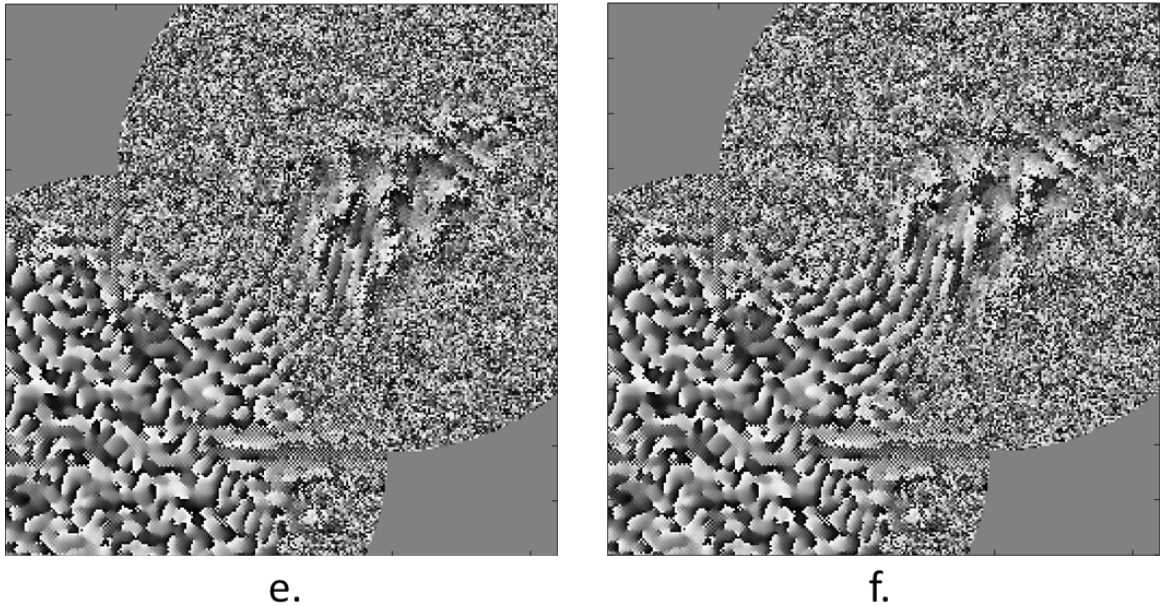


Figure 88: Registered and deconvolved images of a 1951 USAF target with LED illumination a. before and b. after piston phase correction along with the corresponding magnitudes (c and d) and angles (e and f) of their spatial frequency spectra before Wiener deconvolution.

The piston correction for this example is 2.59 radians. Another example of a corrected and deconvolved 1951 USAF target rotated by approximately 90 degrees is given in Figure 89. Its piston error is 2.75 radians.

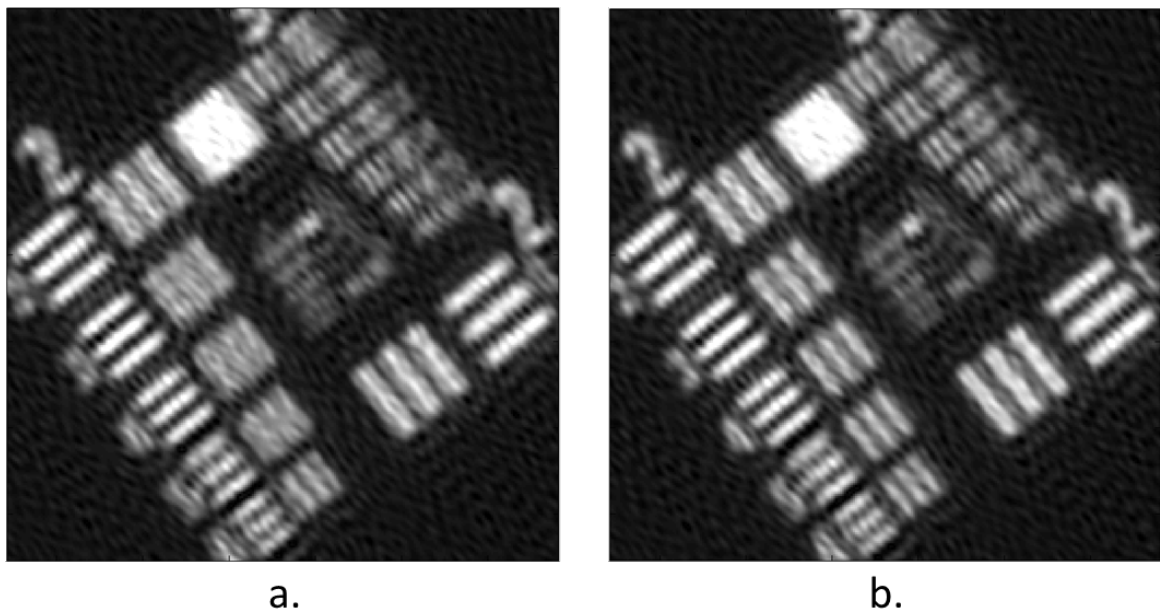


Figure 89: Deconvolved images of the 1951 USAF target rotated at 90 degrees a. before and b. after piston phase corrections.

An NSR of $K = 1000$ was used for the deconvolution and the exposure time was 10 ms. A resolution increase in the direction of the gain is clearly seen in Figure 88 and Figure 89.a and b. Without gain, the highest resolution visible is group 2, element 5 of the target. With gain, group 3, element 5 is visible in one dimension of Figure 88 and Figure 89.b. This corresponds to target resolutions of 6.35 lp/mm without gain and 12.70 lp/mm with gain.

Image magnification adjusts the spatial frequencies in the image domain based on equation (118). A simulated OTF of the imaging system is shown in Figure 90.a, where the spatial frequency cutoff is 9.65 lp/mm without gain and 22.49 lp/mm with gain.

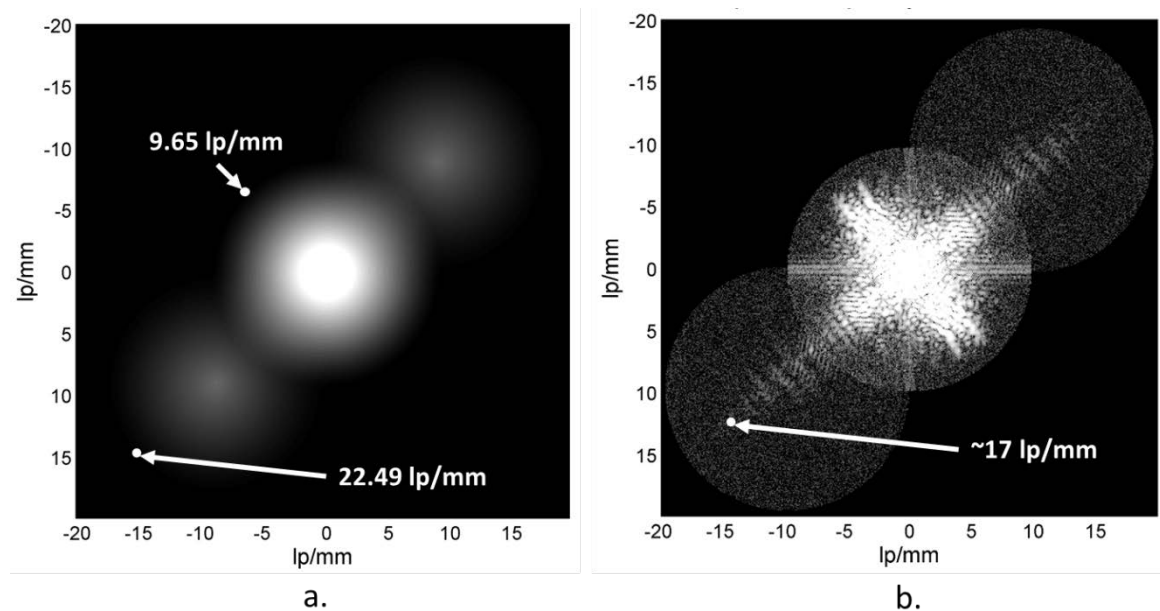


Figure 90: a. Simulated OTF for the imaging system and the b. magnitude of spatial frequency spectrum for the 1951 USAF target rotated at 0 degrees. The maximum spatial frequency cutoffs are indicated in both dimensions along with the approximate spatial frequency at which the signal falls beneath noise levels.

Likewise, the image magnification causes the spatial frequencies for the 1951 USAF target of Figure 88 and Figure 89 to be 8.77 lp/mm without gain and 17.54 lp/mm with gain. The 1951 USAF target's spatial frequency without gain is close to the cutoff in the OTF. In the direction of the gain, however, the cutoff of the OTF is noticeably higher than the target's highest resolution. It appears that the spatial frequency signal in this direction falls beneath

noise levels at approximately 17 lp/mm, which is much closer to the highest gain visible in Figure 88.b and Figure 89.b. Since the images in Figure 88 and Figure 89 show a noticeable improvement in resolution, they can be used to test the possibility of rotational image synthesis.

9.4 Rotational Synthesis

In an imaging system with an entrance pupil comprised of linearly placed apertures, the resolution gain is only found along the axis of aperture placement. Both the two aperture setup in the lab and the six-aperture setup for the simulation contain linearly placed apertures. In order to have gain in all directions, the target can be rotated and multiple images' spatial frequencies can be incoherently synthesized after the phase corrections. Even if the apertures in the pupil function are not linearly placed, the spatial frequency spectrum of a multi-aperture system will rarely be truly circular and rotational synthesis could still be useful.

In the field, it would be more realistic to rotate the imaging system, but it was simpler to rotate the target in the lab. Figure 91 shows images of the 1951 USAF target rotated at 0 and 90 degrees to the direction of the gain. In the previous section, Figure 88.b and Figure 89.b show the deconvolutions of these same corrected images rotated at 0 and 90 degrees, respectively. However, the images before Wiener deconvolution are what is needed for rotational synthesis. It should be possible to synthesize more than two images, but this particular target conveniently has most of its spatial frequency information in only two dimensions.

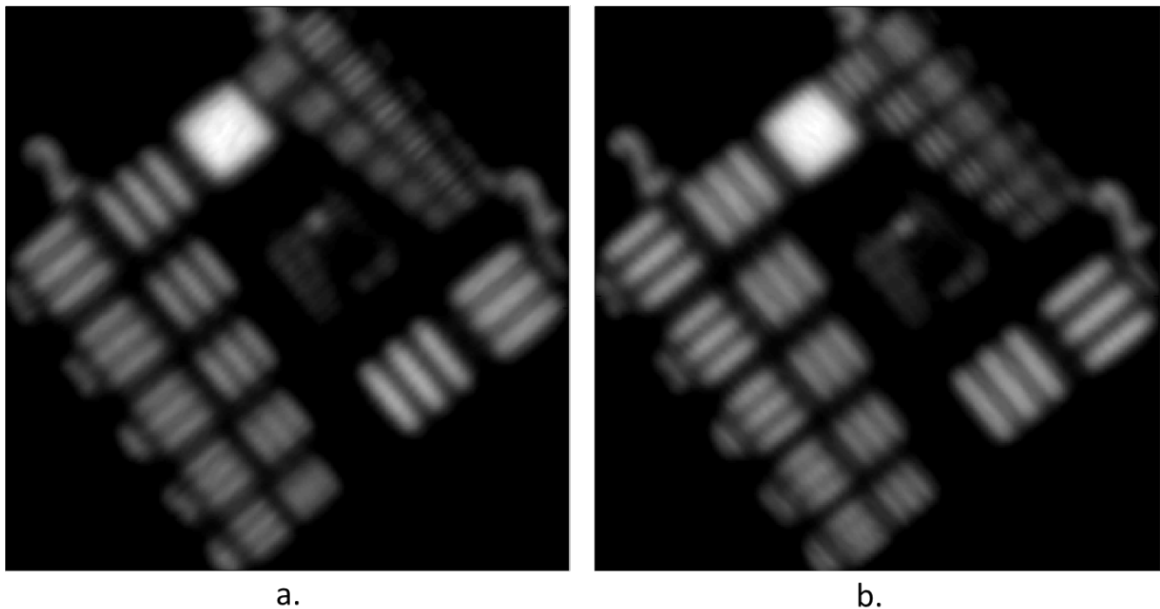


Figure 91: 1951 USAF target rotated at approximately a. 0 and b. 90 degrees. The 90 degree target has been digitally rotated back to match with the first image.

To begin the process of image synthesis, the two rotated images were rotated, registered, and shifted to align with one another. Ideally, these two images should overlay precisely. However, a close comparison of Figure 91.a and b shows that each image is slightly compressed in the direction of the gain. This distortion may have been caused by the target not being exactly orthogonally aligned with the optical axis, magnification differences between the two image axes created by angle of the blaze grating as discussed in Section 8.4, or by other issues in the system. Since the grating angle used to reflect the light towards the exit pupil is different than the angle a mirror would need, the reflection of the blaze grating is imperfect.

Fortunately, this specific setup contains two apertures that are remapped the same distance along the same axis, so the fields interfering at the focal plane from the two apertures both have the same compression. Therefore, the problem only occurs when the target is rotated between frame exposures and is relatively simple to correct

computationally. To do this, one of the images in Figure 91 was rotated 45 degrees so that its axes coincided with those of the pixel field. Then the image processing toolbox in Matlab was used to resize the image to be one pixel shorter in one direction and one pixel longer in the other before rotating it back to the correct angle. However, for other remapping setups in which more than two apertures are remapped into more than one dimension, solutions will need to be found for proper inter-aperture interference at the pupil plane, as discussed in Section 8.4.

After rotating, registering, and dimensionally resizing the phase corrected images, they are ready for image synthesis. Take the FT of each image and incoherently sum the two spatial frequency spectra, as shown in Figure 92.a. The FT of the summed spectrum is the rotationally synthesized image, which is displayed in Figure 92.b.

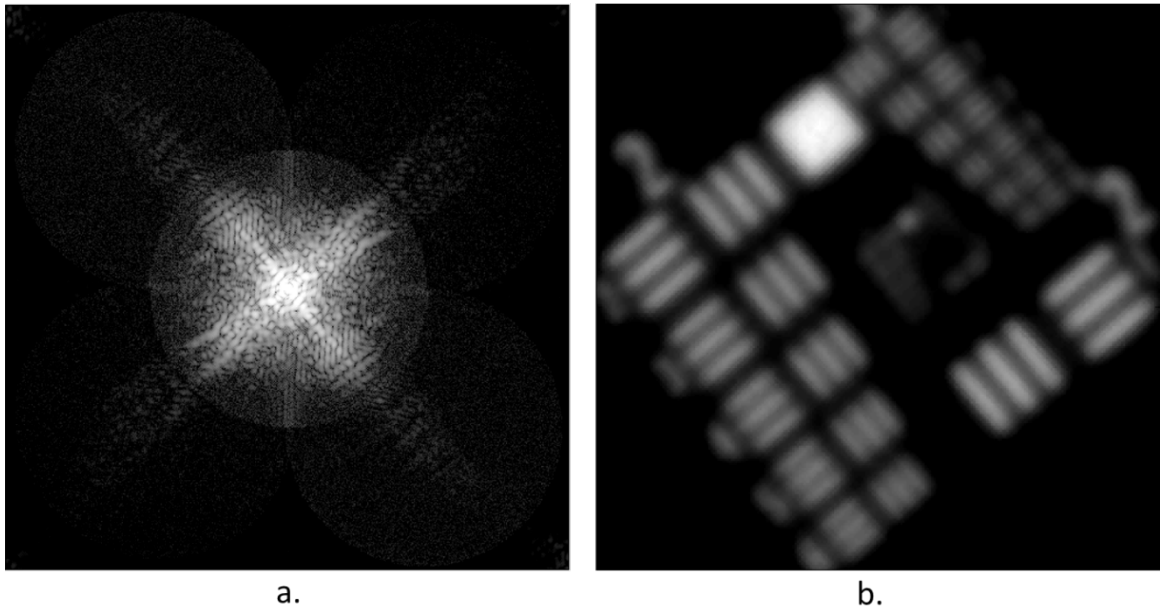


Figure 92: a. Incoherently summed spatial frequency spectra of 2 rotated 1951 USAF images. b. Rotationally synthesized image.

As with the corrected images in Section 9.3, however, the edges seem to lack sharpness and it is difficult to see improvements in resolution.

Wiener deconvolution can again be used to fix this. A PSF for this image is created by taking two copies of the simulated OTF from Figure 78 in Section 9.1. One copy of this OTF is rotated by 90 degrees and the two fields are summed. The FT of the summed OTFs is the full PSF for the rotationally synthesized image. The synthesized OTF and PSF are shown in Figure 93.

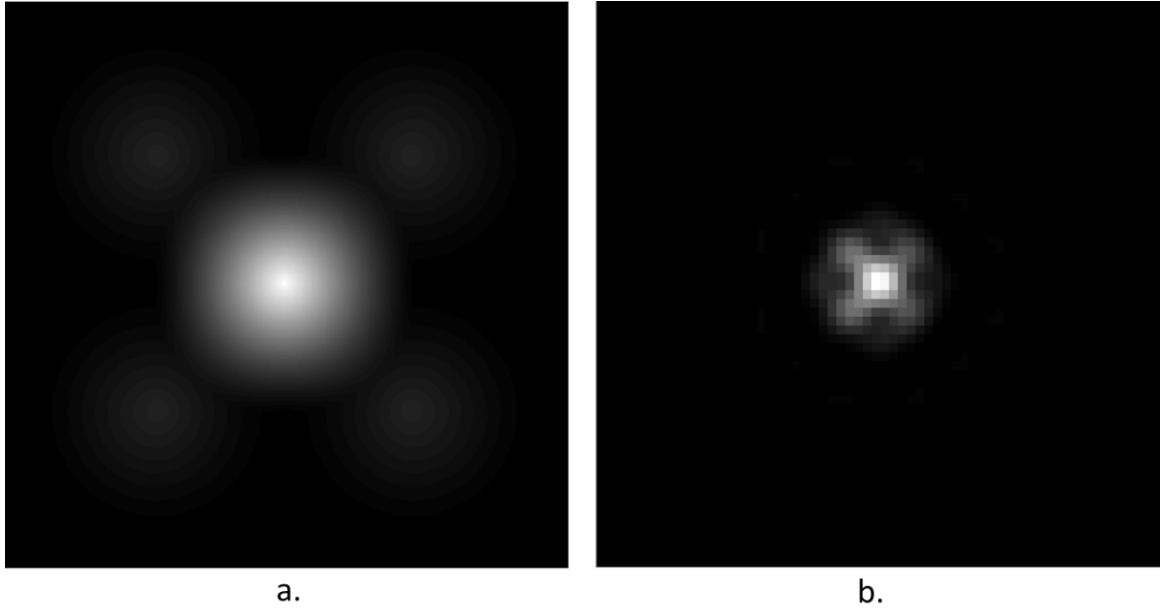


Figure 93: a. OTF and b. PSF for the rotationally synthesized 1951 USAF target.

Deconvolving Figure 92.b using the PSF from Figure 93.b and applying an NSR of $K = 1000$ produces the deconvolved image shown in Figure 94.

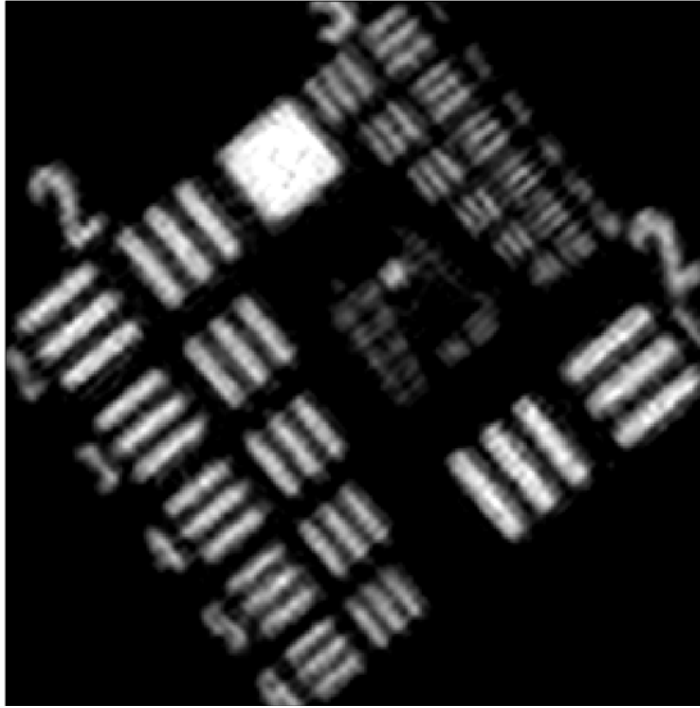


Figure 94: Deconvolved, rotationally synthesized image of 1951 USAF target.

The highest visible spatial frequency in this image is 17.54 lp/mm in both directions. This value is the same as the gain cutoffs for the individual phase corrected images used for the rotational synthesis. Keeping in mind that the 1951 USAF target uses discrete frequencies, the resolution in Figure 94 may still be slightly lower than in Figure 88.b or Figure 89.b, but this can be attributed to additive noise from the spatial frequency summation and artifacts from any residual miss-registration of the images. Overall, the rotational synthesis of the two images is a success.

CHAPTER 10

CONCLUSIONS

Previous aperture phasing techniques for multi-aperture arrays include passive and active systems. Passive systems use adaptive optics and incoherent illumination to correct the phase errors. They can reach diffraction limited resolution for a multi-aperture array as long as the path lengths of the various apertures are aligned to a fraction of the illumination wavelength. Active systems use coherent illumination and images are computationally phased post- collection by reconstructing the complex field. Since coherent illumination is used, the OPD phasing requirement is relaxed and the target depth only has to be smaller than the coherence length of the illumination. In the case of a coherent system with a local oscillator, the exposure time or round trip paths for each aperture need to be less than the coherence length of the illumination. A third method has been presented that can use either passive or active illumination and images can be phase corrected computationally as long as the OPDs between apertures are less than the coherence length of the partially coherent illumination [1]. This technique was expanded in this dissertation.

Phase correction algorithms were developed that can correct the piston, tip, and tilt phase errors of a six aperture array. Each of the apertures are separated via pupil remapping so that the cross-correlations between all of the apertures are non-redundant. Piston

differences between apertures are estimated by multiplying each correlation with the complex conjugate of another correlation that has overlapping spatial frequencies and taking the angle of the sum of each complex product. An MLE method can be used to take all of these piston differences between correlations and estimate the relative piston errors for each of the apertures. Tip and tilt phase differences can be estimated by registering the FTs of two cross-correlations. The registration results in a shift between the images that can be converted into the tips and tilts between the cross-correlations. All of these tip and tilt differences can be used to estimate the tip and tilt of the six apertures in the array using a LS matrix method.

The piston, tip, and tilt correction algorithms were tested using images generated to simulate Fried parameters two, one, one half, and one third the size of an apertures at multiple different SNRs. One hundred images were created for each Fried parameter and SNR. It was found that piston, tip, and tilt corrections provide the best results when the Fried parameter is the size of a single aperture or larger. A 1951 USAF target was corrected at lower SNRs than the dead leaves target, showing that sharper images can be corrected with a lower amount of signal support.

Anisoplanatic corrections were also tested. The phase correction algorithm works best if it only processes one isoplanatic angle at a time. Masking larger, anisoplanatic images to the size of an isoplanatic patch can make this possible. Image masking was tested using a grid target to provide relatively invariant spatial frequency content across the FOV and a flat plate background was added to lower the image sharpness enough to allow changes in the SNR to affect how well phase correction worked. It was found that phase

corrections should be possible for isoplanatic angles that are large enough to contain 18 diffraction limited spots from a single aperture at an SNR of 95.

Comparisons between the piston, tip, and tilt algorithms and blind deconvolution showed that the two techniques are relatively well matched. Blind deconvolution does better when correcting atmospheric turbulence at lower SNRs, but is slightly worse at high SNRs when the means of multiple phase corrected images are used. The phase correction algorithm generally works better when phase errors are caused by hardware misalignments in the absence of atmospheric turbulence.

The pupil remapping used for these piston, tip, and tilt corrections can cause FDC if the aperture fields are remapped by a constant distance. Constant aperture remapping leads to a wavelength dependent shift in the separated cross-correlations that affects the contrast of the shifted spatial frequencies across the FOV of the image. To address this problem, a two-aperture pupil remapping system was created using blazed gratings at an intermediate image plane to cause a wavelength dependent shift in the aperture fields. Since the aperture remapping was dependent on wavelength, the cross-correlation shift remained constant and FDC was avoided. Examples of piston corrected images collected with partially coherent illumination from an LED without FDC were shown in CHAPTER 8.

The aperture arrays used in this research were sets of apertures placed linearly along a single axis, which led to resolution gain in only one direction. To create images with gain in more than one direction, rotational image synthesis is used. This was demonstrated by taking two images of a 1951 USAF resolution target that were rotated by 90 degrees with regard to one another. After piston corrections, the FTs of these two images were

added together, transformed back using an inverse FT, and deconvolved to increase the edge sharpness. The final synthesized image showed nearly as much image gain in two directions that previous examples had shown in only one.

Overall, this research successfully produced a piston, tip, and tilt phase estimation algorithm that can be used to computationally correct phase errors in multi-aperture arrays with hardware misalignments or Fried parameters the size of an individual aperture. Anisoplanatic corrections are also possible as long as the image is masked to the size of one isoplanatic patch at a time and retains enough image support. Blazed gratings can be used to greatly reduce image FDC caused by aperture remapping and partially coherent piston corrections were experimentally demonstrated using such a system. Rotational image synthesis was used to generate images with gain in more than one direction using the same experimental results. This single frame, non-iterative phase correction method offers some advantages over the iterative, multi-frame, and sometimes time-consuming blind deconvolution method of image deblurring, especially in the case of phase errors caused by aperture misalignments, but also for some cases of images affected by atmospheric turbulence. The ability to use this phase correction method with either passive or active illumination at higher bandwidths also increases the imaging capabilities of multi-aperture systems.

10.1 Original Contributions

- Piston phase estimation and correction algorithm for a multi-aperture array
- Tip and tilt phase estimation and correction algorithm for a multi-aperture array
- Looked at how anisoplanatism affects the piston, tip, and tilt correction algorithms

- Studied how the application of different sized masks affects phase correction performance, which could enable corrections for individual isoplanatic patches within anisoplanatic images
- Showed that blazed gratings placed at the intermediate image plane of an imaging system can be used to create wavelength dependent pupil remapping to reduce or even negate the effects of FDC

10.2 Future Work

Several pathways could be followed to increase the effectiveness of the piston, tip, and tilt correction algorithms. Currently, tip and tilt estimation uses only the cross-correlations since the sum of all of the auto-correlations creates a non-linear tip and tilt that cannot be used to give accurate estimates. A solution to this problem may be found by using the tips and tilts estimated with the cross-correlations to create an estimate of the non-linear tip and tilt of the auto-correlation. The piston, tip, and tilt algorithms could also be made more effective if the MLE and LS techniques were expanded to compare the correlations from more than one exposure frame at a time.

Further research could be done on anisoplanatic correction techniques, as well. Masking images to correct for a single isoplanatic angle was shown in this dissertation, but more tests could be run. Masking multiple isoplanatic patches within an anisoplanatic image, correcting them all separately, and registering them to create one corrected, anisoplanatic image would demonstrate the effectiveness of this idea. Also the MLE and LS algorithms could be adjusted to take a multi-conjugate approach into account and lead to more flexible anisoplanatic corrections. This would allow piston, tip, and tilt corrections

in the field at smaller isoplanatic angles from either stronger atmospheric turbulence or longer imaging distances.

Experimentally, a pupil remapping imaging system could be designed and built for an array of more than two apertures. Designing such a system would include looking at ways to work around the image distortions created by the angular orientations of the blazed gratings. The multi-aperture array could be used for field tests in order to compare the simulations with real-world data.

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APPENDIX A

Integration of a Dirac Delta

Begin from equation (40),

$$\mathcal{G}\left(\frac{f_x}{\lambda d_i}, \frac{f_y}{\lambda d_i}\right) = \sum_{m=1}^N \sum_{n=1}^N \left[U_m(f_x, f_y) \otimes U_n^*(-f_x, -f_y) \otimes \delta\left(f_x - \frac{M_p b_{mn}}{\lambda d_i}, f_y - \frac{M_p a_{mn}}{\lambda d_i}\right) \right] e^{-j\varphi_{nm}} \quad (122)$$

Rotate the coordinate system of the Dirac delta so that the f_x and f_y shifts are defined by only one shift in a single direction. The rotation angle is,

$$\theta_{mn} = \tan^{-1}\left(\frac{a_{mn}}{b_{mn}}\right) \quad (123)$$

the coordinate rotation is,

$$\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \begin{pmatrix} \cos \theta_{mn} & -\sin \theta_{mn} \\ \sin \theta_{mn} & \cos \theta_{mn} \end{pmatrix} = \begin{pmatrix} f_x - \frac{a_{mn}}{b_{mn}} f_y \\ \frac{a_{mn}}{b_{mn}} f_x + f_y \end{pmatrix} \quad (124)$$

and the magnitude of the shift is,

$$M_p \sqrt{\left(\frac{a_{mn}}{\lambda d_i}\right)^2 + \left(\frac{b_{mn}}{\lambda d_i}\right)^2} \approx \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i} \quad (125)$$

Using equations (123), (124), and (125), equation (122) becomes,

$$\begin{aligned} & \mathcal{G}\left(\frac{f_x}{\lambda d_i}, \frac{f_y}{\lambda d_i}\right) \\ &= \sum_{m=1}^N \sum_{n=1}^N \left[U_m(f_x, f_y) \otimes U_n^*(-f_x, -f_y) \otimes \delta\left(\left(\frac{a_{mn}}{b_{mn}} f_x + f_y\right) - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i}\right) \right] e^{-j\varphi_{nm}} \end{aligned} \quad (126)$$

This equation uses only one wavelength.

To calculate the frequency response for all wavelengths, integrate over all possible shifts caused by the wavelength bandwidth,

$$\begin{aligned} \hat{\mathcal{G}}\left(\frac{f_x}{\lambda d_i}, \frac{f_y}{\lambda d_i}\right) &= \int_{\sqrt{a_{mn}^2 + b_{mn}^2}/\lambda_1 d_i}^{\sqrt{a_{mn}^2 + b_{mn}^2}/\lambda_2 d_i} \sum_{m=1}^N \sum_{n=1}^N \left[U_m(f_x, f_y) \otimes U_n^*(-f_x, -f_y) \otimes \right. \\ & \quad \left. \delta\left(\left(\frac{a_{mn}}{b_{mn}} f_x + f_y\right) - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i}\right) \right] e^{-j\varphi_{nm}} d\left(\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i}\right) \end{aligned} \quad (127)$$

This can be re-written since the shift is found only in the Dirac delta function,

$$\begin{aligned} \hat{\mathcal{G}}\left(\frac{f_x}{\lambda d_i}, \frac{f_y}{\lambda d_i}\right) &= \sum_{m=1}^N \sum_{n=1}^N \left[U_m(f_x, f_y) \otimes U_n^*(-f_x, -f_y) \otimes \right. \\ & \quad \left. \int_{\sqrt{a_{mn}^2 + b_{mn}^2}/\lambda_1 d_i}^{\sqrt{a_{mn}^2 + b_{mn}^2}/\lambda_2 d_i} \left[\delta\left(f_y' - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i}\right) d\left(\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i}\right) \right] \right] e^{-j\varphi_{nm}} \end{aligned} \quad (128)$$

Now, the integral can be solved independently of the rest of $\hat{\mathcal{G}}\left(\frac{f_x}{\lambda d_i}, \frac{f_y}{\lambda d_i}\right)$,

$$\int_{M_p \sqrt{a_{mn}^2 + b_{mn}^2}/\lambda_1 d_i}^{M_p \sqrt{a_{mn}^2 + b_{mn}^2}/\lambda_2 d_i} \delta\left(f_y' - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i}\right) d\left(\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i}\right) \quad (129)$$

Since integrals are additive, the equation above can be separated into two separate integrals,

$$\begin{aligned}
& \int_{-\infty}^{M_p \sqrt{a_{mn}^2 + b_{mn}^2} / \lambda_2 d_i} \delta \left(f'_y - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i} \right) d \left(\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i} \right) \\
& - \int_{-\infty}^{M_p \sqrt{a_{mn}^2 + b_{mn}^2} / \lambda_1 d_i} \delta \left(f'_y - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i} \right) d \left(\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda d_i} \right)
\end{aligned} \tag{130}$$

This results in two Heaviside functions [21],

$$H \left(\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda_2 d_i} - f'_y \right) - H \left(\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda_1 d_i} - f'_y \right) \tag{131}$$

Rewrite this as,

$$H \left(f'_y - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda_1 d_i} \right) - H \left(f'_y - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda_2 d_i} \right) \tag{132}$$

The distance of this equation from the origin is the mean of the two shifts:

$$\frac{\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda_1 d_i} + \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda_2 d_i}}{2} \approx \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\bar{\lambda} d_i} \tag{133}$$

and the distance of the two Heaviside functions from that mean is,

$$\frac{\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda_1 d_i} - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\lambda_2 d_i}}{2} \approx \frac{M_p \Delta \lambda \sqrt{a_{mn}^2 + b_{mn}^2}}{2 \bar{\lambda}^2 d_i} \tag{134}$$

where $\Delta \lambda = \lambda_2 - \lambda_1$. Substituting these into equation (132) gives,

$$\begin{aligned}
& H \left(f'_y - \left(\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\bar{\lambda} d_i} - \frac{M_p \Delta \lambda \sqrt{a_{mn}^2 + b_{mn}^2}}{2 \bar{\lambda}^2 d_i} \right) \right) \\
& - H \left(f'_y - \left(\frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\bar{\lambda} d_i} + \frac{M_p \Delta \lambda \sqrt{a_{mn}^2 + b_{mn}^2}}{2 \bar{\lambda}^2 d_i} \right) \right)
\end{aligned} \tag{135}$$

Two Heaviside functions form a rectangular function [22]. In this form, it can be seen that two times equation (134) is the width and equation (133) is the location of the center of the function along the rotated axis, f'_y ,

$$\left(\frac{M_p \Delta \lambda}{\bar{\lambda}^2 d_i} \sqrt{a_{mn}^2 + b_{mn}^2} \right) \text{rect} \left[\frac{f'_y - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\bar{\lambda} d_i}}{\frac{M_p \Delta \lambda \sqrt{a_{mn}^2 + b_{mn}^2}}{\bar{\lambda}^2 d_i}} \right] \quad (136)$$

where the constant normalizes the function. This can also be written as a convolution between a rectangular function centered at zero and a Dirac delta function shifted by the mean,

$$\left(\frac{M_p \Delta \lambda}{\bar{\lambda}^2 d_i} \sqrt{a_{mn}^2 + b_{mn}^2} \right) \delta \left(f'_y - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\bar{\lambda} d_i} \right) \otimes \text{rect} \left[\frac{f'_y \bar{\lambda}^2 d_i}{M_p \Delta \lambda \sqrt{a_{mn}^2 + b_{mn}^2}} \right] \quad (137)$$

Substituting the expression above into equation (128) gives,

$$\begin{aligned} \hat{\mathcal{G}} \left(\frac{f_x}{\lambda d_i}, \frac{f_y}{\lambda d_i} \right) &= \sum_{m=1}^N \sum_{n=1}^N \left[1 + \delta_{mn} \left(\frac{M_p \Delta \lambda}{\bar{\lambda}^2 d_i} \sqrt{a_{mn}^2 + b_{mn}^2} - 1 \right) \right] \\ &\quad \left[U_m(f_x, f_y) \otimes U_n^*(-f_x, -f_y) \otimes \right. \\ &\quad \left. \delta \left(f'_y - \frac{M_p \sqrt{a_{mn}^2 + b_{mn}^2}}{\bar{\lambda} d_i} \right) \otimes \text{rect} \left[\frac{f'_y \bar{\lambda}^2 d_i}{M_p \Delta \lambda \sqrt{a_{mn}^2 + b_{mn}^2}} \right] \right] e^{-j\varphi_{nm}} \end{aligned} \quad (138)$$

Returning the Dirac delta to the original coordinate system and writing the rectangular function in terms of the original coordinates results in equation (43).

$$\begin{aligned}
\hat{\mathcal{G}}\left(\frac{f_x}{\lambda d_i}, \frac{f_y}{\lambda d_i}\right) &= \sum_{m=1}^N \sum_{n=1}^N \left[1 + \delta_{mn} \left(\frac{M_p \Delta \lambda}{\bar{\lambda}^2 d_i} \sqrt{a_{mn}^2 + b_{mn}^2} - 1 \right) \right] \\
&\quad \left[U_m(f_x, f_y) \otimes U_n^*(-f_x, -f_y) \otimes \delta\left(f_x - \frac{M_p b_{mn}}{\bar{\lambda} d_i}, f_y - \frac{M_p a_{mn}}{\bar{\lambda} d_i}\right) \right. \\
&\quad \left. \otimes \text{rect}\left(\frac{\left(\frac{a_{mn}}{b_{mn}} f_x + f_y\right) \bar{\lambda}^2 d_i}{M_p \Delta \lambda \sqrt{a_{mn}^2 + b_{mn}^2}}\right) \right] e^{-j\phi_{nm}}
\end{aligned} \tag{139}$$

APPENDIX B

Correlation of Apertures with Phase Tilts

Equation (32) is reproduced below.

$$\mathcal{G}(f_x, f_y) = \sum_{m=1}^N \sum_{n=1}^N [U_m(\xi, \eta) e^{j(\varphi_m + \varphi_{my}\eta)} \otimes U_n^*(-\xi, -\eta) e^{-j(\varphi_n + \varphi_{ny}\eta)}] \quad (141)$$

To find the overall tilt for each correlation in this equation, begin with the correlation between two aperture fields with piston and tilt phases.

$$U_m(\xi, \eta) e^{-j(\varphi_m - \eta\varphi_{my})} \quad (142)$$

$$U_n(\xi, \eta) e^{-j(\varphi_n - \eta\varphi_{ny})} \quad (143)$$

Goodman defines a correlation as a convolution between one function and the complex conjugate of another function with a coordinate inversion. Thus, if a convolution is defined as,

$$f(\xi, \eta) \otimes g(\xi, \eta) \quad (144)$$

the corresponding cross-correlation is,

$$f(\xi, \eta) \otimes g^*(-\xi, -\eta) \quad (145)$$

From equation (145), the cross-correlation of the terms in equations (142) and (143) is,

$$U_m(\xi, \eta) e^{-j(\varphi_m - \eta\varphi_{my})} \otimes U_n^*(-\xi, -\eta) e^{j(\varphi_n + \eta\varphi_{ny})} \quad (146)$$

To simplify, only the η dimension is used. The piston phases are constant and have no dependence on η , so they can be pulled out of the correlation,

$$[U_m(\eta) e^{j\eta\varphi_{my}} \otimes U_n^*(-\eta) e^{j\eta\varphi_{ny}}] e^{-j(\varphi_m - \varphi_n)} \quad (147)$$

For convenience, look at the case where the target is a point source. This allows equations (142) and (143) to be even, circular functions. Since the complex conjugate of an even function with coordinate inversion is equal to the original function, the correlation

and convolution are equivalent. In one dimension, a circular function becomes a rectangular function,

$$[\text{rect}(\eta)e^{j\eta\varphi_{my}} \otimes \text{rect}(\eta)e^{j\eta\varphi_{ny}}]e^{-j(\varphi_m - \varphi_n)} \quad (148)$$

where,

$$\text{rect}(\eta) = \begin{cases} 1 & |\eta| < 1/2 \\ \frac{1}{2} & |\eta| = 1/2 \\ 0 & |\eta| > 1/2 \end{cases} \quad (149)$$

A manual solution to the convolution in equation (148) is complex and requires careful treatment of boundary conditions, which is beyond the scope of this research. Of greater importance is the form of the solution and its treatment of the phase tilts. Mathematica provides a symbolic solution to the convolution given in equation (148).

Using the input:

`Convolve[UnitBox[y]*Exp[I*y*φmY], UnitBox[y]*Exp[I*y*φnY], y, η]`

where UnitBox is the rectangular function, gives the solution,

$$= \begin{cases} \frac{-je^{\frac{j(\varphi_{my} - \varphi_{ny})}{2}} [e^{j\varphi_{my}(1-\eta)} - e^{j\varphi_{ny}(1-\eta)}]}{\varphi_{my} - \varphi_{ny}} & -1 < \eta \leq 0 \\ \frac{-2e^{\frac{j\eta(\varphi_{my} + \varphi_{ny})}{2}} \sin\left[\frac{\varphi_{my} - \varphi_{ny}}{2}(-1 + \eta)\right]}{\varphi_{my} - \varphi_{ny}} & 0 < \eta < 1 \\ 0 & \text{True} \end{cases} \quad (150)$$

These equations can be reworked to give,

$$= \begin{cases} \frac{2e^{\frac{j\eta(\varphi_{my} + \varphi_{ny})}{2}} \sin\left[\frac{\varphi_{my} - \varphi_{ny}}{2}(1 + \eta)\right]}{\varphi_{my} - \varphi_{ny}} & -1 < \eta \leq 0 \\ \frac{2e^{\frac{j\eta(\varphi_{my} + \varphi_{ny})}{2}} \sin\left[\frac{\varphi_{my} - \varphi_{ny}}{2}(1 - \eta)\right]}{\varphi_{my} - \varphi_{ny}} & 0 < \eta < 1 \\ 0 & \text{True} \end{cases} \quad (151)$$

In this form, the phase tilt is the mean of the tilts of the two aperture fields, just as predicted.

If the magnitude is defined as,

$$C_{mn}(\eta; \varphi_{my}, \varphi_{ny}) = \begin{cases} \frac{2 \sin \left[\frac{\varphi_{my} - \varphi_{ny}}{2} (1 + \eta) \right]}{\varphi_m - \varphi_n} & -1 < \eta \leq 0 \\ \frac{2 \sin \left[\frac{\varphi_{my} - \varphi_{ny}}{2} (1 - \eta) \right]}{\varphi_{my} - \varphi_{ny}} & 0 < \eta < 1 \\ 0 & \text{True} \end{cases} \quad (152)$$

Then equation (151) can be plugged into equation (148) and the final convolution is written as,

$$C_{mn}(\eta; \varphi_{my}, \varphi_{ny}) e^{-j \left(\varphi_{nm} - \frac{\eta(\varphi_{my} + \varphi_{ny})}{2} \right)} \quad (153)$$

which is equal to the interior of equation (32).

It is interesting to note that for the small angle approximation, when $\varphi_{my} - \varphi_{ny} \ll 1$,

$$C_{mn}(\xi, \eta) = \begin{cases} 1 + \eta & -1 < \eta \leq 0 \\ 1 - \eta & 0 < \eta < 1 \\ 0 & \text{True} \end{cases} \quad (154)$$

This is the definition of the triangle function and the convolution of two rectangular functions without any phase tilts is also the triangle function.

APPENDIX C

Addition of Tilted Correlations

Equation (32), reproduced below, is the sum of all cross- and auto-correlations between aperture fields in a pupil function.

$$\mathcal{G}(f_x, f_y) = \sum_{m=1}^N \sum_{n=1}^N [U_m(\xi, \eta) e^{j(\varphi_m + \varphi_{my}\eta)} \otimes U_n^*(-\xi, -\eta) e^{-j(\varphi_n + \varphi_{ny}\eta)}] \quad (155)$$

Plugging equation (153) from APPENDIX B into equation (32) results in,

$$\mathcal{G}(f_x, f_y) = \sum_{m=1}^N \sum_{n=1}^N \mathcal{C}_{mn}(\eta; \varphi_{my}, \varphi_{ny}) e^{-j\left(\varphi_{nm} - \frac{\eta(\varphi_{my} + \varphi_{ny})}{2}\right)} \quad (156)$$

The sum of the auto-correlations occurs when $m = n = k$. In this case, the equation becomes,

$$\sum_{k=1}^N \mathcal{C}_{kk}(\xi, \eta) e^{j\eta\varphi_{ky}} \quad (157)$$

Each of the auto-correlations in this sum have an associated tilt, φ_{ky} . The goal is to find the final slope of the phase after all of the auto-correlations have been added together. It would be ideal if the final phase slope were the mean of the tilts from all of the auto-correlations, but this is not necessarily the case. This assumption works for two apertures, but not for three or more. The math for both of these situations is provided below.

When there are two apertures, equation (157) becomes,

$$\mathcal{C}_{11}(\xi, \eta) e^{j\eta\varphi_{1y}} + \mathcal{C}_{22}(\xi, \eta) e^{j\eta\varphi_{2y}} \quad (158)$$

Since $\mathcal{C}_{kk}$ is real and the object of focus is the phase, $\mathcal{C}_{kk}$ can be simplified to 1 for this example.

$$e^{j\eta\varphi_{1y}} + e^{j\eta\varphi_{2y}} \quad (159)$$

The total phase of this formula can be found by utilizing Euler's identity to separate the terms into their real and imaginary components.

$$\cos(\eta\varphi_{1y}) + j \sin(\eta\varphi_{1y}) + \cos(\eta\varphi_{2y}) + j \sin(\eta\varphi_{2y}) \quad (160)$$

The phase of a complex number can be found through,

$$\varphi = \tan^{-1} \left(\frac{\text{imaginary}}{\text{real}} \right) \quad (161)$$

so equation (160) can be used to create,

$$\varphi = \tan^{-1} \left(\frac{\sin(\eta\varphi_{1y}) + \sin(\eta\varphi_{2y})}{\cos(\eta\varphi_{1y}) + \cos(\eta\varphi_{2y})} \right) \quad (162)$$

Through trigonometric identities, this can be changed to,

$$\varphi = \tan^{-1} \left(\frac{2 \sin \left(\eta \frac{\varphi_{1y} + \varphi_{2y}}{2} \right) \cos \left(\eta \frac{\varphi_{1y} - \varphi_{2y}}{2} \right)}{2 \cos \left(\eta \frac{\varphi_{1y} + \varphi_{2y}}{2} \right) \cos \left(\eta \frac{\varphi_{1y} - \varphi_{2y}}{2} \right)} \right) \quad (163)$$

which simplifies as,

$$\varphi = \tan^{-1} \left(\tan \left(\eta \frac{\varphi_{1y} + \varphi_{2y}}{2} \right) \right) \quad (164)$$

Therefore, for two apertures, the final phase is a linear function with a slope that is equal to the mean of the two correlation tilts,

$$\varphi = \eta \frac{\varphi_{1y} + \varphi_{2y}}{2} \quad (165)$$

In comparison, the sum of three aperture tilts can be written as,

$$e^{j\eta\varphi_{1y}} + e^{j\eta\varphi_{2y}} + e^{j\eta\varphi_{3y}} \quad (166)$$

In this case, equation (162) takes the form,

$$\varphi = \tan^{-1} \left(\frac{\sin(\eta\varphi_{1y}) + \sin(\eta\varphi_{2y}) + \sin(\eta\varphi_{3y})}{\cos(\eta\varphi_{1y}) + \cos(\eta\varphi_{2y}) + \cos(\eta\varphi_{3y})} \right) \quad (167)$$

Unfortunately, there is no way to rewrite this equation in a similar fashion to equation

(164). Using the paraxial approximation of $\eta \ll 1$, the phase can be approximated as,

$$\varphi \approx \tan^{-1} \left(\frac{\eta(\varphi_{1y} + \varphi_{2y} + \varphi_{3y})}{1 + 1 + 1} \right) \approx \eta \frac{\varphi_{1y} + \varphi_{2y} + \varphi_{3y}}{3} \quad (168)$$

However, η becomes much larger than the approximation range as it reaches the edges of the apertures, which is where the calculations for tilt corrections take place. The true phase slope across the full aperture can be found by taking the derivative of equation (167) [34],

$$\begin{aligned} \frac{\partial \varphi}{\partial \eta} = & \frac{\varphi_{1y} + \varphi_{2y} + \varphi_{3y} + (\varphi_{1y} + \varphi_{2y}) \cos[\eta(\varphi_{1y} - \varphi_{2y})]}{3 + 2 \cos[\eta(\varphi_{1y} - \varphi_{2y})] + 2 \cos[\eta(\varphi_{1y} - \varphi_{3y})] + 2 \cos[\eta(\varphi_{2y} - \varphi_{3y})]} \\ & + \frac{(\varphi_{1y} - \varphi_{3y}) \cos[\eta(\varphi_{1y} - \varphi_{3y})] + (\varphi_{2y} + \varphi_{3y}) \cos[\eta(\varphi_{2y} - \varphi_{3y})]}{3 + 2 \cos[\eta(\varphi_{1y} - \varphi_{2y})] + 2 \cos[\eta(\varphi_{1y} - \varphi_{3y})] + 2 \cos[\eta(\varphi_{2y} - \varphi_{3y})]} \end{aligned} \quad (169)$$

At $\eta = 0$, this monstrosity becomes the mean of the three aperture tilts, but it is nonlinear everywhere else. Therefore, the phase of the sum of the auto-correlations should not be used when calculating the tilts of the aperture fields.