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Global Modal and Non-Modal Instability Analyses of Shock-Induced Separation Bubbles

Vassilios Theofilis
NUMERICAL MODELLING SL.
CALLE DE MARIANO BARBACID, 1 - 9
BOADILLA DEL MONTE, 28660
ES

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14. ABSTRACT Following an in-depth literature survey, work in this grant was performed along five paths: (i) two-dimensional direct numerical simulations which delivered the steady laminar base flows analyzed, (ii) analytical generation of meshes on which accurate instability analysis results could be obtained, (iii) development of LiGHT (Linear instability for Global, Hypersonic Transition), a massively parallel large eigenvalue and singular value problem solver used to (iv) perform modal and non-modal, 2- and 3D global linear instability analyses and (v) derivation and programming of the interactive boundary layer / triple-deck equations for base flow and instability analysis of supersonic flows.					
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FINAL REPORT

Global Modal and Non-Modal Instability Analyses of Shock-Induced Separation Bubbles

Principal Investigator: Vassilis THEOFILIS, Ph.D.

Ph.D. Students: Nicolas CERULUS, Helio QUINTANILHA Jr.

M.Sc. Student: Nathan DUGGINS

School of Engineering, University of Liverpool
Brownlow Hill, L69 3GH, Liverpool, UK
t: +44 (0) 151 794 4849
eMail: v.theofilis@liverpool.ac.uk

AFOSR Program Officer: Ivett LEYVA, Ph.D.

EOARD Program Officers:

Douglas SMITH, Ph.D. & Russell CUMMINGS, Ph.D.

MARCH 2020

Executive Summary

Following an in-depth literature survey, work in this grant was performed along five paths: (i) two-dimensional direct numerical simulations which delivered the steady laminar base flows analyzed, (ii) analytical generation of meshes on which accurate instability analysis results could be obtained, (iii) development of *LiGHT* (Linear instability for Global, Hypersonic Transition), a massively parallel large eigenvalue and singular value problem solver used to (iv) perform modal and non-modal, 2- and 3D global linear instability analyses and (v) derivation and programming of the interactive boundary layer / triple-deck equations for base flow and instability analysis of supersonic flows.

Task (i) dealt with the generation of the laminar base states to be analyzed. Work was mostly performed using the suite of open source code OpenFOAM, which is based on second-order finite-volume spatial discretization. Extensive validation and verification of the routines used was performed and analytically-known solutions in benchmark compressible problems have been recovered. Subsequently, high-resolution laminar solutions of supersonic flow over the compression ramp were obtained and compared with high-order finite-difference solutions delivered by independently performed simulations on the same geometry.

Task (ii) was born out of the need to generate analytical meshes on which the laminar flows computed on (i) could be interpolated and the instability analyses of task (iv) could be reliably performed. The main driver in this development has been the accurate description of the recompression shock found in the vicinity of the compression corner, which our global instability analyses have demonstrated to be an integral part of both the eigenmode amplitude functions and linear optimal 2- and 3D perturbations developing on the compression ramp.

Task (iii) represents a large portion of the work performed in the present Grant and has dealt with the generation of *LiGHT*, a general-purpose matrix-forming algorithm for the massively parallel numerical solution of multi-dimensional non-symmetric complex generalized eigenvalue problems (EVP) and singular value decomposition (SVD) problems, respectively governing global linear modal and non-modal fluid flow instability. In the supersonic and hypersonic regimes of interest, these EVP and SVD problems result in dense matrices of relatively large, $O(10^4 - 10^5)$, leading dimension. The ScaLAPACK library, available as part of vendor-optimized libraries on most modern supercomputers, has been used to perform a full distributed LU-decomposition, as required by shift-and-invert strategies which promote physically-interesting and suppress spurious eigenmodes. The iterative generation of Krylov subspaces and the computation of the Ritz vectors are performed by linear algebra operations distributed over the available processors, while the serial computation of the Ritz values represents a negligible part of the overall computing effort. The algorithm was validated by forming and solving 2- and 3D eigenvalue problems, results of which are known analytically. In addition, matrices describing real and complex two-dimensional (BiGlobal) and three-dimensional (TriGlobal) EVPs arising in incompressible flow stability have been read in the *LiGHT* solver and the (known) eigenvalues have been accurately computed in parallel. The elements of these matrices were read in parallel, using routines developed by Dr. M. Tufts (AFRL), while computing time provided on the Copper DoD HPCMP Open Research System has been used to port the *LiGHT* code and deliver it to AFRL. Work on the code is presently being completed by the parallel generation of the elements of the compressible linearised Navier-Stokes operator.

Task (iv) addressed the main activity of this Grant, namely modal and non-modal analyses of supersonic flows on the compression corner. Two-dimensional modal analyses showed that the amplitude functions of the (damped) leading perturbations peaked at the vicinity of the recompression shock and the laminar separation bubble. Independently performed collaborative work with the group of Prof. D. Levin highlighted this connection in hypersonic flow over a variety of configurations discussed in the journal and conference papers cited in the present list of activities. Three-dimensional modal global instability analysis in the compression ramp, performed by solving the EVP using the LiGHT code, has delivered analogous findings as regards spanwise periodic linear perturbations. The Reynolds number at which most of the analyses were performed, $Re_{\delta^*} \approx 600$ based on boundary layer displacement thickness at the inflow boundary of the analysis domain, was chosen to be relatively low such that modal perturbations be damped and the role of non-modal linear instability be examined. Transient growth analyses performed by numerical solution of the SVD within the LiGHT code has produced the rather interesting result of relatively large, $O(10^5)$, energy gain of the three-dimensional perturbations analysed. While higher Reynolds numbers will no doubt lead to more and more unstable global modes being amplified exponentially, the implication of our non-modal analysis is that the transient growth scenario discovered herein will also become (even) stronger; if the theoretically known from incompressible analyses dependence $G_{\max} \sim Re^2$ of the maximal energy gain $G_{\max}$ on the Reynolds number Re is confirmed, it becomes clear that transition in the compression ramp can be the result of either modal or non-modal mechanisms. Work to generate sufficient parametric studies to support this conjecture is underway.

Finally, triple-deck theory addressed in Task (v) has been demonstrated to offer invaluable insight into physics of steady and unsteady laminar separation in both subsonic and supersonic flows. Within its limitations, this theory has been used successfully in the past to address two-dimensional instability of supersonic flow over a compression corner at orders of magnitude reduced computing cost compared with full direct numerical simulation or the present global instability analyses. The interactive boundary layer equations governing the two-dimensional laminar base flow have been derived and programmed during this Grant effort. The corresponding three-dimensional equations governing global instability of three-dimensional disturbances have been derived for the first time herein and their numerical solution has been programmed. Presently, base flow solutions obtained by direct numerical simulation in Task (ii) are being analyzed in the framework of triple-deck theory with the aim of producing global linear instability predictions that are uniformly valid at all Reynolds and supersonic Mach numbers.

Activities supported by the Grant

Personnel Supported by the Grant:

- Nicolas Cerulus (US Citizen) *Ph.D. Student*. Fully supported by this Grant as registered postgraduate student at the Department of Mechanical, Materials and Aerospace Engineering of the University of Liverpool (UoL).
- Helio Quintanilha Jr. *Ph.D. Student*. Partially supported by this Grant as registered postgraduate student at the UoL Department of Mechanical, Materials and Aerospace Engineering.
- Nathan Duggins *M.Sc. Student*. Registered at the Master of Science in Advanced Aerospace Engineering course of UoL in 2017, not receiving support from the Grant.
- Vassilis Theofilis. *Professor*.

Activities supported by the Grant:

AFOSR/ONR Joint Annual Review Meeting: U of Colorado, Boulder CO, July 8-12, 2019

- Theofilis V. "Global Modal and Non-Modal Instability Analyses of Shock-Induced Separation Bubbles"

AFOSR/ONR Joint Annual Review Meeting: BRICC, Arlington VA, July 16-19, 2018

- Theofilis V. "Global Modal and Non-Modal Instability Analyses of Shock-Induced Separation Bubbles"

AFOSR Annual Review Meeting: NASA Langley RC, Hampton VA, July 24-27, 2017

- Theofilis V. "Global Modal and Non-Modal Instability Analyses of Shock-Induced Separation Bubbles"
- Theofilis V. "Global Transient Growth Mechanisms in High-Speed Flows with Application to the Elliptic Cone"

Interactions:

- Prof. Pino Martin. Regular online meetings held via Skype. Met at AFOSR Annual Review, APS and AIAA meetings.
- Mr. James Trichilo (Graduate Student of Prof. Martin). Mr. Trichilo visited the University of Liverpool during the first two weeks of March 2017 in order to be introduced to triple-deck theory, and has exchanged information with the Liverpool group since his return to Maryland. The PI met with Mr. Trichilo for the last time at the 2019 AIAA SciTech meeting, prior to the student's departure from the group of Prof. Martin.
- Prof. Deborah Levin. Weekly online meetings held via Skype with Prof. Levin and her graduate students, Mr. Ozgur Tumuklu and Mr. Saurabh Sawant throughout the duration of the grant. Interactions led to the publication of archival and conference publications listed herein. A Grant proposal was submitted to ONR and has been approved, work to start on April 1, 2020.

Dissemination:

Seminar and Lecture Series delivered:

- Theofilis V (2018) "Global Linear Stability Theory in Aerospace Applications", 61st Ludwig Prandtl Memorial Lecture - Extended. TechEdge, Dayton OH, July 12, 2018. Prof. Theofilis presented an extended version of the Ludwig Prandtl Memorial Lecture to AFRL scientists at Wright-Patterson AFB.
- Theofilis V. (2017) "Global Stability Theory". Ohio State University, Columbus, OH, June 2017. Invited by Prof. D. Gaitonde, V. Theofilis held a week-long lecture series at OSU, attended by graduate students and members of the research group of Prof. Gaitonde. One day of the lectures was attended by a group of AFRL scientists led by Dr. M. Visbal.

Invited Presentations:

- Theofilis V (2018) "Global Linear Stability Theory in Aerospace Applications", 61st Ludwig Prandtl Memorial Lecture opened the GAMM 2018 conference, Technical University of Munich, March 19, 2018.
- Levin DA, Tumuklu O, Theofilis V (2018) Hypersonic Flows at Moderate Reynolds Numbers", 31st Rarefied Gas Dynamics symposium, July 23–27, 2018, Glasgow, UK
- Theofilis V (2017) "Massively parallel solution of (global) linear stability (non-symmetric complex generalized)eigenvalue and singular value problems", October 13, Imperial College London, UK
- Theofilis V (2017) "An overview of global stability analysis: highlights, a closure and new frontiers". 70th Annual Meeting of the APS Division of Fluid Dynamics, November 19-21, Denver, CO

Journal publications:

- Cerulus N, Santos R, Quintanilha Jr H, Alves L, Theofilis V (2020) "Modal and nonmodal laminar-turbulent transition scenarios in supersonic compression ramp flows", Journal of Fluid Mechanics (in preparation)
- Tumuklu O, Levin DA, Theofilis V (2019) "Modal analysis with proper orthogonal decomposition of hypersonic separated flows over a double wedge", Phys. Rev. Fluids 4:033403 (Editor's Suggestion)

- Tumuklu O, Levin DA, Theofilis V (2019) "Kinetic modeling of unsteady hypersonic flows over a tick geometry", *Physics of Fluids* 31:056108
- Meeri Kim "Analysis of nonequilibrium effects characterizes shock wave-boundary layer interactions in hypersonic flow", *AIP Scilight*, 440004 (2018)
- Tumuklu O, Theofilis V, Levin DA (2018) "On the unsteadiness of shock-laminar boundary layer interactions of hypersonic flows over a double cone", *Physics of Fluids* 30:106111 (Featured Article)
- Tumuklu O, Levin DA, Theofilis V (2018) "Investigation of unsteady, hypersonic, laminar separated flows over a double cone geometry using a kinetic approach", *Physics of Fluids* 30:046103 (Editor's Pick)
- Taira K, Brunton SL, Dawson STM, Rowley CW, Colonius T, McKeon BJ, Schmidt OT, Gordeyev S, Theofilis V & Ukeiley LS (2017) "Modal Analysis of Fluid Flows: An Overview". *AIAA Journal* 55(12):4013–4041

Conference Presentations:

(In case of multiple authors, the presenter is underlined)

2019

- Cerulus N, Santos R, Quintanilha Jr. H, Alves L, Theofilis V (2019) "Linear Instability in a Supersonic Compression Ramp", 8th IUTAM Laminar-Turbulent Transition Symposium, London, Sept. 2-6, 2019
- Quintanilha Jr H, Santos R, Alves L, Theofilis V (2019) "Linear Instability Mechanisms of Supersonic Flow Past Blunt Bodies", 8th IUTAM Laminar-Turbulent Transition Symposium, London, Sept. 2-6, 2019
- Cerulus N, Theofilis V (2019) "Global Modal and Non-Modal Instability Analyses of Shock-Induced Separation Bubbles", UK Turbulence Consortium Annual Review, London, Sept. 10–11, 2019
- Sawant S, Tumuklu O, Theofilis V, Levin DA (2019) "Linear Instability of Shock-Dominated Laminar Hypersonic Separated Flows", 8th IUTAM Laminar-Turbulent Transition Symposium, London, Sept. 2–6, 2019
- Tumuklu O, Levin DA, Theofilis V. (2019) "Use of particle methods for understanding hypersonic shock boundary layer interactions", *AIP Conference Proceedings* 2132:090002
- Quintanilha Jr H, Theofilis V, Hanifi A (2019) "Global transient-growth analysis of hypersonic flow on the HIFiRE-5 elliptic cone model", *AIAA SciTech*, Jan 7–11 2019, San Diego CA, AIAA Pap. 2019-2148
- Santos R, Alves L, Cerulus N, Theofilis V (2019) "On Two-Dimensional Steady-States of Supersonic Flows Over Compression Ramps and their Global Linear Instability", *AIAA SciTech*, Jan 7–11 2019, San Diego CA, AIAA Pap. 2019-2321

2018

- Cerulus N, Theofilis V (2018) "Global Modal and Non-Modal Instability Analyses of Shock-Induced Separation Bubbles", UK Turbulence Consortium Annual Review, London, Sept. 10–11, 2018
- Karami S, Stegeman P, Schmid PJ, Theofilis V, Soria J (2018) "Linearised dynamics and non-modal instability analysis of an impinging under-expanded supersonic jet", *AIP J. Phys.: Conf. Ser.* 1001:012019
- Tumuklu O, Levin DA, Theofilis V (2018) "Development of New Techniques for Studying Unsteady Behavior in Hypersonic Boundary Layer Interactions", *AIAA Aviation*, June 25-29 2018, Atlanta, GA, AIAA Pap. 2018-3709

- Tumuklu O, Levin DA, Theofilis V (2018) "Effects of Reynolds Number on Laminar Boundary Layer Shock-Interaction Hypersonic Flows on a Double Cone", AIAA Aviation, June 25-29 2018, Atlanta, GA, AIAA Pap. 2018-4032
- Quintanilha Jr H, Kataras PB, Theofilis V, Hanifi A (2018) "Nonmodal stability analysis of the HIFiRE-5 elliptic cone model flow in different flight altitudes", 58th Israel Annual Conference on Aerospace Sciences, March 2018, Tel Aviv and Haifa, Israel
- Quintanilha Jr H, Alves LS (2018) "On the existence of multiple self-excitation frequencies in the absolute instability of interacting mixing-layers and wakes", 31st Congress of the International Council of the Aeronautical Sciences, September 2018, Belo Horizonte, Brazil

2017

- Quintanilha Jr H, Santos R, Alves LS, Theofilis V (2017) "Distributed Solution of Global Eigenvalue Problems on Large Clusters", 23rd AIAA Computational Fluid Dynamics Conference, 5-9 June 2017, Denver, CO
- Tumuklu O, Levin D, Theofilis V (2017) "On the Temporal Evolution in Laminar Separated Boundary Layer Shock-Interaction Flows using DSMC", AIAA Paper 2017-1614
- Theofilis V (2017) "Massively parallel solution of the global linear instability non-symmetric generalized eigenvalue problem". NATO AVT-240 Transition Meeting, 10-11 April 2017, Tullahoma, Tennessee
- Theofilis V (2017) "Global instability analysis of DSMC-computed hypersonic laminar separated flows: shock boundary layer interactions on a double cone at Ma=16". NATO AVT-240 Transition Meeting, 10-11 April 2017, Tullahoma, Tennessee
- Tumuklu O, Perez JM, Levin D, Theofilis V (2017) "On Linear Stability Analyses of Hypersonic Laminar Separated Flows in a DSMC Framework Part I: Base Flow Computations in a Double Cone and a Tick Model", 57th Israel Annual Conference on Aerospace Sciences, March 2017, Tel Aviv and Haifa, Israel.
- Tumuklu O, Perez JM, Theofilis V, Levin D (2017) "On Linear Stability Analyses of Hypersonic Laminar Separated Flows in a DSMC Framework Part II Residuals Algorithm and the Least Damped Global Modes", 57th Israel Annual Conference on Aerospace Sciences, March 2017, Tel Aviv and Haifa, Israel.
- Mathias MS, Medeiros MAF, Theofilis V (2017) "Implementation of a Jacobian-free method for the instability analysis of a compressible flow over an open cavity", Paper COBEM 2017-2889, 24th ABCM International Congress of Mechanical Engineering December 3-8, 2017, Curitiba, PR, Brazil.

Travel Support:

- N. Cerulus used grant support to attend and present at the 8th IUTAM Laminar-Turbulent Transition symposium, London, Sept. 2019; the 2019 SciTech conference, San Diego CA, Jan. 2019; the Autumn School on Aeroacoustics and Thermoacoustics of Propulsion Systems, Stockholm, Oct. 2017; the ERCOF-TAC Spring School in Aeroacoustics, Montestigliano, Italy, April 2017
- H. Quintanilha used grant support to attend and present at the 8th IUTAM Laminar-Turbulent Transition symposium, London, Sept. 2019; the 2019 SciTech conference, San Diego CA, Jan. 2019; the 2018 IACAS conference, Tel Aviv, March 2018;
- V. Theofilis used grant support to attend the NATO AVT-240 Transition Meeting, West Lafayette, Indiana, April 2019 and Tullahoma, Tennessee, April 2017; AIAA SciTech Conference, San Diego CA, 2019; 23rd AIAA Computational Fluid Dynamics Conference, Denver, CO, June 2017; AFOSR Annual Review meetings: Boulder CO, July 2019, Arlington VA, July 2018; Hampton, VA, July 2017; APS Fluid Dynamics meeting, Denver, CO, Nov 2017

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Chapter 1

Introduction

Separation generated in supersonic (and hypersonic) flow by a shock (system) impinging on the boundary layer formed on a flat-plate, that associated with various-shaped deviations from the flat geometry, such as the compression ramp schematically depicted in Fig. 1.1, as well as separation on axisymmetric configurations have all been the subject of intense investigations ever since the early experimental works of Chapman et al. (1957, 1958) recognized the significance of separation to the laminar-turbulent transition process at high speeds. In turn, correct prediction of the instability dynamics of flow separation is key to the exact determination of the relative portions of laminar and turbulent flow on parts of a vehicle traveling at supersonic speeds, and to the consequent correct prediction of heat transfer through the surface of vehicle parts that are critical to aerodynamic performance.

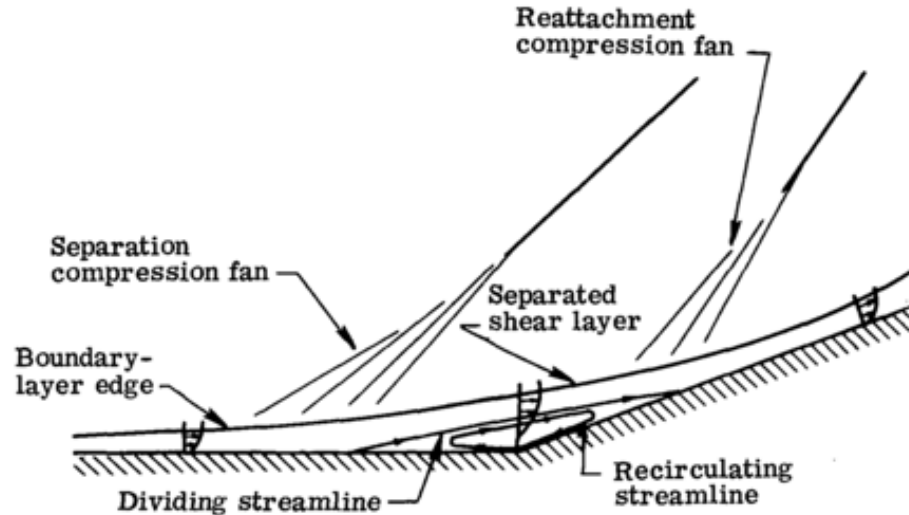


Figure 1.1: Schematic representation of the compression corner (Carter, 1972)

Canonical cases studied in this context include separation induced by shocks impinging on laminar or turbulent boundary layers on the flat plate, separation caused by forward- or backward-facing steps, or roughness elements embedded inside the boundary layer, having a size of the order of the local boundary layer thickness, as well as separation localized at small- and finite-angle, two-dimensional or oblique wedges. Edney (1968) documented

experimentally and classified a number of patterns arising in shock-induced separation, Babinsky and Harvey (2011) present an overview of the observed phenomena, while Gaitonde (2015) recently discussed progress made in understanding this class of flows. A key issue identified is the low-frequency shock oscillation, purported in the literature to be associated with some form of linear flow instability. Understanding the precise nature of this instability, as well as its dependence on the underlying base or mean flow parameters, would close the currently open debate on this topic and also open up avenues to pursue flow control via control of flow instabilities. The latter approach is widely accepted to be the most efficient, from an energy-balance point of view, theoretically founded flow control methodology (Collis et al., 2004) which has been employed successfully in a wide range of applications of aeronautical interest (Joslin and Miller, 2009).

1.1 Classic linear stability analyses

Over the last half-century analysis of instability phenomena arising from shock/laminar boundary-layer interaction has employed classic modal linear stability theory (LST) based on the parallel-flow approximation and solutions of variants of the Rayleigh or the Orr-Sommerfeld equation (Lin, 1955b; Stetson and Kimmel, 1992; Schneider, 2004; Fedorov, 2011) and focused on study of either small-amplitude perturbations in their passage through model shock configurations, or on the region around separation bubbles in isolation from the shock system in the surrounding flow. In all cases the parallel flow assumption implied study of modal amplification of nominally streamwise (and spanwise) homogeneous flows. However, the neglect of a wall-normal base flow velocity, inherent in the parallel-flow approximation, severely limits the usefulness of the results of the classic LST methodology, on account of the strong coupled dependence of flow in the separation region on both the streamwise and the wall-normal spatial coordinate. Nevertheless, application of classic LST to analyze small-amplitude perturbations impinging on a straight shock away from solid surfaces has substantially advanced knowledge in this area, as will be discussed shortly.

On the other hand, boundary-layer linear stability theory, accompanied by experiment and, more recently, by numerical simulation, has been a mainstay of compressible boundary layer laminar-turbulent flow transition research for over half a century. Although compressible boundary layer flow instability was treated as a simple extension of the incompressible regime in early theoretical studies (Lin, 1955b), numerical solutions of the pertinent one-dimensional eigenvalue problem of the Orr-Sommerfeld class by Mack (1969b) have demonstrated the existence of linear disturbances particular to compressible flow which dominate the transition process at high speeds (Mack, 1984; Fedorov, 2011). At supersonic Mach numbers, the amplification of the compressible eigenmodes is of the same order of magnitude as that of their incompressible counterparts, while the compressible modes dominate the transition process in the high supersonic regime. Such disturbances are also found to play a key role in the transition process on axisymmetric bodies (Stetson and Kimmel, 1992; Schneider, 2004).

Owing to their relevance to components of high-speed vehicles, two configurations, the flat plate and the cone of circular cross-section, have received particular attention with respect to the linear instability of their respective (parallel) boundary layers. By virtue of the geometric simplicity of these configurations the assumption of homogeneity along the streamwise and spanwise/azimuthal spatial directions holds in both cases, while in the third, wall normal spatial direction the equations governing linear disturbances are expressed as an one-dimensional eigenvalue problem and are fully resolved. The straightforward nature of the latter problem results in predictions of linear instability characteristics of the boundary layer on flat plates and circular cones which are both reliable and easy to obtain numerically (Macaraeg et al., 1988; Malik, 1990). Hypersonic flow adds yet another level of complexity to compressible flow stability scenarios, on account of the diverse possibilities to incorporate atmospheric chemistry in the analysis and the profound effects of the choices made on both basic and perturbed flow characteristics (Malik and Anderson, 1991; Stuckert and Reed, 1994).

The parallel flow assumption stipulates that two spatial directions be treated as homogeneous (resulting in a wave-like character of linear disturbances) while the third, wall-normal spatial direction, along which the wall-normal base flow velocity component is neglected, is fully resolved. The parallel flow assumption is invalidated

when instability of a shock system in wall proximity is to be analyzed, or when flow over/through geometrically complex configurations is to be addressed. In both situations the ability to treat the streamwise spatial direction as homogeneous and expand it in Fourier modes is lost and one needs to resolve the flowfield in question in a coupled manner. Of course, non-parallel basic states (as well as nonlinear interactions of the modes predicted by classic linear theory) can be studied by Direct Numerical Simulation (DNS) (Fasel, 1976); DNS is also the methodology on which one almost exclusively relies for the prediction of instability characteristics in more complex geometries such as a (separation-inducing) wedge (Adams, 1998; Zhong, 1998). Needless to say, owing to its cost, a DNS methodology is hardly appropriate for parametric studies of instability phenomena. Alternative linear stability theories have been developed in the last decades, all of which have now been applied to high-speed flow.

1.1.1 Extensions of the classic linear theory

Extensions of the classic linear theory in both incompressible and compressible flow have followed four distinct avenues. First, in the context of attached (and to some extent, separated) laminar boundary layer flows the assumption of streamwise homogeneity has been relaxed and the instability of weakly nonparallel boundary layers can now be addressed by using the concept of the Parabolized Stability Equations (PSE) (Herbert, 1997), first applied successfully to the Blasius boundary layer by Bertolotti et al. (1992) and later extended, amongst others, to compressible flows by Haynes and Reed (2000).

Second, the non-normality of the linear operator underlying linear stability investigations has been recognized as the origin of transient growth; this phenomenon has explained shortcomings of the classical linear theory such as subcritical instability of plane Poiseuille and stability of plane Couette and Hagen-Poiseuille flow (Trefethen et al., 1993; Schmid and Henningson, 2001a). Until relatively recently, all transient growth analyses have addressed parallel or weakly nonparallel flows; the work of Abdessemed et al. (2009) is the first non-modal instability analysis of a fully nonparallel flow, that in a low-pressure-turbine blade cascade. Some details on transient growth analysis are discussed in the next Section 1.1.2.

Third, the concept of absolute/convective linear instability (Huerre and Monkewitz, 1990) has been introduced still in parallel flows, in order to classify flow behavior after the introduction of infinitesimal disturbances. Should the disturbance propagate in the downstream direction, growing or decaying in the process, the flow is called convectively unstable. If the disturbance remains at its point of introduction and grows, the flow is called absolutely unstable. Both concepts apply to strictly parallel flows, while absolute instability of weakly nonparallel flows has (confusingly) been termed *global* instability by Chomaz (2005).

Fourth, the currently accepted definition of *global* instability has been introduced, in order to describe linear instability of essentially nonparallel flows (Theofilis, 2011). Global linear stability theory involves relaxation of the parallel flow approximation and full coupled resolution of multiple inhomogeneous directions in the linearized Navier-Stokes operator (incompressible or compressible) in the context of global linear stability theory; the multiple facets of this analysis are summarized in Table 2.1. In order to discern from the type of analysis discussed by Chomaz (2005), the terms *BiGlobal* and *TriGlobal* instability analysis have been introduced (Theofilis, 2003b). BiGlobal analysis proceeds along the same conceptual lines as the classic linear theory, although the key difference is that BiGlobal theory considers an inhomogeneous basic state in two spatial directions; the third direction is treated as periodic and resolved by a Fourier expansion in the instability problem. The coupled resolution of two inhomogeneous spatial directions results in a large partial-derivative eigenvalue problem, the variable coefficients of which are provided by the two-dimensional basic flow. The requirement for accuracy of the coefficients of the BiGlobal eigenvalue problem is typically satisfied by employing two-dimensional DNS for the recovery of the basic state; the high-order computations of shock-induced separation bubble flow by Visbal and Gaitonde (2005) provide such basic states.

Theory	Basic state		Eigenmode	
			Amplitude function	Phase function
TriGlobal	$\bar{q}(x_1, x_2, x_3)$	+	$\hat{q}(x_1, x_2, x_3)$	$\cdot \exp \Theta_{2d}(t) \quad ; \quad \Theta_{2d} = -\Omega t$
Parabolized Stability Equations - 3D	$\bar{q}(x_1'', x_2, x_3)$	+	$\hat{q}(x_1'', x_2, x_3)$	$\cdot \exp \Theta_{2d}''(x_1''); t \quad ; \quad \Theta_{2d}'' = \int_{x=x_0}^{x''} \alpha(\xi) d\xi - \Omega t$
BiGlobal Secondary Theory	$\bar{q}(x_1, x_2, x_3'; t')$	+	$\sum_{n=-\infty}^{\infty} \hat{q}(x_1, x_2, x_3', t')$	$\cdot \exp \Theta_{2d}'(t') \quad ; \quad \Theta_{2d}' = \sigma t$
BiGlobal	$\bar{q}(x_1, x_2)$	+	$\hat{q}(x_1, x_2)$	$\cdot \exp \Theta_{2d}(x_3; t) \quad ; \quad \Theta_{2d} = \beta x_3 - \Omega t$
Parabolized Stability Equations	$\bar{q}(x_1'', x_2)$	+	$\hat{q}(x_1'', x_2)$	$\cdot \exp \Theta_{1d}''(x_1'', x_3; t) \quad ; \quad \Theta_{1d}'' = \int_{x=x_0}^{x''} \alpha(\xi) d\xi + \beta x_3 - \Omega t$
Herbert Secondary Theory	$\bar{q}(x_1', x_2; t')$	+	$\sum_{n=-\infty}^{\infty} \hat{q}(x_1', x_2, t')$	$\cdot \exp \Theta_{1d}'(x_3; t') \quad ; \quad \Theta_{1d}' = \beta x_3 - \sigma t$
Rayleigh / Orr-Sommerfeld	$\bar{q}(x_2)$	+	$\hat{q}(x_2)$	$\cdot \exp \Theta_{1d}(x_1, x_3; t) \quad ; \quad \Theta_{1d} = \alpha x_1 + \beta x_3 - \Omega t$
Internal Gravity Waves / Rayleigh-Taylor/-Benard Instabilities	$\mathbf{0}$	+	$\hat{q}(x_2)$	$\cdot \exp \Theta_{1d}(x_1, x_3; t) \quad ; \quad \Theta_{1d} = \alpha x_1 + \beta x_3 - \Omega t$

Table 1.1: Linear Stability Theory Decompositions. Primes denote periodic variables and double primes denote spatial coordinates which vary weakly compared with their unprimed counterparts (Theofilis, 2011)

1.1.2 Modal and Non-Modal Stability Theory

Linear stability analysis describes the evolution of small perturbations as an initial-value or an eigenvalue problem. Modal analysis considers the temporal or spatial development of the perturbations in the asymptotically large time-limit, $t \rightarrow \infty$, by solving an eigenvalue (EVP) problem. Although the modal scenario is the only relevant to linear flow instability when unstable eigenmodes are identified, situations arise in which all solutions to the eigenvalue problem are damped, and yet the flow becomes (linearly) unstable and transitions to turbulence. The theoretical framework encompasses dynamical systems described by non-normal matrices discretizing the respective linear operators, whereby non-normality is a physical property of the dynamical system, as opposed to a property related with the numerical discretization. The related ideas was first discussed in a fluids context by Trefethen et al. (1993) and are detailed in the book of Schmid and Henningson (2001a) which mainly focuses on incompressible flows. Rather than exponential growth of perturbations at large times, one seeks algebraic transient growth of the perturbation energy at small times. When this transient growth is found to be large, the modal linear instability scenario is by-passed and laminar flow transitions to turbulence through a linear transient growth mechanism. Typically modal / exponential instability is associated with infinitesimally small perturbation environments (hence its relevance to flight) while non-modal / transient growth can be encountered at larger background disturbance amplitudes, as schematically depicted in Figure 1.2.

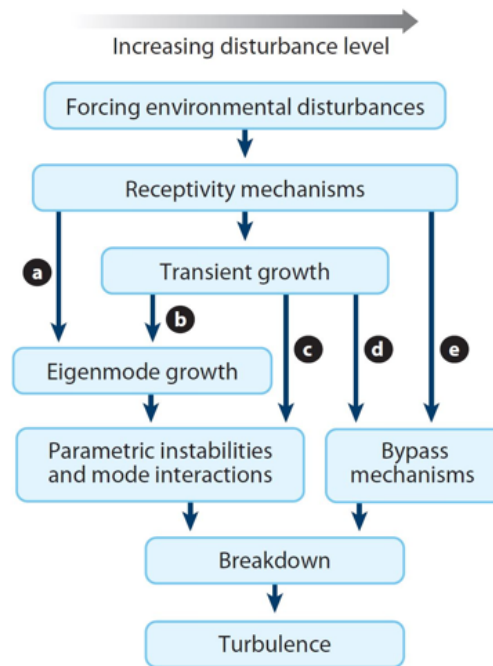


Figure 1.2: Paths to turbulence (Fedorov, 2011)

Hanifi et al. (1996a) were the first to apply transient growth concepts to a compressible boundary layer and demonstrated the relevance of this mechanism to laminar-turbulent transition. It was shown that the maximum transient energy growth (the "gain") scales with Re^2 and increases with Mach number, leading to the formation of structures that are independent of the streamwise direction, while being periodic along the spanwise direction. Additionally the fact that optimal transient growth could be excited by non-linear interactions of oblique first mode waves showed the potential of explaining the difference in timescales between oblique and second mode flow breakdown. Further analyses to investigate the relation between transient and streamwise independent instabilities growth showed that for small times, when viscosity effects are small, the two were in good agreement Hanifi and Henningson (1998). Finally Tempelmann et al. (2012) studied spatial optimal growth over swept flat plates in the compressible regime and demonstrated non-modal / transient growth in boundary layers at conditions of linear stability of the underlying (modal) crossflow perturbations. The disturbances that experienced that growth became crossflow modes downstream, generating streamwise streaks for all the different Mach numbers tested.

1.2 Literature Survey

1.2.1 Modal instability analysis of shock-dominated flows

Classic modal viscous instability analysis of flows including (straight) shocks away from solid surfaces and the inviscid counterpart of the analysis in flows containing shocks attached to a surface has reached a relatively high level of maturity. The key question posed in this context has been shock-wave oscillations, since these can have serious repercussions for high-speed flows over bodies. Such oscillations may be triggered by freestream disturbances of small amplitude ahead of the shock, which can be modally (exponentially) amplified leading to strong nonlinear effects. Originally attached boundary layers (on the surface of the body, behind the shock) can be caused to separate, even leading the boundary layer to transition. Such effects have been observed in supersonic wind tunnel experiments in the presence of sound waves radiated by turbulent tunnel-wall boundary layers or by freestream turbulence, a phenomenon which is actually related with the extremely challenging nature of generating and maintaining a supersonic tunnel in a "quiet" state. In such situations, acoustic wave mainstream disturbances are first processed by the shock wave induced by the body in supersonic stream, and then interact with the boundary layer to cause instability. The shock-body combination is known to produce a wide spectrum of wavelengths from monochromatic waves, i.e. a freestream disturbance with a single wavelength; in particular, the production of shorter wavelength disturbances than would be expected from the free shock-disturbance interaction is noted. Then these shorter wavelength perturbations resulting from the shock-body combination could go on to interact with the boundary layer on the surface of the body, leading to the aforementioned possible boundary-layer separation or instability. In the case of turbulence-shock-body interaction, it is noted that the turbulence itself is composed of a continuous spectrum of wavelengths and the interaction would produce even more short-wavelength disturbances than the single wavelength disturbance described here. Such interactions could possibly enhance mixing of a multi-component flow owing to the amplification and generation of vorticity behind the shock.

Shock-induced separation, or the presence of a wedge in a uniform supersonic stream, both lead to the supersonic flow abruptly changing direction through an oblique shock wave, details of which are determined by the Rankine-Hugoniot conditions. Moore (1954); Ribner (1954) and McKenzie and Westphal (1968) investigated the processing of single-wavelength disturbances by freely propagating shocks, and noted that through the interaction of the shock and the freestream disturbance, single wavelength entropy, acoustic and vorticity disturbances are all produced behind the shock in response to an isolated acoustic disturbance, or to an isolated entropy disturbance, or to an isolated vorticity disturbance. In the presence of separation (or a wedge), the flow behind the shock is no longer comprised of single wavelength acoustic, entropy and vorticity disturbances and limited work appears to have been done on the theoretical/computational solution of the problem of interaction between shock waves and boundary-layer instabilities, although most of this work has focused on the wedge problem (only). Cowley and Hall (1988) presented an asymptotic model, applicable to three-dimensional viscous modes of instability, with appropriately simplified conditions applied on the shock surface (derived from the Rankine-Hugoniot conditions), which was taken to lie just outside of the boundary layer. Chang et al. (1990) considered the viscous small disturbance equations with parallel mean flow assumption, using the full inviscid Rankine-Hugoniot conditions on the shock. The related problem of freestream turbulence amplification caused through interactions with shock waves has been considered by Chang (1957); Anyiwo and Bushnell (1982); Hussaini et al. (1985); Meadows et al. (1991), and Kumar et al. (1989) who confirmed the important result found by Moore (1954); Ribner (1954) and McKenzie and Westphal (1968) that a pure acoustic, vorticity, or entropy wave, upon interacting with a plane shock wave, generally generates all three classes of disturbance downstream.

The interaction of a shear wave with a detonation wave induced by a wedge in a supersonic flow, as well as the response of the oblique detonation to oscillations of the wedge was considered by Lasseigne and Hussaini (1993). In studying the interaction of a shear wave with the detonation, they assumed that a weak steady sinusoidal vorticity wave is obliquely convected through an overdriven detonation attached to a wedge. Their concern was

with the response, measured by the deviation of the detonation position from its unperturbed state and by the vorticity and pressure at the detonation, as the degree of overdrive is increased. They compared the flow in the presence of the wedge with the unobstructed flow field considered in Jackson et al. (1990). In particular, for this special type of disturbance (i.e. steady), they found an infinite discrete spectrum of disturbances to exist behind the shock. Regarding shock stability per se, it is generally accepted that the so-called strong shock solution Liepmann and Roshko (1957a) is unstable (see Levinson (1945); Carrier (1949); Henderson and Atkinson (1976); Rusanov and Sharakshanae (1980) and Salas and Morgan (1983)), whilst the proof for the stability of the weak shock solution has largely been either numerical (Rusanov and Sharakshanae, 1980; Salas and Morgan, 1983) or subject to some (artificial) restrictions. All the aforementioned analyses have focused on straight shock waves attached to wedges. The situation with regard to straight shocks, related with the generation of separation, and curved shocks, forming downstream of the shock-induced separation on the flat plate, is not at all clear at present. The curvature of the shock inevitably leads to vorticity, and hence non-uniform flow behind the shock, which undoubtedly complicates matters. On the other hand, the ability to perform linear global instability analysis of spatially essentially inhomogeneous flows has opened up an entirely new avenue of possibilities for the linear stability analysis of shock-dominated flows; such efforts have commenced in the last decade, as briefly detailed below.

1.2.2 Global linear stability theory

In addressing the question of how a separation bubble, laminar or turbulent, interacts with the shock system that generates it, Trilling (1957) presented a simplified model which, within its assumptions, predicted self-excitation of the separation bubble at $1.57 \leq Ma \leq 1.65$, $Re \sim 1.2 \times 10^5$, thereby classifying the shock-induced separation bubble as an *oscillator* at those conditions. By contrast, Plotkin (1975) proposed a different model to describe instability of the shock position being driven by turbulent boundary layer fluctuations and, based on the agreement of his model with experimental data, asserted that the only role of instability is to dampen shock motion and restore its original position, in other words, the (mean) turbulent separation bubble is an *amplifier*. Neither author has employed linear stability theory in the sense described in the earlier sections.

Global instability analysis of a separated laminar boundary layer flow was presented by Theofilis et al. (2000), who applied the then nascent (modal) global theory to a laminar separation bubble embedded in the incompressible flat-plate boundary layer, treating the separated flow region as an essentially nonparallel, three-dimensional spanwise homogeneous flowfield. It was demonstrated that, beyond a certain level of flow recirculation in the bubble, $O(10\%)$, the laminar separation zone has the capacity to become self-excited and amplify *stationary, spanwise periodic three-dimensional* perturbations, a behavior of the separation zone known as an *oscillator*. This previously unknown modal path to laminar-turbulent transition is complementary to and may co-exist with the well-known *amplifier* function of the laminar separation bubble, whereby two- or three-dimensional disturbances entering the separation zone experience strong exponential amplification through the Kelvin-Helmholtz (KH) instability mechanism at the shear layer.

It should be noted that the absolute/convective instability analysis of Huerre and Monkewitz (1990), applied to a laminar separation bubble, and confusingly also referred to in the literature as *global* instability, can qualitatively predict the oscillator character of the separation zone. However, this theory is based on the assumption of a weakly non-parallel base flow and the level of recirculation that this theory requires for self-excitation of instability is unrealistically high when compared with the predictions delivered by global linear theory based on the solution of the partial-derivative eigenvalue problem: in a given configuration on which both instability approaches have been used, the recirculation level of the steady laminar base flow required for a stationary three-dimensional global mode to amplify in the Huerre and Monkewitz (1990) theory is of $O(30\%)$ of the streamwise component of the free-stream velocity. The corresponding value delivered by global modal analysis is of $O(10\%)$ in the work of Theofilis et al. (2000) and of $O(7.5\%)$ in the more recent analysis of Rodríguez and Theofilis (2010), the latter also confirmed by independently performed direct numerical simulations. The ability to incorporate an arbitrarily large magnitude of wall-normal base flow velocity in the flavor of global theory

introduced by Theofilis et al. (2000), has resulted in instability analysis based on numerical solution of the partial-derivative eigenvalue problem to be adopted as the method of choice to analyze global flow instability in a plethora of separated flow configurations which have since appeared in the literature.

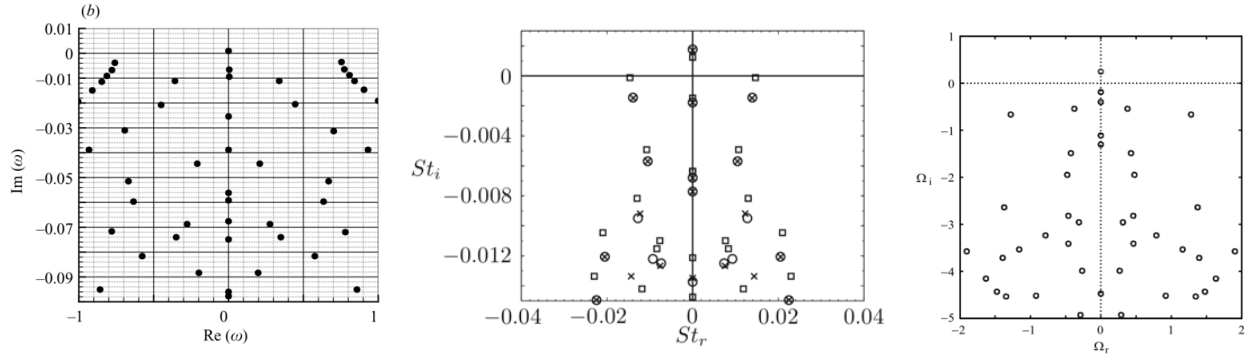


Figure 1.3: Eigenspectra presenting evidence of unstable stationary three-dimensional ($\beta \neq 0$) perturbations in three laminar separation bubbles: *Left*: LSB arising from impinging shock on a flat plate at $M = 2.15$, $Re = 10^5$ (Robinet, 2007); *Middle*: LSB arising from impinging shock on a flat plate at $M = 2.28$, $Re = O(10^6)$ (Nichols et al., 2017); *Right*: LSB arising from (Howarth) linear deceleration profile on a flat plate in incompressible flow at $Re = 10^3$ (Theofilis et al., 2000)

The first modal global linear instability analysis of interest to shock-induced separation has been the two-dimensional global mode analyses of Crouch et al. (2007, 2009) who monitored the separated flow generated by two-dimensional shock/turbulent boundary-layer interaction over an airfoil at transonic conditions. Results obtained by these authors could successfully associate the buffeting phenomenon observed on airfoils with the frequency of the most-amplified *two-dimensional* global mode of the underlying turbulent mean flow, the latter obtained by solution of the Reynolds Averaged Navier Stokes equations. On the other hand, the work of Robinet (2007) has been the first documented global linear instability analysis of a separation bubble generated by shock wave / laminar boundary layer interaction. It was asserted that the analog of the stationary three-dimensional global mode discovered by Theofilis et al. (2000) in incompressible flow is also the most amplified instability at the supersonic conditions examined. In other words, Robinet (2007) sustains that the separation bubble in this flow acts as an oscillator. A similar result regarding the existence of self-excited stationary three-dimensional eigenmodes has been obtained in the recent work of Nichols et al. (2017), who analyzed a mean turbulent, as opposed to a steady laminar separation bubble. Nevertheless, the analogies in the amplified spectra corresponding to both supersonic and the incompressible separated flow, shown in Fig. 1.3 is striking.

The clearest evidence to-date regarding the existence of self-excited *oscillator*-type spanwise-periodic disturbances generated inside the laminar separation zone and extending along the downstream ramp wall comes from the DNS work of Egorov et al. (2011). Figure 1.4 displays the vicinity of the ramp, seen from above, and clearly shows three-dimensional structures appearing at a finite time in the simulation. Detailed analysis of the DNS signal was shown at Mach number value $Ma = 5$ and Reynolds numbers around $Re = 6 \times 10^4$. The growth in time of these structures, referred to as *deviations*, has been documented in terms of the time-dependence of the r.m.s. values of the velocity components and the thermodynamic variables. On a semi-logarithmic scale they all exhibit linear growth of their magnitude over several decades, thereby confirming exponential amplification of a self-excited global mode.

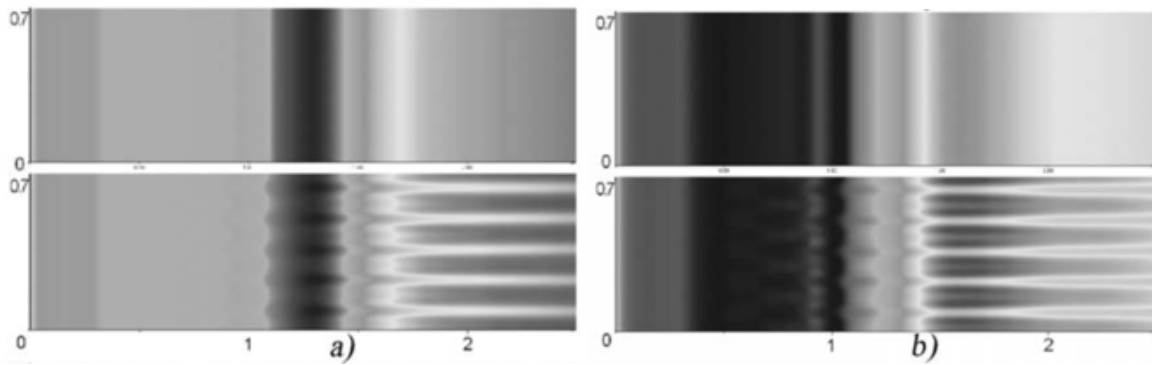


Figure 1.4: Spanwise periodic structures developing at the compression ramp during transition of laminar flow at $Ma = 5$, $Re = 6 \times 10^4$ in terms of skin friction (left) and heat flux (right) (Egorov et al., 2011). The upper images correspond to the initial conditions ($t = 0$), while the lower images correspond to (non-dimensional) time $t = 16$.

As regards global non-modal analyses of separated flows, Abdessemed et al. (2009) performed a complete study of all possible linear instability mechanisms in a cascade of Low Pressure Turbine (LPT) blades in the incompressible regime and produced the world-first demonstration of laminar separation amplifying algebraic, transiently growing non-modal instabilities, besides exponentially amplifying both incoming traveling two-dimensional (KH) and stationary three-dimensional perturbations, respectively through the amplifier and oscillator modal laminar-turbulent transition scenarios discussed earlier. The recent work of He et al. (2017) showed that the same physical mechanisms are present in the massively separated flow associated with stalled airfoils, where it was found that transient growth converts optimal perturbations into the wake mode associated with KH instability. In an axisymmetric (stenotic) geometry, energy gains $G = O(10^{40})$ have been documented (Blackburn et al., 2008), a fact which underlines the ability of transient growth to lead flow to transition by-passing the modal instability scenario.

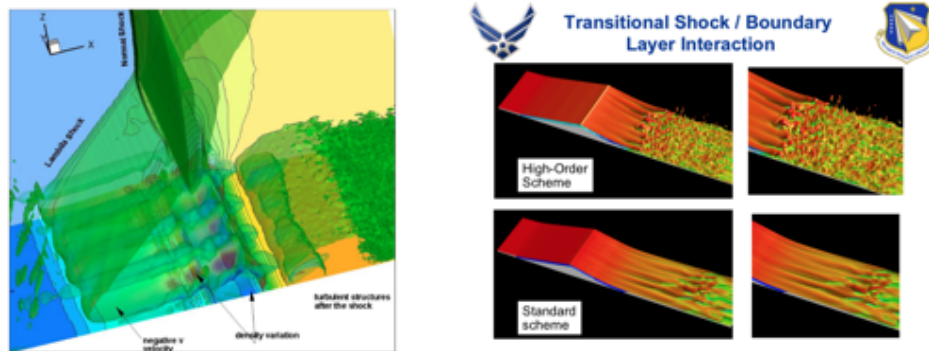


Figure 1.5: *Left*: Three-dimensional spanwise periodic structures and transition to turbulence in $Ma = 2$, $Re = 10k$ SWLBI separation (Ekaterinaris, 2004). *Right*: Same at $Ma = 2$, $Re = 300k$ (Visbal and Gaitonde, 2005) and Visbal (2007), personal communication

The key issue of the relationship between optimal transiently growing perturbations and asymptotic-time, exponentially growing modes has not been addressed in supersonic laminar or turbulent shock-induced separated

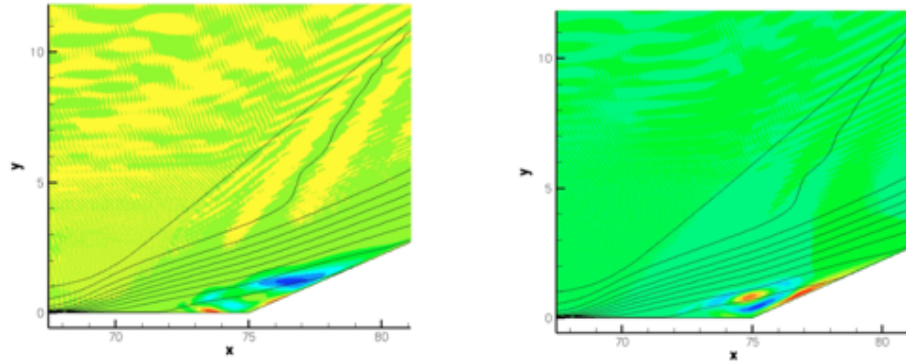


Figure 1.6: Amplitude functions of the streamwise (left) and spanwise (right) perturbation velocity components obtained in preliminary global linear instability analyses of the time- and spanwise-averaged LES data of Poggie et al. (2015) at $\beta = 1$, $L_z = 2\pi$.

boundary layer flow, making this a top priority in the present effort. Concretely, it is worth examining whether the transient growth mechanism, whose signature in incompressible boundary layer flow is algebraic growth of streamwise aligned, spanwise-periodic structures, is also the mechanism responsible for analogous structures being clearly visible in Figure 1.5, taken from high-fidelity laminar-turbulent transition simulations of SLBLI configurations (Ekaterinaris, 2004; Visbal and Gaitonde, 2005).

First global modal instability analyses of the time- and spanwise-averaged turbulent flowfields obtained by Poggie et al. (2015) have been performed. Preliminary results of the amplitude functions of an amplified stationary global mode identified at spanwise wavelength of $L_z = 2\pi$ scaled with the inflow boundary layer displacement thickness, are shown in Figure 1.6. They exhibit the typical characteristics of amplitude functions of separation bubbles, with high-gradient regions appearing near the wall and in the center of the recirculation zone, in addition to the existence of a non-vanishing spanwise perturbation velocity component although the mean flow analyzed has been averaged in the spanwise direction and is nominally two-dimensional. Reconstruction of the composite field resulting from linear superposition of the spanwise homogeneous mean flow and the spanwise periodic perturbation would result in qualitative reproduction of the spanwise periodic patterns observed in experiment. However, it would only make sense to construct such a composite field once both modal and non-modal analyses have been performed at all spanwise wavenumbers, a key target of the present effort.

1.2.3 Large Eddy and Direct Numerical Simulations

Touber and Sandham (2009) have employed three-dimensional large-eddy simulation to examine flow instability in SWTBLI and found the amplifier scenario associated with two-dimensional instability to be the prevailing laminar-turbulent transition mechanism. The latter result could also be interpreted as a-posteriori justification of the purely two-dimensional analysis performed by Crouch and co-workers. This research left open the question whether separation bubbles generated by shock boundary layer interaction amplify distinct branches of eigenmodes in laminar and turbulent flow, and make the separation zone behave as an oscillator in laminar and as an amplifier in turbulent flow. Large-scale simulations following the work of Touber and Sandham (2009) have accumulated additional evidence on the existence of global instability in the separation zone associated with shock wave turbulent boundary layer interaction. Pirozzoli et al. (2010) have analyzed their impinging shock LES database by computing Koopman modes (Rowley et al., 2015) and performing linear global stability analyses. They identified two distinct modes, one linked to the turbulent structures in the incoming boundary layer, the other associated with a cyclic expansion and contraction of the separation region. On the other hand, Grilli et al. (2012) analyzed the flow dynamics of SWTBLI by Dynamic Mode Decomposition (Schmid, 2010)

and identified a low-frequency mode associated with the pulsation of the separation bubble and accompanied by a forward-backward motion of the shock. More recent work of Priebe et al. (2016) also points to the latter phenomenon.

Evidence is accumulated in the literature both in favor and against this interpretation of the instability character of the separation zone. Priebe and Martin (2012) have argued that low frequency unsteadiness observed in shock/turbulent boundary layer interaction over a wedge may be associated with global flow instability and presented as evidence the double-minimum shape of the skin-friction coefficient dependence on the chordwise spatial coordinate. On the other hand, in their quest to identify a unified model explaining shock unsteadiness in turbulent separated flow via flow instability mechanisms, Poggie et al. (2015) examined unsteadiness phenomena observed in a multitude of experiments and flight tests, as well as in a high-resolution large-eddy simulation that these authors performed. Although they explicitly state that the *oscillator* and *amplifier* mechanisms may co-exist in the flows examined, Poggie et al. (2015) come to the conclusion that the two-dimensional *amplifier* unsteadiness scenario prevails in all experimental conditions they have monitored and also in their large-eddy simulations, the conditions of which closely match those of the above-mentioned earlier LES work of Toubert and Sandham (2009).

The demonstration of existence of amplified global modes in the laminar/transitional DNS results of Egorov et al. (2011) was discussed in Section 1.2.2. Unfortunately, these authors do not quantify the domains used in terms of boundary layer properties, but it is evident from the images presented that the spanwise extent of the domain in which simulations were performed is sufficiently wide to capture several periodicity lengths of the global mode. More recent DNS of shock wave turbulent boundary layer interaction by Jammalamadaka et al. (2014), who generated separation from an incipient state to a finite level of recirculation by systematically increasing the shock incidence angle, clearly linked the increase of the incidence angle with the skin-friction coefficient dependence on incidence angle undergoing a qualitative change from a single- to a double minimum. In both the qualitative description of Priebe and Martin (2012) and the results of Jammalamadaka et al. (2014), the separation zone is seen to maintain the primary separation point (straight line in three spatial dimensions) fixed, while secondary reattachment, secondary separation and another secondary reattachment appears in an unsteady (time-periodic) manner. The increasing incidence angle in the DNS of Jammalamadaka et al. (2014) can be interpreted as leading to higher recirculation levels in the bubble which, despite the very different underlying base flow conditions, could in turn be conjectured to be linked with the secondary reattachment-separation-reattachment pattern, which was found to be the signature of spanwise-periodic stationary amplified global mode of the (laminar) separation bubble in the analysis of Theofilis et al. (2000).

Of interest in the DNS of Jammalamadaka et al. (2014) is the study of three-dimensionality of the separation zone resolved. These authors compute the spanwise skin-friction coefficient, $C_{f,z} = 2\sigma_{yz}/(\rho U_\infty)$, and state that: The incoming boundary layer is statistically homogeneous in the spanwise direction and more-or less 'uniform' but we observe several patches of high $C_{f,z}$ values in the separation region as well as in the regions downstream of the reattachment point. This indicates that the instantaneous flow is highly three-dimensional not only in the separation region but also downstream of the reattachment point, where the boundary layer is recovering back to an equilibrium state.

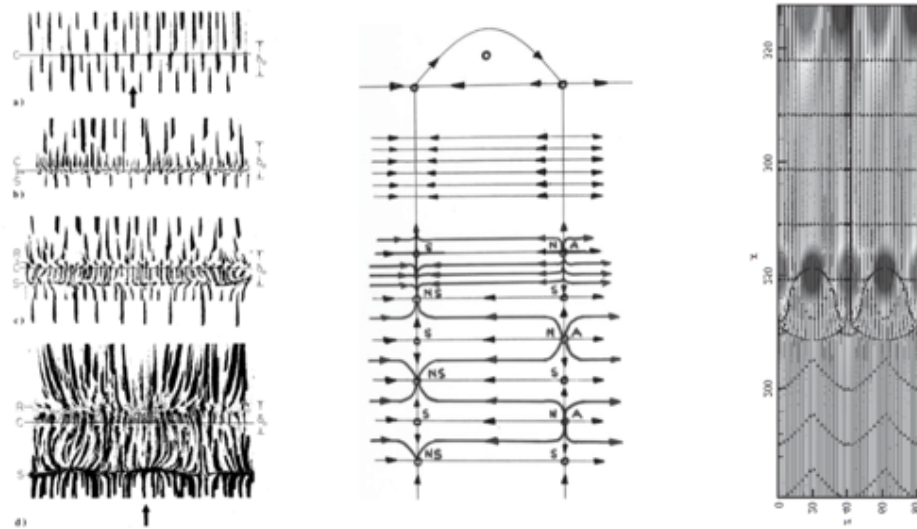


Figure 1.7: *Left*: Flow visualization in the experiments of Settles et al. (1979) in which the progressive three-dimensionalization of the separated flowfield around the wedge is made visible as a function of the wedge angle, from 8deg in (a) to 24deg in (d). *Middle*: Sketched wall-streamline pattern arising from global instability of the stationary three-dimensional global eigenmode of laminar separation in the incompressible flat-plate boundary layer (Theofilis et al., 2000). *Right*: Computed wall-shear distribution of the composite field resulting from linear superposition of the most unstable three-dimensional stationary global mode of steady laminar 2D separation bubble in an incompressible flat-plate boundary layer (Rodríguez and Theofilis, 2010)

1.2.4 Experiments

Further evidence in support of the association of instability in the shock-induced turbulent separation bubble on a wedge with the amplifier mechanism via a three-dimensional unsteady modification of the nominally two-dimensional flow is offered by the following re-interpretation of the experimental results of Settles et al. (1979), from which Figure 1.7 (left) is extracted. Shown is a sequence of wall streak patterns as a function of wedge angle. Flow is from the lower to the upper part of the image and results are presented for four wedge angles, (a): 8deg, (b): 16deg, (c): 20deg and (d): 24deg. The locations of the corner (C), separation (S) and reattachment (R) are also marked on the images. What can be seen in the experimental results is that, while flow at low wedge angles is along the streamwise direction and is spanwise homogeneous, a spanwise periodic pattern develops as the wedge angle increases and is clearly visible at the results obtained at the highest wedge angle of 24deg. The periodicity length of the pattern developing is of the order of the local boundary layer thickness, $O(\delta^*)$. This feature is strongly reminiscent of the characteristics of the stationary global mode of laminar separation found by Theofilis et al. (2000) and further analyzed by Rodríguez and Theofilis (2010).

Besides the qualitative analogies of the wall-shear patterns, a further fact which strengthens the conjecture that the observation of Settles et al. (1979) may be linked with linear amplification of the stationary three-dimensional global mode of separation is that spanwise periodicity of the flow field only sets in beyond a certain wedge angle value. This is consistent with the known result of global instability analysis of incompressible laminar separation bubbles, namely that a certain level of recirculation is required in the bubble in order for the stationary global instability mechanism to amplify and the bubble to acquire the amplifier mechanism in addition to its ever-present oscillator nature. It is also known that a certain spanwise width of the domain is required at a given flow condition in order for multiples of amplified spanwise-periodic structures of the stationary global mode to fit in the domain and manifest themselves in the wall-shear pattern. It might then be argued that the reason

	Approach	L_z/δ^*
Settles et al. (1979)	Experiment	6.6
Touber and Sandham (2009)	LES	1.4
Pirozzoli et al. (2010)	LES	2
Egorov et al. (2011)	DNS	n/a
Priebe and Martin (2012)	DNS	2
Grilli et al. (2012)	LES	4
Pasquariello et al. (2013)	LES	2.62
Jammalamadaka et al. (2014)	DNS	3.48

Table 1.2: Spanwise extent of the domains considered in DNS and LES of shock / turbulent boundary layer interactions and comparison with that in the experimental work of Settles et al. (1979)

why the amplifier mechanism does not always appear in SWTBLI flow is that the simulations performed are too narrow in the spanwise direction for the global mode to fit in the simulation domain. A compilation of the spanwise extent of several present-day direct numerical and large-eddy simulations of the flow in question, as well as that estimated from the results of Settles et al. (1979) are shown in Table 1.2; the ratio of spanwise extent to the boundary layer thickness is not available (n/a) in the work of Egorov et al. (2011).

Chapter 2

Theory

The appropriate framework for linear instability analysis of shock-induced separated boundary layer flow in a planar (and axisymmetric) compression ramp configuration is provided by BiGlobal theory (Theofilis, 2003b, 2011). By contrast to the classic linear stability theory (Mack, 1984), which rests on solution of variants of the compressible Orr-Sommerfeld equation and led to the discovery of Mack modes in compressible boundary layers, the Parabolized Stability Equations (PSE; Herbert, 1997), or the triple-deck concept in conjunction with absolute/convective instability ideas, all of which are strictly valid for parallel or weakly non-parallel flows, BiGlobal theory permits inclusion of key elements of the basic flow in question into the instability problem: shocks, attached, or detached, straight or curved, and expansions, intimately linked with both the generation of the separation bubble and of steady laminar structures developing in the downstream part of the separation zone (Visbal and Gaitonde, 2005).

The placement of global linear theory in the context of linear and nonlinear instability analyses is exposed in Table 2.1, in which the symbols will be defined shortly. Conceptually analogous with the classical linear instability concept, which gives rise to the Orr-Sommerfeld Equation (OSE) and the variants of the OSE approach in compressible flow, BiGlobal theory decomposes all flow quantities into steady laminar basic state and unsteady linear perturbations. The key difference between the BiGlobal- and the classic linear instability theories lies in the dimensionality of the basic state and that of the amplitude functions of the perturbations associated with this basic state: while classic theory considers both of these to be functions of one spatial direction (the wall-normal, in case of separated flows), BiGlobal theory fully resolves in a coupled manner a two-dimensional plane (that of the streamwise and wall-normal spatial directions) on which the basic state is defined. BiGlobal theory is also different from the classic PSE concept. PSE is applicable to weakly nonparallel flows alone, in which the wall-normal basic flow velocity component is introduced in the analysis in a consistent inverse-Reynolds-number small-parameter expansion framework. As such, PSE is strictly valid for attached boundary layer flows although successful analyses have been reported in the literature (e.g. Theofilis et al., 2000) of PSE analyses of laminar separation bubble flows. On the other hand, results of BiGlobal theory encompass those of both the classic linear analysis and classic PSE in the case of the basic state being parallel or weakly-nonparallel; in addition, BiGlobal theory has the capacity to reveal instabilities that are particular to nonparallel states alone.

2.1 The linearized Navier-Stokes equations (LNSE)

The equations governing the evolution of infinitesimal-amplitude disturbances studied here start with the compressible Navier-Stokes equations. In this part of the analysis, Cartesian coordinate system will be employed since the primary objective is presenting the idea of the analysis itself; the equations solved in the generalized coordinate system used in the compression ramp, as well as variants of the transformation of the physical ramp onto a computational rectangular domain will be discussed in Section 2.6.

The governing equations read

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (2.1)$$

$$\frac{\partial(\rho \mathbf{V})}{\partial t} + \nabla \cdot (\rho \mathbf{V} \mathbf{V}) = -\nabla p + \frac{1}{Re} \nabla \cdot \sigma \quad (2.2)$$

$$\rho \frac{\partial T}{\partial t} - Ec \frac{\partial p}{\partial t} = Ec(\mathbf{V} \cdot \nabla p) - \rho \mathbf{V} \nabla \cdot T + \frac{1}{RePr} \nabla \cdot (k \nabla T) + \frac{Ec}{Re} \left(\frac{1}{2} (\nabla \mathbf{V} + \nabla \mathbf{V}) : \sigma \right) \quad (2.3)$$

$$p = \frac{\rho T}{\gamma Ma^2} \quad (2.4)$$

where,

$$\sigma = [\mu(\nabla \mathbf{V} + (\nabla \mathbf{V})^T)] + \nabla[\lambda(\nabla \cdot \mathbf{V})]$$

is the viscous stress tensor, $\mathbf{V} = \{u, v, w\}^T$ is the velocity vector, ρ is the density, p is the pressure, T is the temperature, μ is the first coefficient of viscosity, λ is the second coefficient of viscosity and k is the thermal conductivity. The equation of state of perfect gases with constant $\gamma = 1.4$ is also presented. The Prandtl number Pr is chosen as constant and equals 0.72 and Stokes' hypothesis has been applied to calculate λ .

The linearized Navier-Stokes equations (LNSE) result from the substitution of a linear decomposition of all flow quantities, $\mathbf{q} = (u, v, w, T, p)^T$ into a steady laminar base or mean flow and a small amplitude disturbance, according to

$$\mathbf{q}(\mathbf{x}, t) = \bar{\mathbf{q}}(\mathbf{x}) + \epsilon \tilde{\mathbf{q}}(\mathbf{x}, t), \quad \text{with } \epsilon \ll 1. \quad (2.5)$$

The $O(1)$ resulting equations are those governing the base state and are satisfied by construction, the $O(\epsilon^2)$ equations are neglected, on account of smallness of the perturbation amplitudes, while the $O(\epsilon)$ equations are the LNSE, that need to be solved, either as an eigenvalue problem (modal analysis) or as an initial value problem (non-modal analysis).

2.2 Local and global linear stability analysis

The decomposition (2.5) is valid irrespective of the spatial dimensionality of the base flow and small perturbations. The classification of Table 2.1 shows the decomposition in a local and a global linear stability analysis context.

2.2.1 Local instability

In local and nonlocal analysis, as defined in Table 2.1, only one spatial direction is taken to be inhomogeneous, in both the base flow and the amplitude functions. Parallel shear flows (Schmid and Henningson, 2001b; Schmidt and Rist, 2011) can be analyzed using this concept, in which the velocity field comprises a streamwise and a spanwise component, both of which are functions of the transverse spatial direction only, while the transverse/wall-normal base flow velocity component is neglected. In axisymmetric geometries, the base flow and amplitude functions dependent only on the radial coordinate. Applying separation of variables, the two

	Denomination	Base Flow assumptions	Amplitude function	Phase function	
	TriGlobal	—	$\overline{\mathbf{q}}(x, y, z)$	$\hat{\mathbf{q}}(x, y, z)$	ωt
Global	PSE-3D	$\partial_z \overline{\mathbf{q}} \ll \partial_y \overline{\mathbf{q}}$	$\overline{\mathbf{q}}(x, y, Z)$	$\hat{\mathbf{q}}(x, y, Z)$	$\int_0^Z \beta(\zeta) d\zeta - \omega t$
	BiGlobal	$\partial_z \overline{\mathbf{q}} = 0$	$\overline{\mathbf{q}}(x, y)$	$\hat{\mathbf{q}}(x, y)$	$\beta z - \omega t$
Nonlocal	PSE	$\partial_z \overline{\mathbf{q}} \ll \partial_y \overline{\mathbf{q}}; \partial_x \overline{\mathbf{q}} = 0$	$\overline{\mathbf{q}}(y, Z)$	$\hat{\mathbf{q}}(y, Z)$	$\alpha x + \int_0^Z \beta(\zeta) d\zeta - \omega t$
Local	OSE	$\partial_x \overline{\mathbf{q}} = \partial_z \overline{\mathbf{q}} = 0$	$\overline{\mathbf{q}}(y)$	$\hat{\mathbf{q}}(y)$	$\alpha x + \beta z - \omega t$

Table 2.1: Classification of linear stability theories

homogeneous directions can be decomposed in the Fourier space and the linearized equations of motion may be re-written as a system of linear ordinary differential equations, as discussed by Mack (1984), one form of which is the celebrated Orr-Sommerfeld equations (OSE).

Mathematically, assuming x and z as the homogeneous spatial directions, such that the base flow is only dependent on the y spatial coordinate, modal perturbations are of the form

$$\tilde{\mathbf{q}}(x, y, z, t) = \hat{\mathbf{q}}(y) \exp[i(\alpha x + \beta z - \omega t)], \quad (2.6)$$

where $L_x = \frac{2\pi}{\alpha}$ and $L_z = \frac{2\pi}{\beta}$ are periodicity lengths along the homogeneous spatial directions, while α and β are the corresponding wavenumbers. Substituting this Ansatz into the compressible LNSE leads to the one-dimensional eigenvalue problem presented and solved by Mack (1969a, 1984). In the incompressible limit, the resulting system, after the substitution of the Ansatz (2.6), can be reduced into two coupled ordinary differential equations, by eliminating the pressure variable and using the definition of normal vorticity,

$$\begin{aligned} [(-i\omega + i\alpha\bar{u})(D_{yy}^2 - k^2) - i\alpha\bar{u}_{yy} - \frac{1}{Re}(D_{yy}^2 - k^2)]\hat{v} &= 0 \\ [(-i\omega + i\alpha\bar{u}) - \frac{1}{Re}(D_{yy}^2 - k^2)]\hat{\eta} &= -i\beta\bar{u}_y\bar{v}, \end{aligned} \quad (2.7)$$

where $k^2 = \alpha^2 + \beta^2$. The first of these is the Orr-Sommerfeld equation, while the second is known as the Squire equation; they have been used to analyze linear modal instability of shear flows since the beginning of the XX century (Drazin and Reid, 1981; Lin, 1955a). Despite the simplicity of these equations, they provide the exact eigenmodes of strictly parallel flows and are reasonable approximations for quasi-parallel, slowly diverging flows, such as the boundary layer or a free shear layer. The $Re \rightarrow \infty$ limit of the Orr-Sommerfeld was derived by Lord Rayleigh and describes inviscid instability of flows with inflectional base flow profiles.

Linear perturbations may be taken grow in time or space; the respective temporal analysis framework is assuming real wavenumber parameters and solves for a complex eigenvalue ω in the temporal *linear* eigenvalue problem

$$\mathcal{A}\hat{\mathbf{q}} = \omega\mathcal{B}\hat{\mathbf{q}}, \quad (2.8)$$

while the spatial EVP assumes a real perturbation frequency ω and solves the *nonlinear* eigenvalue problem

$$\mathcal{A}\hat{\mathbf{q}} = \sum_{k=1}^2 \alpha^k \mathcal{B}_k \hat{\mathbf{q}}, \quad (2.9)$$

for a complex eigenvalue α . Equation (2.9) can be converted into a linear EVP using a companion matrix method (Bridges and Morris, 1984; Theofilis, 1995; Heeg and Geurts, 1997).

2.2.2 BiGlobal instability

If the base flow is now assumed to be dependent on two spatial directions, say x and y , BiGlobal linear instability theory can be applied; the reader is referred to the review of Theofilis (2003a) for more details. A harmonic dependence of the small-amplitude perturbations is assumed along the homogeneous spatial direction, z , with periodicity length $L_z = 2\pi/\beta$, permitting further expansion of (2.5) using

$$\tilde{\mathbf{q}}(x, y, z, t) = \hat{\mathbf{q}}(y, z) = \exp[i(\beta z - \omega t)]. \quad (2.10)$$

Substituting the Ansatz (2.10) in the compressible LNSE, the resulting components of the temporal operators $\mathcal{A}$ and $\mathcal{B}$ are

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} & \mathcal{A}_{14} & \mathcal{A}_{15} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} & \mathcal{A}_{24} & \mathcal{A}_{25} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} & \mathcal{A}_{34} & \mathcal{A}_{35} \\ \mathcal{A}_{41} & \mathcal{A}_{42} & \mathcal{A}_{43} & \mathcal{A}_{44} & \mathcal{A}_{45} \\ \mathcal{A}_{51} & \mathcal{A}_{52} & \mathcal{A}_{53} & \mathcal{A}_{54} & \mathcal{A}_{55} \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{T} \\ \hat{p} \end{pmatrix} = \omega \begin{pmatrix} \mathcal{B}_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_{22} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{B}_{33} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{B}_{44} & \mathcal{B}_{45} \\ 0 & 0 & 0 & \mathcal{B}_{54} & \mathcal{B}_{55} \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{T} \\ \hat{p} \end{pmatrix} \quad (2.11)$$

In the present context, a temporal analysis framework is followed, in which the spanwise wavenumber $\beta = \frac{2\pi}{L_z}$ is a real parameter, while the sought complex eigenvalue is ω .

2.3 Numerical solution of the global EVP - Modal analysis

Explicit expressions for the entries of the coefficient matrices $\mathcal{A}$ and $\mathcal{B}$ appearing in the compressible BiGlobal EVP (2.11) have been presented by (e.g. Theofilis and Colonius, 2004; Theofilis et al., 2004; Paredes, 2014). Solution methods for the eigenvalue problem are discussed in detail by Theofilis (2003b); unless use of the direct QZ algorithm is warranted in order to recover the full eigenspectrum, at a cost scaling with the cube of the total number of nodes used in the spatial discretization, the methods of choice are based on Krylov iteration and the (implicitly restarted) Arnoldi method (IRAM).

The solution of a new problem typically employs first the direct QZ algorithm, in order to obtain first estimates of the parameters leading to instability regions. For this step a shared-memory machine is necessary in order to keep in memory the four large arrays (each having a size equal to that discretizing the EVP) required by this algorithm; experience shows that in single-domain spectral simulations, which permit recovery of results at a minimum resolution, and at typical Reynolds numbers for instability to occur, at least 4 Gb of main memory is necessary in order to store each of these four arrays. Once the instability region has been identified, the analysis may be completed by switching to the iterative IRAM approach, in which one array only need to be stored. At this stage, a parallel computation is warranted, typically in order to scan efficiently the large parameter ranges involved, or in order to distribute subdomains of a multidomain decomposition of the EVP (De Vicente et al., 2006). The solution procedure followed in the present context will be discussed in detail in Section 4.

2.4 Numerical solution of the global IVP - Nonmodal analysis/transient growth

The decomposition of any flow quantity ($\mathbf{q}$) into a steady laminar base or mean flow plus a small amplitude disturbance

$$\mathbf{q}(\mathbf{x}, t) = \bar{\mathbf{q}}(\mathbf{x}) + \epsilon \tilde{\mathbf{q}}(\mathbf{x}, t), \quad \text{with } \epsilon \ll 1, \quad (2.12)$$

substituted into the dimensionless compressible Navier-Stokes equations gives rise to the well-known compressible linearized Navier-Stokes equations (LNSE). These can be written as an initial-value-problem (IVP) in the form of

$$\mathcal{B} \frac{\partial \tilde{\mathbf{q}}}{\partial t} = \mathcal{A} \tilde{\mathbf{q}} \quad (2.13)$$

where the matrices $\mathcal{A}$ and $\mathcal{B}$ contain the spatial discretization, the base flow and its derivatives. Since a steady base flow is considered, there is no coupling between time and space coordinates and a Fourier decomposition in time can be made ($\tilde{\mathbf{q}} = \exp(-i\omega t)\hat{\mathbf{q}}$), leading to

$$\frac{\partial \tilde{\mathbf{q}}}{\partial t} = \mathcal{L} \tilde{\mathbf{q}} \quad (2.14)$$

where $\tilde{\mathbf{q}}(x, y, z, t) = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{T}, \tilde{P})^T$ is the vector containing the disturbance functions of velocity, temperature and pressure and $\mathcal{L} = \mathcal{B}^{-1}\mathcal{A}$ is a linear operator mapping the flow field to its time derivative and contain the appropriated boundary conditions. Note that the matrix $\mathcal{B}$ is non-singular in the compressible regime, so it is possible to compute its inverse. The solution of the IVP is given in general form by

$$\tilde{\mathbf{q}} = \exp(t\mathcal{L})\tilde{\mathbf{q}}_0, \quad (2.15)$$

where $\tilde{\mathbf{q}}_0$ is the initial condition. It is important to notice that, so far, no approximations regarding the shape of the disturbances or their temporal behavior have been introduced. The matrix exponential $\mathcal{L}$ can also be seen as the stability operator after the linearization step described before.

Two main methodologies can be used to compute linear optimal disturbances. The first consists of explicit construction of the spatial propagator operator (Theofilis, 2011), performance of a singular value decomposition (SVD) of the propagator and computation of the (transient/algebraic) amplification of the initial perturbations, the initial energy of which is normalized to unity, until a maximum amplification at a given so-called optimal time, t_{opt} . The second method (Luchini, 2000; Corbett and Bottaro, 2001; Luchini and Bottaro, 2014) is based on repeated iterative solutions using the direct linearized Navier-Stokes equations (LNSE) of motion in the forward (in time or space) and the adjoint LNSE in the backward iteration; the absence of adjoint equations in the compression ramp problem and the availability of the LiGHT SVD solver led us to use the first approach in the framework of the present effort.

2.4.1 Quantitative description of energy growth - the Gain function

Defining stability as the amplification of initial disturbance energy over a prescribed time interval (Schmid and Henningson, 2001b; Schmid, 2007) and optimizing it over all permissible initial conditions the *gain function* $G(t)$ may be defined as the maximum optimal energy amplification

$$G(t) = \max_{\tilde{\mathbf{q}}_0} \frac{E(\tilde{\mathbf{q}}(t))}{E(\tilde{\mathbf{q}}_0)}, \quad (2.16)$$

($E\tilde{\mathbf{q}}$) being the energy of the disturbance. Substituting (2.15) into the above definition of energy amplification yields to

$$G(t) = \max_{\tilde{\mathbf{q}}_0} \frac{\|\tilde{\mathbf{q}}(t)\|_E^2}{\|\tilde{\mathbf{q}}_0\|_E^2} = \max_{\tilde{\mathbf{q}}_0} \frac{\|\exp(t\mathcal{L})\tilde{\mathbf{q}}_0\|_E^2}{\|\tilde{\mathbf{q}}_0\|_E^2} = \|\exp(t\mathcal{L})\|_E^2 \quad (2.17)$$

The norm of the disturbance will be defined shortly. This energy norm of the matrix exponential represents therefore the largest amplification of energy that any infinitesimal initial disturbance can experience given a time interval. In general, the matrix exponential is hard to compute, and an eigenvalues decomposition of $\mathcal{L}$ is used as proxies

$$\|\exp(t\mathcal{L})\|_E^2 = \|\exp(t\mathcal{V}\Lambda\mathcal{V}^{-1})\|_E^2 = \|\mathcal{V}\exp(t\Lambda)\mathcal{V}^{-1}\|_E^2 \quad (2.18)$$

where $\mathcal{V}$ is the matrix containing the normalized eigenvectors and Λ is a matrix containing the eigenvalues in its diagonal. The above expression is very important to understand the difference between the present analysis

and a classical stability analysis. The eigenvalues of Λ represent only part of $G(t)$, and this will predict indeed the behavior of the disturbances only if the similarity transformation give by the eigenvectors does not change the value of the norm. This is true only when the eigenvectors are orthogonal to each other. System matrices resulting in non orthogonal eigenvectors are called non-normal matrices, whereas matrices with orthogonal eigenvectors are called normal. Schmid and Henningson (2001b) summarized this presenting upper and lower bounds (estimates) of the operator exponential norm. They showed that since the energy cannot decay at a faster rate than that given by the least stable eigenvalue the lower bound is given by this mode. For the upper bound, the eigenvalue decomposition 2.18 is used evoking the triangle inequality to split the norm

$$\exp(2t\lambda_{max}) \leq \|\exp(t\mathcal{L})\|_E^2 \leq \|\mathcal{V}\|_E^2 \|\mathcal{V}^{-1}\|_E^2 \exp(2t\lambda_{max}). \quad (2.19)$$

Two cases can be distinguished. If $\|\mathcal{V}\|_E^2 \|\mathcal{V}^{-1}\|_E^2 = 1$ then the upper and lower bound are the same. In this case, the energy amplification is indeed governed by the least stable eigenvalue λ_{max} . However, if this value is much larger then 1, the upper and lower bound differ significantly and the energy amplification is governed by the least stable eigenvalue only for asymptotic period of time.

2.4.2 Definition of the energy norm

Recapitulating the start of the analysis, (2.16) requires the definition of an inner product and its associated norm. These quantities constitute the fundamental measure and orthogonality describing the transient behavior. As pointed out by Hanifi et al. (1996b), the energy density of a disturbance is a natural and physically meaningful measure of its size in the context of an incompressible flow. However, no obvious choice exists when dealing with compressible flows. For this reason, they re-derived the norm proposed by Mack (1969a) in the context of modal instability, requiring pressure work to be conservative. Chu (1965) also described this energy norm, and the possibility to use different $\tilde{\mathbf{q}}$ vectors. Mathematically, the inner product can be written as

$$(\tilde{\mathbf{q}}_i, \tilde{\mathbf{q}}_j) = \int_0^\infty \tilde{\mathbf{q}}_i^H \mathcal{M} \tilde{\mathbf{q}}_j dy \quad (2.20)$$

whereas the associated norm is

$$2E = (\tilde{\mathbf{q}}, \tilde{\mathbf{q}}) = \|\tilde{\mathbf{q}}\|_E^2. \quad (2.21)$$

If the disturbance vector is $\tilde{\mathbf{q}} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{T}, \tilde{p})$, then Tumin and Reshotko (2003) showed that

$$\mathcal{M} = \begin{bmatrix} \bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & \bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{\bar{\rho}}{(\gamma(\gamma-1)\bar{T}M^4)} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the overbar indicates base flow quantities. The relation between the above energy norm and the standard Euclidean L_2 norm can be established by the following transformation (Reddy and Henningson, 1993; Schmid and Henningson, 2001b; Trefethen and Embree, 2005):

$$E(\tilde{\mathbf{q}}) = \|\tilde{\mathbf{q}}\|_E^2 = \langle \tilde{\mathbf{q}}, \tilde{\mathbf{q}} \rangle_E = \tilde{\mathbf{q}}^H \mathcal{M} \tilde{\mathbf{q}} = \tilde{\mathbf{q}}^H F^H F \tilde{\mathbf{q}} = \langle F\tilde{\mathbf{q}}, F\tilde{\mathbf{q}} \rangle_2 = \|F\tilde{\mathbf{q}}\|_2^2. \quad (2.22)$$

The energy weight matrix is defined in (2.4.2). This matrix is positive definite since it contains positive quantities (energy norm). Thus, it can be decomposed via a Cholesky decomposition, generating F and F^H . The energy norm of the disturbance is then equivalent to take the Euclidean L_2 norm of the vector $F\tilde{\mathbf{q}}$. For matrices, using the definition of a vector-induced norm

$$\|\mathcal{L}\|_E = \max_{\tilde{\mathbf{q}}} \frac{\|\mathcal{L}\tilde{\mathbf{q}}\|_E}{\|\tilde{\mathbf{q}}\|_E} = \max_{\tilde{\mathbf{q}}} \frac{\|F\mathcal{L}F^{-1}F\tilde{\mathbf{q}}\|_2}{\|F\tilde{\mathbf{q}}\|_2} = \|F\mathcal{L}F^{-1}\|_2. \quad (2.23)$$

Therefore, for matrices, a simple similarity transformation using the Cholesky decomposition relates the energy norm to the Euclidean L_2 norm.

2.4.3 Transient growth calculation

In order to make further simplification of the linearized governing equations presented before (2.14), an expansion of the vector quantities $\tilde{\mathbf{q}}$ in the vector eigenfunctions of the linear initial value problem is necessary. Considering that the flow analyzed is inhomogeneous in the spanwise and wall-normal spatial directions (z,y), and homogeneous in the streamwise (x), a Fourier transformation can be used to write:

$$\tilde{\mathbf{q}}(x, y, z, t) = \hat{\mathbf{q}}(z, y) \exp(i(\alpha x - \omega t)) \quad (2.24)$$

where α is the wavenumber in the streamwise direction and ω is a complex number. Introducing this decomposition into the IVP (2.14) give rise to a generalized eigenvalue problem of the form

$$\mathcal{L}\hat{\mathbf{q}} = i\omega\hat{\mathbf{q}} \quad (2.25)$$

where the entries of the matrix $\mathcal{L}$ can be easily recovered from the linearized equations after introducing (2.24). In this case, a temporal analysis is chosen and the flow is classified as linearly unstable if the imaginary part of the eigenvalue ω is positive. Since a decision about the scalar product and norm has already been made, the transient growth analysis can be further investigated. Following close the analysis performed by Hanifi et al. (1996c), we can restricting our analysis to the space spanned by the first N eigenfunctions of (2.25)

$$\text{span}\{\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2, \hat{\mathbf{q}}_3, \dots, \hat{\mathbf{q}}_N\} \quad (2.26)$$

the state vector $\tilde{\mathbf{q}}$ can be expressed in terms of this basis:

$$\tilde{\mathbf{q}} = \sum_{k=1}^N \mathcal{K}_k(t) \hat{\mathbf{q}}_k, \quad (2.27)$$

where

$$\mathcal{K}_k(t) = \mathcal{K}_k(0) \exp(-i\omega_k t) \quad (2.28)$$

with $\mathcal{K}_k$ as the time-dependent expansion coefficients. Thus, the eigenvalues represent the linear evolution operator projected onto the space spanned by the first N eigenvectors. Restricting the transient growth analysis to this space and applying the scalar product and the energy norm defined previously by equations (2.20 - 2.21) leads to

$$G(t) = G(\alpha, Re, M, t) = \max_{\tilde{\mathbf{q}}_0} \frac{\|\tilde{\mathbf{q}}(t)\|_E^2}{\|\tilde{\mathbf{q}}(0)\|_E^2} = \|\exp(-it\mathcal{L})\|_E^2 = \|F\Lambda F^{-1}\|_2^2. \quad (2.29)$$

where

$$\Lambda = \text{diag}(\exp(-it\omega_1), \exp(-it\omega_2), \dots, \exp(-it\omega_N)) \quad \mathcal{A} = F^H F \quad (2.30)$$

and $\mathcal{A}$ defined as the scalar product presented in (2.20). The maximum gain is thus obtained as the square of the largest single value.

2.4.4 Linear optimal disturbances

As discussed before, for non-normal systems exhibiting substantial potential to transient growth, it is important to calculate the initial condition that optimizes the energy amplification factor G_{max} . This initial condition can be different for different times and the resulting curve $G(t)$ should be thought of as the envelope of the energy evolution of individual initial conditions. Nevertheless, to determine the initial condition that reach the maximum possible amplification at a given time t^* a singular value decomposition (SVD) of the expression in (2.16) can be performed:

$$\mathcal{B}V = U\Sigma \quad (2.31)$$

where $\mathcal{B} = F\Lambda F^{-1}$ and V and U are unitary matrices with ortho-normalized columns and Σ is a diagonal matrix consisting of the singular values, ordered in size, i.e. $\sigma_1 \geq \sigma_2 \geq \dots \sigma_N$. The dominant singular value is equivalent to the energy norm of $\exp(-t\omega_k)$. Thus, extracting the principal column vectors of V and U leads to

$$\mathcal{B}v_1 = \sigma_1 u_1. \quad (2.32)$$

As pointed out by Schmid and Henningson (1994), this equation can be interpreted as a mapping $\mathcal{B}$ of an input vector v_1 onto an output vector u_1 stretched by a factor of the largest single value. Therefore, it is straightforward to identify the optimal initial condition as the principal right singular value vector and the principal left singular vector as the resulting state-vector. Furthermore, since equation (2.16) is advanced in time, it is clear that every step will produce its own optimal disturbance. Finally, to recover the initial expansion coefficients of equation (2.27), which give the optimal disturbances associated with a particular amplification, it is necessary to write v_1 in the basis $\{\hat{\mathbf{q}}_1, \dots, \hat{\mathbf{q}}_N\}$ as:

$$v_1 = F\mathcal{K}_0. \quad (2.33)$$

Therefore, the computational steps required to recover the optimal initial condition is a singular value decomposition (SVD) of the matrix exponential evaluated at different specified time t^* .

2.5 Iterative eigenvalue problem computation

The iterative numerical solution of the large-scale eigenvalue problems encountered in this analysis implements Krylov subspace iteration based on serial and parallel versions of the Arnoldi algorithm. This approach delivers a number of eigenvalues in the vicinity of a specific target eigenvalue and is substantially less expensive than the classic QZ method (Arnoldi, 1951; Saad, 1980). The number of eigenvalues delivered is determined by the Krylov subspace dimension, a free parameter in the analysis, whose limit is the leading dimension of matrices $\mathbf{A}$ and $\mathbf{B}$. In this eigenvalue problem, a so-called shift-and-invert strategy is also implemented, in which the problem solve is (Theofilis, 2003a)

$$\mathbf{C}\mathbf{X} = \mu\mathbf{X}, \quad \mathbf{C} = (\mathbf{A} - \sigma\mathbf{B})^{-1}\mathbf{B}, \quad \mu = \frac{1}{\omega - \sigma} \quad (2.34)$$

where μ is the new eigenvalue searched, ω is the shift parameter and the Krylov subspace is constructed by

$$K_m = \text{span}(\mathbf{v}, C.\mathbf{v}, C^2.\mathbf{v}, \dots, C^{m-1}.\mathbf{v}) \quad (2.35)$$

where m is the dimension of the subspace.

The vectors $\mathbf{v}$ forming the basis are ordered by columns into the matrix $\mathbf{V}_m$. The projection of the linear operator on that basis yields the relation

$$\mathbf{V}_m^H C \mathbf{V}_m = \mathbf{H}_m \quad (2.36)$$

where $\mathbf{H}_m$ is an upper triangular Hessenberg matrix. The restrictions of the eigenvalue problem to the Krylov subspace K_m is

$$\mathbf{H}_m \mathbf{V}_m^H \hat{\mathbf{q}} = \mu \mathbf{V}_m^H \hat{\mathbf{q}} \quad (2.37)$$

The eigenvalues of the Hessenberg matrix $\mathbf{H}_m$ are approximations to the m largest eigenvalues of the original problem.

With regard to the eigenvector, if $y_i^{(m)}$ is the eigenvector corresponding to the i -th eigenvalue of the $\mathbf{H}_m$ the corresponding approximation to the eigenvector of the original problem, known as Ritz vector, is

$$\hat{\mathbf{q}}_i = \mathbf{V}_m \mathbf{y}^{(m)} \quad (2.38)$$

2.6 Instability analysis in generalized coordinates

A key element of the instability analysis in the compression ramp is the non-trivial geometry, which introduces an additional level of complexity in the solution algorithm, owing to the need to discretize a spatial domain in which the ramp corner is a geometric singularity. Three alternatives have been implemented and tested in the framework of the present effort, namely a discontinuously patched domain (Anderson Jr, 1995), the analytic transformations discussed by Adams (1998, 2000); Martin et al. (2003) and Pagella et al. (2004), as well as a newly-developed mapping appropriate for global instability analysis; details follow.

2.7 Compression ramp coordinates transformation

The generalized three-dimensional coordinate system (ξ, η, ζ) can be related to the Cartesian coordinates (x, y, z) by the transformation

$$\xi = \xi(x, y, z), \quad \eta = \eta(x, y, z), \quad \zeta = \zeta(x, y, z) \quad (2.39)$$

To express the first order partial derivatives, the chain rule can be used

$$\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} + \zeta_x \frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta} + \zeta_y \frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial z} = \xi_z \frac{\partial}{\partial \xi} + \eta_z \frac{\partial}{\partial \eta} + \zeta_z \frac{\partial}{\partial \zeta} \quad (2.40)$$

The metrics appearing in these equations $(\xi_x, \xi_y, \xi_z, \eta_x, \eta_y, \eta_z, \zeta_x, \zeta_y, \zeta_z)$ can be obtained directly if an explicit expression for ξ, η and ζ is provided. If not, they are determined solving the system

$$\begin{pmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{pmatrix}^{-1} = J \begin{pmatrix} y_\eta z_\zeta - y_\zeta z_\eta & -(x_\eta z_\zeta - x_\zeta z_\eta) & x_\eta y_\zeta - x_\zeta y_\eta \\ -(y_\xi z_\zeta - y_\zeta z_\xi) & x_\xi z_\zeta - x_\zeta z_\xi & -(x_\xi y_\zeta - x_\zeta y_\xi) \\ y_\xi z_\eta - y_\eta z_\xi & -(x_\xi z_\eta - x_\eta z_\xi) & x_\xi y_\eta - x_\eta y_\xi \end{pmatrix} \quad (2.41)$$

where J is the Jacobian of the transformation

$$J = \frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)} = \frac{1}{\frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)}} = \frac{1}{x_\xi(y_\eta z_\zeta - y_\zeta z_\eta) - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi) + x_\zeta(y_\xi z_\eta - y_\eta z_\xi)}. \quad (2.42)$$

For cases where the transformation is the direct result of the grid generation scheme, the metrics can be computed numerically. In the case of second order derivatives in terms of computational coordinate, the result

from the system 2.41 can be derived in respect to the appropriated variable leading to the second derivative metric. Particularly, the derivative of ξ_x in respect to x is

$$\begin{aligned} \frac{\partial^2 \xi}{\partial x^2} = J \left[-\frac{\partial z}{\partial \eta} \left(\frac{\partial \eta}{\partial x} \frac{\partial^2 y}{\partial \eta \partial \zeta} + \frac{\partial \xi}{\partial x} \frac{\partial^2 y}{\partial \xi \partial \zeta} + \frac{\partial \zeta}{\partial x} \frac{\partial^2 y}{\partial \zeta^2} \right) + \frac{\partial y}{\partial \eta} \left(\frac{\partial \eta}{\partial x} \frac{\partial^2 z}{\partial \eta \partial \zeta} + \frac{\partial \xi}{\partial x} \frac{\partial^2 z}{\partial \xi \partial \zeta} + \frac{\partial \zeta}{\partial x} \frac{\partial^2 z}{\partial \zeta^2} \right) + \right. \\ \left. \frac{\partial z}{\partial \zeta} \left(\frac{\partial \zeta}{\partial x} \frac{\partial^2 y}{\partial \eta \partial \zeta} + \frac{\partial \xi}{\partial x} \frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial x} \frac{\partial^2 y}{\partial \eta^2} \right) - \frac{\partial y}{\partial \zeta} \left(\frac{\partial \zeta}{\partial x} \frac{\partial^2 z}{\partial \eta \partial \zeta} + \frac{\partial \xi}{\partial x} \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial x} \frac{\partial^2 z}{\partial \eta^2} \right) \right] + \\ \left(\frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \zeta} \frac{\partial z}{\partial \eta} \right) \left(\frac{\partial J}{\partial \zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial J}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial J}{\partial \xi} \frac{\partial \xi}{\partial x} \right). \end{aligned} \quad (2.43)$$

Second order derivatives can be expressed in terms of the computational coordinate system, using Einstein notation, as

$$\frac{\partial f}{\partial c_i} = \chi_{c_i}^j \frac{\partial f}{\partial \chi^j} = J \frac{\partial}{\partial \chi^j} \left(\frac{\chi_{c_i}^j f}{J} \right) - J f \left[\frac{\partial}{\partial \chi^j} \frac{\chi_{c_i}^j}{J} \right], \quad i = 1, 2, 3 \quad (2.44)$$

where f is an arbitrary function, $(c_1, c_2, c_3) = (x, y, z)$ and $(\chi_1, \chi_2, \chi_3) = (\xi, \eta, \zeta)$. Substituting the metrics 2.41 into the term with square brackets of equation 2.44, it can be showed that this term is zero. The mix derivatives present in the Navier Stokes equations can be expressed as

$$\frac{\partial^2 f}{\partial c_i \partial c_j} = \chi_{c_i}^k \chi_{c_j}^l \frac{\partial^2 f}{\partial \chi^k \partial \chi^l} + J \frac{\partial}{\partial \chi^k} \left(\frac{\chi_{c_i}^k \chi_{c_j}^l}{J} \right) \frac{\partial f}{\partial \chi^l} \quad (2.45)$$

Particularly, the wall-normal second order derivatives is

$$\begin{aligned} \frac{\partial^2}{\partial y^2} = & \xi_y^2 \frac{\partial^2}{\partial \xi^2} + \eta_y^2 \frac{\partial^2}{\partial \eta^2} + \zeta_y^2 \frac{\partial^2}{\partial \zeta^2} \\ & + 2\xi_y \eta_y \frac{\partial^2}{\partial \xi \partial \eta} + 2\xi_y \zeta_y \frac{\partial^2}{\partial \xi \partial \zeta} + 2\eta_y \zeta_y \frac{\partial^2}{\partial \eta \partial \zeta} \\ & + J \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_y^2}{J} \right) + \frac{\partial}{\partial \eta} \left(\frac{\xi_y \eta_y}{J} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\xi_y \zeta_y}{J} \right) \right] \frac{\partial}{\partial \xi} \\ & + J \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_y \eta_y}{J} \right) + \frac{\partial}{\partial \eta} \left(\frac{\eta_y^2}{J} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\eta_y \zeta_y}{J} \right) \right] \frac{\partial}{\partial \eta} \\ & + J \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_y \zeta_y}{J} \right) + \frac{\partial}{\partial \eta} \left(\frac{\eta_y \zeta_y}{J} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\zeta_y^2}{J} \right) \right] \frac{\partial}{\partial \zeta} \end{aligned} \quad (2.46)$$

the span-wise second order derivatives is

$$\begin{aligned}
 \frac{\partial^2}{\partial x^2} = & \xi_x^2 \frac{\partial^2}{\partial \xi^2} + \eta_x^2 \frac{\partial^2}{\partial \eta^2} + \zeta_x^2 \frac{\partial^2}{\partial \zeta^2} \\
 & + 2\xi_x \eta_x \frac{\partial^2}{\partial \xi \partial \eta} + 2\xi_x \zeta_x \frac{\partial^2}{\partial \xi \partial \zeta} + 2\eta_x \zeta_x \frac{\partial^2}{\partial \eta \partial \zeta} \\
 & + J \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_x^2}{J} \right) + \frac{\partial}{\partial \eta} \left(\frac{\xi_x \eta_x}{J} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\xi_x \zeta_x}{J} \right) \right] \frac{\partial}{\partial \xi} \\
 & + J \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_x \eta_x}{J} \right) + \frac{\partial}{\partial \eta} \left(\frac{\eta_x^2}{J} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\eta_x \zeta_x}{J} \right) \right] \frac{\partial}{\partial \eta} \\
 & + J \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_x \zeta_x}{J} \right) + \frac{\partial}{\partial \eta} \left(\frac{\eta_x \zeta_x}{J} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\zeta_x^2}{J} \right) \right] \frac{\partial}{\partial \zeta}
 \end{aligned} \tag{2.47}$$

and the mixed derivative is

$$\begin{aligned}
 \frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x} = & \xi_x \xi_y \frac{\partial^2}{\partial \xi^2} + \eta_x \eta_y \frac{\partial^2}{\partial \eta^2} + \zeta_x \zeta_y \frac{\partial^2}{\partial \zeta^2} \\
 & + (\xi_x \eta_y + \xi_y \eta_x) \frac{\partial^2}{\partial \xi \partial \eta} + (\xi_x \zeta_y + \xi_y \zeta_x) \frac{\partial^2}{\partial \xi \partial \zeta} + (\eta_x \zeta_y + \eta_y \zeta_x) \frac{\partial^2}{\partial \eta \partial \zeta} \\
 & + J \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_x \xi_y}{J} \right) + \frac{\partial}{\partial \eta} \left(\frac{\xi_x \eta_y}{J} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\xi_x \zeta_y}{J} \right) \right] \frac{\partial}{\partial \xi} \\
 & + J \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_y \eta_x}{J} \right) + \frac{\partial}{\partial \eta} \left(\frac{\eta_x \eta_y}{J} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\eta_x \zeta_y}{J} \right) \right] \frac{\partial}{\partial \eta} \\
 & + J \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_y \zeta_x}{J} \right) + \frac{\partial}{\partial \eta} \left(\frac{\eta_y \zeta_x}{J} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\zeta_x \zeta_y}{J} \right) \right] \frac{\partial}{\partial \zeta}
 \end{aligned} \tag{2.48}$$

Applying the transformation (2.40) and (2.44) to the compressible Navier-Stokes equations leads to the compressible generalized Navier-Stokes equations. Once the Jacobian is provided (analytically or numerically) linear stability of flow around or through any complex geometry can be analyzed.

The mesh on which OpenFOAM base flow direct numerical simulations are performed is generated numerically using GMSH (Geuzaine and Remacle, 2009a). Key features of this mesh are a flat plate with a rounded leading edge, which avoids a singularity at this location and the discontinuous patching of the flat plate and the ramp meshes. These features are either unnecessary or even detrimental for the subsequent performance of global instability analysis; it is expected that the analysis will be performed both more efficiently and more accurately on an analytically-known two-dimensional mesh; this section is devoted to the discussion of alternatives as well as the procedure to interpolate data from the DNS / base flow calculation grid onto the mesh on which global instability analysis will be performed.

The wedge geometry resembles that of an expansion corner, with the ramp following the corner angled upwards rather than downwards. Anderson Jr (1995) has discussed a discontinuous mesh for this class of geometries; this grid, with a minor modification, has been examined first in the context of the present effort. In view of the geometric singularity present in the discontinuous grid, variants of the grids used in the analyses of Adams (1998, 2000), Martin et al. (2003) and Pagella et al. (2004) have also been employed. Finally, global instability analysis results have been obtained on a novel grid, details of which will be discussed in what follows.

BiGlobal instability analysis has been performed on a number of alternative choices of the two-dimensional rectangular computational grids, (ξ, η) , based on (independent) analytical transformations of the standard Chebyshev-Gauss-Lobatto (CGL) spectral collocation grid,

$$x(j) = \cos\left(\frac{j\pi}{N}\right), \quad j = 0, 1, 2, \dots, N \tag{2.49}$$

onto the ξ and η coordinates. A tensor-product base grid can be constructed as

$$\xi = \begin{bmatrix} -1 \\ \vdots \\ 1 \end{bmatrix}, \eta = \begin{bmatrix} -1 \\ \vdots \\ 1 \end{bmatrix}, (\xi, \eta) = \begin{bmatrix} (-1, -1) & \dots & (-1, 1) \\ \vdots & \ddots & \vdots \\ (1, -1) & \dots & (1, 1) \end{bmatrix} \quad (2.50)$$

in which clustering of the gridpoints in both spatial directions follows the basic distribution of (2.49). In mapping the rectangular Cartesian computational domain (2.50) onto physical space, two aspects need to be considered, namely, firstly, the description of the geometrical singularity of the ramp corner by some smooth analytic function and, secondly, the ability to cluster points in regions of high base flow gradients, where the amplitude functions are expected to peak. Both of these considerations have been addressed independently in the works of Adams (1998, 2000); Martin et al. (2003) and Pagella et al. (2004). Each of these approaches was designed to perform direct numerical or large-eddy simulations and was found to have certain drawbacks in the context of the present linear instability analysis, in terms of grid alignment; this led to the construction of a new mapping transformation, details of which will be provided shortly.

2.7.1 Basic mapping of the computational grid

The standard CGL grid is first mapped onto $[0, 1]$ using a simple linear transformation

$$\xi_{mapped} = \frac{1 + \xi}{2} \quad (2.51)$$

with the derivative matrices;

$$D^{(1)}(\xi_{mapped}) = 2D^{(1)}, \quad D^{(2)}(\xi_{mapped}) = 4D^{(2)} \quad (2.52)$$

This is sufficient for the stream-wise direction, however to correctly represent the boundary layer a boundary layer mapping is used.

Boundary layer mapping

In order to adequately describe velocity and temperature gradients inside the boundary layer, points are clustered near the wall using a boundary layer mapping. The mapping applied in the wall normal direction η is

$$\eta_{mapped}(j) = l_{BL} \frac{1 - \eta(j)}{1 + s_{BL} + \eta(j)} \quad (2.53)$$

where the function s_{BL} is a scaling function and l_{BL} is a parameter depending on the wall normal extent of the domain, defined by;

$$s_{BL} = 2 \cdot l_{BL}, \quad l_{BL} = \frac{\eta_{half}}{(1 - 2 \cdot \eta_{half})} \quad (2.54)$$

where Y_{half} is the coordinate below which half of the points are to be clustered, allowing control over the density of points near the wall. Similarly to a linear transformation the derivative matrices are altered to incorporate the mapping as follows;

$$D^{(1)}(\eta_{mapped}) = -l_{BL} \frac{s_{BL} + 2}{(l_{BL} + \eta)^2} \cdot D^{(1)} \quad (2.55a)$$

$$D^{(2)}(\eta_{mapped}) = \left(-l_{BL} \frac{s_{BL} + 2}{(l_{BL} + \eta)^2} \right)^2 \cdot D^{(2)} + 2 \cdot l_{BL} \frac{s_{BL} + 2}{(l_{BL} + \eta)^3} \cdot D^{(1)} \quad (2.55b)$$

Once the above mentioned mappings in the computational domain have been performed, a mapping of the latter domain onto the physical compression ramp geometry is still required. Anderson Jr (1995) presented a

discontinuous transformation, while Adams (1998, 2000), Martin et al. (2003) and Pagella et al. (2004) have all used alternative continuous geometry mappings that avoid the corner singularity. Finally, a new transformation related with that of Pagella et al. (2004) but better suited for global instability analysis work will be discussed in what follows.

2.7.2 A discontinuous transformation (Anderson Jr, 1995)

Along the wall-normal direction the grid preceding the corner can remain unchanged, however downstream of the corner it should change according to the angle of the corner, gridpoints being displaced upwards accordingly. One such mesh for the related expansion corner problem has been presented by Anderson Jr (1995). In this setup a invariant grid to the computational grid is used upstream of the corner and then directly downstream of the corner a function in terms simply of downstream location and angle of the corner can define the grid downstream. By simple modification the grid for a compression ramp can be shown to be defined by;

$$y = \begin{cases} H \cdot \eta & x \leq 0 \\ H \cdot \eta + x \cdot \tan(\phi) & x \geq 0 \end{cases} \quad (2.56)$$

where H controls the wall normal extent of the domain. A simple modification of this mesh, in which $H = Y_{max}$, permits the free-stream boundary to increase linearly upwards past the corner. Although such a domain definition is beneficial in terms of allowing the reduction of the domain upstream of the corner, far away from the wall, it also has an inherent sharp discontinuity at the corner. This was found to cause numerical instabilities, which led to abandoning use of the simple discontinuous transformation in the analyses that follow.

2.7.3 The transformation of Adams (1998, 2000); Martin et al. (2003) and Pagella et al. (2004)

The key idea of these works is to use the following mappings between computational and physical space

$$x = (X_{min} + (X_{max} - X_{min})\xi) - X_c \quad (2.57)$$

and

$$y = a \left((X_{min} + (X_{max} - X_{min})\xi) + \frac{\ln(\cosh(x_k))}{c} - \frac{\ln(\cosh(x_\eta))}{c} \right) + Y_{max} \cdot \eta \quad (2.58)$$

where, X_{max} is the outlet location, X_{min} is the inlet location and X_c is the location of corner. However, instead of using (2.57-2.58) directly to define the grid, these equations are used to interpolate between the minimum and maximum values of the grid according to the transformations

$$x(\xi, \eta) = (1 - r(\eta)) \cdot x_w(\xi) + r(\eta) \cdot x_\infty(\xi) \quad (2.59a)$$

$$y(\xi, \eta) = (1 - r(\eta)) \cdot y_w(\xi) + r(\eta) \cdot y_\infty(\xi), \quad (2.59b)$$

$c[\eta]$ now being defined as $c[\eta] = C_{wall}$ at the wall and $c[\eta] = C_\infty$ at the freestream boundary. In this form $r(\eta)$ is an interpolation factor which interpolates the grid along the wall normal direction from the wall (subscript w) and the upper free-stream boundary (subscript ∞). The interpolation factor is defined as follows;

$$r(\eta) = \frac{e^{d \cdot \eta'} - 1}{e^d - 1}, \quad \eta' = \frac{\eta}{(M - 1)\Delta\eta} \quad (2.60)$$

$\Delta\eta$ is the spacing of the mesh and M the number of elements, finally d is an imposed factor to space the grid in a desired manner, small d values approach a linearly spaced grid. It is important to note that unlike the

procedure developed for this study, the grid shown above requires the imposition of a c value at the wall and free-stream. y_w & y_∞ are obtained by evaluating (2.58) for $\eta = 0$ & $\eta = Y_\infty$ respectively. Unlike the original grid, (2.62) is defined in terms of X_{max} as opposed to L , that is defined as the length of the domain. This change is implemented to counteract the inaccuracy caused by having $X_{min} \neq 0$ on a much smaller grid in the physical domain.

There are several advantages to the transformation proposed by Pagella et al. (2004) mainly the simplicity of a and c as they simply become constant parameters that are imposed at the beginning of the simulation. However, the main drawback is the use of the interpolation factor, as it relies on the spacing of the grid. Using this on a nonuniform grid makes the evaluation of $\Delta\eta$ non-trivial and an adaptation of the grid to allow ((?)) is warranted; this is discussed in the next section.

2.7.4 A novel analytic mapping of the compression ramp

The transformation mapping used in the present work adapts the basic idea of the Pagella et al. (2004) grid in that, the wall-normal gridpoint distribution is essentially the same as (2.58)

$$y = a[\eta] \left((X_{min} + (X_{max} - X_{min})\xi) + \frac{\ln(\cosh(x_k))}{c[\eta]} - \frac{\ln(\cosh(x_\eta))}{c[\eta]} \right) + Y_{max} \cdot \eta \quad (2.61)$$

the difference being that, in the context of the present work, instead of the values a and c being imposed constants, the first of them is taken to be a function of the angle of the ramp, ϕ , while c is used to control the radius of curvature of the grid around the corner; both of a and c are now dependent on the wall normal computational coordinate, concretely

$$a[\eta] = -\frac{c[\eta] \tan(\phi + k_\eta \cdot Y_{max} \cdot \eta) \cdot X_r}{-c[\eta] \cdot X_{max} - \ln(\cosh(c[\eta] \cdot X_r)) + \ln(\cosh(x_\eta))} \quad (2.62)$$

$$c[\eta] = \eta(C_\infty - C_{wall}) + C_{wall} \quad (2.63)$$

where, C_{wall} & C_∞ are imposed parameters that are inversely proportional to the radius of curvature at the wall and free stream boundaries respectively. The constant k_η controls the upstream wall normal extent, Y_{max} is the wall normal extent of the inlet and the remaining parameters are defined as;

$$x_k = c[\eta] \cdot x, \quad x_\eta = c[\eta] \cdot X_c, \quad X_r = X_{max} - X_c \quad (2.64)$$

Setting $k_\eta = 0$ maintains a constant wall normal extent of the domain. For the following simulations it was found that the re-compression shock was poorly resolved by a grid with this imposition of k_η . To improve this, it was proposed that the outlet dimension be increased in such a way to ensure that the ξ coordinate direction be transformed such that several transformed y gridlines align with the shock. This was achieved by designing a variable outlet setup as follows.

Variable outlet dimension

Using a non-zero value of k_η , the angle of the corner can be maintained at the wall while it grows at the free stream boundary, allowing control of the outlet dimension, independently of the inlet. The function k_η used was

$$k = A_k \left(\frac{Y_{out} - \tan(\phi) \cdot (X_{max} - X_c)}{Y_{max}} \right) + B_k, \quad (2.65)$$

with values of A_k and B_k used to align the wall-normal gridlines on the ramp with the mean recompression shock direction.

Derivatives and Jacobians

The Jacobian matrix which permits the transformation of variables and calculation of derivatives on the computational grid is defined by

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (2.66)$$

Defining these four derivatives allows for the calculation for the derivatives of the baseflow quantities which is essential for the stability analysis. The derivatives in the x transformation is straightforward due to it being only dependent linearly on ξ

$$\frac{\partial x}{\partial \xi} = X_{max} - X_{min}, \quad \frac{\partial x}{\partial \eta} = 0 \quad (2.67)$$

The relative complexity of (2.61) makes calculation of the y -derivatives not straightforward. Through the use of the chain and quotient rules these derivatives can be defined as

$$\frac{\partial y}{\partial \xi} = a[\eta] \left(\frac{\frac{\partial x}{\partial \xi} + \tanh(x_k) \frac{\partial x_k}{\partial \xi}}{c[\eta]} \right) \quad (2.68)$$

$$\begin{aligned} \frac{\partial y}{\partial \eta} = Y_{max} + \frac{1}{c[\eta]^2} & \left(c[\eta] \left((X_{min} + (X_{max} - X_{min})\xi) \cdot c[\eta] + \ln(\cosh(x_k)) \right. \right. \\ & \left. \left. - \ln(\cosh(x_\eta)) \right) \frac{\partial a}{\partial \eta} + a[\eta] \left(\ln(\cosh(x_\eta)) - \ln(\cosh(x_k)) \frac{\partial c}{\partial \eta} \right. \right. \\ & \left. \left. + c[\eta] \left(\tanh(x_k) \frac{\partial x_k}{\partial \eta} - \tanh(x_\eta) \frac{\partial x_\eta}{\partial \eta} \right) \right) \right) \end{aligned} \quad (2.69)$$

It now remains to define the required derivatives of the parameters defined by ((2.62) - (2.64)). Using a similar process as before, one obtains

$$\frac{\partial c}{\partial \eta} = (C_\infty - C_{wall}), \quad \frac{\partial x_\eta}{\partial \eta} = \frac{\partial c}{\partial \eta} X_c, \quad \frac{\partial x_k}{\partial \xi} = c[\eta] \frac{\partial x}{\partial \xi}, \quad \frac{\partial x_k}{\partial \eta} = \frac{\partial c}{\partial \eta} x \quad (2.70)$$

$$\begin{aligned} \frac{\partial a}{\partial \eta} = \frac{1}{A_d^2} & \left(X_r \left(c[\eta] \tan(\phi + k_\eta \cdot Y_{max} \cdot \eta) \frac{\partial A_d}{\partial \eta} - A_d \left(\tan(\phi + k_\eta \cdot Y_{max} \cdot \eta) \frac{\partial c}{\partial \eta} \right. \right. \right. \\ & \left. \left. + k_\eta \cdot Y_{max} \cdot c[\eta] \cdot \sec^2(\phi + k_\eta \cdot Y_{max} \cdot \eta) \right) \right) \end{aligned} \quad (2.71)$$

Here, A_d is defined as the denominator of (2.62);

$$a[\eta] = -c[\eta] \cdot X_{max} - \ln(\cosh(c[\eta] \cdot X_r)) + \ln(\cosh(x_\eta)) \quad (2.72)$$

and its derivative;

$$\frac{\partial A_d}{\partial \eta} = - \left((X_{max} + X_r \tanh(X_r \cdot c[\eta])) \frac{\partial c}{\partial \eta} \right) + \tanh(x_\eta) \frac{\partial x_\eta}{\partial \eta}. \quad (2.73)$$

Lastly, the second and cross derivatives are then calculated numerically.

For each of the grids defined by the transformations described in §2.7, there are key parameters that are defined to ensure the mesh generated accurately represents the physical case of a compression ramp with an angle of 10° used in this study. These parameters are outlined in Table 2.2.

Parameter:	Discontinuous grid	Pagella et al. (2004) grid	New grid
X_{min}	0.033	0.033	0.033
X_{max}	0.0105	0.0105	0.0105
X_c	0.06	0.06	0.06
Y_{max}	0.02	0.02	0.015
Y_{out}	—	—	0.0455
ϕ	10°	10°	10°
C_{wall}	—	1,873	1,873
C_∞	—	100	35
d	—	1	—

Table 2.2: Parameters used in the grids discussed in §2.7

Figure 2.1 shows representations of the grids described by the transformations in §2.7 and the parameters described in Table 2.2. The visual difference between the three grids can be appreciated in these figures. The key differences between the discontinuous grid, Fig. 2.1(a), and that developed by Pagella et al. (2004), Fig. 2.1(b), are, firstly, that the corner in the latter grid is smoothened and derivatives along the streamwise direction are continuous and, secondly, that in the second grid no discontinuity along the streamwise direction is to be found at any wall-normal location. The main difference between the newly-developed grid and that of Pagella et al. (2004) is that, in the former grid, the wall-normal gridlines on the ramp are aligned with the recompression shock direction, a fact clearly borne out in Fig. 2.2, which shows superposition of one component of the base flow vector and a (intentionally coarse) mesh constructed using the theory discussed in Section 2.7.4.

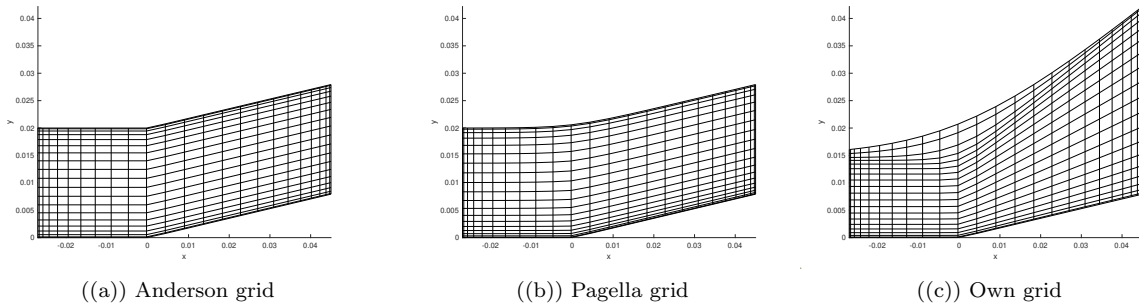


Figure 2.1: Three alternative grids employed for the numerical solution of the compressible BiGlobal EVP in the compression ramp

In generating this grid, an estimate of the shock angle is provided by the oblique shock polars (e.g. White, 2017). At the conditions at hand, the stream deflection angle downstream of the oblique shock and the Mach number of the flow upstream are estimated as

$$\delta = \tan^{-1} \left(\frac{U_y}{U_x} \right) \approx 9.337^\circ, \quad M_1 = \frac{U_x}{\sqrt{\gamma RT}} \approx 2.9 \quad (2.74)$$

As the streamline angle, δ falls between angles given by the oblique shock polars, linear interpolation is used to estimate a shock angle of $\theta = 27.53^\circ$. Comparing this with that obtained from the baseflow by simply measuring the location of the shock where it is shown to be, approximately $\theta_{BF} = 24.31^\circ$, a discrepancy that

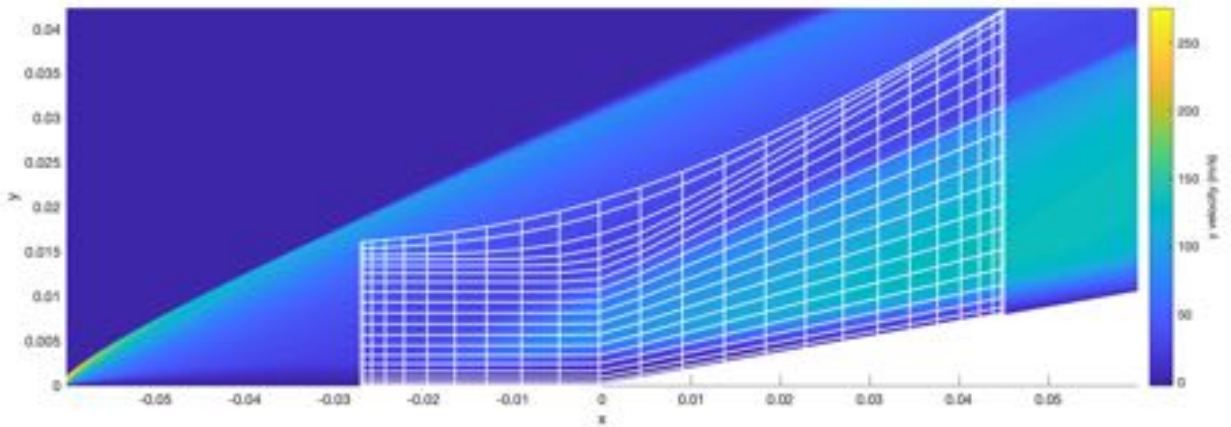


Figure 2.2: Overlay of the new grid from figure 2.1(c) superimposed upon the transverse velocity component of the baseflow starting at the beginning of the flat plate.

can be attributed to the relatively low Reynolds number in the direct numerical simulation.

2.8 Global Stability Analysis with Shock Boundary Conditions

2.8.1 Linearized Shock Boundary Conditions

A typical situation arising in supersonic flow is that of eigenfunctions that are highly oscillatory in the free stream. Standard Dirichlet boundary conditions can not capture such modes correctly and boundary conditions based on the Rankine-Hugoniot relations need to be derived. Due to the inviscid nature of the shock, the derivation of these boundary conditions rests on the Euler equations, which in conservative form can be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = 0, \quad (2.75)$$

where

$$Q = (\rho, \rho u, \rho v, \rho w, e)^T$$

$$E = (\rho u, \rho u^2 + p, \rho uv, \rho uw, (e + p)u)^T \quad (2.76)$$

$$F = (\rho v, \rho uv, \rho v^2 + p, \rho vw, (e + p)v)^T$$

$$G = (\rho w, \rho uw, \rho vw, \rho w^2 + p, (e + p)w)^T$$

and ρ , p , T and e are the density, pressure, temperature and total energy per volume unit, respectively. Here, (x, y, z) are the Cartesian coordinates and (u, v, w) the associated velocity components. In order to transform equation (2.75) into a (generally curved) coordinate system of the shock, a transformation of the Cartesian coordinate system to a general two- (or three-)dimensional frame of reference is required, as illustrated in figure 2.3.

2.8.2 One-dimensional limit

For simplicity in the presentation, and in order to generate validation data for the global stability analysis solver, the one-dimensional analysis of Chang et al. (1990) and Esfahanian (1991) is presented next. The instantaneous position of the shock is given by $y_s = f(x, z, t)$ in Cartesian coordinates and the mean shock position is $\bar{y}_s = \bar{f}(x)$. The local slope (α) is given by the angle between the transformed system and the Cartesian system, $\alpha = \tan(\theta) = \frac{df}{dx}$.

The coordinate transformation is given by

$$\xi = \frac{x}{b} \quad (2.77)$$

$$\eta = b(y - f(x, z, t))$$

where $b = \cos \theta$. Applying this transformation to equation (2.75), the following expression results.

$$\left(\frac{\partial f}{\partial t}\right) \frac{\partial Q}{\partial \eta} + \left(\frac{1}{b}\right) \frac{\partial E}{\partial \xi} + \left(\frac{\partial f}{\partial x}\right) \frac{\partial E}{\partial \eta} - \frac{\partial F}{\partial \eta} + \left(\frac{\partial f}{\partial z}\right) \frac{\partial G}{\partial \eta} = 0 \quad (2.78)$$

Integrating this equation between $\eta = -\epsilon$ and $\eta = +\epsilon$ and taking the limit when $\epsilon \rightarrow 0$, equation (2.78) becomes

$$\frac{\partial f}{\partial t}[Q] + \frac{\partial f}{\partial x}[E] - [F] + \frac{\partial f}{\partial z}[G] = 0, \quad (2.79)$$

where $[\cdot]$ denotes the jump of any quantity across the shock. For example, for a random quantity ϕ , $[\phi] = \phi_1 - \phi_2$, where the subscript 1 denotes the conditions immediately before the shock and 2 denotes the conditions immediately after the shock.

Equation (2.79) is the unsteady Rankine-Hugoniot condition which governs the unsteady motion of a shock wave.

The instantaneous values of the density (ρ_2), velocities (u_2, v_2, w_2), Temperature (T_2), pressure (p_2) behind the shock, as well as the shock position itself (f), can be decomposed in a mean and a fluctuation part as

$$\begin{aligned} \rho_2 &= \bar{\rho}_2 + \rho'_2 \\ u_2 &= \bar{u}_2 + u'_2 \\ v_2 &= \bar{v}_2 + v'_2 \\ w_2 &= \bar{w}_2 + w'_2 \\ T_2 &= \bar{T}_2 + T'_2 \\ p_2 &= \bar{p}_2 + p'_2 \\ f &= \bar{f} + f'. \end{aligned} \quad (2.80)$$

Assuming that the mean flow is "quasi-parallel" in the limit studied by Chang et al. (1990) and Esfahanian

(1991) the disturbances behind the shock are expressed as normal modes

$$\begin{aligned}
 \rho' &= \hat{\rho}(y)e^{i(\alpha x + \beta z - \omega t)} \\
 u' &= \hat{u}(y)e^{i(\alpha x + \beta z - \omega t)} \\
 v' &= \hat{v}(y)e^{i(\alpha x + \beta z - \omega t)} \\
 w' &= \hat{w}(y)e^{i(\alpha x + \beta z - \omega t)} \\
 T' &= \hat{T}(y)e^{i(\alpha x + \beta z - \omega t)} \\
 p' &= \hat{p}(y)e^{i(\alpha x + \beta z - \omega t)} \\
 f' &= \hat{f}(y)e^{i(\alpha x + \beta z - \omega t)}
 \end{aligned} \tag{2.81}$$

where α and β are the wave number in the streamwise and spanwise directions, respectively and ω is the frequency. Substituting the above expressions into equation (2.79), the Rankine-Hugoniot equations can be written as follows:

Continuity equation

$$\begin{aligned}
 \hat{f}(i\alpha(\overline{u_1\rho_1} - \overline{u_2\rho_2}) + i\beta(\overline{w_1\rho_1} - \overline{w_2\rho_2}) + i\omega(\overline{\rho_2} - \overline{\rho_1})) - \\
 \bar{f}'(\widehat{\rho_2 u_2} + \widehat{\rho_2 v_2} + \widehat{\rho_2 w_2}) = 0
 \end{aligned} \tag{2.82}$$

x-momentum equation

$$\begin{aligned}
 \hat{f}(i\alpha(\overline{p_1} - \overline{p_2} + \overline{\rho_1 u_1^2} - \overline{\rho_2 u_2^2}) + i\beta(\overline{\rho_1 u_1 w_1} - \overline{\rho_2 u_2 w_2}) + i\omega(\overline{u_2 \rho_2} - \overline{\rho_1 u_1})) - \\
 \bar{f}'(\widehat{\rho_2 u_2^2} + 2\widehat{u_2 u_2 \rho_2} + \widehat{p_2}) + \widehat{u_2 v_2 \rho_2} + \widehat{v_2 u_2 \rho_2} + \widehat{\rho_2 u_2 v_2} = 0
 \end{aligned} \tag{2.83}$$

y-momentum equation

$$\begin{aligned}
 \hat{f}(i\alpha(\overline{u_1 v_1 \rho_1} - \overline{u_2 v_2 \rho_2}) + i\beta(\overline{v_1 w_1 \rho_1} - \overline{v_2 w_2 \rho_2}) + i\omega(\overline{v_2 \rho_2} - \overline{v_1 \rho_1})) - \\
 \bar{f}'(\widehat{u_2 v_2 \rho_2} + \widehat{v_2 u_2 \rho_2} + \widehat{\rho_2 u_2 v_2}) + \widehat{\rho_2 v_2^2} + 2\widehat{v_2 v_2 \rho_2} + \widehat{p_2} = 0
 \end{aligned} \tag{2.84}$$

z-momentum equation

$$\begin{aligned}
 \hat{f}(i\alpha(\overline{u_1 w_1 \rho_1} - \overline{u_2 w_2 \rho_2}) + i\beta(\overline{p_1} - \overline{p_2} + \overline{w_1^2 \rho_1} - \overline{w_2^2 \rho_2}) + i\omega(\overline{w_2 \rho_2} - \overline{w_1 \rho_1})) - \\
 \bar{f}'(\widehat{u_2 w_2 \rho_2} + \widehat{w_2 u_2 \rho_2} + \widehat{\rho_2 u_2 w_2}) + \widehat{v_2 w_2 \rho_2} + \widehat{w_2 v_2 \rho_2} + \widehat{\rho_2 v_2 w_2} = 0
 \end{aligned} \tag{2.85}$$

Energy equation

$$\begin{aligned}
 \hat{f}(i\alpha(\overline{e_1 u_1} + \overline{p_1 u_1}) - \overline{e_2 u_2} - \overline{p_2 u_2}) + i\beta(\overline{e_1 w_1} - \overline{e_2 w_2} + \overline{p_1 w_1} - \overline{p_2 w_2}) + i\omega(\overline{e_2} - \overline{e_1}) - \\
 \bar{f}'(\widehat{e_2 u_2} + \widehat{u_2 e_2} + \widehat{p_2 u_2} + \widehat{u_2 p_2}) + \widehat{e_2 v_2} + \widehat{v_2 e_2} + \widehat{p_2 v_2} + \widehat{v_2 p_2} = 0
 \end{aligned} \tag{2.86}$$

These five equations contain seven unknowns, since the location of the shock perturbation has been introduced as a disturbance variable $\hat{f}$. In order to close the system in compressible flow an equation of state relates thermodynamic variables. Furthermore, $\hat{f}$ can be expressed using the remaining disturbance variables and either of the continuity or a momentum equation. The above equations, alongside the standard compressible linearized stability equations (Mack, 1984) which are valid in the interior of the domain, and the wall boundary condition complete the description of the eigenvalue problem to be solved.

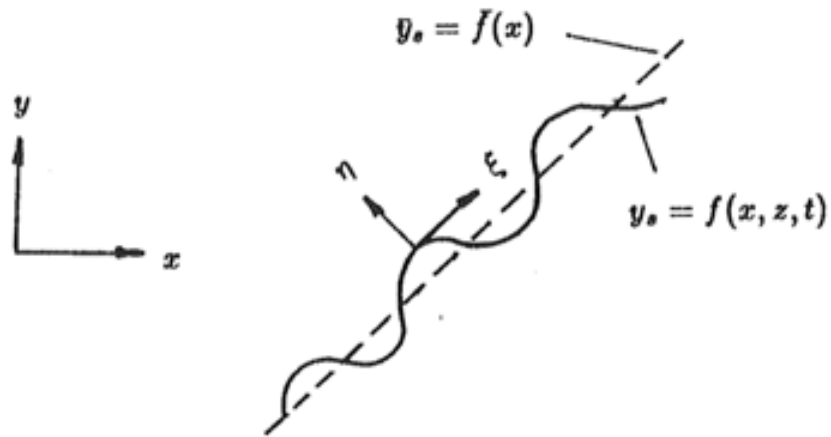


Figure 2.3: Schematic of the instantaneous position of unsteady shock motion in the (ξ, η, t) and (x, y, t) systems, adapted from Chang et al. (1990)

Chapter 3

Compressible Direct Numerical Simulations With OpenFOAM[®]

All laminar base flows analyzed herein have been computed numerically using the open-source simulation code OpenFOAM (Weller et al., 1998). This suite of routines employs second-order finite-volume spatial discretization to compute fluxes and second-order explicit time-advancement schemes. The quality by which steady laminar flows are computed is of paramount concern in the context of linear stability analysis, and thus a fair amount of effort has been spent validating the OpenFOAM code against analytically-known literature results. In the next sections two such validation cases are presented; subsequently, details laminar flows obtained in the compression ramp are discussed in detail.

3.1 Validation and Verification of OpenFOAM

3.1.1 The parabolic cylinder (Anderson, 1989)

When computing flow over the compression corner, two options exist in order to compute the incoming flow on the flat plate boundary layer preceding the wedge. One option is to use uniform supersonic flow as inflow condition and compute flow over the entire configuration composed of the flat plate and the compression corner, including the leading edge of the flat plate in this computation. The other option is to use a similarity solution for the flat-plate boundary layer as an inflow condition at some arbitrarily chosen location on the flat plate, thereby substantially reducing the size of the computational domain and the resulting cost of the simulation compared with the first option. It is planned to follow both approaches and compare the results amongst each other, as well as with those produced by the high-order Maryland code.

Following the first approach, the first concern arising when flow over the entire length of the plate is computed is to address the issue of the leading-edge singularity. In order to avoid a true numerical singularity which can compromise the quality of the numerical solution in the entire flowfield, it is planned to address flow around a finite-thickness plate with rounded leading edge, such that the leading-edge shock stands off at a finite distance from the plate. The ability of OF to address this problem accurately in supersonic flow is first demonstrated next. To this end, inviscid flow over a two-dimensional parabolic cylinder body, for which analytical results are known (Anderson, 1989; Brooks and Powers, 2004), serves to assess resolutions necessary in the neighborhood of the leading edge of the flat plate in the compression ramp. Following a discussion by Marcantonio et al. (2012) the *rhoCentralFoam* solver of OpenFOAM has been used, all flow quantities being dimensional. In this respect it is interesting to recapitulate here the scalings used in this OF solver.

Physical parameters are set following the ideal gas law for air, $p = \rho R^* T$, in which the incoming stream

pressure and temperature are set to $p_{inf} = 105660 \text{ Pa}$ and $T_{inf} = 300 \text{ K}$, respectively. Using the specific gas constant $R^* = 287.056 \text{ Jkg}^{-1}\text{K}^{-1}$ the incoming air density is determined and, alongside a ratio of specific heats $\gamma = 1.4$, the speed of sound $a = \sqrt{\gamma R^* T}$ can also be calculated. The target Mach number $Ma = 3.5$ is then used to compute the incoming flow speed $U_{inf} = 1215.16 \text{ m/s}$. Other parameters set in the solver are a molar weight $m = 28.9 \text{ kg/mol}$, implying treatment of air as diatomic gas, a specific heat at constant pressure, $c_p = 1005 \text{ Jkg}^{-1}\text{K}^{-1}$ and flow enthalpy $h_f = 5 \times 10^6 \text{ Jkg}^{-1}$. Some experimentation has been conducted with the solvers used in the code, before settling for the set-up shown in the appendix. Simulations were performed with a $\text{CFL} \leq 0.1$, which determined an adjustable time-step of $O(10^{-11})\text{s}$.

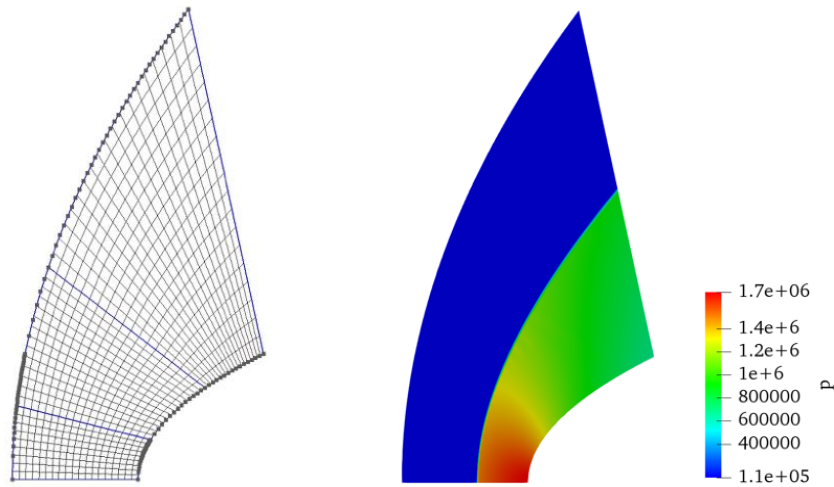


Figure 3.1: Structured mesh used for the solution of the parabolic cylinder problem (left) and pressure distribution at steady state (right)

The parabolic cylinder geometry is analytical, $f(x) = x^{\frac{1}{2}}$, and the solution domain is constructed using GMSH (Geuzaine and Remacle, 2009b) and is decomposed in three sub-domains, such that the structured elements constructed be approximately rectangular, as schematically shown in Figure 3.1. In the left part of this figure it can be seen that both the cylinder surface and the inflow boundaries have been constructed analytically (using cubic splines) and that a common wall-normal discretization and independent spatial discretizations in the direction along the surface are used; a rather low resolution is used in this image, for visualization purposes. In the right part of the same image, the pressure distribution at steady state is shown. Steadiness is monitored by probing solution values as a function of time at arbitrarily chosen locations in the flowfield.

Figure 3.2 shows a typical evolution of the flow variables at a probe placed at $(x_p, y_p) = (-0.1, 0.1)$. It is immediately obvious that, while velocity appears to attains a steady state value at a rather short time after the shock passage and rebound on the body surface, it takes substantially longer for flow quantities to relax to true steady states. Table 3.1 shows the values of these steady states at a time $t = 0.1\text{s}$, a time at which converged of the first three significant digits can be ascertained on the two higher-density grids. The conclusion drawn here is that a (rather dense) grid comprising $O(10^5)$ elements is needed in order to resolve this flow, a value which is consistent with that presented by Marcantonio et al. (2012), and can be used as guidance for the resolutions needed in the vicinity of the rounded leading edge of the flat plate preceding the compression ramp. Such solutions will be obtained in the next reporting period.

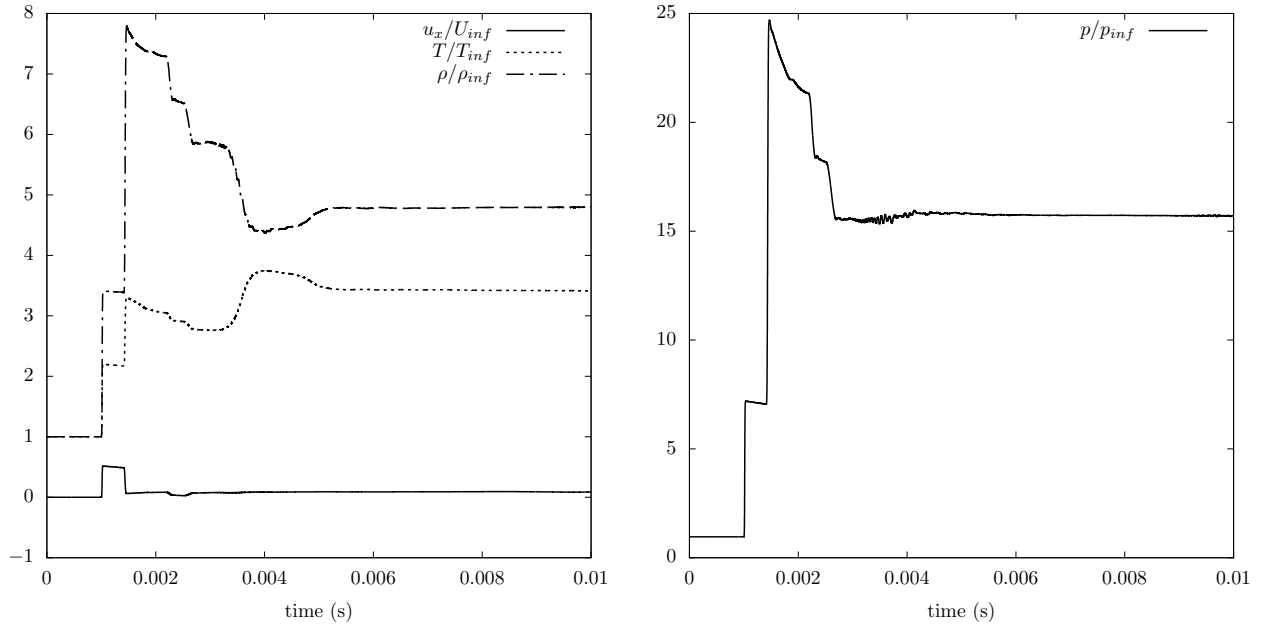


Figure 3.2: Development in time of scaled streamwise velocity, temperature and density (left) and pressure (right) at field point $(x_0, y_0) = (-0.1, 0.1)$ in the inviscid flow over the parabolic cylinder body at $Ma = 3.5$, using the inflow values $U_{inf} = 1215.16 \text{ m/s}$, $T_{inf} = 300 \text{ K}$, $\rho_{inf} = 1.1764 \text{ kg/m}^3$ and $p_{inf} = 105660 \text{ Pa}$ to scale the respective quantities

Elements	$p_{\max}/p_{\infty}$	u_p (m/s)	p_p (Pa)	ρ_p (kg/m ³)	T_p (K)
13061	15.965	117.565	1652969	5.61110	1026.08
53121	16.096	110.735	1658497	5.63049	1025.97
92501	16.220	110.241	1661359	5.63695	1026.56
294241	16.243	110.092	1661655	5.64067	1026.06

Table 3.1: Convergence history of the inviscid flow over the parabolic cylinder body at $Ma = 3.5$. Results obtained using the midpoint integration rule and a full TVD scheme ($\Gamma = 1$) are shown at $t = 0.1 \text{ s}$ and a probe placed at $(x_p = -0.1, y_p = 0.1)$

3.1.2 Flow on a diamond wing (White, 2017)

3.1.3 Background

This validation case consists of analyzing steady potential two-dimensional flow around a diamond wing by employing Prandtl-Meyer theory and comparing the analytically-known (inviscid and irrotational) results with those obtained in numerical simulation performed on the same configuration using the viscous compressible OpenFOAM solver that will be employed to obtain steady laminar base flows in the compression ramp; some details follow.

Inviscid isentropic flow at sharp compression and expansion corners was first studied in the Ph.D. thesis of Meyer (1908), one of Ludwig Prandtl's first students, who arrived at the analytical predictions of the celebrated Prandtl-Meyer theory. Full details can be found in a number of classic references on compressible fluid flow (Shapiro, 1953; Liepmann and Roshko, 1957b). The case solved presently has been defined analytically by White (2017) and is shown in graphical form in Figure 3.3 while Table 3.2 contains the parameters used.

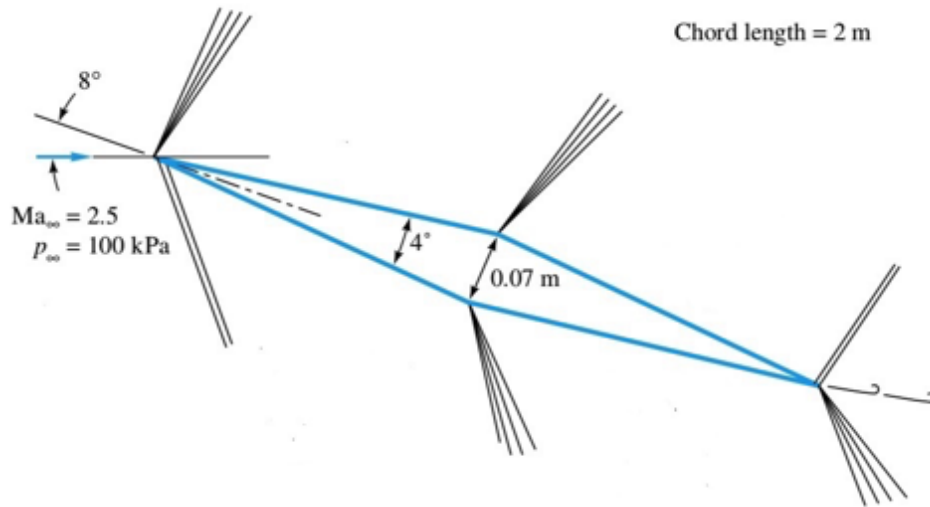


Figure 3.3: Setup of the case solved (White, 2017)

Chord, c (m)	2
Minor diagonal (m)	0.07
Angle of Attack, α (deg)	8
Mach number in region ①, M_∞	2.5
Pressure in region ①, p_∞ (kPa)	100

Table 3.2: Definition of geometry and inflow conditions in the example shown in Figure 3.3

3.1.4 Isentropic compression

For completeness, several relationships needed to resolve isentropic compressions and expansions are summarized here. At a compression corner the $\beta - \theta - Ma$ relationship between the shock angle, β , the deflection angle, θ ,

and the Mach number, M_1 , ahead of the shock (e.g. Liepmann and Roshko, 1957b)

$$\tan \theta = \frac{2 \cot \beta (M_1^2 \sin^2 \beta - 1)}{M_1^2 (\gamma + \cos 2\beta) + 2}, \quad (3.1)$$

in which γ is the ratio of specific heats of the gas, permits solving for either of the two angles when the other angle and the Mach number are known. Equation (3.1) can be solved graphically using the oblique shock polars (e.g. Liepmann and Roshko, 1957b; Anderson, 1989) or numerically, either to evaluate the deflection angle θ when M_1 and β are known, or using Newton's method to compute β for a given pair of known θ and M_1 .

The normal shock relations

$$M_{2,n}^2 = \frac{1 + \frac{\gamma-1}{2} M_{1,n}^2}{\gamma M_{1,n}^2 - \frac{\gamma-1}{2}}. \quad (3.2)$$

and

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1} (M_{1,n}^2 - 1). \quad (3.3)$$

can be used to calculate the Mach number, $M_{2,n}$, and pressure, $p_{2,n}$, past a shock, when their counterparts $M_{1,n}$ and $p_{1,n}$ ahead of the shock are known.

Note that, in oblique shock theory, the normal components of the Mach number are used in (3.2-3.3). These are related with the Mach numbers, M_1 and M_2 , ahead and past the oblique shock by the relations

$$M_{n,1} = M_1 \sin \beta, \quad (3.4)$$

$$M_{n,2} = M_2 \sin(\beta - \theta). \quad (3.5)$$

Again, simple codes can be used to calculate the post-shock conditions $M_{2,n}$ and $p_{2,n}$ according to (3.2-3.5) once the corresponding pre-shock conditions, as well as the shock and deflection angles are known.

3.1.5 Isentropic expansion

At an expansion corner the Prandtl-Meyer function, $\nu(M)$,

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan \sqrt{\frac{\gamma-1}{\gamma+1} (M^2 - 1)} - \arctan \sqrt{M^2 - 1}, \quad (3.6)$$

alongside the equation

$$\nu(M_2) = \nu(M_1) + \omega_{1,2} \quad (3.7)$$

relate the turning angle, $\omega_{1,2}$, by which potential flow turns isentropically, with the Prandtl-Meyer function ν evaluated at the Mach numbers M_1 , prior to the expansion, and M_2 , after the expansion. Equation (3.6) may be used to evaluate the Prandtl-Meyer function when the Mach numbers either side of the expansion fan are known, while the system (3.6-3.7) may be inverted numerically to obtain the Mach number when ν either side of the fan, as well as the turning angle $\omega_{1,2}$ are known.

Stagnation pressure remains constant at an isentropic expansion; this permits relating the pressure past the expansion, p_2 , to that ahead of the expansion, p_1 , in terms of the Mach numbers either side of the expansion through

$$\frac{p_2}{p_1} = \left(\frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2} \right)^{\frac{\gamma}{\gamma-1}}. \quad (3.8)$$

Both the forward and the rearward Mach lines, namely the first and last Mach lines in the expansion fan, can be computed from the well-known expression for the respective Mach angles, μ_1 and μ_2 ,

$$\sin \mu_1 = \frac{1}{M_1} \quad \text{and} \quad \sin \mu_2 = \frac{1}{M_2}. \quad (3.9)$$

Finally, forces acting on an airfoil (be it diamond-shaped or otherwise) can be computed from the difference of pressure between the pressure and suction side. If the airfoil is placed at an angle of attack α to the oncoming supersonic stream, the resultant force, F , is equal to the pressure difference times the wing area (the chord in the case of a two-dimensional airfoil) and points along the normal to the chord. This force can be decomposed into lift, L , and drag¹, D , according to

$$L = F \cos \alpha, \quad (3.10)$$

$$D = F \sin \alpha. \quad (3.11)$$

From these expressions lift and drag coefficients may be calculated, making use of the perfect-gas law identity

$$\frac{1}{2} \rho U_\infty^2 = \frac{1}{2} \gamma p M^2$$

to arrive at

$$c_L = \frac{L}{\frac{1}{2} \gamma p_\infty M_\infty^2 b c}, \quad \text{and} \quad (3.12)$$

$$c_D = \frac{D}{\frac{1}{2} \gamma p_\infty M_\infty^2 b c}, \quad (3.13)$$

where b is the wing span.

Ackeret (1925) developed handier to use formulae for the lift and the wave drag coefficients. Their simplified version is

$$c_L = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}, \quad (3.14)$$

$$c_D = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}} \quad (3.15)$$

and shows that (a) lift in supersonic flow is substantially smaller than in subsonic flow - only to be compensated by high velocities - and (b) that drag increases with the square of the angle of attack.

3.1.6 Prandtl-Meyer theory results

Theoretical results for flow quantities in regions $\textcircled{\text{I}}$ - $\textcircled{\text{V}}$ defined in Figure 3.4 are obtained as follows.

Region I

- Equation (3.6) is used to compute the Prandtl-Meyer function $\nu_I(M_\infty)$ in Region I.

Region II

Between Regions I and II an expansion fan is set up, with a turning angle $\Delta\omega_{I,II} = 6^\circ$.

¹the supersonic *wave drag* (Liepmann and Roshko, 1957b)

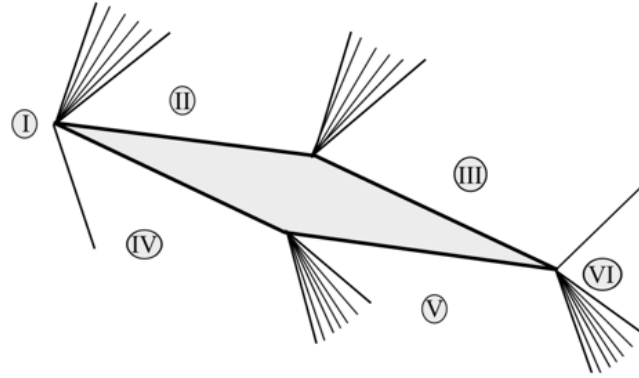


Figure 3.4: Definition of the domains in which flow quantities are to be analyzed

- Equation (3.7) is employed to calculate $\nu_{II} = \nu_I(M_\infty) + \Delta\omega_{I,II}$ in Region II.
- The result and (3.6) are used to calculate the Mach number M_2 in Region II.
- Equation (3.8), the known Mach number and pressure in Region I and the calculated Mach number in Region II are used to compute pressure p_2 in Region II.

Region III

Between Regions II and III an expansion fan is also set up, with a turning angle $\Delta\omega_{II,III} = 4^\circ$. The conditions in Region II have been computed in the previous step.

- The procedure followed in Region II is repeated to compute the conditions in Region III.

Region IV

Region IV is different from the ones addressed previously, in that flow from Region I is compressed into Region IV.

- For the given deflection angle $\theta = 10^\circ$ and $M_1 = 2.5$, the shock angle β is calculated by solving (3.1).
- With β known, equation (3.4) is used to compute the normal component of the Mach number in Region I ahead of the shock.
- The normal shock relation (3.2) is next used to compute the normal component of the Mach number in Region IV.
- With β , θ and $M_{2,n}$ known, equation (3.5) delivers the Mach number in Region IV.
- Finally, equation (3.3) may be used to compute the pressure in Region IV.

Region V

Between Regions IV and V an expansion fan is set up, with a turning angle $\Delta\omega_{IV,V} = 4^\circ$. The conditions in Region IV have been computed in the previous step.

- The procedure followed in Region II is repeated to compute the conditions in Region V. The results are summarized in Table 3.3.

	Region							
	Ⓘ	Ⓜ	$\Delta\omega$	ⓂⓂ	$\Delta\omega$	ⓂⓂ	$\Delta\omega$	Ⓜ
Ma	2.5	2.77	6	2.97	4	2.09	4	2.24
p (kPa)	100	65.91		48.88		186.39		146.90
ν (deg)	39.12	45.12		49.12		28.83		32.83
μ (deg)	23.58	21.16		19.68		28.59		26.51

Table 3.3: Table of results of Prandtl-Meyer theory on the diamond wing defined in Figure 3.3

3.1.7 OpenFOAM[®] simulation parameters

The numerical work employed the *rhoCentralFoam* compressible solver of the open source package OpenFOAM[®], run in inviscid mode by setting the flow dynamic viscosity $\mu = 0$. The boundary domains, defined in the `0/U`, `0/p` and `0/T` simulation files, are `inflow`, `outflow`, `farfield`, `wall`, `frontAndBack`. The boundary conditions are set in the `0/U`, `0/p` and `0/T` simulation files; entries controlling the Mach number and the angle of attack used in the simulation are changed as follows.

- The Mach number is a free parameter in the simulations. Once set, the ideal gas law

$$p = \rho RT, \quad (3.16)$$

in which $R = 287 \frac{J}{kg}$ is the ideal gas constant for air can be used to relate the thermodynamic quantities.

- The values of pressure and temperature are set as free parameters in the simulation and (3.16) can be used to determine the density.
- The speed of sound is then determined consistently as

$$a^2 = \frac{\gamma p}{\rho}. \quad (3.17)$$

- The velocity vector may be set as boundary condition at the inflow boundary, according to the chosen Mach number and angle of attack. For example, for $a = 340 \frac{m}{s}$, $Ma = 3$ flow and $\alpha = 10^\circ$ one would set

```
inflow
{
  type fixedValue;
  value uniform (1004.50 177.12 0);
}
```

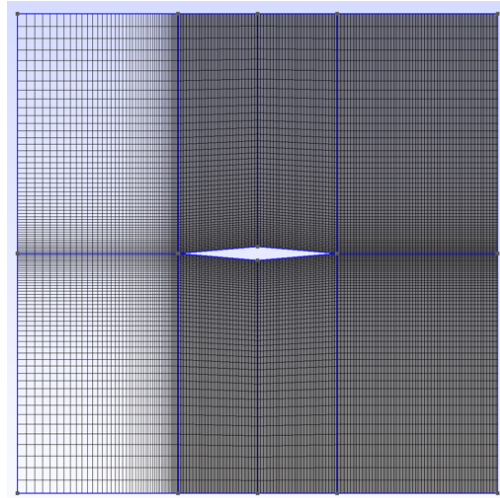


Figure 3.5: Schematic of a .geo file and the associated structured mesh for the diamond wing

3.1.8 OpenFOAM[®] simulation results

The geometry of the diamond-shaped airfoil has been meshed in GMSH[®], as schematically shown in Figure 3.5. Grid independence was obtained on structured meshes comprising upwards of 750,000 elements to discretize two-dimensional space around the diamond wing. Simulations were run with flow starting from rest and were continued until steady state was obtained; the transverse velocity component and magnitude of the density gradient are shown in Fig. 3.6. In both of these images the shocks and expansions emanating from the vertices of the diamond wing are clearly visible. Given that the inviscid results are uniform within the five regions ① - ⑤, numerical values were extracted using the [Plot Over Line](#) filter of [ParaView](#), as shown in Fig. 3.7. Results were obtained along lines parallel to the wing chord at a fixed distance from the wing along both the pressure and the suction side. The pressure values along the line shown in the left part of Fig. 3.7 are presented in the right part of the same figure and it can be seen that pressure remains constant along most of the respective regions. The values obtained in the simulation for the flow Mach number and pressure in each of the five regions are presented in Table 3.4 alongside the results of Prandtl-Meyer theory shown in Table 3.3. As can be seen the agreement between the analytical results and those obtained by inviscid simulation using the OF software is excellent.

	Region				
	①	②	③	④	⑤
Ma (theory)	2.5	2.77	2.97	2.09	2.24
Ma (simulation)	2.5	2.77	2.96	2.08	2.23
p (kPa, theory)	100	65.91	48.88	186.39	146.90
p (kPa, simulation)	100	65.92	48.88	186.40	146.92

Table 3.4: Comparison of Prandtl-Meyer theory and OpenFOAM simulation results on the diamond wing defined in Figure 3.3

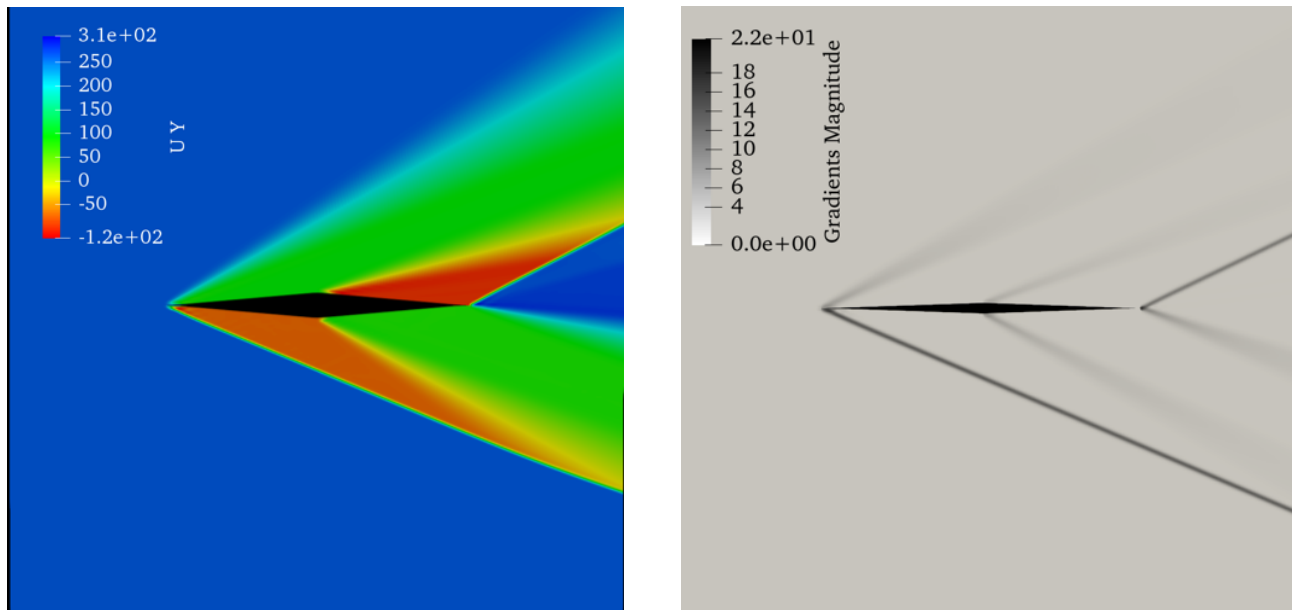


Figure 3.6: Steady state results over the diamond wing. *Left:* Transverse velocity component. *Right:* Density gradient.

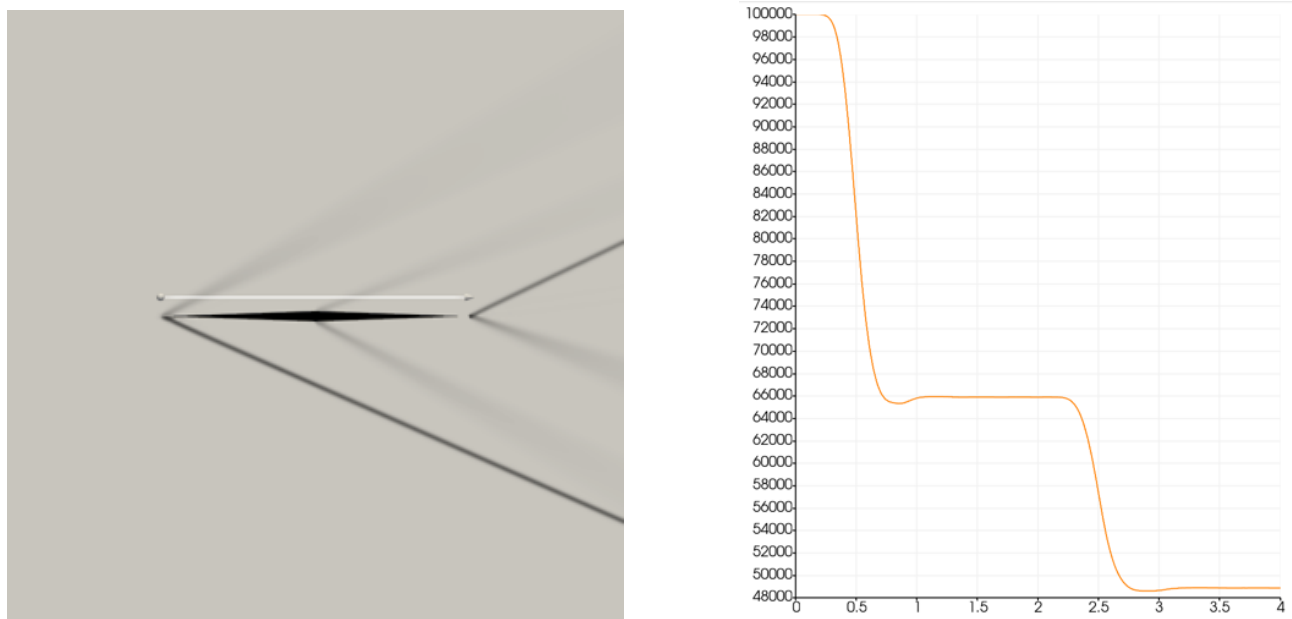


Figure 3.7: *Left:* Line along which simulation data have been extracted. *Right:* Pressure along this line.

3.2 Two-dimensional viscous flow over the compression ramp

The *rhoCentralFoam* solver has next been used to resolve viscous laminar flow over a two-dimensional compression ramp at the conditions of Carter (1972); the free-stream parameter values used in the present simulations are shown in Table 3.5. A systematic grid convergence study was performed in order to generate reference solutions for this geometry. Experience gained in the parabolic cylinder as regards meshing parameters at the supersonic conditions monitored has guided the construction of a simulation domain in which a rounded leading edge preceded the flat plate, in order to eliminate singularities that could condition solution convergence; Fig. 3.8 presents details of the geometry. A coarse version of the structured grid used for the OpenFOAM ramp simulations is shown in Figure 3.9 for illustration purposes. Sutherland's law

$$\mu = C_s \times \left(\frac{T^{3/2}}{T + T_s} \right) \quad \text{where} \quad C_s = \mu_0 \times \left(\frac{T_0 + S}{T_0^{3/2}} \right) \quad , \quad (3.18)$$

has been used to calculate the dynamic viscosity with constant values $C_s = 1.4586 \times 10^{-6} Pa \, K^{-\frac{1}{2}}$ and $T_s = 110.4C$; the remaining free-stream parameter values used in the present simulations are shown in Table 3.5.

Ma	p_∞ (Pa)	u_∞ (m/s)	T_∞ (K)	ρ_∞ (kg/m ³)	μ_∞ (kg/(m s))
3.0	101325	885.25	216.7	1.629	1.42×10^{-5}

Table 3.5: Free stream parameters used in the compression ramp simulations

Table 3.6 presents the steady streamwise velocity component obtained with $O(0.7 \times 10^6)$, $O(3 \times 10^6)$ and $O(7 \times 10^6)$ grid elements at three probe locations inside the boundary layer, namely $(x_1 = 5 \times 10^{-5}, y_1 = 10^{-5})$, prior to the laminar separation bubble, $(x_2 = 10^{-3}, y_2 = 10^{-5})$, inside the laminar separation zone at the ramp junction, and $(x_3 = 1.2 \times 10^{-3}, y_3 = 10^{-4})$, on the downstream ramp wall. These results show that the solution is not yet converged, especially prior to the separation bubble. The reason, however, does not seem to be attributed to poor spatial resolution, as will be discussed shortly. Table 3.7 shows that integral quantities, such as the location of primary separation and primary reattachment, are predicted reasonably well on either of the three grids defined in this table; Fig. 3.10 also shows the distribution of skin friction coefficient with distance from the ramp corner on each of these grids and offers visual confirmation of the apparent spatial convergence of the solutions; Table 3.7 presents the corresponding quantitative results, using a different sequence of three grids of increasing density.

Elements	P_1	P_2	P_3
756,834	579.93003	-36.791174	846.937145
3,074,270	528.98182	-39.992594	848.179798
6,949,543	523.92066	-40.230205	848.400341

Table 3.6: Convergence history of viscous laminar flow in the compression ramp at $Ma = 3.0$. Steady results for the streamwise component of the base flow velocity, $\bar{u}$, presented at three probe locations inside the boundary layer, $P_1(x_1 = 5 \times 10^{-5}, y_1 = 10^{-5})$ prior to the laminar separation bubble, $P_2(x_2 = 10^{-3}, y_2 = 10^{-5})$ inside the laminar separation zone at the ramp junction, and $P_3(x_3 = 1.2 \times 10^{-3}, y_3 = 10^{-4})$ on the downstream ramp wall

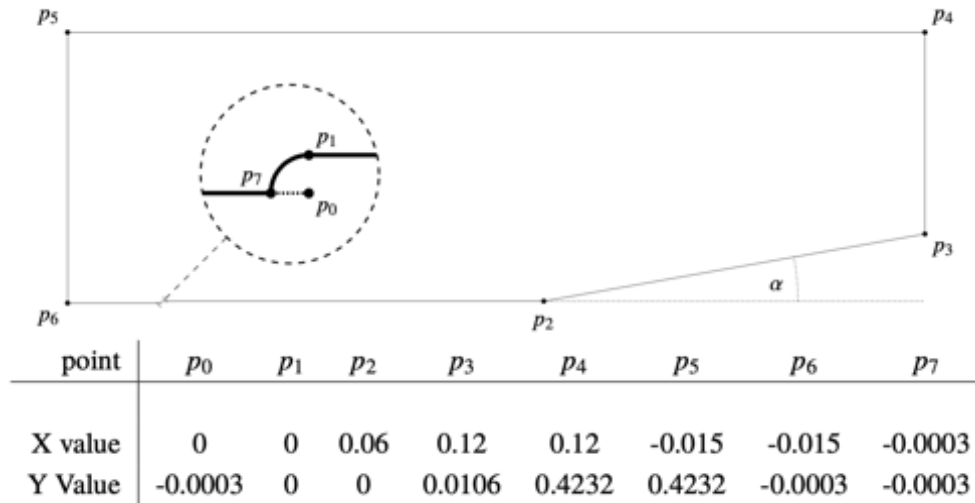


Figure 3.8: Schematic of the compression corner geometry used with a ramp angle $\alpha = 10$ deg, highlighting the leading edge region. The table defines the coordinates (in m) of the points indicated in the graph.

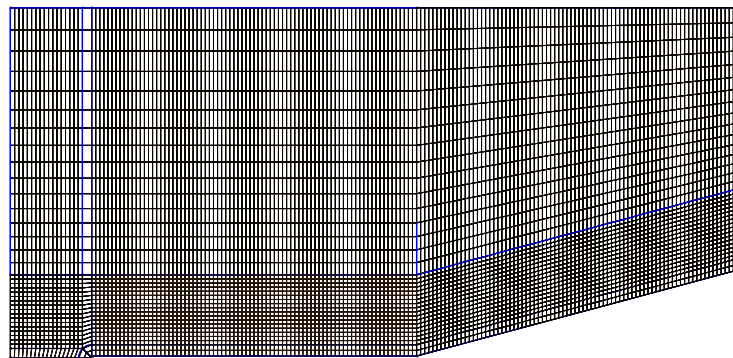


Figure 3.9: Structured mesh used for the solution of flow over a compression ramp. For clarity, a coarse resolution is shown.

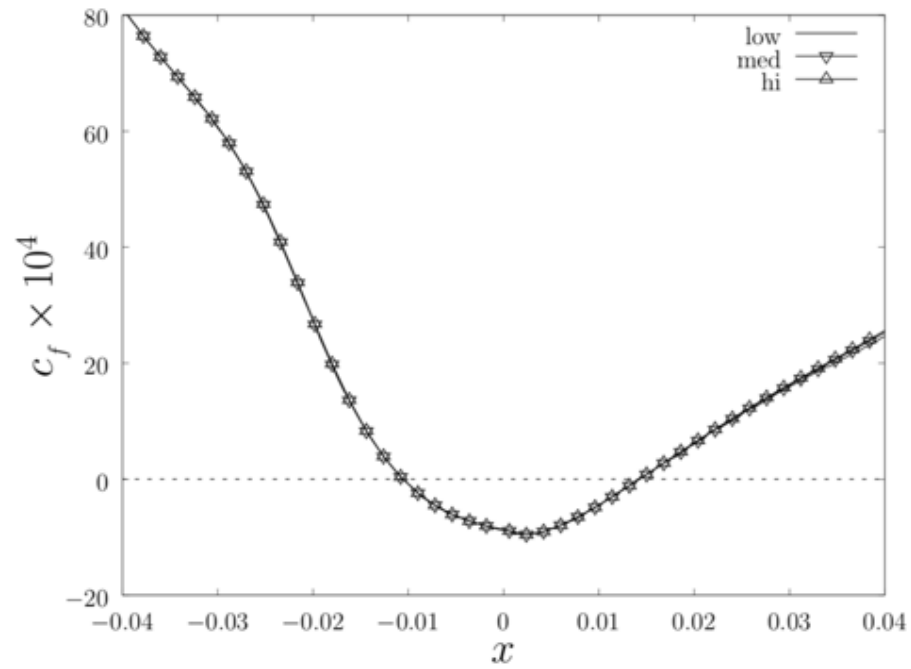


Figure 3.10: Skin friction coefficient evolution along the ramp, showing independence of base flow results on spatial resolution. The three grids shown in Table 3.6 are denoted as *low*, *medium* and *high* spatial resolution

Total elements	Separation location ($x_{\text{plate}}/L, y/L$)	Reattachment location ($x_{\text{ramp}}/L, y/L$)	$(c_f)_{\min}$ $\times 10^4$
2, 282, 160	(−0.1752, 0)	(0.2357, 0.2179)	−9.51
5, 872, 868	(−0.1761, 0)	(0.2375, 0.2182)	−9.56
9, 191, 310	(−0.1759, 0)	(0.2370, 0.2181)	−9.55

Table 3.7: Separation location, reattachment location and minimum skin friction coefficient of three different meshes; lengths have been scaled with the flat plate length, $L = 0.06mm$, with the corner being the origin.

Fig. 3.11 shows snapshots of velocity magnitude gradient as the simulation evolves towards a steady state. While intense activity takes place throughout the entire calculation domain during the early stages of the simulation, as time advances the only features remaining clearly visible are the leading edge (LE) and the recompression shock, as well as the separated laminar boundary layer at the ramp. Although it takes relatively little time for the LE shock and the boundary layer to settle to a steady state, the motion of the recompression shock determines the time for overall temporal convergence of the simulation.

Figure 3.12 shows short and long time histories to the same velocity component before, at and after the separation bubble. After a nonlinear initial stage caused by inaccurate initial conditions, the signal becomes smooth after approximately $10\mu s$. However, between $75\mu s < t < 150\mu s$, depending on the measurement location, disturbances can be observed to grow and saturate, reaching a statistically stationary steady-state whose unsteady signal oscillates with a high frequency. This can be observed in spatial profiles as well. Figure 3.13 shows the streamwise velocity component and pressure variations normal to the wall at different streamwise locations along the ramp. Although the former is reasonably smooth, the latter does present small amplitude high wavenumber oscillations. In other words, it was not possible to reach a true steady-state using OpenFOAM under the temporal and spatial resolutions, as well as parametric conditions, employed. Figure 3.14 presents evidence of this statement in terms of the gradient of the velocity magnitude of the computed solution, in which well-resolved small-amplitude oscillations are visible to emanate at the shock and the vicinity of the leading-edge and extend over the entire solution domain. It is conceivable that these oscillations have physical origin and it is also likely that the resolutions issues encountered both in the OpenFOAM simulations discussed are related with these oscillations being better resolved and certainly remaining active in the solution as resolution is increased.

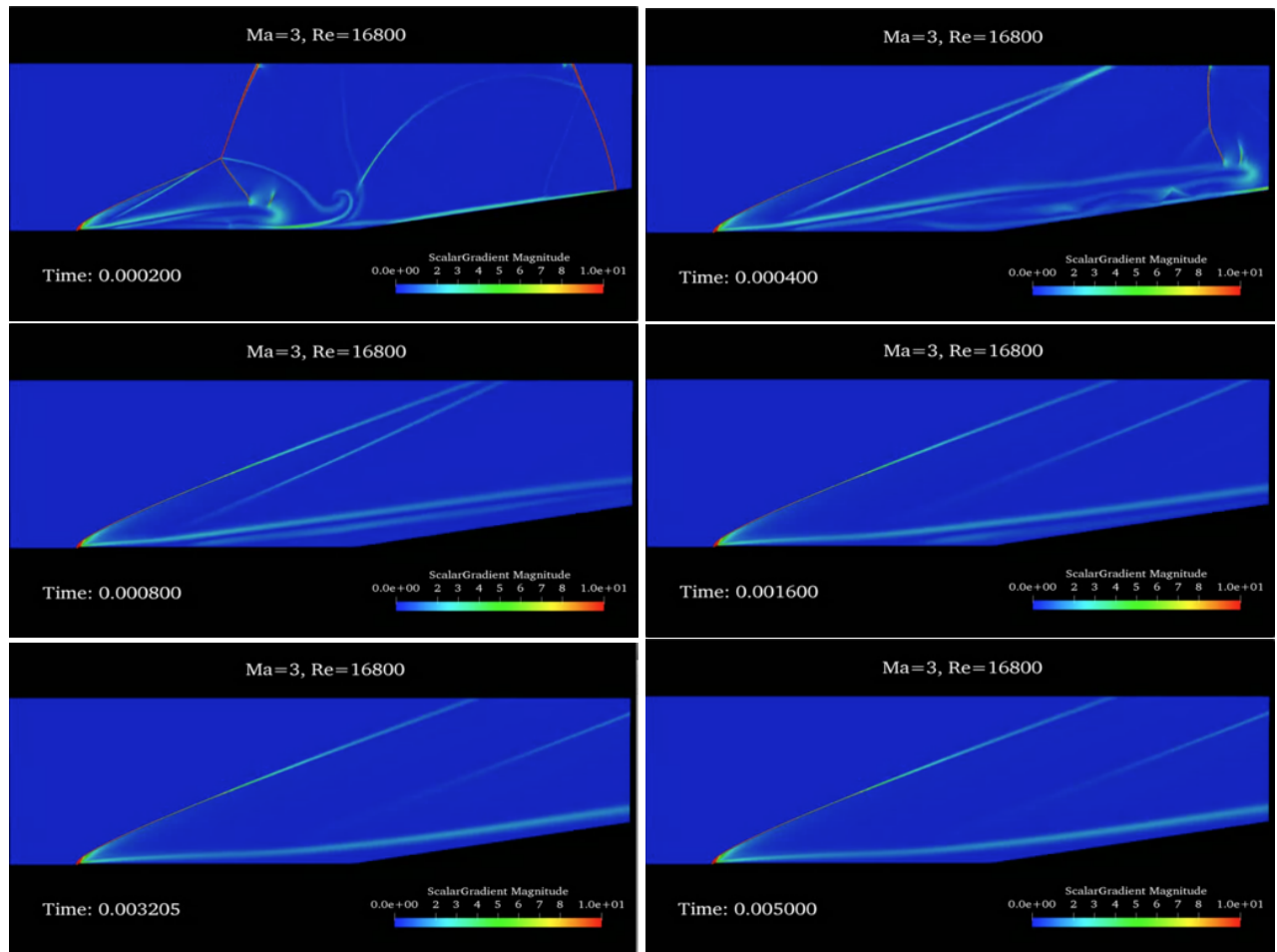


Figure 3.11: Evolution of the gradient of density field at $Ma = 3, Re = 16800$ at times $t = 2, 4, 8, 16, 32$ and $50 \times 10^{-4}s$

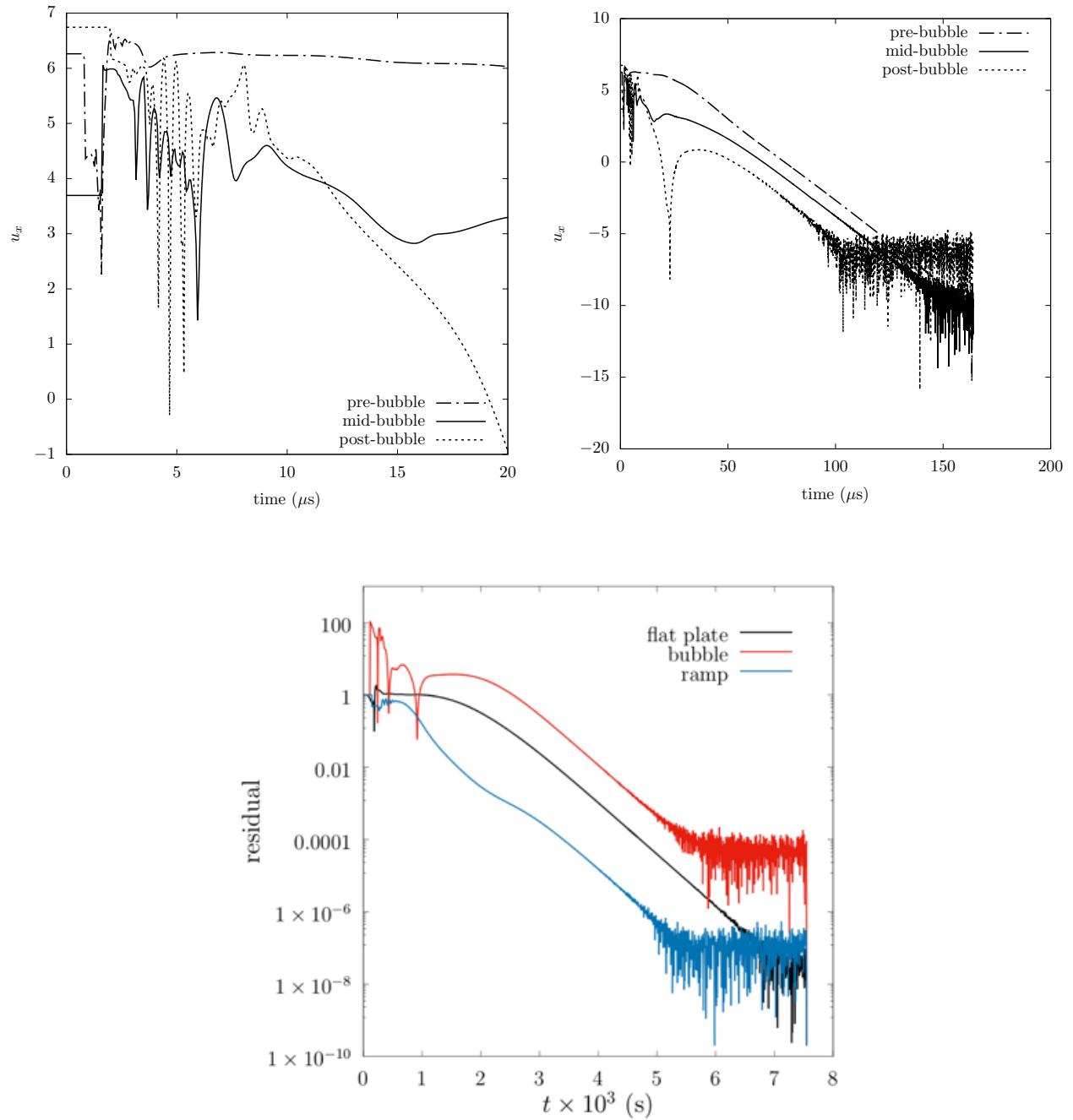


Figure 3.12: *Upper*: Short- and long-time development in time of the streamwise velocity component at three probe locations, prior to the bubble $(x_0, y_0) = (-0.1, 0.1)$, at the ramp junction $(x_0, y_0) = (-0.1, 0.1)$ and at a location after the bubble, $(x_0, y_0) = (-0.1, 0.1)$. *Lower*: Temporal convergence of residuals at highest spatial resolution.

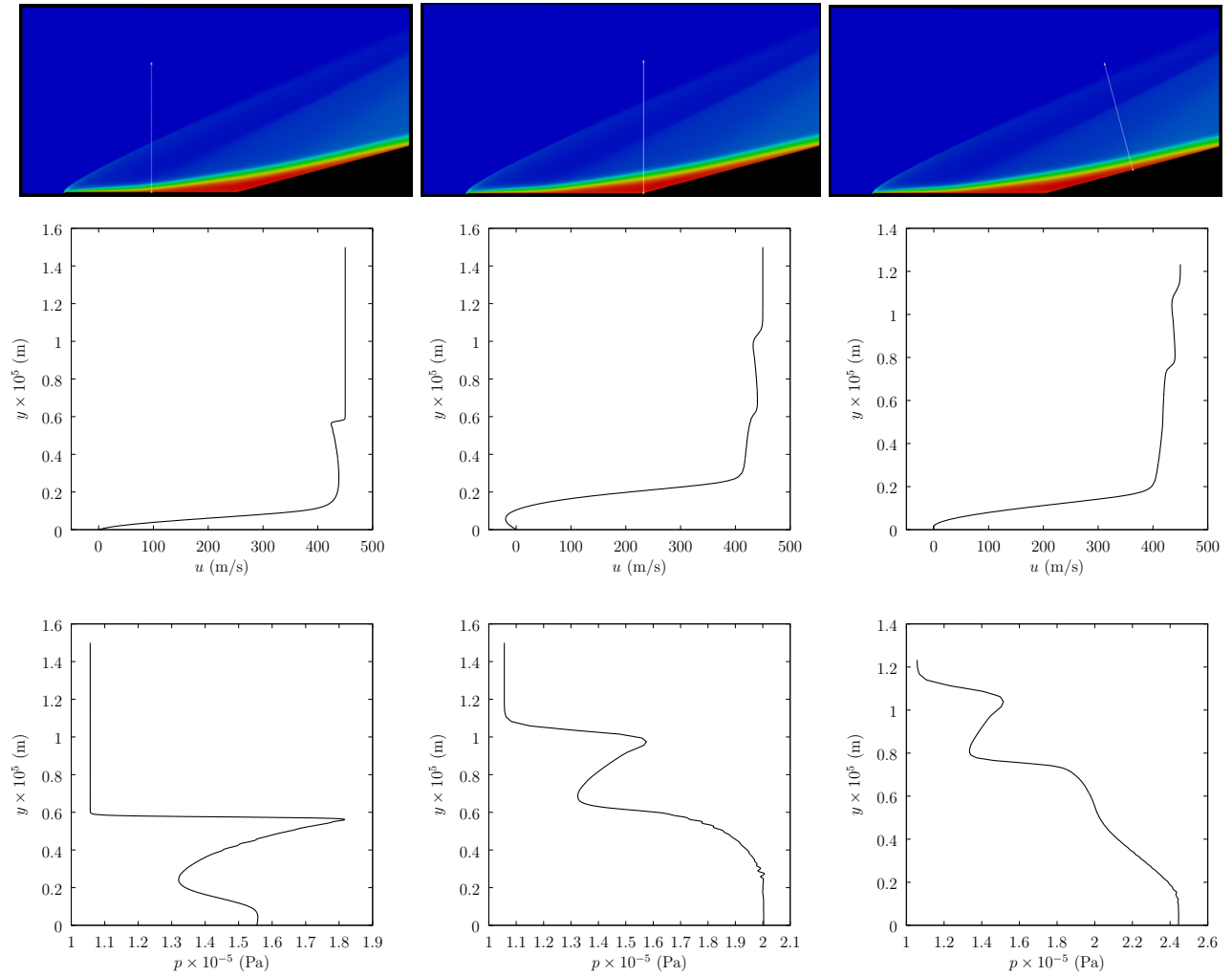


Figure 3.13: *Upper row:* Locations on the ramp where profiles were extracted. *Middle row:* Streamwise velocity component profiles. *Lower row:* Pressure profiles.

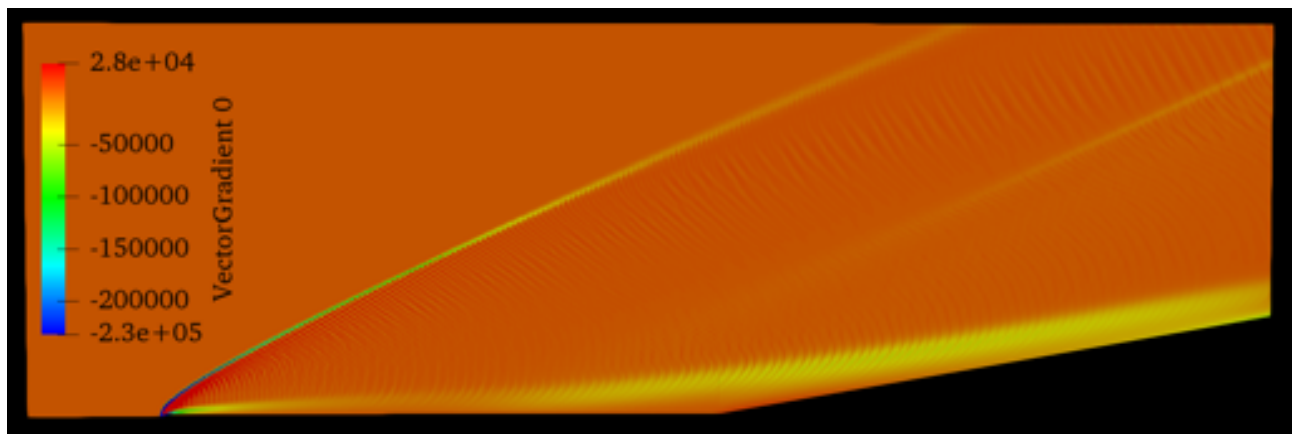


Figure 3.14: Gradient of the velocity magnitude of the steady state solution obtained by OpenFOAM in Fig. 3.13, indicating traces of (well-resolved) small-amplitude perturbations in the entire simulation domain.

Chapter 4

Numerical Methods

4.1 The Linear Global Hypersonic Transition - LiGHT code

4.1.1 Introduction and motivation

Global linear stability analysis is applicable to flow from the incompressible to the hypersonic regime. In a matrix-forming context seen from the perspective of a discretized LNSE operator, the sparsity pattern of the corresponding matrix is affected by the flow limit addressed, the form of the operator solved and the numerical methodology employed to perform spatial discretization. Considering incompressible flow in primitive variables and computing spatial derivatives by narrow-stencil finite-difference schemes leads to relatively sparse matrices. By contrast, global instability analysis of incompressible flows performed by solution of the generalized Orr-Sommerfeld equation (Tatsumi and Yoshimura, 1990), in which cross-differentiation terms arising from elimination of pressure arise, leads to fully dense matrices. The same is true when global instability analysis of compressible flow is performed in primitive variables, combined with spectral collocation spatial discretization, as schematically shown in Figure 4.1.

The question whether dense or sparse linear algebra techniques are better suited to solve the same (incompressible, temporal) BiGlobal EVP in a multi-domain spectral collocation context has been addressed by De Vicente et al. (2011), who showed that a (relatively low) threshold of number of processors exists above which dense parallel solution can be obtained within lower wall-clock times, compared with a serial sparse approach. However, the parallel version of the sparse library used for that work encountered a performance bottleneck in the solution of the linear system and direct comparisons between parallel sparse and dense matrix storage were not performed. The sparsity pattern of the matrix discretizing the LNSE operator can increase substantially when high-order finite-differences with spectral-like resolution, such as those introduced into linear stability analysis by Paredes et al. (2013), are employed to perform spatial discretization. In this situation, the computational cost of the LU decomposition of a matrix with substantially larger leading dimension, compared with one based on spectral discretization of the same accuracy, is compensated by employing LU decomposition based on a larger number of nodes necessary to match the accuracy properties of the spectral method.

Another relevant aspect in this context is whether modal or non-modal linear global instability analysis is performed. Modal analysis deals with solution of any of the aforementioned eigenvalue problems and provides predictions regarding the long-time development of small-amplitude perturbations superposed upon two or three-dimensional flow. By contrast, non-modal, short-time linear instability analysis may be performed in either a BiGlobal or a TriGlobal context, by performing singular value decomposition (SVD) of the matrix discretizing the LNSE. The most straightforward approach to compute the SVD of a matrix is computation of the eigenvalue decomposition of the matrix times its transposed complex conjugate (Theofilis, 2011). When this approach is followed, any benefits of sparse storage of the original (direct) matrix, or its adjoint, are lost,

since the multiplication operation destroys the sparsity pattern. It thus becomes evident that a need exists for algorithms for the efficient solution of the non-symmetric, complex generalized eigenvalue problem.

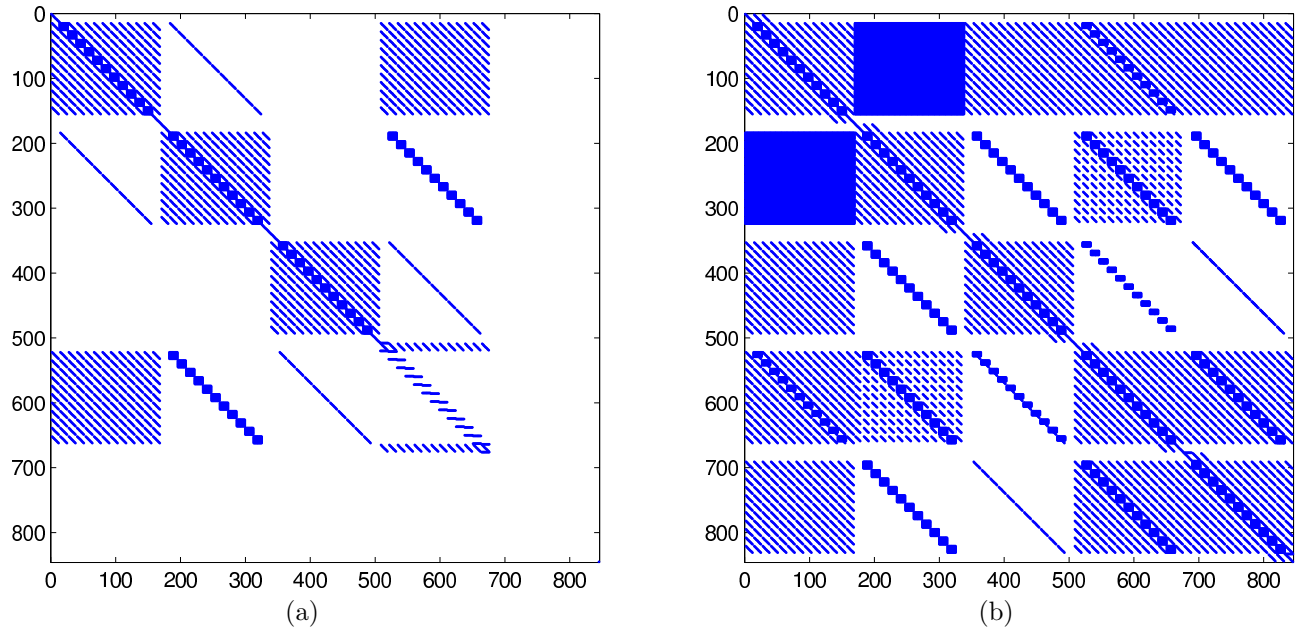


Figure 4.1: Sparsity patterns using spectral collocation spatial discretization of (a): the incompressible temporal BiGlobal EVP (b): the compressible temporal BiGlobal EVP

To deal with the above issues, a matrix-forming algorithm is under development, for the massively parallel solution of multi-dimensional non-symmetric eigenvalue problems (EVPs) governing global linear fluid flow instability. In the supersonic and hypersonic flow regimes of interest to the present effort, as well as to that discussed in FA9550-15-1-0387, these eigenvalue problems result in dense matrices. The ScaLAPACK library, available as part of vendor-optimized libraries on most modern supercomputers, is used to perform a full distributed LU-decomposition, as required by shift-and-invert strategies which promote physically-interesting and suppress spurious eigenmodes. The iterative generation of Krylov subspaces and the computation of the Ritz vectors are performed by linear algebra operations distributed over the available processors, while the serial computation of the Ritz values represents a negligible part of the overall computing effort. In this reporting period, the algorithm has been validated by solving two- and three-dimensional eigenvalue problems, results of which are known analytically, as well as BiGlobal EVPs in the incompressible regime. Some details of progress so far follow.

4.1.2 Block-cyclic distribution of a matrix

A key feature of the present algorithm is that it solves generalized eigenvalue problems described by matrices with large leading dimension that cannot be allocated or stored in main memory. Instead, matrix elements are distributed individually over the available number of processors as they are formed. The procedure followed for the formation of the matrix elements resulting from spatial discretization of the linearized Navier-Stokes equations will be discussed in the next section. In this section the block-cyclic distribution scheme used to assign a matrix element directly to the appropriate process is presented. The block-cyclic distribution has been found to be essential for optimum performance of the subsequent linear algebra operations (Dongarra and Blackford, 1996) and is the scheme also followed presently.

Let the (row, column) coordinates of an element of the (global) matrix be described by the pair of indices $(i_{\text{glob}}, j_{\text{glob}})$, with $0 \leq i_{\text{glob}} \leq M - 1$ and $0 \leq j_{\text{glob}} \leq N - 1$. The block-cyclic distribution scheme is discussed for a rectangular global matrix, $M \neq N$, although the eigenvalue problem is subsequently solved for square matrices, $M = N$. However, the ability to distribute rectangular matrices will be useful in future extensions of the present work, dealing with the singular value decomposition of matrices arising in non-modal global linear instability analysis (Theofilis, 2011).

Let the number of available processes be arranged into a two-dimensional grid, $P \times Q$, and let (p, q) describe the coordinates of the processor within which global element $(i_{\text{glob}}, j_{\text{glob}})$ is found, with $0 \leq p < P - 1$ and $0 \leq q < Q - 1$. Again, the discussion of block-cyclic distribution is offered for $P \neq Q$, although solution of the eigenvalue problem in ScaLAPACK will require $P = Q$.

Finally, let $r \times c$ be the row and column blocking factors, respectively, which determine the size of the blocks within each process, and let (b, d) be the local block coordinates within process identified by the indices (p, q) . The local coordinates $(i_{\text{loc}}, j_{\text{loc}})$ of global element $(i_{\text{glob}}, j_{\text{glob}})$ within process (p, q) and block (b, d) are determined by the following mapping of the global index i_{glob} to a triplet $\langle p, b, i_{\text{loc}} \rangle$ and of global index j_{glob} to a triplet $\langle q, d, j_{\text{loc}} \rangle$, i.e.

$$(i_{\text{glob}}, j_{\text{glob}}) \mapsto \langle (p, q), (b, d), (i_{\text{loc}}, j_{\text{loc}}) \rangle. \quad (4.1)$$

Here

$$(p, q) = \left(\left\lfloor \frac{i_{\text{glob}}}{r} \right\rfloor \bmod (P), \left\lfloor \frac{j_{\text{glob}}}{c} \right\rfloor \bmod (Q) \right), \quad (4.2)$$

$$(b, d) = \left(\left\lfloor \frac{i_{\text{glob}}}{rP} \right\rfloor, \left\lfloor \frac{j_{\text{glob}}}{cQ} \right\rfloor \right), \quad (4.3)$$

$$(i_{\text{loc}}, j_{\text{loc}}) = (i_{\text{glob}} \bmod (r), j_{\text{glob}} \bmod (c)), \quad (4.4)$$

$m \bmod (n)$ is the modulo function and $\lfloor m \rfloor$ is the integer floor function.

The inverse mapping of the triplet $\langle (p, q), (b, d), (i_{\text{loc}}, j_{\text{loc}}) \rangle$ to the global indices $(i_{\text{glob}}, j_{\text{glob}})$ is defined by

$$\langle (p, q), (b, d), (i_{\text{loc}}, j_{\text{loc}}) \rangle \mapsto (r(p + bP) + i_{\text{loc}}, c(q + dQ) + j_{\text{loc}}) \quad (4.5)$$

$$\equiv (i_{\text{glob}}, j_{\text{glob}}) \quad (4.6)$$

An element $(i_{\text{glob}}, j_{\text{glob}})$ in a global rectangular matrix of dimension (M, N) acquires local coordinates $(i_{\text{loc}}, j_{\text{loc}})$ which depend on the number of processes, (P, Q) , used to distribute the global matrix and on the (row, column) blocking factor values (r, c) . Knowledge of these data permits computation of the local indices using equations (4.2-4.4). Conversely, if the local indices, as well as (P, Q) and (r, c) are known, equations (4.2-4.3) and (4.5) can be used to identify the global array indices pertinent to the element in question.

An example of block-cyclic distribution of a square array with leading dimension $M = N = 16$ over four processes arranged in a $(P, Q) = (2 \times 2)$ process grid and using a blocking factor $r = c = 5$ is shown in Table 4.1. The left figure shows the matrix elements as seen from a global perspective, while the right figure shows the elements held by each process, identified by a unique color for each process.

4.1.3 Formation of the discretized LNSE using Kronecker operations

Turning to the procedure followed in order to create the elements of the matrix discretizing the linearized Navier-Stokes equations, or indeed the spatial discretization of any linear PDE-based system, the procedure followed to set up the matrix in parallel heavily relies on Kronecker matrix operations, specifically addition and multiplication.

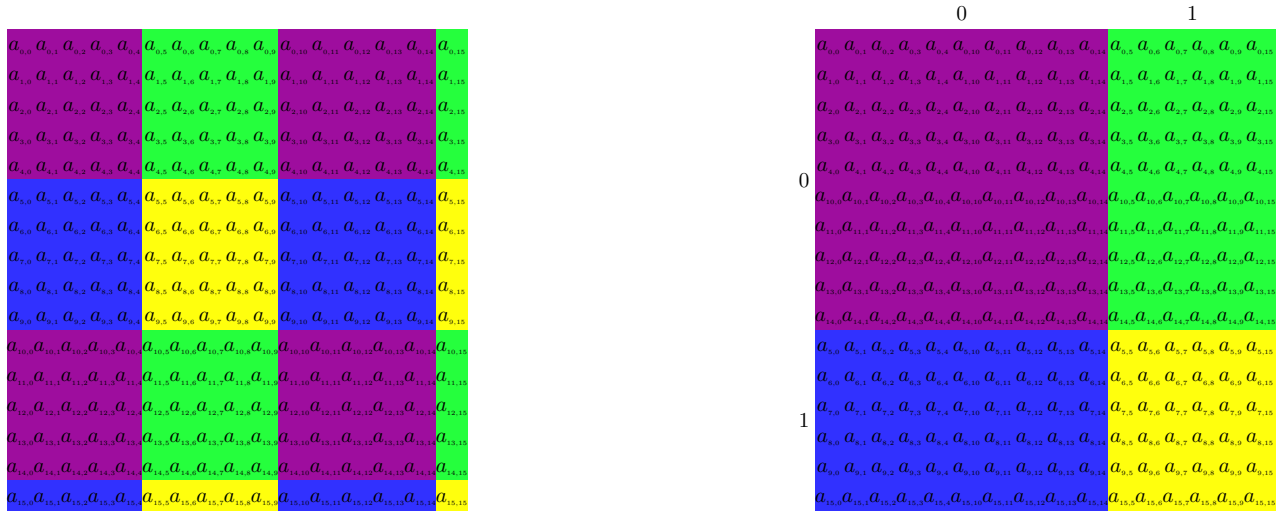


Table 4.1: Distribution of a 16×16 array over a 2×2 processor grid using a block-cyclic scheme with a 5×5 blocking factor

The Kronecker product plays a key role in forming the matrix eigenvalue problem in multiple spatial dimensions, since its associative multiplication property makes the extension of the Kronecker product to higher dimensions straightforward. It also permits a simple construction of the linearized Navier-Stokes operator, including introduction of the base flow and its derivatives, as will be discussed shortly. Formation of the linearized operator is also transparent to the details of the differentiation matrix used, making a Kronecker-product based strategy both flexible and general: distinct spatial differentiation schemes and the corresponding differentiation matrices may be defined independently along each direction, using the Kronecker product to construct tensor product grids in two or three spatial dimensions. In case a spatial direction is inhomogeneous, it may be discretized by typical (Chebyshev or Legendre) spectral collocation differentiation matrices based on Jacobi polynomials, although any differentiation operator that can be written in matrix form, such as low- or high order standard or compact finite-differences, may be used. If the spatial direction in question is homogeneous, Fourier collocation derivative matrices may be used to discretize it, although a flow instability problem in which homogeneous spatial directions exist can be reduced to one described by differential equations of lower order. The generation of the linearized operator using Kronecker products is detailed next.

Kronecker product of matrices

If $\mathbf{A} \in \mathbb{C}^{P \times Q}$, $\mathbf{B} \in \mathbb{C}^{R \times S}$, the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is defined as the rectangular matrix $\mathbf{C} \in \mathbb{C}^{PR \times QS}$

$$\mathbf{C} \equiv \mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1q}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2q}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}\mathbf{B} & a_{p2}\mathbf{B} & \cdots & a_{pq}\mathbf{B} \end{pmatrix}.$$

In compact form, the elements $a_{p,q}$, $b_{r,s}$ and $c_{\mu,\nu}$ of matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are related by the identity

$$c_{\mu,\nu} = c_{\{(p,q);(r,s)\}} = c_{R(p-1)+r,S(q-1)+s} = a_{p,q}b_{r,s}. \quad (4.7)$$

indicating that element (μ, ν) is found at position (r, s) of block (p, q) . The following properties are of interest,

where $+$ is the common addition:

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}, \quad (4.8)$$

$$(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}, \quad (4.9)$$

$$(k\mathbf{A}) \otimes \mathbf{B} = \mathbf{A} \otimes (k\mathbf{B}) = k(\mathbf{A} \otimes \mathbf{B}), \quad (4.10)$$

$$(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) \quad (4.11)$$

Kronecker addition of matrices

In the particular case of square matrices $\mathbf{A}$ and $\mathbf{B}$ with leading dimensions P and R respectively, Kronecker addition, $\oplus$, is defined as

$$\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_R + \mathbf{I}_P \otimes \mathbf{B}, \quad (4.12)$$

where $\mathbf{I}_R$ and $\mathbf{I}_P$ are the identity matrices with leading dimensions P and R , respectively.

4.1.4 Validations

The Laplace EVP in two dimensions

The two-dimensional Laplace eigenvalue problem

$$\nabla_{2D}^2 f + \lambda f = 0, \quad (4.13)$$

where $\nabla_{2D}^2 = \partial_{xx} + \partial_{yy}$, subject to homogeneous Dirichlet boundary conditions, is discretized as

$$\delta_{xx} \oplus \delta_{yy} \equiv \mathbf{D}_{xx} \otimes \mathbf{I}_y + \mathbf{I}_x \otimes \mathbf{D}_{yy} + \lambda \mathbf{I}_x \otimes \mathbf{I}_y = 0. \quad (4.14)$$

Here $\mathbf{D}_{xx}$ and $\mathbf{D}_{yy}$ represent second-order differentiation matrices along the spatial directions x and y , respectively, while $\mathbf{I}_x$ and $\mathbf{I}_y$ are the corresponding identity matrices of appropriate dimensions. Rows are modified in the matrices in order to impose the boundary conditions prior to forming the Kronecker product. An example of the sparsity pattern of the discrete Laplacian resulting from discretization of (4.14) by dense differentiation matrices as well as 8 and 12 points along the x and y spatial directions, respectively, is shown in Figure 4.2.

The analytic eigensystem, found by separation of variables (Morse and Feshbach, 1953), in two spatial dimensions and a domain of extent $a \times b$, subject to homogeneous Dirichlet boundary conditions is

$$\lambda_{ij} = \left(\frac{i\pi}{a}\right)^2 + \left(\frac{j\pi}{b}\right)^2, \quad i, j = 1, 2, \dots \quad (4.15)$$

The Laplace EVP in three dimensions

The corresponding three-dimensional Laplace eigenvalue problem

$$\nabla_{3D}^2 f + \lambda f = 0, \quad (4.16)$$

where $\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}$ is discretized as

$$\mathbf{D}_{xx} \otimes \mathbf{I}_y \otimes \mathbf{I}_z + \mathbf{I}_x \otimes \mathbf{D}_{yy} \otimes \mathbf{I}_z + \mathbf{I}_x \otimes \mathbf{I}_y \otimes \mathbf{D}_{zz} + \lambda \mathbf{I}_x \otimes \mathbf{I}_y \otimes \mathbf{I}_z = 0. \quad (4.17)$$

The Laplace eigenvalue problem (4.16) is solved in a parallelepiped $\Omega = \{x \in [0, a]\} \times \{y \in [0, b]\} \times \{z \in [0, a]\}$ subject to homogeneous Dirichlet boundary conditions.

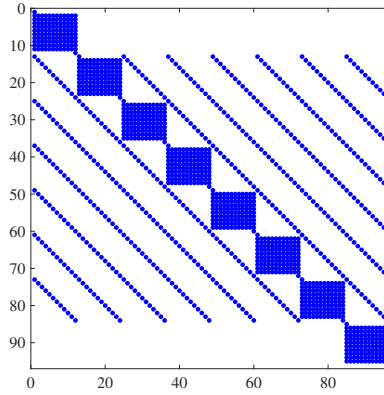


Figure 4.2: Sparsity pattern of the discrete two-dimensional Laplacian operator in (4.14)

The analytical solution can also be found using separation of variables; the eigenvalues of (4.16) are

$$\lambda_{ijk} = \left(\frac{i\pi}{a}\right)^2 + \left(\frac{j\pi}{b}\right)^2 + \left(\frac{k\pi}{c}\right)^2, \quad i, j, k = 1, 2, \dots \quad (4.18)$$

while the corresponding eigenfunctions are

$$\Phi_{ijk}(x, y, z) = \frac{2^{\frac{3}{2}}}{\sqrt{abc}} \sin\left(\frac{i\pi x}{a}\right) \sin\left(\frac{j\pi y}{b}\right) \sin\left(\frac{k\pi z}{c}\right), \quad (4.19)$$

the constant in (4.19) ensuring mutual orthogonality of the eigenfunctions.

4.1.5 Eigenvalue problem results

Parallel matrix formation

The two- and three-dimensional Laplace eigenvalue problems have been discretized by spectral collocation methods in square and cubic domains, discretizing the spatial directions using Chebyshev Gauss Lobatto nodes and the resolutions shown in Table 4.2. The leading eigenvalues, (4.15) and (4.18), of the respective problems were obtained with machine accuracy, while, as expected, the number of converged eigenvalues obtained is a function of the Krylov subspace dimension utilized; two sample eigenfunctions are shown in Figure 4.3.

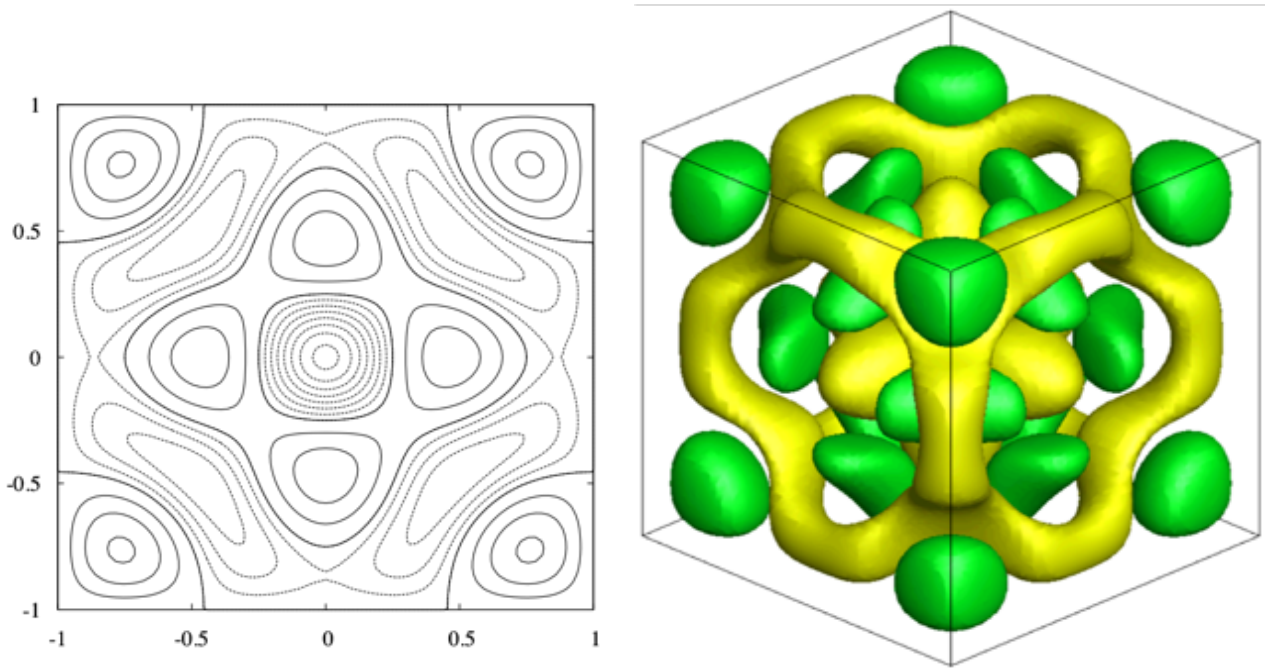


Figure 4.3: Amplitude functions obtained by numerical solution of the Laplace eigenvalue problems (4.14) in two spatial dimensions (*left*) and (4.17) in three spatial dimensions (*right*.)

Computation of the eigensystem of matrices read from disk

While the previous subsection demonstrates the ability of the LiGHT code to consistently construct and solve the matrix eigenvalue problem in parallel, it is also interesting to assess the ability of the code to compute the eigensystem of arbitrary matrices available in discretized form. In this respect, two example cases have been considered, one corresponding to aeroelastic instability (Timme et al., 2012; Timme and Badcock, 2011) and one arising in our own global stability analysis work (Theofilis, 2017).

In the first example, the leading dimension of the matrix is $\dim(A) = 16390$, the number of non-zero elements is $nz = 627979$ and the sparsity pattern is shown in the left part of Figure 4.4, while in the right part a comparison of the results obtained using a modest Krylov subspace dimension $m_{\text{Kryl}} = 50$ are superposed upon those of Timme et al. (2012), delivered by sparse matrix eigenvalue computations. Interestingly, the eigenspectra of the two approaches in the vicinity of the neutral line, here denoted as $\lambda_i = 0$, are identical both in terms of their amplification, λ_i and their frequency, λ_r . Additional converged eigenvalues can be obtained by increasing the subspace dimension in which LiGHT solves the eigenvalue problem, but this was not deemed necessary for the purposes of the present validation.

Finally, the matrix discretizing the TriGlobal eigenvalue problem governing instability in a cubic lid-driven cavity, recently discussed in terms of accurate wall boundary conditions based on the linearized pressure Poisson equation, is formed and stored on disk. Its well-known eigenvalues, obtained through the sparse-matrix techniques and compared with benchmark results in the literature, as discussed by Theofilis (2017), serve to demonstrate the ability of LiGHT to solve the TriGlobal linear eigenvalue problem in this context. Figure 4.5 shows results obtained at $Re = 1000$, showing the (three-dimensional) amplitude functions of the perturbation eigenvector $(\hat{u}(x, y, z), \hat{v}(x, y, z), \hat{w}(x, y, z), \hat{p}(x, y, z))^T$, which are indistinguishable from those of the reference works; further details are provided by Quintanilha et al. (2017).

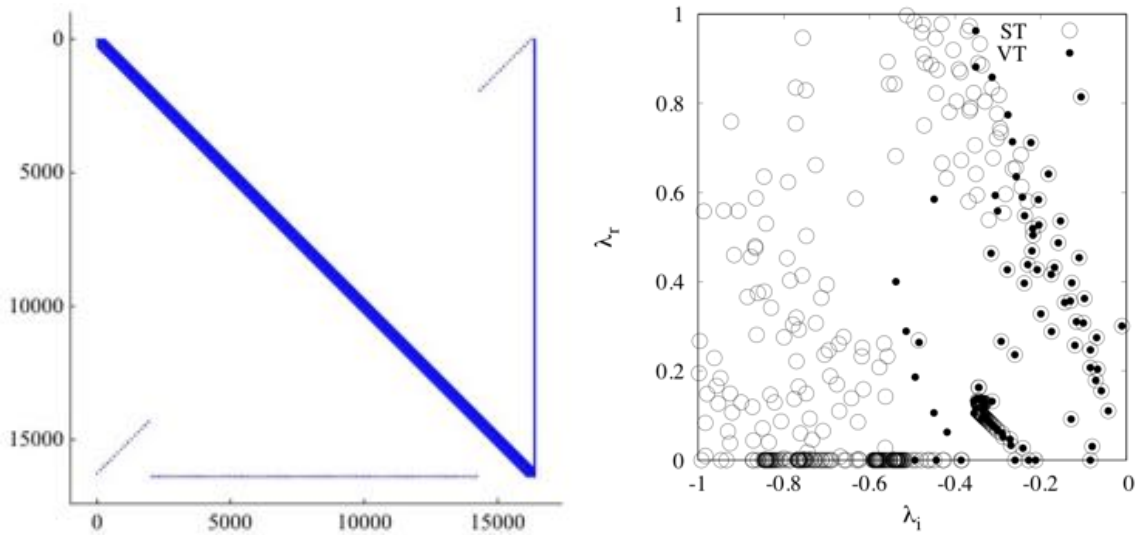


Figure 4.4: *Left:* Sparsity pattern of the aeroelasticity matrix read from disk, the leading eigensystem of which was computed with LiGHT. *Right:* Comparison of LiGHT results, shown as full symbols, with those obtained using sparse matrix methods by Timme et al. (2012), shown as open symbols.

4.1.6 Scalability of the LiGHT code

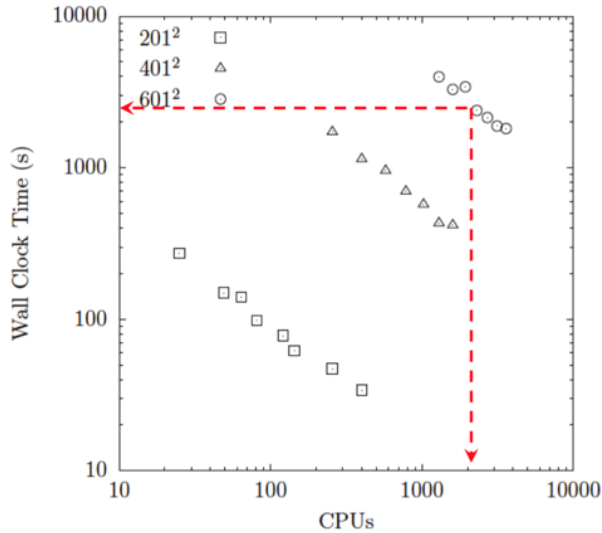


Figure 4.6: Scaling of the LiGHT code

matrix is distributed using distinct numbers of blocks over processor grids of different sizes, as shown in Table 4.2. Once the matrix is LU-decomposed by ScaLAPACK, the distributed Arnoldi algorithm is used to construct a Krylov subspace of dimension $m_K = 200$ and compute the leading Ritz values.

A number of facts can be read in the results of Figure 4.6, the most significant of which is that, within

Having demonstrated accuracy, the question naturally arising is whether the constructed algorithm for the solution of large eigenvalue and singular value problems is competitive from a point of view of efficiency. Scalability tests were performed on *Copper*, the DoD Open Research computer, on which both strong and weak scaling were examined. Figure 4.6 shows the Wall Clock Time, in seconds taken for the computation of the eigensystem of a number of matrices of different sizes, as a function of the number of processors utilized. The corresponding raw data are shown in Table 4.2, where the time to set up the matrix by distributing its elements, LU decompose it in parallel and perform a fixed number of Arnoldi iterations, $m_K = 200$, are quoted as function of three resolutions and solutions on different number of processors. Shown also are the leading dimension, $\dim(A)$, of each of the matrices corresponding to these resolutions, as well as the matrix size resulting from double precision real arithmetic. Analogous scaling results have not yet been obtained for complex arithmetic. The

	Processor Grid	# CPUs	Element Distribution (s)	LU Decomposition (s)	Arnoldi Iteration (s)
201 ²	8 × 8	64	182	140	80
	12 × 12	144	178	62	57
	16 × 16	256	172	47	50
	20 × 20	400	171	34	47
401 ²	16 × 16	256	2800	1722	331
	20 × 20	400	2761	1142	256
	24 × 24	576	2735	956	229
	28 × 28	784	2710	698	206
	32 × 32	1024	2703	572	195
	36 × 36	1296	2690	431	195
	40 × 40	1600	2681	418	192
601 ²	36 × 36	1296	13878	3985	514
	44 × 44	1936	13802	3421	505
	48 × 48	2304	13767	2389	435
	52 × 52	2704	13740	2144	433
	56 × 56	3136	13721	1885	416
	60 × 60	3600	13703	1814	409

Table 4.2: Scaling results for the solution of the two-dimensional Laplace EVP, discretized by 201², 401² and 601² spectral collocation nodes and 128² blocks for the block cyclic distribution of matrix $\mathbf{A}$. *Resolution* 201²: $\dim(\mathbf{A})=40401$, $\text{size}(\mathbf{A})=26.12$ GB, $\min(\#\text{CPUs})=14$. *Resolution* 401²: $\dim(\mathbf{A})=160801$, $\text{size}(\mathbf{A})=413.71$ GB, $\min(\#\text{CPUs})=220$. *Resolution* 601²: $\dim(\mathbf{A})=361201$, $\text{size}(\mathbf{A})=2.087$ TB, $\min(\#\text{CPUs})=1115$.

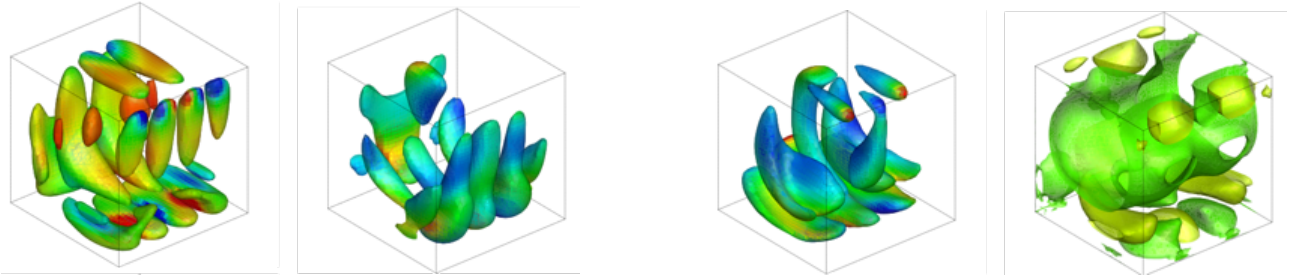


Figure 4.5: Eigenvector pertaining to the leading eigenmode in the solution of the TriGlobal eigenvalue problem governing instability in the cubic lid-driven cavity at $Re = 1000$, as obtained by parallel eigensystem computation of the related stability matrix read from disk. From left to right, $(\hat{u}(x, y, z), \hat{v}(x, y, z), \hat{w}(x, y, z), \hat{p}(x, y, z))^T$.

the ranges examined, the LiGHT code exhibits almost perfect linear scaling, independently of the size of the LU-decomposed matrix or the number of processors used. This result is attributed to the maturity of the ScaLAPACK library (Dongarra and Blackford, 1996; Blackford et al., 1997), as well as to its optimal tuning on the Cray XE6 architecture of Copper. An equally important from a practical point of view result is indicated by the colored arrows: the time required to solve a two-dimensional (BiGlobal) eigenvalue problem discretized by 601^2 Chebyshev Gauss-Lobatto spectral collocation nodes on $O(10^3)$ processors is $O(10^3)$ seconds. In the concrete example indicated on the figure, the global eigenvalue problem is solved on 2048 processors in little over half hour wall clock time.

This result can be better appreciated if the leading dimension, 601^2 , of the matrix devoted to the discretization of the (single) two-dimensional Laplace eigenvalue problem were to be distributed to discretize the (five) compressible linearized Navier-Stokes equations. The number of nodes that would be available for the discretization of each spatial direction in the latter problem, N_2 , can easily be computed from the relation

$$601^2 = 5 \times N_2^2, \quad (4.20)$$

yielding $N_2 \approx 270$ spectral collocation nodes per spatial direction, a resolution that is adequate for most BiGlobal eigenvalue problems encountered in the literature (see, e.g. Paredes et al., 2016). Indeed, the same number of nodes distributed to solve an incompressible TriGlobal eigenvalue problem would make $N_3 \approx 45$ nodes available to discretize each spatial direction (estimated from the relation $601^2 = 4 \times N_3^3$) which, again, suffices to analyze stability of flows with $Re = O(10^2)$.

Table 4.2 offers detailed insight in the performance of the solver and documents the time required by each of the steps of the solution process. The most interesting result shown in the disproportionately large time spent in the creation of the matrix, which is, as expected, a function of the size of the distributed matrix but, unexpectedly, appears to be independent of the number of processors utilized. As can be seen, at low resolutions the generation of the elements and their distribution over different processors is of the same order of magnitude as the time taken by the ScaLAPACK library to LU-decompose the matrix. However, while the latter time decreases linearly with increasing number of processors, as already seen in Figure 4.6, the former time remains practically constant such that, at the higher number of processors used, the time taken to generate and distribute elements becomes an order of magnitude larger than that required for the LU decomposition¹. On the other hand, unlike the experience of De Vicente et al. (2011) with the SuperLU package, the wall time needed for each parallel Arnoldi iteration (essentially a back-substitution and solution of a linear system) is substantially smaller than that of matrix decomposition, justifying the choice of ScaLAPACK for the solution of the problem at hand. In summary, the efficiency of the LiGHT code for the parallel construction and solution of the eigenvalue problem has been demonstrated and reasonable computing times have been obtained, despite

¹Dr. Matthew Tufts (AFRL) has resolved the issue of excessive time taken for the generation and distribution of matrix elements by implementing a fully parallel approach which has been adopted in the main trunk of the LiGHT code

treating the discretized matrix as dense. Precisely this attribute offers the necessary flexibility to solve any matrix eigenvalue problem, without reference to a specific structure of the matrix. Consequently, eigenvalue problems from the incompressible to the hypersonic regime may be solved by the same code, after appropriately modifying the (base flow) coefficients. In addition, spatial discretized algorithms may be interchanged and a wide spectrum of boundary conditions implemented, without performance penalty. All these properties of the LiGHT code will be exploited in the instability analyses planned in the framework of the present effort.

4.2 Numerical integrity of the instability analysis results in the compression ramp

4.2.1 Effect of grid alignment

In Section 2.7.4 a discussion was started regarding the effect of grid alignment on the base flow computation. Here this point is elaborated upon by showing details of this effect on both the base flow and eigenmode results. Fig. 4.7 presents direct numerical simulation results obtained on a typical member of the class of grids of S. 2.7.3; in all of these grids the gridline direction on the ramp forms a finite angle with the recompression shock. Superposed upon the grid in this figure is the streamwise derivative of the streamwise base flow velocity component, computed in the DNS and interpolated on this grid from the original DNS results delivered by OpenFOAM.

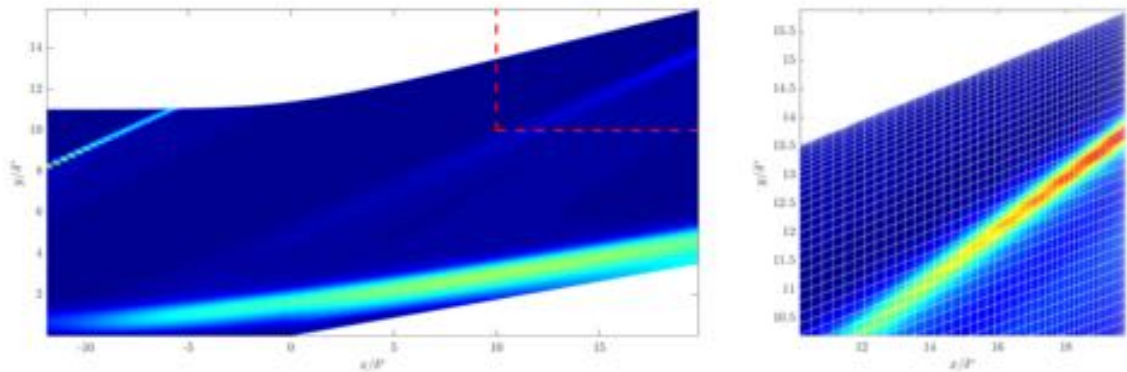


Figure 4.7: Effect of misalignment of gridlines and recompression shock on the base flow velocity gradient in the compression ramp

Although the qualitative features of the overall structure of the base flow can be seen to be correctly captured by this type of grids, a "staircase" effect can be seen on the leading edge shock, a piece of which is seen in the NW corner of the figure. More importantly, the same staircase effect affects the entire (interpolated) recompression shock. In the global instability analyses that follow, the leading edge shock has been found to not play a role in the amplitude functions and has, on occasion, been eliminated altogether from the instability calculations.

However, the misalignment of the grid and the recompression shock does generate a source of inaccuracies in the global eigenspectrum results and has been the main reason for the development and utilization of the grid presented in § 2.7.4.

An example of the latter effect of grid misalignment on the structure of the amplitude functions is shown in Fig. 4.8. In the left two columns of this figure the magnitude of the amplitude functions of the leading global mode at $Ma = 3$, $Re_{\delta^*} = 636$, $\beta = 0.57$ is shown, as obtained on a relatively high resolution of 141^2 spectral nodes mapped according to § 2.7.3, while in the right two columns the same results obtained using the grid of 2.7.4 are shown. In each set of results both the overall view of the amplitude function, as well as a magnified view of the region in the vicinity of the recompression shock at the NE corner is shown, superposed upon the respective underlying grid. Few quantitative differences can be perceived in the overall view of the results (first and third column); however, the close-up views (second column) clearly reveal the inaccuracies caused by the grid misalignment with the recompression shock (second column) as well as the disappearance of this numerical artifact when the gridlines on the ramp are aligned with the shock (fourth column). Henceforth, all global instability results to be shown in § 5 were obtained on appropriately designed grids following the theory of § 2.7.4.

4.2.2 Effect of boundary conditions

A fair amount of numerical experimentation has been devoted to understanding the effect of the boundary conditions to close the eigenvalue problem § 2.3 and the initial value problem 2.4. For all results shown in what follows homogeneous Dirichlet boundary conditions have been imposed on all eigenvector components at the inflow boundary, while linear extrapolation from the interior of the domain has been used at the outflow boundary. At the wall homogeneous Dirichlet conditions are used on all but the pressure perturbation. Different permutations of boundary conditions have been used at the wall (South) for the pressure perturbation and at the far field (North) boundaries for all components of the eigenvector; our findings are summarised in Fig. 4.9. In the left subfigure a macroscopic view of the global eigenspectrum obtained at $Ma = 3$, $Re_{\delta^*} = 636$, $\beta = 0.57$ is shown, a superficial interpretation of which could be that different modes are clearly clustered in their respective region and the results are independent of the boundary conditions utilised to close the EVP. However, closer inspection in the vicinity of each eigenmode reveals a very different picture. The right subplot in Fig. 4.9 presents a magnified view of the same results around the red box drawn in the left subplot, i.e. in the vicinity of the leading global mode. In this zoomed view one notices that what appeared to be a single eigenvalue actually corresponds to three clearly defined groups of eigenvalues, denoted as 1 (first), 2, 3, 4 (second) and 5, 6, 7 (third group). The eigenvalue denoted by 1 is obtained by imposing homogeneous Dirichlet boundary conditions at the wall and a Neumann condition at the far-field boundary. The group of eigenvalues 2, 3, 4 corresponds to results obtained by imposing Neumann or linear extrapolation boundary conditions at the far-field, while group 5, 6, 7 is obtained by imposing homogeneous Dirichlet conditions at this boundary.

Inspection of the amplitude functions, shown in Fig. 4.10 reveals the following. Boundary condition 1 results in an unphysical peak at the NE corner of the domain; this result is not reproduced by any other combination of boundary conditions and can easily be discarded. Conditions 2 – 4 deliver converged eigenvectors, the amplitude functions of which show propagation of waves across the NE corner of the far-field domain boundary. Finally, conditions 5 – 7 result in eigenvectors with well-resolved peaks within the calculation boundary. Given that from a physical point of view, these boundary conditions describe self-excited structures within the compression ramp, and at the same time preclude perturbations from the outside to penetrate into the calculation domain, all results shown in Section § 5 have been obtained with these boundary conditions.

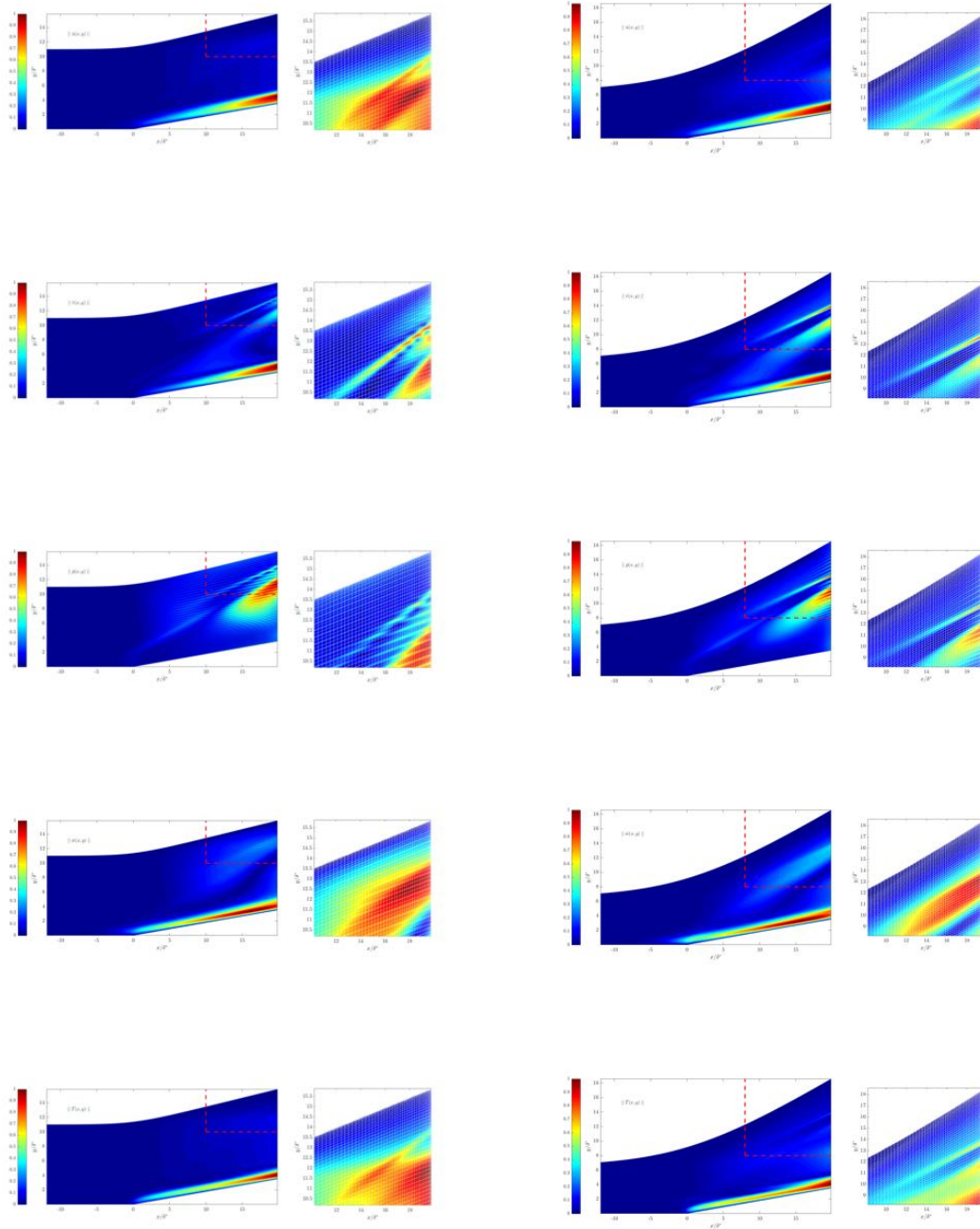


Figure 4.8: *Left column:* Global mode eigenvector component $\hat{v}(x, y)$ obtained on the Pagella grid shown in § 2.7.3. *Right column:* same on the grid of § 2.7.4

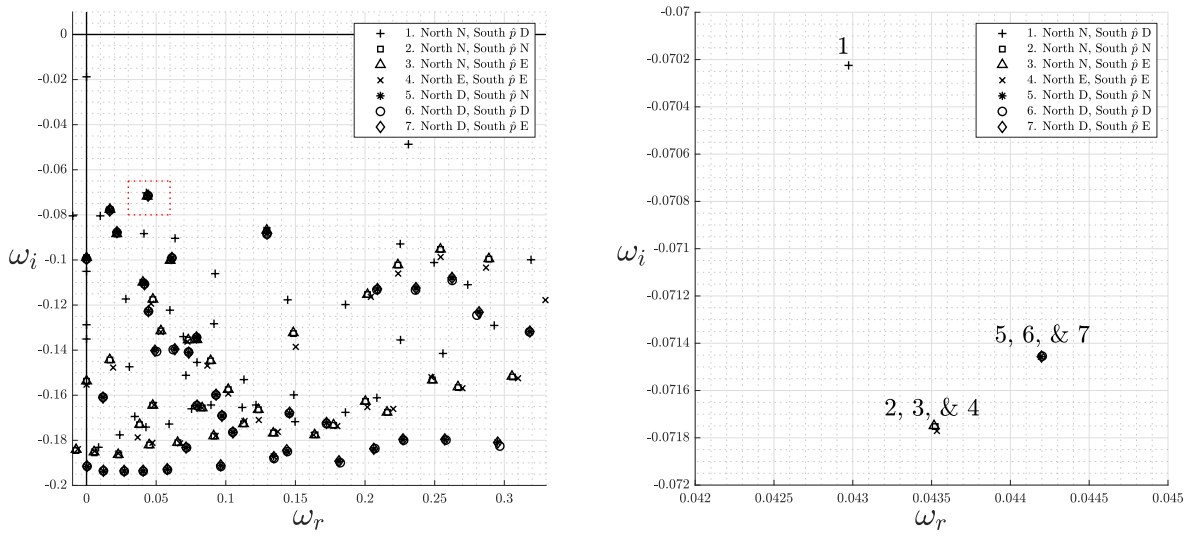


Figure 4.9: Eigenspectra obtained using different boundary conditions to close the eigenvalue problem at $Ma = 3$, $Re_{\delta^*} = 636$, $\beta = 0.57$ using the same resolution, 141^2 , on the grid of § 2.7.4. *Left subfigure:* global view; *Right subfigure:* magnified view of the red rectangle indicated in the left subfigure.

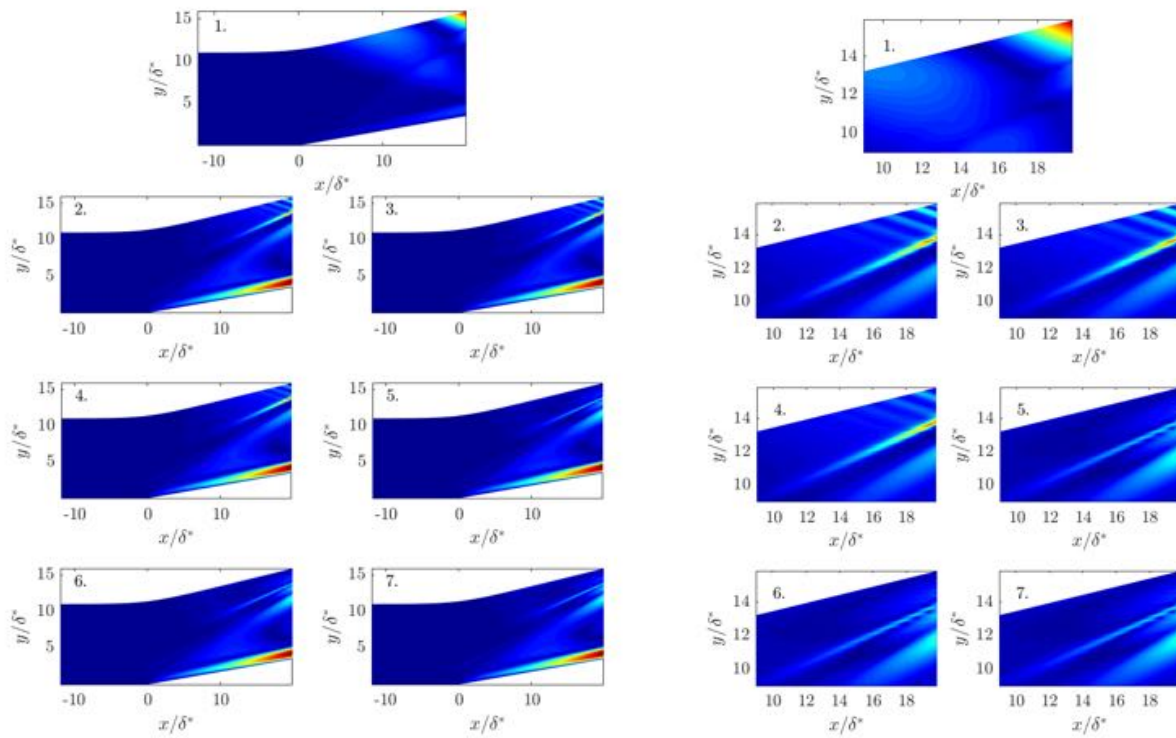


Figure 4.10: Amplitude functions of the $\hat{v}(x, y)$ disturbance velocity component (1–7 in the left column) and detail of the same (1–7 in the right column) for all boundary conditions permutations defined in Fig. 4.9

Chapter 5

Global instability analyses

5.1 Modal analysis - eigenvalue problem results

Despite the obvious questions posed by the temporal content still present in the two-dimensional base flows computed, it is interesting and instructive to inquire the shape of the global mode that is being damped before a steady state is reached. Motivation for such an analysis is twofold: on the one hand, it is expected to shed light on the origin of the oscillations observed in the steady states computed; on the other hand, it may also contribute to the discussion on the role played by steady laminar separation bubbles as an oscillators or amplifiers (Trilling, 1957; Plotkin, 1975; Theofilis et al., 2000; Theofilis, 2011; Poggie et al., 2015).

5.1.1 Residuals algorithm for two-dimensional global mode computation

The least-damped two-dimensional global mode of the compression ramp can be computed using the residuals algorithm (Theofilis, 2000). Close to convergence, any flow field quantity in the time-accurate OpenFOAM field, $\mathbf{q}(x_i, t)$, can be written in terms of the steady two-dimensional base flow, $\bar{\mathbf{q}}(x_i)$, toward which the simulation marches and small-amplitude residuals, $\tilde{\mathbf{q}}(x_i, t)$, *i.e.*,

$$\mathbf{q}(x_i, t) = \bar{\mathbf{q}}(x_i) + \epsilon \tilde{\mathbf{q}}(x_i, t) \quad (5.1)$$

where $\mathbf{q} = (\rho, u_i, p, T)^T$ with $\epsilon \ll 1$. In the case of the two-dimensional simulations performed in Section 3.2 the residuals $\tilde{\mathbf{q}}$ can be expanded as:

$$\tilde{\mathbf{q}}(x_i, t) = \hat{\mathbf{q}}(x_i) e^{\sigma t} \quad (5.2)$$

where $\hat{\mathbf{q}}(x_i)$ is the amplitude function and σ is the damping rate.

In order for the theory to be applicable, the DNS computation needs to have entered a regime of linear decay of residuals; this is clearly the case in the results of Fig. 3.12, in which it can be seen that, after the initial transients have subsided, a unique value of the logarithmic derivative of the signal may be extracted from any of the DNS results at any location in the two-dimensional domain. The Ansatz (5.1) is then written at two randomly chosen times, t_1 and t_2 , after the simulation has entered the linear regime, e.g. at $t > 4ms$ in the case of the simulation described by Fig. 3.12. The damping rate σ is estimated from the logarithmic derivative of the signal, such that the factor $e^{\sigma t}$ is known at the two times. With $\mathbf{q}(x_i, t_1)$ and $\mathbf{q}(x_i, t_2)$ known, the 2×2 system is then solved for the prediction of the unknown steady laminar base state, $\bar{\mathbf{q}}(x_i)$, and the corresponding amplitude functions of the least damped two-dimensional global mode, $\hat{\mathbf{q}}(x_i)$.

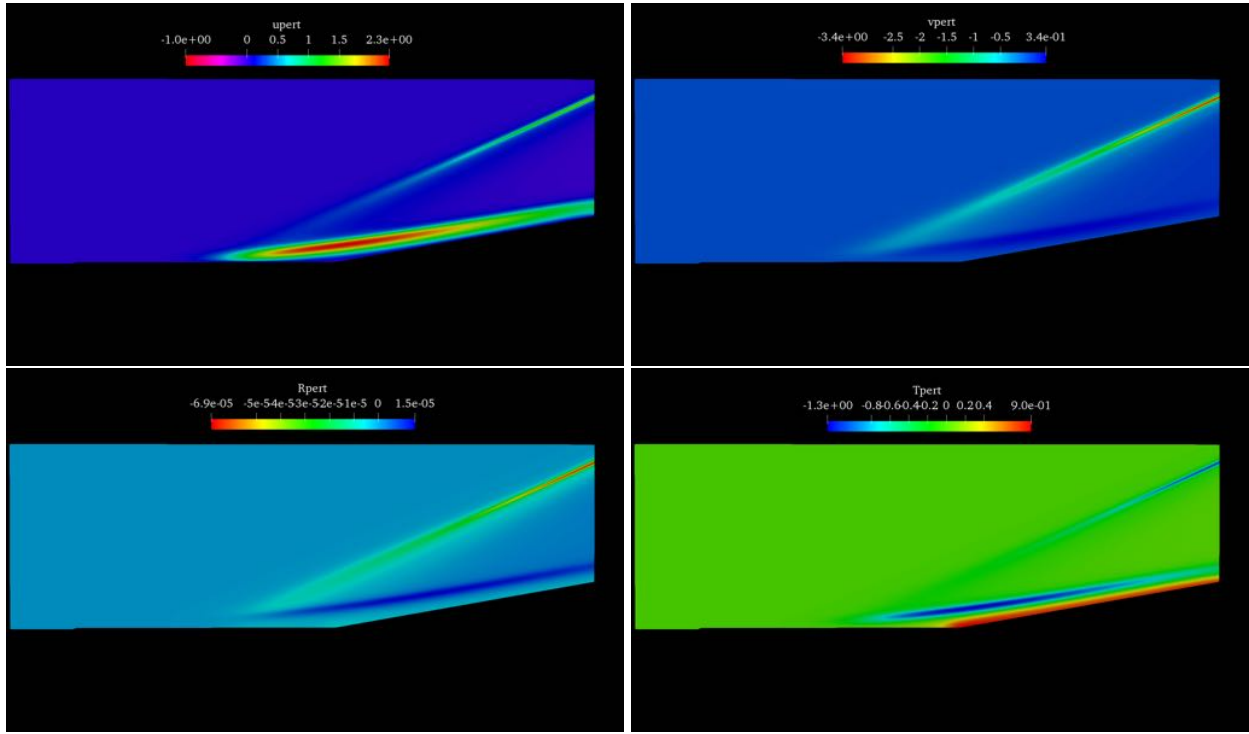


Figure 5.1: Amplitude functions of the least-damped two-dimensional global mode at the conditions examined by Carter (1972). *Upper left*: streamwise velocity component, $\hat{u}(x, y)$; *Upper right*: transverse velocity component, $\hat{v}(x, y)$; *Lower left*: perturbation density $\hat{\rho}(x, y)$; *Lower right*: perturbation temperature $\hat{T}(x, y)$;

5.1.2 Two-dimensional global mode amplitude functions

Figure 5.1 presents the amplitude functions $\hat{\mathbf{q}}(x_i)$ of the least-damped global mode in terms of the two velocity components, the density and temperature perturbations. Results are shown in the vicinity of the compression corner since, significantly, it was found that the leading edge region, including the leading edge shock, is steady and does not contribute to the amplitude functions of the global mode. In contrast, what is clearly visible in the results presented, is the intimate connection of the recompression shock and the laminar separation bubble: *the peaks of the amplitude function of the global mode are to be found at these locations*. In other words, instability in the laminar separation will affect shock motion and vice-versa. This result is in line with the findings of the celebrated work of Crouch et al. (2007) on the origins of buffeting in turbulent transonic flow on two-dimensional wings, where shock oscillations has been associated with linear amplification of a global mode having its peak at precisely the shock and (mean) separation location. The same result regarding the inter-dependence of shock motion and global instability in a laminar separation bubble has also been documented in the analysis of $Ma = 16$ hypersonic flow in the double-cone junction by Tumuklu et al. (2018). The present results indicate that instability analysis of supersonic flow in the compression corner need to consider the recompression shock as an integral part of the analysis.

5.1.3 Three-dimensional global modes

Attention is next turned to the case of three-dimensional BiGlobal instability, indicated by $\beta \neq 0$ in (2.10). For reference in what follows, Fig. 5.2 presents the streamwise component of the base flow velocity $\bar{u}(x, y)$ at $Ma = 3$, $Re_L = 28000$, which translates into $Re_{\delta^*} \approx 600$ in terms of the boundary layer displacement thickness at the inflow of the domain in which the instability analysis has been performed and, firstly, at a given spanwise wavenumber parameter value, $\beta = 0.01$. A grid sequencing study has delivered a number of converged stationary (S) and traveling (T) modes, shown in Fig. 5.3. It can be seen that, at these conditions, the stationary mode S1 is the only amplified global instability, followed by the damped traveling mode T1 and the remaining members of the eigenspectrum. In this figure, it can also be seen that resolutions upwards of 60^2 CGL nodes are adequate for the accurate description of the leading eigenspectrum members.

The amplitude functions of the first four modes of Fig. 5.3 are shown in Figs. 5.4-5.7. A number of observations can be made in these results. Firstly, as it has also been seen in the two-dimensional ($\beta = 0$) damped global mode results of Section 5.1.2 the recompression shock is an integral part of the amplitude functions of all leading flow perturbations. Although not all components of a given eigenvector peak in the vicinity of the recompression shock, at least one velocity component and one thermodynamic component of the perturbation eigenvector always peaks at this location. The implication is clear, namely that instability analysis of the compression ramp flow needs to include (and adequately resolve) the neighborhood of the recompression shock, as well as the vicinity of the boundary layer on this non-trivial geometry. In this respect, the advantage offered by the novel transformation introduced in Section 2.7.4 now becomes clear, namely the ability to align the wall-(near-)parallel gridlines on the ramp with the recompression shock direction, the latter indicated by the dashed line in Fig. 5.2. This alignment has been found to be essential in order to be able to properly resolve the eigenvectors and produce converged results such as those of Fig. 5.3.

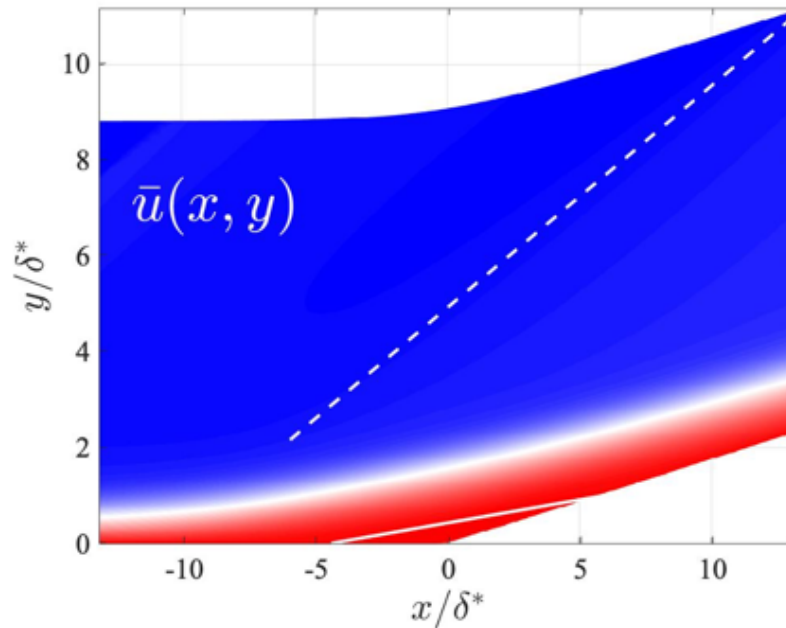


Figure 5.2: Base flow streamwise velocity component at $Ma = 3$, $Re_L = 28000$ ($Re_{\delta^*} = 600$). The solid line indicates the boundary of the laminar separation bubble and the dashed line indicates the location of the recompression shock

An additional observation that is immediately made in the results of Figs.5.4-5.7 regards the peaks of the components of the eigenvectors being confined in the region between the (separated) boundary layer and the recompression shock, the latter acting as a natural boundary along the wall-normal direction beyond which no activity of the perturbation eigenvector is seen at these conditions. Perturbations have invariably been found to peak within the laminar separation bubble at the compression corner, at the recompression shock region and on the downstream ramp wall. Interestingly, in all modes examined, the streamwise perturbation velocity component, $\bar{u}(x, y)$, dominates over its transverse, $\bar{v}(x, y)$, or its spanwise, $\bar{w}(x, y)$ counterparts. Ultimately, the results obtained indicate that additional peaks of the perturbation eigenvector are to be expected on the downstream ramp wall, a conjecture that is currently being examined by longer-domain simulations.

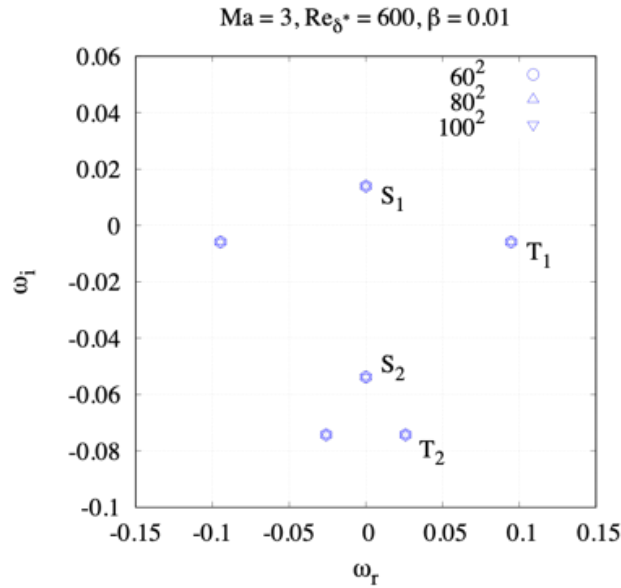


Figure 5.3: Convergence of the BiGlobal eigenspectrum at $Ma = 3$, $Re_{\delta^*} = 600$, $\beta = 0.01$

In order to address the question of the dependence of the above results on the spanwise wavenumber, a large number of analyses has been performed in the range of $0 \leq \beta \leq 1$, the results of which are summarized in Fig. 5.8, which shows the complex ω values obtained for all spanwise wavenumbers in this range. The two leading stationary and traveling modes shown earlier are indicated in this figure. Fig. 5.9 elaborates on the S1 and T1 modes, showing only results pertinent to the range $0 \leq \beta \leq 0.33$. The two conclusions that may be drawn from these results are that, firstly, at these conditions, the dominant flow instability is the two-dimensional stationary mode ($\beta = 0$) already discussed in Section 5.1.2 and, secondly, that the leading three-dimensional instability ($\beta \neq 0$) is a traveling mode which has spanwise periodicity length $L_z = 2\pi/\beta \approx 2\pi/0.12 \approx 52$ (in terms of δ^*); this global mode is expected to become unstable at higher Reynolds numbers. The amplitude functions of the S1 and T1 modes are shown by the respective magnitudes of the eigenvector components in Figs. 5.10 and 5.11, and share the same qualitative characteristics as those discussed earlier in Figs.5.4-5.7.

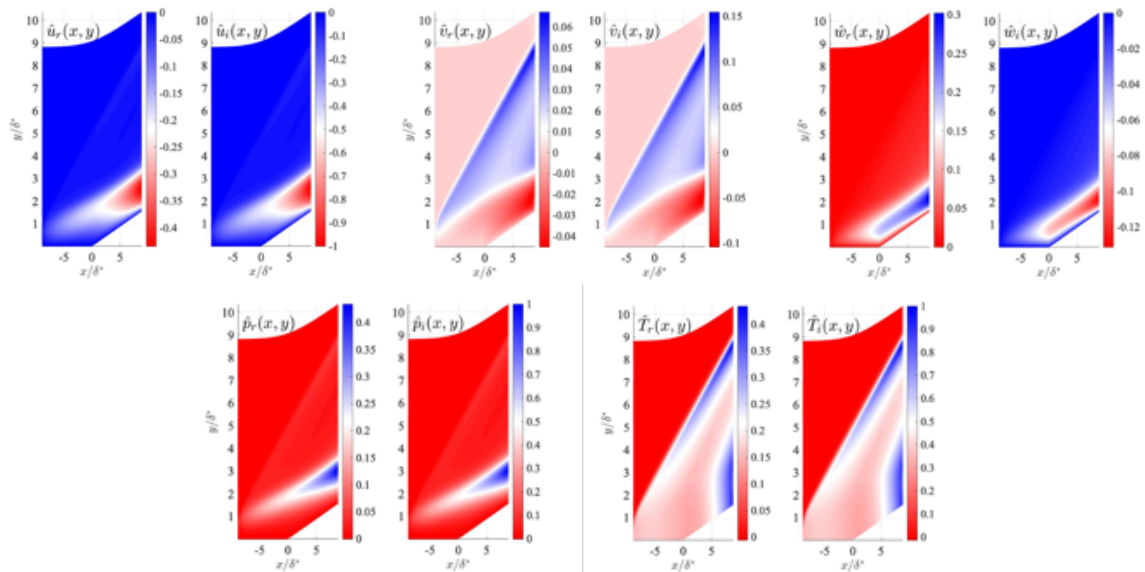


Figure 5.4: Amplitude functions of the leading stationary eigenmode S1 shown in Fig. 5.3

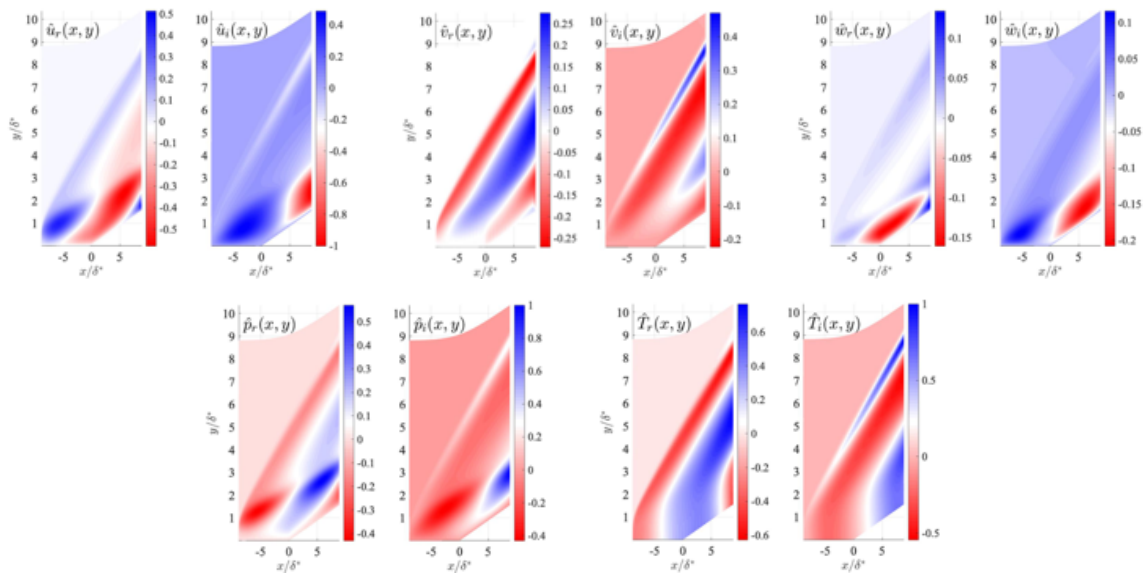


Figure 5.5: Amplitude functions of the leading traveling eigenmode T1 shown in Fig. 5.3

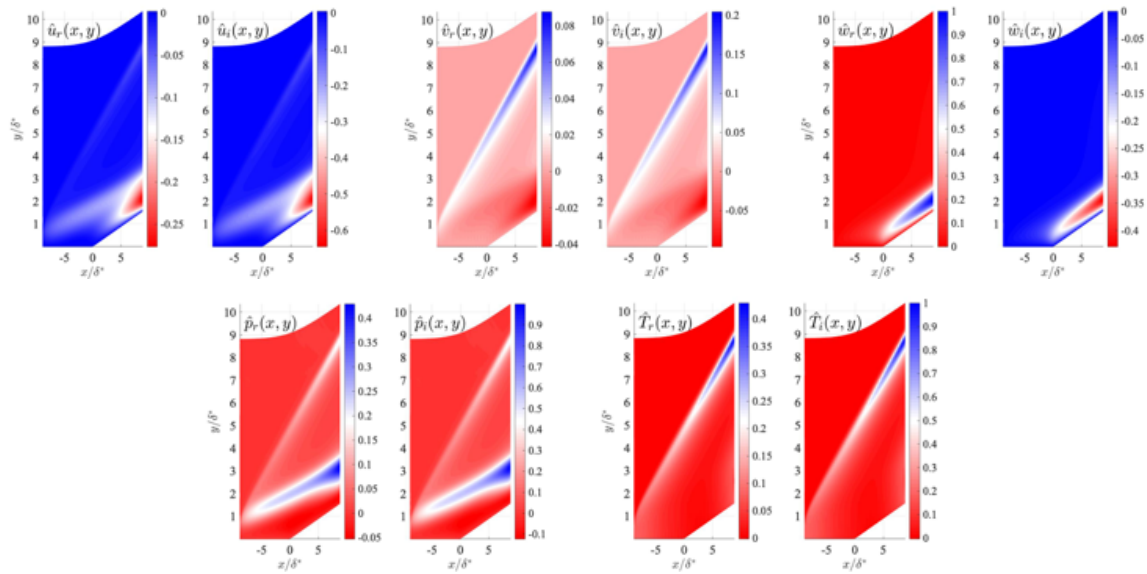


Figure 5.6: Amplitude functions of the second stationary eigenmode S2 shown in Fig. 5.3

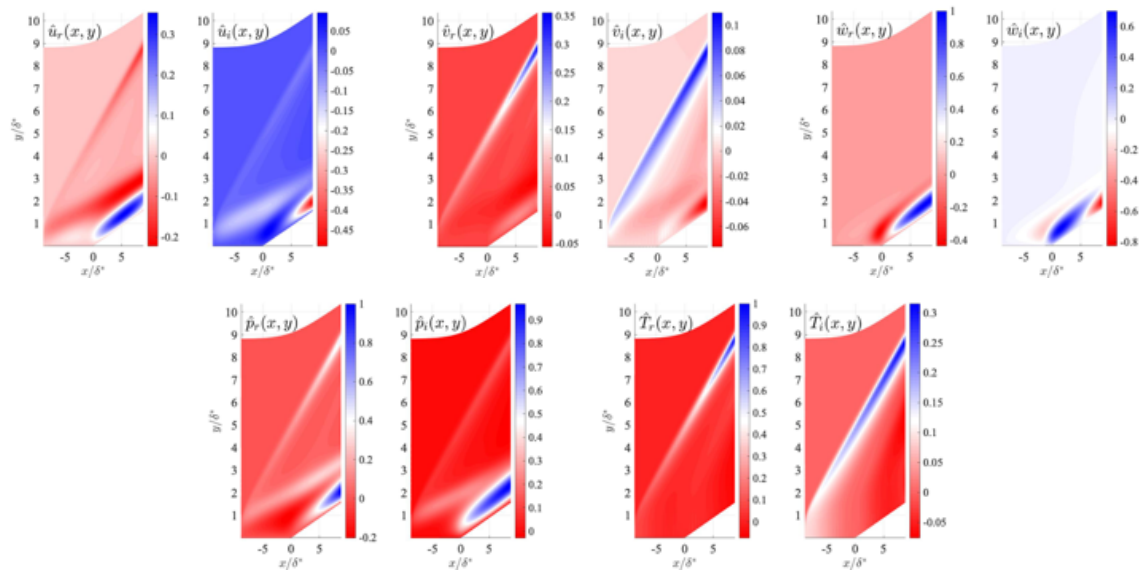


Figure 5.7: Amplitude functions of the second traveling eigenmode T2 shown in Fig. 5.3

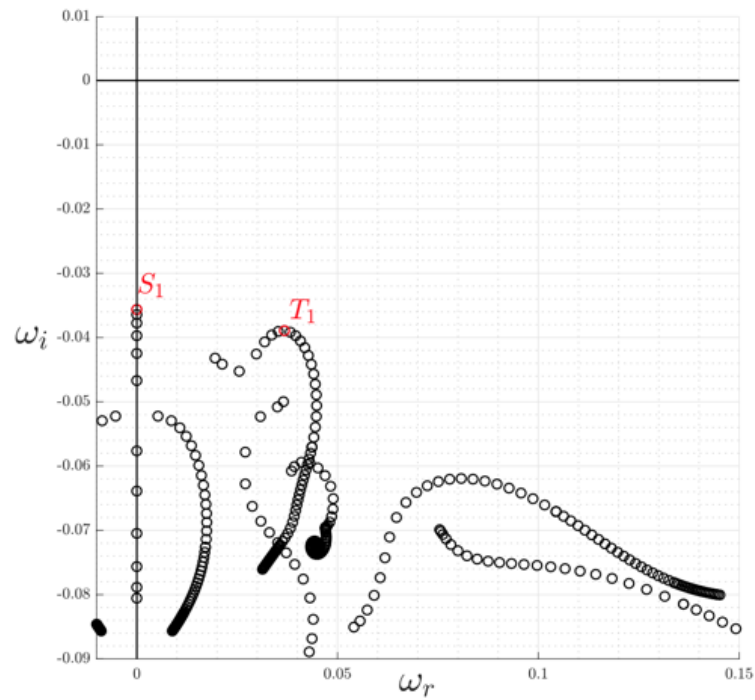


Figure 5.8: Damping rate ω_i dependence on frequency ω_r in the range $0 \leq \beta \leq 1$

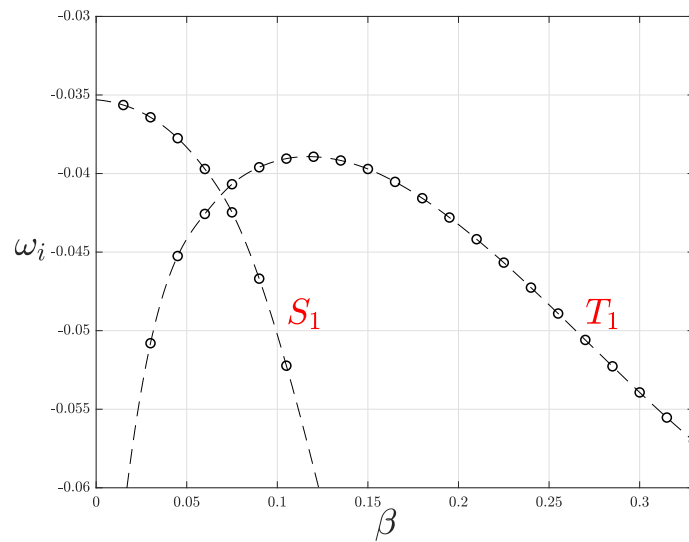


Figure 5.9: Damping rate ω_i dependence on spanwise wavenumber β in the range $0 \leq \beta \leq 0.33$

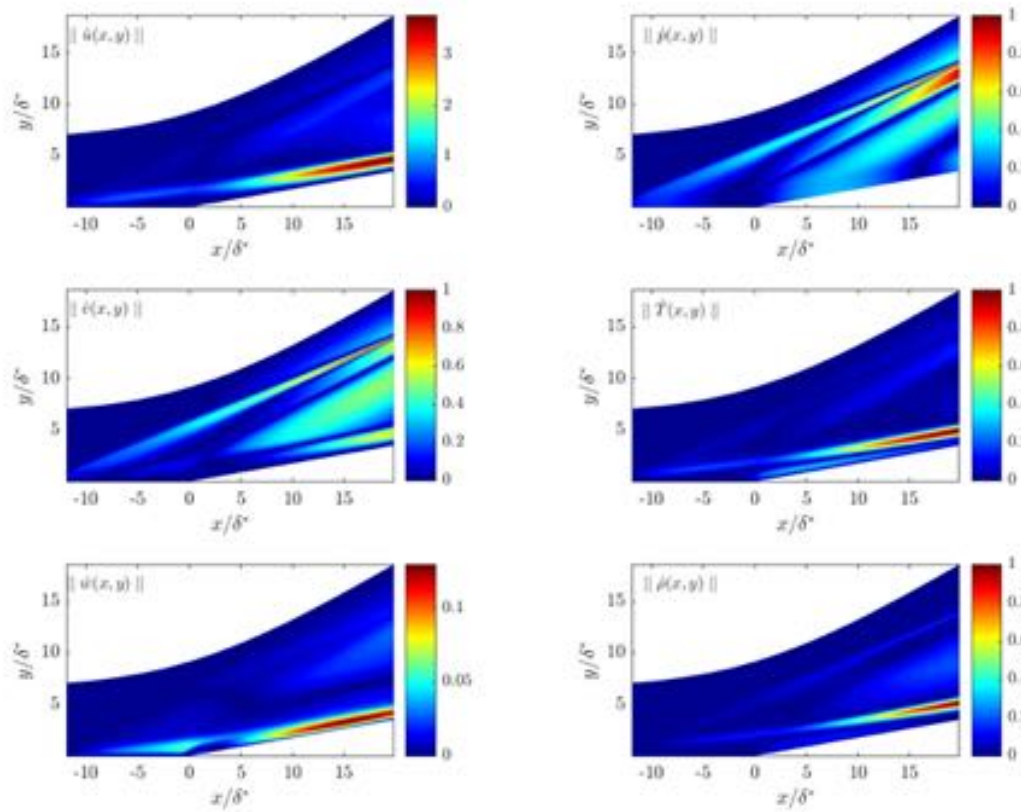


Figure 5.10: Amplitude function of the leading 3D stationary mode; at $\beta = 0.015, \omega_i = -0.035$

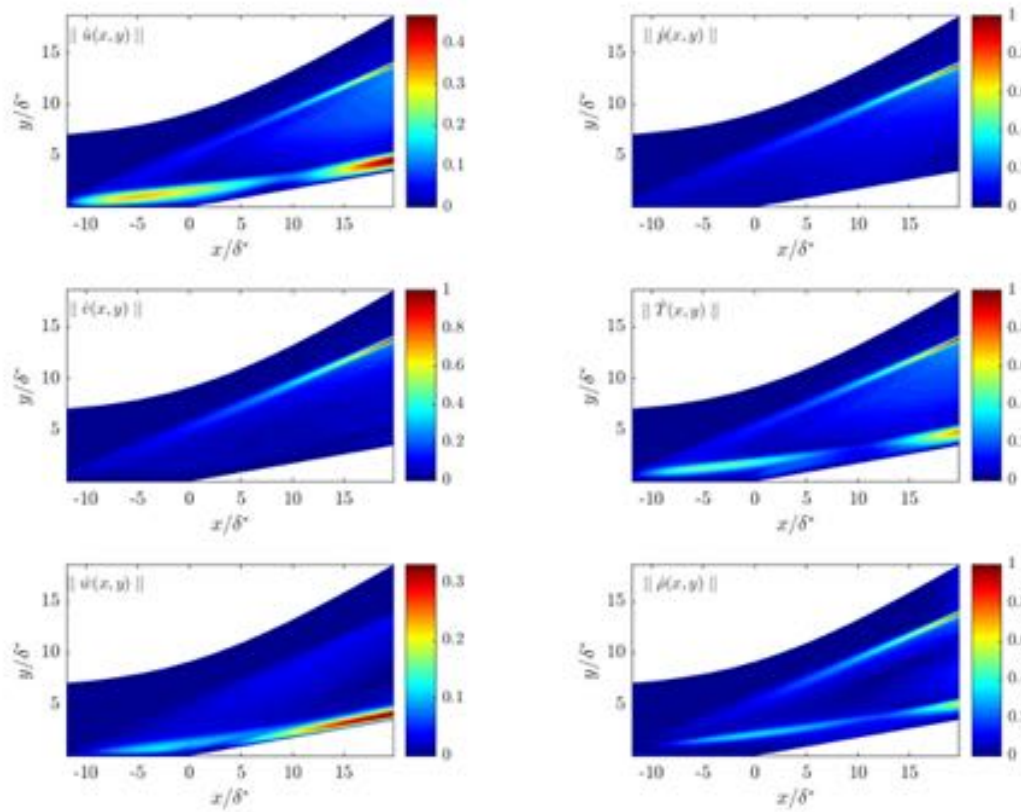


Figure 5.11: Amplitude function of the leading 3D traveling mode; at $\beta = 0.12, \omega_i = -0.039$

5.1.4 Reynolds number effect

Finally, the effect of the Reynolds number on the leading eigenmodes has been addressed. Fig. 5.12 presents the amplitude function of the spanwise perturbation velocity component, $\hat{w}(x, y)$, of the leading stationary eigenmode at three Reynolds numbers, $Re_L = 16, 800, 100,000$ and $250,000$. The main feature that can be appreciated in these results is the increasingly larger number of peaks of ever finer structures that appear in the amplitude function of the global modes as the Reynolds number increases. Notwithstanding differences between global and POD modes (Taira et al., 2017) the sequence shown in Fig. 5.12 is reminiscent of the POD modes in Fig. 5.13 obtained by Mustafa et al. (2019) in their analysis of (turbulent, three-dimensional) experimental data on the compression corner, at practically the same Mach number $Ma = 2.8$ and a Reynolds number $Re_\theta = 1750$, that is orders of magnitude larger than those considered in the present analysis.

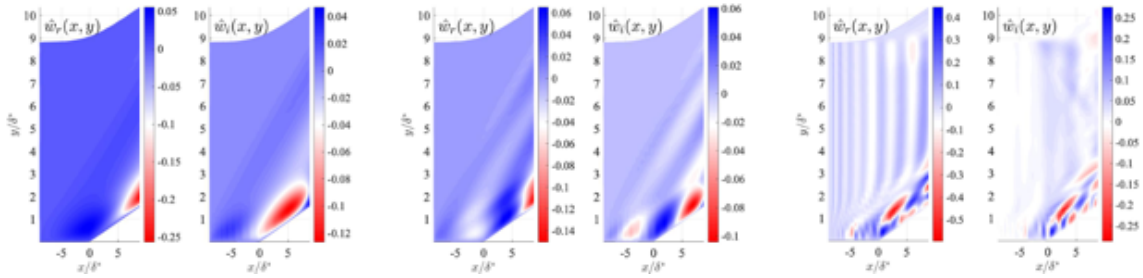


Figure 5.12: Effect of Re on the structure of the amplitude function of the leading stationary eigenmode S1. From left to right: $Re_L = 16, 800, 100,000$ and $250,000$

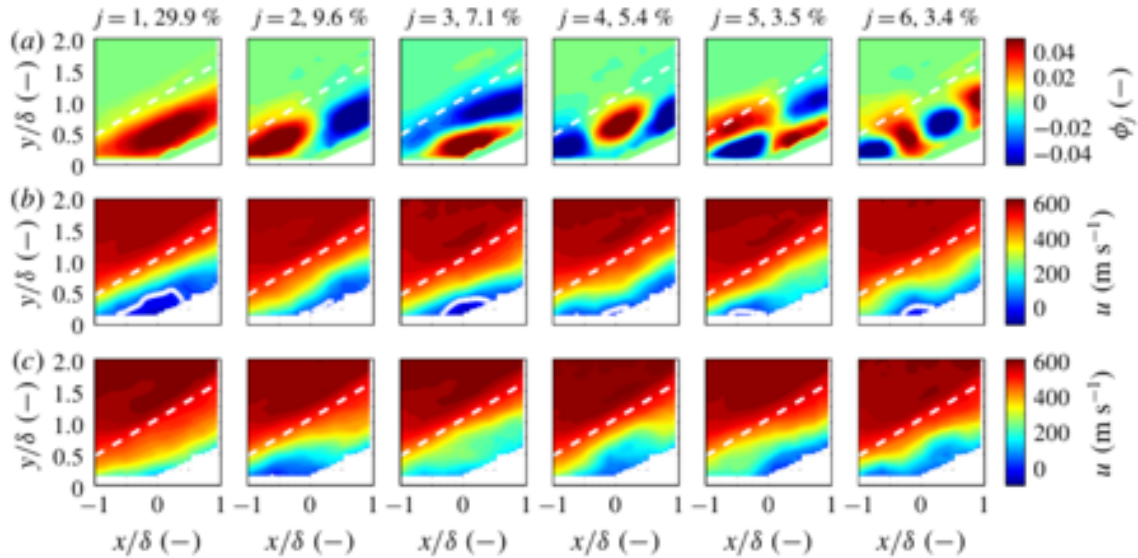


Figure 5.13: Experimental results on POD modes in a compression ramp at $Ma = 2.8, Re_\theta = 1750$ (Mustafa et al., 2019)

5.2 Non-modal analysis - transient growth

Instability in the compression ramp flow can have either modal or non-modal origins. In the previous section § 5.1 it was seen that at the conditions at which most of the present analyses were performed, $Ma = 3, Re_{\delta^*} = 636$, the flow is (globally) stable from a modal analysis perspective. This does not preclude, in principle, the same flow to amplify algebraically linear perturbations according to a non-modal / transient growth scenario; this is the subject of the analyses performed in this section.

5.2.1 Transient energy gain

The energy gain function defined by equation (2.16) and calculated in practice by equations (2.29–2.30) has been computed in the compression ramp at $Ma = 3, Re_{\delta^*} = 636, \beta = 0.118$ on 141^2 spectral nodes mapped according to § 2.7.4. Fig. 5.14 presents results at three Krylov subspace dimensions, $m_{\text{Kryl}} = 2000, 3000$ and 4000 , on the basis of which two major observations are made.

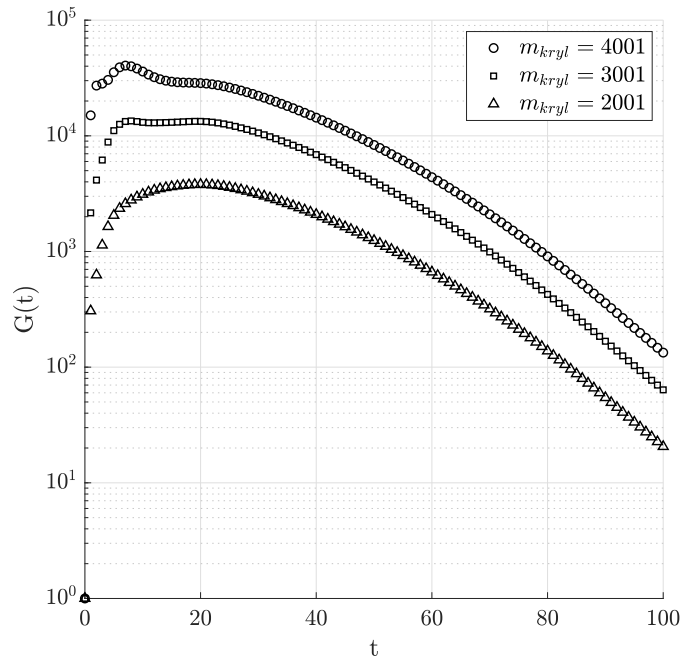


Figure 5.14: Transient energy gain function dependence on time at $Ma = 3, Re_{\delta^*} = 636, \beta = 0.118$ and $m_{\text{Kryl}} = 2000, 3000, 4000$

Firstly, a rather strong energy growth is obtained at these conditions: starting at normalized energy level of unity, $G(t)$ grows algebraically by approximately five orders of magnitude at these conditions, before decaying as predicted by the linear modal theory discussed in the previous section. As a matter of fact, the maximum value of the gain function, G_{max} , increases when the Krylov subspace dimension is enlarged, i.e. more and more stable global eigenmodes are included in the analysis. A systematic increase of m_{Kryl} up to the limit of the feasible resolutions considered in the present study yields a tendency of the results obtained to converge at asymptotically large times, while the peak value of $G(t)$ keeps increasing. On the other hand, the time at which transient energy growth peaks is well defined in all simulations at $t_{\text{opt}} \approx 8$. The levels of G_{max} obtained

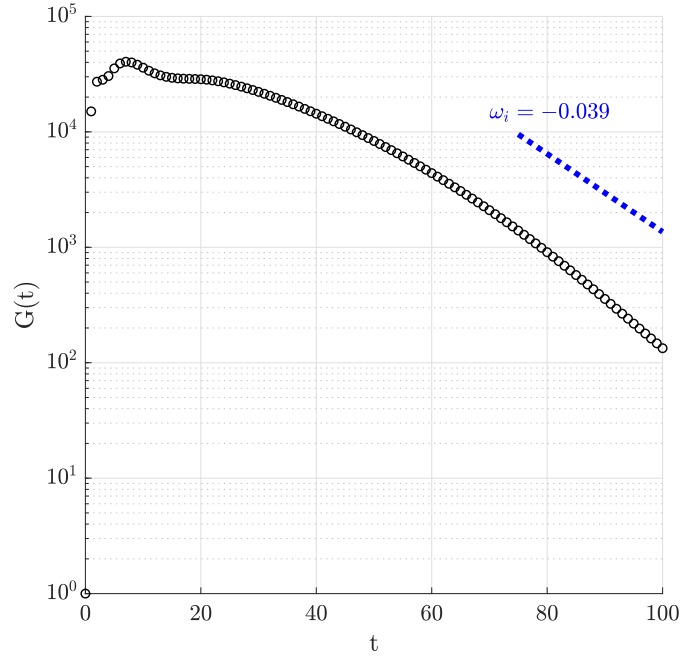


Figure 5.15: Gain function dependence on time at $Ma = 3$, $Re_{\delta^*} = 636$, $\beta = 0.118$ and $m_{Kryl} = 4000$; superposed is the result of numerical solution of the eigenvalue problem

are already remarkable, since the spanwise wavenumber was chosen to correspond to the maximally amplified *three-dimensional* global mode. On theoretical grounds, it is expected that $G_{\max}$ will grow beyond this already high value when $\beta \rightarrow 0$; these calculations are underway and will be reported elsewhere.

Secondly, all three solutions of the IVP at asymptotically large times decay at the same rate, as expected from equation (2.21) and the fact that all perturbations at these conditions are linearly (modally) stable. Fig. 5.15 shows this result by superposing the IVP solution at the highest $m_{Kryl} = 4000$ and the dashed line corresponding to the damping rate of the leading global mode at these conditions, $\omega_i \approx -0.038$. This result implies that, as expected on theoretical grounds from equations (2.19) and (2.21), the (very expensive) solution of the IVP at asymptotically large times yields the exact same prediction for the least damped linear perturbation as that delivered by solution of the EVP (2.11) for the least damped global mode; we will return to this point in Section 5.2.4.

5.2.2 Linear optimal perturbations at $t_{\text{opt}} \approx 8$

The linear optimal perturbations at conditions of maximal transient energy growth, computed by equations (2.31–2.33), are plotted in Fig. 5.16; the corresponding real and imaginary parts are shown in Figs. 5.17 and 5.18. The most striking aspect of these results is that the peaks of the linear optimals are to be found inside the boundary layer on the inclined wall of the compression ramp, with very weak traces of the recompression shock and the separation bubble being visible. Of course this result needs to be contrasted against $G(t)$ results at different β values, which are being generated. Only after the corresponding $G_{\max}(\beta)$ values have been computed will a full picture of transient growth emerge and definitive conclusions will be drawn on the physical mechanisms of linear instability in the compression ramp. However, $G_{\max} = O(10^4 - 10^5)$ values obtained at this β value already point to the potential of linear transient growth to lead flow into transition.

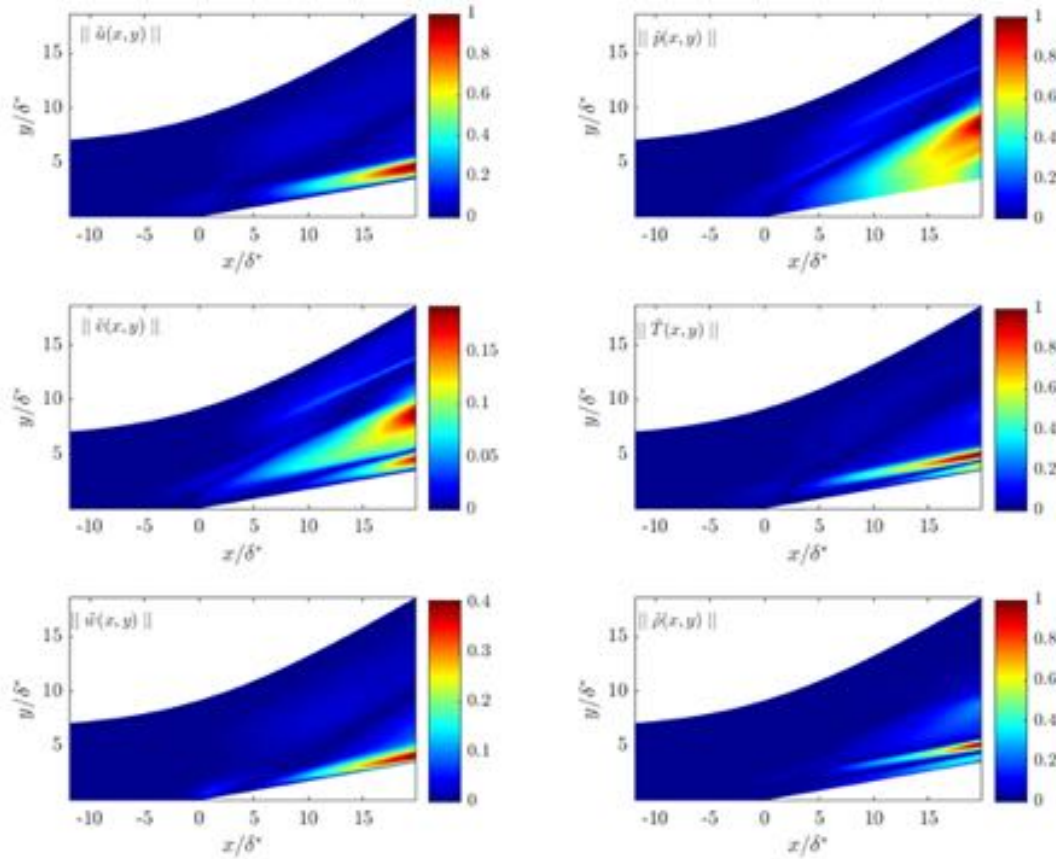


Figure 5.16: Magnitude of the amplitude functions of the linear optimal perturbations at the time $t_{\text{opt}} \approx 8$, when $G(t)$ reaches it maximum value, $G(t) = O(10^5)$

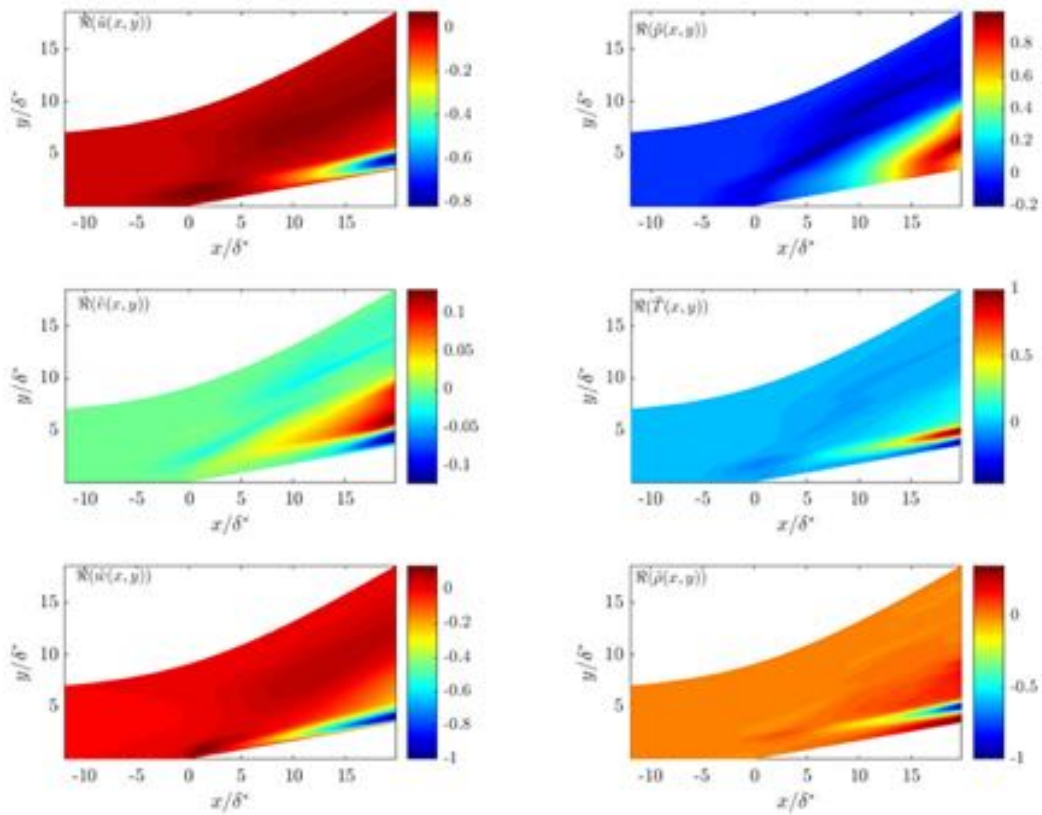


Figure 5.17: Real part of the amplitude functions of the linear optimal perturbations at the time $t_{\text{opt}} \approx 8$, when $G(t)$ reaches it maximum value, $G(t) = O(10^5)$

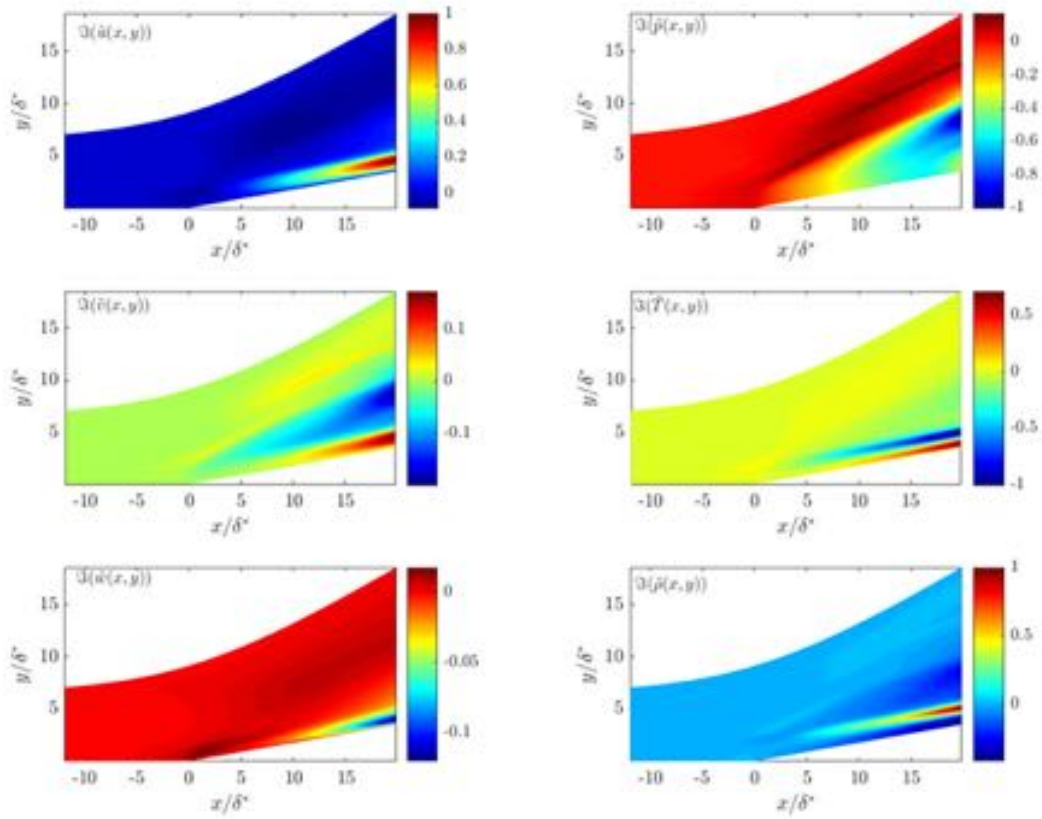


Figure 5.18: Imaginary part of the amplitude functions of the linear optimal perturbations at the time $t_{\text{opt}} \approx 8$, when $G(t)$ reaches it maximum value, $G(t) = O(10^5)$

5.2.3 Optimal initial linear perturbations

The magnitude of the linear optimal initial perturbations at $t = 0$, that lead to maximum transient energy growth, also computed by equations (2.31–2.33), are plotted in Fig. 5.19; the corresponding real and imaginary parts are shown in Figs. 5.20 and 5.21. Consistently with the results shown in Figs. 5.16–5.18, all components of the linear perturbations at $t = 0$ that lead $G(t)$ to reach its maximum value, $G(t) = O(10^5)$, at $t \approx 8$ peak on the downstream ramp wall, with the laminar separation bubble at the junction or the recompression shock being barely visible in these amplitude functions. Again, the relative importance of this result will only become clear when the parametric study in wavenumbers has been completed.

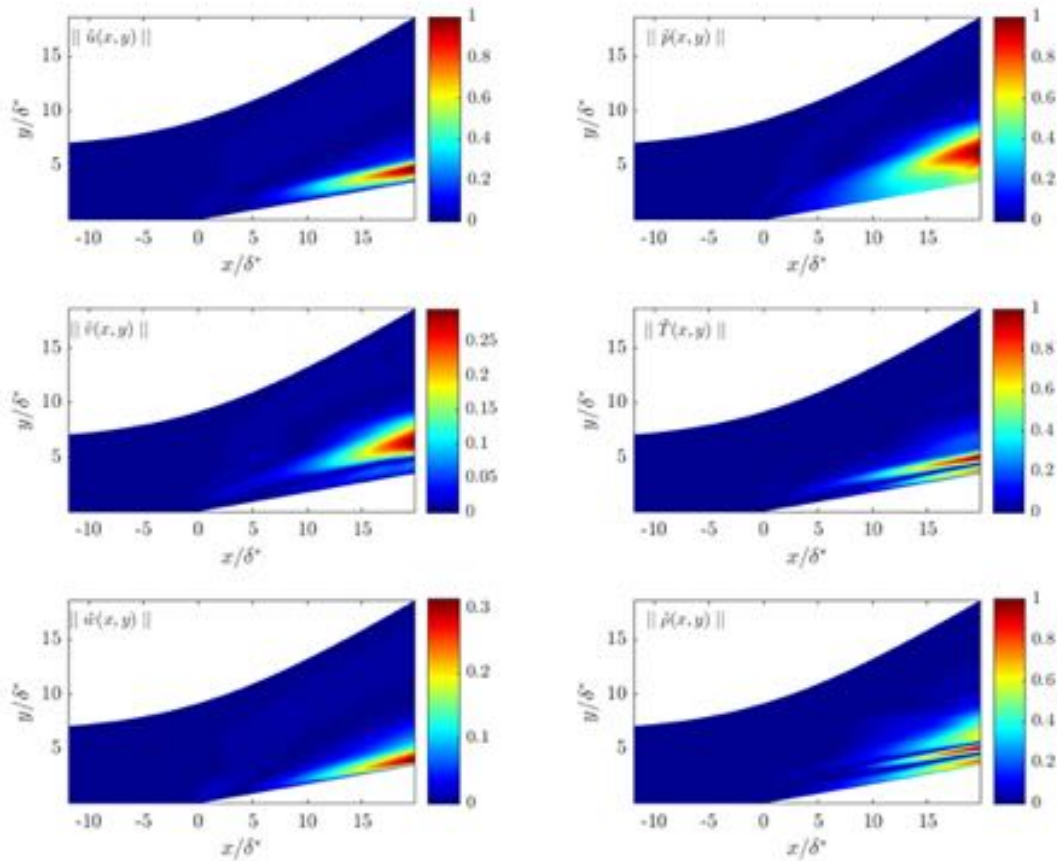


Figure 5.19: Magnitude of the amplitude functions of the linear initial optimal perturbations at $t = 0$

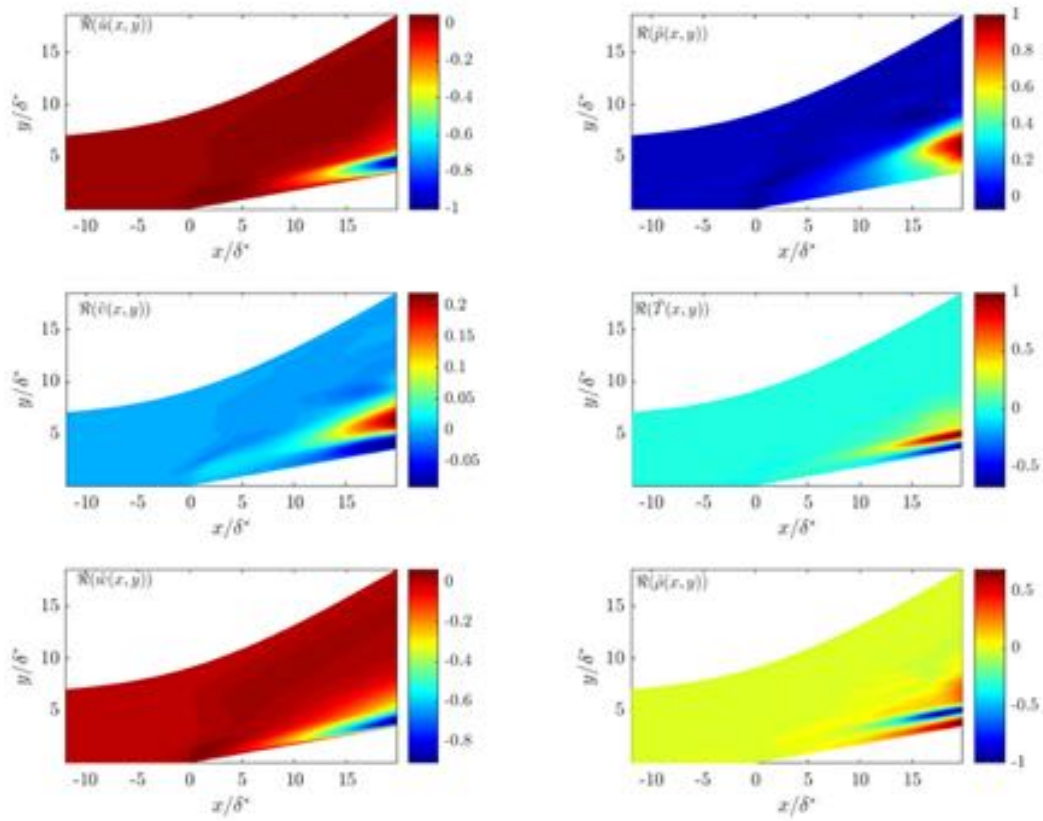


Figure 5.20: Real part of the amplitude functions of the linear initial optimal perturbations at $t = 0$

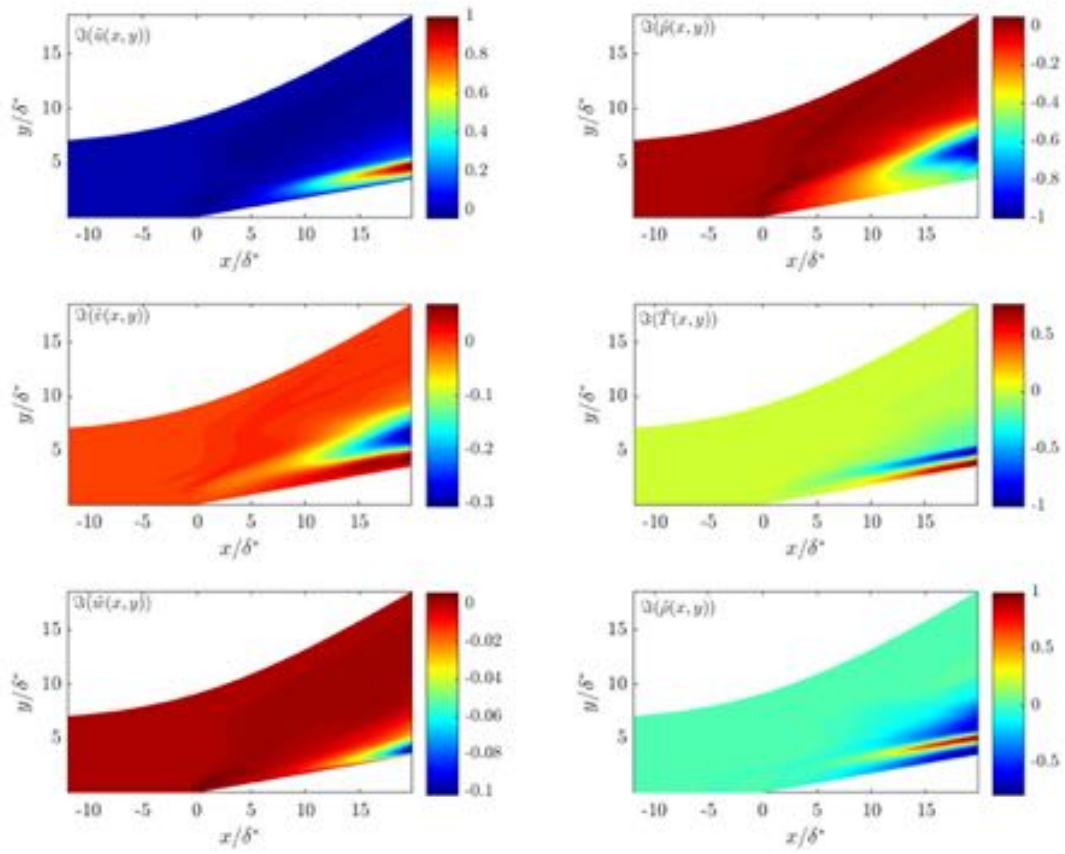


Figure 5.21: Imaginary part of the amplitude functions of the linear initial optimal perturbations at $t = 0$

5.2.4 Linear perturbations at asymptotically large times

Figures 5.22, 5.23 and 5.24 respectively show the magnitude, real and imaginary parts of the components of the linear perturbation vector at the asymptotically large time $t = 100$, at which the initial value problem (2.31-2.33) has been solved. As could be anticipated by the agreement of the slope of the $\lim_{t \rightarrow \infty} G(t)$ curve shown in Fig. 5.15 with the damping rate, ω_i of the least damped global mode delivered by solution of the eigenvalue problem (2.11) shown in §5.2.1, the structures shown in these figures correspond to the *amplitude functions of the least damped three-dimensional global mode* of the flow at these conditions. In these results, qualitative features of the amplitude functions seen in § 5.1, such as the peaks of the amplitude functions in the vicinity of the recompression shock and the laminar separation bubble, are clearly visible.

In other words, the linear initial optimal perturbations of § 5.2.2, which grew algebraically to the linear optimals shown in § 5.2.3, have evolved into the least-damped global mode predicted by the modal global analysis. The excellent agreement between on the one hand results delivered by numerical solution of the IVP following non-modal theory and on the other hand solutions of the eigenvalue problem (2.11) offers quantitative cross-verification of the present implementation of both linear modal and non-modal theories.

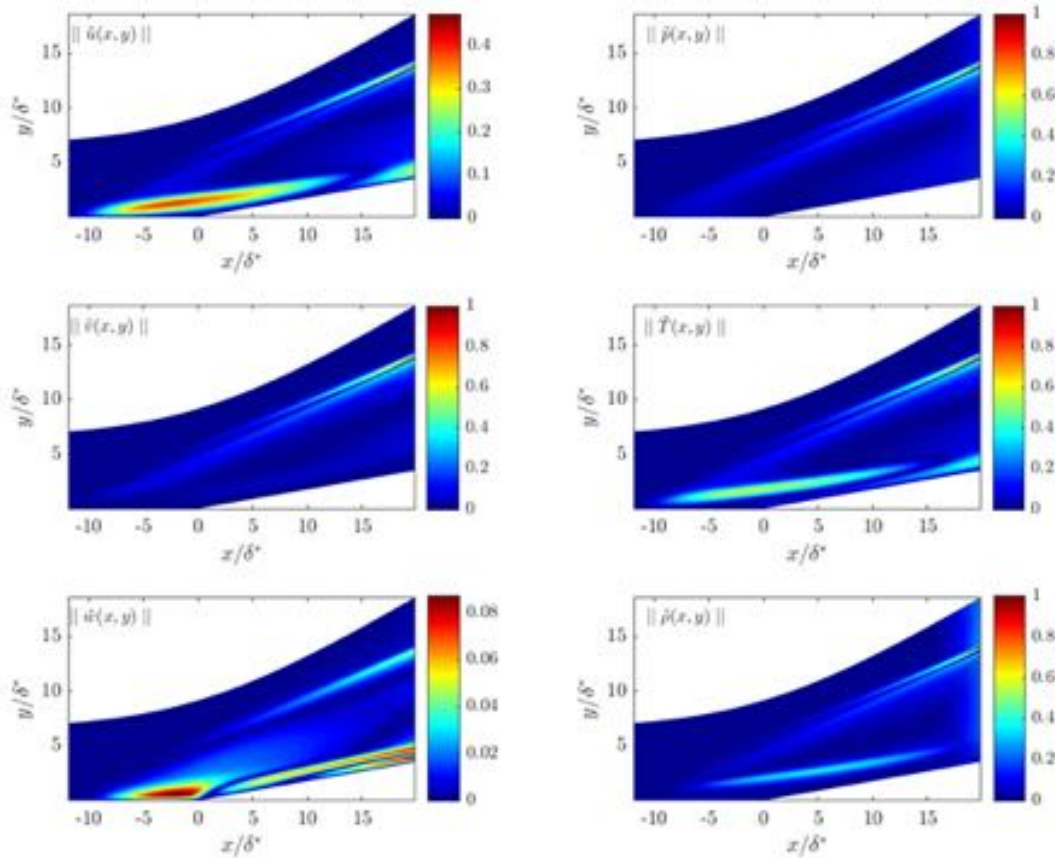


Figure 5.22: Magnitude of the amplitude functions at $t = 100$

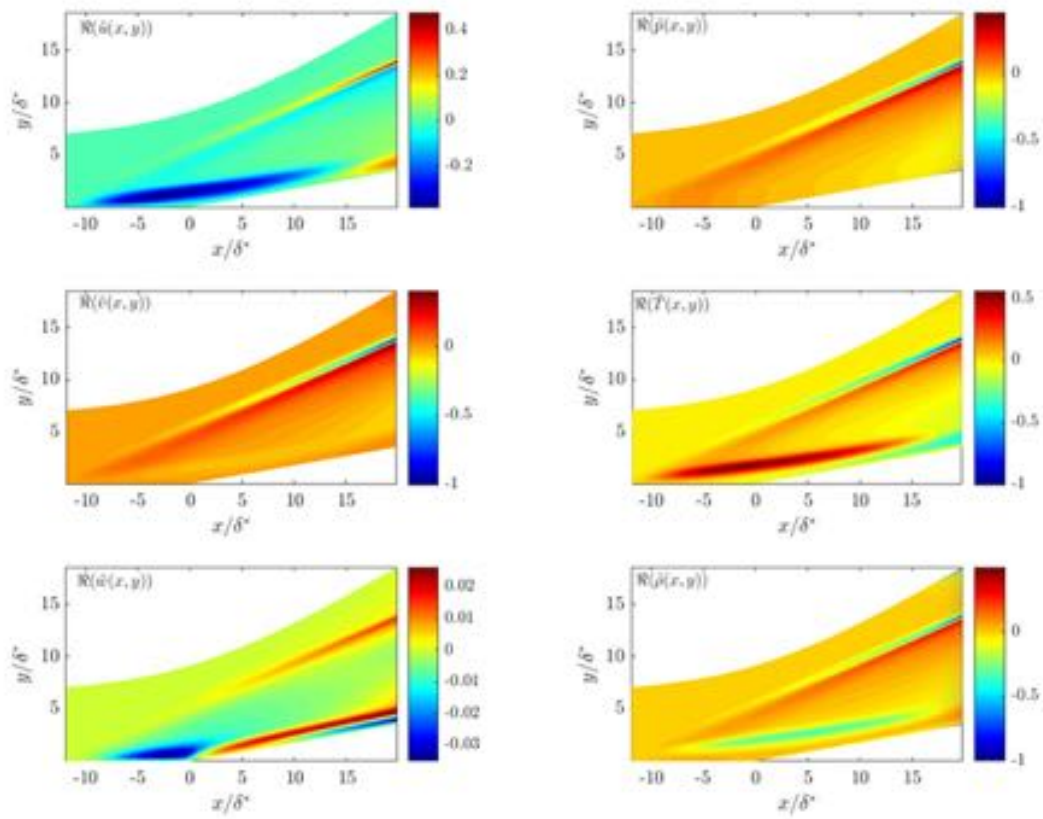


Figure 5.23: Real part of the amplitude functions at $t = 100$

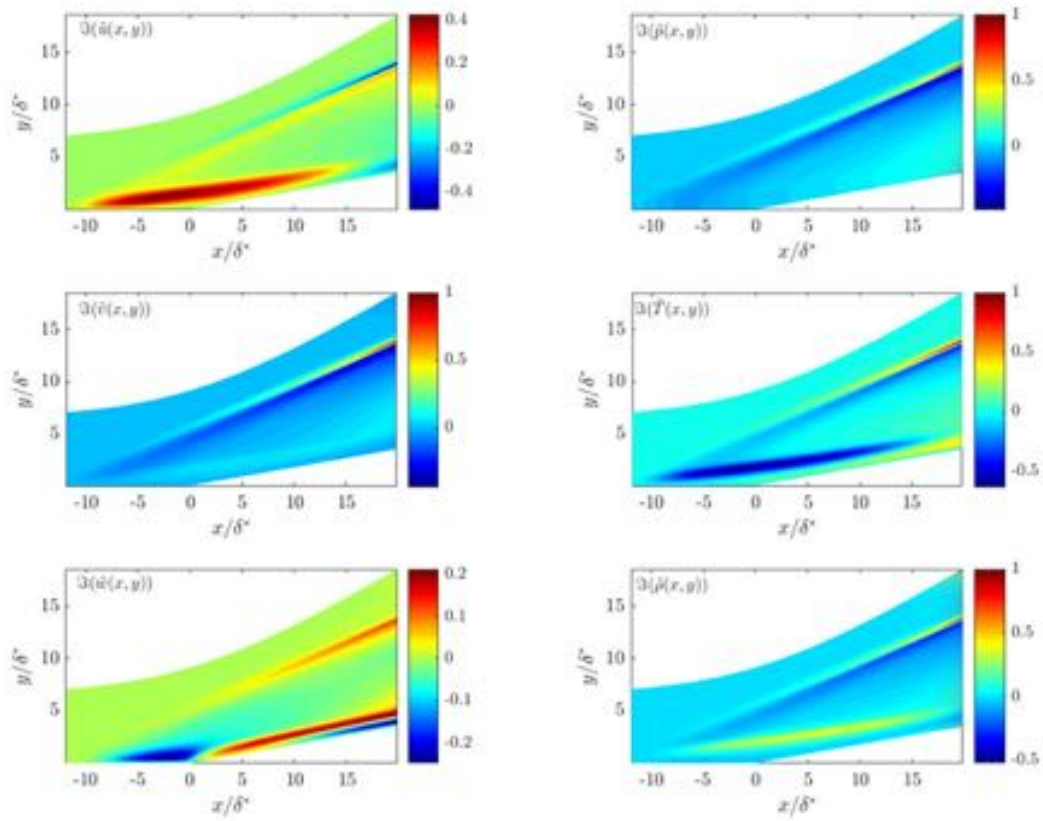


Figure 5.24: Imaginary part of the amplitude functions at $t = 100$

Chapter 6

Interactive boundary layer and the triple-deck theory

A widely used analysis methodology to study instability of steady and unsteady shock boundary layer interaction is based on the triple-deck concept (Neiland, 1969; Stewartson and Williams, 1969; Messiter, 1970). Early successful applications of the triple-deck theory included analysis of the formation process of laminar separation bubbles at a compression corner (Rizzetta et al., 1978a; Ruban, 1978) and the development of an absolute instability in the sense of Huerre and Monkewitz (1990) inside the bubble (Cassel et al., 1995; Korolev et al., 2002; Fletcher et al., 2004). However, in the latter work the separation zone is considered as a sequence of weakly nonparallel, one-dimensional profiles, the discussion of instability is confined within two spatial dimensions and the effect of three-dimensionality on flow instability is neglected.

Smith and Khorrami (1991) were the first to identify the limitations of employing triple-deck theory for the description of flow in the compression ramp and associated breakdown in the ability of the theory to arrive at useful predictions with the physical phenomenon of local breakdown of the solution procedure at a finite value of the control parameter, the latter being the ramp angle in that work; Inger (2011) discusses this point further. Cowley and Hall (1990) pointed out the limitations that non-parallelism of separated flow in the compression corner poses for the triple-deck analysis; nevertheless, within the range of validity of the theory, these authors were able to analyze interaction of Tollmien-Schlichting waves with separated flow in a laminar hypersonic boundary layer over the compression corner.

As direct numerical solutions of the full compressible equations of motion became more prevalent towards the end of last century, progressively diminishing efforts were devoted to analysis of separation and its instability by means of interactive boundary layer concept. A notable exception is the relatively recent combined triple-deck and DNS work of Mengaldo et al. (2015) in which excellent comparisons were shown between predictions of theory and computations on the wall-shear distribution in the wake of an isolated two-dimensional roughness element, work that to a certain extent has revived interest in the predictive capabilities of triple-deck theory. It is interesting to note in this context that the focus in the work of Mengaldo et al. (2015) has been on the subsonic regime owing to the state of development of the high-order DNS algorithm used, rather than an inherent limitation of the theory, which is actually fully described (Stewartson, 1974). A comparison between triple-deck predictions and numerical simulation results in the compression ramp has also appeared recently in the literature (Grisham et al., 2018), contradicting the good agreement between DNS and triple-deck theory and motivating the present analytical effort.

6.1 Theoretical background

Triple-deck analysis extends the classic boundary layer structure proposed by Prandtl (1904) to describe a fluid domain close to a solid surface, as schematically depicted in Fig. 6.1. Three sub-domains (*decks*) are considered, defined by different powers of a small expansion parameter¹, $\varepsilon \equiv Re^{-\frac{1}{8}}$. The width of the *upper deck* scales as $O(Re^{-\frac{3}{8}})$ and describes inviscid flow. The *middle deck* scales as $O(Re^{-\frac{4}{8}})$, corresponding to the Prandtl (1904) two-layer structure, and flow in this deck is viscous. The new element in triple-deck theory is the *lower deck*, which also describes viscous flow at an even smaller scale compared to the middle deck, $O(Re^{-\frac{5}{8}})$, but, unlike the situation in the Prandtl (1904) boundary layer equations, flow separation can now take place. This is in contrast to the similar or non-similar boundary layer equations, in which a strong *prescribed* adverse pressure gradient in the free-stream, that would cause flow separation, instead leads to breakdown of the Prandtl (1904) equations when the (in)famous Goldstein (1948) singularity arises. Successful applications of triple-deck theory to the prediction of incompressible laminar separation is the work of Cebeci and Stewartson (1983) which reproduced the DNS results of Briley (1971) on the laminar separation bubbles arising by the Howarth linear deceleration profile, as well as the more recent above-mentioned work of Mengaldo et al. (2015) which demonstrated excellent agreement between theoretical and numerical simulation results for the laminar separation bubble forming behind a hump embedded in a compressible laminar boundary layer.

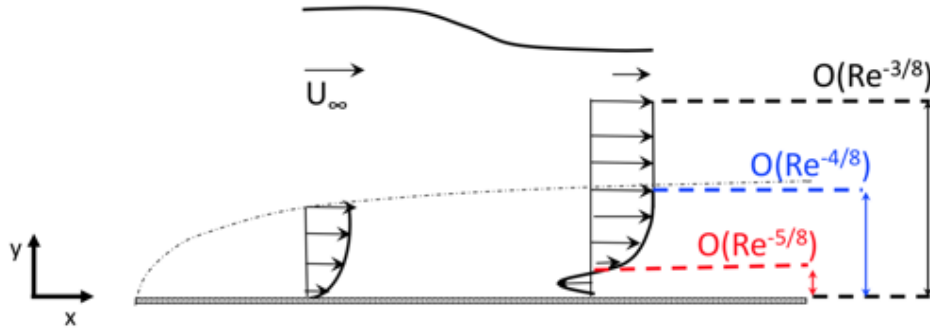


Figure 6.1: Triple-deck structure of the boundary layer on a flat plate

6.2 The compressible laminar triple-deck equations

The triple-deck scaling discussed above carries over to the ramp problem, sketched in Fig. 6.2. The key element of the theory is use of matched asymptotic expansions (Van Dyke, 1975) to formally expand all variables in the full compressible laminar Navier-Stokes equations. Of primary concern is the inner layer, which is a fundamentally incompressible region and the only sub-domain within the triple-deck model to contain any effect of viscosity. In what follows, it will be shown that the length scales associated with the inner layer will reduce the (compressible) Navier-Stokes equations to Prandtl's (incompressible) boundary layer equations.

6.2.1 Definition of scales

Let ρ_w and μ_w denote dimensionless density and viscosity at the wall, by which the original dimensional variables, denoted with a tilde in what follows, are scaled. Furthermore, introduce a nondimensionalisation via the associated free-stream variable, such that density and viscosity do not deviate from their value at the wall:

¹the origin of this scale will be explained in what follows

$$\frac{\tilde{\rho}}{\rho_{\infty}} = \rho_w \quad (6.1)$$

$$\frac{\tilde{\mu}}{\mu_{\infty}} = \mu_w \quad (6.2)$$

The variables λ and β are introduced, and correspond to the numerical coefficient of wall shear stress for the Blasius boundary layer, and the standard supersonic parameter respectively.

$$\lambda = 0.33206 \quad (6.3)$$

$$\beta = \sqrt{M_{\infty}^2 - 1} \quad (6.4)$$

Define the fundamental parameter ε , the relation of which with the flow Reynolds number will be determined as part of the analysis of the momentum equation in what follows. The permutations of ρ_w , μ_w , λ , β and ε are used to scale the dimensional variables in the compressible Navier-Stokes equations.

$$\frac{\tilde{x}}{L} = \rho_w^{-\frac{1}{2}} \mu_w^{-\frac{1}{4}} \lambda^{-\frac{5}{4}} \beta^{-\frac{3}{4}} \varepsilon^3 x \quad (6.5)$$

$$\frac{\tilde{y}}{L} = \rho_w^{-\frac{1}{2}} \mu_w^{\frac{1}{4}} \lambda^{-\frac{3}{4}} \beta^{-\frac{1}{4}} \varepsilon^5 y \quad (6.6)$$

$$\frac{\tilde{u}}{U_{\infty}} = \rho_w^{-\frac{1}{2}} \mu_w^{\frac{1}{4}} \lambda^{\frac{1}{4}} \beta^{-\frac{1}{4}} \varepsilon u \quad (6.7)$$

$$\frac{\tilde{v}}{U_{\infty}} = \rho_w^{-\frac{1}{2}} \mu_w^{\frac{3}{4}} \lambda^{\frac{3}{4}} \beta^{\frac{1}{4}} \varepsilon v \quad (6.8)$$

$$\frac{\tilde{p} - p_{\infty}}{\rho_{\infty} U_{\infty}^2} = \mu_w^{\frac{1}{2}} \lambda^{\frac{1}{2}} \beta^{-\frac{1}{2}} \varepsilon^2 p \quad (6.9)$$

It should be noted here that the original paper of Stewartson (1974) defines no characteristic time scale; the scale used below is defined as the time taken for a perturbation in u to travel over x .

$$\frac{\tilde{t} U_{\infty}}{L} = \mu_w^{-\frac{1}{2}} \lambda^{-\frac{3}{2}} \beta^{-\frac{1}{2}} \varepsilon^2 t \quad (6.10)$$

It is easily seen from the above scales that the time scale for perturbations in u over x is the same for those in v over y . Since the inner layer is incompressible, consider the original dimensional Navier-Stokes equations in the absence of body forces;

$$\tilde{\rho} \left(\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{u}}{\partial \tilde{y}} \right) = - \frac{\partial \tilde{p}}{\partial \tilde{x}} + \tilde{\mu} \left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \right) \quad (6.11)$$

$$\tilde{\rho} \left(\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{y}} \right) = - \frac{\partial \tilde{p}}{\partial \tilde{y}} + \tilde{\mu} \left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} \right) \quad (6.12)$$

And defining a material derivative;

$$\tilde{\rho} \frac{D \tilde{u}}{D \tilde{t}} = \tilde{\rho} \left(\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{u}}{\partial \tilde{y}} \right) \quad (6.13)$$

$$\tilde{\rho} \frac{D \tilde{v}}{D \tilde{t}} = \tilde{\rho} \left(\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{y}} \right) \quad (6.14)$$

6.2.2 The Streamwise Momentum Equation

Substitution of the triple-deck length scales into the material derivative of the x -equation shows that each term is identical in scale.

$$\bar{\rho} \frac{D\tilde{u}}{D\tilde{t}} = \frac{\mu_w^{\frac{3}{4}} \rho_w^{\frac{1}{2}} \beta^{\frac{1}{4}} \lambda^{\frac{7}{4}} \rho_\infty U_\infty^2}{\varepsilon L} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) \quad (6.15)$$

Furthermore, the pressure gradient term is identical in scale to the material derivative.

$$\frac{\partial \tilde{p}}{\partial \tilde{y}} = \frac{\mu_w^{\frac{3}{4}} \rho_w^{\frac{1}{2}} \beta^{\frac{1}{4}} \lambda^{\frac{7}{4}} \rho_\infty U_\infty^2}{\varepsilon L} \frac{\partial p}{\partial x} \quad (6.16)$$

And finally, the scales of the Laplacian are:

$$\tilde{\mu} \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = \frac{\mu_\infty \mu_w^{\frac{7}{4}} \rho_w^{\frac{1}{2}} \beta^{\frac{5}{4}} \lambda^{\frac{11}{4}} U_\infty}{\varepsilon^5 L^2} \frac{\partial^2 u}{\partial x^2} \quad (6.17)$$

$$\tilde{\mu} \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} = \frac{\mu_\infty \mu_w^{\frac{3}{4}} \rho_w^{\frac{1}{2}} \beta^{\frac{1}{4}} \lambda^{\frac{7}{4}} U_\infty}{\varepsilon^9 L^2} \frac{\partial^2 v}{\partial x^2} \quad (6.18)$$

Re-scaling the entire equation by the coefficient of the new material derivative yields:

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\mu_\infty \mu_w \beta \lambda}{\varepsilon^4 L \rho_\infty U_\infty} \frac{\partial^2 u}{\partial x^2} + \frac{\mu_\infty}{\varepsilon^8 L \rho_\infty U_\infty} \frac{\partial^2 u}{\partial y^2} \quad (6.19)$$

Guided by the last term in (6.19), the small parameter ε is defined as

$$\varepsilon = Re^{-\frac{1}{8}}, \quad (6.20)$$

where

$$Re = \frac{\rho_\infty U_\infty L}{\mu_\infty}, \quad (6.21)$$

such that the coefficient of the last viscous term in (6.19) becomes unity. Consequently,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\beta \lambda \mu_w}{\sqrt{Re}} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (6.22)$$

Finally, β , λ and μ_w are all order $O(1)$ constants for a supersonic flow, allowing u_{xx} to be discarded on the grounds that $Re \gg 1$:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \quad (6.23)$$

6.2.3 The Wall Normal Momentum Equation

The inner layer scales are substituted into the Navier-Stokes y -momentum equation in exactly the same manner as before. The resulting material derivative is:

$$\bar{\rho} \frac{D\tilde{v}}{D\tilde{t}} = \frac{\mu_w^{\frac{5}{4}} \rho_w^{\frac{1}{2}} \beta^{\frac{3}{4}} \lambda^{\frac{9}{4}} \rho_\infty U_\infty^2 \varepsilon}{L} \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) \quad (6.24)$$

The pressure gradient over y :

$$\frac{\partial \tilde{p}}{\partial \tilde{y}} = \frac{\mu_w^{\frac{1}{4}} \rho_w^{\frac{1}{2}} \lambda^{\frac{5}{4}} \rho_\infty U_\infty^2}{\beta^{\frac{1}{4}} \varepsilon^3 L} \frac{\partial p}{\partial y} \quad (6.25)$$

And the Laplacian:

$$\tilde{\mu} \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} = \frac{\mu_\infty \mu_w^{\frac{9}{4}} \rho_w^{\frac{1}{2}} \beta^{\frac{7}{4}} \lambda^{\frac{13}{4}} U_\infty}{\varepsilon^3 L^2} \frac{\partial^2 v}{\partial x^2} \quad (6.26)$$

$$\tilde{\mu} \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} = \frac{\mu_\infty \mu_w^{\frac{5}{4}} \rho_w^{\frac{1}{2}} \beta^{\frac{3}{4}} \lambda^{\frac{9}{4}} U_\infty}{\varepsilon^7 L^2} \frac{\partial^2 v}{\partial y^2} \quad (6.27)$$

Combining into a single equation and dividing by the coefficient of the material derivative as before yields:

$$\frac{Dv}{Dt} = -\frac{1}{\beta \lambda \mu_w \varepsilon^4} \frac{\partial p}{\partial y} + \frac{\beta \lambda \mu_w \mu_\infty}{\rho_\infty U_\infty L \varepsilon^4} \frac{\partial^2 v}{\partial x^2} + \frac{\mu_\infty}{\rho_\infty U_\infty L \varepsilon^8} \frac{\partial^2 v}{\partial y^2} \quad (6.28)$$

Introducing the scale for ε the wall-normal momentum equation becomes

$$\frac{Dv}{Dt} = -\frac{\sqrt{Re}}{\beta \lambda \mu_w} \frac{\partial p}{\partial y} + \frac{\beta \lambda \mu_w}{\sqrt{Re}} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \quad (6.29)$$

Once again, β , λ and μ_w are all order $O(1)$ constants and $Re \gg 1$, meaning that the only non-negligible term is the pressure gradient term. Thus, the the y -momentum equation is identically satisfied, reducing the Navier-Stokes equations into the incompressible boundary layer equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (6.30)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \quad (6.31)$$

$$\frac{\partial p}{\partial y} = 0 \quad (6.32)$$

6.2.4 The Compression Ramp

The triple-deck scaling discussed above carries over to the ramp problem, sketched in Fig. 6.2. The key element of the theory is use of matched asymptotic expansions (Van Dyke, 1975) to formally expand all variables in the full Navier-Stokes equations according to (Ruban, 1978; Rizzetta et al., 1978a; Cassel et al., 1995; Grisham et al., 2018)

$$\frac{\tilde{x}}{L} = \rho_w^{-\frac{1}{2}} \mu_w^{-\frac{1}{4}} \lambda^{-\frac{5}{4}} \beta^{-\frac{3}{4}} Re^{-\frac{3}{8}} x, \quad (6.33)$$

$$\frac{\tilde{y}}{L} = \rho_w^{-\frac{1}{2}} \mu_w^{\frac{1}{4}} \lambda^{-\frac{3}{4}} \beta^{-\frac{1}{4}} Re^{-\frac{5}{8}} [y - f(x)], \quad (6.34)$$

$$\frac{\tilde{u}}{U_\infty} = \rho_w^{-\frac{1}{2}} \mu_w^{\frac{1}{4}} \lambda^{\frac{1}{4}} \beta^{-\frac{1}{4}} Re^{-\frac{1}{8}} u, \quad (6.35)$$

$$\frac{\tilde{v}}{U_\infty} = \rho_w^{-\frac{1}{2}} \mu_w^{\frac{3}{4}} \lambda^{\frac{3}{4}} \beta^{\frac{1}{4}} Re^{-\frac{3}{8}} \left[v - \frac{df}{dx} \right], \quad (6.36)$$

$$\frac{\tilde{p} - p_\infty}{\rho_\infty U_\infty^2} = \mu_w^{\frac{1}{2}} \lambda^{\frac{1}{2}} \beta^{-\frac{1}{2}} Re^{-\frac{1}{4}} p, \quad (6.37)$$

6.2.5 The Prandtl (1938) transposition theorem

The transposition theorem, which was put forward by Prandtl (1938) in the context of the classic incompressible boundary layer, was later extended to compressible and three-dimensional boundary layers by Glauert (1957) and has been incorporated in the scalings (6.34) and (6.36) is discussed next. The theorem states that the boundary layer equations (6.40-6.58) remain invariant under a transformation of the independent variables²

$$\hat{x} = x, \quad (6.44)$$

$$\hat{y} = y - f(x), \quad (6.45)$$

provided that the velocity components are consistently transformed ('transposed') using

$$\hat{u} = u, \quad (6.46)$$

$$\hat{v} = v - f'(x)u. \quad (6.47)$$

It is straightforward to show that substitution of (6.46-6.47) into (6.40-6.57) and use of the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \hat{x}} + \frac{\partial}{\partial \hat{y}} \left(-\frac{df}{dx} \right), \quad (6.48)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \hat{y}} \quad (6.49)$$

results in

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{u}}{\partial \hat{y}} \left(-\frac{df}{dx} \right) + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{df}{dx} \frac{\partial \hat{u}}{\partial \hat{y}} = \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \quad (6.50)$$

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \implies \\ \frac{\partial \hat{u}}{\partial t} + \hat{u} \left[\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{u}}{\partial \hat{y}} \left(-\frac{df}{dx} \right) \right] + [\hat{v} + f'(x)\hat{u}] \frac{\partial \hat{u}}{\partial \hat{y}} &= -\frac{\partial p}{\partial \hat{x}} + \frac{\partial p}{\partial \hat{y}} \left(-\frac{df}{dx} \right) + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} \implies \\ \frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} &= -\frac{\partial p}{\partial \hat{x}} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} \end{aligned} \quad (6.51)$$

The usefulness of the Prandtl transposition theorem in the compression ramp flow resides in the ability to incorporate an explicit functional description of the wall, $f(x)$, in the triple-deck analysis. In the form used by earlier researchers, the transposition function $f(x)$ is introduced to incorporate a compression ramp placed at an angle $\pi\alpha$ relative to the free-stream, scaled by an appropriate permutation of μ_w , λ , β and ε according to

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \alpha x & \text{if } x > 0 \end{cases} \quad (6.52)$$

$$\tilde{\alpha} = \mu_w^{\frac{1}{2}} \lambda^{\frac{1}{2}} \beta^{\frac{1}{2}} \varepsilon^2 \alpha. \quad (6.53)$$

Here, the this model is extended and the wall is described by

$$f(x) = \frac{1}{2} \alpha \left[x + (x^2 + r^2)^{\frac{1}{2}} \right], \quad (6.54)$$

where α is the scaled ramp angle defined in (6.39) and r is a small radius parameter of a circle inscribed in the wedge, permitting to avoid a singularity in the streamwise derivative of $f(x)$ that appears in (6.47).

²(6.45) also holds for unsteady displacements of the wall described by an analytic function $f(x, t)$

6.2.6 Numerical implementation

Numerical solutions of the triple-deck equations for the compression ramp problem have been obtained, following the works of Rizzetta et al. (1978a), Cassel et al. (1995) and Korolev et al. (2002). Prior to discussing the (non-dimensional) results obtained, it is instructive to decipher the realm of applicability of the triple-deck theory, the fundamental scaling of which relates the length l for which the theory is applicable was put forward by Lighthill (1953)

$$\frac{l}{L} = O(Re^{-\frac{3}{8}}), \quad (6.55)$$

where $L = O(1)$ is a dimensional physical length characteristic of the problem in question. Of interest here are values that the scaled ramp angle α takes when its physical-space, dimensional counterpart, $\tilde{\alpha}$, related to α through equation (6.39), takes $O(1)$ values typical of those in applications.

Re	l/L	α	
		$\tilde{\alpha} = 15^\circ$	$\tilde{\alpha} = 30^\circ$
10^3	0.0750	1.0743	2.1485
10^4	0.0316	1.9103	3.8207
10^5	0.0133	3.3971	6.7943

Table 6.1: Typical values of the expected extent of the laminar separation zone, l , scaled with $L = 1m$, and of the scaled ramp angle α corresponding to two values $\tilde{\alpha} = 15^\circ$ and $\tilde{\alpha} = 30^\circ$, at $Ma = 3$, in the Reynolds number range $10^3 \leq Re \leq 10^5$

Table 6.1 presents values of l and the scaled angle, α , for $\tilde{\alpha} = 15^\circ$ and 30° at $Ma = 3$ and $10^3 \leq Re \leq 10^5$. Here the (dimensionless) constant value $\mu_w = 2$, obtained in a numerical solution of the equations of motion at $Ma = 3$ has been used, alongside the constants $\lambda = 0.332$ (Inger, 2011) and $\beta = (M_\infty^2 - 1)^{\frac{1}{2}} = \sqrt{8}$. It can be seen that the extent of the separated interaction region inside the boundary layer decreases with increasing Reynolds number, although it still remains of $O(1mm)$ if a scale $L = 1m$ is used, implying that useful predictions can be obtained by triple-deck theory in the laminar regime, the streamwise extent of which has been found in the simulations to also be of $O(1mm)$ in the examined Mach and Reynolds number range.

On the other hand, the scaled angles in this range are $0 \leq \alpha \leq 10$, which is also in line with the assumptions of triple-deck theory. Results have been obtained for a number of scaled ramp angles within this range, as shown in Fig. 6.3. Consistently with earlier findings (e.g. Rizzetta et al., 1978a), it can be seen that in the range $0 \leq \alpha \leq 10$ laminar separation is formed around the ramp corner, $x = 0$. The extent of separation bubble increases and the minimum value of the skin friction coefficient decreases with increasing ramp angle. At low values of α separation is steady, while at $\alpha \approx 5$, corresponding to $\tilde{\alpha} \approx 30^\circ$ at the Reynolds number in question, two-dimensional instability develops within the laminar separation bubble portion upstream of the compression corner. However, from a physical point of view, the question whether unsteadiness of the two-dimensional flow through the mechanism described by the results of Fig. 6.3 will precede or succeed three-dimensional BiGlobal instability of the entire laminar separation bubble. The theoretical framework to answer this question in a triple-deck context will be discussed in the next Section 6.2.7, while linear global instability analysis of the DNS-obtained laminar base flows will be addressed in Section 5.

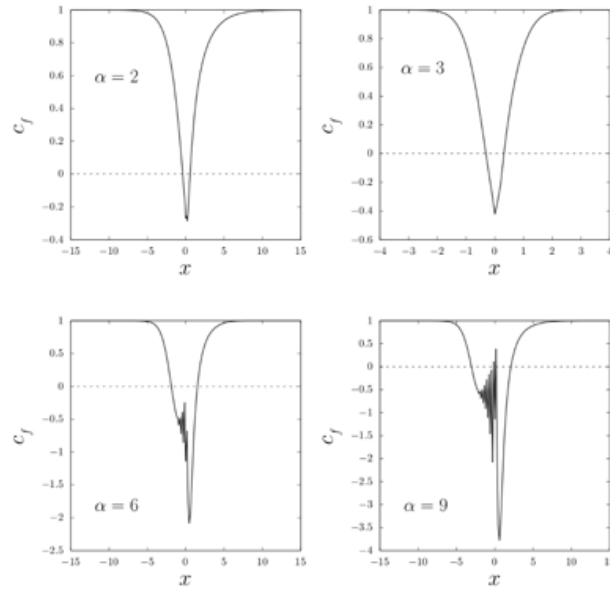


Figure 6.3: Development of two-dimensional global instability in the laminar separation bubble at the compression corner, manifested as 2D unsteadiness of separated flow upstream of the compression corner

6.2.7 Three-dimensional instability

The three-dimensional equations governing flow in the triple-deck framework are (Smith et al., 1977; Stewartson, 1981)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (6.56)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \quad (6.57)$$

$$\frac{\partial p}{\partial y} = 0 \quad (6.58)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial y^2} \quad (6.59)$$

The boundary conditions satisfied by the flow variables are (Stewartson, 1981; Rizzetta et al., 1978b)

At $x \rightarrow -\infty$:

$$u - y \rightarrow A(x, z) \quad (6.60)$$

$$w \rightarrow \frac{1}{z} B(x, z) \quad (6.61)$$

$$p \rightarrow 0 \quad (6.62)$$

At $x \rightarrow +\infty$:

$$u - y \rightarrow A(x, z) + \lambda x \quad (6.63)$$

$$w \rightarrow \frac{1}{z}B(x, z) \quad (6.64)$$

$$p \rightarrow \lambda + c \quad (6.65)$$

6.2.8 Instability analysis

It seems natural to perform the analysis of small-amplitude perturbations by decomposing the boundary conditions applicable to the full system (6.56-6.59) in a manner consistent with both the triple-deck theory discussed in the previous sections and the global linear theory Ansatz (2.10) of section § 5. Accordingly, the far-field dependence of u, v and w in (6.60-6.65) is decomposed in $O(1)$ and small-amplitude contributions

$$u \rightarrow \bar{u}(x, y) + \varepsilon \hat{u}(x, y) \exp[i(\beta z - \omega t)], \quad (6.66)$$

$$v \rightarrow \bar{v}(x, y) + \varepsilon \hat{v}(x, y) \exp[i(\beta z - \omega t)], \quad (6.67)$$

$$w \rightarrow \bar{w}(x, y) + \varepsilon \hat{w}(x, y) \exp[i(\beta z - \omega t)], \quad (6.68)$$

where $\bar{u}(x, y) = y + \bar{A}(x)$ and the case of no spanwise base flow $\bar{w} \equiv 0$ is considered.

Small perturbations to (6.56-6.59) are then analyzed by introducing the linear decomposition $\mathbf{q}(x, y, z, t) = \bar{\mathbf{q}}(x, y) + \varepsilon \hat{\mathbf{q}}(x, y) \exp[i(\beta z - \omega t)]$ and a base flow $(\bar{u}, \bar{v}, 0)(x, y)$ to arrive at

$$\hat{u}_x + \hat{v}_y + i\beta \hat{w} = 0, \quad (6.69)$$

$$-i\omega \hat{u} + \bar{u} \hat{u}_x + \bar{u}_x \hat{u} + \bar{v} \hat{u}_y + \bar{u}_y \hat{v} = -\hat{p}_x + \hat{u}_{yy}, \quad (6.70)$$

$$\hat{p}_y = 0, \quad (6.71)$$

$$-i\omega \hat{w} + \bar{u} \hat{w}_x + \bar{v} \hat{w}_y = -i\beta \hat{p} + \hat{w}_{yy} \quad (6.72)$$

$$\begin{bmatrix} -\mathcal{D}_{yy} + \bar{u}\mathcal{D}_x + \bar{v}\mathcal{D}_y + \bar{u}_x\mathcal{I} & \bar{u}_y & 0 & \mathcal{D}_x \\ 0 & 0 & 0 & \mathcal{D}_y \\ 0 & 0 & -\mathcal{D}_{yy} + \bar{u}\mathcal{D}_x + \bar{v}\mathcal{D}_y & i\beta \\ \mathcal{D}_x & \mathcal{D}_y & i\beta & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{p} \end{bmatrix} = \omega \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{p} \end{bmatrix} \quad (6.73)$$

In our current work portions of the base flow fields obtained by direct numerical simulation in section § 3.2 and are consistent with the near-wall structure of the streamwise velocity component within the framework of triple-deck theory are extracted from the simulation results and analyzed by numerical solution of (6.73); results will be reported in due course.

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Appendix A

Coefficients of the equation (2.11)

A.1 Continuity

$$\mathcal{A}_{51} = \bar{\rho}D_x + \bar{\rho}_x$$

$$\mathcal{A}_{52} = \bar{\rho}D_y + \bar{\rho}_y$$

$$\mathcal{A}_{53} = i\beta\bar{\rho}$$

$$\mathcal{A}_{54} = -\frac{\bar{\rho}}{\bar{T}}\bar{u}D_x - \frac{\bar{\rho}}{\bar{T}}\bar{v}D_y + \frac{\bar{\rho}}{\bar{T}}(-\bar{u}_x - \bar{v}_y - i\beta\bar{w}) + \frac{\bar{\rho}}{\bar{T}^2}(\bar{u}\bar{T}_x + \bar{v}\bar{T}_y) - \frac{1}{\bar{T}}(\bar{u}\bar{\rho}_x + \bar{v}\bar{\rho}_y) \quad (\text{A.1})$$

$$\mathcal{A}_{55} = \gamma Ma^2 \frac{\bar{u}}{\bar{T}}D_x + \gamma Ma^2 \frac{\bar{v}}{\bar{T}}D_y - \frac{\gamma Ma^2}{\bar{T}^2}(\bar{u}\bar{T}_x + \bar{v}\bar{T}_y) + \frac{\gamma Ma^2}{\bar{T}}(\bar{u}_x + \bar{v}_y + i\beta\bar{w})$$

$$\mathcal{B}_{54} = -\frac{i\bar{\rho}}{\bar{T}}$$

$$\mathcal{B}_{55} = \gamma Ma^2 \frac{i}{\bar{T}}$$

A.2 Streamwise momentum equation

$$\begin{aligned}
\mathcal{A}_{11} &= -\frac{4}{3} \frac{\bar{\mu}}{Re} D_{xx} + \left[-\frac{4}{3} \frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_x + \bar{\rho}\bar{u} \right] D_x - \frac{\bar{\mu}}{Re} D_{yy} + \left[-\frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_y + \bar{\rho}\bar{v} \right] D_y \frac{\beta^2 \bar{\mu}}{Re} + \bar{\rho}\bar{u}_x + i\beta\bar{\rho}\bar{w} \\
\mathcal{A}_{12} &= -\frac{1}{3} \frac{\bar{\mu}}{Re} D_{xy} - \frac{1}{Re} \frac{\bar{\mu}}{dT} \bar{T}_y D_x + \frac{2}{3} \frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_x D_y + \bar{\rho}\bar{u}_y \\
\mathcal{A}_{13} &= -\frac{i}{3} \frac{\beta}{Re} \bar{\mu} D_x + \frac{2}{3} \frac{i\beta}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_x \\
\mathcal{A}_{14} &= -\frac{1}{Re} \frac{d\bar{\mu}}{dT} \left(\frac{4}{3} \bar{u}_x - \frac{2}{3} \bar{v}_y \right) D_x - \frac{1}{Re} \frac{d\bar{\mu}}{dT} (\bar{u}_y + \bar{v}_x) D_y \\
&\quad - \frac{1}{Re} \frac{d\bar{\mu}}{dT} \left(i\beta\bar{w}_x + \frac{4}{3} \bar{u}_{xx} + \bar{u}_{yy} + \frac{1}{3} \bar{v}_{xy} \right) - \frac{1}{Re} \frac{d^2\bar{\mu}}{dT^2} \left(\frac{4}{3} \bar{T}_x \bar{u}_x + \bar{T}_y (\bar{u}_y + \bar{v}_x) - \frac{2}{3} \bar{T}_x \bar{v}_y \right) \\
&\quad - \frac{\bar{\rho}}{\bar{T}} (\bar{u}\bar{u}_x + \bar{v}\bar{u}_y) \\
\mathcal{A}_{15} &= D_x + \frac{\gamma Ma^2}{\bar{T}} (\bar{u}\bar{u}_x + \bar{v}\bar{u}_y) \\
\mathcal{B}_{11} &= i\bar{\rho}
\end{aligned} \tag{A.2}$$

A.3 Wall-normal momentum equation

$$\begin{aligned}
\mathcal{A}_{21} &= -\frac{1}{3} \frac{\bar{\mu}}{Re} D_{xy} + \frac{2}{3} \frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_y D_x - \frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_x D_y + \bar{\rho} \bar{v}_x \\
\mathcal{A}_{22} &= -\frac{\bar{\mu}}{Re} D_{xx} + \left[-\frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_x + \bar{\rho} \bar{u} \right] D_x - \frac{4}{3} \frac{\bar{\mu}}{Re} D_{yy} + \left[-\frac{4}{3} \frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_y + \bar{\rho} \bar{v} \right] D_y \\
&\quad + \frac{\beta^2 \bar{\mu}}{Re} + \bar{\rho} \bar{v}_y + i\beta \bar{\rho} \bar{w} \\
\mathcal{A}_{23} &= -\frac{i}{3} \frac{\beta}{Re} \bar{\mu} D_y + \frac{2}{3} \frac{i\beta}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_y \\
\mathcal{A}_{24} &= -\frac{1}{Re} \frac{d\bar{\mu}}{dT} (\bar{u}_y + \bar{v}_x) D_x - \frac{1}{Re} \frac{d\bar{\mu}}{dT} \left(\frac{4}{3} \bar{v}_y - \frac{2}{3} \bar{u}_x \right) D_y \\
&\quad - \frac{1}{Re} \frac{d\bar{\mu}}{dT} \left(i\beta \bar{w}_y + \frac{4}{3} \bar{v}_{yy} + \frac{1}{3} \bar{u}_{xy} + \bar{v}_{xx} \right) - \frac{1}{Re} \frac{d^2 \bar{\mu}}{dT^2} \left(\frac{4}{3} \bar{T}_y \bar{v}_y + \bar{T}_x (\bar{u}_y + \bar{v}_y) - \frac{2}{3} \bar{T}_y \bar{u}_x \right) \\
&\quad - \frac{\bar{\rho}}{\bar{T}} (\bar{u} \bar{v}_x + \bar{v} \bar{v}_y) \\
\mathcal{A}_{25} &= D_y + \frac{\gamma Ma^2}{\bar{T}} (\bar{u} \bar{v}_x + \bar{v} \bar{v}_y) \\
\mathcal{B}_{22} &= i\bar{\rho}
\end{aligned} \tag{A.3}$$

A.4 Spanwise momentum equation

$$\begin{aligned}
\mathcal{A}_{31} &= -\frac{i}{3} \frac{\beta}{Re} \bar{\mu} D_x - \frac{i\beta}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_x + \bar{\rho} \bar{w}_x \\
\mathcal{A}_{32} &= \mathcal{A}_{31} = -\frac{i}{3} \frac{\beta}{Re} \bar{\mu} D_y - \frac{i\beta}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_y + \bar{\rho} \bar{w}_y \\
\mathcal{A}_{33} &= -\frac{\bar{\mu}}{Re} D_{xx} - \frac{\bar{\mu}}{Re} D_{yy} + \left[-\frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_x + \bar{\rho} \bar{u} \right] D_x + \left[-\frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{T}_y + \bar{\rho} \bar{v} \right] D_y \\
&\quad + \frac{4}{3} \frac{\beta^2 \bar{\mu}}{Re} + i\beta \bar{\rho} \bar{w} \\
\mathcal{A}_{34} &= -\frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{w}_x D_x - \frac{1}{Re} \frac{d\bar{\mu}}{dT} \bar{w}_y D_y \\
&\quad - \frac{1}{Re} \frac{d\bar{\mu}}{dT} \left(\bar{w}_{xx} + \bar{w}_{yy} - \frac{2}{3} i\beta (\bar{u}_x + \bar{v}_y) - \frac{1}{Re} \frac{d^2 \bar{\mu}}{dT^2} (\bar{T}_x \bar{w}_x + \bar{T}_y \bar{w}_y) - \frac{\bar{\rho}}{\bar{T}} (\bar{u} \bar{w}_x + \bar{v} \bar{w}_y) \right) \\
\mathcal{A}_{35} &= i\beta + \frac{\gamma Ma^2}{\bar{T}} (\bar{u} \bar{w}_x + \bar{v} \bar{w}_y) \\
\mathcal{B}_{33} &= i\bar{\rho}
\end{aligned} \tag{A.4}$$

A.5 Energy equation

$$\begin{aligned}
\mathcal{A}_{41} &= \left[-2 \frac{(\gamma-1)Ma^2}{Re} \bar{\mu} \left(\frac{4}{3} \bar{u}_x - \frac{2}{3} \bar{v}_y \right) + \gamma Ma^2 \bar{p} \right] D_x - 2 \frac{(\gamma-1)Ma^2}{Re} \bar{\mu} (\bar{u}_y + \bar{v}_x) D_y \\
&\quad - 2 \frac{(\gamma-1)Ma^2}{Re} \bar{\mu} \bar{w}_x + Ma^2 \bar{p}_x \\
\mathcal{A}_{42} &= -2 \frac{(\gamma-1)Ma^2}{Re} \bar{\mu} (\bar{u}_y + \bar{v}_x) D_x + \left[-2 \frac{(\gamma-1)Ma^2}{Re} \bar{\mu} \left(-\frac{2}{3} \bar{u}_x + \frac{4}{3} \bar{v}_y \right) + \gamma Ma^2 \bar{p} \right] D_y \\
&\quad - 2 \frac{(\gamma-1)Ma^2}{Re} \bar{\mu} \bar{w}_y + Ma^2 \bar{p}_y \\
\mathcal{A}_{43} &= -2 \frac{(\gamma-1)Ma^2}{Re} \bar{\mu} \bar{w}_x D_x - 2 \frac{(\gamma-1)Ma^2}{Re} \bar{\mu} \bar{w}_y D_y + \frac{4}{3} \frac{i\beta(\gamma-1)Ma^2}{Re} \bar{\mu} (\bar{u}_x + \bar{v}_y) + i\beta\gamma Ma^2 \bar{p} \\
&\quad (A.5) \\
\mathcal{A}_{44} &= -\frac{\bar{k}}{RePr} D_{xx} - \frac{2}{RePr} \frac{d\bar{k}}{dT} \bar{T}_x D_x - \frac{\bar{k}}{RePr} D_{yy} - \frac{2}{RePr} \frac{d\bar{k}}{dT} \bar{T}_y D_y \\
&\quad + \frac{\beta^2 \bar{k}}{RePr} - \frac{1}{RePr} \frac{d\bar{k}}{dT} (\bar{T}_{xx} + \bar{T}_{yy}) - \frac{1}{RePr} \frac{d^2 \bar{k}}{dT^2} (\bar{T}_x^2 + \bar{T}_y^2) \\
&\quad + \frac{(\gamma-1)Ma^2}{Re} \frac{d\bar{\mu}}{dT} \left(\frac{2}{3} (\bar{u}_x + \bar{v}_y)^2 - (\bar{u}_y + \bar{v}_x)^2 - 2(\bar{u}_x^2 + \bar{v}_y^2) - \bar{w}_x^2 - \bar{w}_y^2 \right) \\
\mathcal{A}_{45} &= Ma^2 \bar{u} D_x + Ma^2 \bar{v} D_y + i\beta Ma^2 \bar{w} + \gamma Ma^2 (\bar{u}_x + \bar{v}_y) \\
\mathcal{B}_{45} &= iMa^2
\end{aligned}$$