



Characterization of perturbation dynamics in unsteady flows

Kunihiko Taira
FLORIDA STATE UNIV TALLAHASSEE

04/08/2020
Final Report

DISTRIBUTION A: Distribution approved for public release.

Air Force Research Laboratory
AF Office Of Scientific Research (AFOSR)/ RTA1
Arlington, Virginia 22203
Air Force Materiel Command

DISTRIBUTION A: Distribution approved for public release

REPORT DOCUMENTATION PAGE					Form Approved OMB No. 0704-0188	
The public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing the burden, to Department of Defense, Executive Services, Directorate (0704-0188). Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number. PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ORGANIZATION.						
1. REPORT DATE (DD-MM-YYYY) 24-08-2020		2. REPORT TYPE Final Performance		3. DATES COVERED (From - To) 01 Dec 2017 to 30 Nov 2019		
4. TITLE AND SUBTITLE Characterization of perturbation dynamics in unsteady flows				5a. CONTRACT NUMBER		
				5b. GRANT NUMBER FA9550-18-1-0040		
				5c. PROGRAM ELEMENT NUMBER 61102F		
6. AUTHOR(S) Kunihiko Taira, Kourosh Shoele, Louis Cattafesta				5d. PROJECT NUMBER		
				5e. TASK NUMBER		
				5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) FLORIDA STATE UNIV TALLAHASSEE 874 TRADITIONS WAY TALLAHASSEE, FL 32306 - 0001 US				8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) AF Office of Scientific Research 875 N. Randolph St. Room 3112 Arlington, VA 22203				10. SPONSOR/MONITOR'S ACRONYM(S) AFRL/AFOSR RTA1		
				11. SPONSOR/MONITOR'S REPORT NUMBER(S) AFRL-AFOSR-VA-TR-2020-0150		
12. DISTRIBUTION/AVAILABILITY STATEMENT A DISTRIBUTION UNLIMITED: PB Public Release						
13. SUPPLEMENTARY NOTES						
14. ABSTRACT We report on a joint theoretical, computational, and experimental effort to analyze perturbation dynamics about unsteady base states caused by a vortical wake, unsteady separation, and unsteady surface motion. From this project, we have developed analysis techniques to characterize perturbation dynamics about unsteady flows and demonstrated their uses for bluff body flow and fluid-structure interaction problems. Because the analyses of such problems require handling of high-degree of freedom, model reduction techniques were considered through modal analysis, phase-reduction, and randomized techniques. The current work paves the way for advancing our knowledge of unsteady flows with potential applications for accurate estimation, modeling, and control.						
15. SUBJECT TERMS unsteady aerodynamics, resolvent analysis, global stability analysis, fluid-surface interaction, turbulent flow						
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT UU	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON ABATE, GREGG	
a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified			19b. TELEPHONE NUMBER (Include area code) 703-588-1779	

FINAL REPORT: Characterization of Perturbation Dynamics in Unsteady Flows

for Air Force Office of Scientific Research
(FA9550-18-1-0040, Dec 01, 2017 to Nov 30, 2019)
Program Managers: Drs. Gregg Abate & Douglas Smith

Authors: Kunihiko Taira^{†1,2}, Kourosh Shoele^{‡2}, Louis N. Cattafesta^{¶2},
Kai Zhang^{1,2}, Tso-Kang Wang², Ross Richardson²,
Chi-An Yeh^{1,2}, and Jean H. M. Ribeiro^{1,2}

¹Department of Mechanical and Aerospace Engineering, University of California, Los Angeles, CA 90095, USA

²Department of Mechanical Engineering, Florida State University, Tallahassee, FL 32310, USA

Abstract: We report on a joint theoretical, computational, and experimental effort to analyze perturbation dynamics about unsteady base states caused by a vortical wake, unsteady separation, and unsteady surface motion. From this project, we have developed analysis techniques to characterize perturbation dynamics about unsteady flows and demonstrated their uses for bluff body flow and fluid-structure interaction problems. Because the analyses of such problems require handling of high-degree of freedom, model reduction techniques were considered through modal analysis, phase-reduction, and randomized techniques. The current work paves the way for advancing our knowledge of unsteady flows with potential applications for accurate estimation, modeling, and control.

CONTENTS

1	Introduction	2
2	Randomized linear stability analysis	3
2.1	Linear stability analysis for time-varying flows	4
2.2	Randomized linear stability analysis	5
2.3	Randomized singular value decomposition for nonmodal stability analysis	5
2.4	Exact dynamic mode decomposition for modal stability analysis	6
2.5	Example: linear stability analysis of pulsatile Poiseuille flow	8
2.6	Summary	10
3	Randomized resolvent analysis	10
3.1	Introduction	10
3.2	Approach	12
3.3	Randomized resolvent analysis of turbulent post-stall flow	17
3.4	Summary	24

† PI, ktaira@seas.ucla.edu

‡ co-PI, kshoele@fsu.edu

¶ co-PI, lcattafesta@eng.famu.fsu.edu

4	Resolvent analysis on the origin of transonic buffet	25
4.1	Introduction	25
4.2	Problem setup	26
4.3	Results	28
4.4	Summary	32
5	Phase reduction analysis	33
5.1	Introduction	33
5.2	Phase-reduction analysis	34
5.3	Phase-response analysis of unsteady cylinder wake	37
5.4	Summary	41
6	Dynamics of a separation bubble subject to unsteady conditions: The computational Work	42
6.1	Introduction	42
6.2	Methodology	42
6.3	Results	44
6.4	Summary	53
7	Dynamics of a separation bubble subject to unsteady conditions: The Experimental Work	54
7.1	Approach and Methodology	54
7.2	Experimental Implementation	55
7.3	Results	60
7.4	Summary	67
8	Concluding remarks	67
9	Publication Record	67

1. Introduction

Hydrodynamic stability analysis has enabled the fluid mechanics community gain deep insights into fundamental physics of transition, mixing, and separation in various types of fluid flows. The findings from this field of research have allowed us to reveal the emergence of flow unsteadiness and predict dominant frequencies and coherent structures, which are useful for designing fluids-based engineering systems in a range of aerospace, chemical, and marine applications (Schmid & Henningson 2001; Drazin & Reid 1981; Theofilis 2011a; Joslin & Miller 2009a). These stability analysis techniques are founded on dynamical systems with extensive theoretical and computational undertakings. The development of mathematical tools and improvements in computational resources have been crucial in enhancing the capabilities of stability analysis (Theofilis 2011a; Trefethen & Embree 2005).

While the community has made significant strides in analyzing fluid flows with respect to steady base flows, examination of the complex dynamics with respect to time-varying base states remains a challenge. The difficulty stems from the lack of theories and analysis techniques available for time-varying base states or any guidelines on how to choose the time-varying base state itself. At the moment, the de facto tool available for dynamical systems or unsteady fluid flows is the use of Floquet analysis (Floquet 1883; Perko 2013; Jordan & Smith 1999).

We aim to revisit and extend the applications of unsteady analysis techniques to flows with large unsteadiness in the time-varying base state. To achieve this goal, we proposed a complementary theoretical, computational, and experimental examination of perturbation dynamics about high-amplitude time-varying base states. In particular, we considered the resolvent analysis and phase reduction analysis as well as randomized techniques.

This present effort was undertaken over a performance period of 12 months plus an additional 12 months of NCE. We have developed computational and experimental capabilities to establish the foundation to study a range of unsteady fluid flows. We leveraged the available resources of the PIs and the accumulated expertise to ensure accelerated research progress within the limited performance period. The time-varying dynamics of particular interest in this study were with respect to the unsteady base states caused by vortical wakes, unsteady separation, and unsteady surface motion (fluid-structure interaction).

One of the unique aspects of this work was to uncover perturbation dynamics shared between the unsteady fluid flow and structural dynamics. This is particularly important as we consider ways to enhance (or mitigate) the constructive (destructive) interaction between the flow and the structure (Cheng *et al.* 2017; Zhou *et al.* 2017). Through this supported project, we have been able to expand the fluid-structure interaction analysis capabilities to understand both the fluid flows and structures in unsteady operating conditions.

In what follows, we describe key outcomes from our work. Each section essentially constitutes a report/manuscript for each sub-effort. They can be read in any order. In Sections 2 and 3, we describe the use of randomized techniques to perform linear stability and resolvent analysis, which are suitable for analyzing large scale problems. We also consider its application in Section 4 to uncover the source of buffeting. In section 5, we introduce the use of phase-reduction analysis to uncover synchronization physics of fluid flows about unsteady base states. A fluid-structure interaction analysis at FSU from computational and experimental approaches are described in detail in Sections 6 and 7. Brief remarks are provided in Section 8 with a list of publications in Section 9.

2. Randomized linear stability analysis

Hydrodynamic stability analysis has enabled fluid mechanics community to gain deep insights into fundamental physics of transition, mixing and separation in various types of fluid flows. The findings from this field of research have allowed us to reveal the emergence of flow unsteadiness and predict dominant frequencies and coherent structures, which are useful for designing fluids-based engineering systems in a range of aerospace, chemical, and marine applications (Schmid & Henningson 2012; Drazin & Reid 2004; Joslin & Miller 2009*b*; Criminale *et al.* 2018).

Linear stability analysis of fluid flows can be divided into two major branches, modal stability analysis (Theofilis 2003, 2011*b*), and nonmodal stability analysis (Farrell & Ioannou 1996*a,b*; Schmid 2007). Modal stability analysis employs the assumption of an exponential time dependence of the perturbation developing on the base flow. This allows the transformation of the initial-value problem into an eigenvalue problem. The computed eigenvalues are interpreted as the growth rates with the corresponding frequencies. The base flow is labeled unstable if one of the growth rates is larger than zero. On other hand, the nonmodal approach is more general in that the exponential growth assumption is not adopted, and the evolution of the perturbation is treated as an initial-value problem as is. This type of analysis allows to track the short-time transient amplification of perturbation energy, which could lead to the destabilization of the system even though the base flow is linearly stable in the asymptotic sense. For this reason, the nonmodal stability analysis is also referred to as transient growth analysis.

A majority of the linear stability analysis have been conducted for the time-invariant base flows. For this type of flows, both the modal and nonmodal approaches can be applied to study the asymptotic and transient perturbation dynamics. However, configurations of practical interest exist where the base flow also varies with time. For the analysis of linear time-varying system, much of the physically relevant perturbation dynamics occurs on timescales

shorter than would be required for achieving $t \rightarrow \infty$ asymptotics, so that the framework of transient growth analysis can be adopted. On the other hand, modal stability theory can not be generalized to such flows because the normal modes do not exist when the system is inhomogeneous in time (Farrell & Ioannou 1996b). The only exception is the case of time-periodic flows, where the system can be transformed to a time-invariant one via the Floquet transform, and the conventional modal stability analysis (referred to as Floquet stability analysis) can be applied.

Both the modal and nonmodal stability analysis requires handling the linearized NS operator. The conventional matrix forming method constructs the linearized NS operator explicitly, which could be computationally expensive for large systems. This situation is aggravated for the time-varying flows, for which linearized NS operator needs to be constructed at each time step. In contrast to the huge computational cost associated with the linear stability analysis, only the leading eigenmodes or singular vectors are of interest in general.

Recently, randomized numerical linear algebra has emerged as an effective approach to reduce a large matrix to its low-order representation (Halko *et al.* 2011; Drineas & Mahoney 2016; Tu *et al.* 2014; Tropp *et al.* 2017), for applications such as big data compression and data transfer. Inspired by these works, we proposed a framework based on the randomized linear algebra to perform nonmodal (transient growth) analysis for base flows with arbitrary time dependency and modal (Floquet) stability analysis for time-periodic flows.

2.1. Linear stability analysis for time-varying flows

The development of small-amplitude perturbations superimposed upon a given flow is described the linearized Navier Stokes equation, which is written compactly as

$$\frac{d\mathbf{q}}{dt} = \mathbf{L}(t)\mathbf{q}, \quad \mathbf{q} = \mathbf{q}_0. \quad (2.1)$$

Here, $\mathbf{q}$ represents the perturbation with initial condition $\mathbf{q}_0$, and $\mathbf{L}(t)$ denotes the linearized Navier-Stokes (LNS) operator constructed from the base flow at time t . In general, arbitrary time dependency is assumed for $\mathbf{L}(t)$. The solution to equation (2.1) is given as

$$\mathbf{q}(t) = \mathbf{A}(t, t_0)\mathbf{q}_0, \quad (2.2)$$

where $\mathbf{A}(t, t_0)$ is the state transition matrix (STM), which advances the state of perturbation at t_0 to t .

The nonmodal stability analysis seeks the maximum growth of the perturbation from an optimal initial condition. This can be done by the singular value decomposition (SVD) of the STM,

$$\mathbf{A}(t, t_0) = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*. \quad (2.3)$$

Here, $\mathbf{\Sigma}$ is a diagonal matrix containing the growth of the perturbation (gain) ordered from largest to smallest. $\mathbf{V}$ provides a set of optimal initial conditions corresponding to the gain. $\mathbf{U}$ stores the output at t given the initial condition $\mathbf{V}$.

In the special case of time-periodic base flows with period T , i.e., $\mathbf{L}(t) = \mathbf{L}(t + T)$, modal (Floquet) stability analysis can also be applied by assuming the solution of the form $\mathbf{q}(t) = \exp(\lambda_j t)\tilde{\mathbf{q}}_j(t) + \text{c.c.}$, where $\tilde{\mathbf{q}}_j(t) = \tilde{\mathbf{q}}_j(t+T)$. Then, equation (2.2) can be transformed into an eigenvalue problem as

$$\mathbf{A}(T)\tilde{\mathbf{q}}_j = \mu_j\tilde{\mathbf{q}}_j, \quad \mu_j \equiv \exp(\lambda_j T), \quad (2.4)$$

in which μ_j is the Floquet multiplier, λ_j is the Floquet exponent, and the eigenmodes of $\mathbf{A}(T)$ are the Floquet modes evaluated at a specific temporal phase. Here, we use $\mathbf{A}(T)$ to denote $\mathbf{A}(t_i, t_j)$ with $t_i - t_j = T$. If there are any eigenvalues μ_j with $|\mu_j| > 1$, then the

system is asymptotically unstable. On the other hand, if every eigenvalue satisfies $|\mu_j| < 1$, then the system is linearly stable.

2.2. Randomized linear stability analysis

Essentially, the transient growth analysis of the time-varying flow requires the singular value decomposition of the STM $\mathbf{A}(t, t_0)$, while the Floquet analysis of periodic flows requires the eigen-decomposition of the STM $\mathbf{A}(T)$. One of the most common methods to compute the STM is a finite product of operator exponentials, i.e.,

$$\mathbf{A}(t, t_0) = \lim_{\Delta t \rightarrow 0} e^{\mathbf{L}(t_K)\Delta t} e^{\mathbf{L}(t_{K-1})\Delta t} \dots e^{\mathbf{L}(t_1)\Delta t} e^{\mathbf{L}(t_0)\Delta t}, \quad K = (t - t_0)/\Delta t. \quad (2.5)$$

where $\mathbf{L}(t_i)$ is the linearized NS operator constructed at time t_i . Note that since in general $\mathbf{L}(t_i)$ does not commute with $\mathbf{L}(t_j)$, so $e^{\mathbf{L}(t_i)\Delta t} e^{\mathbf{L}(t_j)\Delta t} \neq e^{[\mathbf{L}(t_i)\Delta t + \mathbf{L}(t_j)\Delta t]}$. There exists two major difficulties in directly forming the STM. Firstly, the calculation of $e^{\mathbf{L}(t_i)\Delta t}$ is cumbersome, despite the various methods to compute the matrix exponential (Moler & Van Loan 2003). Secondly, even after the matrix exponential is obtained, each matrix multiplication typically requires $2N^3 - N^2$ operations (N is the dimension of the matrix), which is prohibitive for large systems.

For the first difficulty, we note that each matrix exponential in equation (2.5) $e^{\mathbf{L}(t_i)\Delta t}$ advances the solution from t_i to t_{i+1} . Consider the Taylor expansion of the matrix exponential

$$e^{\mathbf{L}\Delta t} = \mathbf{I} + \mathbf{L}\Delta t + \frac{(\mathbf{L}\Delta t)^2}{2!} + \frac{(\mathbf{L}\Delta t)^3}{3!} + \dots \quad (2.6)$$

The matrix exponential acts like an explicit integration scheme with infinite-order of accuracy. Thus, we can use the popular integration techniques such as Euler, Runge-Kutta methods with small enough time-step to approximate the matrix exponential.

For the second difficulty, we proposed to use the randomized linear algebra for both the nonmodal and modal stability analysis of time-varying flows, as described in the following.

2.3. Randomized singular value decomposition for nonmodal stability analysis

We are inspired by the randomized SVD (rSVD) algorithm (Halko *et al.* 2011; Erichson *et al.* 2016) to circumvent the huge computational cost associated with the SVD on the state transition matrix $\mathbf{A}(t, t_0)$. The original rSVD algorithm is schematically shown in figure 1. The key of the randomized technique is to introduce a random matrix $\mathbf{X}$ of the size $N \times M$, where $M \ll N$, to obtain the sketched matrix $\mathbf{Y}$. Then, the orthogonal basis of $\mathbf{Y}$, stored in $\mathbf{Q}$, are premultiplied with the STM $\mathbf{A}$ to obtain $\mathbf{B}$, whose SVD results in M singular values and right singular vectors. The left singular vectors can be obtained by the multiplication of $\mathbf{Q}$ and $\tilde{\mathbf{U}}$.

We consider the state transition matrix $\mathbf{A}$ to be unavailable in an explicit manner. We notice that the first step in figure 1 $\mathbf{A}\mathbf{X} = \mathbf{Y}$ is equivalent to integrating equation (2.1) with the random initial conditions stored in the columns of $\mathbf{X}$. After the QR decomposition of $\mathbf{Y}$, the procedure $\mathbf{Q}^T \mathbf{A} = \mathbf{B}$ can be transposed as $\mathbf{A}^T \mathbf{Q} = \mathbf{B}^T$. Although we do not have direct access to $\mathbf{A}^T$, again the action of $\mathbf{A}^T$ on $\mathbf{Q}$ is equivalent to integrating the transposed system of equation (2.1) backward in time using the end condition stored in $\mathbf{Q}$. This can be proved simply by transposing equation (2.5) as

$$\mathbf{A}(t, t_0)^T = \lim_{\Delta t \rightarrow 0} e^{\mathbf{L}(t_0)^T \Delta t} e^{\mathbf{L}(t_1)^T \Delta t} \dots e^{\mathbf{L}(t_{K-1})^T \Delta t} e^{\mathbf{L}(t_K)^T \Delta t}, \quad (2.7)$$

and noting $[e^{(\mathbf{L}\Delta t)}]^T = e^{\mathbf{L}^T \Delta t}$. In summary, the rSVD-inspired transient growth analysis is presented in algorithm 1.

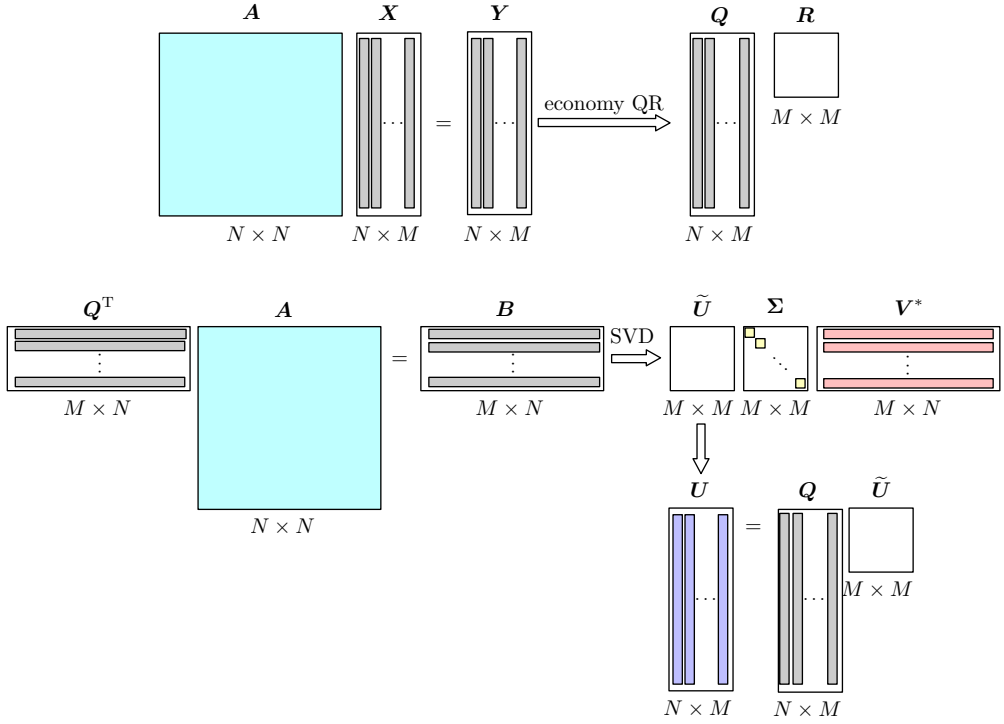


FIGURE 1. A scheme of the randomized singular value decomposition algorithm. Adapted from Erichson *et al.* (2016).

Algorithm 1 Randomized nonmodal stability analysis

```

1: function RNONMODAL
2:    $\mathbf{X} \leftarrow \text{randn}(N, M)$  ▷ random normal matrix generator
3:    $\mathbf{Y} \leftarrow \mathbf{X}$  ▷ integrate using equation (2.1)
4:    $(\mathbf{Q}, \sim) \leftarrow \text{qr}(\mathbf{Y}, 0)$  ▷ economy-sized QR decomposition
5:    $\mathbf{B} \leftarrow \mathbf{Q}$  ▷ integrate using the (2.7)
6:    $(\tilde{\mathbf{U}}, \Sigma, \mathbf{V}) \leftarrow \text{svd}(\mathbf{B}', \text{'econ'})$  ▷ reduced SVD decomposition
7:    $\mathbf{U} \leftarrow \mathbf{Q}\tilde{\mathbf{U}}$  ▷ recover the left singular vectors
8:   return  $(\mathbf{U}, \Sigma, \mathbf{V})$ 

```

2.4. Exact dynamic mode decomposition for modal stability analysis

The modal stability analysis requires an eigenvalue analysis of the STM. For this purpose, we prescribe M random initial conditions $\mathbf{q}_j(t_0)$, where $j = 1, 2, \dots, M$. Similar to the randomized nonmodal analysis, we integrate the linear time-varying system (2.1) with these initial conditions to the time of interest t . The resulting solutions are denoted as $\mathbf{q}_j(t)$. In order to obtain the eigenvalues and eigenmodes of the STM $\mathbf{A}(t, t_0)$, we construct two matrices as

$$\mathbf{X} = [\mathbf{q}_1(t_0) \quad \mathbf{q}_2(t_0) \quad \dots \quad \mathbf{q}_M(t_0)], \quad \mathbf{Y} = [\mathbf{q}_1(t) \quad \mathbf{q}_2(t) \quad \dots \quad \mathbf{q}_M(t)] \quad (2.8)$$

in which $\mathbf{X}$ stores the M initial conditions $\mathbf{q}_j(t_0)$, and $\mathbf{Y}$ stores the states of perturbations at time t . The two matrices are related by

$$\mathbf{Y} = \mathbf{A}(t, t_0)\mathbf{X}. \quad (2.9)$$

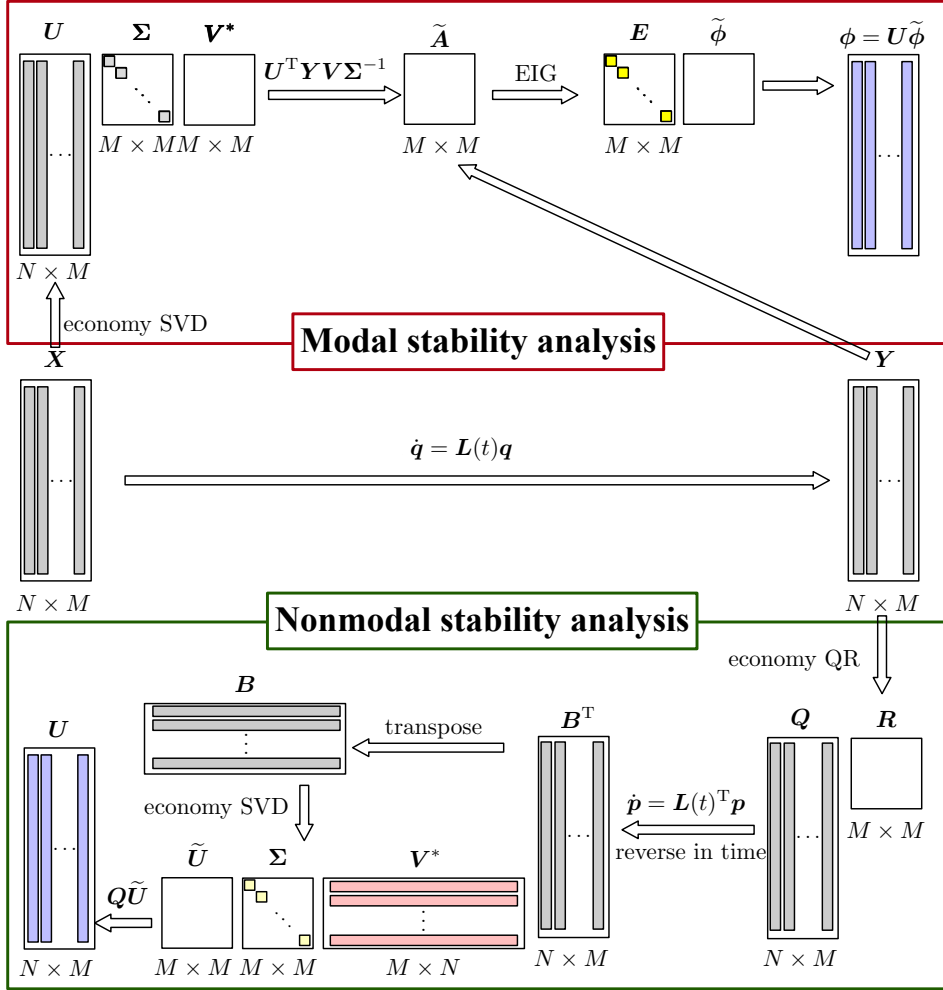


FIGURE 2. Scheme of modal and nonmodal stability analysis for time-varying system.

The leading eigenvalues of the STM $\mathbf{A}$ can be recovered by leveraging the simulated data $\mathbf{X}$ and $\mathbf{Y}$ using the exact DMD algorithm (Rowley *et al.* 2009; Schmid 2010; Tu *et al.* 2014; Kutz *et al.* 2016), as summarized in table 2. Similar methods have been applied to a number of structural engineering applications (Liu 1997, 1999). Although this algorithm should work for arbitrary pairs of t_1 and t_2 , the eigenvalues of $\mathbf{A}(t_2, t_1)$ retain their meaning only when $t_2 - t_1 = T$ in a time-periodic flow.

Algorithm 2 Randomized modal stability analysis

```

1: function RMODAL
2:    $\mathbf{X} \leftarrow \text{randn}(\mathbf{N}, \mathbf{M})$  ▷ generate  $k$  different initial conditions
3:    $\mathbf{Y} \leftarrow \mathbf{X}$  ▷ integrate according to equation (2.1)
4:    $(\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}) \leftarrow \text{svd}(\mathbf{X}, \text{'econ'})$  ▷ reduced SVD decomposition
5:    $\tilde{\mathbf{\Phi}} \leftarrow \mathbf{U}' * \mathbf{Y} * \mathbf{V} * \text{inv}(\mathbf{\Sigma})$  ▷ form the companion matrix
6:    $[\mathbf{y}, \mathbf{\lambda}] = \text{eig}(\tilde{\mathbf{\Phi}})$  ▷ eigen-decomposition of the companion matrix
7:    $\phi \leftarrow \mathbf{U} * \mathbf{y}$  ▷ recover the eigenvectors
8:   return  $(\mathbf{\lambda}, \phi)$ 

```

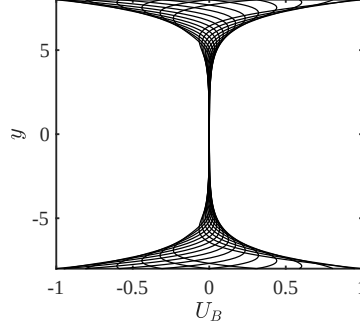


FIGURE 3. The base flow of the pulsatile Poiseuille flow. The black lines represents the instantaneous velocity profiles.

A scheme of the proposed randomized methods for both the nonmodal and modal stability analysis of time-varying flows is depicted in figure 2. Both analysis start with the introduction of a set of random initial conditions. These initial conditions are then integrated through the linearized NS equation 2.1 to obtain the solution at time t . Leveraging these data, both the nonmodal and modal stability analysis can be sought in an inexpensive manner.

2.5. Example: linear stability analysis of pulsatile Poiseuille flow

2.5.1. Problem setup

Let us apply the proposed randomized methods for the linear stability analysis of the pulsatile poiseuille flow (Blennerhassett & Bassom 2006). Consider the motion induced in a Newtonian fluid by synchronously oscillating two infinite parallel plates, a distance $2d$ apart, with the velocity $U_0 \cos(\omega t)$. The normalized based flow is obtained by scaling length with $\sqrt{2\nu/\omega}$, velocity components with U_0 , and time with $\tau = \omega t$. The Reynolds number is defined as $Re = U_0/\sqrt{2\nu\omega}$. If the movement of the plates is in the x -direction with the y -direction normal to the bounding surface, the undisturbed base flow takes the form

$$U_B(y, \tau) = \text{Real} \left\{ \frac{\cosh(y + iy)}{\cosh(h + ih)} e^{i\tau} \right\}, \quad (2.10)$$

where $h = d\sqrt{\omega/(2\nu)}$. Such base flow is schematically shown in figure 7.

The linear stability analysis of the base flow is found by considering a disturbed flow in the form $(u, v, w) = (U_B, 0, 0) + \epsilon \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}, 0 \right)$ where $\Psi(x, y, \tau) = e^{i\alpha x} \psi(y, \tau)$. The governing equation for the disturbance is given as

$$\frac{\partial \psi}{\partial \tau} = \mathbf{L}(t)\psi, \quad (2.11)$$

where $\mathbf{L}(t) = \left(\frac{\partial^2}{\partial y^2} - \alpha^2 \right)^{-1} \left\{ \left[\frac{1}{2} \left(\frac{\partial^2}{\partial y^2} - \alpha^2 \right) - i\alpha Re U_B \right] \left(\frac{\partial^2}{\partial y^2} - \alpha^2 \right) + i\alpha Re U_B'' \right\}$. In order to construct the time-varying linear operator, Chebyshev mesh with 101 nodes in the y -direction is used. The size of the linear operator $\mathbf{L}(t)$ becomes 100×100 at each timestep.

We prescribe M random initial conditions and integrate equation (2.11). The results for modal and nonmodal analyses are reported below.

2.5.2. Nonmodal stability analysis

Since the size of the linear operator is still manageable, we can directly calculate the state transition matrix $\mathbf{A}(t, t_0)$ by calling the Matlab function `expm` at each time step and

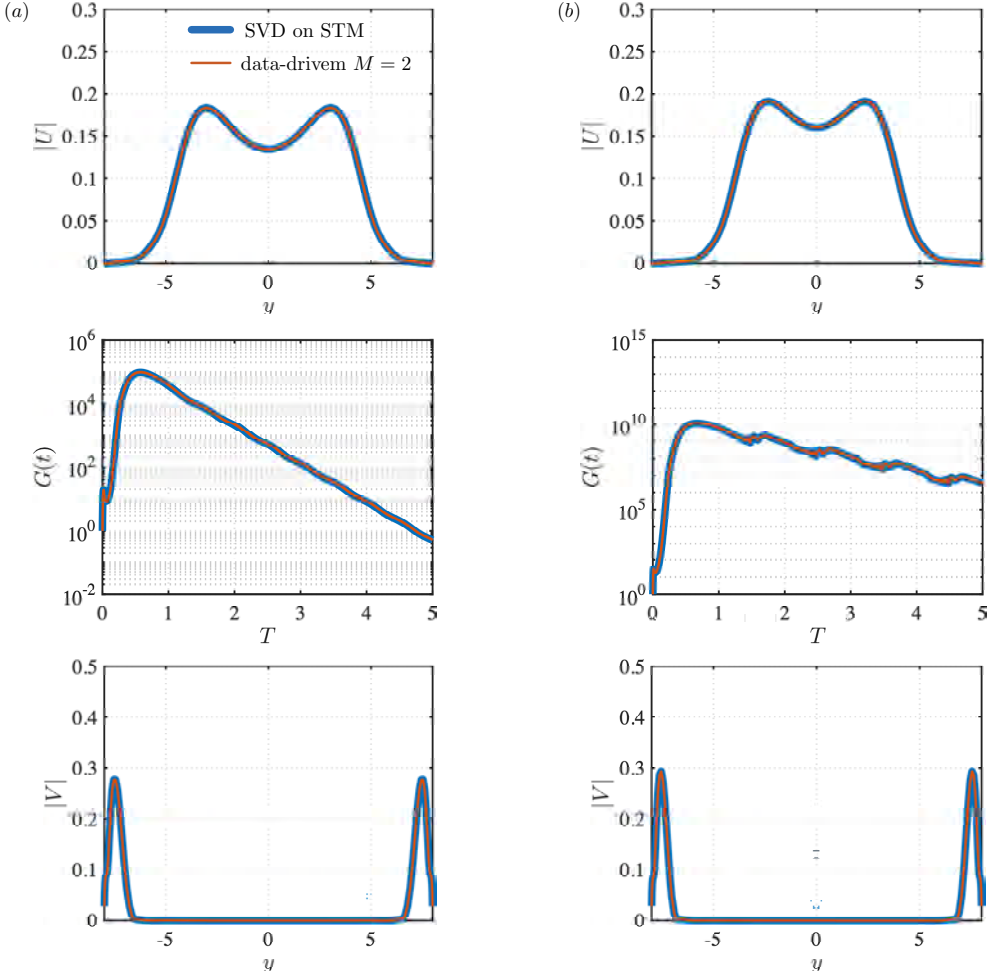


FIGURE 4. SVD of $\mathbf{A}(t, t_0)$ with (a) $(\alpha, h, Re) = (0.3, 8, 400)$ and (b) $(\alpha, h, Re) = (0.3, 8, 700)$. In both cases, the flow is asymptotically stable. Transient growth can be as large as 10^{10} . Thick blue lines are obtained by SVD on the state transition matrix directly calculated by equation (2.5). Thin red lines are obtained by the data-driven methods using two sets of simulations. Magnitude of $\mathbf{U}$ and $\mathbf{V}$ is shown for the time at highest gain.

the multiply them together, as shown in equation (2.5). After the state transition matrix is obtained, we calculate the singular value decomposition of $\mathbf{A}(t, t_0)$. This result is used as the reference to verify the proposed data-driven method. The comparison between the two methods is shown in figure 4, for two Reynolds numbers. A comparison of the leading singular vectors and singular values are provided in figure 4. In the data-driven method, only two (and at least two) sets of simulations is needed to achieve almost perfect match with the direct method. The agreement between the data-driven method and the direct method is robust with respect to the number of simulations M (increasing M does not deteriorate the results).

2.5.3. Modal stability analysis

As a reference, we first calculate the Floquet exponents by the conventional Floquet stability analysis. The eigenvalues are compared with those obtained from the present data-driven method, as shown in figure 5. Here, only two sets of simulations are used to accurately predict the two leading eigenvalues.

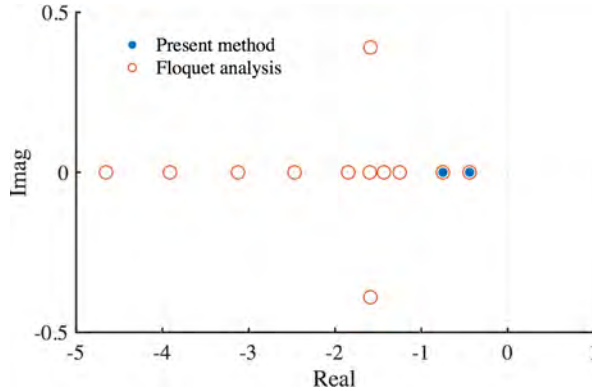


FIGURE 5. Floquet exponents for the case of $(\alpha, h, Re) = (0.3, 8, 400)$. The two leading eigenvalues from the data-driven method with $M = 2$ agree well with the conventional Floquet stability analysis. However, further increasing M is not able to recover the remaining of the eigenvalues.

2.6. Summary

In this report, we introduced a framework for the linear stability analysis of time-varying flows using randomized linear algebra. The present methods begin with the temporal integration of the linearized NS equations using a set of random vectors as initial conditions. For nonmodal stability analysis, the leading singular value and vectors of the state transition matrix are recovered by the randomized SVD (rSVD) algorithm. For modal (Floquet) stability analysis, the leading Floquet exponents and the modes are computed from the exact DMD algorithm. The proposed approach is tested for both the transient growth and Floquet stability analyses of the pulsatile Poiseuille flow. The results from the randomized methods agree well with the direct methods.

3. Randomized resolvent analysis

3.1. Introduction

One of the central questions in turbulence and flow control is concerned with the evolution of perturbations. Gaining detailed understanding of the perturbation dynamics in turbulent flows is a daunting task, due to the complex nonlinear dynamics that takes place over a broad range of spatial and temporal scales. To modify the global flow characteristics with control, it is necessary that the nonlinear interactions involving the actuation input (perturbation) become appropriately large to alter the base flow. For flow control, we need not track all possible ways in which the actuation input can modify the flow, but instead we can focus on the dominant directions in which the perturbation can be amplified. This notion has led to the modal analysis based approaches (Holmes *et al.* 2012; Schmid 2010; Taira *et al.* 2017, 2019), including the global stability analysis (Theofilis 2011b) and the resolvent analysis (Trefethen *et al.* 1993).

In the presence of sustained perturbations or forcing inputs, the linear system response can be described by the transfer function from control theory. This linear analysis is greatly simplified when the input to the system is sinusoidal and leads to the well-known Bode plots that reveal the gain and phase response of the system over a range of forcing frequencies. The transfer function that relates the system input to the output is called the resolvent and its analysis has been extended to fluid flows by Trefethen *et al.* (1993). The resolvent analysis is based on the pseudospectral analysis and has been used to study the transient energy growth

(Trefethen *et al.* 1993) as well as the harmonic response of the system (Jovanović & Bamieh 2005). These initial studies of resolvent analysis were performed about stable laminar flows.

An extension to resolvent analysis to study turbulent flows was presented by McKeon & Sharma (2010). They considered the nonlinear advection term to be the self-sustained input within the natural feedback loop of the fluid flow. This viewpoint has enabled the use of time-averaged base flows to reveal the input-output dynamics of turbulent flows. Moreover, discounting or finite-time horizon based extension of the resolvent analysis has enabled resolvent analysis to study flows with unstable base states (Jovanović 2004; Yeh & Taira 2019). Because resolvent analysis can determine the most amplified forcing and response directions, it serves as a powerful analytical tool to find effective active and passive flow control techniques (Yeh & Taira 2019; Nakashima *et al.* 2017).

The resolvent analysis needs two key ingredients: (i) the base flow and (ii) the linearized Navier–Stokes operator. It is known that the accuracy of the base flow and the spatial discretization of the linear operators is critical for extracting response characteristics correctly (Yeh & Taira 2019). The need for accurate discretization of the linearized Navier–Stokes operators calls for sufficient grid resolution and appropriate computational domain size. As such, the discrete resolvent operator becomes large with size $m \times m$. Here, m is essentially the number of variables times the size of the grid, which can easily be upward of $\mathcal{O}(10^6)$ for turbulent flows (Kajishima & Taira 2017). For resolvent analysis, the singular value decomposition (SVD) needs to be performed on the large resolvent operator with a taxing operation count of $\mathcal{O}(m^3)$. To enable resolvent analysis at high Reynolds numbers, we must find a relief to perform SVD of the resolvent operator.

Although the resolvent analysis is performed on a very large matrix, only the leading forcing and response modes are generally sought. Based on the amount of necessary matrix data used to perform the analysis, the desired output is only a very small fraction of the input data size. For this reason, it would be natural to consider that all elements of the resolvent matrix are not necessary to determine the leading resolvent modes. Within the resolvent framework, the core of the computations lies with the SVD. In order to handle a large operator for SVD to find the leading modes, we can consider subsampling the matrix of interest and perform the SVD on the low-order representation of this large matrix. Randomized numerical linear algebra has recently emerged as an effective technique to reduce a large matrix to its low-order representation (Halko *et al.* 2011; Drineas & Mahoney 2016; Tropp *et al.* 2017), for applications including big data compression and data transfer. The key idea is to pass a randomly generated low-rank test matrix through the large matrix to obtain the so-called sketch of the full matrix. This sketch is low-rank but holds key information about the full matrix and can be used to derive appropriate bases to represent the full matrix in a low-dimensional manner (Clarkson & Woodruff 2009; Woolfe *et al.* 2008; Halko *et al.* 2011; Tropp *et al.* 2017). Randomized techniques can be incorporated into SVD to achieve tremendous computational and memory savings (Woolfe *et al.* 2008; Rokhlin *et al.* 2010; Halko *et al.* 2011).

In recent years, modal analyses are tackling flows over complex geometries and high-Reynolds number flows with increasingly large degrees of freedom (Taira *et al.* 2017, 2019; Jeun *et al.* 2016; Schmidt *et al.* 2018; Dwivedi *et al.* 2019; Ribeiro & Wolf 2017; Ricciardi *et al.* 2018). To further aid this endeavor, randomized SVD have been incorporated into data-based modal analysis techniques, including the proper orthogonal decomposition (Rokhlin *et al.* 2010) and dynamic mode decomposition (Erichson *et al.* 2019). For global operator-based analyses of high-Reynolds-number flows (Jeun *et al.* 2016; Schmidt *et al.* 2018; Dwivedi *et al.* 2019), the leading singular values and modes can be determined with significant reduction in computational costs with the aid of randomized techniques. The randomized technique

presented in this study can greatly expand the applicability of the resolvent analysis to high-Reynolds number flows with multiple inhomogeneous spatial directions.

In fact, randomized SVD has been adopted in the one-dimensional resolvent analysis of turbulent channel flow by Moarref *et al.* (2013). Since the details on the use of the randomized techniques for large-scale resolvent analysis is not available, one of our objectives is to provide detailed guidance on the use of randomized resolvent analysis for multi-dimensional turbulent base flows. Moreover, the error bound of randomized SVD is generally not guaranteed and needs to be further characterized. We also note that the sketching strategy can be further improved with insights from the base flow. In the present study, we aim to address these issues on the use of randomized SVD in the context of resolvent analysis for fluid flows.

In what follows, we present the randomized resolvent analysis and demonstrate its use for turbulent flow over a canonical airfoil. In section 3.2, we introduce the randomized resolvent analyses and discuss how randomized SVD can be implemented without the explicit generation of inverse matrices. We also propose an alternative route of recovering the response modes (left singular vectors) with significantly enhanced accuracy. In section 3.3, the present randomized technique is applied to the resolvent analysis of turbulent flow over a NACA 0012 airfoil. We assess the level of error and the convergence behavior with respect to resolvent modes and gains obtained from the use of randomized SVD. To further enhance the accuracy of randomized resolvent analysis, we introduce a physics-informed sampling technique that leverages the insights from the base flow. At last, we provide concluding remarks in section 5.4.

3.2. Approach

3.2.1. Full resolvent analysis

Let us consider the flow state $\mathbf{q} \in \mathbb{R}^m$ as a sum of the time-invariant base state $\bar{\mathbf{q}}$ and the statistically stationary fluctuating component $\mathbf{q}'$. With this Reynolds decomposed flow variable and appropriate discretization, we can express the discrete Navier–Stokes equations as

$$\frac{\partial \mathbf{q}'}{\partial t} = \mathbf{L}_{\bar{\mathbf{q}}} \mathbf{q}' + \mathbf{f}', \quad (3.12)$$

where $\mathbf{L}_{\bar{\mathbf{q}}} \in \mathbb{R}^{m \times m}$ is the linearized Navier–Stokes operator about the base state $\bar{\mathbf{q}}$ and $\mathbf{f}'$ collects the nonlinear terms and the external forcing inputs. We gather the nonlinear terms as external forcing in the turbulent mean flow following the perspective of McKeon & Sharma (2010), Farrell & Ioannou (1994), and Schmid (2007). For the traditional resolvent analysis, $\bar{\mathbf{q}}$ is chosen to be the stable laminar equilibrium state such that $\mathbf{f}'$ can be considered as the forcing input to the system with the nonlinear term neglected (Jovanović & Bamieh 2005). More recently, turbulent mean flows has been used for $\bar{\mathbf{q}}$ with $\mathbf{f}'$ representing the nonlinear terms as sustained forcing input within the natural feedback system (McKeon & Sharma 2010).

We can consider the Fourier transform $[\mathbf{q}'(\mathbf{x}, t), \mathbf{f}'(\mathbf{x}, t)] = \int_{-\infty}^{\infty} [\hat{\mathbf{q}}_{\omega}(\mathbf{x}), \hat{\mathbf{f}}_{\omega}(\mathbf{x})] e^{-i\omega t} d\omega$ and express the relationship between $\mathbf{q}'$ and $\mathbf{f}'$ in frequency space as

$$-i\omega \hat{\mathbf{q}}_{\omega} = \mathbf{L}_{\bar{\mathbf{q}}} \hat{\mathbf{q}}_{\omega} + \hat{\mathbf{f}}_{\omega}, \quad (3.13)$$

where ω is the frequency. Note that spatial Fourier transform can also be incorporated if directional homogeneity is present. For stable base flows, ω can be chosen to be real. To extend resolvent analysis to unstable base flows, we can consider the use of finite-time/discounted analysis (Jovanović 2004; Yeh & Taira 2019) by choosing a complex frequency $\omega = \omega_r + i\beta$, where both ω_r and β are real and β discounts the modal growth rate of $\mathbf{L}_{\bar{\mathbf{q}}}$. The input-output

relationship between $\hat{\mathbf{f}}$ and $\hat{\mathbf{q}}$ can be found from (3.13) as

$$\hat{\mathbf{q}}_\omega = \mathbf{A} \hat{\mathbf{f}}_\omega, \quad (3.14)$$

where

$$\mathbf{A} = [-i\omega \mathbf{I} - \mathbf{L}_{\hat{\mathbf{q}}}]^{-1} \in \mathbb{C}^{m \times m} \quad (3.15)$$

is referred to as the *resolvent operator* (Trefethen *et al.* 1993; Jovanović & Bamieh 2005; McKeon & Sharma 2010). It serves as a transfer function that amplifies (or attenuates) the harmonic forcing input $\hat{\mathbf{f}}_\omega$ and maps it to the response $\hat{\mathbf{q}}_\omega$. The goal of resolvent analysis is to identify the dominant directions along which $\hat{\mathbf{f}}_\omega$ can be most amplified through $\mathbf{A}$ to form the corresponding responses in $\hat{\mathbf{q}}_\omega$. This question is addressed by the SVD of

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^*, \quad (3.16)$$

where $\mathbf{V}^*$ denotes the Hermitian of $\mathbf{V}$. Resolvent analysis interprets left and right singular vectors $\mathbf{U} = [\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \dots, \hat{\mathbf{u}}_m] \in \mathbb{C}^{m \times m}$ and $\mathbf{V} = [\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \dots, \hat{\mathbf{v}}_m] \in \mathbb{C}^{m \times m}$ respectively as response modes and forcing modes, with the magnitude-ranked singular values $\mathbf{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_m) \in \mathbb{R}^{m \times m}$ being the amplification (gain) for the corresponding forcing-response pair. For unstable base flows, it is important that a finite-time window is chosen with β larger than the highest growth rate such that the resolvent analysis reveals the input-output relationship on a shorter time scale than that of the base flow instability (Jovanović 2004; Yeh & Taira 2019).

Performing the SVD of $\mathbf{A} \in \mathbb{C}^{m \times m}$ requires an theoretical operation count of $\mathcal{O}(m^3)$. In practice, some algorithms can reduce this operation count when only a few singular values are to be recovered, while still being computationally taxing for large m (Morgan & Zeng 2006). Such cases are encountered in high-Reynolds number flows and bi/tri-global analysis settings. However, we note that many applications of resolvent analysis call only for the dominant forcing and response modes $[\hat{\mathbf{v}}_1, \hat{\mathbf{u}}_1]$ associated with the highest gain σ_1 . This is appropriate when the first gain σ_1 is much larger than the rest of the gains $\sigma_{j>1}$ and shows a quick roll off. Such condition is related to non-normality of linear operator $\mathbf{L}_{\hat{\mathbf{q}}}$, which is encountered for flows with strong shear and separation Schmid & Henningson (2012).

When the contributions from the higher-order modes are neglected, it is referred to as the rank-1 assumption for which the flow response from forcing $\hat{\mathbf{f}}$ is approximated as $\hat{\mathbf{q}} \approx \hat{\mathbf{u}}_1 \sigma_1 \langle \hat{\mathbf{v}}_1, \hat{\mathbf{f}} \rangle$, provided that $\sigma_1 \gg \sigma_2$ and $\hat{\mathbf{f}}$ has reasonable magnitude along $\hat{\mathbf{v}}_1$. For seeking only the dominant modal insights from resolvent analysis, we discuss a remedy for performing large-scale resolvent analysis (Jeun *et al.* 2016; Schmidt *et al.* 2018) in a computationally tractable manner below.

3.2.2. Randomized resolvent analysis

For a flow that have dominant structures, we consider a low-rank representation of the resolvent operator $\mathbf{A}$. That is, instead of directly performing SVD of $\mathbf{A}$ and obtaining the leading-mode representation, we seek a low-rank representation of $\mathbf{A}$ and perform the SVD of the low-rank version of $\mathbf{A}$. We can consider finding an appropriate low-dimensional basis to project the large resolvent operator on a suitable subspace to derive the low-rank resolvent approximation.

The action of a full matrix on a vector should reveal some insights on which components are modified in the dominant directions. In the case of flow that can be described with the rank-1 approximation, there should be a low number of dominant directions. This very point can be taken advantage of through what is known as *sketching* in numerical linear algebra. Sketching refers to a procedure in which a tall and skinny test matrix $\mathbf{\Omega} \in \mathbb{R}^{m \times k}$ (or $\mathbb{C}^{m \times k}$),

where $k \ll m$, is passed through $\mathbf{A}$

$$\mathbf{Y} = \mathbf{A}\mathbf{\Omega}. \quad (3.17)$$

Here, matrix $\mathbf{Y} \in \mathbb{C}^{m \times k}$ is called the *sketch* of the input matrix $\mathbf{A}$ (Woolfe *et al.* 2008; Halko *et al.* 2011; Tropp *et al.* 2017). The test matrix $\mathbf{\Omega}$ can be constructed using random values with Gaussian distribution (Woolfe *et al.* 2008) and be weighted by any input matrix insight, as will be discussed in section 3.3.4 for a physics-inspired random test matrix. As the sketch holds the dominant influence of $\mathbf{A}$, we can consider orthonormalizing $\mathbf{Y}$ using a QR decomposition to form the orthonormal basis with $\mathbf{Q} \in \mathbb{C}^{m \times k}$ upon which we can project the full matrix $\mathbf{A}$ to derive its low-rank approximation. In this way, it is possible to approximate $\mathbf{A}$ for a rank $k \ll m$ as long as this approximation preserves the features of the leading modes.

Given this $\mathbf{Q}$, a low-rank approximation of $\mathbf{A}$ can be found as $\mathbf{A} \approx \mathbf{Q}\mathbf{Q}^* \mathbf{A}$ (Halko *et al.* 2011). We can view this as a low-rank decomposition of $\mathbf{A} \approx \mathbf{Q}\mathbf{B}$, where $\mathbf{B} = \mathbf{Q}^* \mathbf{A} \in \mathbb{C}^{k \times m}$. It is this reduced matrix $\mathbf{B}$ upon which we can perform the SVD

$$\mathbf{B} = \tilde{\mathbf{U}} \mathbf{\Sigma} \mathbf{V}^* \quad (3.18)$$

Hence, as a low-rank approximation, we now have

$$\mathbf{A} \approx \mathbf{Q} \tilde{\mathbf{U}} \mathbf{\Sigma} \mathbf{V}^* \quad (3.19)$$

where we can consider $\mathbf{U} \approx \mathbf{Q} \tilde{\mathbf{U}}$. This process is the *randomized SVD* (Halko *et al.* 2011), where the sketch $\mathbf{Y}$ was used to derive $\mathbf{Q}$. This approximation almost always satisfies $\|\mathbf{A} - \mathbf{Q}\mathbf{Q}^* \mathbf{A}\| \leq (1 + 9\sqrt{k+p}\sqrt{m}) \sigma_{k+1}$, where p is the oversampling parameter, which is applied to build an approximation of rank k while projecting the matrix $\mathbf{A}$ to the low-dimensional subspace with $(k+p)$ vectors. With this overall approach, the computational cost for SVD is reduced to $\mathcal{O}(mk^2)$ instead of $\mathcal{O}(m^3)$ for the full SVD. In our implementation, we use $\mathbf{v}_1$ from (3.19) and retrieve the leading singular value and left singular vector through

$$\mathbf{A} \hat{\mathbf{v}}_1 = \hat{\mathbf{u}}_1 \sigma_1. \quad (3.20)$$

The singular value and vector can be separated by noticing that $\|\hat{\mathbf{u}}_1\| = 1$. Applied to resolvent analysis, the last equation provides more accurate leading singular value σ_1 and left singular vector $\hat{\mathbf{u}}_1$ compared to the original randomized SVD algorithm (Halko *et al.* 2011). The same operation can be used to recover the higher-order modes, with better accuracy than using the original algorithm (Halko *et al.* 2011). For applications where high-order modes and orthogonality are desired, we can solve for $\mathbf{U}\mathbf{\Sigma}$ and compute its SVD. In the present randomized resolvent analysis, we emphasize that only the discrete linear operator $\mathbf{L}_{\bar{\mathbf{q}}}$ is needed for sketching $\mathbf{Y}$ and to find the reduced matrix $\mathbf{B}$. Unlike the original resolvent method, matrix linear solvers are used to avoid calling for the inverse within the resolvent operator. The resulting algorithm constitutes the *randomized resolvent analysis* summarized in Algorithm 1.

To utilize the randomized SVD for resolvent analysis, we must be aware that the resolvent operator $\mathbf{A} = [-i\omega \mathbf{I} - \mathbf{L}_{\bar{\mathbf{q}}}]^{-1}$ contains an inverse operation in its definition, which need not be numerically performed. We do not intend to perform an inverse operation within $\mathbf{A}$ in the present work. In the full resolvent analysis, when the matrices become too large and the inverse can not be performed (which is likely the case for 2D and 3D problems), one can focus on modes corresponding to the smallest singular values of $\mathbf{A}^{-1}$ to find those for the largest singular values of $\mathbf{A}$. Similar approaches have avoided the inverse computation, including the work by Jeun *et al.* (2016).

Both full resolvent and randomized resolvent analyses are shown schematically in figure 6. Notice that we are not interested in all singular values and vectors of $[-i\omega \mathbf{I} - \mathbf{L}_{\bar{\mathbf{q}}}]^{-1}$, but

only in a few subset of the largest σ_j and their corresponding $\hat{\mathbf{u}}_j$ and $\hat{\mathbf{v}}_j$. In the randomized resolvent analysis, we can approximate a low-rank representation of it using $[-i\omega\mathbf{I} - \mathbf{L}_{\bar{\mathbf{q}}}]$. Figure 6 shows an adaptation of the procedure from Halko *et al.* (2011) in order to compute the largest singular values of the resolvent without performing its inverse. In the randomized resolvent analysis, we solve a linear system with $[-i\omega\mathbf{I} - \mathbf{L}_{\bar{\mathbf{q}}}]$, the columns of the random matrix Ω form the right-hand side and the sketch columns of $\mathbf{Y}$ are the unknowns. By doing so, we sketch $[-i\omega\mathbf{I} - \mathbf{L}_{\bar{\mathbf{q}}}]^{-1}$ without finding the actual inverse matrix. The same procedure is performed to project the matrix to the low-dimensional subspace. The matrices are rearranged in a way that the projection is performed using $[-i\omega\mathbf{I} - \mathbf{L}_{\bar{\mathbf{q}}}]$, but results in the low-dimensional projection of $[-i\omega\mathbf{I} - \mathbf{L}_{\bar{\mathbf{q}}}]^{-1}$ instead. Figure 6 also illustrates the procedures for recovering the left singular vectors and the singular values from the original algorithm (Halko *et al.* 2011) and from the present implementation.

Algorithm 3 Randomized Resolvent Analysis

Require: discrete linear operator $\mathbf{L}_{\bar{\mathbf{q}}} \in \mathbb{C}^{m \times m}$

```

1: function RANDOMIZED_RESOLVENT( $\omega, k$ )
2:    $\Omega \leftarrow \text{randn}(m, k)$   $\triangleright$  random normal matrix generator
3:    $\mathbf{Y} \leftarrow [-i\omega\mathbf{I} - \mathbf{L}_{\bar{\mathbf{q}}}] \backslash \Omega$   $\triangleright$  solve linear system for  $\mathbf{Y}$ ,  $\mathcal{O}(m^2k)$ 
4:    $(\mathbf{Q}, \sim) \leftarrow \text{qr}(\mathbf{Y}, 0)$   $\triangleright$  economy-sized QR decomposition,  $\mathcal{O}(mk^2)$ 
5:    $\mathbf{B} \leftarrow \mathbf{Q}^* / [-i\omega\mathbf{I} - \mathbf{L}_{\bar{\mathbf{q}}}]$   $\triangleright$  solve linear system for  $\mathbf{B}$ ,  $\mathcal{O}(m^2k)$ 
6:    $(\sim, \sim, \mathbf{V}) \leftarrow \text{svd}(\mathbf{B}, \text{'econ'})$   $\triangleright$  reduced SVD decomposition,  $\mathcal{O}(mk^2)$ 
7:    $\mathbf{U}^\Sigma \leftarrow [-i\omega\mathbf{I} - \mathbf{L}_{\bar{\mathbf{q}}}] \backslash \mathbf{V}$   $\triangleright$  recover  $\mathbf{U}^\Sigma$ ,  $\mathcal{O}(m^2k)$ 
8:   for  $j \leftarrow 1$  to  $k$  do
9:      $\Sigma_{j,j} \leftarrow \text{norm}(\mathbf{U}_{1:m,j}^\Sigma, 2)$   $\triangleright$  recover singular values  $\Sigma$ 
10:     $\mathbf{U}_{1:m,j} \leftarrow \mathbf{U}_{1:m,j}^\Sigma / \Sigma_{j,j}$   $\triangleright$  normalize  $\mathbf{U}$ 
11: return  $(\mathbf{U}, \Sigma, \mathbf{V})$ 

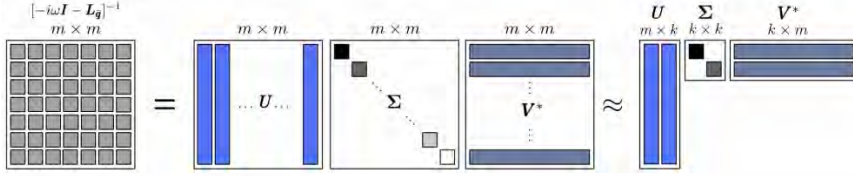
```

Oversampling and power iteration schemes: Randomized algorithms can incorporate two additional procedures to improve performance and accuracy. Namely, they are oversampling (Erichson *et al.* 2016) and power or subspace iterations (Rokhlin *et al.* 2010; Halko *et al.* 2011; Gu 2015). Oversampling sketches the input matrix using $(k + p)$ vectors (with p extra vectors) and increases the low-dimensional subspace to accurately recover a smaller quantity of singular values k . For the randomized resolvent analysis, oversampling has the same outcome, in practice, of selecting a larger k and the influence of k will be discussed at the end of section 5.3.3. When k becomes large, it should be noticed that the memory consumption increases. Even for large sparse matrices, the sketch matrix and the subsequently reduced matrices that are formed are generally dense, which adds a computational burden.

The second procedure is the power or subspace iterations. These methods are a powerful tool when the singular values of the matrix decay slowly. For example, this type of spectral behavior appeared in the input-output analysis performed by Jeun *et al.* (2016) for jet flows. The method consists of performing additional iterations after the sketch $\mathbf{Y}$ is evaluated. It should however be realized that such procedure calls for additional linear solvers, which is the most time consuming operation in Algorithm 1. For power iterations, one must compute the adjoint $\mathbf{A}\mathbf{A}^*$, where $\mathbf{A}^*$ is the Hermitian of $\mathbf{A}$, q times and solve the linear system k times. For subspace iterations, additional *QR* decompositions are necessary, and the number of additional linear systems to be solved will be q times k . In practice, small values of q improves the accuracy of the results substantially. More information on the general applicability of the subspace and power iterations are discussed by Halko *et al.* (2011) and Erichson *et al.* (2016). In section 3.3.4, we present another option for improving accuracy of the overall technique in a computationally inexpensive manner by constructing a physics-informed random test matrix Ω .

(a) Full Resolvent Analysis

resolvent operator



(b) Randomized Resolvent Analysis

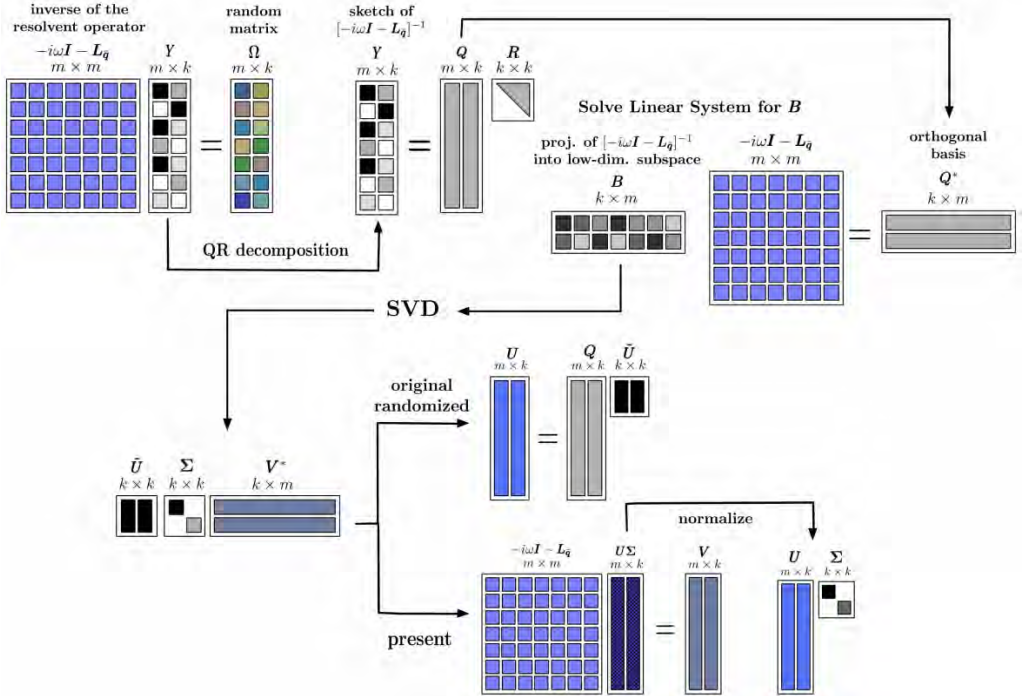
Solve Linear System for Y 

FIGURE 6. Schematics of the (a) full resolvent and (b) randomized resolvent analyses. For the full resolvent analysis, a direct SVD is applied. For the randomized resolvent analysis, the orthogonal basis Q is computed to project the operator into the low-dimensional subspace, where the SVD is performed on the reduced matrix B . Two options to recover the left singular vectors and singular values are provided, using the original randomized approach Halko *et al.* (2011) and the present implementation. Orthogonal vectors are represented by long bars.

Test matrix Ω : The standard choice for the test matrix Ω is the random matrix generated with a normal distributions. Such matrix is known to present excellent performance and accuracy Tropp *et al.* (2017). For some cases, especially for very large matrices or when the singular values present slow decay, larger values of k may be necessary to better approximate the matrix in the low-dimensional subspace. When large values of k are used, orthonormalization of the columns of the test matrix can be considered to improve numerical stability (Demmel *et al.* 2007; Halko *et al.* 2011). The test matrix can also be generated using a Rademacher distribution (Clarkson & Woodruff 2009). It is also possible to build an ultrasparse matrix with Rademacher distribution in the non-sparse entries which allows for the control of cost, stability and reliability in the operations (Tropp *et al.* 2017; Clarkson & Woodruff 2009).

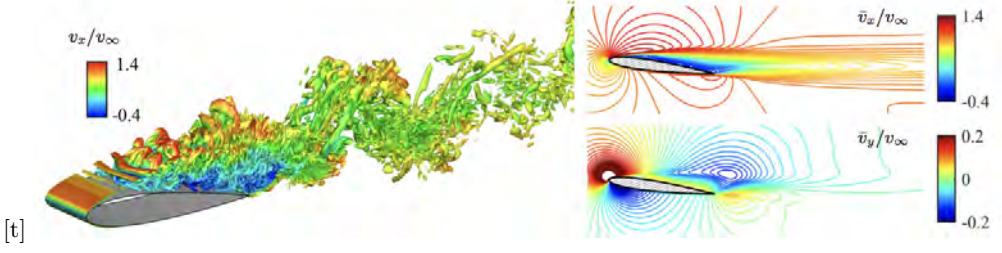


FIGURE 7. The instantaneous (left) and time/spanwise-averaged (right) flows over a NACA 0012 airfoil at $Re_{L_c} = 23,000$. The instantaneous flow visualization shows the isosurface of Q -Criterion ($QL_c^2/v_\infty^2 = 50$) colored by the instantaneous streamwise velocity.

When randomized SVD is applied, there is no a priori knowledge of the structure of the matrix. However, in the present application, we know how the resolvent operator is constructed. This theoretical insight can be used to build a random test matrix that outperforms the standard normal distribution matrix. We later propose a physics-informed test matrix Ω that can focus our sketching operation for regions of physical importance. In our application, the dominant directions are related to regions with the presence of high shear. The results from this approach will be discussed in Section 3.3.4.

3.3. Randomized resolvent analysis of turbulent post-stall flow

We demonstrate the use of randomized resolvent analysis on turbulent flow over a NACA 0012 airfoil. In this example, the randomized resolvent analysis will be applied a resolvent operator of size $m \times m$, where $m \simeq 7 \times 10^5$, to reveal the dominant gain and modal structures with a thin sketching matrix having as little as $k = 10$ columns. The convergence of the gain and resolvent modes will also be reported with respect to the size of the sketching matrix. Influence of the ratio between the first and the second singular values of the resolvent operator will also be examined.

3.3.1. Problem setup

We consider the spanwise-periodic turbulent flow over a NACA 0012 airfoil at an angle of attack of 9° , a chord-based Reynolds number of $Re_{L_c} \equiv v_\infty L_c / \nu_\infty = 23,000$ and a free stream Mach number of $M_\infty \equiv v_\infty / a_\infty = 0.3$. Here, v_∞ is the free-stream velocity, L_c is the chord length, a_∞ is the free-stream sonic speed, and ν_∞ is the kinematic viscosity. The time- and spanwise-averaged turbulent flow is considered as the base flow for the full and randomized resolvent analyses. For this 2D base flow, we adopt the bi-global setting that decomposes $\mathbf{q}'$ into spanwise Fourier modes with the wavenumber k_z .

To obtain the base flow, large-eddy simulation (LES) is performed using a finite-volume compressible flow solver *CharLES* (Khalighi *et al.* 2011; Brès *et al.* 2017), which is second-order accurate in space and third-order accurate in time. Vremen's sub-grid scale model (Vremen 2004) is utilized in the LES. The LES is performed on a C-shaped mesh with the domain extent of $x/L_c \in [-19, 26]$, $y/L_c \in [-20, 20]$ and $z/L_c \in [-0.1, 0.1]$ in the streamwise, transverse and spanwise direction, respectively, with the airfoil leading edge at $x/L_c = y/L_c = 0$. Dirichlet boundary condition is specified at the far-field boundary as $(\rho, v_x, v_y, v_z, T) = (\rho_\infty, v_\infty, 0, 0, T_\infty)$, where ρ is the density, v_x , v_y and v_z are respectively the streamwise, transverse and spanwise velocity, and T is the temperature. Over the airfoil, the no-slip adiabatic boundary condition is prescribed. Along the outlet boundary, a sponge layer (Freund 1997) is applied with a running-averaged state being the target state. The simulation has been validated with respect to the time-averaged pressure, lift and drag over the airfoil. The turbulent separated flow over the airfoil is visualized in figure 7. The

visualization of the instantaneous flow shows the laminar separation from the leading edge. We have found that the shear layer physics dominates the pseudospectral behavior of the linearized Navier–Stokes operator, as shear is the main source of nonnormality in the operator. Further details regarding the computational setup, flow physics, and resolvent analysis based flow control of this setup are reported in Yeh & Taira (2019).

The full and randomized resolvent analyses are performed on a separate mesh from that used in the LES. This mesh has a 2D rectangular domain with the extent of $x/L_c \in [-15, 16]$ and $y/L_c \in [-12, 12]$, comprising approximately 0.15 million cells. Compared to the LES mesh, the mesh for resolvent analysis is coarser over the airfoil and in the wake, but is much finer in the upstream of the airfoil in order to resolve the forcing mode structures. The time- and spanwise-averaged flow $\bar{q}$ obtained from LES is interpolated onto this mesh. At the far-field boundary and over the airfoil, Dirichlet conditions are set for density and velocities and Neumann condition is prescribed for pressure in q' . At the outlet boundary, Neumann condition is set for all flow variables. With these boundary conditions for q' and the base flow $\bar{q}$, we construct the linearized Navier–Stokes operator $L_{\bar{q}}(k_z)$ for a chosen k_z . The size of $L_{\bar{q}}$ and the resolvent operator is approximately 0.75 million $\times$ 0.75 million.

For this large operator, we summarize in Table 1 the computational costs of performing resolvent analysis using the Krylov-based Arnoldi-iteration method with a range of parameter setups (i.e., number of singular values (n_{ev}), Krylov subspace dimension ($\dim(\mathcal{S})$), and tolerance) and compare them with those for the present randomized algorithm. The former was conducted by simply calling the svds command in MATLAB. It requires almost 80 gigabytes of memory and takes approximately 30 to 70 minutes (single-core) for each SVD. The high-memory demand necessitates the use of high performance computing resource to conduct the full resolvent analysis. In contrast, the randomized resolvent approach (Algorithm 1) achieves significant reductions in computational time and memory consumption. The present method only requires a third of the memory usage of the Arnoldi-iteration and cuts down the computational time by an order of magnitude. We also note that, in Algorithm 1, the linear systems solvers are the operations with higher computational cost. Since all the three linear systems solvers are conducted for the same operator, the LU decomposition of $[-i\omega I - L_{\bar{q}}]$ is performed in the beginning of the algorithm and is passed through the three solvers. This decomposition becomes the main source of the memory consumption.

3.3.2. Results

We perform the full and randomized resolvent analyses for spanwise wavenumbers of $k_z L_c = 0$ and 20π . Since the base flow is found to be unstable (Yeh & Taira 2019), the finite-time approach is adopted with $v_\infty/\beta L_c = 3$ to ensure that the resolvent analysis is performed on a shorter time scale than that associated with the leading growth rate of the instability. Initially, for the randomized analysis, we consider $k = 10$ for the width of the test matrix Ω with random Gaussian distribution. Later in Section 3.3.4, we show that this value of k can be further reduced without compromising accuracy.

The leading response and forcing modes obtained from both full and randomized analyses are compared in figure 8 for representative frequencies St and spanwise wavelength $k_z L_c$. Although $k/m = 1.3 \times 10^{-5}$, we observe excellent agreement between the modes from the full resolvent analysis and the randomized algorithm. We observe that randomized forcing and response modes are very similar to full resolvent ones. Only at $St = 1$ and $k_z L_c = 20\pi$ we observe the appearance of spatially distributed errors in the background, which we refer to as background noise.

The forcing modes are recovered directly from the SVD of the low-dimensional subspace projection. As they are used to recover the response modes using the linear operator, the accuracy of the forcing modes affect the results of the response ones. In the particular case of

Iteratively restarted Arnoldi method (MATLAB svds)

n_{ev}	$\dim(S)$	tolerance	time (sec)	memory
10	30	1E-14	4185	78.6 GB
5	15	1E-14	2764	78.6 GB
2	6	1E-14	1486	78.6 GB
10	30	1E-05	2245	78.6 GB
10	15	1E-14	4194	78.6 GB

Randomized resolvent (present)

n_{ev}	k	time (sec)	memory
2	2	354	28.2 GB
5	5	462	28.4 GB
10	10	615	28.8 GB

TABLE 1. Comparison of the computational time and memory consumption for the implicitly restarted Arnoldi iteration (svds in MATLAB) with the present randomized approach for different parameter setups with the number of singular values (n_{ev}), Krylov subspace dimension ($\dim(S)$), and tolerance.

$St = 1$ and $k_z L_c = 20\pi$, when forcing mode is affected by noise, the randomized approach returns some structures emanating from the trailing edge in the response mode $\hat{\mathbf{u}}_1^{\text{rand}}$, which was not present from the full resolvent analysis. This behavior is related to leakage from high-order modes, as σ_1 and σ_2 are close in the energy spectrum. This remarkable level of agreement over all frequencies and spanwise wavenumbers ensures that the randomized approach presented in Algorithm 1 can help extract insights into the spatial structures to identify regions of sensitivity and guide flow control efforts.

These results were obtained using the present implementation that extends the original randomized SVD algorithm. In figure 9, using the original randomized algorithm (Halko *et al.* 2011) within the resolvent analysis to recover the left singular vectors and singular values, the response modes contain background noise. Here, we use Algorithm 1 up to line 5, then SVD is performed as $\mathbf{B} = \tilde{\mathbf{U}} \Sigma \mathbf{V}^*$ and $\mathbf{U}$ is recovered a posteriori using the original procedure for the randomized SVD with the resolvent analysis (Halko *et al.* 2011; Moarref *et al.* 2013), by $\mathbf{U} = \mathbf{Q} \tilde{\mathbf{U}}$. Yeh & Taira (2019) showed that the present problem setup presents a peak in the singular values near $St = 6$ for both spanwise wavenumbers, influenced by the eigenmodes associated with the shear-layer structure over the separation bubble. These eigenmodes are highly nonnormal and induce high-energy amplification through pseudo-resonance (Trefethen *et al.* 1993). For the frequencies in a narrow region near $St = 6$, both implementations present similar results. Far from this band, the response modes obtained by the original procedure (Halko *et al.* 2011) are contaminated by random background noise or leakage from higher-order modes, as shown in figure 9 for $St = 0.5$ and 15 and $k_z L_c = 0$. In these critical cases, the original randomized approach may not provide meaningful insights into flow physics. The present implementation shown in Algorithm 1 improves solving for the response modes. Algorithm 1 does not enhance the forcing modes, as they are already accurate. When utilizing this technique to generate reduced-order models, one may perform steps 11 and 12

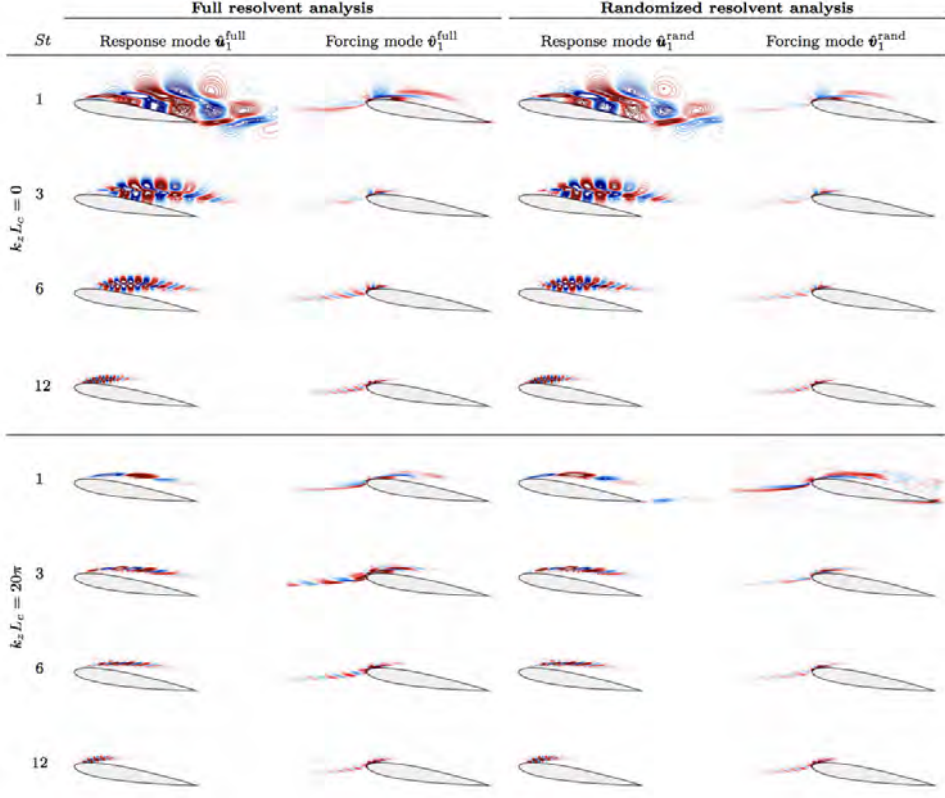


FIGURE 8. The leading response ($\hat{\mathbf{u}}_1$) and forcing ($\hat{\mathbf{v}}_1$) modes from the full and randomized resolvent analyses, using Algorithm 1, for $k_z L_c = 0$ and 20π at representative frequencies St . Modes are visualized with the streamwise velocity component with contour levels of $[-0.6, 0.6]$.

in Algorithm 1 to orthogonalize the left singular vectors. Considering the results obtained by our implementation, the modes are found very accurately for almost all frequencies and wavenumbers. To provide a concrete assessment, we quantitatively assess the accuracy of the randomized resolvent analysis.

The agreement between the full and randomized analyses with respect to the gain (leading singular value) and modes over a range of frequencies is presented in figure 10. When Algorithm 1 is applied to recover left singular vectors and singular values, the randomized analysis accurately captures the trend of gain distribution over $1 \lesssim St \lesssim 15$ in figure 10(a,b) for both wavenumbers. At the low and high-frequency ends, the gain shows deviations. The resemblance of the modal structure is quantified in figure 10(c,d) with the cosine similarities, i.e., the inner products, $\langle \hat{\mathbf{u}}_1^{\text{full}}, \hat{\mathbf{u}}_1^{\text{rand}} \rangle$ and $\langle \hat{\mathbf{v}}_1^{\text{full}}, \hat{\mathbf{v}}_1^{\text{rand}} \rangle$. As singular vectors are normalized, the cosine similarity of 1 suggests that perfect match is attained between the modes from full and randomized resolvent analyses. Since these modes are complex, the cosine similarity removes dependence on the phase difference. For almost the entire range of frequencies the cosine similarities are near unity, which means the agreement between full and randomized modes is excellent. When this value is reduced, the modes may be affected by noise, as seen for $\hat{\mathbf{v}}_1^{\text{rand}}$ at $St = 1$ and $k_z L_c = 0$ in figure 9. For the original approach, using the randomized SVD algorithm (Halko *et al.* 2011), the modes have good agreement for a narrow band of frequencies only. By comparing the results from figures 8 and 9 to the values in figure

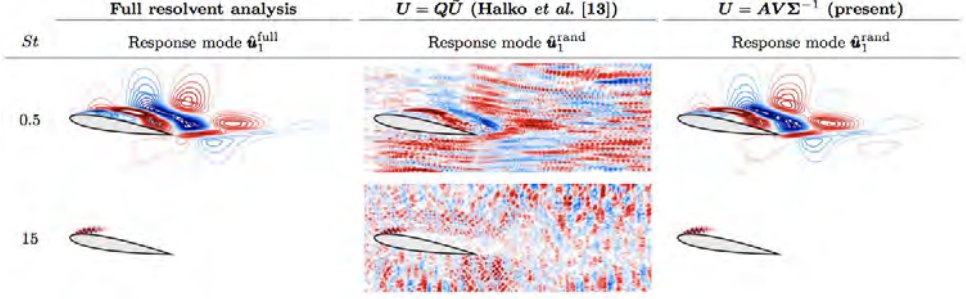


FIGURE 9. Comparison of the leading response modes ($\hat{\mathbf{u}}_1$) recovered from the randomized resolvent analyses using $U = Q\tilde{U}$ (the original approach in Halko *et al.* (2011)) and $U = AV\Sigma^{-1}$ (present, see equation (3.20)). The response modes from the full analysis are also shown for reference. Results are shown for $k_z L_c = 0$ at representative frequencies St . Modes are visualized with the streamwise velocity component with contour levels of $[-0.6, 0.6]$.

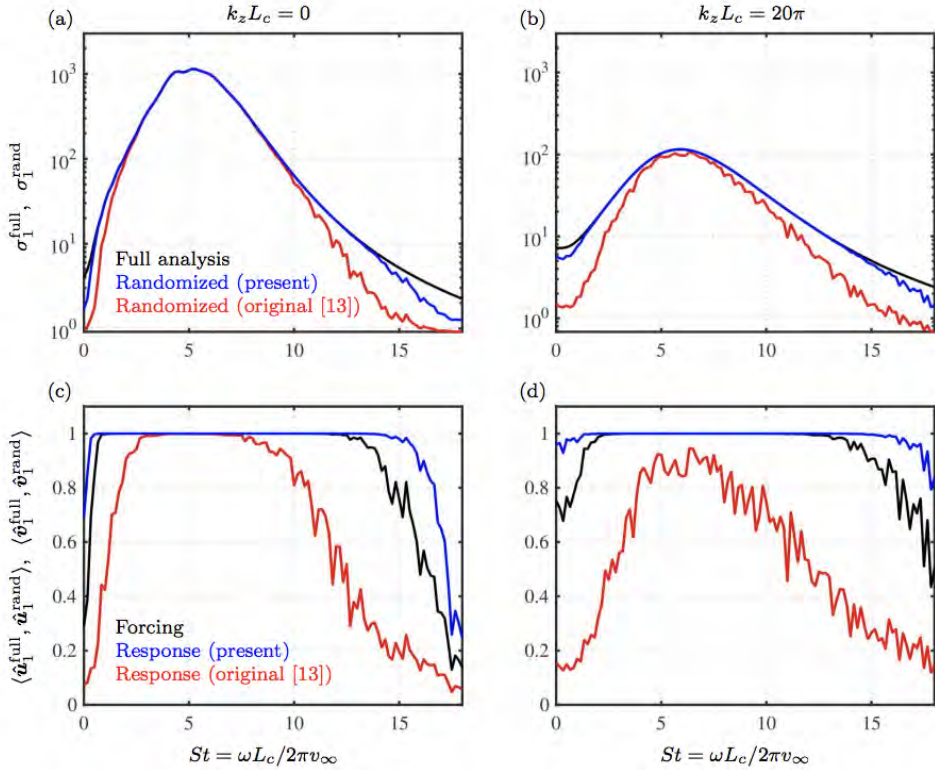


FIGURE 10. (a,b) The leading amplification for the full and randomized resolvent analyses. (c-d) Cosine similarities for the leading response $\langle \hat{\mathbf{u}}_1^{\text{full}}, \hat{\mathbf{u}}_1^{\text{rand}} \rangle$ and forcing $\langle \hat{\mathbf{v}}_1^{\text{full}}, \hat{\mathbf{v}}_1^{\text{rand}} \rangle$ modes. Improvements of accuracy in the present randomized analysis can be observed by comparing the σ_1^{rand} and $\hat{\mathbf{u}}_1^{\text{rand}}$ recovered from equation (3.20) to those from the original approach of Halko *et al.* (2011) (note that forcing modes obtained from both approaches are identical). Results for spanwise wavenumber of $k_z L_c = 0$ and 20π are shown in the left and right columns, respectively.

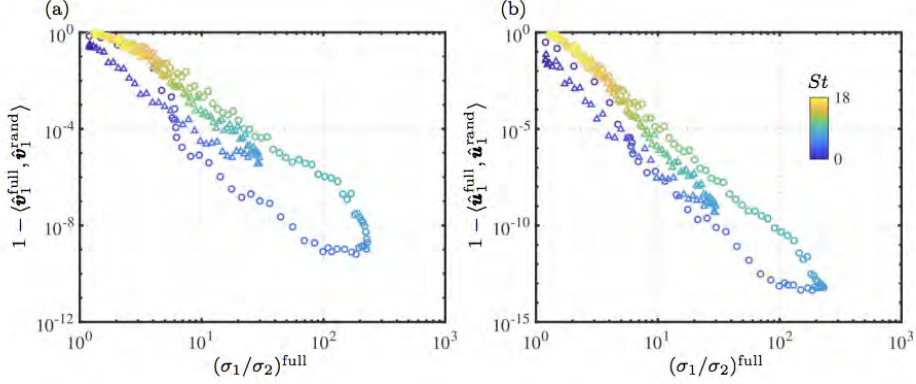


FIGURE 11. Error based on cosine similarity for the leading (a) forcing modes ($1 - \langle \hat{\mathbf{v}}_1^{\text{full}}, \hat{\mathbf{v}}_1^{\text{rand}} \rangle$) and (b) response modes ($1 - \langle \hat{\mathbf{u}}_1^{\text{full}}, \hat{\mathbf{u}}_1^{\text{rand}} \rangle$) over the leading gap from the full resolvent analysis. The symbols $\circ$ and $\triangle$ represent results from $k_z L_c = 0$ and 20π , respectively, colored by St .

10(c,d), we observe that the noise affects the modes when cosine similarity is below 0.5. For frequencies and wavenumbers with cosine similarity up to 0.8 or higher, there is no noise and the results for full and randomized resolvent agree well. For this reason, it is desirable to search for solutions that provide a reliable agreement up to this scale to a broad range of frequencies and both wavenumbers. With the high-gain frequency range well captured, randomized resolvent analysis has demonstrated its capability of predicting the dominant pathway for energy amplification over the spectral space with reduced computational cost.

As stated in section 3.2.2, the use of low-rank approximation in the randomized approach is built upon the assumption of the low-rank nature of the resolvent operator. The randomized resolvent analysis shows its strength when the singular values exhibit fast decay, as evident from figure 11. The accuracy of the modal structure captured by randomized analysis is examined with respect to the ratio of the leading and second singular values, $(\sigma_1/\sigma_2)^{\text{full}}$ from the full resolvent analysis. The error in the modal structures exhibits a decreasing trend as this ratio increases. When this ratio is close to unity, the randomized technique may not accurately separate the first and second modes. In fact, the aforementioned trailing edge structure that appeared in the randomized response mode for $k_z L_c = 20\pi$ and $St = 1$ is caused by the leakage of the structures from the second response mode (see figure 8). When the ratio $(\sigma_1/\sigma_2)^{\text{full}}$ is above 30, the error decreases to $\lesssim 10^{-5}$ for forcing modes and $\lesssim 10^{-8}$ for response modes.

Next, we study the influence of the width of the test matrix k on the error in the leading singular values and modes, as presented in figure 12. When the value of k is varied from 2 to 500, the error from the use of randomized analysis decreases. For three representative frequencies, we observe the same rate of convergence $\approx \mathcal{O}(k)$ for both the gain and cosine similarity. As stated in section 3.2.2, increasing k has the same practical effect of oversampling, in the present application. For this flow, we observe that $k = 10$ is sufficient to achieve sufficient accuracy with $\lesssim 1\%$ error, which is remarkably low when compared to the high dimensionality of the resolvent operator.

The computational cost of the randomized resolvent technique can be further reduced. For instance, the biconjugate gradient stabilized (BiCGStab) or the generalized minimum residual (GMRES) methods can be utilized to solve linear systems with appropriate preconditioners (e.g., incomplete LU and Jacobian).

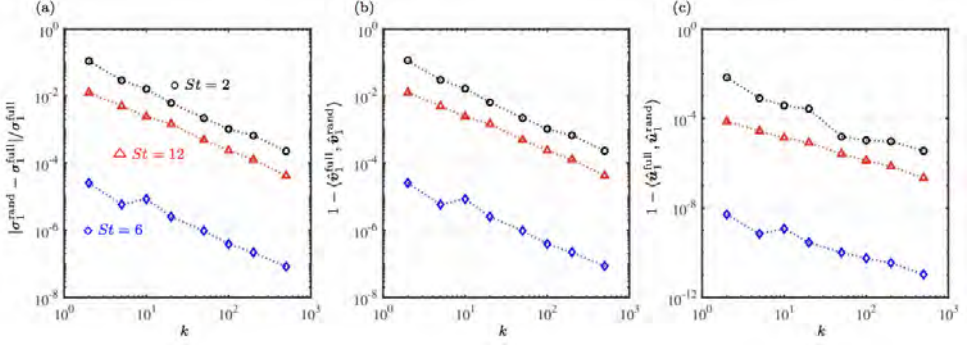


FIGURE 12. Influence of test matrix size k on the accuracy of (a) leading singular value, (b) forcing mode, and (c) response mode at $k_z L_c = 20\pi$. All exhibit $\mathcal{O}(k)$ convergence.

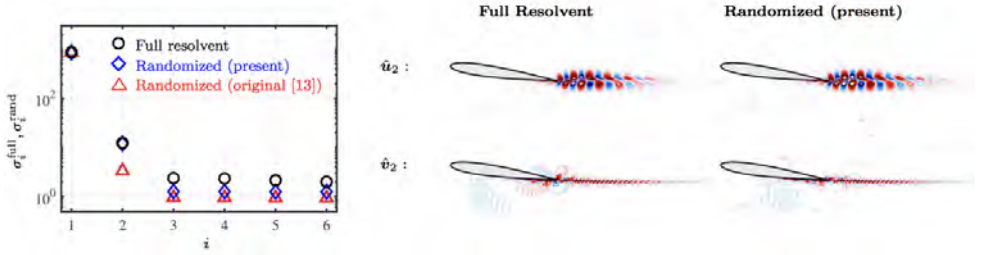


FIGURE 13. Recovery of the higher-order resolvent gains ($\sigma_{i=1-6}^{\text{rand}}$) and modes ($\hat{v}_2$ and $\hat{u}_2$) for $St = 4$ and $k_z L_c = 0$. The resolvent gains obtained from the present randomized analysis ($\diamond$, recovered by equation (3.20)) and those from the original approach of Halko *et al.* (2011) ($\triangle$) are compared to those from the full resolvent analysis ($\circ$).

3.3.3. Higher-order modes

Let us discuss the performance of the randomized technique with respect to the high-order modes. For some cases, the second largest singular value may also be spaced apart from the higher-order singular values and be determined accurately. In figure 13, we show for $k_z L_c = 0$ and $St = 4$ a case where both the leading and second singular values are spread from the rest of the singular values. In this case, the randomized algorithms accurately capture the second modes. The flow structures at the trailing edge are perceived in the response modes. The forcing modes appear over the pressure side near the trailing edge. For the results obtained from Algorithm 1, the modes are the same for randomized resolvent and for the full resolvent. However, for the resolvent analysis using the randomized SVD algorithm (Halko *et al.* 2011) within the resolvent analysis, the secondary singular values are poorly captured and the modes are polluted by background noise and leakage from other modes (not shown here). For this reason, when applying randomized resolvent, the present implementation shown in Algorithm 1 must be considered as they can approximate the detached high-order modes accurately.

3.3.4. Choice of the test matrix Ω

For the randomized resolvent analysis with a test matrix size of $k = 10$, Gaussian, orthonormal, Rademacher and ultrasparse Rademacher test matrices provide similar results and no observable difference in computational savings. By using the implementation shown in Algorithm 1, all test matrices present similar accuracy as shown in figures 10 and 11. For very low and very high St numbers in the range of frequencies analyzed in this work, where the singular values decay slowly, one can increase the size k or apply subspace or power iterations.

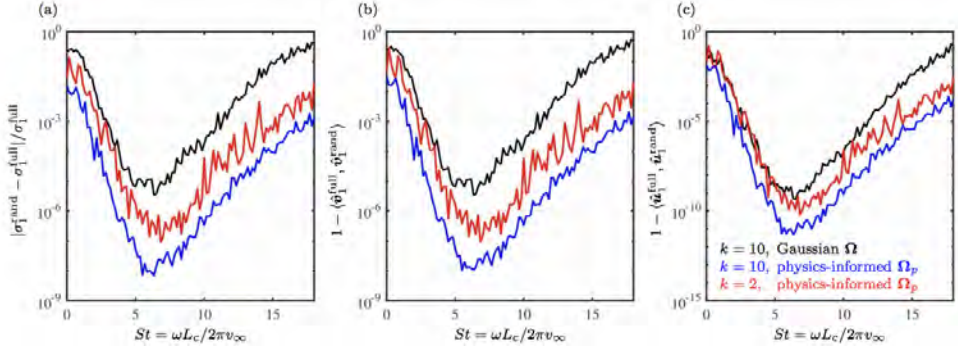


FIGURE 14. Influence of test matrices on the relative error for (a) gain distribution and cosine similarities for (b) forcing and (c) response modes, $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{u}}_1$, at $k_z L_c = 20\pi$. Results are shown for Gaussian random normal distribution test matrix Ω and physics-informed random test matrices $\Omega_p = \text{diag}(\Phi)\Omega$ with sizes $k = 10$ and $k = 2$.

However, it is possible to obtain more accurate results by constructing a random test matrix Ω that incorporates physical insights from the base flow.

While the random test matrix is effective in yielding accurate results, we can consider constructing a test matrix that can generate the entries in a smart manner by incorporating the knowledge of the base flow. We know that regions of strong shear are important in amplifying forcing inputs. Moreover, regions with minimal velocity gradients are not that important. For these reasons, the velocity gradient at each grid point can be used to scale the test matrix. Here, we propose a physics-informed test matrix, Ω_p , scaled by the 2-norm of the velocity gradient, $\|\nabla \mathbf{v}\|_2$, where $\mathbf{v}$ is the velocity vector. We construct a scaling factor $\Phi_j = \|\nabla \mathbf{v}_j\|_2$ at each grid point j . The scaling vector Φ has to be stacked according to the number of variables to reach the size m of the linear operator. The physics-informed test matrix then becomes

$$\Omega_p = \text{diag}(\Phi)\Omega. \quad (3.21)$$

The results based on the physics-informed test matrix are shown in figure 14. While the results obtained from the use of a normal distribution test matrix have shown excellent accuracy, the use of physics-informed test matrix further improves the accuracy by a few orders of magnitude for the considered frequencies and spanwise wavenumbers. More importantly, we achieve results with comparable or higher level of accuracy using a extremely low width of the test matrix of $k = 2$. This results in a considerable reduction in computation time (see Table 1). Using $k = 10$, linear systems are solved at least 30 times. Now, using $k = 2$, only 6 linear solvers are needed to obtain the same accuracy, which is achieved only with a physics-informed scaling of the test matrix. By combining randomized numerical linear algebra and some physical insights, we are now empowered to perform the input-output analysis for ever more complex 2D and 3D turbulent base flows on a standard computer, or on a high-performance computing cluster to expand the envelope of resolvent analysis.

3.4. Summary

Resolvent analysis has proven to be a powerful technique to reveal the input-output characteristics of fluid flows. However, the computational cost and memory allocation of the resolvent analysis can be taxing for high-Reynolds number flows, making it prohibitive to be applied to complex turbulent base flows. The major computational cost of the analysis is associated with the SVD of the resolvent operator. To remove this bottleneck, the randomized approach has been adopted to reduce the computational cost of SVD by considering the low-

rank approximation of the resolvent operator. This was achieved by constructing the low-rank basis based on the insights gained from the sketch of the resolvent, which is obtained from a linear system solver. For flows with fast singular value decays, e.g., flows with strong shear and separation, the randomized resolvent analysis reveals its power to accurately capture the response and forcing modes as well as the gain. Moreover, we consider the use of the velocity gradient to scale the random test matrix. Such scaling enhanced the accuracy of the randomized resolvent analysis. The necessary computation time was significantly reduced as the number of linear systems to be solved is considerably smaller. To demonstrate the capability of the randomized resolvent analysis, we analyzed the turbulent separated flow over a NACA 0012 airfoil. Excellent agreement of the leading forcing and response modes and the gains were shown between the full and randomized resolvent methods. By incorporating the knowledge of the base flow in terms of its velocity gradient into the randomized test matrix, additional speed up and accuracy enhancement were achieved. With the computational cost and memory allocation being relieved with the randomized approach, the application of the resolvent analysis can be significantly extended to higher-Reynolds number 2D and 3D base flows.

4. Resolvent analysis on the origin of transonic buffet

4.1. Introduction

Transonic flow over an airfoil exhibits complex dynamics from the interplay among the unsteady shock motion, flow separation due to the shock/boundary-layer interaction, and the generation and propagation of pressure waves. Under certain range of angles of attack and Mach number, the shock wave oscillates vigorously creating a highly unsteady flow (Lee 2001). This shock wave oscillation phenomenon known as transonic buffet is caused by a self-sustaining aerodynamic instability. The large-scale motion of the shock induces violent fluctuations in aerodynamic forces and leads to structural vibrations. The level of unsteadiness induced by transonic buffet can be so large that it can compromise the integrity of aircraft structure and flight safety. For these reasons, transonic buffet is a major limiting factor for aircraft flight envelope and has been a focus of many studies, as summarized recently by Giannelis *et al.* (2017).

Various experimental and numerical studies have been devoted to the characterization of transonic buffet. Past studies have revealed that typical oscillating frequency of shock waves is very low with the chord-based Strouhal number being approximately 0.06 (Deck 2005; Jacquin *et al.* 2009). This low frequency is distinguished from other unsteady phenomena around the airfoil related to the velocity fluctuations in the separation region and vortex shedding in the wake (Sartor *et al.* 2014). More recently, some researchers have reported that a transonic buffet on a three-dimensional wing has span-wise instability with a much higher frequency where $0.2 < St < 0.6$ (Dandois 2016; Ohmichi *et al.* 2018).

Separation on the airfoil induced by shock-wave/boundary layer interaction (SWBLI) plays an important role in transonic buffet. Early experiments (Seegmiller *et al.* 1978; Levy 1978) showed that the onset of shock oscillation occurred when the separation generated at the shock foot reaches the trailing edge of the airfoil. Recent numerical studies (Iovnovich & Raveh 2012; Grossi *et al.* 2014; Fukushima & Kawai 2018) reported that unsteadiness of the separation is closely tied to buffet. When the shock moves downstream from its upstream position, the shock becomes weaker and the flow behind the shock remains attached. In contrast, when the shock moves upstream from its downstream location, the shock becomes stronger, and causes large-scale separation.

A number of studies have analyzed the self-sustained mechanism of shock wave oscillation.

Lee (2001) argued that the pressure wave propagating between the shock wave and the trailing edge of the airfoil plays a critical role feedback to maintain shock oscillation. More recently, global linear stability analysis performed by Crouch *et al.* (2009) extracted the growth rate of the buffet fluctuation with respect to the time-averaged flow field based on the eigenvalue problem of linearized Navier-Stokes operator. They examined the linear stability of transonic flow around a two-dimensional NACA 0012 airfoil obtained from Reynolds-averaged Navier-Stokes (RANS) simulations. Analyzing the critical angle of attack and Mach number of the buffet onset, the critical values reasonably agreed with the experimental results by McDevitt & Okuno (1985). Nonetheless, the origin of the transonic buffet is still under debate, with open questions on validity of the linear stability formalism for base flows that are not the exact solution of the Navier-Stokes equations.

Recently, Sartor *et al.* (2014) employed global stability, adjoint, and resolvent analysis to examine transonic buffet flow around a NACA0012 airfoil at $Re = 3 \times 10^6$. At the buffet frequency, they used resolvent analysis to find that the essential origin of shock unsteadiness is at the shock foot region on the suction side of the airfoil. There are also reports including Nitzsche (2009) that identified various sources that can trigger buffet such as the motion of flap, the pitch oscillation, and the translation of the airfoil. Based on these observations, various methods has been developed to control buffet, including vortex generators (McCormick 1993) and trailing-edge deflection techniques (Gao *et al.* 2017).

The objective of the present study is to systematically determine the origin of two-dimensional transonic buffet flow around a NACA0012 airfoil at a low-Reynolds number of $Re = 2000$ in the absence of any modeling terms. We reveal the input-output relationship for transonic buffet with the resolvent analysis, which can extract not only the frequency response but also identify the forcing (input) and response (output) modes of flow systems (Trefethen *et al.* 1993; Jovanović & Bamieh 2005; McKeon & Sharma 2010; Yeh & Taira 2019; Taira *et al.* 2017, 2019). To validate that the identified source of buffet is indeed the driving mechanism for sustained oscillations, we perform direct numerical simulations (DNS) to show that buffet can appear even in low Reynolds number flows under appropriate conditions. This analysis not only provides the fundamental insights into transonic buffet but also serves as a building block for low-Reynolds number compressible aerodynamics ($Re \approx O(10^3 \sim 10^5)$). This area of aerodynamics has traditionally been overlooked but is now becoming important for developing high-efficiency wings and propulsion systems for unmanned aircraft on Mars (Anyoji *et al.* 2015; Munday *et al.* 2015; Koning *et al.* 2019).

4.2. Problem setup

4.2.1. Problem description

We consider two-dimensional laminar transonic flows over a NACA 0012 airfoil at a free-stream Mach number $M_\infty \equiv U_\infty/a_\infty = 0.85$, a chord-based Reynolds numbers of $Re_{L_c} \equiv U_\infty L_c/\nu_\infty = 2\,000$, and an angle of attack of $\alpha = 3^\circ$. Here, a_∞ is the free-stream sonic speed, L_c is the chord length, and ν_∞ is the free-stream kinetic viscosity. Prandtl number is set to $Pr = 0.7$. While not reported, we also considered $\alpha = 1^\circ$ and 5° . As we obtained analogous results for these angles of attack, we only present the representative case of $\alpha = 3^\circ$ herein. Shock waves form around the airfoil under these conditions, making this base flow an appropriate candidate for this study.

4.2.2. Flow simulation

We simulate transonic flows over the airfoil by solving the compressible Navier-Stokes equations in a two-dimensional setting using the finite-volume solver *CharLES*, which is second-order accurate in space and third-order accurate in time (Brès *et al.* 2017). We employ

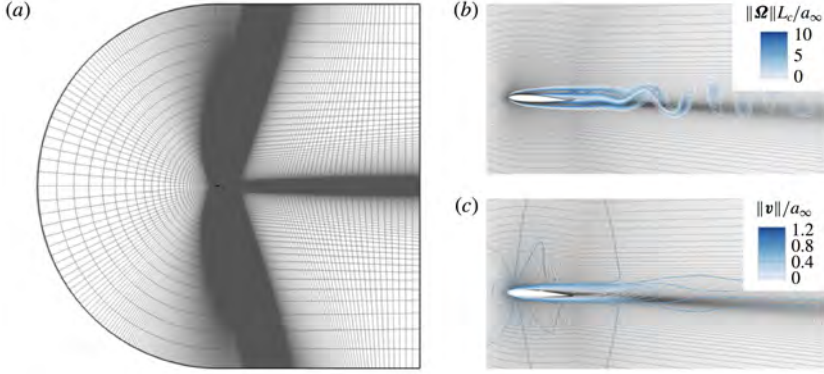


FIGURE 15. The computational grid used for numerical simulation and resolvent analysis. (a) The whole computational domain. The enlarged views around the airfoil are shown with the (b) instantaneous magnitude of normalized vorticity and (c) time-averaged velocity fields.

a second-order fully unstructured essentially non-oscillatory (ENO) method (Shi *et al.* 2002) for shock capturing. A C-shaped hexahedral mesh is used for the simulations, as shown in figure 15. The computational domain has an extent of $x_c/L_c \in [-50, 50]$ and $y_c/L_c \in [-50, 50]$, where x_c and y_c represent the chord-wise and chord-normal directions, respectively. The airfoil is positioned with its leading edge at the origin. The domain is discretized into 70 000 cells with 100 nodes on each side of the airfoil, 90 nodes along the wake and 80 nodes in the normal direction where the minimum wall-adjacent Δy is set to be $\Delta y/L_c = 1.42 \times 10^{-3}$. A grid convergence study has been conducted with respect to the time-averaged flow to ensure sufficient spatial resolution for this computational grid. At the far-field boundary, the free-stream condition is prescribed as $[\rho, v_{x_c}, v_{y_c}, T] = [\rho_\infty, U_\infty \cos \alpha, U_\infty \sin \alpha, T_\infty]$, where ρ is density and T is temperature. The free stream angle is changed through the prescription of the far-field flow velocities v_{x_c} and v_{y_c} in x_c and y_c directions, respectively. No-slip adiabatic condition is prescribed over the airfoil. Along the outlet boundary, a sponge layer (Freund 1997) is applied over $x_c/L_c \in [40, 50]$ with the target state being the running-averaged flow over $tU_\infty/L_c = 1$. The time integration is performed at a constant Courant–Friedrichs–Lewy number of 1.

4.2.3. Resolvent analysis

The objective of the present study is to reveal the input–output relationship for two-dimensional transonic buffet and identify the origin of the energetic shock-wave oscillations. In this study, we focus our resolvent analysis to two-dimensional modes which are essential in all buffet phenomena (Iovnovich & Raveh 2012; Sartor *et al.* 2014; Ohmichi *et al.* 2018). Note that resolvent analysis itself is not limited to the two-dimensional flow. Further resolvent analysis of transonic buffet over the three-dimensional wings could extract the spanwise modes, which have been reported in previous studies (Iovnovich & Raveh 2015; Ohmichi *et al.* 2018).

We consider the Reynolds decomposition of the flow variable $\mathbf{q}(\mathbf{x}, t) = [\rho, v_x, v_y, T]$ into the sum of time-averaged base state $\bar{\mathbf{q}}(\mathbf{x})$ and the statistically stationary fluctuating component $\mathbf{q}'(\mathbf{x}, t)$, which enables us to express the Navier–Stokes equation as

$$\partial_t \mathbf{q}' = \mathbf{L}_{\bar{\mathbf{q}}} \mathbf{q}' + \mathbf{B} \mathbf{f}', \quad (4.22)$$

where $\mathbf{L}_{\bar{\mathbf{q}}}$ is the linearized Navier–Stokes operator about the mean flow $\bar{\mathbf{q}}$ (McKeon & Sharma 2010) and $\mathbf{B}$ is the input matrix that can spatially window the forcing input $\mathbf{f}'$. We construct the discrete linear operator $\mathbf{L}_{\bar{\mathbf{q}}}$, incorporating the boundary conditions of $[\rho', v'_{x_c}, v'_{y_c}, \nabla_n T'] =$

0 over the airfoil and the far field, and $\nabla_n[\rho', v'_{x_c}, v'_{y_c}, T'] = 0$, at the computational outlet, where ∇_n denotes the surface-normal gradient. The spacial discretization for $\mathbf{L}_{\bar{q}}$ is performed on the same mesh used in the flow simulation, as shown in figure 15. When the time-averaged flow is chosen as the base state in resolvent analysis, the finite-amplitude nonlinear terms with respect to $\mathbf{q}'$ are incorporated in $\mathbf{f}'$, which can be interpreted as a sustained forcing input within the natural feedback system (McKeon & Sharma 2010). Moreover, we consider an observable output vector $\mathbf{y}$ given by $\mathbf{y} = \mathbf{C}\mathbf{q}'$, such that spatial windowing $\mathbf{C}$ can be applied in general (Jeun *et al.* 2016; Schmidt *et al.* 2018).

With the Fourier representation $\mathbf{q}'(\mathbf{x}, t) = \int_{-\infty}^{\infty} \hat{\mathbf{q}}_{\omega}(\mathbf{x}) e^{-i\omega t} d\omega$, equation (4.22) can be expressed in frequency space as

$$-i\omega \hat{\mathbf{q}}_{\omega} = \mathbf{L}_{\bar{q}} \hat{\mathbf{q}}_{\omega} + \mathbf{B} \hat{\mathbf{f}}_{\omega} \quad \text{with} \quad \hat{\mathbf{y}}_{\omega} = \mathbf{C} \hat{\mathbf{q}}_{\omega}. \quad (4.23)$$

The input–output relationship between the input (forcing) $\hat{\mathbf{f}}_{\omega}$ and the output (response) $\hat{\mathbf{y}}_{\omega}$ at a specified frequency ω becomes

$$\hat{\mathbf{y}}_{\omega} = \mathbf{R}_{\bar{q}}(\omega) \hat{\mathbf{f}}_{\omega}, \quad \text{where} \quad \mathbf{R}_{\bar{q}}(\omega) = \mathbf{C} [-i\omega \mathbf{I} - \mathbf{L}_{\bar{q}}]^{-1} \mathbf{B}. \quad (4.24)$$

Here, $\mathbf{R}_{\bar{q}}(\omega)$ is referred to as the resolvent (operator). It serves as a transfer function that amplifies (or attenuates) the harmonic forcing input $\hat{\mathbf{f}}_{\omega}$ and maps it to the response $\hat{\mathbf{y}}_{\omega}$. In this study, the base flows are found to be stable according the eigenvalues of the linear operators $\mathbf{L}_{\bar{q}}$. Therefore, we consider real-valued frequency ω in constructing $\mathbf{R}_{\bar{q}}(\omega)$ for our resolvent analysis (Jovanović 2004; Yeh & Taira 2019).

Resolvent analysis identifies the dominant directions along which $\hat{\mathbf{f}}_{\omega}$ can be most amplified through $\mathbf{R}_{\bar{q}}(\omega)$ to form the corresponding response $\hat{\mathbf{y}}_{\omega}$. This is accomplished by performing the singular value decomposition (SVD) of the resolvent

$$\mathbf{R}_{\bar{q}}(\omega) = \mathbf{Y} \mathbf{\Sigma} \mathbf{F}^*, \quad (4.25)$$

where $\mathbf{F}^*$ denotes the Hermitian of $\mathbf{F}$. Resolvent analysis interprets left and right singular vectors $\mathbf{Y} = [\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \dots, \hat{\mathbf{y}}_m]$ and $\mathbf{F} = [\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \dots, \hat{\mathbf{f}}_m]$, respectively, as response modes and forcing modes, with the magnitude-ranked singular values $\mathbf{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_m)$ being the amplification (gain) for the corresponding forcing–response pair at frequency ω . We examine the modal structures of the dominant forcing and response modes $[\hat{\mathbf{f}}_1, \hat{\mathbf{y}}_1]$ to study the origin of transonic buffet over this canonical airfoil.

The application of spatial window to the global response $\hat{\mathbf{q}}_{\omega}$ can be implemented by specifying $\mathbf{C}$ as a diagonal weight matrix with unit weights inside the window and zeros otherwise. This window is chosen to highlight the location over the suction surface where oscillatory motion of the shock wave appears and limits the output to be within inside of window, as is illustrated in figure 17. While matrix $\mathbf{B}$ for forcing can be designed in a similar manner, we use $\mathbf{B} = \mathbf{I}$ such that there is no restriction on the region of forcing to identify the source of buffet. The use of window $\mathbf{C}$ in the resolvent analysis can help reveal the optimal energy amplification from forcing to a local response in the shock region over the suction surface of the airfoil.

4.3. Results

4.3.1. Base flow

We simulate the unsteady two-dimensional flow over a NACA 0012 airfoil at $\alpha = 3^\circ$ for $M_{\infty} = 0.85$ and $Re_{L_c} = 2000$. An instantaneous vorticity field from DNS is shown in figure 16(a). The laminar vortex sheets generated from the pressure and suction sides of the wing rolls into vortices in the wake. This representative transonic flow at low Reynolds number exhibits a distinct shedding frequency of $St \equiv f L_c / U_{\infty} = 1.0$. This frequency can be detected

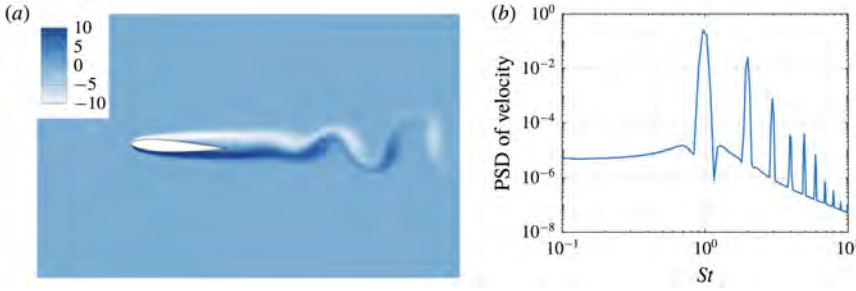


FIGURE 16. Unsteady transonic flow over a NACA0012 airfoil. (a) Instantaneous vorticity field ($\Omega_z L_c / a_\infty$). (b) Power spectral density of v_y / a_∞ probed at $(x, y) / L_c = (2.10, -0.11)$.

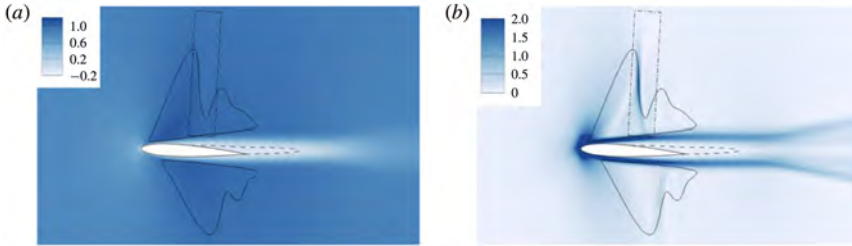
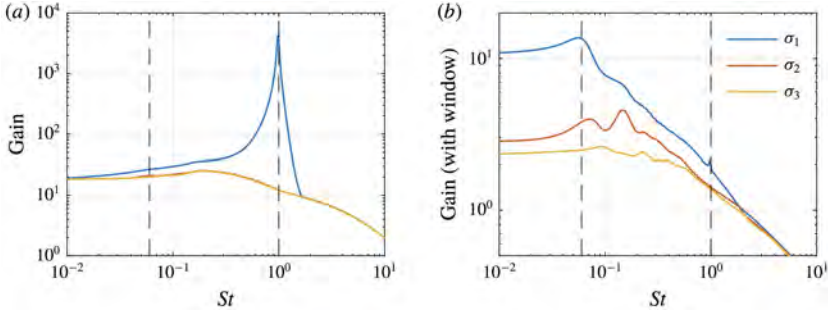
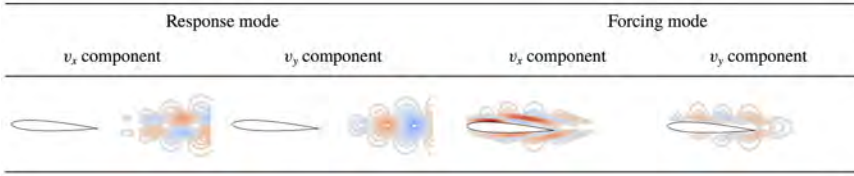


FIGURE 17. Time-averaged flow field around the NACA0012 airfoil. (a) Streamwise velocity (v_x / a_∞). (b) Numerical schlieren with density gradient magnitude ($\|\nabla \rho\| L_c / \rho_\infty$). Regions enclosed by the solid and broken curves represent the supersonic and recirculation regions, respectively. The dashed-dotted box depicts the spatial window for the filtered resolvent analysis.

from the power spectral density (PSD) of the velocity probe at $(x, y) / L_c = (2.10, -0.11)$, along with its harmonics, in figure 16(b). For this flow, we do not directly observe buffet over the airfoil. Nitzsche (2009) investigated the frequency response of shock waves to small perturbations and found that the external forcing to transonic flow around an airfoil can trigger shock wave oscillations even if the base flow is stable. He has reported that the maximum gain was observed close to the buffet onset angle. The present resolvent analysis holds great potential to reveal the optimal input-output relationship with respect to the onset of buffet.

To perform resolvent analysis, we consider the time-averaged flow to be the base state. The temporal averaging is performed over $tU_\infty / L_c \in [0, 350]$ to ensure statistical convergence. This long window for average allows us to set the minimum frequency in the resolvent analysis at 2.9×10^{-3} , which is sufficiently small compared with the dominant buffet frequency of $St \approx 0.07$ (Jacquin *et al.* 2009; Sartor *et al.* 2014). The time-averaged stream-wise velocity and the numerical schlieren fields are shown in figure 17. The solid and broken curves in figures 17(a) and (b) indicate the contour lines for $M = 1$ and $v_x = 0$, respectively. The area enclosed in the solid curve corresponding to the supersonic region, while the region surrounded by the broken curves represents the separated flow region. Weak shock waves are observed in the supersonic regions. The supersonic region on the suction side is further divided into two stages of acceleration–deceleration phases according to the λ -type structure of the shock waves (Delery 1985). The first stage corresponds to the initial flow acceleration near the leading edge. The flow surpasses the sonic speed and decelerates after the first shock wave. Over the airfoil, the flow undergoes the second stage of acceleration and forms the second shock structure near the trailing edge. Strong inverse pressure gradient due to λ -type shock waves are observed aft of the first acceleration.

FIGURE 18. (a) Original and (b) windowed resolvent gains over St .FIGURE 19. Response and forcing modes normalized by the velocity magnitude for $St = 1$.

4.3.2. Resolvent analysis

As transonic buffet is characterized by the energetic shock-wave oscillation over the airfoil, the cause of buffet may be uncovered by closely studying the input–output process at the relevant fluctuation frequency using the resolvent analysis described in Section 4.2.3. Let us first present in figure 18(a) the distribution of the leading gain σ_1 over a range of frequencies from resolvent analysis without windowing. The leading gain σ_1 shows a distinct peak at $St = 1$ corresponding to the von Kármán shedding frequency. However, there is no other distinct peaks appearing in the gain distribution for σ_1 . This is expected since the dominant physics for this low Reynolds number base flow is the wake shedding. We do not observe noticeable peaks even for subdominant modes.

Let us present the dominant resolvent modes for $St = 1$ without windowing. Since the gain peaks at the shedding frequency, response modes exhibit the modal shapes that are representative of convective dynamics in the wake, as visualized in figure 19. The forcing modes are present in the vicinity of the airfoil where shear is large, as it can be observed from figure 17. This is in line with our understanding of the formation of wake vortices from the roll up of shear layers. These observations are also in agreement with modal analysis for subsonic flows over airfoils (Zhang & Samtaney 2016; Yeh & Taira 2019).

Next, let us consider the gain distribution from the windowed resolvent analysis. Here, the output is windowed by operator $\mathbf{C}$ with a spatial profile that is restricted to the region in the vicinity of the shock, as shown in figure 17. With the output of resolvent analysis focusing on the fluctuations close to the standing shock, we find that the dominant gain σ_1 shows a peak at the buffet frequency of $St = 0.06$, which is in agreement with past numerical and experimental findings (Deck 2005; Jacquin *et al.* 2009; Dandois 2016; Sartor *et al.* 2014), which were conducted at high Reynolds numbers of $Re_{Le} > O(10^6)$. For subdominant gains, we observe peaks appearing at superharmonics that capture smaller-scale oscillations. We can observe a minor peak in σ_1 at $St = 1$. However, the overall amplification process for the shock region is focused toward the low-frequency components at the buffet frequency. This suggests that the amplification physics for buffet is buried underneath the global input-output dynamics at this low Reynolds number. We also note that the conclusive assessment on the

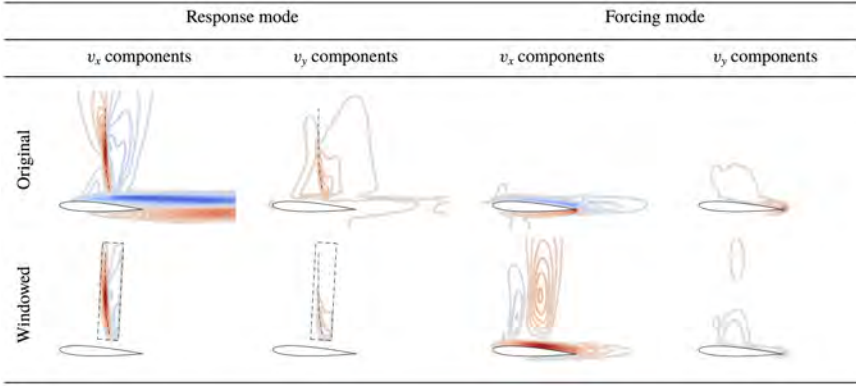


FIGURE 20. The original (top) and the windowed (bottom) resolvent modes normalized by the velocity magnitude for $St = 0.06$. The dashed-dotted lines depicts the position of the shock on the suction side.

buffet physics is robust against the choice of the response window. A prominent peak at the buffet frequency $St = 0.06$ with similar forcing structure is still revealed using a different window that covers the entire suction surface over $x_c \in [0, 1]$ and $y_c \in [0, \infty)$.

We visualize the response and forcing modes that correspond to the buffet frequency of $St = 0.06$ as identified from the gain distribution with windowed resolvent analysis. These modes in figure 20 are quite different from the modes at $St = 1$. While the response modes for $St = 1$ exhibit convective oscillations in the wake region, the response modes for $St = 0.06$ are predominantly supported in the region of the standing shock. The windowed analysis is able to highlight the response in the region of the shock. For these reasons, the response modes can be seen as the oscillatory mechanism for buffet.

The input to generate the buffet unsteadiness can be identified by studying the forcing modes. We can observe from figure 20 that the forcing modes from both resolvent analysis with and without windowing show the boundary layer around the airfoil to be the source of buffet. In particular, from the windowed forcing mode, we find that the region corresponding to the shock foot is the most sensitive region to instigate the buffet phenomenon, which is in agreement with Sartor *et al.* (2014) who have carried out resolvent analysis at $Re = 3 \times 10^6$. In a number of past studies (Deck 2005; Grossi *et al.* 2014; Fukushima & Kawai 2018), the SWBLI at the shock foot exhibited turbulent boundary layer separation, which was suspected to play a role in the emergence of buffet. The current result by the windowed resolvent analysis suggests that the low-frequency buffet can be stimulated even though the boundary layer is entirely laminar at a significantly lower Reynolds number of $Re = 1000$. Interestingly, the response modes do not show fluctuations about the weaker pressure-side shock wave observed in the numerical schlieren of figure 17(b).

Let us also bring attention to the vicinity of the trailing edge for the original forcing modes in figure 20. We can observe that there is a compact region at the trailing edge that appear in the original forcing modes. Since the dominant response mode cover the wing surface, its fluctuation alters the circulation of the airfoil, making the trailing edge to act as sensitive location due to its singular nature from the cusp geometry. For this reason, we see the trailing edge also supporting the fluctuations in the original forcing mode. This observation is in tune with the report of Nitzsche (2009) and past control techniques using trailing-edge deflector to effectively attenuate buffet (Gao *et al.* 2017). The blockage of flow around the trailing edge by the deflector can obstruct the change in circulation in a kinematic manner. The current resolvent analysis with windowing is able to identify these sources of buffet in a clear

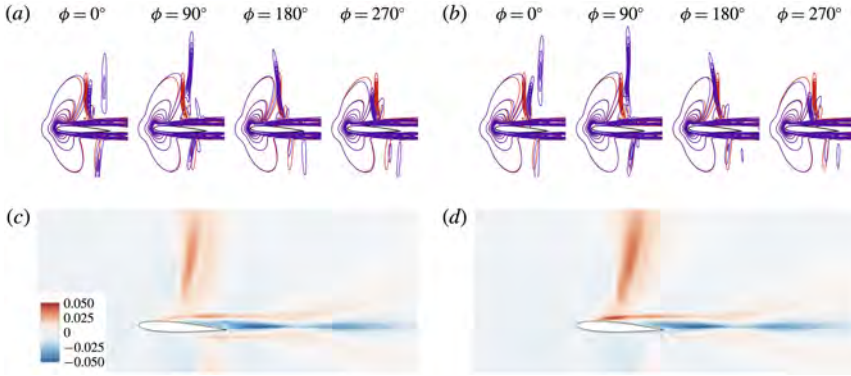


FIGURE 21. Transonic flows over the airfoil with perturbations added in the form of the dominant forcing modes from original (left column) and windowed resolvent analysis (right column). The motions of the shock wave are visualized in (a-b) with the blue contour lines in four phases ϕ_f with respect to the forced buffet cycle, on top of the red lines for unforced case. Both contour lines display levels of $\|\nabla \rho\| L_c / \rho_\infty$. Visualized in (c-d) are differences of the rms of streamwise velocity fields $\Delta v_{\text{rms},x} / a_\infty$, where $\Delta v_{\text{rms},x} \equiv v_{\text{rms},x,\text{forced}} - v_{\text{rms},x,\text{unforced}}$.

manner. While the present discussion is only concerned with $\alpha = 3^\circ$, we note that similar observations are made for other angles of attack of $\alpha = 1^\circ$ and 5° .

To corroborate our findings from resolvent analysis, we perform companion simulations of transonic flows over the airfoil with perturbations added in the shape of forcing modes. We add the forcing to the flow as a body force, $\mathbf{f}_{\text{body}}$, with the spatial profiles given by the dominant forcing modes at $St = 0.06$ (see figure 20). The amplitude of forcing is chosen such that the momentum coefficient $C_\mu \equiv \|\mathbf{f}_{\text{body}}\| / (\frac{1}{2} \rho_\infty U_\infty^2 L_c) = 0.003$, which is low for this viscous flow problem (Munday & Taira 2018). These perturbations with the spatial profiles of the primary forcing modes without and with windowing do indeed stimulate the emergence of buffet over the airfoil, as shown in figure 21 (a-b). Even at this low Reynolds number, the standing shock is found to oscillate violently with large-amplitude motion over the airfoil. The amplification in the oscillation from the forcing input to the flow response (including nonlinear effects) is assessed using $\Gamma = \kappa \Delta \|\mathbf{u}_{\text{rms}}\| / \|\mathbf{f}_{\text{body}}\|$, where $\Delta \|\mathbf{u}_{\text{rms}}\| \equiv \|\mathbf{u}_{\text{rms}}\|_{\text{forced}} - \|\mathbf{u}_{\text{rms}}\|_{\text{unforced}}$ and $\kappa \equiv \rho_\infty f^+$, showing higher amplification of $\Gamma = 9.4$ using the forcing mode from the original resolvent analysis, compared to $\Gamma = 7.0$ using that from the windowed analysis. The higher amplification agrees with the higher gain at $St = 0.06$ obtained from the original resolvent analysis (see figure 18), which measures the global response to the forcing input. Although the use of windowed resolvent forcing profile shows lower amplification, the shock oscillation coupled with the shear layer fluctuation at the shock foot is more energetic, as observed in figure 21 (d). These results validate the insights gained from resolvent analysis. Perturbations at the foot of the standing shock becomes amplified through the linear dynamics about the base flow to oscillate the shock in a violent manner, even at a Reynolds number that is commonly not associated with buffet.

4.4. Summary

We examined the origin of two-dimensional transonic buffet over a NACA0012 airfoil at a Reynolds number of 2000 through resolvent analysis. While the transonic base state at this Reynolds number exhibits unsteadiness only due to the von Kármán shedding, we show that the amplification mechanism for buffet is present in the global dynamics. The response and forcing modes from the windowed resolvent analysis revealed that the source of transonic buffet lies at the shock foot. Perturbations within the boundary layer at the shock foot can

be amplified to produce oscillations about the standing shock over the airfoil with a low frequency of $St = 0.06$. We also noted that such perturbations are closely tied to the change in the flow around the trailing edge suggesting the effectiveness of trailing-edge buffet control devices. The identified buffet mechanism was validated by companion DNS that generated buffet by perturbing the flow through the forcing modes. The results show that even at a Reynolds number much lower than what is traditionally associated with transonic buffet, we are able to instigate buffet through the hidden amplification mechanisms. This mechanism does not require the flow to be at high Reynolds number. The findings from the present study offers fundamental insights not only into the origin of buffet but also for low-Reynolds-number compressible aerodynamics with the growing interest in the development of Martian aircraft.

5. Phase reduction analysis

5.1. Introduction

Synchronization is a phenomenon where a system with oscillatory dynamics exhibits its oscillations in unison (Winfree 1967; Guckenheimer 1975; Kuramoto 1984; Strogatz 2000; Pikovsky *et al.* 2003). Such phenomenon was first reported by Huygens, the inventor of the pendulum clock, who noticed that pendulums of two different clocks on a beam always swung in phase or antiphase (Pikovsky *et al.* 2003). Since then, there have been a variety of studies on uncovering the synchronization mechanisms of oscillators, including oscillatory chemical reactions (Kuramoto 1984), synchronous flashing of fireflies (Buck 1988), and jet lag through inter-neuronal communications (Yamaguchi *et al.* 2013).

Even if the overall physics involving synchronization is complex and high-dimensional, the analysis of synchronization can be simplified immensely for weak perturbations by considering its phase dynamics. Such formulation allows us to focus solely on the scalar phase of the oscillator instead of having to be concerned of the full state dynamics. This approach is known as the phase-reduction analysis and enables an elegant formulation to examine a set of interacting oscillators (Winfree 1967; Ermentrout & Terman 2010; Kuramoto 1984; Nakao 2016). The phase-based analysis reveals its strength in analyzing networked dynamics and the emergence of synchronized motion and patterns.

While the phase-reduction analysis has been applied to numerous problems in biological (Winfree 1967; Ermentrout & Terman 2010) and chemical systems (Kuramoto 1984), there has been limited use in the field of unsteady fluid dynamics, especially with regard to synchronization. Some of the few uses of phase-based analysis to fluid mechanics have been concerned with Hele–Shaw flows (Kawamura & Nakao 2013, 2015). In the present work, we apply the phase-based analysis to full nonlinear periodic wake flows to examine synchronization. The discussion herein provides an alternative perspective on examining fluid flows through a single scalar phase variable, which enables significant simplification to the theoretical analysis and reduction in computational and experimental efforts to examine lock on.

In what follows, we present the phase-reduction analysis (Nakao 2016) in the context of periodic fluid flows. The perspective to examine fluid flows with phase dynamics is provided in section 5.2, with focus on analyzing synchronization of fluid flows to external excitation. The phase-reduction analysis is then applied to an example of unsteady cylinder wake with periodic forcing in section 5.3. The condition for synchronization is determined from phase-response analysis revealing the Arnold tongue for lock on. Concluding remarks are offered in section 5.4. The approach described below can be performed through simulations and experiments, making it attractive for analyzing a range of fluid flow problems, including unsteady flows, active flow control, fluid-structure interactions, and reacting flows.

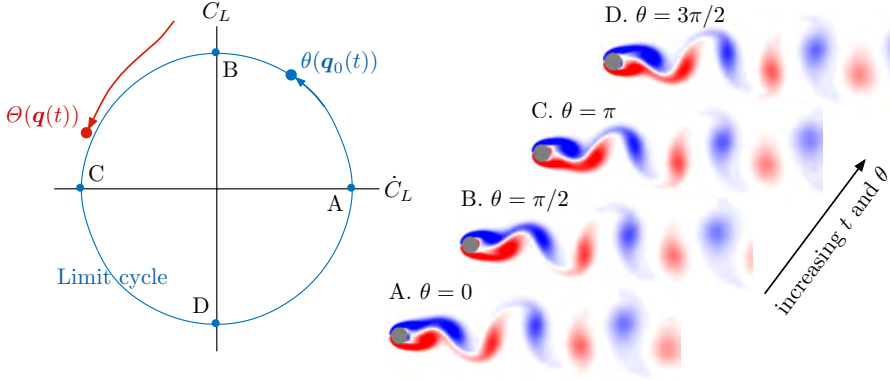


FIGURE 22. Definition of phase θ and phase function Θ with C_L and $\dot{C}_L$ for cylinder flow at $Re = 100$. Vorticity fields are shown at representative phases.

5.2. Phase-reduction analysis

5.2.1. Phase-reduction analysis of periodic flows

Let us consider the dynamics of incompressible flow represented as

$$\dot{\mathbf{q}} = \mathbf{F}(\mathbf{q}), \quad (5.26)$$

where $\mathbf{q}$ is the flow variable that has a stable periodic flow (limit cycle) $\mathbf{q}_0$ satisfying

$$\mathbf{q}_0(\mathbf{x}, t + T) = \mathbf{q}_0(\mathbf{x}, t). \quad (5.27)$$

For this limit cycle, the periodicity is denoted by T that gives the natural frequency of $\omega_n = 2\pi/T$. As expressed in equation (5.27), flows are generally described in terms of spatial and temporal variables. For incompressible flow, we can take $\mathbf{q}$ to be the velocity vector. In this work, we instead describe the state of the oscillatory flow using the phase θ of the limit cycle such that

$$\dot{\theta}(t) = \omega_n, \quad \text{where } \theta(t) \in [-\pi, \pi]. \quad (5.28)$$

The main idea here is to reduce the description of the flow from the flow state $\mathbf{q}(\mathbf{x}, t)$ to a single scalar phase variable θ (Winfree 1967; Guckenheimer 1975; Kuramoto 1984). If we have a periodic flow, the phase can point back to the full periodic flow field $\mathbf{q}_0(\theta)$. The description of the phase is given in figure 22 for an example of periodic cylinder flow, which will be examined in detail later in section 5.3. In this example, the phase dynamics of the flow can be understood by examining the lift C_L on the cylinder, which holds the same synchronization properties as that of the flow. The limit cycle of periodic cylinder flow is shown in blue over the $\dot{C}_L$ - C_L phase space, with the vorticity fields visualized at representative phases.

The above characterization of the phase is limited to flow states that reside on the limited cycle $\mathbf{q}_0$. We can extend the concept of phase in the basin of the limit cycle and define a phase function $\Theta(\mathbf{q}(\mathbf{x}, t))$ that returns the phase, i.e., $\theta = \Theta(\mathbf{q})$. The phase function can capture the phase dynamics off the limit cycle as illustrated in figure 22. By combining equations (5.26) and (5.28), we observe that the phase function satisfies

$$\dot{\theta}(t) = \dot{\Theta}(\mathbf{q}) = \nabla_{\mathbf{q}}\Theta(\mathbf{q}) \cdot \dot{\mathbf{q}} = \nabla_{\mathbf{q}}\Theta(\mathbf{q}) \cdot \mathbf{F}(\mathbf{q}) = \omega_n. \quad (5.29)$$

This phase function enables the phase to be defined over the entire phase space. A surface of constant phase is referred to as an isochron and is directed perpendicular to $\nabla\Theta(\mathbf{q})$ (Guckenheimer 1975). The notion of isochron is closely related to the eigenfunction of the Koopman operator (Mauroy & Mezić 2012).

5.2.2. Phase-reduction analysis of perturbed flows

Now, let us consider perturbing the governing dynamics by

$$\dot{\mathbf{q}} = \mathbf{F}(\mathbf{q}) + \varepsilon \mathbf{f}(t), \quad (5.30)$$

where we have added a small perturbation $\varepsilon \mathbf{f}(t)$ with $\varepsilon \ll 1$ and $\|\mathbf{f}\| = 1$. The dynamics of the phase under perturbation can be determined through

$$\dot{\theta}(t) = \dot{\Theta}(\mathbf{q}(t)) = \nabla_{\mathbf{q}} \Theta(\mathbf{q}) \cdot \dot{\mathbf{q}}(t) = \nabla_{\mathbf{q}} \Theta(\mathbf{q}) \cdot [\mathbf{F}(\mathbf{q}) + \varepsilon \mathbf{f}(t)]. \quad (5.31)$$

We can evaluate $\nabla_{\mathbf{q}} \Theta$ along the limit cycle with $\mathbf{q}_0$, assuming that the higher-order terms are negligible since the added perturbation is weak. Recalling equation (5.29), we then arrive at

$$\dot{\theta}(t) = \omega_n + \varepsilon \mathbf{Z}(\theta) \cdot \mathbf{f}(t), \quad (5.32)$$

where $\mathbf{Z}(\theta) \equiv \nabla_{\mathbf{q}} \Theta(\mathbf{q})|_{\mathbf{q}=\mathbf{q}_0}$ is called the phase-sensitivity function. Given the perturbation function, we can determine the influence of the perturbation function on the phase dynamics through equation (5.32) if $\mathbf{Z}(\theta)$ is known.

This phase-sensitivity function can be determined in two ways (Nakao 2016). We can find $\mathbf{Z}(\theta)$ by examining the phase response to impulse perturbations (direct method) or considering the adjoint dynamics (adjoint method; Ermentrout & Terman (2010)). In this work, we consider the former impulse response approach that can be used in both computational and experimental studies. We consider here the influence of a weak impulse perturbation $\mathbf{I} = I\delta(t - t_0)h(\mathbf{x})\hat{\mathbf{e}}_f$ on the state variable $\mathbf{q}$, where $\hat{\mathbf{e}}_f$ is the unit vector in the forcing direction and $h(\mathbf{x})$ describes the spatial profile of the impulse. This impulse can be added to the governing dynamics such that

$$\dot{\mathbf{q}} = \mathbf{F}(\mathbf{q}) + \mathbf{I} \quad (5.33)$$

at some phase θ (at a chosen t_0). With the introduction of this impulse, the phase dynamics becomes affected causing a shift in the phase along the limit cycle state at steady oscillating conditions.

The phase change at asymptotic state in response to the impulse is known as the phase-response function $g(\theta; I\hat{\mathbf{e}}_f)$ and is dependent on the phase at which the impulse is added as well as its location and strength, as illustrated in figure 23(a). Given a weak perturbation, the phase-sensitivity function can be evaluated as

$$Z_j(\theta) = \lim_{I \rightarrow 0} \frac{g(\theta; I\hat{\mathbf{e}}_f)}{I} \approx \frac{g(\theta; I\hat{\mathbf{e}}_f)}{I}. \quad (5.34)$$

By introducing the impulse perturbations over a range of θ , the phase-sensitivity function can be determined, as shown in figure 23(b). If the phase dynamics of the state variable $\mathbf{q}$ is followed by an observable (sensor output), the observable such as lift in figure 23(a) can be used to determine the phase response. This enables experimental efforts with limited measurements determine the phase-sensitivity function without difficulty.

5.2.3. Synchronization of phase dynamics to external oscillation

With the oscillatory dynamics of fluid flow reduced to its phase dynamics, we can further analyze the condition for which the flow exhibits synchronization to an external periodic perturbation $\mathbf{f}(t)$ with a frequency ω_f (and period $T_f = 2\pi/\omega_f$) that is close to ω_n . To facilitate our analysis, we consider the relative phase $\phi(t)$ between the phase $\theta(t)$ of the flow and the phase of the perturbation $\omega_f t$

$$\phi(t) \equiv \theta(t) - \omega_f t. \quad (5.35)$$

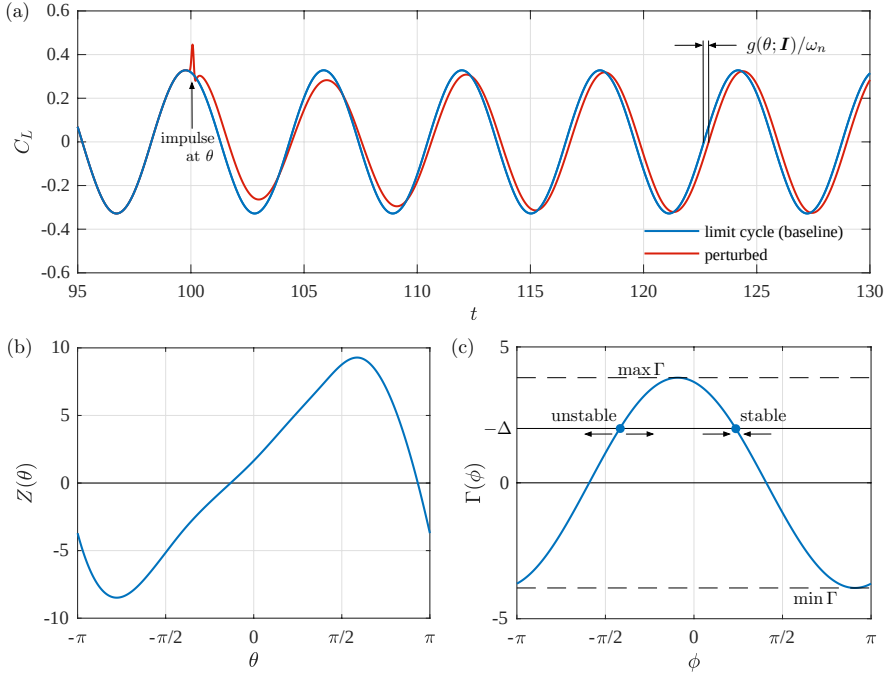


FIGURE 23. (a) Determining the phase-response function $g(\theta; \mathbf{I})$ from impulse input. (b) Phase-sensitivity function. (c) Phase-coupling function and stability condition for phase difference ϕ between the system dynamics and oscillatory actuation. For graphical clarity, $I = 0.025$ is used here. All subsequent phase-reduction analysis uses $I = 0.01$.

Using equation (5.32), we find the time rate of change of the relative phase is

$$\dot{\phi}(t) = \omega_n - \omega_f + \varepsilon \mathbf{Z}(\phi(t) + \omega_f t) \cdot \mathbf{f}(t) = \varepsilon [\Delta + \mathbf{Z}(\phi(t) + \omega_f t) \cdot \mathbf{f}(t)], \quad (5.36)$$

where the difference in the natural frequency ω_n and the perturbation frequency ω_f is expressed as $\varepsilon \Delta = \omega_n - \omega_f$ with $\Delta = \mathcal{O}(1)$.

By examining the stability characteristics, we can determine when the oscillatory dynamics of the flow can synchronize to external oscillation. That is, when $|\dot{\phi}| \rightarrow 0$, the oscillation frequency of the flow converges towards the actuator frequency. At this moment, equation (5.36) describes the dynamics of the phase difference $\phi(t)$ in a non-autonomous manner. To perform a stability analysis, it is desirable to have equation (5.36) expressed in autonomous form. By realizing that $\phi(t)$ varies slowly compared to $\omega_f t$, we can approximate the right-hand side of equation (5.36) by its average over a period of actuation (Kuramoto 1984; Ermentrout & Terman 2010) to arrive at

$$\dot{\phi}(t) = \varepsilon [\Delta + \Gamma(\phi)], \quad (5.37)$$

where

$$\Gamma(\phi) = \frac{1}{T_f} \int_t^{t+T_f} \mathbf{Z}(\phi + \omega_f \tau) \cdot \mathbf{f}(\tau) d\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{Z}(\phi + \psi) \cdot \mathbf{f}(\psi/\omega_f) d\psi. \quad (5.38)$$

The function $\Gamma(\phi)$ is 2π -periodic and is known as the phase-coupling function. With equation (5.37), we now have a one-dimensional autonomous evolution equation for ϕ .

Through inspection of equation (5.37), we find that the dynamics exhibits stable behavior if $\min_{\phi} \Gamma(\phi) < -\Delta < \max_{\phi} \Gamma(\phi)$, which translates to

$$\varepsilon \min_{\phi} \Gamma(\phi) < \omega_f - \omega_n < \varepsilon \max_{\phi} \Gamma(\phi). \quad (5.39)$$

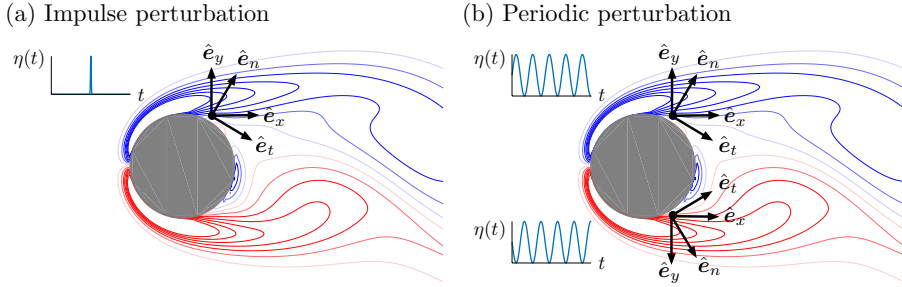


FIGURE 24. Cylinder flow forcing setup to analyze (a) the impulse response with top actuator only and (b) the effect of periodic forcing with top and bottom actuators. Selected forcing directions are x direction (x), y direction (y), tangential (t), and normal (n).

The phase-coupling function $\Gamma(\phi)$ is shown for the example of cylinder flow in figure 23(c) with an illustration of the stability condition. The value of the right hand side of equation (5.37) describes the directions of the solution with respect to the equilibrium points, which are indicated by the arrows in figure 23(c). These conditions determine the region of stability over the $\omega_f - \varepsilon$ space, known as the Arnold tongue (Arnol'd 1997). This will be shown later in Section 5.3.3 (see figure 26). The decay of frequency difference between the natural limit cycle and the forcing input constitutes the synchronization of the dynamics (they can have a constant offset in phase under synchronization). Strictly speaking, the above condition for synchronization holds for small forcing input $\varepsilon \ll 1$. While the above analysis assumes that ω_f and ω_n to be close to one another for the averaging operation to be a valid approximation, the analysis can be easily extended to consider subharmonic and superharmonic frequencies.

For fluid flows, such synchronization amounts to the so called lock on. The above result enables us to determine the combination of actuation frequency and amplitude for the flow to synchronize with periodic excitations. As long as we can examine the system response against impulse inputs, the overall formulation can be performed through numerical simulations or experiments, which is attractive for fluid mechanics.

5.3. Phase-response analysis of unsteady cylinder wake

5.3.1. Model problem

We apply the phase-reduction analysis to examine the synchronization of circular cylinder wake to periodic forcing. The base periodic flow is taken to be the two-dimensional incompressible flow over a circular cylinder at a diameter-based Reynolds number of $Re \equiv U_\infty d / \nu = 100$, where U_∞ , d , and ν denote the characteristic freestream velocity, cylinder diameter, and kinematic viscosity of the flow, respectively. The dynamics of the flow (equation (5.26)) is governed by the non-dimensional incompressible Navier–Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + Re^{-1} \nabla^2 \mathbf{u} + \varepsilon \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0, \quad (5.40)$$

where $\mathbf{u}$ is the velocity and p is the pressure. In this equation, the forcing (actuation) input $\varepsilon \mathbf{f}$ is taken to be a localized force (momentum injection)

$$\varepsilon \mathbf{f} = \delta(\mathbf{x} - \mathbf{x}_f) \eta(t) \hat{\mathbf{e}}_f \quad (5.41)$$

having a magnitude of $\varepsilon \ll 1$, position at $\mathbf{x}_f$, and forcing direction specified by a unit vector $\hat{\mathbf{e}}_f$. The Dirac delta in space is approximated using the three-cell discrete delta function (Roma *et al.* 1999). The overall setups of the forcing input for the two cases considered below are illustrated in figure 24.

	C_L	C_D	St
Present	± 0.328	1.345 ± 0.009	0.165
Liu <i>et al.</i> (1998)	± 0.339	1.35 ± 0.012	0.165
Linnick & Fasel (2005)	± 0.337	1.34 ± 0.009	0.165
Canuto & Taira (2015)	± 0.329	1.34 ± 0.0091	0.167

TABLE 2. Comparison of lift, drag, and Strouhal number for circular cylinder flow at $Re = 100$.

The function $\eta(t)$ specifies the temporal form of the forcing input as

$$\eta(t) = \begin{cases} I\delta(t) \approx \frac{I}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right] & \text{for impulse perturbation} \\ \frac{\varepsilon}{2} [1 + \sin(\omega_f t + \psi)] & \text{for periodic perturbation} \end{cases} \quad (5.42)$$

The impulse perturbation, approximated as a narrow Gaussian in time (with $\sigma = 0.005$), is introduced to the flow to determine the phase-response, phase-sensitivity, and phase-coupling functions. The sinusoidal forcing function is chosen to examine the lock-on behavior of the flow to periodic flow control input. While the impulse perturbation is introduced only at the top separation point (58.4° from the aft separation point, positioned $3\Delta x$ away from the surface), we add the sinusoidal forcing input at the top and bottom separation points with a phase difference of ϕ (top: $\psi = 0$ and bottom: $\psi = \pi$). The actuation is introduced near the separation points to effectively modify the wake dynamics. However, the perturbations can be added to any locations in the flow field for this analysis. The forcing amplitudes are chosen such that $I = 0.01$ and $\varepsilon \in [0, 0.1]$. The corresponding steady and oscillatory momentum coefficients for the periodically forced cases are less than 0.02 and 0.007, respectively, for $\varepsilon \leq 0.1$ (Munday & Taira 2013).

5.3.2. Computational Setup

The immersed boundary projection method simulates the unsteady flow on a Cartesian grid with the cylinder represented through the boundary forcing (Taira & Colonius 2007). For the present study, the fast version of the immersed boundary projection method is chosen (Colonius & Taira 2008) with the computational domain extending over $(x/d, y/d) \in [-31, 33] \times [-32, 32]$. The grid near the cylinder has $\Delta x_{\min}/d = \Delta y_{\min}/d = 0.02$ and the time step is chosen to meet $U_\infty \Delta t / \Delta x_{\min} = 0.5$. The computational algorithm and setup have been validated extensively in past studies, including a similar study on control of cylinder flow (Munday & Taira 2013). The present study captures the baseline flow accurately and agrees with findings from three independent numerical codes, as summarized in Table 2.

5.3.3. Analysis of cylinder wake

In this section, we determine the phase-response, phase-sensitivity, and phase-coupling functions to characterize the cylinder flow dynamics in terms of phase on the $\dot{C}_L$ - C_L space. Once these functions are determined, we can find the conditions for synchronization of the cylinder wake to periodic perturbations.

We first characterize the phase response of cylinder wake to weak impulse perturbations added near the top separation point. By evaluating the change in phase, we can determine the phase-response function $g(\theta; I\hat{e}_f)$. The magnitude of the impulse is set to $I = 0.01$, which is small in magnitude compared to that of the force on the cylinder. Here, we consider adding the impulse in four different directions (x , y , surface-tangential, and surface-normal directions) from the top separation point, as illustrated in figure 24. These directions are chosen to form two pairs of orthogonal directions, such that they can serve as bases for subsequent analyses.

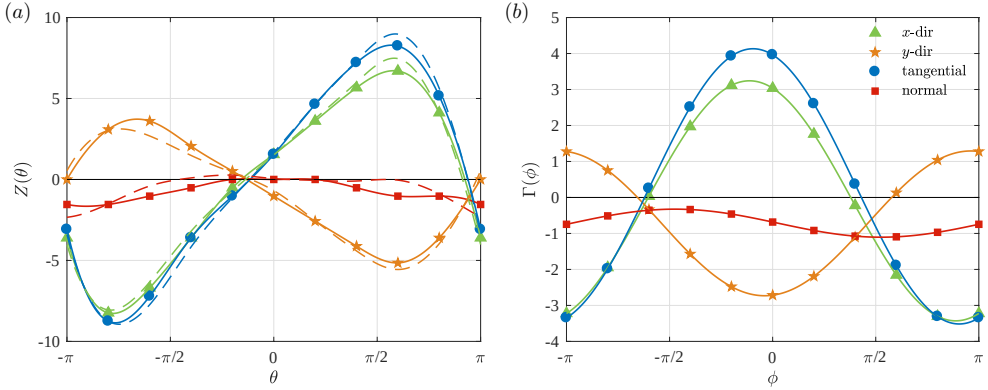


FIGURE 25. (a) Phase-sensitivity function $Z(\theta)$ and (b) phase-coupling function $\Gamma(\phi)$ for cylinder wake and periodic forcing input. The dashed lines for the phase-sensitivity function show approximations from complementary bases. Symbols represent where actual simulations were performed.

Based on the impulse responses, we can compile in figure 25(a) the phase-sensitivity function $Z(\theta)$ evaluated with equation (5.34) over the phase space for the four directions, shown by the solid lines. Symbols represent where actual simulations were performed; 10 simulations were performed for each direction. Impulse added in the tangential and x directions lead to phase advancement. The tangential forcing exhibits the largest phase sensitivity among the four directions examined. Forcing in the y direction causes phase delay and normal forcing does not result in much of a phase change. We also show with dashed lines in figure 25(a) the phase-sensitivity functions found by rotating the coordinate systems. That is, x and y components are evaluated based on tangential and normal components, and vice versa. The close agreements between the solid and dashed lines suggest that the chosen impulse magnitude is adequately small for the linear approximations made in equations (5.32) and (5.34) to hold. Hence, it suffices to perform the impulse-response analysis in two independent forcing directions to map out the phase-sensitivity function over the phase space. In this example, the phase sensitivity function quantifies the sensitivity of the shear layer dynamics aft of the separation point (as well as the overall wake vortex dynamics) to the added impulse perturbation.

With the phase-sensitivity function $Z(\theta)$ available, the phase-coupling function $\Gamma(\phi)$ can be found through equation (5.38). For the present periodic forcing case, we need to sum the influence of the top and bottom actuators, which essentially doubles the magnitude of $\eta(t)$ in the computation of $\Gamma(\phi)$. The phase-coupling functions for the four representative directions are shown in figure 25(b). As expected, large variations in $\Gamma(\phi)$ over the relative phase ϕ are given by the periodic perturbations in the tangential and x directions. Forcing in the y direction has opposing effects with reduced variation. Normal forcing seems to be the least influential among these four directions. It is interesting to note that $Z(\theta)$ has a zero average over θ for the tangential forcing case, which removes the influence of non-zero mean of the forcing input.

Recall that greater variations in the phase-coupling function widens the regions for achieving synchronization of the wake dynamics to actuation fluctuation, as stated in condition (5.39). For example, tangential and normal direction forcing yield $(\min \Gamma, \max \Gamma) = (-3.52, 4.13)$ and $(-1.10, 0.33)$, respectively. We can take these minimum and maximum values for the synchronization condition and compare them with the results from DNS (Munday & Taira 2013), as summarized in figure 26. In the work of Munday & Taira (2013), the cylinder wake

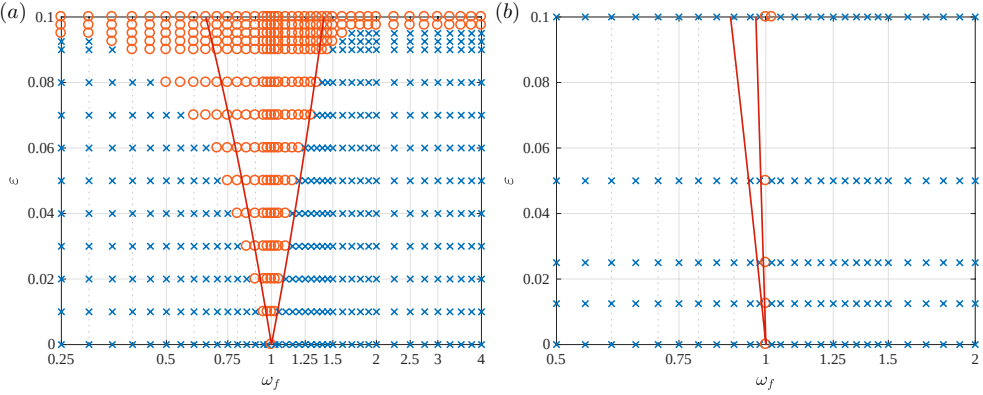


FIGURE 26. Synchronization of cylinder wake to periodic actuation for (a) tangential and (b) normal forcing inputs. Synchronized and unsynchronized cases from DNS (Munday & Taira 2013) are shown by $\circ$ and $\times$, respectively. The criteria for synchronization (5.39) correspond to the region bounded by the red lines.

is considered to be synchronized (locked on) to the actuation frequency when the lift power spectra exhibits a single peak at the actuation frequency.

For the tangential forcing cases shown in figure 26(a), the condition for synchronization from the phase-response analysis agrees well with the full nonlinear wake simulations capturing the region of the Arnold tongue, especially for low forcing amplitude ε until nonlinear effects become apparent at higher $\varepsilon \gtrsim 0.07$. As mentioned earlier, the phase-sensitivity function for tangential forcing removes the influence of the non-zero forcing input for synchronization. In fact, the region of lock on is not altered much for a zero mean forcing case, as presented in the DNS findings (Munday & Taira 2013). For $\varepsilon \gtrsim 0.097$, the wake becomes synchronized to the oscillatory forcing for all considered actuation frequencies, which is beyond the validity of the present phase-reduction analysis.

For the normal forcing case, we observe that the Arnold tongue is much narrower than that of the tangential forcing case, as evident from figure 26(b). The phase-reduction analysis predicts the narrow width of synchronization region well, but slightly biases the tongue towards the lower frequency, which becomes obvious for higher forcing amplitudes. This deviating trend from normal forcing is likely caused by the inaccuracy in the evaluation of the small-amplitude phase-sensitivity function as well as by the nonlinear effects.

The identification of the Arnold tongue from the phase-response analysis is attractive since we only require a limited number of simulations to determine the phase-response function over the phase $\theta \in [-\pi, \pi]$. Instead of having to perform a large number of full DNS (or experiments) as a parametric study to locate the Arnold tongue, we can analytically determine the boundary of synchronization from the phase-coupling function. Moreover, for two-dimensional flow, we can consider two independent forcing directions to determine the impulse response for any forcing direction, as depicted in figure 27. By constructing the phase-sensitivity function over the forcing direction α based on the tangential and normal components or the x and y components, we can determine the minimum and maximum values of Γ as a function of α , as shown in figure 27. In this diagram, we shade out the directions that would be directed below the tangential direction (into the body) as well as the upstream direction (acting against the free stream). Restricting our interest to downstream forcing between $\alpha \in [0, \pi/2]$, we find that synchronization can be achieved for the widest range of ω_f with tangential forcing ($\alpha \approx 2^\circ$). On the other hand, normal forcing would be the most difficult to achieve synchronization with its narrowest bound over α . We emphasize that the

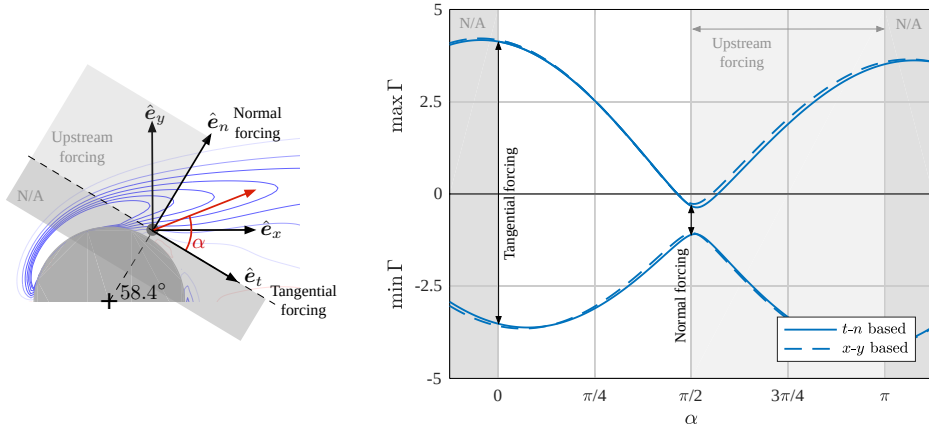


FIGURE 27. Minimum and maximum values of the phase-coupling function Γ over the forcing angle α measured the tangential direction. Extrema based on the bases of x and y directions as well as tangential and normal directions are shown. Forcing directions and the angle α are depicted on the left.

bounds for synchronization has been determined with only a limited number of simulations and does not require an extensive parameter sweep, which significantly reduces the need for computational (and experimental) resources.

While synchronization in this particular example leads to drag increase for the majority of the synchronized cases, the phase-response analysis shows tremendous potential to identify the condition of lock on for a range of fluid flow problems. In this example, we know from our previous work (Munday & Taira 2013) that the addition of non-zero mean in tangential forcing can achieve drag reduction and with minimal effect of altering the lock-on region. In fact, the zero-mean case also shows good agreement with the predicted condition for synchronization. There are other cylinder flow examples that can achieve drag reduction, as in the case of synchronizing the wake to sinusoidal rotational of the cylinder (Tokumaru & Dimotakis 1991).

For the presently considered periodically forced cylinder wake, we do not observe subharmonic or superharmonic synchronizations from DNS. There are of course other fluid flows that exhibit such behavior. Phase-response analysis can be extended to determine the condition for synchronization near harmonic frequencies.

5.4. Summary

We have applied the phase-reduction analysis to periodic fluid flows for studying the phase dynamics and synchronization properties. This was achieved by capturing the oscillatory physics through its scalar phase dynamics. With such formulation, we were able to determine the phase response to impulse perturbations, which formed the basis to evaluate the influence of external perturbations on the oscillatory flow dynamics. These insights reveal the Arnold tongue over which the flow can exhibit synchronization to periodic excitation. As an example, we have considered the unsteady wake dynamics behind a circular cylinder and characterized the synchronization properties for momentum-based forcing. Comparison with DNS showed close agreement with the phase-based analysis. The present study also confirmed that the tangential forcing direction is the optimal direction to achieve lock on for the widest range of actuation frequency. This phase-based analysis can be performed in numerical simulations and experiments, holding promising potential to uncover synchronization properties for a

large class of problems, including wake dynamics, flow control, vortex-induced vibration, fluid-structure interaction, and reacting flows, without the need for extensive parametric sweeps.

6. Dynamics of a separation bubble subject to unsteady conditions: The computational Work

6.1. Introduction

Unsteady flow separation has adverse effects such as reduced lift and increased drag, affecting the performance of airfoils and leading to increased noise. The canonical fluid dynamic phenomenon of flow separation has been one of the focal points in the community since the pioneering work by Anderson (2005). Different computational techniques (Sears & Telionis 1975; Aram *et al.* 2010a; Raju *et al.* 2009) have since been proposed to improve understanding of the inherent unsteady dynamics of a separation flow. However, the majority of recent research has targeted the flow over rigid boundaries. Practically compliant surfaces often appear along with separation flow as observed on the flexible wing tips of birds or fish fins (Heathcote *et al.* 2008). Additionally, modern aircraft design often make use of lighter materials in order to improve agility and fuel efficiency (Patil & Hodges 2006) which also increases the compliance and thus the risk of structural instability. Hence it is essential to improve our understanding of how separated flows interact with compliant surfaces.

Past numerical studies help explain how the compliant surface could reduce the drag and increase efficiency of aeronautic vehicles (Carpenter & Garrad 1985, 1986). Compliance also has an important impact on the boundary layer stability due to the loss of lift and transition process as shown in many computational works (Rempfer *et al.* 2003; Fukagata *et al.* 2008; Kim & Choi 2014; Tsigklifis & Lucey 2017; Xia *et al.* 2017). However, as the critical step that leads to complete separation, the knowledge of a separation bubble subject to compliant surface is lacking. Studies with specific geometries that mimic bat wings at high angle of attack (Rojratsirikul *et al.* 2009, 2010; Gordnier 2009) or UAV membrane wings (Zhang *et al.* 2012; Timpe *et al.* 2012, 2013; Arce *et al.* 2013) have explored how the compliance alter the flow around the wings. However, most of them were limited to particular geometries and/or flow conditions.

Another risk to the safety and stability of aeronautical vehicles is the sudden change in the weather condition. Turbulent flow and transient flow condition such as gust can exert large forces on the airfoil and impact the stability of the vehicles. The gust produces a sudden rise of loading and is seen as an onset to fluttering instability (Fuller 1995; Wright & Cooper 2008; Cooper *et al.* 2012). Here we try to capture the effect of the gust by creating a sharp change in the flow boundary condition.

In this study, we aim to create a benchmark study and gain fundamental knowledge for the system consists of a separation bubble subject to unsteady conditions in a controlled environment. The interplay between the separation bubble and unsteady flow or solid boundary conditions are explored through numerical simulations and could serve as a basis for developing viable control strategy to stabilize or increase the efficiency of aeronautic vehicles.

6.2. Methodology

In this section we will discuss how we build a numerical framework to systematically study unsteady separation-structure interactions. The numerical setup is built upon the previous work by Seo *et al.* (2018) and Connell & Yue (2007). A schematic of the computational domain is shown in Fig. 28(a). The length scale used for non-dimensionalization of the fluid problem is chosen to be the thickness of the incoming boundary layer profile, δ .

A Blasius boundary layer velocity profile is used as the inflow condition to incorporate

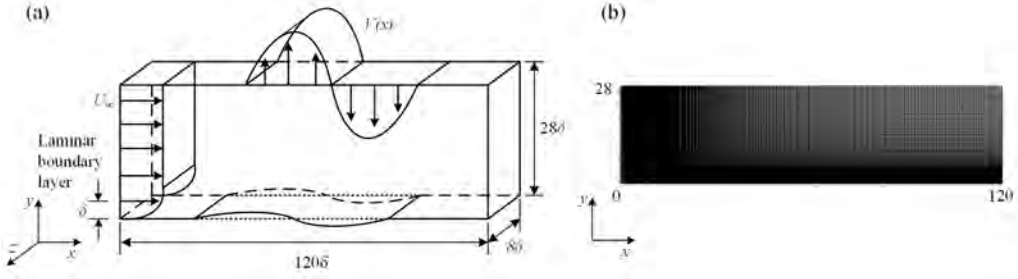


FIGURE 28. (a) Schematic drawing of the computational domain and (b) the computational grid in x-y plane. The grid size is $(Nx, Ny, Nz) = (983, 239, 33)$.

the effect of the elliptic flat plate leading-edge in the corresponding experiments. Periodic boundary conditions are assumed in the spanwise direction. In order to control the flow separation, the top boundary is subject to suction upstream and blowing at downstream location to induce a separation bubble. Experimentally this is achieved by a recirculation duct installed on the top wall. The suction and blowing (s/b) are modeled with a spatially sinusoidal velocity profile, in which the vertical velocity profile is specified as (Aram *et al.* 2010b)

$$V(x) = -V_{sb} * \sin(2\pi(x - x_s)/L_s) * \exp\{-a_s(2(x - x_s)/L_s)^{b_s}\}, \quad (6.43)$$

where $V_{sb} = 0.35$ is the peak s/b strength relative to the free stream, and $L_s = 55.3$, $x_s = 70.2$, $a_s = 10$, and $b_s = 50$ are parameters for spatial distribution of the vertical imposed velocity based on experimental measurements. The streamwise velocity on the top boundary is determined accordingly using the zero vorticity condition. In order to observe how a separation bubble react to transient flow condition, the s/b amplitude is modified with a Heaviside function. The rising then plateau s/b amplitude models a gust passing through, creating strong load and inducing separation flow. Different plateau period is tested to understand how the transition time affects the formation of the separation bubble.

A three-dimensional Cartesian method is employed to solve the governing equations for the incompressible flow (Mittal *et al.* 2008):

$$\nabla \cdot \vec{u} = 0, \quad \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} + \frac{\nabla p}{\rho} = \nu \nabla^2 \vec{u}, \quad (6.44)$$

where $\vec{u}$ is the velocity vector, p is the pressure, ρ is the fluid density, and ν is the fluid kinematic viscosity. Figure 28(b) shows an example of the computational grid, which has higher resolution in the inflow and bottom boundary to resolve the separation region.

On the bottom boundary, we adopt the method proposed by Connell & Yue (2007) to implement a compliant structure that interacts with the fluid. We consider a highly inextensible thin body, i.e., the structure has a large modulus of elasticity and infinitesimal thickness, the structure can that be considered as a membrane with length L . The non-dimensionalized equation of motion of the membrane is written as:

$$\mu \frac{\partial^2 \vec{x}}{\partial t^2} - \frac{\partial}{\partial s} [(T_s - T_0) \frac{\partial \vec{x}}{\partial s}] + K_B \frac{\partial^4 \vec{x}}{\partial s^4} = \vec{f} \quad (6.45)$$

where

$$\mu = \frac{\rho_s h}{\rho_f L}, \quad K_S = \frac{Eh}{(1 - \nu_s^2) \rho_f U^2 L}, \quad \text{and} \quad K_B = \frac{EI}{(1 - \nu_s^2) \rho_f U^2 L^3} \quad (6.46)$$

are the structure-to-fluid mass ratio, stretching rigidity, and bending rigidity, respectively. T_0 represents the pre-tension applied to the streamwise boundaries and T_s represents the

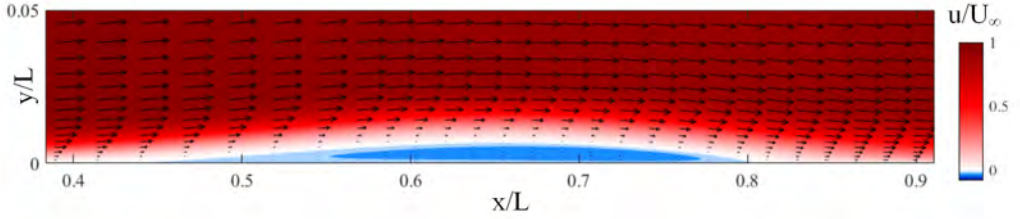


FIGURE 29. Time-averaged horizontal velocity contour with superimposed velocity vectors.

nonlinear stretching term dictates by the stretching stiffness K_s . With the proper choices of the boundary conditions and discretization resolution, we can employ an iterative solver for the governing equations. For the linear two-way coupling simulation, we expand the structural equation and the higher-order nonlinear terms disappear in the linear limit.

We start from one-way coupling by putting the pressure distribution of a separation bubble induced on a rigid surface to the membrane so that we can quickly sweep through different membrane properties and find the region of interest where the membrane strongly reacts to the forcing. Then we look at the reverse one-way coupling, applying the membrane motion from selected cases to the flow in the form of velocity boundary conditions to see how the flow responds to small amplitude boundary motions. Two-way coupling is then looked into with the method of linear FSI. Rising s/b amplitude cases are examined upon a rigid surface to further inspect the transient effect. The simulations were performed on the FSU Research Computing Center (FSU RCC) cluster.

6.3. Results

In this section we will discuss the baseline case with a separation bubble formed on the rigid surface first, then demonstrate how the physical parameters of the compliance surface modulate the separation bubble. Lastly we will discuss how transient flow conditions induce and influence the separation bubble.

A separation bubble is induced on the rigid surface with the position, s/b amplitude and width matched with corresponding experimental setup. The corresponding Reynolds number is $Re_\delta = 864.4$ based on the free stream velocity and boundary layer thickness. Figure 29 shows the time-averaged flow with obvious reverse flow that indicates the existence of a separation bubble. The shedding frequency is represented with the Strouhal number, defined with the shear layer frequency f , separation bubble length L_{sep} and free stream velocity U_∞ as $St = fU_\infty/L_{sep}$. The shear layer frequency f is acquired from the velocity fluctuations within the shear layer in the simulations. The separation bubble length defined in Seo *et al.* (2018) is applied here, with the separation point located at the zero crossing point of the skin friction coefficient C_f and reattachment point at the maximum pressure coefficient C_p on the surface, as shown in Fig. 30. The Strouhal number of the simulation is determined to be $St_{num} = 1.76$ which agrees well with the value from the experiment $St_{exp} = 1.74$.

For the compliant structure we matched the dimensionless mass ratio, stiffness, and the geometry with the corresponding membrane used in the experiments. The pre-tension is calculated from the elastic modulus and the pre-straining ϵEA_c where ϵ is the tensile strain and A_c is the cross-section area.

Due to the large parametric space of FSI simulations, conducting fully coupled simulations for many combinations of the parameters is not feasible. We start dissecting the problem by separating the response of structure and fluid in the first step. Results shown below will demonstrate how structures with different material properties and boundary conditions react to the pressure profile induced atop the rigid surface.

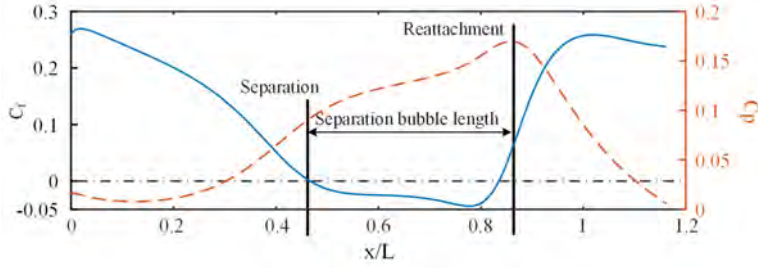


FIGURE 30. An instantaneous spanwise average C_f and C_p curve shown with the criteria for deciding separation bubble length in numerical simulations.

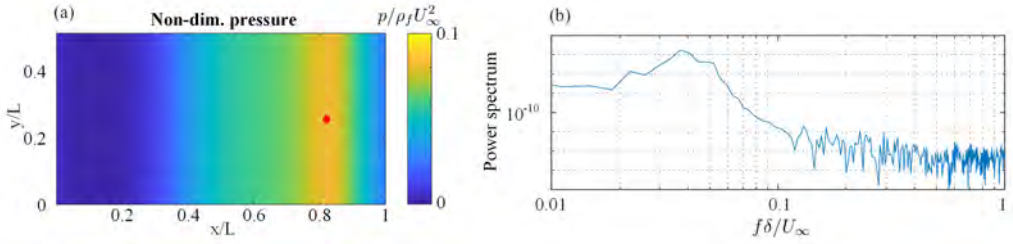


FIGURE 31. (a) An instantaneous snapshot of the pressure at the bottom surface from the baseline case and (b) the power spectrum of the vertical flow velocity at the red dot position.

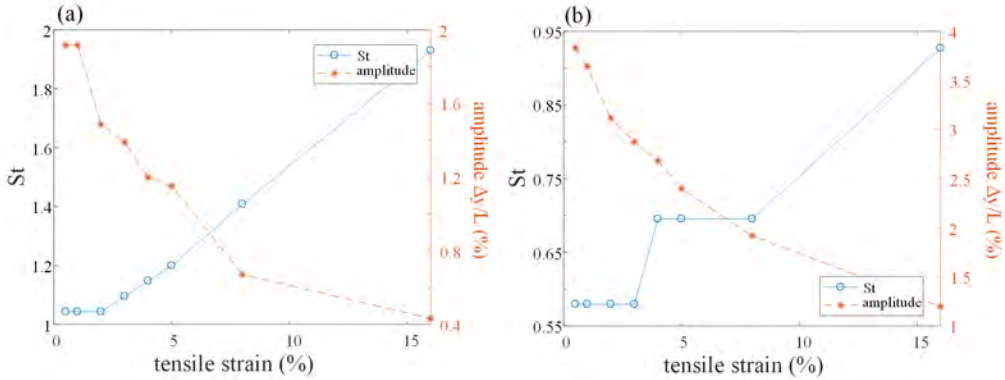


FIGURE 32. The Strouhal number and amplitude of vibration motion for (a) neoprene, fixed bc and (b) neoprene, free bc.

6.3.1. Structure Subject to Flow Pressure

In this section we show the one-way coupling results where only the flow pressure applied to the surface with no feedback. The pressure profile taken from the baseline simulation is shown in Fig. 31 with the vertical flow velocity spectra at the specified point on the surface. The dimensionless frequency of the pressure fluctuation is 0.037. Surface displacement and velocity are recorded and then inspected via the FFT to identify the dominant frequency of the motion. Four different cases with various tension levels are inspected: neoprene/silicone rubber with fixed/free boundary conditions.

Figure 32 shows the Strouhal number St_s of the surface motion and peak deflection amplitude relative to the tensile strain which is the elongation ratio of the membrane in the streamwise direction. The St_s is calculated with membrane length $St_s = f_s L / U_\infty$ where f_s is

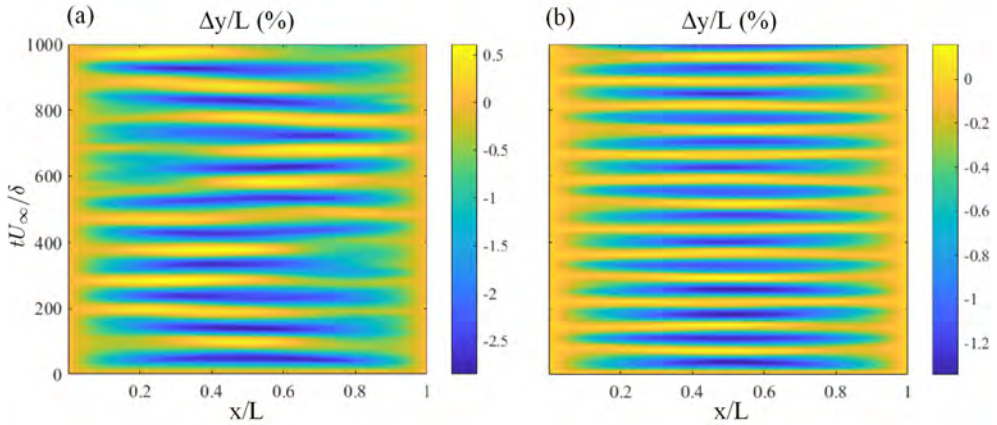


FIGURE 33. Contour level of vertical displacement at the spanwise centerline for (a) neoprene, fixed bc, 2% tensile strain and (b) neoprene, fixed bc, 8% tensile strain in numerical simulations.

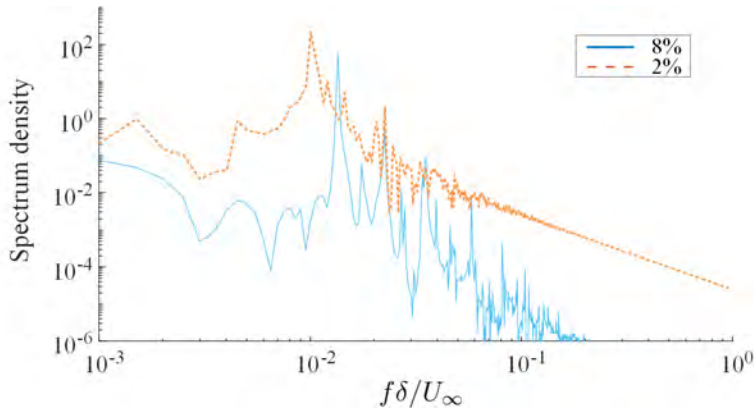


FIGURE 34. Spectra of vertical displacement at the spanwise centerline for neoprene with fixed bc and subject to 2% and 8% tensile strain in numerical simulations.

the dominant frequency. As the tensile strain increases the Strouhal number can be seen also increases and the amplitude of the motion decreases for both free and fixed boundary condition cases. Different from the natural response which scales with $\sqrt{T}$ here we have oscillating fluid loading exerted on the surface and the characteristics of the motion is altered. An interesting observation is that beyond a certain tensile strain the slope of the deflection amplitude changes. Here we pick out several specific cases to understand why this happens. Figure 33 shows the contour plots of the normalized centerline deflection amplitude. Demonstrated here are the neoprene membrane with fixed boundary conditions under 2% and 8% tensile strain respectively. Under the 2% tensile strain the center line clearly presents both a first order mode and a lower frequency mode, relative to the first mode dominated 8% case. The power spectrum shown in Fig. 34 confirms this observation, where in the 2% tensile strain case additional low-frequency peaks are present aside from the dominant mode. We also see that as the tensile strain increases the higher order modes shift to higher frequencies, which are less likely to be excited by the low-frequency fluid pressure loading.

To further demonstrate the compliant surface motion, series of snapshot are shown in Fig. 35. We see that at 8% tensile strain the vertical deflection is first-mode dominant and negatively biased due to the persistent pressure exerted by the separation bubble. The 2%

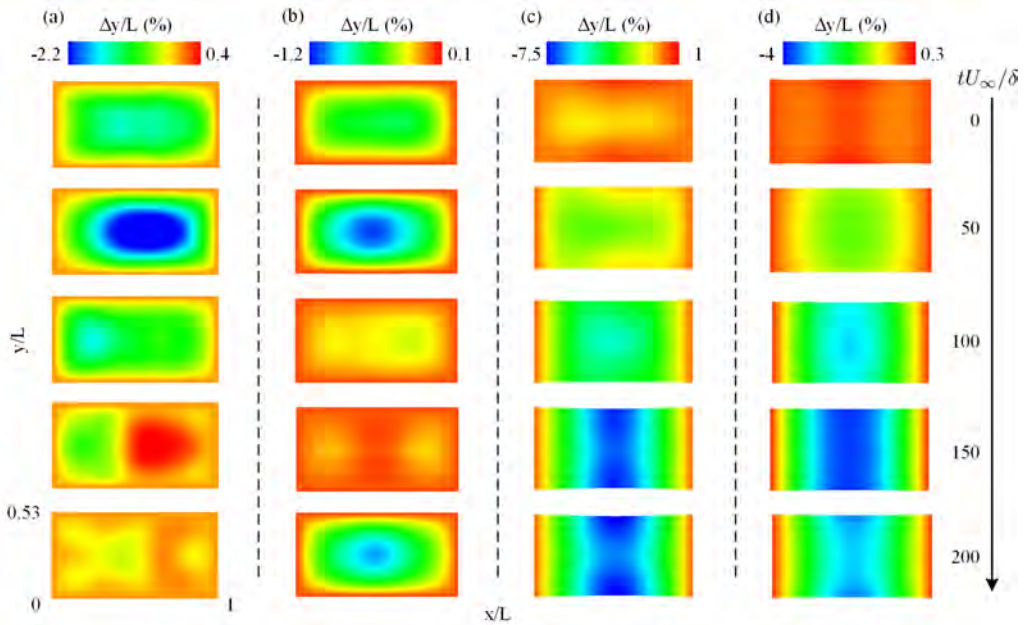


FIGURE 35. Instantaneous snapshots of the vertical displacement of (a) neoprene, fixed bc, 2% tensile strain, (b) neoprene, fixed bc, 8% tensile strain, (c) neoprene, free bc, 2% tensile strain, (d) neoprene, free bc, 8% tensile strain in numerical simulations.

tensile strain case instead shows a more dynamic response. We can conclude that as the tensile strain increases, the amplitude of the motion reduces and the higher order modes are not excited as much. The linearly growing frequency and displacement amplitude slope change can thus both be explained by the absence of higher order modes at the higher tensile strain. Free boundary condition cases also show similar response aside from the frequencies of the motion is lower and the amplitude is larger. Once the tensile strength increases beyond a certain value, the membrane motion becomes dominated by the first-order mode and the amplitude decreases.

Figure 36 shows that the silicone rubber exhibits similar behavior as the neoprene rubber cases, with the frequency increasing linearly while the amplitude decreases with an increase in the tension. In the fixed boundary condition cases, we see the same trend as the neoprene membrane cases. However, in the free boundary cases, there is a region where the amplitude increases as the tensile strain increases. The reason behind this increase of amplitude is unclear since at this range of the tensile strain the motion is mostly first-mode dominant.

6.3.2. Flow Subject to Structure Velocity

Now we will examine how the separation bubble respond to an oscillating velocity boundary condition acquired from the flow-to-structure one-way coupling. With small amplitude and low frequency structural motion, the structural deflection can be neglected and only the velocity of the membrane is imposed to the bottom of the computational domain where the membrane is installed. This provides a simple and efficient way to examine how a compliant surface can impact the separation bubble dynamics.

Two test cases with different amplitudes and frequencies will be demonstrated in this section. The vibration amplitudes and frequencies of the silicone membrane with the free spanwise condition subject to 12% and 20% tensile strain are used to impose velocity boundary

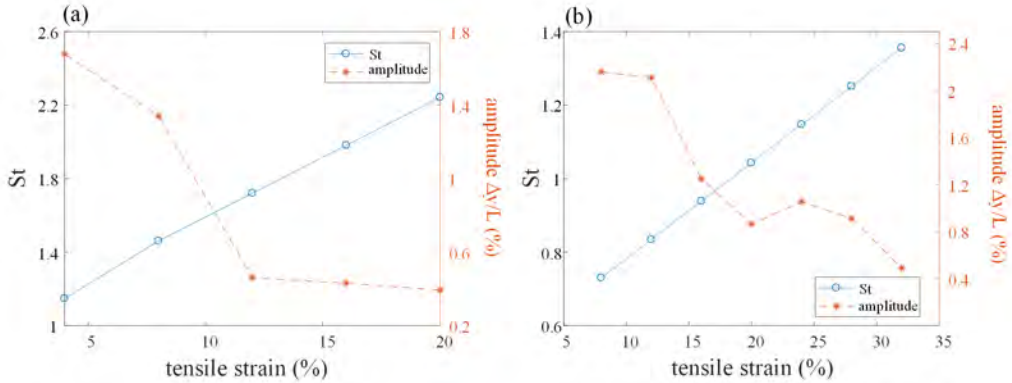


FIGURE 36. The Strouhal number and relative amplitude of vibration motion for (a) silicone, fixed bc and (b) silicone, free bc in numerical simulations.

conditions at the bottom of the computational domain. The free stream velocity, s/b profile, and the computational domain dimensions are identical to the baseline case.

A series of snapshots of the vertical and horizontal flow velocity with the imposed velocity boundary conditions will be shown to illustrate the results. We start from the 20% tensile strain case, shown in Fig. 37. At the beginning of the snapshot series, the bottom boundary has negative velocity, corresponding to the membrane deflecting downward. The downward boundary velocity counteracts the effect of the suction and suppresses the formation of the separation bubble. Then when the membrane starts to bounce back, the velocity turns positive, which induces separation. The separation region is separated into two regions, with one small separation bubble closer to the leading edge of the membrane and a large one closer to the suction and blowing region. Once the membrane reaches peak deflection the velocity reverses again and the separation bubble collapses. The large separation bubble breaks into smaller vortices before dissipating and convecting out of the domain. Note that the evolution of the separation bubble is inherently unsteady over time due to the interactions between cycles.

The same process is also observed in the 12% tensile strain case, shown in Fig. 38, starting from the suppression of separation bubble. Then the velocity boundary condition reverses to positive and induces separation, which then breaks into two regions due to s/b . The separation bubble then breaks down and dissipates as the membrane moves downward again. Since under the 12% tensile strain the vibration amplitude of the structure is much larger, the separation bubble induced also has a larger amplitude.

The simulations shown exhibits a rich interaction between the separation bubble and the compliant surface. With the diverse dynamic behavior, it is difficult to identify the characteristics of the flow and structure motion with traditional tools like FFT. In the future we will apply tools such as modal analysis to better dissect the system in order to gain more understanding. Full-scale FSI studies are also planned to provide high-fidelity simulations for various flow and structure conditions.

6.3.3. Linear Fluid-Structure Interaction

The dynamic interaction between the flexible membrane and the separation bubble is further explored with the linear two-way coupling simulation. At each time step, the fluid pressure is exerted onto the structure as external forces, and the induced velocity of the structure is used as the boundary condition for the flow problem. Within the small disturbance limit the

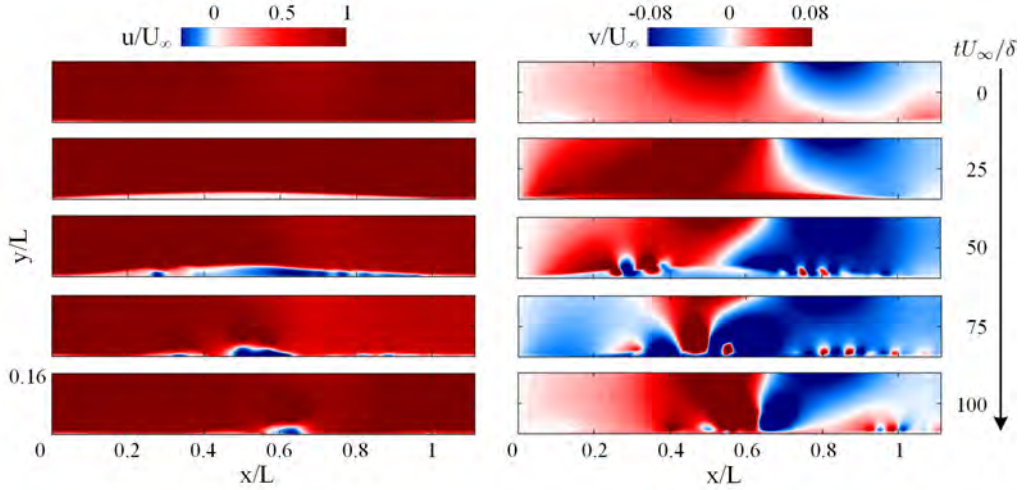


FIGURE 37. The horizontal and vertical velocity with the bottom boundary imposed with the vertical velocity of a silicone rubber subject to 20% tensile strain in numerical simulations. The period of the motion is $t\delta/U_\infty = 100$.

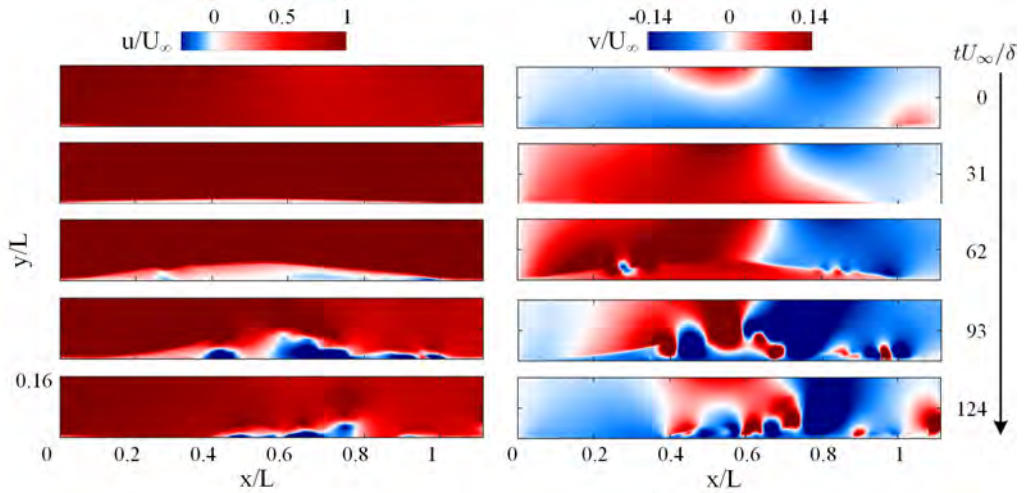


FIGURE 38. The horizontal and vertical velocity with the bottom boundary imposed with the vertical velocity of a silicone rubber subject to 12% tensile strain in numerical simulations. the period of the motion is $t\delta/U_\infty = 125$.

linear model can capture how the flexible membrane and separation bubble respond to and affect each other.

Below we will show results of using the properties of a neoprene membrane with different tensile strain applied and flow condition. The computational domain size is 58.5δ by 15δ by 2δ . Suction and blowing boundary condition spans from 10δ to 40δ . The membrane is subject to periodic boundary condition in the wall direction, and located from 2δ to 24δ at the bottom spanning the whole width of the channel, making the membrane lie within the first half of the separation bubble. Note that to eliminate the strong transient effect that could unstabilize the linear system we introduce the interaction between fluid and structure gradually by weighting the velocity and external force exerted.

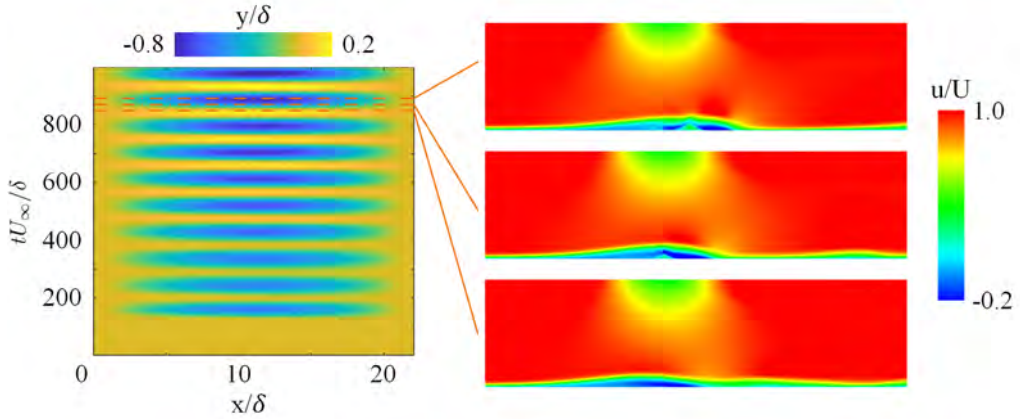


FIGURE 39. The spacetime plot of the membrane centerline vertical deflection with corresponding streamwise velocity field. The membrane is subject to 1% tensile strain and s/b amplitude is equal to $0.5U_\infty$.

First we look at a case with peak s/b velocity equal to $0.5U_\infty$ and the membrane is stretched with 1% tensile strain. Figure 39 shows the space-time plot of the deflection of the membrane at midline position and snapshots of the flow field at key time steps. It is clear that at this parameter combination the membrane behaves very similar to the results from one-way coupling which exhibits only the first-order membrane modes. The flow also responds similarly to the membrane motion as one-way coupling case with the growing and collapsing of the separation bubble.

However, once we reduce the tensile strain to 0.5%, the system acts very different. The space-time plot in Fig. 40 shows that after the transient first-mode dominant period the membrane starts to exhibit higher-order mode motion. Contrary to what we found in one-way coupling cases that the vibration frequency reduces along with the reduction of the tensile strain, here we see at the presence of higher-order modes the structure motion has a higher frequency. The flow responds to the structural motion in the same principle: separation is induced where the membrane is moving upward and suppressed when subjecting to downward boundary velocity. Together with the complex membrane motion, multiple separation bubbles could exist at the same moment. These separation bubbles in turn exerts forces to the membrane in different spots and further disturb the interaction.

Next we increased the s/b strength from $0.5U_\infty$ to $0.65U_\infty$. With the increased s/b strength the pressure from the separation bubble also increases. Since the membrane in this case is located in the first half of the separation bubble the trailing section of the membrane endures larger force and creates asymmetric bending, which leads to the appearance of higher-order modes. With the stronger s/b, a large separation bubble is always present but due to the non-periodic membrane motion it breaks down in different section.

In this section we present the preliminary results of the two-way coupling with the linear FSI method. We observed very diverse responses from the fluid and structure and further investigation is needed to gain more complete knowledge to the complex interaction.

6.3.4. Transient flow condition

The Heaviside s/b profile used to model the sudden increase of load due to flow conditions like turbulence or gust passing through an airfoil. To avoid the instability caused by sharp changes, the s/b amplitude rises linearly within 1/100th the plateau time and the transitions between the rising, plateauing, and dropping are smoothly connected by 4th-order polynomi-

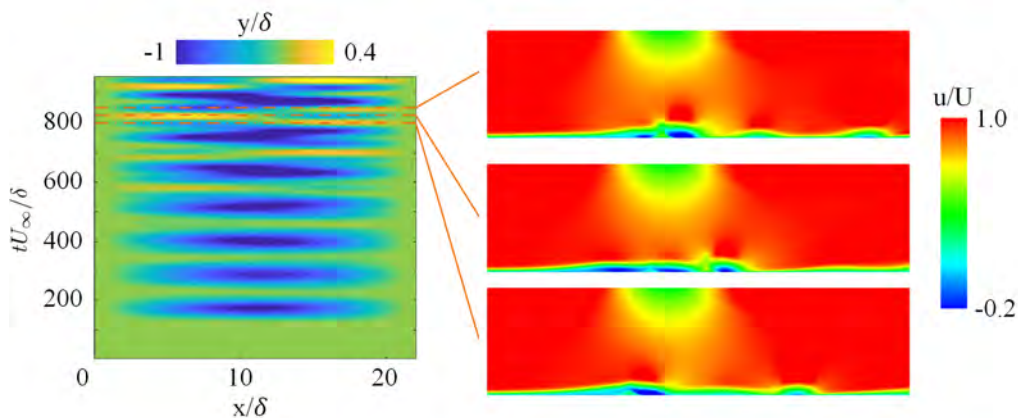


FIGURE 40. The spacetime plot of the membrane centerline vertical deflection with corresponding streamwise velocity field. The membrane is subject to 0.5% tensile strain and s/b amplitude is equal to $0.5U_\infty$.

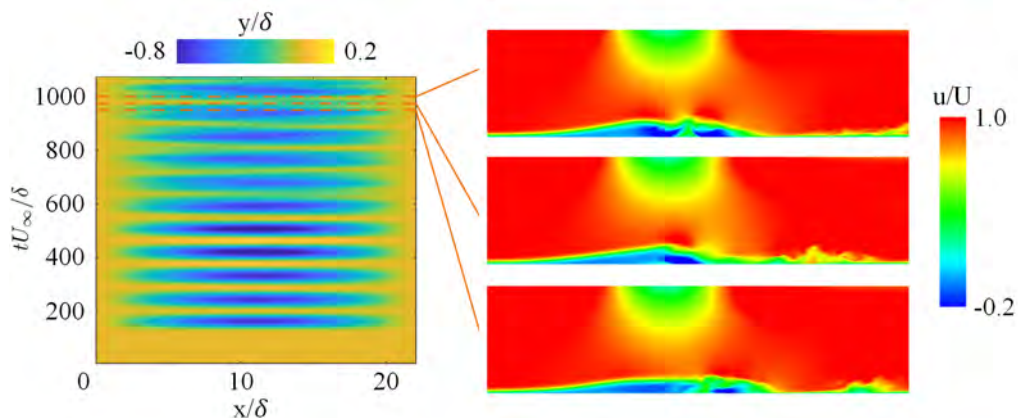


FIGURE 41. The spacetime plot of the membrane centerline vertical deflection with corresponding streamwise velocity field. The membrane is subject to 1% tensile strain and s/b amplitude is equal to $0.65U_\infty$.

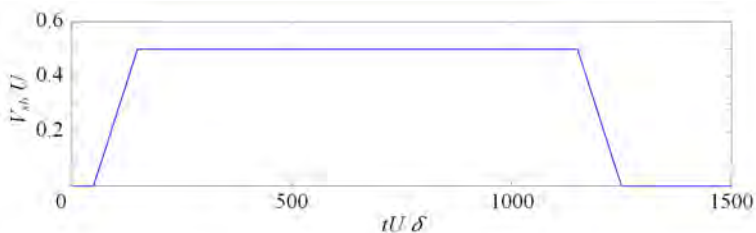


FIGURE 42. The s/b amplitude profile used to model the transient flow disturbance.

als. A sample profile is shown in Fig. 42 with the rising and plateau time equal to 100 and 1000 convective time unit, respectively. The s/b amplitude is normalized by the free stream velocity.

Figure 43 shows a series of snapshots of the spanwise average streamwise velocity. In

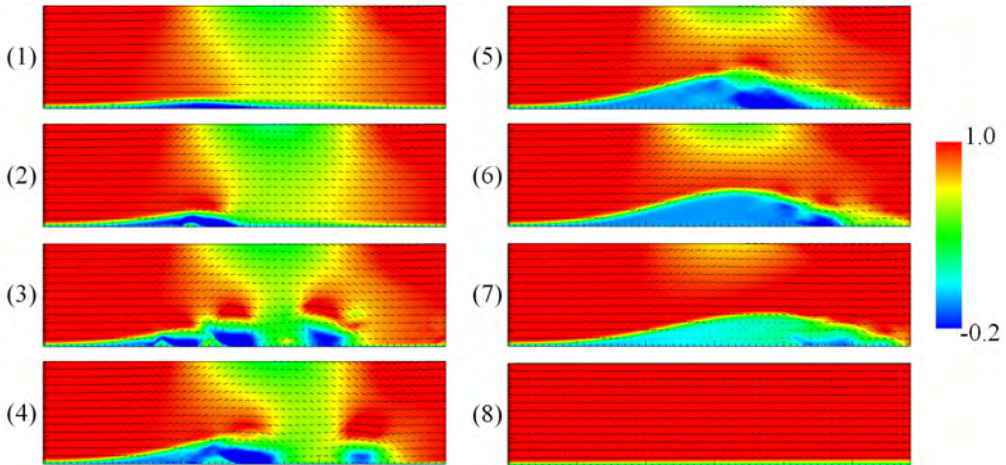


FIGURE 43. Snapshots of streamwise velocity overlaid with velocity vectors of a separation bubble subject to unsteady flow condition shown in Fig. 42.

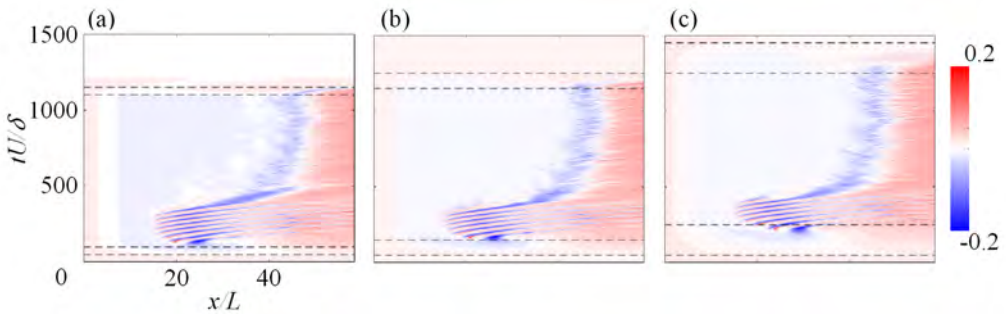


FIGURE 44. Space-time plot of the spanwise averaged skin friction coefficient on the bottom of the domain. The rising time are (a) 50, (b) 100, and (c) 200 unit convective time.

this case, the s/b amplitude profile is the same as shown in Fig. 42. Four phases of flow development are observed. First, when the s/b starts rising, a small separation bubble forms closer to the leading edge. Then at certain point this small separation bubble rolls up to become large vortex structures traveling downstream. Eventually one of the large vortice structures is stretched and developed into a large separation bubble with shear layer in the trailing edge. Once the s/b amplitude starts to decrease, the large separation bubble is quickly washed out by the freestream. To better depict how the rising time affects the development of the separation bubble, space-time plots with different rising periods are shown in Fig. 44. In these plots, at each cross section in x-direction is the spanwise average skin friction coefficient at the bottom of the simulation domain. From the bottom to top, the four broken lines represents the moment when the s/b starts/ends rising and starts/ends dropping, respectively. Interestingly, the length of the rising period does not affect the development phases very much, where in all three cases shown the four phases described above are clearly present. Practically, this means that any disturbance over the surface, as long as it reaches the onset amplitude and persists long enough, will induce the instability in the flow. Also from the y-direction cross section in these plots it is observed that the developing separation bubble has two characteristic frequencies: one from the roll-up phase and another stems from the shear layer after the separation bubble is established.

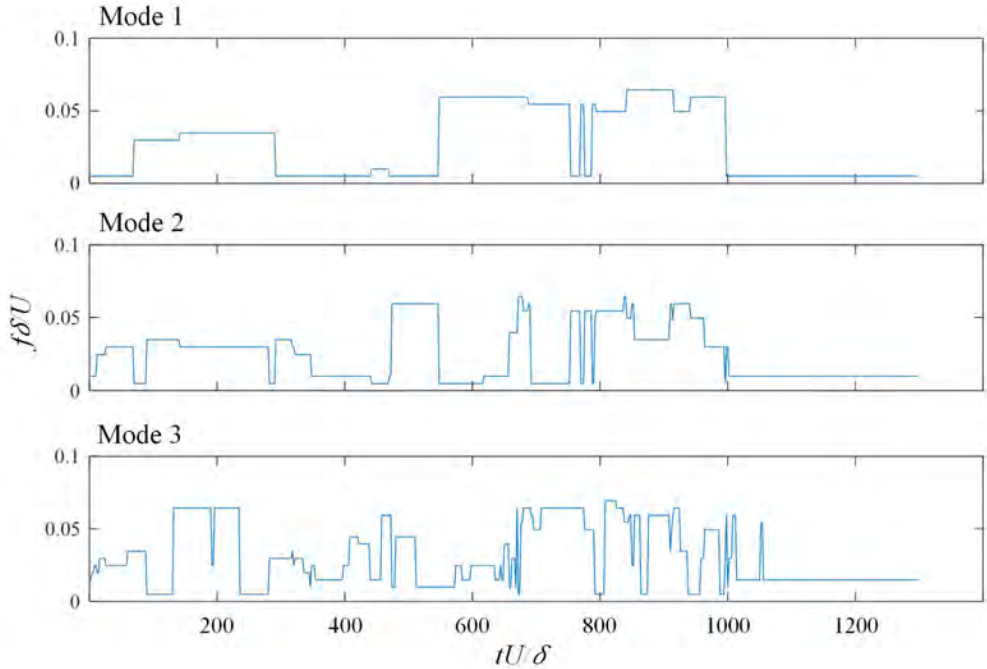


FIGURE 45. Frequency evolution of the first three WDMD modes.

Windowed dynamic mode decomposition (WDMD) (Zhang *et al.* 2019) is applied to reveal more behind the formation of the separation bubble. WDMD is a modal analysis technique aim to extract frequency information along with spatial modes from a gradually changing system. The idea is to apply dynamic mode decomposition with a moving window to capture the transient effect in the flow field. Figure 45 and 46 shows the WDMD results of the same case as shown in Fig. 12 with the interrogation window width $w = 20\delta/U$. From the frequency plot of the first three modes the different phases of the separation bubble formation are exposed. Figure 45 shows that the WDMD modes capture the change of the frequency from the roll-up phase to the shear layer frequency well with a gap between two distinctive frequencies. Figure 46 shows the corresponding mode shapes with the largest magnitude at different specified moments and as expected the big roll-up vortices and the shear layer are observed in different . However, WDMD could not tell us much about the transition between them from the mode shape and frequency due to the short transition period.

The results shown in this section reveal that the rising time of the flow disturbance is irrelevant in the formation of instability. The roll-up phase is an onset phenomena before large-scale separation happens. With better analysis tools we mat be able to detect this phase and apply necessary control before separation begins to enhance structural stability.

6.4. Summary

In this work the interaction between a separation bubble and unsteady boundary conditions was investigated with numerical experiments. Structural simulations reveals the material properties and tensile strain effectively modulate the frequency and amplitude of the membrane vibration. Imposing the structural velocities on the flow with only first mode motion shows a rich response from the separation bubble. The linear coupled FSI simulation shows how rich the dynamics can be with different flow and structure condition. Unsteady flow conditions

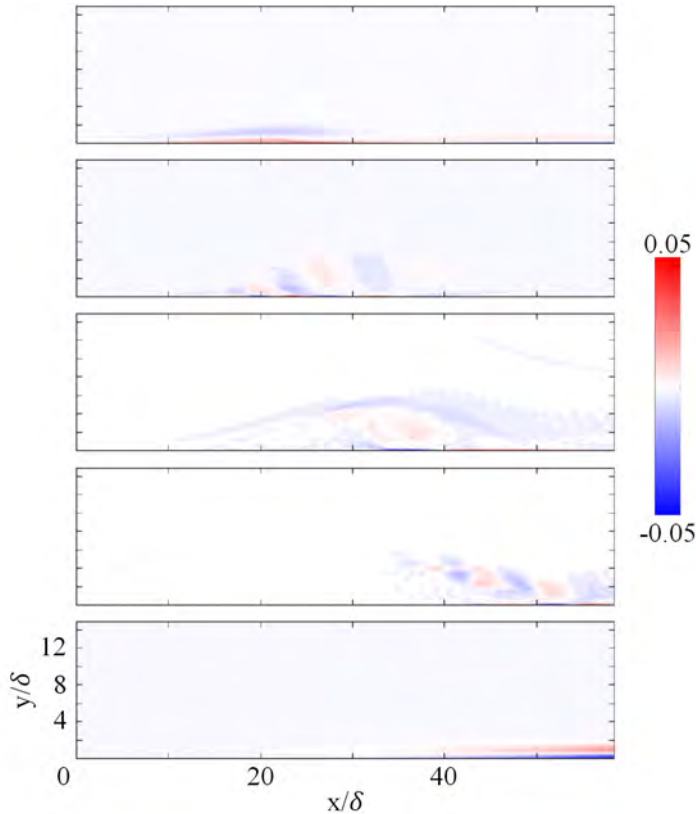


FIGURE 46. Evolution of the mode shape of the first WDMD mode. The five phases can be observed here from top to bottom: small separation bubble, roll-up, transition, shear layer of the large separation bubble, and the BL after the separation bubble is washed out.

expose how the separation bubble is generated and identify the critical point of the transition. From all the diverse physics we observed, we conclude that more research has to be done to fully understand the complex coupled dynamics of the unsteady separation bubble system.

Continuation of this work will consist of modal analysis techniques to take a deeper look into the underlying dynamic of both the structure motion and unsteady separation bubble. Correlation methods will also be used to understand the coupled dynamics of the system. High fidelity nonlinear FSI simulations will provide a more accurate model of the dynamic fluid-structure interaction and will also provide an opportunity to compare the approximation of the simplified model used in these works. In conclusion, new analysis tools are in urgent need to reveal the mystery underlying these complex flow and structure phenomenon to improve the controllability of the increasingly ubiquitous complex unsteady systems.

7. Dynamics of a separation bubble subject to unsteady conditions: The Experimental Work

7.1. Approach and Methodology

An experimental approach has been taken to investigate the perturbations dynamics of unsteady flows with time-varying base states, in particular, unsteady separation. The facility used for this study is the Florida State University Flow Control (FSFC) wind tunnel. This wind

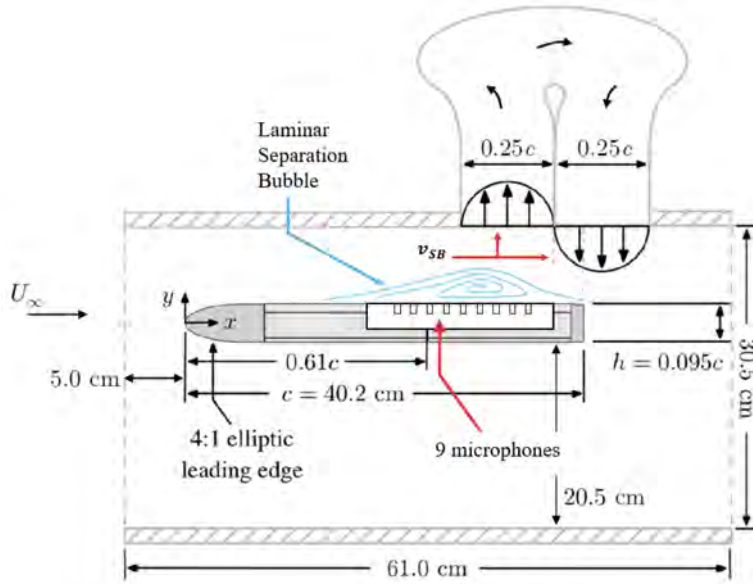


FIGURE 47. Schematic of flat plate model with rigid surface and 9 surface mounted microphones

tunnel is equipped with a recirculation duct atop its test section. A DC axial fan inside the duct imposes a zero-net mass flux (ZNMf) suction/blowing (s/b) boundary condition on the top wall of the test section causing flow separation and reattachment on an elliptic leading edge flat plate model inside. A schematic of the experimental configuration and relevant dimensions can be seen in Fig. 47. It has been shown under steady suction and blowing that a statistically stationary laminar separation bubble (LSB) forms on the top surface of the model Griffin *et al.* (AIAA-2013-2968); Deem *et al.* (AIAA-2017-3309, 2018).

Unsteady Separation

To investigate the dynamics of unsteady separation alterations to the previously mentioned experimental setup were required. Two approaches to introduce unsteadiness into the experimental domain were implemented. The first, creating a time-varying pressure gradient by controlling the s/b velocity boundary condition $v_{SB}(t)$. The second, approach by introducing a compliant boundary beneath the region of the separation bubble through the implementation of an elastic membrane in place of the rigid top surface of the flat plate model.

7.2. Experimental Implementation

7.2.1. Approach 1: Time varying pressure gradient

The s/b boundary condition imposed by the fan is modulated as a function of time $v_{SB}(t)$ to introduce unsteadiness into the system. A PI control scheme was implemented to modulate the rotation rate of the fan. By doing so, the magnitude of the adverse pressure gradient in the test section imposed by the s/b varies in an oscillatory manner. After implementation, it was quickly determined that the achievable time scales of s/b oscillations $\mathcal{O}(1)$ Hz with control were not of the same order of magnitude as the characteristic time scales in the flow $\mathcal{O}(10^2)$ Hz. In order to target the higher frequency dynamics of the separation bubble another method capable of response times of the same order of magnitude of the separation bubble was explored.

Transient dynamics

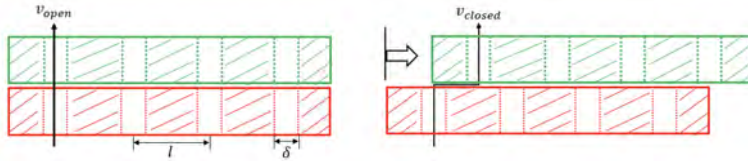


FIGURE 48. Schematic of shutter operation

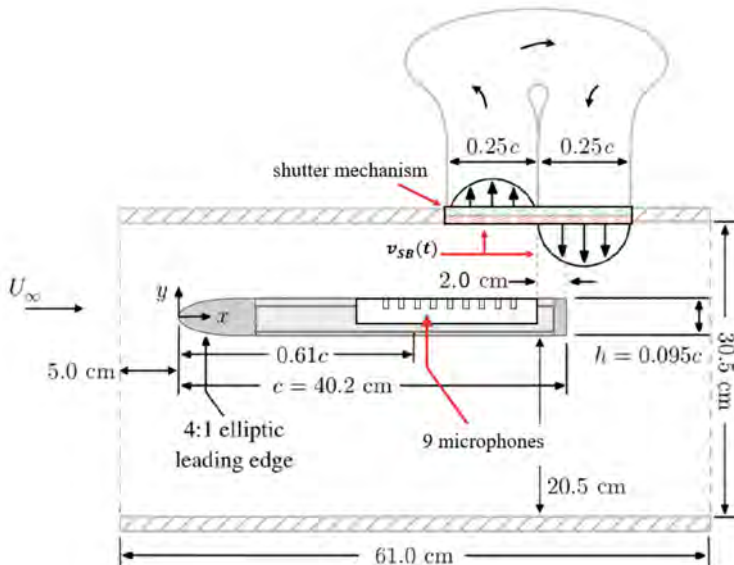


FIGURE 49. Schematic of shutter mechanism in the FSFC wind tunnel

To analyze the dynamics of a system it is common to monitor the response of that system subject to a step-input. This led to the design and implementation of a rapid duct closing mechanism. To do this a spring-actuated shutter mechanism was designed to impulsively start/stop flow in the recirculation duct and thus impose separation or reattachment. Since the characteristic frequency of the flow associated with the Kelvin-Helmholtz instability of the separating shear layer is on the order of 100 Hz, the design target was to turn on/off the s/b boundary condition at a time scale on the order of 10 ms or less (i.e. faster than the characteristic time scales of the flow) Deem *et al.* (AIAA-2017-3309). Two shutters made from Delrin sheets (low-friction) were used to do this. The shutters contained 225 slots in each with a thickness (δ) of 0.030 in and a distance between slot centerlines (l) of 0.100 in. The width of the slots was minimized to have a very short travel distance to open/close the shutters in minimal time. Also allowing sufficient cross-sectional area to not drastically increase the pressure drop the fan must overcome to move air through the recirculation duct. The opening/closing of the shutter is done by initially having the slot aligned and then linearly translating one shutter over the other until the slots are closed. One shutter is fixed while the other is able to move with one degree of freedom. A schematic to better explain the methodology can be seen in Fig. 48. The physical shutter is positioned at a location directly above the s/b boundary on the top wall of the test section. The shutter position in the wind-tunnel can be seen in the schematic of Fig. 49. The red shutter is fixed, and the green shutter translates.

Upon implementation, when the shutter mechanism was actuated from the open to closed

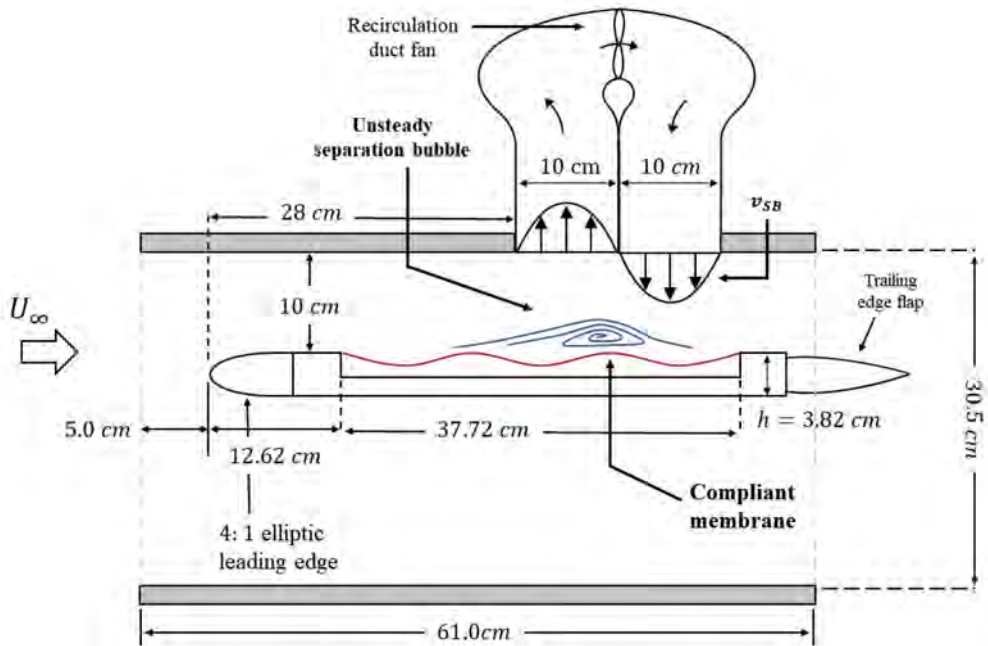


FIGURE 50. Schematic of wind tunnel and model with compliant surface.

position, the resulting pressure drop across the closed shutter was not significant enough to retard the s/b , therefore the separation bubble persisted. After the outcome of the first approach was concluded to be unsuccessful at introducing the desired transient response the second approach was initiated.

7.2.2. Approach 2: Compliant boundary

In place of the rigid top surface of the elliptic leading edge flat plate, modification were made to incorporate a variable tension elastic membrane to allow for unsteady boundary motion throughout the region of the separation bubble, as seen in Fig. 50.

Along with these modifications the model was elongated in the streamwise direction with the addition of a trailing edge flap to mitigate any wake effects. The membrane starts at a distance of 12.62 cm from the leading edge of the model and has a streamwise length of 37.72 cm. The membrane has a span of 20 cm, centered about the mid-span of the model. The design permits adjustment of a uniform tension in the streamwise direction. This is done by extending the membrane through the top surface and attaching it to tensioning rods which are inserted inside the model. Custom clamps were made to ensure a smooth transition from the rigid surface to the compliant membrane while allowing the membrane to be pinned at its intersection with the rigid parts of the model. For all cases discussed hereafter, the membrane is fixed at the leading and trailing edges. The boundary conditions at edges in the z direction (into the board) are either pinned or free conditions. Two different types of material are used for the compliant membrane: neoprene and silicone. Their material properties are given in Table 3. These two materials were chosen due to relatively significant differences in their elastic modulus and non-dimensional parameters of interest. An important parameter to consider is the ratio of the free stream velocity to the wave speed in the solid, known as the reduced velocity. When the reduced velocity approaches unity, the relative speed in the solid is of the same order as the free stream, which could provide a suitable condition for

TABLE 3. Properties of membrane materials

Material	Elastic Modulus [Pa]	Density [kg/m^3]	Thickness [mm]	Length [cm]	Width [cm]
Neoprene	1.7×10^6	1200	0.4	37.72	20
Silicone	6.67×10^5	685	0.25	37.72	20

fluid-solid dynamic coupling. Based on the general material properties of neoprene compared to other elastomers, it permits a reduced velocity closer to unity. Silicone has been observed in prior research to exhibit large deflection and dynamic motion due to its extreme compliance Timpe *et al.* (2012, 2013); Zhang *et al.* (2012). The elastic modulus for both materials was approximated using the shore A durometer (ASTM D2240) relationship for elastomers Gent (1958). The tension of the membrane is calculated from the elastic modulus and pre-straining $T = \epsilon EA_c$, where $\epsilon = \Delta l/l$ is the tensile strain and A_c is the cross-sectional area of the membrane.

7.2.3. Wind Tunnel Facility and Experimental Details

As mentioned previously the Florida State Flow Control (FSFC) wind tunnel was used for the experimental studies. The FSFC wind tunnel has an inlet contraction of 9:1 with a square cross-section of 30.5×30.5 cm, and a length of 61.0 cm. The leading edge of the flat plate model is a 4:1 ellipse and the trailing edge is a symmetric Wortmann FX 76-120 airfoil with a chord length of 15.24 cm and maximum thickness of 3.82 cm, equivalent to the thickness of the flat plate model. The camber line of the flap is positioned to be parallel with the top of the rigid flat plate model to mitigate wake effects during experiments. The flat plate model used in this study spans the entire width of the test section, has a total length of 68.25 cm and thickness of 3.82 cm.

The mean velocity profile for the baseline case was measured using PIV snapshots at the location of the start of the membrane on the rigid flat plate model, which is approximately 12.62 cm from the leading edge. The model is at a slight positive angle of attack of 0.35 degrees to achieve nominally zero pressure gradient over the top surface of the model. The mean free stream velocity, boundary layer thickness and shape factor at this location are approximately $U_\infty = 3.7$ m/s, $\delta = 3.6$ mm, and $H = 2.7$ respectively. The corresponding Reynolds number based on boundary layer thickness is $Re_\delta = 864.4$. Comparing the shape factor to that of a Blasius boundary layer, $H = 2.59$, indicates that the flow is laminar at the location of the start of the membrane on the rigid flat plate model. The velocity profile at compared that of a Blasius boundary layer can be seen in Fig. 51.

The s/b boundary condition imposed at the top of the test section is controlled by an MM28080H series DC axial fan located at the apex of the recirculation duct. The location of the s/b with respect to the model position can be seen in Fig. 50. The mean s/b velocity profile was measured using PIV snapshots. The profile of the s/b slightly offset from the top of the test section has a nominally parabolic profile with a maximum velocity of 0.35, non-dimensionalized by the free stream velocity.

Particle Image Velocimetry

The baseline separation bubble formed on the rigid surface model was analyzed via 2-component planar PIV. The measurement plane is oriented in the x-y plane, slightly offset from the midspan of the model. The flow is seeded with olive oil particles with a diameter of approximately $1 \mu m$ from a TSI 9307-6 atomizer. Particles are illuminated with a Nd-YAG

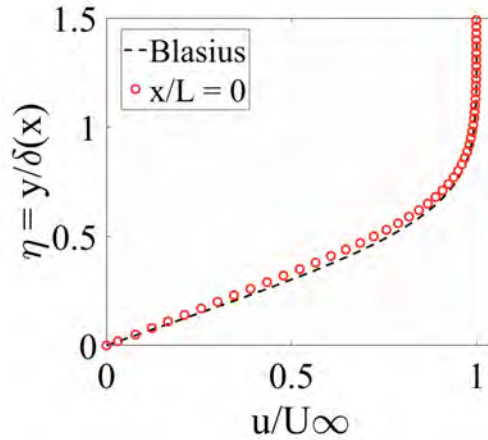


FIGURE 51. Velocity profile in the boundary layer at the start of the membrane

laser operating in a double pulse configuration at a rate of 15 Hz with 150 μ s time delay between pulses. Images of the illuminated particles are acquired with an Imager sCMOS camera with a resolution of 2560×2160 pixels. A total of 1000 images are acquired. Image processing was done using LaVision DaVis 8.4.0 using multi-pass cross correlation with an initial interrogation window size of 64×64 pixels and a final window size of 24×24 pixels with a 75% overlap. The resulting vector resolution is 2.09 vectors per mm.

The separation bubble formed on the model with a compliant membrane was analyzed via 2-component time-resolved PIV (TR-PIV). The measurement plane is oriented in the x-y plane at the same location as the baseline case. Particles are illuminated by a high rep rate Nd-YAG laser operating in a single pulse configuration at a rate of 5000 Hz. Images of the illuminated particles are acquired with a Phantom v2012 high speed camera with a resolution of 1280×800 pixels. A total of 10000 images are acquired. Image processing was done using LaVision DaVis 8.4.0 using multi-pass cross correlation with an initial interrogation window size of 48×48 pixels and a final window size of 16×16 pixels with a 75% overlap. The resulting vector resolution for the cases with a neoprene and silicone membrane are 1.87 vectors per mm and 1.90 vectors per mm, respectively.

Digital Image Correlation (DIC) The full-field three dimensional motion of the elastic surface was captured via stereo DIC. This non-intrusive, optical measurement technique allows for full field measurements of material strain and displacement. Two high speed Phantom v2012 cameras were positioned nominally 60 degrees apart from each other, and roughly 45 degrees off the normal of the elastic surface. A random speckle pattern was applied to the membrane with white gloss paint allowing the cameras to capture its motion. Two, high intensity, pulsed linear LED arrays, were used to illuminate the speckle pattern applied to the membrane surface. A total of approximately 1700 images were acquired at a rate of 200 Hz. Image processing was done using LaVision DaVis 8.4.0 StrainMaster software using least squares matching with a subset size of 21 pixels and step size of 7 pixels.

Spectral Proper Orthogonal decomposition (SPOD) To better understand which coherent structures are the most energetic components of the dynamics, SPOD was utilized to decompose the dynamics of flow field and surface motion. This method approaches Proper orthogonal decomposition (POD) from the frequency domain and leads to mode shapes associated with a particular frequency. For statistically stationary systems SPOD identifies coherent structures that evolve in both space and time Towne *et al.* (2018).

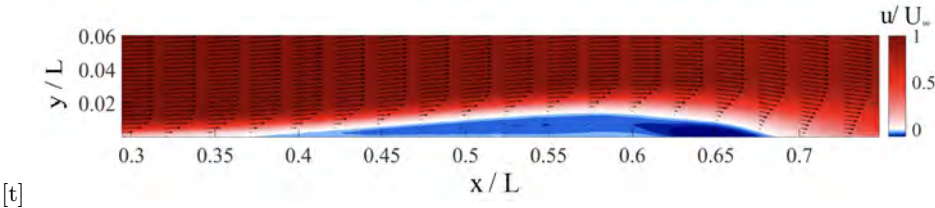


FIGURE 52. Time-averaged horizontal velocity contour with superimposed velocity vectors.

TABLE 4. Pre-strains and boundary conditions tested for membrane materials in experiments

Test Cases	Fixed: pre-strain [%]	Free: pre-strain [%]
Neoprene	1, 2, 3, 4, 5	3, 4, 5
Silicone	8, 10, 12	12, 14, 16, 20

7.3. Results

7.3.1. Rigid Surface

The separation bubble induced with steady s/b over a rigid surface serves as the baseline case. The free stream velocity and boundary layer thickness at the start of the membrane are $U_\infty = 3.7$ m/s and $\delta = 3.6$ mm, respectively. The corresponding Reynolds number based on the boundary layer thickness is $Re_\delta = 864.4$. Figure 52 shows the time-averaged streamwise velocity component over the rigid surface model. The Strouhal number with respect to the shear layer shedding frequency is $St_{exp} = 1.74$, where the dominant frequency was extracted from the 9 surface mounted microphones seen in Fig. 47. The time averaged height and length of the separation bubble are 5.3 mm and 121.3 mm respectively, where the length is defined as the distance between first and last location where $\bar{u} = 0$ just of the surface of the model in the time averaged velocity field Deem *et al.* (2018).

7.3.2. Compliant Surface

For the experiments presented here, two compliant membrane materials with different properties and dimensions (Table 3) were used to replace the baseline rigid top flat plate of the model. The boundary conditions and pre-tension of each membrane were used to alter the dynamics of the laminar separation bubble passively. A series of pre-tensioning levels and boundary conditions were used for both membranes, and TR-PIV was used to observe the flow field while DIC was used to measure membrane motion. The membrane conditions tested can be seen in Table 4.

The following discussion is related to the experiments with a neoprene membrane fixed on all boundaries, subject to 1% and 4% pre-tensioning, as well as two silicone membrane cases of: a) 12% pre-tensioning fixed on all boundaries and b) 20 % pre-tensioning with free boundary conditions along its spanwise edges.

Time Averaged Flow Fields

With the addition of a compliant membrane, the separation bubble displays different behavior compared to the baseline case. Here, the presence of a membrane in the place of the rigid plate results in dynamic surface deformation from the unsteady flow induced pressure loads and displays both steady and unsteady responses. In all tested cases, a noticeable change in the separation bubble size was observed. For the tests with a neoprene membrane

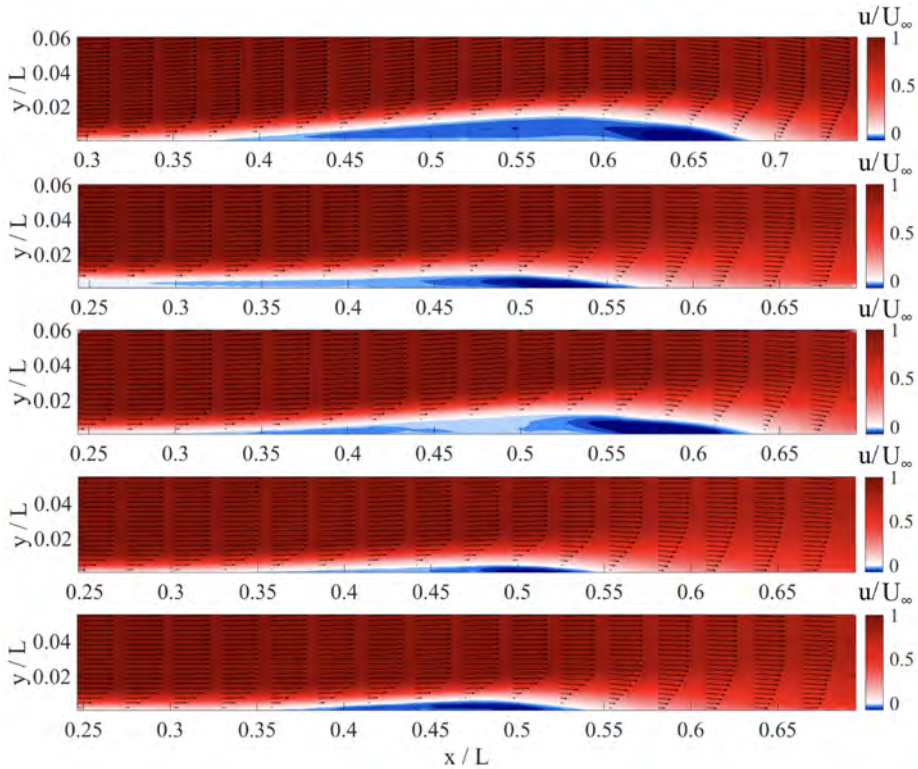


FIGURE 53. Time-averaged streamwise velocity: Baseline, Neoprene 1%, Neoprene 4%, Silicone 12%, and Silicone 20% pre-tension (top to bottom respectively).

and a pre-tension of 1%, the change in the size of the separation bubble can be seen in the streamwise averaged velocity component displayed in Fig. 53. In this case, the dominant mechanism which is responsible for the majority of changes in the separation bubble is the mean deflection of the membrane. When the membrane is under low pre-tensioning forces the natural frequency of the system is small relative to the convective time scale of the boundary layer and the flow is primarily affected by the instantaneous position of the membrane. In the case of lower pre-tension, there is less coupling between the fluid and solid dynamics since the dominant time scales of the flow and solid phases are very different compared to larger pre-tension of the membrane. However, since there is a large adverse pressure gradient imposed by the suction and blowing top boundary, the weakly tensioned membrane undergoes a relatively large magnitude mean deflection. Due to this large deformation response of the membrane, the flow field changes and the modified cross-sectional area effectively reduces the strength of the local adverse pressure gradient and generates a smaller separation bubble. This reduction in the size of separation region due to a compliant membrane is also seen in membrane airfoils and manifests as a change in the airfoil camber due to the mean deformation of the membrane Rojratsirikul *et al.* (2008, 2009). For the neoprene membrane with 4% pre-tension, the time-averaged separation bubble has grown back closer to its baseline size. The silicone cases exhibit a similar trend in the separation bubble size as pre-tensioning level increases. However both of the silicone cases have smaller separation bubbles compared to the neoprene membrane as the silicone membrane is more compliant than the neoprene. Consequently it deflects even more due to the pressure loading of the flow field. This results in the reduced size of the separation bubble.

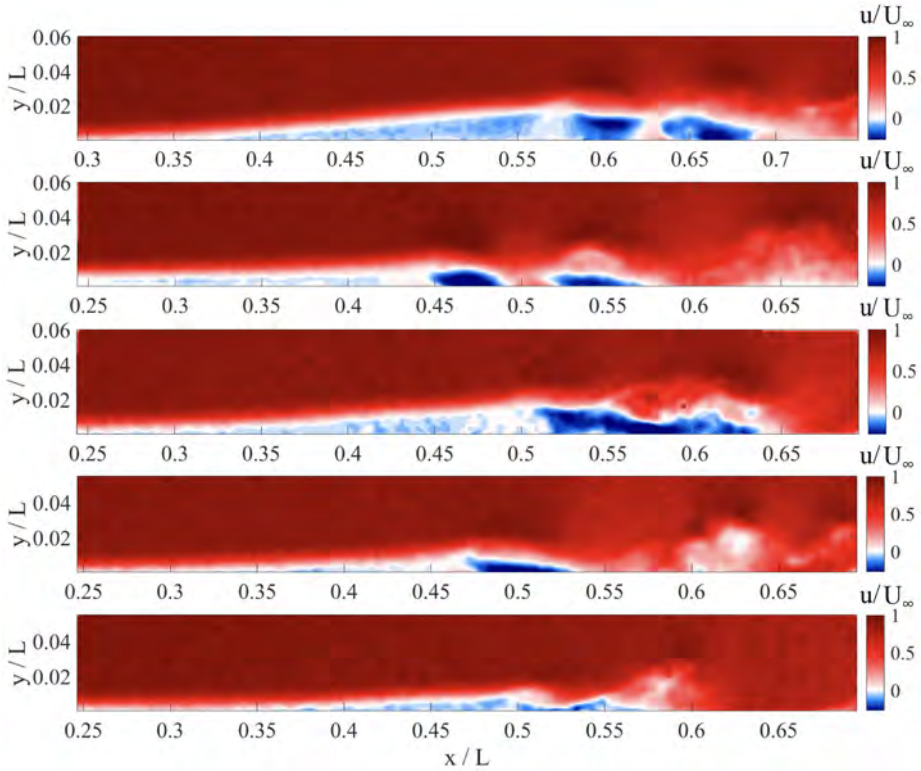


FIGURE 54. Instantaneous streamwise velocity component: Baseline, Neoprene 1%, Neoprene 4%, Silicone 12%, and Silicone 20% pre-tension (top to bottom respectively).

Instantaneous Velocity Fields and Vortex Shedding

The instantaneous vector fields show some interesting features of the flow fields. The instantaneous snapshots shed light on the localized magnitude of the velocity components associated with the separation bubble and the convecting vortices. Figures 54 and 55 show instantaneous snapshots of the streamwise and vertical velocity component for various cases of interest.

From these, it is observed that the low-frequency modulation of the separation bubble, sometimes referred to as the breathing mode, interacts with the membrane motion. Depending on the local motion of the membrane in the region near the bubble formation, the separation bubble periodically ingests and expels fluid. The expulsion of fluid from the recirculation region interacts with the shear layer and forms coherent vortical structures. The vortices are consequently shed from the bubble and convect downstream. This phenomenon was seen in all cases, but the size and rate at which these structures detach and convect from the region of separation differed. The observed periodicity in the instantaneous snapshots, in fact, shows clearly the presence of these structures. It is also observed that the unsteady interaction between cycles cause changes in the rate of vortex shedding.

Figure 56 shows the streamwise and vertical velocity components during one vortex shedding cycle of the separation bubble. This was for the case of a silicone membrane with a 12% pre-tensioning level and fixed boundary conditions on all edges. We observe similar features as seen in the simulation: the separation is stabilized when the membrane is deflected downward, then destabilized when the membrane starts to deflect upward. The separation bubble breaks down and sheds strong vortices in each cycle. The frequency of this cycle for

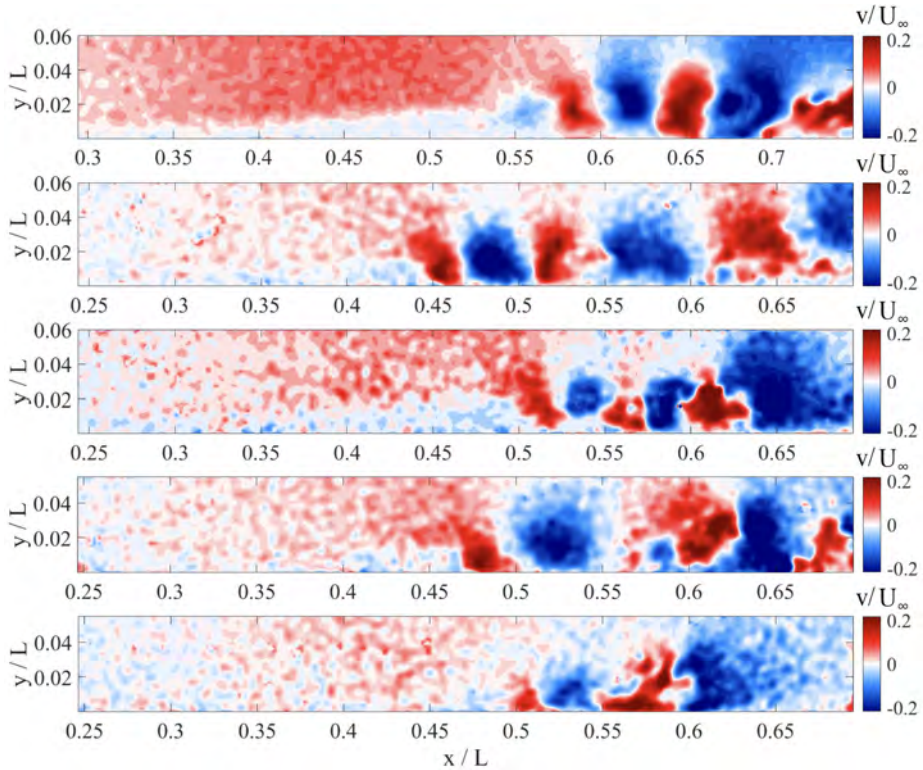


FIGURE 55. Instantaneous vertical velocity component: Baseline, Neoprene 1%, Neoprene 4%, Silicone 12%, and Silicone 20% pre-tension (top to bottom respectively).

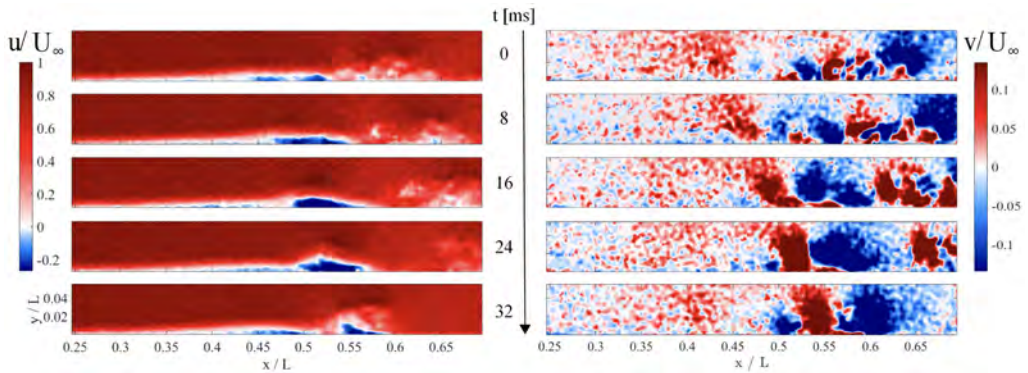


FIGURE 56. Separation bubble shedding cycle, streamwise and vertical velocity: Silicone 12% pre-tension, fixed on all sides.

this case was approximately 24 Hz. Similar patterns observed here validate the simulation results qualitatively.

Membrane Motion

By replacing the baseline rigid surface with a compliant membrane an additional degree of freedom has been added to the system. Due to the fluidic pressure loading, the compliant membrane exhibits a relatively large amplitude mean deflection and dynamic surface motion. Focusing on the case of the silicone membrane with 12% pre-tension and fixed boundaries, the time averaged deflection in the z -direction (surface normal) can be seen in Fig. 57.

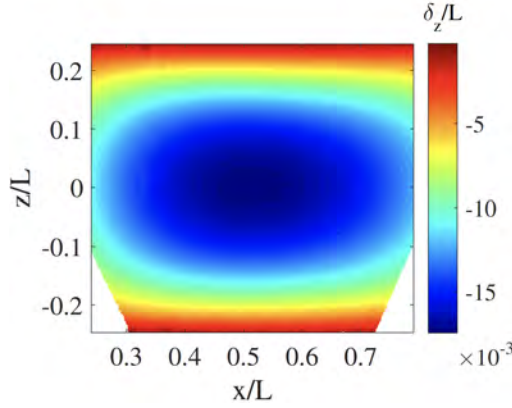


FIGURE 57. Time-averaged in-plane membrane displacement, Silicone 12 % pre-strain.

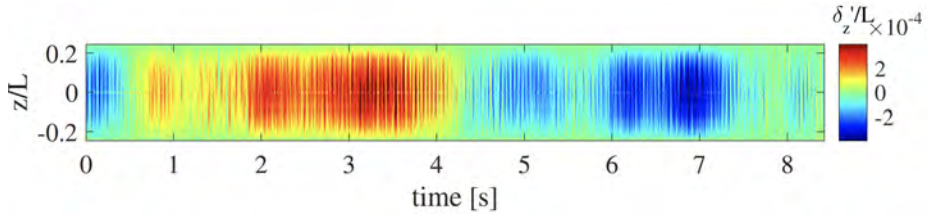


FIGURE 58. Fluctuations of membrane displacement across mid-span over time, Silicone 12 % pre-strain.

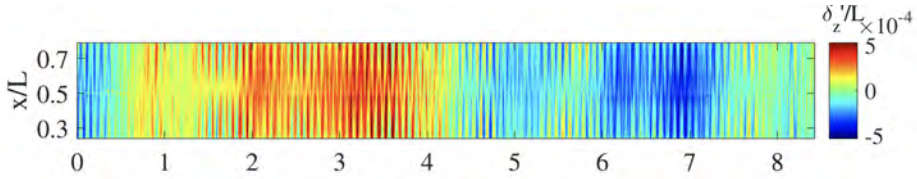
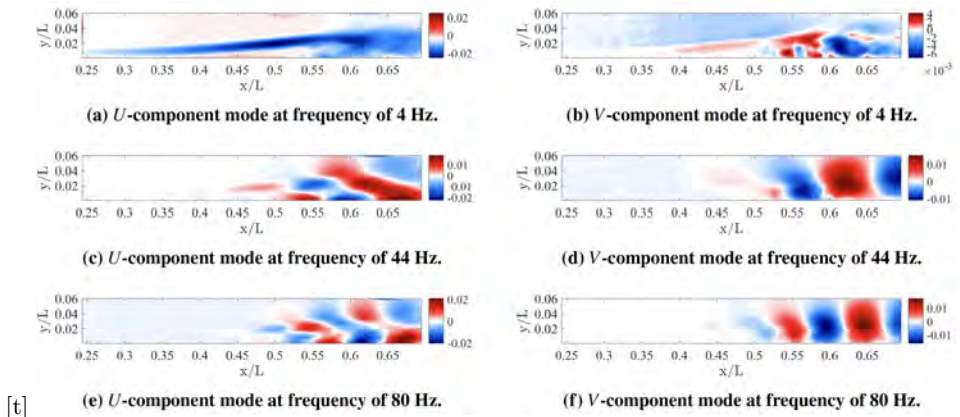


FIGURE 59. Fluctuations of membrane displacement down the streamwise centerline over time, Silicone 12 % pre-strain.

FIGURE 60. SPOD mode shapes and oscillation frequency corresponding to the 3 most energetic coherent structures in the streamwise (*U*) and vertical (*V*) velocity components of the flow field over the Neoprene membrane with 4% pre-tension and fixed boundaries.

The adverse pressure gradient imposed by the s/b boundary condition creates a significant pressure difference across the membrane causing it to "bow" inwards.

To initially assess the unsteady surface motion, the mean-subtracted instantaneous membrane motion is plotted. A slice of data through the center of the membrane in both the streamwise and spanwise directions is taken from each vector field and used to create the space-time plots in Fig. 58 and 59 below. It can be seen from the figures that there are distinct patterns to the motion in these regions. The surface fluctuations across the span are relatively in phase and exhibit low frequency changes while motion in the streamwise direction contains additional higher frequency content. The "zig-zag" like features observed in Fig. 59 reveal higher order modal behavior. There appears to be a significant phase lag between the upstream and downstream fluctuations of the membrane.

Modal Analysis

To further analyze the unsteady dynamic motion of the separation bubble as well as the compliant membrane the modal analysis technique SPOD was employed. This data analysis tool decomposes the data snapshots into energy-ranked modes that oscillate at a particular frequency Towne *et al.* (2018). The modal shapes and oscillation frequency reveal dominating characteristics of the dynamics behind the separation bubble and compliant membrane motion.

The laminar separation bubble flow that developed over the neoprene membrane with 14% pre-tension and silicone membrane with 12% pre-tension is analyzed using this technique. The 10000 total images were broken into blocks of 2500 snapshots. The blocks are overlapped by 75% and a Hamming window function is applied when calculating the temporal discrete fourier transform (DFT). This results in a frequency resolution of 4 Hz. First looking at the case of the neoprene membrane, the modes found to have the most energy pertain to a dimensional frequencies of approximately 4, 44 and 80 Hz. The mode shape pertaining to each frequency of motion can be seen in Fig. 60. The shapes seen in the V -component of the mode clearly resemble convective structures pinching off the shear layer and shedding downstream. Very similar features are seen in the instantaneous images of Fig. 55. Comparing these to the SPOD modes calculated from the flow field over the silicone membrane, Fig. 61, some differences can be observed. The three most dominant frequencies in the flow field for the case of the silicone membrane are 48 Hz, 64 Hz and 92 Hz. The mode shape of the 48 Hz motion are qualitatively nearly identical to those found for the neoprene membrane at 44 Hz. The separation bubble over the silicone membrane also has energy at a higher frequency of 92 Hz. The higher frequency content could be attributed to the reduced mass and higher pre-tension of the silicone membrane compared to the neoprene. This would cause it to have a higher natural frequency of oscillation.

SPOD was also performed on the out-of-plane displacement fields of the membrane motion. Approximately 1700 images were broken into records of 100 snapshots. The temporal DFT of each block was computed using 75% overlap and Hamming window function resulting in a frequency resolution of 2 Hz. The most energetic modes at frequencies of 2, 12, 22 and 32 Hz can be seen in Fig. 62. The lower frequency mode shapes with oscillations of 2 and 12 Hz resemble the first two principle modes of a clamped membrane. Almost all dominant frequency components of the membrane are lower than what was determined from the flow fields. This further emphasizes the separation of the characteristic time-scales between the separation bubble dynamics and the membrane. Although it has lower energy, the mode shape with a frequency of 22 Hz was invested due to the fact that the same frequency was also measured when doing an impulse test on the silicone membrane. It is also the case that the SPOD mode found in the flow field over the silicone membrane has significant energy content at approximately 48 Hz, which is close to the first harmonic of the 22 Hz membrane motion. A potential driving mechanism between the separation-bubble compliant membrane

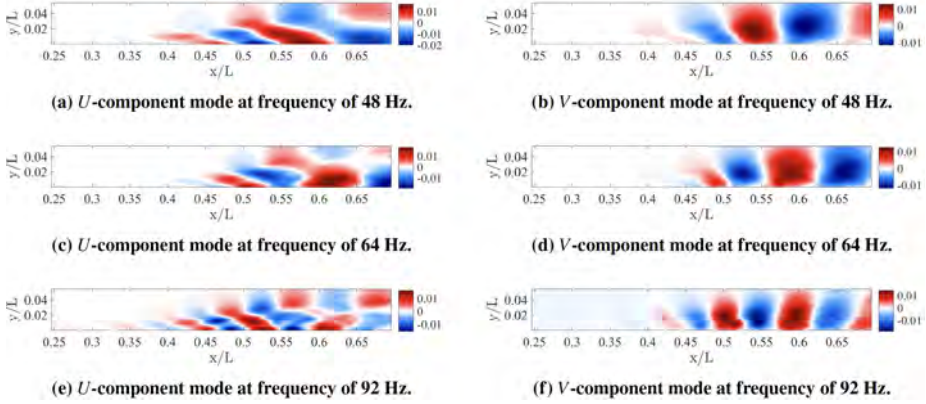


FIGURE 61. SPOD mode shapes and oscillation frequency corresponding to the 3 most energetic coherent structures in the streamwise (U) and vertical (V) velocity components of the flow field over the Silicone membrane with 12% pre-tension and fixed boundaries.

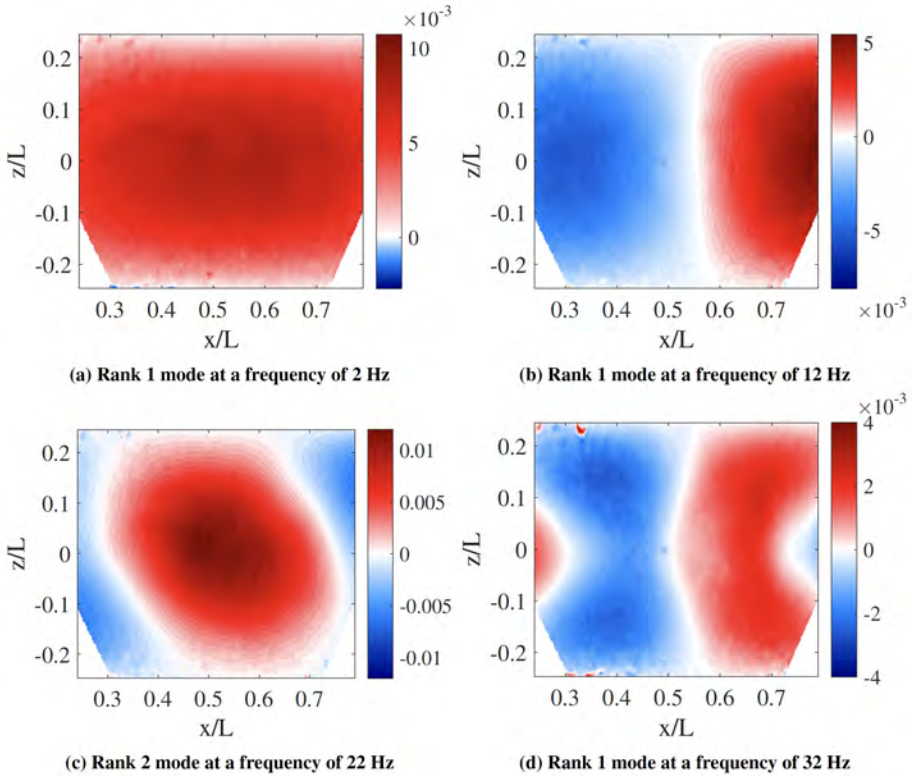


FIGURE 62. SPOD mode shapes and oscillation frequency corresponding to the 4 most energetic, out-of-plane, coherent membrane structures.

interaction could be due to a dynamic coupling between the lower and higher order modes of the membrane and separation bubble respectively. The lower frequency motion of the membrane seems to intermittently couple with the shedding of vortices from the separation bubble. Harmonic frequencies of the membrane motion seem to be captured in the dominant SPOD modes of the flow field.

7.4. Summary

In this work the interaction between a separation bubble and compliant membrane was investigated from an experimental standpoint. In the experiments it was observed that the compliant surface deformation has a large impact on the behavior of the separation bubble by affecting the size, rate of shear layer shedding, and bubble dynamics. Due to the compliance of the membranes the mean and fluctuating pressure load from the separation bubble causes a relatively large mean deflection of the membrane accompanied by dynamic surface motion about the mean. To better understand the coupled dynamic phenomena between the separation bubble and the compliant surface, modal analysis tools were used to extract high energy coherent structures and their corresponding frequency of motion. Dominant frequencies of membrane and flow field motion seem to reveal energy transfer via the excitation of harmonics between domains.

8. Concluding remarks

From this one-year effort (and additional year of NCE), we have been able to develop a number of theoretical, computational, and experimental techniques to analyze the perturbations dynamics for unsteady flows. The use of randomized numerical linear algebra was leveraged to handle large-scale operators seen in high-Reynolds number flows and time-varying base state analyses. We have incorporated the randomized techniques into stability and resolvent analysis efforts. One major application of the randomized analysis technique is the buffeting analysis of transonic flow over an airfoil. We further considered the use of phase-reduction analysis, which projects the governing equations into the phase dynamics relation. This technique is sensor-based and allows for its use in computations and experiments. Synchronization of the wake dynamics to unsteady flow control input was shown. We have also made significant strides in establishing the computational and experimental capabilities of studying fluid-structure interaction problems. This setup enables upcoming studies on complex multi-physics problems related to aeroelasticity and flow control. The output from the present effort has established some of the foundations needed to pursue unsteady flow analysis, which goes beyond traditional analysis techniques that rely on the assumption of steady and quasi-steady analysis.

9. Publication Record

Journal papers

- K. Taira and H. Nakao, "Phase-Response Analysis of Synchronization for Periodic Flows," *Journal of Fluid Mechanics*, 846, R2, 2018
- B. L. O. Ramos, W. R. Wolf, C.-A. Yeh, and K. Taira, "Active Flow Control for Drag Reduction of a Plunging Airfoil under Deep Dynamic Stall," *Physical Review Fluids*, 4, 074603, 2019
- Y. Kojima, C.-A. Yeh, K. Taira, and M. Kameda, "Resolvent Analysis on the Origin of Two-Dimensional Transonic Buffet," *Journal of Fluid Mechanics*, 885, R1, 2020
- J. H. M. Ribeiro, C.-A. Yeh, and K. Taira, "Randomized Resolvent Analysis," *Physical Review Fluids*, 5, 033902, 2020
- K. Taira, M. S. Hemati, S. L. Brunton, Y. Sun, K. Duraisamy, S. Bagheri, S. T. M. Dawson, and C.-A. Yeh, "Modal Analysis of Fluid Flows: Applications and Outlook," *AIAA Journal*, 58(3), 998-1022 (invited), 2020

Conference papers

- B. L. O. Ramos, W. R. Wolf, C.-A. Yeh, and K. Taira, "High-Fidelity Simulation and

Flow Control of a Plunging Airfoil under Deep Dynamic Stall," AIAA Science and Technology Forum and Exposition, San Diego, CA, Jan. 7-11, 2019

- R. Richardson, T.-K. Wang, L. N. Cattafesta and K. Shoele, "Dynamics of a separation bubble subject to compliant surface motion," AIAA Aviation 2019 Forum, Dallas, TX, Jun. 17-21, 2019

References

- ANDERSON, J. D. 2005 Ludwig prandtl's boundary layer. *Physics Today* **58** (12), 42–48.
- ANYOJI, M., NUMATA, D., NAGAI, H. & ASAI, K. 2015 Effects of Mach number and specific heat ratio on low-Reynolds-number airfoil flows. *AIAA J.* **53** (6), 1640–1654.
- ARAM, E., MITTAL, R. & CATTAFESTA, L. 2010a Simple representations of zero-net mass-flux jets in grazing flow for flow-control simulations. *Int. J. Flow Ctrl.* **2** (2).
- ARAM, E., MITTAL, R., GRIFFIN, J. & CATTAFESTA, L. 2010b Towards effective znmf jet based control of a canonical separated flow. *AIAA Paper 2010-4705*.
- ARCE, M., TIMPE, A., UKEILEY, L., HUBNER, J. & IFJU, P. 2013 Passively compliant membranes in low aspect ratio wings. *AIAA Paper 2013-681*.
- ARNOL'D, V. I. 1997 *Mathematical methods of classical mechanics*. Springer.
- BARKLEY, D. & HENDERSON, R. D. 1996 Three-dimensional floquet stability analysis of the wake of a circular cylinder. *Journal of Fluid Mechanics* **322**, 215–241.
- BLACKBURN, H. M. & LOPEZ, J. M. 2003 On three-dimensional quasiperiodic floquet instabilities of two-dimensional bluff body wakes. *Physics of Fluids* **15** (8), L57–L60.
- BLACKBURN, H. M. & SHEARD, G. J. 2010 On quasiperiodic and subharmonic floquet wake instabilities. *Physics of Fluids* **22** (3), 031701.
- BLENNERHASSETT, P. J. & BASSOM, A. P. 2006 The linear stability of high-frequency oscillatory flow in a channel. *J. Fluid Mech.* **556**, 1–25.
- BRÈS, G. A., HAM, F. E., NICHOLS, J. W. & LELE, S. K. 2017 Unstructured large-eddy simulations of supersonic jets. *AIAA Journal* pp. 1164–1184.
- BUCK, J. 1988 Synchronous rhythmic flashing of fireflies. II. *Quart. Rev. Bio.* **63** (3), 265–289.
- CANUTO, D. & TAIRA, K. 2015 Two-dimensional compressible viscous flow around a circular cylinder. *J. Fluid Mech.* **785**, 349–371.
- CARPENTER, P. W. & GARRAD, A. D. 1985 The hydrodynamic stability of flow over kramer-type compliant surfaces. Part 1. tollmien-schlichting instabilities. *J. Fluid Mech.* **155**, 465–510.
- CARPENTER, P. W. & GARRAD, A. D. 1986 The hydrodynamic stability of flow over Kramer-type compliant surfaces. Part 2. flow-induced surface instabilities. *J. Fluid Mech.* **170**, 199–232.
- CHENG, CHUN, LI, SHUNMING, WANG, YONG & JIANG, XINGXING 2017 Force and displacement transmissibility of a quasi-zero stiffness vibration isolator with geometric nonlinear damping. *Nonlinear Dynamics* **87** (4), 2267–2279.
- CLARKSON, K. L. & WOODRUFF, D. P. 2009 Numerical linear algebra in the streaming model. In *Proceedings of the forty-first annual ACM symposium on Theory of computing*, pp. 205–214.
- COLONIUS, T. & TAIRA, K. 2008 A fast immersed boundary method using a nullspace approach and multi-domain far-field boundary conditions. *Comput. Methods Appl. Mech. Engrg.* **197** (25-28), 2131–2146.
- CONNELL, B. S. H. & YUE, D. K. P. 2007 Flapping dynamics of a flag in a uniform stream. *J. Fluid Mech.* **581**, 33–67.
- COOPER, J., KHODAPARAST, H., RICCI, S., GEORGIOU, G., VIO, G., TRAWAGLINI, L. & DENMER, P. 2012 Rapid prediction of worst case gust loads. *AIAA Paper 2012-2040*.
- CRIMINALE, W. O., JACKSON, T. L. & JOSLIN, R. D. 2018 *Theory and computation in hydrodynamic stability*. Cambridge University Press.
- CROUCH, J. D., GARBARUK, A., MAGIDOV, D. & TRAVIN, A. 2009 Origin of transonic buffet on aerofoils. *J. Fluid Mech.* **628**, 357–369.

- DANDOIS, J. 2016 Experimental study of transonic buffet phenomenon on a 3D swept wing. *Phys. Fluids* **28** (1), 016101.
- DECK, S. 2005 Numerical simulation of transonic buffet over a supercritical airfoil. *AIAA J.* **43** (7), 1556–1566.
- DEEM, E., CATTAFESTA, L., ZHANG, H., ROWLEY, C., HEMATI, M., CADIEUX, F. & MITTAL, R. 2017 Identifying dynamic modes of separated flow subjection to ZNMF-based control from surface pressure measurements. *AIAA 2017 Aviation*.
- DEEM, E. A., CATTAFESTA, L. N., YAO, H., HEMATI, M., ZHANG, H. & ROWLEY, C. W. 2018 Experimental implementation of modal approaches for autonomous reattachment of separated flows. *AIAA Paper 2018-1052*.
- DEEM, E. A., CATTAFESTA, L. N., ZHANG, H., ROWLEY, C. W., HEMATI, M., CADIEUX, F. & MITTAL, R. AIAA-2017-3309 Identifying dynamic modes of separated flow subject to znmf-based control from surface pressure measurements. *AIAA Paper 2017-3309*.
- DELERY, J. M. 1985 Shock wave/turbulent boundary layer interaction and its control. *Prog. Aerosp. Sciences* **22** (4), 209–280.
- DEMME, J., DUMITRIU, I. & HOLTZ, O. 2007 Fast linear algebra is stable. *Num. Math.* **108** (1), 59–91.
- DRAZIN, P. G. & REID, W. H. 1981 *Hydrodynamic stability*. Cambridge Univ. Press.
- DRAZIN, P. G. & REID, W. H. 2004 *Hydrodynamic stability*. Cambridge university press.
- DRINEAS, P. & MAHONEY, M. W. 2016 RandNLA: randomized numerical linear algebra. *Commun. ACM* **59** (6), 80–90.
- DWIVEDI, A., SIDHARTH, G. S., NICHOLS, J. W., CANDLER, G. V. & JOVANOVIĆ, M. R. 2019 Reattachment streaks in hypersonic compression ramp flow: an input–output analysis. *J. Fluid Mech.* **880**, 113–135.
- ERICHSON, N. B., MATHELIN, L., KUTZ, J. N. & BRUNTON, S. L. 2019 Randomized dynamic mode decomposition. *SIAM J. Appl. Dyn. Syst.* **18** (4), 1867–1891.
- ERICHSON, N. B., VORONIN, S., BRUNTON, S. L. & KUTZ, J. N. 2016 Randomized matrix decompositions using R. *arXiv preprint arXiv:1608.02148*.
- ERMENTROUT, G. B. & Terman, D. H. 2010 *Mathematical foundations of neuroscience*. Springer.
- FARRELL, B. F. & IOANNOU, P. J. 1994 Variance maintained by stochastic forcing of non-normal dynamical systems associated with linearly stable shear flows. *Physical Rev. Let.* **72** (8), 1188.
- FARRELL, B. F. & IOANNOU, P. J. 1996a Generalized stability theory. Part I: Autonomous operators. *J. Atmospheric Sci.* **53** (14), 2025–2040.
- FARRELL, B. F. & IOANNOU, P. J. 1996b Generalized stability theory. Part II: Nonautonomous operators. *J. Atmospheric Sci.* **53** (14), 2041–2053.
- FLOQUET, G 1883 Sur les équations différentielles linéaires à coefficients périodiques. *Annales de l'École Normale Supérieure* **12** (1883), 47–88.
- FREUND, J. B. 1997 Proposed inflow/outflow boundary condition for direct computation of aerodynamic sound. *AIAA J.* **35** (4), 740–742.
- FUKAGATA, K., KERN, S., CHATELAIN, P., KOUMOUTSAKOS, P. & KASAGI, N. 2008 Evolutionary optimization of an anisotropic compliant surface for turbulent friction drag reduction. *J. Turbul.* **9**, 35.
- FUKUSHIMA, Y. & KAWAI, S. 2018 Wall-modeled large-eddy simulation of transonic airfoil buffet at high Reynolds number. *AIAA J.* **56** (6), 2372–2388.
- FULLER, J. R. 1995 Evolution of airplane gust loads design requirements. *J. Aircraft* **32** (2), 235–246.
- GAO, C., ZHANG, W., KOU, J., LIU, Y. & YE, Z. 2017 Active control of transonic buffet flow. *J. Fluid Mech.* **824**, 312–351.
- GENT, A. N. 1958 On the relation between indentation hardness and Young’s modulus. *Rubber Chem. Technol.* **31** (4), 896–906.
- GIANNELIS, N. F., VIO, G. A. & LEVINSKI, O. 2017 A review of recent developments in the understanding of transonic shock buffet. *Prog. Aerosp. Sciences* **92**, 39–84.
- GOPALAKRISHNAN MEENA, MURALIKRISHNAN, TAIRA, KUNHIKO & ASAI, KEISUKE 2017 Airfoil wake modification with gurney flap at low-reynolds number. In *Submitted*.

- GORDNIER, R. E. 2009 High fidelity computational simulation of a membrane wing airfoil. *J. Fluids Struct.* **25** (5), 897–917.
- GRIFFIN, J. C., OYARZUN, M., CATTAFESTA, L. N., TU, J. H. & ROWLEY, C. W. AIAA-2013-2968 Control of a canonical separated flow. *AIAA Paper 2013-2968*.
- GROSSI, F., BRAZA, M. & HOARAU, Y. 2014 Prediction of Transonic Buffet by Delayed Detached-Eddy Simulation. *AIAA J.* **52** (10), 2300–2312.
- GU, M. 2015 Subspace iteration randomization and singular value problems. *SIAM J. Sci. Comput* **37** (3), A1139–A1173.
- GUCKENHEIMER, J. 1975 Isochrons and phaseless sets. *J. Math. Bio.* **1** (3), 259–273.
- HALKO, N., MARTINSSON, P.-G. & TROPP, J. A. 2011 Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. *SIAM Rev.* **53** (2), 217–288.
- HE, W., GIORIA, R. S., PÉREZ, J. M. & THEOFILIS, V. 2017 Linear instability of low reynolds number massively separated flow around three naca airfoils. *Journal of Fluid Mechanics* **811**, 701–741.
- HEATHCOTE, S., WANG, Z. & GURSUL, I. 2008 Effect of spanwise flexibility on flapping wing propulsion. *J. Fluids Struct.* **24** (2), 183–199.
- HOLMES, P., LUMLEY, J. L., BERKOOZ, G. & ROWLEY, C. W. 2012 *Turbulence, coherent structures, dynamical systems and symmetry*. Cambridge university press.
- HUANG, NORDEN E, SHEN, ZHENG, LONG, STEVEN R, WU, MANLI C, SHIH, HSING H, ZHENG, QUANAN, YEN, NAI-CHYUAN, TUNG, CHI CHAO & LIU, HENRY H 1998 The empirical mode decomposition and the hilbert spectrum for nonlinear and non-stationary time series analysis. In *Proceedings of the Royal Society of London A: mathematical, physical and engineering sciences*, , vol. 454, pp. 903–995. The Royal Society.
- IOVNOVICH, M. & RAVEH, D. E. 2012 Reynolds-averaged Navier-Stokes study of the shock-buffet instability mechanism. *AIAA J.* **50** (4), 880–890.
- IOVNOVICH, M. & RAVEH, D. E. 2015 Numerical study of shock buffet on three-dimensional wings. *AIAA J.* **53** (2), 449–463.
- JACQUIN, L., MOLTON, P., DECK, S., MAURY, B. & SOULEVANT, D. 2009 Experimental study of shock oscillation over a transonic supercritical profile. *AIAA J.* **47** (9), 1985–1994.
- JEUN, J., NICHOLS, J. W. & JOVANOVIĆ, M. R. 2016 Input-output analysis of high-speed axisymmetric isothermal jet noise. *Phys. Fluids* **28** (4), 047101.
- JORDAN, DOMINIC WILLIAM & SMITH, PETER 1999 *Nonlinear ordinary differential equations: an introduction to dynamical systems*, , vol. 2. Oxford University Press, USA.
- JOSLIN, R. D. & MILLER, D. N., ed. 2009a *Fundamentals and applications of modern flow control*. AIAA.
- JOSLIN, R. D. & MILLER, D. N. 2009b *Fundamentals and applications of modern flow control*. American Institute of Aeronautics and Astronautics.
- JOVANOVIĆ, M. R. 2004 Modeling, analysis, and control of spatially distributed systems. PhD thesis, University of California, Santa Barbara, Santa Barbara, CA.
- JOVANOVIĆ, M. R. & BAMIEH, B. 2005 Componentwise energy amplification in channel flows. *J. Fluid Mech.* **534**, 145–183.
- KAJISHIMA, T. & TAIRA, K. 2017 *Computational fluid dynamics: incompressible turbulent flows*. Springer.
- KAWAMURA, Y. & NAKAO, H. 2013 Collective phase description of oscillatory convection. *Chaos* **23** (4), 043129.
- KAWAMURA, Y. & NAKAO, H. 2015 Phase description of oscillatory convection with a spatially translational mode. *Physica D* **295**, 11–29.
- KHALIGHI, Y., HAM, F., NICHOLS, J., LELE, S. & MOIN, P. 2011 Unstructured large eddy simulation for prediction of noise issued from turbulent jets in various configurations. *AIAA Paper 2011-2886*.
- KIM, E. & CHOI, H. 2014 Space–time characteristics of a compliant wall in a turbulent channel flow. *J. Fluid Mech.* **756**, 30–53.
- KONING, W. J. F., JOHNSON, W. & GRIP, H. F. 2019 Improved Mars helicopter aerodynamic rotor model for comprehensive analyses. *AIAA J.* **57** (9), 3969–3979.
- KURAMOTO, Y. 1984 *Chemical oscillations, waves, and turbulence*. Springer.

- KUTZ, J. N., BRUNTON, S. L., BRUNTON, B. W. & PROCTOR, J. L. 2016 *Dynamic mode decomposition: data-driven modeling of complex systems*. SIAM.
- LEE, B. H. K. 2001 Self-sustained shock oscillations on airfoils at transonic speeds. *Prog. Aerosp. Sci.* **37** (2), 147–196.
- LEVY, L. L. 1978 Experimental and computational steady and unsteady transonic flows about a thick airfoil. *AIAA J.* **16** (6), 564–572.
- LINNICK, M. N. & FASEL, H. F. 2005 A high-order immersed interface method for simulating unsteady incompressible flows on irregular domains. *J. Comput. Phys.* **204** (1), 157–192.
- LIU, C., ZHENG, X. & SUNG, C. H. 1998 Preconditioned multigrid methods for unsteady incompressible flows. *J. Comput. Phys.* **139** (1), 35–57.
- LIU, K. 1997 Identification of linear time-varying systems. *J. Sound Vib.* **206** (4), 487–505.
- LIU, K. 1999 Extension of modal analysis to linear time-varying systems. *J. Sound Vib.* **226** (1), 149–167.
- LIU, Q., SUN, Y., CATTAFESTA, L. N., UKEILEY, L. S. & TAIRA, K. 2018 Resolvent analysis of compressible flow over a long rectangular cavity. AIAA 2018 SciTech meeting, accepted.
- MAUROY, A. & MEZIĆ, I. 2012 On the use of fourier averages to compute the global isochrons of (quasi) periodic dynamics. *Chaos* **22** (3), 033112.
- MCCORMICK, D. C. 1993 Shock/boundary-layer interaction control with vortex generators and passive cavity. *AIAA J.* **31** (1), 91–96.
- MCDVITT, J. B. & OKUNO, A. F. 1985 Static and dynamic pressure measurements on a NACA 0012 airfoil in the Ames high Reynolds number facility. *Tech. Rep.* NASA TP-2485. National Aeronautics and Space Administration.
- MCKEON, B. J. & SHARMA, A. S. 2010 A critical-layer framework for turbulent pipe flow. *J. Fluid Mech.* **658**, 336–382.
- MITTAL, R., DONG, H., BOZKURTAS, M., NAJJAR, F. M., VARGAS, A. & VON LOEBBECKE, A. 2008 A versatile sharp interface immersed boundary method for incompressible flows with complex boundaries. *J. Comput. Phys.* **227** (10), 4825–4852.
- MOARREF, R., SHARMA, A. S., TROPP, J. A. & MCKEON, B. J. 2013 Model-based scaling of the streamwise energy density in high-reynolds-number turbulent channels. *J. Fluid Mech.* **734**, 275–316.
- MOLER, C. & VAN LOAN, C. 2003 Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM Rev.* **45** (1), 3–49.
- MORGAN, R. B. & ZENG, M. 2006 A harmonic restarted Arnoldi algorithm for calculating eigenvalues and determining multiplicity. *Lin. Alg. Appl.* **415** (1), 96–113.
- MUNDAY, P. M. & TAIRA, K. 2013 On the lock-on of vortex shedding to oscillatory actuation around a circular cylinder. *Phys. Fluids* **25** (1), 013601.
- MUNDAY, P. M. & TAIRA, K. 2014 Separation control on NACA 0012 airfoil using momentum and wall-normal vorticity injection. In *AIAA Paper 2014-2685*, p. 2685.
- MUNDAY, P. M. & TAIRA, K. 2018 Effects of wall-normal and angular momentum injections in airfoil separation control. *AIAA J.* **56** (5), 1830–1842.
- MUNDAY, P. M., TAIRA, K., SUWA, T., NUMATA, D. & ASAI, K. 2015 Nonlinear lift on a triangular airfoil in low-Reynolds-number compressible flow. *J. Aircraft* **52** (3), 924–931.
- NAKAO, H. 2016 Phase reduction approach to synchronisation of nonlinear oscillators. *Contemporary Phys.* **57** (2), 188–214.
- NAKASHIMA, S., FUKAGATA, K. & LUHAR, M. 2017 Assessment of suboptimal control for turbulent skin friction reduction via resolvent analysis. *J. Fluid Mech.* **828**, 496–526.
- NITZSCHE, J. 2009 A numerical study on aerodynamic resonance in transonic separated flow. *International Forum on Aeroelasticity and Structural Dynamics* pp. 1–15.
- NOACK, B. R. & ECKELMANN, H. 1994 A global stability analysis of the steady and periodic cylinder wake. *Journal of Fluid Mechanics* **270**, 297–330.
- OHMACHI, Y., ISHIDA, T. & HASHIMOTO, A. 2018 Modal decomposition analysis of three-dimensional transonic buffet phenomenon on a swept wing. *AIAA J.* **56** (10), 3938–3950.
- PATIL, M. J. & HODGES, D. H. 2006 Flight dynamics of highly flexible flying wings. *J. Aircraft* **43** (6), 1790–1799.
- PERKO, LAWRENCE 2013 *Differential equations and dynamical systems*, , vol. 7. Springer Science & Business Media.

- PIKOVSKY, A., KURTHS, J., ROSENBLUM, M. & KURTHS, J. 2003 *Synchronization: a universal concept in nonlinear sciences*. Cambridge university press.
- RAJU, R., ARAM, E., MITTAL, R. & CATTAFESTA, L. 2009 Simple models of zero-net mass-flux jets for flow control simulations. *Int. J. Flow Ctrl.* **1** (3), 179–197.
- REMPFER, D., PARSONS, L., XU, S. & LUMLEY, J. 2003 Theoretical approaches to the effect of wall compliance on turbulent flow. In *Flow past Highly Compliant Boundaries and in Collapsible Tubes*, pp. 231–252. Springer.
- RIBEIRO, J. H. M. & WOLF, W. R. 2017 Identification of coherent structures in the flow past a naca0012 airfoil via proper orthogonal decomposition. *Phys. Fluids* **29** (8), 085104.
- RICCIARDI, T. R., WOLF, W. R. & SPETH, R. 2018 Acoustic prediction of LAGOON landing gear: cavity noise and coherent structures. *AIAA J.* **56** (11), 4379–4399.
- ROBICHAUX, J., BALACHANDAR, S. & VANKA, S. P. 1999 Three-dimensional floquet instability of the wake of square cylinder. *Physics of Fluids* **11** (3), 560–578.
- ROJRATSIKUL, P., WANG, Z. & GURSUL, I. 2008 Unsteady aerodynamics of membrane airfoils. *AIAA Paper 2008-613*.
- ROJRATSIKUL, P., WANG, Z. & GURSUL, I. 2009 Unsteady fluid–structure interactions of membrane airfoils at low reynolds numbers. *Exp. fluids* **46** (5), 859–872.
- ROJRATSIKUL, P., WANG, Z. & GURSUL, I. 2010 Effect of pre-strain and excess length on unsteady fluid–structure interactions of membrane airfoils. *J. Fluids Struct.* **26** (3), 359–376.
- ROKHLIN, V., SZLAM, A. & TYGERT, M. 2010 A randomized algorithm for principal component analysis. *SIAM J. Matrix Anal. Appl.* **31** (3), 1100–1124.
- ROMA, A. M., PESKIN, C. S. & BERGER, M. J. 1999 An adaptive version of the immersed boundary method. *J. Comput. Phys.* **153** (2), 509–534.
- ROWLEY, C. W., MEZIĆ, I., BAGHERI, S., SCHLATTER, P. & HENNINGSON, D. S. 2009 Spectral analysis of nonlinear flows. *J. Fluid Mech.* **641**, 115–127.
- SARTOR, F., METTOT, C. & SIPP, D. 2014 Stability, receptivity, and sensitivity analyses of buffeting transonic flow over a profile. *AIAA J.* **53** (7), 1980–1993.
- SCHMID, P. J. 2007 Nonmodal stability theory. *Annu. Rev. Fluid Mech.* **39**, 129–162.
- SCHMID, P. J. 2010 Dynamic mode decomposition of numerical and experimental data. *J. Fluid Mech.* **656**, 5–28.
- SCHMID, P. J. & HENNINGSON, D. S. 2001 *Stability and transition in shear flows*. Springer.
- SCHMID, P. J. & HENNINGSON, D. S. 2012 *Stability and transition in shear flows*. Springer.
- SCHMIDT, O. T., TOWNE, A., RIGAS, G., COLONIUS, T. & BRÈS, G. A. 2018 Spectral analysis of jet turbulence. *J. Fluid Mech.* **855**, 953–982.
- SEARS, W. R. & TELIONIS, D. P. 1975 Boundary-layer separation in unsteady flow. *SIAM J. Appl. Math.* **28** (1), 215–235.
- SEEGMILLER, H. L., MARVIN, J. G. & LEVY JR., L. L. 1978 Steady and unsteady transonic flow. *AIAA J.* **16** (12), 1262–1270.
- SEO, J. H., CADIEUX, F., MITTAL, R., DEEM, E. & CATTAFESTA, L. 2018 Effect of synthetic jet modulation schemes on the reduction of a laminar separation bubble. *Phys. Rev. Fluids* **3**, 033901.
- SHI, J., HU, C. & SHU, C. W. 2002 A technique of treating negative weights in WENO schemes. *Journal of Computational Physics* **175** (1), 108–127.
- STROGATZ, S. H. 2000 From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators. *Physica D* **143** (1–4), 1–20.
- SUN, Y., TAIRA, K., CATTAFESTA, L. N. & UKEILEY, L. S. 2017 Biglobal instabilities of compressible open-cavity flows. *Journal of Fluid Mechanics* **826**, 270–301.
- TAIRA, K., BRUNTON, S. L., DAWSON, S. T. M., ROWLEY, C. W., COLONIUS, T., MCKEON, B. J., SCHMIDT, O. T., GORDEYEV, S., THEOFILIS, V. & UKEILEY, L. S. 2017 Modal analysis of fluid flows: An overview. *AIAA J.* **55** (12), 4013–4041.
- TAIRA, K. & COLONIUS, T. 2007 The immersed boundary method: a projection approach. *J. Comput. Phys.* **225** (2), 2118–2137.
- TAIRA, K., HEMATI, M. S., BRUNTON, S. L., SUN, Y., DURAISAMY, K., BAGHERI, S., DAWSON, S. T. M. & YEH, C.-A. 2019 Modal Analysis of Fluid Flows: Applications and Outlook. *AIAA J.* pp. 1–25.

- THEOFILIS, V. 2003 Advances in global linear instability analysis of nonparallel and three-dimensional flows. *Prog. Aerosp. Sci.* **39** (4), 249–315.
- THEOFILIS, V. 2011*a* Global linear instability. *Annu. Rev. Fluid Mech.* **43**, 319–352.
- THEOFILIS, V. 2011*b* Global linear instability. *Annu. Rev. Fluid Mech.* **43**, 319–352.
- TIMPE, A., UKEILEY, L., ZHANG, Z., HUBNER, J. & IFJU, P. 2012 Flow and structure measurements of a passively compliant wing. *AIAA Paper 2012-1210*.
- TIMPE, A., ZHANG, Z., HUBNER, J. & UKEILEY, L. 2013 Passive flow control by membrane wings for aerodynamic benefit. *Exp. Fluids* **54** (3), 1471.
- TOKUMARU, P. T. & DIMOTAKIS, P. E. 1991 Rotary oscillation control of a cylinder wake. *J. Fluid Mech.* **224**, 77–90.
- TOWNE, A., SCHMIDT, O. T. & COLONIUS, T. 2018 Spectral proper orthogonal decomposition and its relationship to dynamic mode decomposition and resolvent analysis. *J. Fluid Mech.* **847**, 821–867.
- TREFETHEN, LLOYD NICHOLAS & EMBREE, MARK 2005 *Spectra and pseudospectra: the behavior of nonnormal matrices and operators*. Princeton University Press.
- TREFETHEN, L. N., TREFETHEN, A. E., REDDY, S. C. & DRISCOLL, T. A. 1993 Hydrodynamic stability without eigenvalues. *Science* **261** (5121), 578–584.
- TROPP, J. A., YURTSEVER, A., UDELL, M. & CEVHER, V. 2017 Practical sketching algorithms for low-rank matrix approximation. *SIAM J. Matrix Anal. Appl.* **38** (4), 1454–1485.
- TSIGKLIFIS, K. & LUCEY, A. D. 2017 The interaction of Blasius boundary-layer flow with a compliant panel: global, local and transient analyses. *J. Fluid Mech.* **827**, 155–193.
- TU, J. H., ROWLEY, C. W., LUCHTENBURG, D. M., BRUNTON, S. L. & KUTZ, J. N. 2014 On dynamic mode decomposition: Theory and applications. *J. Comput. Dyn.* **1** (2), 391–421.
- VREMAN, A. W. 2004 An eddy-viscosity subgrid-scale model for turbulent shear flow: Algebraic theory and applications. *Phys. Fluids* **16** (10), 3670–3681.
- WINFREE, A. T. 1967 Biological rhythms and the behavior of populations of coupled oscillators. *J. Theo. Bio.* **16** (1), 15–42.
- WOOLFE, F., LIBERTY, E., ROKHLIN, V. & TYGERT, M. 2008 A fast randomized algorithm for the approximation of matrices. *Appl. Comput. Harmon. Anal.* **25** (3), 335–366.
- WRIGHT, J. R. & COOPER, J. E. 2008 *Introduction to aircraft aeroelasticity and loads*. John Wiley & Sons.
- XIA, Q.-J., HUANG, W.-X. & XU, C.-X. 2017 Direct numerical simulation of turbulent boundary layer over a compliant wall. *J. Fluids Struct.* **71**, 126–142.
- YAMAGUCHI, Y., SUZUKI, T., MIZORO, Y., KORI, H., OKADA, K., CHEN, Y., FUSTIN, J.-M., YAMAZAKI, F., MIZUGUCHI, N., ZHANG, J., DONG, X., TSUJIMOTO, G., OKUNO, Y., DOI, M. & OKAMURA, H. 2013 Mice genetically deficient in vasopressin V1a and V1b receptors are resistant to jet lag. *Science* **342** (6154), 85–90.
- YEH, C.-A., MUNDAY, P. & TAIRA, K. 2017 Use of local periodic heating for separation control on a NACA 0012 airfoil. In *AIAA Paper 2017-1451*.
- YEH, C.-A. & TAIRA, K. 2019 Resolvent-analysis-based design of airfoil separation control. *J. Fluid Mech.* **867**, 572–610.
- YOON, D.-H., YANG, K.-S. & CHOI, C.-B. 2010 Flow past a square cylinder with an angle of incidence. *Physics of Fluids* **22** (4), 043603.
- ZHANG, H., ROWLEY, C. W., DEEM, E. A. & CATTAFESTA, L. N. 2019 Online dynamic mode decomposition for time-varying systems. *SIAM J. Appl. Dyn. Syst.* **18** (3), 1586–1609.
- ZHANG, W. & SAMTANEY, R. 2016 Biglobal linear stability analysis on low-Re flow past an airfoil at high angle of attack. *Phys. Fluids* **28** (4), 044105.
- ZHANG, Z., HUBNER, J., TIMPE, A., UKEILEY, L., ABUDARAM, Y. & IFJU, P. 2012 Effect of aspect ratio on flat-plate membrane airfoils. *AIAA Paper 2012-1084*.
- ZHOU, JIAXI, XIAO, QINGYU, XU, DAOLIN, OUYANG, HUAJIANG & LI, YINGLI 2017 A novel quasi-zero-stiffness strut and its applications in six-degree-of-freedom vibration isolation platform. *Journal of Sound and Vibration* **394**, 59–74.