GBTL v3.0: Primitives for Optimized mxm

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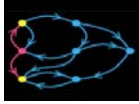

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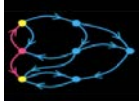


GraphBLAS Base Operations

(C API Version 1.3)

Operation	Math	Out	Inputs
mxm	$\mathbf{C}\langle\neg/s\mathbf{M},z\rangle = \mathbf{C} \odot (\mathbf{A}^{\mathbf{T}} \oplus.\otimes \mathbf{B}^{\mathbf{T}})$	**C**	¬, s, **M**, z, ⊙, **A**, **T**, ⊕.⊗, **B**, **T**
mxv (vxm)	$\mathbf{c}\langle\neg/s\mathbf{m},z\rangle = \mathbf{c} \odot (\mathbf{A}^{\mathbf{T}} \oplus.\otimes \mathbf{b})$	**c**	¬, s, **m**, z, ⊙, **A**, **T**, ⊕.⊗, **b**
eWiseMult	$\mathbf{C}\langle\neg/s\mathbf{M},z\rangle = \mathbf{C} \odot (\mathbf{A}^{\mathbf{T}} .\otimes \mathbf{B}^{\mathbf{T}})$	**C**	¬, s, **M**, z, ⊙, **A**, **T**, ⊗, **B**, **T**
eWiseAdd	$\mathbf{C}\langle\neg/s\mathbf{M},z\rangle = \mathbf{C} \odot (\mathbf{A}^{\mathbf{T}} .\oplus \mathbf{B}^{\mathbf{T}})$	**C**	¬, s, **M**, z, ⊙, **A**, **T**, ⊕, **B**, **T**
reduce (row)	$\mathbf{c}\langle\neg/s\mathbf{m},z\rangle = \mathbf{c} \odot [\oplus_j \mathbf{A}^{\mathbf{T}}(:,j)]$	**c**	¬, s, **m**, z, ⊙, **A**, **T**, ⊕
apply	$\mathbf{C}\langle\neg/s\mathbf{M},z\rangle = \mathbf{C} \odot f(\mathbf{A}^{\mathbf{T}})$	**C**	¬, s, **M**, z, ⊙, **A**, **T**, *f*()
transpose	$\mathbf{C}\langle\neg/s\mathbf{M},z\rangle = \mathbf{C} \odot \mathbf{A}^{\mathbf{T}}$	**C**	¬, s, **M**, z, ⊙, **A** (**T**)
extract	$\mathbf{C}\langle\neg/s\mathbf{M},z\rangle = \mathbf{C} \odot \mathbf{A}^{\mathbf{T}}(\mathbf{i},\mathbf{j})$	**C**	¬, s, **M**, z, ⊙, **A**, **T**, **i**, **j**
assign	$\mathbf{C}\langle\neg/s\mathbf{M},z\rangle (\mathbf{i},\mathbf{j}) = \mathbf{C}(\mathbf{i},\mathbf{j}) \odot \mathbf{A}^{\mathbf{T}}$	**C**	¬, s, **M**, z, ⊙, **A**, **T**, **i**, **j**
build (meth.)	$\mathbf{C} = \mathbb{S}^{mxn}(\mathbf{i},\mathbf{j},\mathbf{v},\odot)$	**C**	⊙, m, n, **v**, **i**, **j**
extractTuples (meth.)	$(\mathbf{i},\mathbf{j},\mathbf{v}) = \mathbf{A}$	**i**,**j**,**v**	**A**

Notation: **i,j** – index arrays, **v** – value array, **m** – 1D mask, **c,b** – 1D vector (column), **M** – 2D mask, **A,B,C** – 2D matrix,
T – transpose, ¬ - complement, s – structure-only, z – replace
⊙ accumulate monoid/binary function, ⊕.⊗ semiring,
blue – optional parameters, red – optional modifiers (using Descriptors in the C API)

Part 3: GBTL v 3.0 Release

Changelog (https://github.com/cmu-sei/gbtl)

2020-06 Scott McMillan smcmillan@sei.cmu.edu

*** Version 3.0 Release. C++17 compiler required to build. This version is NOT backward compatible with Version 2.0** for the following reasons:

- namespace name "GraphBLAS" has been replaced with "grb".
- The boolean "replace_flag" has change types to an enum with MERGE and REPLACE (corresponding false and true respectively).
- Some of the names of the monoid and semiring template classes have been changed to more closely match C API 1.3 names (e.g. MaxSelect1stSemiring is now MaxFirstSemiring)
- Order of parameters in markov cluster has changed (swapped convergence_threshold and max_iters)

*** API updated to implement new functionality included in GraphBLAS C API Specification version 1.3:**

- add getVersion() method (reports the C API equivalent version)
- add removeElement() method to Matrix and Vector classes.
- add resize() method to Matrix and Vector classes.
- add Exclusive-NOR (XNOR) boolean binary operator
- add bitwise OR/AND/XOR/XNOR binary operators for integer types
- add all predefined monoids and semirings.
- **add structure-only view for masks (composable with complement).**
- add kronecker matrix multiply operation.
- add apply operation that takes a binary operator and a scalar constant for one of its inputs (Note: this requires constexpr processing and not recommended for C++ due to support of lambdas and bind objects in the original variants).
- add init() and wait(obj) that are NO-OP functions

*** Modern C++ idioms from C++14 and C++17 Standards included**

- removed tag_type from Matrix, Vector and "view" classes in favor of modern traits idioms external to the classes.
- Using SFINAE techniques for template function overload resolution and also template type restriction (e.g. std::enable_if, etc) both for operators and operations.
- used tag dispatch to resolve overloaded template functions (e.g. apply and assign operations)
- removed typedefs (like result_type) from within operators to be consistent with C++17 changes to <functional> and to support use of lambdas in place of predefined operator functors. Replaced uses with decltype() and declval() where necessary.
- Replaced typedefs with using.
- range based for-loops used where possible in implementation
- structured bindings used to access elements of tuples.
- Replaced calls to std::vector's insert() and push_back() methods with emplace() and emplace_back() where possible.
- Removed GraphBLAS::BinaryOp_Bind2nd functor in favor of C++ STL's std::bind().
- Replaced std::conditional<>::type with std::conditional_t<>, std::is_same<>::value with std::is_same_v<>, etc.

* Refactored view classes: Changed design of view classes to lightweight wrappers and added support for structure-only views. The "backend" view classes have been removed. Nearly all friend declarations are no longer necessary and have been removed.

* "sequential" platform

- Numerous performance improvements applied.
- modified to replace code paths that accessed columns of the row-major storage matrix with row access only (improves performance).
- backend NoMask class removed.
- backend View classes removed.
- backend utility functions removed (refactored into classes).

*** New platform: "optimized_sequential" included that is a work in progress to improve the performance of the "sequential" platform. Currently mxm operation is implemented with more efficient code branches (72 of them).**

*** Algorithms:**

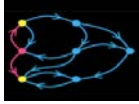
- Updated algorithms replace use of GraphBLAS::BinaryOp_Bind2nd with std::bind().
- Updated algorithms to use structure only masks where needed and removed 1-based indexing.
- Improved performance of index-of() operation for single-source parent BFS using a dense vector (of index values) and eWiseMult (with First or Second operator).
- Added single-source level BFS algorithm
- Added Louvain clustering algorithm
- Removed an erroneous implementation of maxflow algorithm (maxflow_ford_fulk2)
- Two new triangle counting algorithm variants:
  1) |L .* (L +.* U)|,
  2) |L .* (L +.* L)| (no transpose).
- Removed all experimental (FLAME-based) approaches to triangle counting.
- Switched to lambda (as a binary operator to eWiseMult) for computing random values in MIS algorithm so that random number generator objects are local variables.
- Changed how nsver is subtracted from result vector in BC algorithm (vertex_betweenness_centrality_batch_alt_trans_v2())
- Added direct translation of all algorithms in the appendix of the C API Specification v 1.3.0.

*** Test and demo programs:**

- augmented tests for operators (new and old) found in algebra.hpp
- augmented tests for mxm to cover all combinations of input transposes and mask views. Augmented other tests to cover more combinations.
- added tests for vector/matrix methods and new operations.
- removed tests for TransposeView (aka single argument transpose) and ComplementView.
- triangle_count_demo now sorts the vertices by degree (flame versions removed).
- new demo programs for appendix_algorithms, louvain clustering
- addition of mxm, mxv, and vxm timing demos to measure performance.

* Removing additional compiler warnings during build (some still exist from algebra.hpp).

GBTL C++ Signatures: MxM



$$C\langle \neg/sM, z \rangle = C \odot (A^T \oplus.\otimes B^T)$$

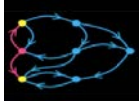
```
template<typename CMatrixT,
         typename MaskT,
         typename AccumT,
         typename SemiringT,
         typename AMatrixT,
         typename BMatrixT>
void mxm(CMatrixT       &C,
         MaskT    const &M,
         AccumT          accum,
         SemiringT       op,
         AMatrixT const &A,
         BMatrixT const &B,
         ReplaceEnum     z = MERGE);
```

Matrices:

- Templated 2-D “containers”:
  - Scalar template parameter for stored value type
  - Template parameters that *could* control type of data structures
  - Future: templatized by the index type
- Destination (C) is an in/out parameter
- Mask matrix
  - Is optional (use grb::NoMask)
  - Can be interpreted as a binary matrix or as structure only
- Input matrices: A and B

```
template <typename ScalarT, typename... StorageTagsT>
class Matrix;
```

GBTL C++ Signatures: MxM



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```
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         typename MaskT,
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         typename SemiringT,
         typename AMatrixT,
         typename BMatrixT>
void mxm(CMatrixT       &C,
         MaskT    const &M,
         AccumT          accum,
         SemiringT       op,
         AMatrixT const &A,
         BMatrixT const &B,
         ReplaceEnum     z = MERGE);
```
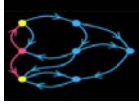

```
// call site
mxm(C, NoMask(), NoAccum(), ArithmeticSemiring<float>(),
    transpose(A), transpose(B));
```

Transpose:

- A and B input matrices can be optionally transposed
- Specified at the call site with lightweight wrappers

GBTL C++ Signatures: MxM



```
template<typename CMatrixT,
         typename MaskT,
         typename AccumT,
         typename SemiringT,
         typename AMatrixT,
         typename BMatrixT>
void mxm(CMatrixT       &C,
         MaskT    const &M,
         AccumT          accum,
         SemiringT       op,
         AMatrixT const &A,
         BMatrixT const &B,
         ReplaceEnum     z = MERGE);
```

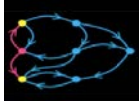
$$C\langle \neg/sM, z \rangle = C \odot (A^T \oplus.\otimes B^T)$$

Operators:

- "Semiring" ($\oplus.\otimes$) is a pair of operators templatized by two input and one output type
  - "Multiply" ($\otimes$) is a binary operator compatible templatized by two different input types and one output type
  - "Add" ($\oplus$) is a commutative monoid operator (binaryop with identity) templatized by one type for both inputs and outputs
  - E.g. Arithmetic semiring: $\oplus$ = +, $\otimes$ = x
- Accumulate ($\odot$) with the current output matrix:
  - Is optional (use grb::NoAccumulate)
  - Binary operator (e.g. std::plus<float>, grb::Plus<float, int, float>)
  - If it is not commutative and/or has no identity, unintuitive results can happen.

```
template<class D1, class D2=D1, class D3=D1>
class ArithmeticSemiring
{
   D3 add(D3 a, D3 b) const;
   D3 mult(D1 a, D2 b) const;
   D3 zero() const;
};
```

GBTL C++ Signatures: MxM



$$C\langle \neg/s M, z \rangle = C \odot (A^T \oplus . \otimes B^T)$$

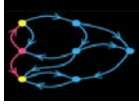
```
template<typename CMatrixT,
         typename MaskT,
         typename AccumT,
         typename SemiringT,
         typename AMatrixT,
         typename BMatrixT>
void mxm(CMatrixT       &C,
         MaskT    const &M,
         AccumT          accum,
         SemiringT       op,
         AMatrixT const &A,
         BMatrixT const &B,
         ReplaceEnum     z = MERGE);
```

```
// call site
mxm(C, complement(structure(M)), NoAccum(),
    ArithmeticSemiring<float>(),
    A, B);
```

Mask and its modifiers:

- Controls which values are written to the output matrix
  - Stucture mask (s):
    - if a value is stored at M(i,j), then C(i,j) can be written.
  - Value mask (default):
    - if a value is stored at M(i,j) *and* its evaluates to true, then C(i,j) can be written.
  - Complement (¬):
    - Inverts the logic of either of the first two
- Structure and complement specified at the call site with wrappers
  - none (value mask)
  - structure(M)
  - complement(M) – complement of value mask
  - complement(structure(M)) – complement of structure mask

GBTL C++ Signatures: MxM



$$C\langle \neg/sM, z \rangle = C \odot (A^T \oplus.\otimes B^T)$$

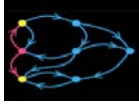
```
template<typename CMatrixT,
         typename MaskT,
         typename AccumT,
         typename SemiringT,
         typename AMatrixT,
         typename BMatrixT>
void mxm(CMatrixT       &C,
         MaskT    const &M,
         AccumT          accum,
         SemiringT       op,
         AMatrixT const &A,
         BMatrixT const &B,
         ReplaceEnum     z = MERGE);
```

```
// call site
mxm(C, M, NoAccum(), ArithmeticSemiring<float>(),
    A, B, REPLACE);
```

Replace (z) vs. Merge :

- Only when mask is present
- Controls what happens to non-masked values
  - Replace (z): The value in C will be removed (annihilated).
  - Merge (default): The location in C will be untouched (stored value or not).

Definition of mxm (from spec)



$\mathbf{C\langle \neg/s\mathbf{M},z\rangle = C \odot (A^T \oplus.\otimes B^T)}$

$\mathbf{T = A^T \oplus.\otimes B^T}$

$$T_{ij} = \bigoplus_{k \in \mathbf{ind}(\widetilde{\mathbf{A}}(i,:)) \cap \mathbf{ind}(\widetilde{\mathbf{B}}(:,j))} (\widetilde{\mathbf{A}}(i,k) \otimes \widetilde{\mathbf{B}}(k,j)),$$

After optional transpose

Treated as sparse matrices

$\mathbf{Z = [C \odot\ ]\ T}$

$$Z_{ij} = \widetilde{\mathbf{C}}(i,j) \odot \widetilde{\mathbf{T}}(i,j), \text{ if } (i,j) \in (\mathbf{ind}(\widetilde{\mathbf{T}}) \cap \mathbf{ind}(\widetilde{\mathbf{C}})),$$

$$Z_{ij} = \widetilde{\mathbf{C}}(i,j), \text{ if } (i,j) \in (\mathbf{ind}(\widetilde{\mathbf{C}}) - (\mathbf{ind}(\widetilde{\mathbf{T}}) \cap \mathbf{ind}(\widetilde{\mathbf{C}}))),$$

$$Z_{ij} = \widetilde{\mathbf{T}}(i,j), \text{ if } (i,j) \in (\mathbf{ind}(\widetilde{\mathbf{T}}) - (\mathbf{ind}(\widetilde{\mathbf{T}}) \cap \mathbf{ind}(\widetilde{\mathbf{C}}))),$$

Only if $\odot$ is defined

$\mathbf{C\langle \neg/s\mathbf{M},z\rangle = Z}$

After optional structure or complement

Replace → $\mathbf{L}(\mathsf{C}) = \{(i,j,Z_{ij}) : (i,j) \in (\mathbf{ind}(\widetilde{\mathbf{Z}}) \cap \mathbf{ind}(\widetilde{\mathbf{M}}))\}.$

Merge → $\mathbf{L}(\mathsf{C}) = \{(i,j,C_{ij}) : (i,j) \in (\mathbf{ind}(\mathsf{C}) \cap \mathbf{ind}(\widetilde{\neg\mathbf{M}}))\} \cup \{(i,j,Z_{ij}) : (i,j) \in (\mathbf{ind}(\widetilde{\mathbf{Z}}) \cap \mathbf{ind}(\widetilde{\mathbf{M}}))\}.$

Optimizing mxm in GBTL

$$C\langle \neg/sM, z \rangle = C \odot (A^T \oplus.\otimes B^T)$$
$$T = A^T \oplus.\otimes B^T$$
$$Z = [C \odot]\ T$$
$$C\langle \neg/sM, z \rangle = Z$$

- 72 different code paths investigated:
  - Transpose either input matrix (4 options, columns)
  - Optional write mask with optional complement or structure (5 options)
    - NoMask, Mask, StructMask, CompMask, CompStructMask
  - Optional accumulate with output matrix (2 options)
  - Merge/Replace semantics (2 options, when mask used)

¬/sM	⊙	z	A*B	A*B^T	A^T*B	A^T*B^T
NoMask	NoAccum	X	C := A*B	C := A*B′	C := A'*B	C := A'*B'
NoMask	Accum	X	C := C + (A*B)	C := C + (A*B′)	C := C + (A'*B)	C := C + (A'*B′)
Mask	NoAccum	REPLACE	C := M .* (A*B)	C := M .* (A*B′)	C := M .* (A'*B)	C := M .* (A'*B′)
Mask	NoAccum	MERGE	C := [¬M .* C] U {M .* (A*B)}	C := [¬M .* C] U {M .* (A*B′)}	C := [¬M .* C] U {M .* (A'*B)}	C := [¬M .* C] U {M .* (A'*B′)}
Mask	Accum	REPLACE	C := [M .* C] + {M .* (A*B)}	C := [M .* C] + {M.* (A*B′)}	C := [M .* C] + {M.* (A'*B)}	C := [M .* C] + {M.* (A'*B′)}
Mask	Accum	MERGE	C := [¬M .* C] U {[M .* C] + M .* (A*B)}	C := [¬M .* C] U {[M .* C] + M.* (A*B′)}	C := [¬M .* C] U {[M .* C] + M.* (A'*B)}	C := [¬M .* C] U {[M .* C] + M.* (A'*B′)}
StructMask	NoAccum	REPLACE	C := s(M) .* (A*B)	C := s(M) .* (A*B′)	C := s(M) .* (A'*B)	C := s(M) .* (A'*B′)
StructMask	NoAccum	MERGE	C := [¬s(M) .* C] U {s(M) .* (A*B)}	C := [¬s(M) .* C] U {s(M) .* (A*B′)}	C := [¬s(M) .* C] U {s(M) .* (A'*B)}	C := [¬s(M) .* C] U {s(M) .* (A'*B′)}
StructMask	Accum	REPLACE	C := [s(M) .* C] + {s(M) .* (A*B)}	C := [s(M) .* C] + {s(M).* (A*B′)}	C := [s(M) .* C] + {s(M).* (A'*B)}	C := [s(M) .* C] + {s(M).* (A'*B′)}
StructMask	Accum	MERGE	C := [¬s(M) .* C] U {[s(M) .* C] + s(M) .* (A*B)}	C := [¬s(M) .* C] U {[s(M) .* C] + s(M).* (A*B′)}	C := [¬s(M) .* C] U {[s(M) .* C] + s(M).* (A'*B)}	C := [¬s(M) .* C] U {[s(M) .* C] + s(M).* (A'*B′)}
CompMask	NoAccum	REPLACE	C := ¬M .* (A*B)	C := ¬M .* (A*B′)	C := ¬M .* (A'*B)	C := ¬M .* (A'*B′)
CompMask	NoAccum	MERGE	C := [M .* C] U¬M .* (A*B)	C := [M .* C] U {¬M .* (A*B′)}	C := [M .* C] U {¬M .* (A'*B)}	C := [M .* C] U {¬M .* (A'*B′)}
CompMask	Accum	REPLACE	C := (¬M .* C) + ¬M .* (A*B)	C := {(¬M .* C) + ¬M .* (A*B′)}	C := {(¬M .* C) + ¬M .* (A'*B)}	C := {(¬M .* C) + ¬M .* (A'*B′)}
CompMask	Accum	MERGE	C := [M .* C] U {(¬M .* C) + ¬M .* (A*B)}	C := [M .* C] U {(¬M .* C) + ¬M .* (A*B′)}	C := [M .* C] U {(¬M .* C) + ¬M .* (A'*B)}	C := [M .* C] U {(¬M .* C) + ¬M .* (A'*B′)}
CompStructMask	NoAccum	REPLACE	C := ¬s(M) .* (A*B)	C := ¬s(M) .* (A*B′)	C := ¬s(M) .* (A'*B)	C := ¬s(M) .* (A'*B′)
CompStructMask	NoAccum	MERGE	C := [s(M) .* C] U¬s(M) .* (A*B)	C := [s(M) .* C] U {¬s(M) .* (A*B′)}	C := [s(M) .* C] U {¬s(M) .* (A'*B)}	C := [s(M) .* C] U {¬s(M) .* (A'*B′)}
CompStructMask	Accum	REPLACE	C := (¬s(M) .* C) + ¬s(M) .* (A*B)	C := {(¬s(M) .* C) + ¬s(M) .* (A*B′)}	C := {(¬s(M) .* C) + ¬s(M) .* (A'*B)}	C := {(¬s(M) .* C) + ¬s(M) .* (A'*B′)}
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$$T = A \oplus.\otimes B$$
$$Z = [C \odot]\ T$$
$$C\langle M\rangle = Z$$

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Mask	NoAccum	REPLACE	C := M .* (A*B)	C := M .* (A*B′)	C := M .* (A'*B)	C := M .* (A'*B′)
Mask	NoAccum	MERGE	C := [¬M .* C] U {M .* (A*B)}	C := [¬M .* C] U {M .* (A*B′)}	C := [¬M .* C] U {M .* (A'*B)}	C := [¬M .* C] U {M .* (A'*B′)}
Mask	Accum	REPLACE	C := [M .* C] + {M .* (A*B)}	C := [M .* C] + {M.* (A*B′)}	C := [M .* C] + {M.* (A'*B)}	C := [M .* C] + {M.* (A'*B′)}
Mask	Accum	MERGE	C := [¬M .* C] U {[M .* C] + M .* (A*B)}	C := [¬M .* C] U {[M .* C] + M.* (A*B′)}	C := [¬M .* C] U {[M .* C] + M.* (A'*B)}	C := [¬M .* C] U {[M .* C] + M.* (A'*B′)}
StructMask	NoAccum	REPLACE	C := s(M) .* (A*B)	C := s(M) .* (A*B′)	C := s(M) .* (A'*B)	C := s(M) .* (A'*B′)
StructMask	NoAccum	MERGE	C := [¬s(M) .* C] U {s(M) .* (A*B)}	C := [¬s(M) .* C] U {s(M) .* (A*B′)}	C := [¬s(M) .* C] U {s(M) .* (A'*B)}	C := [¬s(M) .* C] U {s(M) .* (A'*B′)}
StructMask	Accum	REPLACE	C := [s(M) .* C] + {s(M) .* (A*B)}	C := [s(M) .* C] + {s(M).* (A*B′)}	C := [s(M) .* C] + {s(M).* (A'*B)}	C := [s(M) .* C] + {s(M).* (A'*B′)}
StructMask	Accum	MERGE	C := [¬s(M) .* C] U {[s(M) .* C] + s(M) .* (A*B)}	C := [¬s(M) .* C] U {[s(M) .* C] + s(M).* (A*B′)}	C := [¬s(M) .* C] U {[s(M) .* C] + s(M).* (A'*B)}	C := [¬s(M) .* C] U {[s(M) .* C] + s(M).* (A'*B′)}
CompMask	NoAccum	REPLACE	C := ¬M .* (A*B)	C := ¬M .* (A*B′)	C := ¬M .* (A'*B)	C := ¬M .* (A'*B′)
CompMask	NoAccum	MERGE	C := [M .* C] U¬M .* (A*B)	C := [M .* C] U {¬M .* (A*B′)}	C := [M .* C] U {¬M .* (A'*B)}	C := [M .* C] U {¬M .* (A'*B′)}
CompMask	Accum	REPLACE	C := (¬M .* C) + ¬M .* (A*B)	C := {(¬M .* C) + ¬M .* (A*B′)}	C := {(¬M .* C) + ¬M .* (A'*B)}	C := {(¬M .* C) + ¬M .* (A'*B′)}
CompMask	Accum	MERGE	C := [M .* C] U {(¬M .* C) + ¬M .* (A*B)}	C := [M .* C] U {(¬M .* C) + ¬M .* (A*B′)}	C := [M .* C] U {(¬M .* C) + ¬M .* (A'*B)}	C := [M .* C] U {(¬M .* C) + ¬M .* (A'*B′)}
CompStructMask	NoAccum	REPLACE	C := ¬s(M) .* (A*B)	C := ¬s(M) .* (A*B′)	C := ¬s(M) .* (A'*B)	C := ¬s(M) .* (A'*B′)
CompStructMask	NoAccum	MERGE	C := [s(M) .* C] U¬s(M) .* (A*B)	C := [s(M) .* C] U {¬s(M) .* (A*B′)}	C := [s(M) .* C] U {¬s(M) .* (A'*B)}	C := [s(M) .* C] U {¬s(M) .* (A'*B′)}
CompStructMask	Accum	REPLACE	C := (¬s(M) .* C) + ¬s(M) .* (A*B)	C := {(¬s(M) .* C) + ¬s(M) .* (A*B′)}	C := {(¬s(M) .* C) + ¬s(M) .* (A'*B)}	C := {(¬s(M) .* C) + ¬s(M) .* (A'*B′)}
CompStructMask	Accum	MERGE	C := [s(M) .* C] U {(¬s(M) .* C) + ¬s(M) .* (A*B)}	C := [s(M) .* C] U {(¬s(M) .* C) + ¬s(M) .* (A*B′)}	C := [s(M) .* C] U {(¬s(M) .* C) + ¬s(M) .* (A'*B)}	C := [s(M) .* C] U {(¬s(M) .* C) + ¬s(M) .* (A'*B′)}

Optimizing mxm in GBTL

Mask+Accum+Merge+NoTranspose, assuming rowMajor storage

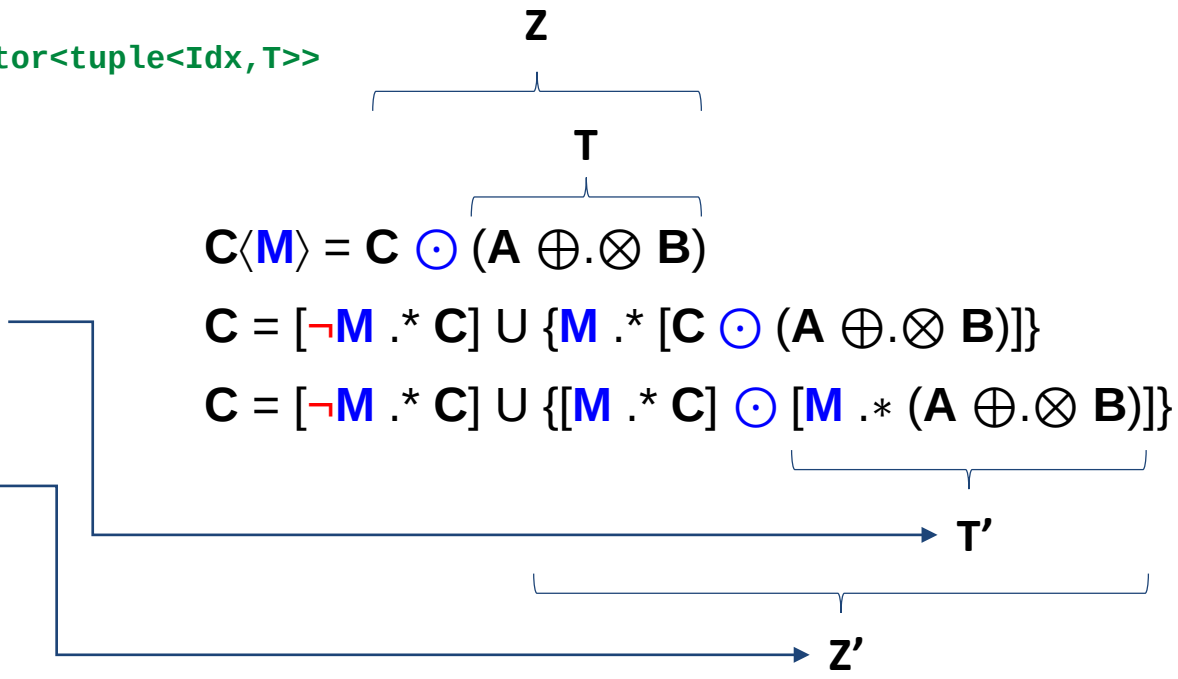
```
SparseRow<T> t, z, c; // like vector<tuple<Idx,T>>

for i=0..A.rows {
  t.clear()

  for (k, a_ik) in A[i] {
    // masked_axpy
    t = t ⊕ [M[i].*(a_ik ⊗ B[k])]
  }

  // masked_accum
  z = M[i].*C[i] ⊙ t

  // masked_merge
  c = (!M[i].*C[i]) U z
  C[i] = c  // set_row
}
```

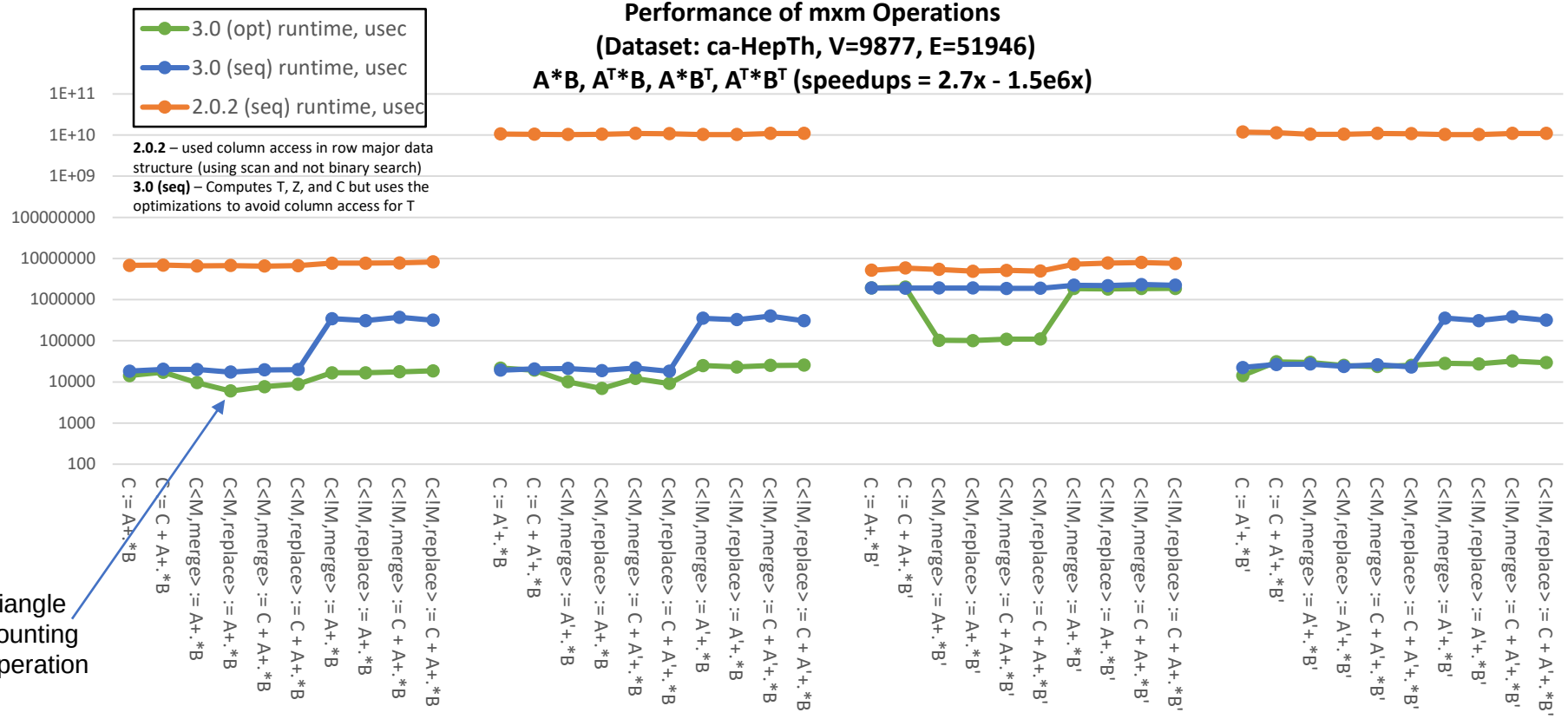


Recurring Sparse Primitives

- Sparse “axpy” using a semiring :
  - Sparse set union with “plus” operation, $\oplus$, where elements overlap, one set is scaled.
  - $\mathbf{t} = \mathbf{t} \oplus (a_{ik} \otimes \mathbf{B}[k])$
- Masked sparse “axpy” using a semiring:
  - Like “sparse axpy” where elements are filtered by the binary mask, M:
  - $\mathbf{t} = \mathbf{M}[i] .* [\mathbf{t} \oplus (a_{ik} \otimes \mathbf{B}[k])]$
- Sparse accumulate:
  - Sparse set union applying an accumulate operation, $\odot$, where elements intersect
  - $\mathbf{C}[i] = \mathbf{C}[i] \odot \mathbf{t}$
- Masked sparse accumulate:
  - $\mathbf{C}[i] = (\mathbf{M}[i] .* \mathbf{C}[i]) \odot \mathbf{t}$
  - where **t** has already been masked by **M**[i] (as in masked sparse axpy).

Backups

GBTL 3.0 Performance Comparison (mxm, 40 paths)



triangle counting operation

Math	C := A (+.*) B	C := C (+) A (+.*) B	C := M .* [A (+.*) B]	C := [!M.*C] U {M.*[A(+.*)B]}	C := [M.*C] (+) {M.*[A(+.*)B]}	C := [!M.*C] U {[M.*C] (+) M.*[A(+.*)B
Func name	**NoMask_NoAccum**	**NoMask_Accum**	**Mask_NoAccum, REPLACE**	**Mask_NoAccum, MERGE**	**Mask_Accum, REPLACE**	**Mask_Accum, MERGE**
Inputs	A, B, (+.*)	C, A, B, (+.*) (+)	M, A, B, (+.*)	M, A, B, (+.*)	C, M, A, B, (+.*), (+)	C, M, A, B, (+.*), (+)
Outputs	C	C	C	C	C	C
early exits	if (A | B are empty)	if (A | B are empty)	if (A|B|M are empty)	if M.empty	if M.empty	if M.empty
	clear C	return	clear C	return	clear C	return
axpy kernel	for i=0..A.rows	for i=0..A.rows	for i=0..A.rows	for i=0..A.rows	for i=0..A.rows	for i=0..A.rows
					if M[i] empty skip i	if M[i] empty skip i
	T.clear	T.clear	T.clear	T.clear	T.clear	T.clear
			if M[i] not empty	if M[i] not empty		
	for a_ik in A[i]	for a_ik in A[i]	for a_ik in A[i]	for a_ik in A[i]	for a_ik in A[i]	for a_ik in A[i]
	if B[k] not empty	if B[k] not empty				
	//axpy	//axpy	//masked_axpy(Mi)	//masked_axpy(Mi)	//masked_axpy(Mi)	//masked_axpy(Mi)
Compute T	T += (a_ik*B[k])	T += (a_ik*B[k])	T += M[i] .* a_ik*B[k]	T += M[i] .* a_ik*B[k]	T += M[i] .* a_ik*B[k]	T += M[i] .* a_ik*B[k]
					// maskedAccum	// masked_accum
Compute Z					Z = M[i].*C[i] + T	Z = M[i].*C[i] + T
				//maskedMerge		//masked_merge
		if T not empty		Z = (!M[i].*C[i]) U T		Z = (!M[i].*C[i]) U Z
Compute C	C[i] = T //setRow	C[i] += T //mergeRow	C[i] = T //setRow	C[i] = Z // setRow	C[i] = Z // setRow	C[i] = Z // setRow