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**LEADER SELECTION IN COMPLEX NETWORKS  
FOR CONTROLLABILITY AND ENERGY  
EFFICIENCY**

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## ABSTRACT

This report investigates the controllability and energy-related controllability of complex networks. Specifically, our objective is to establish controllability characteristics of signed complex networks, where the network units interact with each other via linear consensus dynamics and the network admits positive and negative edges to capture cooperative and competitive interactions among these units. The network units can be classified into leaders and followers. For network controllability, graphical characterizations of the controllability of signed networks are first developed based on the investigation of the interactions between network topology and unit dynamics. Since signed path and cycle graphs are basic building blocks for a variety of networks, the developed topological characterizations are then exploited to develop leader selection methods for signed path and cycle graphs to ensure the network controllability. Heuristic algorithms are also developed showing how the leader selection methods developed for path and cycle graphs can be potentially extended to more general signed networks. As the control energy metrics, the Gramian-based control energy measures are exploited to quantify the difficulty of the control problem on signed networks in terms of the required control energy and system performance. Fundamental relationships between these measures and the network topology are developed via graph Laplacian to characterize the energy-related controllability. It is revealed that, for the structurally unbalanced signed graphs, the energy-related controllability is closely related to the diagonal entries of the inverse of the graph Laplacian. It is also discovered that the structurally balanced signed graphs and their corresponding unsigned graphs have the same energy-related controllability.

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# 1. INTRODUCTION

## 1.1 Research Background

Complex networks, consisting of interacting elements linked together with processing units, are ubiquitous across many disciplines in modern science and engineering. For example, in social networks, the units are persons or communities and the links between them are social interactions [1]. In power networks, units are electrical generators/substations/transformers and links among units are power lines [2]. Interactions among units in the networks bring new challenges of analyzing the dynamical properties of the networks, such as opinion formation in social networks [3] and formation/consensus in swarm robots [4]. Because of the tremendous potential of networked systems, growing research has been devoted to investigating the structural and functional properties of complex networks. In such applications, one popular approach is to cast the network control problem [5] on complex networks into a leader-follower framework, wherein units receiving external inputs work as leaders to dictate the overall behavior of the network by influencing the rest of the unit, i.e., followers via the connectivity characteristics of the network. Leader-follower systems that coordinate and cooperate over information-exchange networks have been increasingly applied in both science and engineering. Such systems can appear in formation control problems [6]. For example, some robots in a robot team are controlled directed by external signals while the other robots update their states through the communication within the team [4].

Complete state controllability (or simply controllability) describes the ability of external inputs to move the internal state of a system from any initial state to any other final state in a finite time interval [7]. Network controllability is interpreted as the ability to steer a networked system to the desired behavior via external controls, which is of fundamental significance to achieving system functionalities [8,9]. In some applications, networked control systems are required to be maneuvered by external inputs to complete desired tasks and functions, where network controllability ensures the goal of driving a networked control system to any desired state. Classical control theory provides abundant tools, including the Popov-Belevitch-Hautus (PBH) rank condition [10] or the Kalman rank condition [11], via linear algebra and matrix analysis to assess the controllability of a system. However, due to the complexity and higher dimensionality of complex networks, adopting classical ways to check network controllability is of great computation and is obscure for designing networked control systems. The controllability has to be re-investigated via the classical methods through the control theory whenever small changes of network links and/or units occur. On the other hand, it is difficult to numerically determine the Kalman criterion for large networks because the computing is ill-conditioned and is very sensitive to roundoff errors [12].

Compared to traditional ways of checking network controllability, researchers have been trying to investigate the controllability of complex networks by adopting a topological perspective [13–15]. Complex networks mapped by graphs provide insights on interactions among individuals as well as structures of systems. The evolution of network controllability via topology remarkably facilitates the analysis and design of fully controllable networked systems. More importantly, existing algorithms developed on the detection and exploration of topological properties of complex networks can be borrowed to help dig the characteristics of network controllability.

Along with the controllability of complex networks, an additional important aspect is the feasibility of implementing a control approach. This is particularly critical in a situation where a network might be controllable by a selected leader group but the system performance might be infeasible to achieve in practice [16]. For example, a designed control input may exceed the limitations of the actuators. To this effect, an investigation jointly considering the control energy metrics [17] and the controllability, referred to as the energy-related controllability, is necessary for complex networks.

Given the importance of controllability and energy-related controllability of complex networks, this report is motivated by exploring the following two perspectives:

1. Network Controllability: the identification of a small subset of units as leaders such that the selected leaders can drive the network to the desired behavior,
2. Control Energy: minimizing the total energy required to steer the network to a target state by the selected leaders.

## 1.2 Literature Review

Controllability on cooperative networks has been extensively studied in the literature. Leader-follower controllability was considered for the first time in [18], where the network controllability was characterized based on the spectral analysis of the system matrix. Graph-theoretic approaches were then explored to provide characterizations of network controllability. For instance, it was established in [19] that the symmetry of leader groups can potentially lead to uncontrollability. Graphical and topological characterizations of network controllability were investigated in [20] and [21]. Graph-distance based lower bounds on the rank of the controllability matrix were developed in [22]. Sufficient and necessary conditions for network controllability were developed for tree topologies [13], grid graphs [14], and path and cycle graphs [15]. Other than graphical characterizations of network controllability, structural properties of cooperative networks were also exploited from matrix-theoretical perspectives in the works of [23–27] to reveal the connections between network controllability and underlying graphs. Other representative works that investigate network controllability include the results in [28–30]. Besides characterizing network controllability, various methods, e.g., combinatorial [31] and heuristic [32] selection methods, were developed to select leaders to ensure the controllability of given networks. Some representative works regarding leader selection for network controllability include [33–35]. However, most of the aforementioned reports focused on the characterization of network controllability and leader selection on cooperative networks (i.e., unsigned graphs), without considering networks with potential antagonistic interactions (i.e., signed graphs). Therefore, most existing analysis tools (e.g., graph symmetry) and leader selection approaches cannot be applied to signed networks, in which some negative weighted links exist.

In [16], leader selection in complex networks was investigated, where, in addition to ensuring the network controllability, the control energy was also taken into account. In [36], two types of problems were studied in designing leader-follower networked systems by either selecting the minimum number of leaders or selecting a predefined number of leaders with the minimum energy

cost to ensure the controllability of dynamic networks. Various metrics of network controllability have been developed to characterize control energy. Among the most widely used are the Gramian-related metrics [16,17] which introduce energy notions to network control paradigms. The properties of controllability Gramian, such as its minimum eigenvalue, the trace of its inverse, and the condition number, have been extensively explored in the works of [16,17,37] to characterize the energy-related performance in network control. The control energy can also be characterized via spectral analysis of system matrices which have been found to hold a relationship with the controllability Gramian. In [38] and [39], it was discovered that minimal control energy was related to the distribution of the eigenvalues of the system matrix. In [40], the leading right and left eigenvectors of the system matrix were found to play a crucial role in quantifying how much each node contributes to the network in terms of controllability and control energy. Other representative approaches include the optimization-based leader group selection for minimal control energy [41–44], the graph-theoretical characterizations of energy-constrained controllability [45–47], and the network design methods for improved controllability and energy efficiency [48]. Such recent advances in network science have provided control formalisms that include energy considerations to derive feasible solutions to multi-agent control systems. In contrast to past studies, the research presented in this report investigated the energy-related controllability of signed graphs with Laplacian dynamics.

### 1.3 Overview

Motivated by recent advances, the research presented in this report investigated the leader-follower controllability on signed networks. Specifically, we considered signed networks which admit both positive (i.e., collaborative or cooperative between units) and negative (i.e., competitive between units) links (i.e., edges). The signed network is capable of representing a variety of practical network applications, such as social networks, fault-tolerant networks, and secure networks, where the networks may have both friendly and adversarial interactions. Motivated by the broad range of potential applications, it is of particular interest to identify some key nodes (i.e., leaders) in a signed network to drive the network to the desired behavior, even in the presence of antagonistic interactions. In other words, part of this report focuses on leader selection to ensure the controllability of signed networks. In particular, graph-inspired topological characterizations of the leader-follower controllability of signed networks were developed. Such characterizations investigated the interaction between the underlying network topology and unit dynamics to pave the way for leader selection on signed networks. Since signed path and cycle graphs are basic building blocks for a variety of networks, the revealed topological characterizations were then exploited to develop the leader selection methods for signed path and cycle graphs. Along with illustrative examples, the heuristic algorithms were developed showing how leader selection methods developed for path and cycle graphs can be potentially extended for more general signed networks.

In the second part of this report, the energy-related controllability of signed networks is revealed through graph topology, where the network units interact via linear consensus rule [49] and the network allows positive and negative edges to capture cooperative and competitive interactions among network units. Specifically, energy-related measures [16,17], namely, average controllability, average control energy, and volumetric control energy, were considered. These

measures were then characterized in relation to signed graph Laplacian to gain topological insights into the energy-related controllability of complex networks. Also, the controllability Gramian-based measures [16,17] were exploited to quantify the difficulty of the control problem on signed networks in terms of the required control energy. Fundamental relationships between these measures and network topology were developed via graph Laplacian to characterize energy-related controllability. It is revealed that, for structurally unbalanced signed graphs, the energy-related controllability is closely related to the diagonal entries of the inverse of the graph Laplacian. It is also discovered that structurally balanced signed graphs and their corresponding unsigned graphs have the same energy-related controllability. It shall be noted that a signed graph is called structurally balanced if and only if the node set can be partitioned into two subsets, where nodes within one subset share positive edges and nodes within another subset share negative edges. Otherwise, a signed graph is structurally unbalanced [49].

## 2. CONTROLLABILITY ENSURED LEADER GROUP SELECTION ON SIGNED NETWORKS

Despite extensive study of controllability of unsigned networks, relatively few research effort has focused on signed networks. As presented in this section, we developed the leader selection rules for signed path and cycle networks and provided constructive approaches to select leaders for network controllability. Since some networks can be considered as a combination of path and cycle networks, the developed leader selection rules on path and cycle networks can be potentially extended to more complex and sophisticated networks. In contrast to most existing matrix-theoretical approaches to characterize network controllability, graph-inspired understandings of network controllability were given. Specifically, this section investigates the relationship between the network controllability and the underlying topology, revealing how leader-to-leader and leader-to-follower connections affect the controllability of a signed network with Laplacian dynamics. Such graphical characterizations of network controllability can provide more intuitive and effective means in selecting leaders for network controllability.

### 2.1 Problem Formulation

Consider a networked control system represented by an undirected signed graph as  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  where the node set  $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$  and the edge set  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  represent the network units and the interactions (i.e., communication channels) between pairs of units, respectively. An undirected edge  $(v_i, v_j) \in \mathcal{E}$  indicates that nodes  $v_i$  and  $v_j$  are able to interact with each other (e.g., the mutual information exchange). It shall be noted that a directed edge  $(v_i, v_j)$  means the information can be passed only from node  $v_i$  to node  $v_j$ . In this report, we only consider undirected graphs. The potential interactions among units are captured by the adjacency matrix  $\mathcal{A} = \mathbb{R}^{n \times n}$  where  $a_{ij} \neq 0$  if  $(v_i, v_j) \in \mathcal{E}$  and  $a_{ij} = 0$  otherwise. No self-loop is considered, i.e.,  $a_{ii} = 0 \forall i = 1, \dots, n$ . Different from classical unsigned graphs that contain non-negative, the adjacency matrix,  $a_{ij} \in \{\pm 1\}$  in this work admits positive or negative weights to capture collaborative or competitive relationships between units, thus resulting in a signed graph  $\mathcal{G}$ . Specifically, nodes  $v_i$

and  $v_j$  are called positive neighbors of each other if  $a_{ij} = 1$  and negative neighbors if  $a_{ij} = -1$ , where positive neighborhood indicates cooperative interactions while negative neighborhood indicates non-cooperative (i.e., competitive) interactions, respectively.

A path of length  $k - 1$  in  $\mathcal{G}$  is a concatenation of distinct edges  $\{(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k)\} \subseteq \mathcal{E}$ . A cycle is a path with identical starting and end node, i.e,  $v_1 = v_k$ . Graph  $\mathcal{G}$  is connected if there exists a path between any pair of nodes in  $\mathcal{V}$ . The neighbor set of  $v_i$  is defined as  $\mathcal{N}_i = \{v_j | (v_i, v_j) \in \mathcal{E}\}$ , and the degree of  $v_i$ , denoted as  $d_i \in \mathbb{Z}^+$  (i.e. the set of positive integers) is defined as the number of its neighbors. The degree of node  $v_i$  can be calculated as  $d_i = |\mathcal{N}_i| = \sum_{j \in \mathcal{N}_i} \text{abs}(a_{ij})$ , where  $|\mathcal{N}_i|$  denotes the cardinality of  $\mathcal{N}_i$  and  $\text{abs}(a_{ij})$  denotes the absolute value of  $a_{ij}$ . On the other hand, the signed graph Laplacian of  $\mathcal{G}$  can be defined as  $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D} - \mathcal{A}$ , where  $\mathcal{D} \triangleq \text{diag}\{d_1, \dots, d_n\}$  is a diagonal matrix. Due to the consideration of negative weights, unlike unsigned graphs,  $-\mathcal{L}(\mathcal{G})$  is no longer a Metzler matrix<sup>1</sup> and its row/column sums are not necessarily zero.

Let  $x(t) = [x_1(t), \dots, x_n(t)]^T \in \mathbb{R}^n$  denote the stacked system states<sup>2</sup>, where each entry  $x_i(t) \in \mathbb{R}$  represents the state of node  $v_i$ . Suppose the system states evolve according to the following Laplacian dynamics [50],

**Equation 1.** *Laplacian dynamics*

$$\dot{x}(t) = -\mathcal{L}(\mathcal{G}) x(t), \quad (1)$$

where the graph Laplacian  $\mathcal{L}(\mathcal{G})$  indicates that each node  $v_i$  updates its state  $x_i$  only taking into account the states of its neighboring nodes, i.e.,  $v_j \in \mathcal{N}_i$ .

Suppose the node set  $\mathcal{V}$  is split into a leader set  $\mathcal{V}_l \subset \mathcal{V}$  and a follower set  $\mathcal{V}_f \subset \mathcal{V}$  with  $\mathcal{V}_l \cup \mathcal{V}_f = \mathcal{V}$  and  $\mathcal{V}_l \cap \mathcal{V}_f = \emptyset$ , thus forming a typical leader-follower network. Without loss of generality, assume that the first  $m$  nodes form the follower set  $\mathcal{V}_f = \{v_1, \dots, v_m\}$ , while the remaining nodes form the leader set  $\mathcal{V}_l = \{v_{m+1}, \dots, v_n\}$ . Let  $x(t) = [x_f^T(t), x_l^T(t)]^T \in \mathbb{R}^n$  be the aggregated system states, where  $x_f(t) \in \mathbb{R}^m$  and  $x_l(t) \in \mathbb{R}^{n-m}$  represent the aggregated states of followers and leaders, respectively. Similar to [2], the graph Laplacian in (1) can be partitioned as

**Equation 2.** *Graph Laplacian*

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{L}_f(\mathcal{G}) & \mathcal{L}_{fl}(\mathcal{G}) \\ \mathcal{L}_{lf}(\mathcal{G}) & \mathcal{L}_l(\mathcal{G}) \end{bmatrix} \quad (2)$$

<sup>1</sup> Metzler matrices are matrices with nonnegative off-diagonal entries [63].

<sup>2</sup> If multi-dimensional system states (e.g.,  $x_i \in \mathbb{R}^m$ ) are considered, the Laplacian dynamics in (1) can be trivially extended to  $\dot{x}(t) = -\mathbb{L}(\mathcal{G}) x(t)$ , where  $\mathbb{L}(\mathcal{G})$  is augmented to  $\mathbb{L}(\mathcal{G}) \triangleq \mathcal{L}(\mathcal{G}) \otimes I_m$  by the  $m$ -dimensional identity  $I_m$  and the matrix Kronecker product  $\otimes$ . Without loss of generality, the subsequent development will focus on the case that  $x_i \in \mathbb{R}$  for ease of presentation.

with  $\mathcal{L}_f(\mathcal{G}) \in \mathbb{R}^{m \times m}$ ,  $\mathcal{L}_{fl}(\mathcal{G}) = \mathcal{L}_{lf}^T(\mathcal{G}) \in \mathbb{R}^{m \times (n-m)}$ , and  $\mathcal{L}_l(\mathcal{G}) \in \mathbb{R}^{(n-m) \times (n-m)}$ . Based on (1) and (2), the dynamics of the followers become

**Equation 3.** *Dynamics of the followers*

$$\dot{x}_f(t) = -\mathcal{L}_f(\mathcal{G}) x_f - \mathcal{L}_{fl}(\mathcal{G}) x_l(t), \quad (3)$$

where  $x_l(t)$  can be viewed as the exogenous control signal dictated by the leaders. In leader-follower networks, leaders are tasked to direct the overall behavior of the network by influencing the followers. The dynamics in (3) signify that the followers are influenced or controlled by the leaders via the structure of the network, where the exogenous signal becomes the leader's control input.

## 2.2 Main Results

### 2.2.1 Topological Characterization of Leader-Follower Controllability

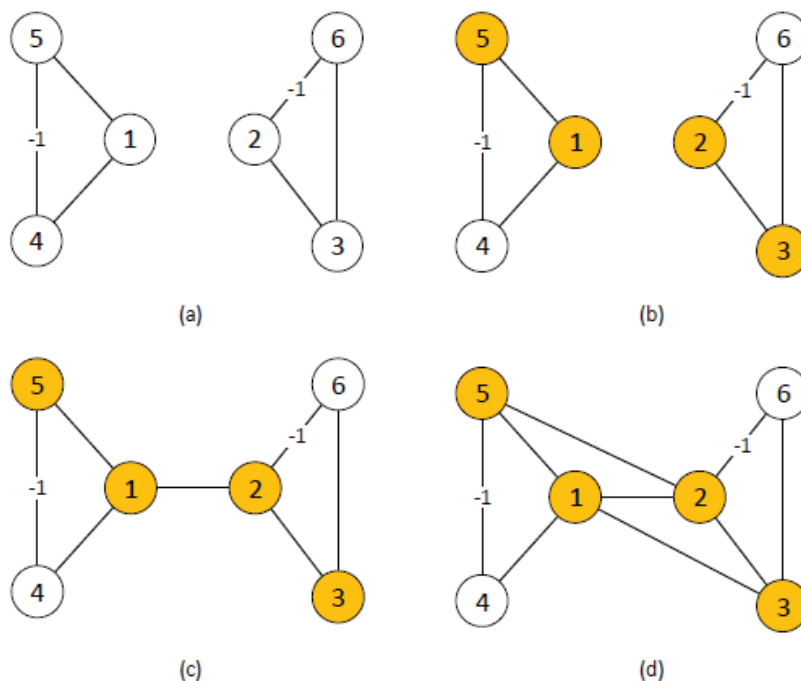
This section of the report shows that how leader-to-leader and leader-to-follower connections affect the controllability of a signed network. Meanwhile, the network controllability is revealed to be invariant under alterations of leader-to-leader connections, which is described by Proposition 1.

**Proposition 1.** [51] *Provided a set of controllable signed graphs  $\mathcal{G}_i, i = \{1, \dots, k\}$ , evolving according to (1),  $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0, \mathcal{A}_0)$  remains controllable if  $\mathcal{G}_0$  is constructed such that 1)  $\mathcal{V}_0 = \bigcup_{i=1}^k \mathcal{V}_i$ , 2)  $\mathcal{E}_0 = \bigcup_{i=1}^k \mathcal{E}_i \cup \mathcal{E}'$  where  $\mathcal{E}' \subseteq \prod_{i=1}^k \mathcal{V}_{li}$ , indicating the additional edges  $\mathcal{E}'$  are restricted to leader-to-leader connections within the leader sets  $\mathcal{V}_{li}$ , and 3)  $\mathcal{A}_0$  indicates the weights associated with the edges in  $\mathcal{E}_0$ .*

The proof of this proposition can be found in [51]. Proposition 1 provides a constructive topological design approach in generating a combined graph that preserves network controllability from a set of controllable sub-graphs. Besides constructing a controllable graph, Proposition 1 also provides insights on leader selection to render network controllability. For instance, if a given graph can be partitioned into a set of connected sub-graphs, as long as the selected leaders ensure controllability for each sub-graph and are connected following the rules in Proposition 1, the given graph is guaranteed to be controllable by Proposition 1. This idea will be further explored in the subsequent sections. Here we use an example to show the application of Proposition 1 in determining network controllability.

**Example 1.** Fig. 1 shows how a controllable leader-follower networked system described by the graph can be constructed from a set of controllable sub-graphs. Fig. 1(a) contains two sub-graphs, which become controllable if the nodes  $\{1,5\}$  and  $\{2,3\}$  are selected as leaders, which generate the exogenous signal to control the followers, respectively, as shown in Fig. 1(b). The combined

graph in Fig. 1(c) is constructed by connecting the two leaders  $\{1,2\}$ . It can be verified that the combined graphs in Fig. 1(c) remains controllable since its construction follows the rules in Proposition 1. It is worth pointing out that only sufficient conditions to preserve network controllability are developed in Proposition 1. There might exist different ways of connecting leaders to preserve network controllability. As a different construction, the combined graph in Fig. 1(d) is constructed by including three new edges (i.e.,  $(1, 2)$ ,  $(1, 3)$ , and  $(5, 2)$ ), which is also controllable according to Proposition 1.



**Figure 1.** Constructing a controllable graph from controllable sub-graphs

### 2.2.2 Leader Selection for Signed Path and Cycle Graphs

Based on the developed topological characterizations of the controllability of signed networks in subsection 2.2.1, this subsection presents the theorems, developed and proved in our published work [51], which provides the sufficient conditions on selecting leader nodes for the controllability of signed networks. Specifically, conditions on the signed path, cycle, tree graphs are given by the following theorems. To illustrate the idea of Theorem 3, Example 2 is provided.

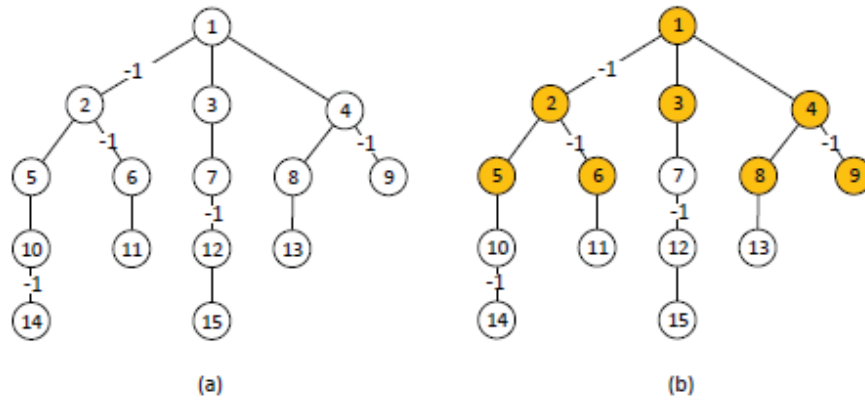
**Theorem 1.** A signed path graph  $\mathcal{G}_p = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  with followers evolving according to (3) is controllable if one of the end nodes (i.e.,  $v_1$  or  $v_n$ ) is selected as a leader.

**Theorem 2.** A signed path graph  $\mathcal{G}_p$  with followers evolving according to (3) is controllable if multiple adjacent nodes in  $\mathcal{G}_p$  are selected as leaders.

**Theorem 3.** Suppose that a signed tree  $\mathcal{G}_t$  can be partitioned into a set of signed path graphs  $\{\mathcal{G}_{pi}\}, i \in \{1, \dots, m\}$  with  $\mathcal{G}_t = \cup_{i=1}^m \mathcal{G}_{pi}$ . The tree  $\mathcal{G}_t$  is controllable, if selected leaders ensure the controllability of each path graph in  $\{\mathcal{G}_{pi}\}$  and  $\mathcal{G}_t$  is reconstructed by only connecting leaders from each path graph.

**Theorem 4.** Provided that all followers evolve according to (3), a signed cycle graph  $\mathcal{G}_c$  is controllable if any two adjacent nodes in  $\mathcal{G}_c$  are selected as leaders.

**Example 2.** Consider a signed tree shown in Fig. 2 (a), where the roles (i.e., leaders or followers) are not assigned yet. To selected leaders that ensure the network controllability, the tree can be first partitioned into 5 paths, i.e.,  $\{5, 10, 14\}$ ,  $\{6, 11\}$ ,  $\{3, 7, 12, 15\}$ ,  $\{8, 13\}$ , and  $\{2, 1, 4, 9\}$ . If the leaders are selected as  $\{1, 2, 3, 4, 5, 6, 8, 9\}$ , then each of the path graphs is controllable by Theorem 1. Since the path graphs are connected in a way that only leaders from each path are connected to form the tree, the tree is controllable based on Theorem 2 and Proposition 1. Therefore, it can be verified that the selected leaders ensure the leader-follower controllability of the tree graph.



**Figure 2.** A signed tree graph

### 2.2.3 Leader Selection on General Signed Graphs

This subsection shows how the leader selection rules developed for signed path and cycle graphs can be potentially extended to more general signed networks. Since path and cycle graphs are basic building blocks for general signed graphs, motivated by this idea, the heuristic leader selection algorithm was developed. As illustrated in Table 1, we start the leader selection procedure by identifying nodes in a given signed graph whose node degree is greater than two. Since nodes in either path and cycle graphs have a degree at most two, the reason to identify nodes with a degree more than two is to find out those nodes that potentially connect path or cycle graphs. Those high

degree nodes will facilitate the partition of the signed graph into a variety of path and cycle graphs, where existing graph partition techniques are applicable (e.g., graph partition into paths in [52], cycles in [53], and both paths and cycles in [54]). Once the graph is partitioned into a set of path and cycle graphs, the leader selection rules developed in Theorems 1-4 and Proposition 1 can be immediately applied to generate a controllable leader-follower network. To illustrate the leader selection algorithm, Example 3 is provided.

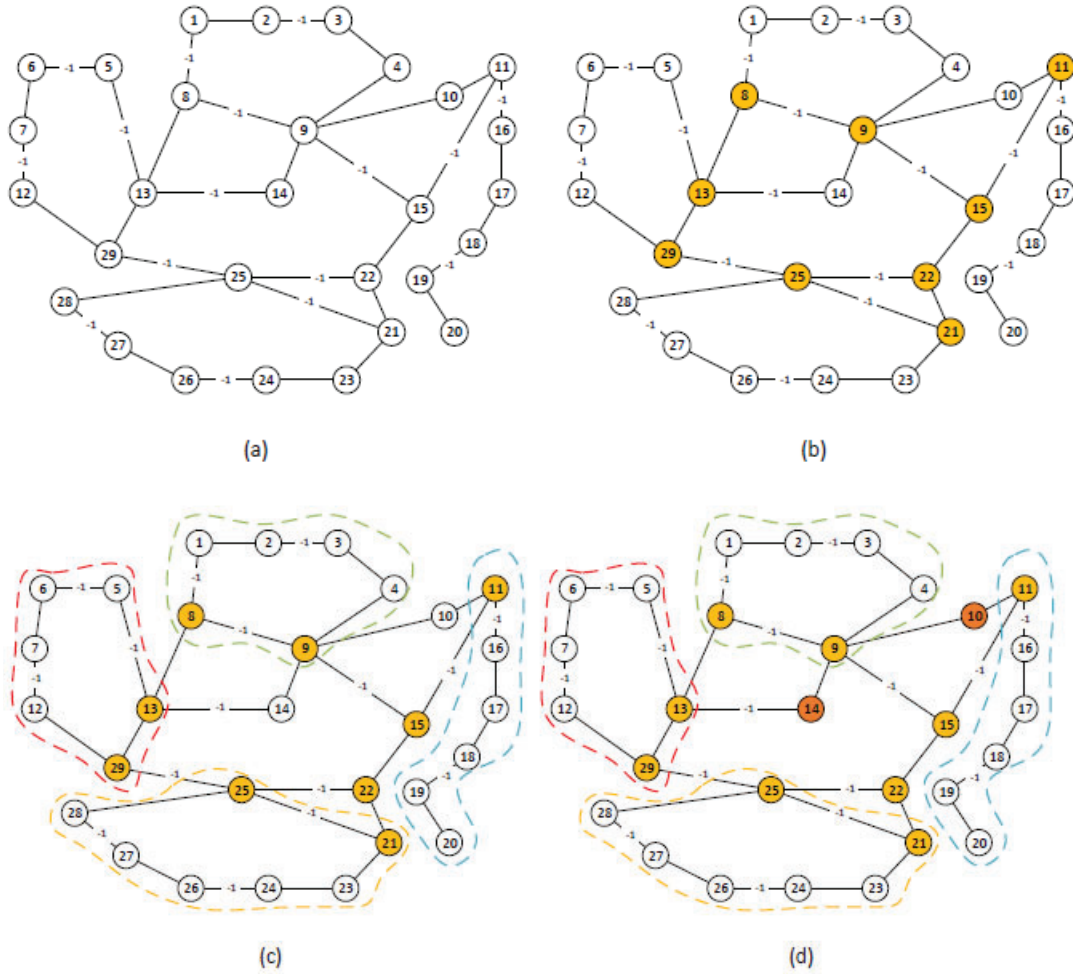
**Example 3.** Consider the signed graph in Fig. 3(a) where the objective is to select a set of leaders such that the leader-follower network is controllable. Following the selection procedure in Table 1, the leader nodes (i.e., high degree nodes  $\{8, 9, 11, 13, 15, 21, 22, 25, 29\}$ ) are first identified in Fig. 3(b). Based on the selected leader nodes and graph partition techniques in [52–54], the signed graph is partitioned into a set of path and cycle graphs, which are shown in dashed lines in Fig. 3(c). The leader selection rules developed in Theorem 1-4 are then applied to ensure that each path and cycle graph are controllable. For instance, the selected nodes  $\{8, 9\}$  ensure the controllability of the cycle graph formed by  $\{1, 2, 3, 4, 8, 9\}$ . The leader set is then updated to include more nodes (i.e., addition leaders  $\{10, 14\}$  in Fig. 3(d)) whenever necessary such that the individual path and cycle graphs are connected satisfying Proposition 1, which ensures the controllability of the original signed graph.

**Table 1.** Leader Selection Algorithm for Signed Graph

---

1:	<b>Procedure</b> INPUT: (Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ ); OUTPUT: The set of leaders $\mathcal{V}_l$
2:	Calculate the node degree $d_i$ for each node $v_i \in \mathcal{V}$ ;
3:	Select nodes $v_i$ with $d_i > 2$ to form $\mathcal{V}_l$ ;
4:	Use graph partition techniques (e.g., [38–40]) to partition $\mathcal{G}$ into cycles and paths based on the selected high degree nodes in $\mathcal{V}_l$ ;
5:	<b>for</b> each cycle or path <b>do</b>
6:	<b>if</b> the cycle or path is controllable <b>then</b>
7:	Keep the selected leaders in $\mathcal{V}_l$ ;
8:	<b>else</b>
9:	Apply Theorems 1-4 to select appropriate nodes as leaders and update $\mathcal{V}_l$ ;
10:	<b>end if</b>
11:	<b>end for</b>
12:	Update $\mathcal{V}_l$ based on Proposition 1 to ensure the network controllability;
13:	Output $\mathcal{V}_l$ ;
14:	<b>end procedure</b>

---



*Figure 3. A signed graph with 29 nodes*

### 2.3 Summary

Leader selection on signed networks for ensured-controllability was presented in this section. The mathematical proofs of this section can be found in [51]. We developed graph-inspired topological characterizations of the controllability of signed networks based on the leader selection methods for signed path and cycle graphs. Heuristic algorithms were also developed showing how leader selection methods developed for path and cycle graphs can be potentially extended to more general signed networks. Although the effectiveness of the developed leader selection rules was demonstrated via examples, there might exist different leader sets that are capable of ensuring network controllability with additional constraints (e.g., minimal leader number). Future research will consider extending the results in this work taking into account additional constraints.

### 3. ENERGY-RELATED CONTROLLABILITY OF SIGNED NETWORKS

This section investigates the energy-related controllability of complex networks. Specifically, the objective is to establish controllability characteristics on signed complex networks, where the network units interact via linear consensus dynamics [49] and the network admits positive and negative edges to capture cooperative and antagonistic interactions among these units. The network units can be classified into leaders and followers. To this end, controllability Gramian-based measures [16,17] were exploited to quantify the difficulty of the control problem on signed networks in terms of the required control energy. Fundamental relationships between these measures and network topology were developed via graph Laplacian to characterize the energy-related controllability. It is revealed that, for structurally unbalanced signed graphs, the energy-related controllability is closely related to the diagonal entries of the inverse of the graph Laplacian matrix [49]. It is also discovered that structurally balanced signed graphs and their corresponding unsigned graphs have the same energy-related controllability.

#### 3.1 Problem Formulation

Here considers a networked control system represented by an undirected signed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  as defined in Section 2.1. The dynamics of the networked system described in this section is the same as equation (1). However, the external control inputs are applied to some key nodes (i.e., leaders) to drive the network to the desired state. Without losing generality, it is assumed that the first  $k$  nodes form a leader set as  $\mathcal{V}_l = \{v_1, \dots, v_k\} \subseteq \mathcal{V}$  where  $k \geq 1$ . The leaders can be endowed with external controls, and the system dynamics in (1) can be rewritten as

*Equation 4. System dynamics with external inputs*

$$\dot{x}(t) = -\mathcal{L}(g)x(t) + \mathcal{B}_{\mathcal{K}}u(t) \quad (4)$$

where  $\mathcal{B}_{\mathcal{K}} = [e_1, \dots, e_k] \in \mathbb{R}^{n \times k}$  is the input matrix with basis vectors  $e_i, i = 1, \dots, k$ , indicating that the  $i$ th node (one of the leaders) is endowed with external controls  $u(t) \in \mathbb{R}^k$ .

The dynamics of (4) indicates that the network behavior not only is driven by the graph Laplacian  $\mathcal{L}$ , but also depends on the input matrix  $\mathcal{B}_{\mathcal{K}}$  via the leader set  $\mathcal{V}_l$ . Different leader sets can result in different  $\mathcal{B}_{\mathcal{K}}$ , leading to drastic differences in the capability of controlling a network, which is elucidated by introducing the definition of leader-follower controllability. Compared to network dynamics in (3), the discussion in [55] indicates the two different models, (3) and (4), share the same controllability if they have the same group of leaders. In other words, if the selected leaders yield a controllable leader-follower network with dynamics (3), the controllability result holds for the same set of leaders on a network with dynamics (4).

A network with dynamics in (4) is controllable if the controllability matrix  $\mathcal{C}_{\mathcal{K}} = [\mathcal{B}_{\mathcal{K}} \quad -\mathcal{L}\mathcal{B}_{\mathcal{K}} \quad \dots \quad (-1)^{n-1}\mathcal{L}^{n-1}\mathcal{B}_{\mathcal{K}}]$  has a full row rank [11]. Hence, in theory, a network can be controllable with an appropriate selection of leader nodes (and consequently  $\mathcal{B}_{\mathcal{K}}$ ). However, it does not tell how

difficult it is to control the network in practice, i.e., how much energy is needed to drive the network to the target state. To provide energy-related quantification of network control, the total control energy over the time interval  $[0, t]$  is given by

*Equation 5. The total control energy*

$$E(t) = \int_0^t \|u(\tau)\|_2^2 d\tau, \quad (5)$$

where  $\|u(\tau)\|_2$  represents the Euclidean norm of  $u$ . Assuming the initial state  $x(0) = 0$  and the optimal control  $u(t)$  in [56], the minimum control energy [7,16,17] required to drive the system in (4) from  $x(0)$  to a desired target state  $x_g$  is [7]

*Equation 6. The minimum control energy*

$$E(t) = x_g^T \mathcal{W}_{\mathcal{K}}^{-1}(t) x_g \quad (6)$$

where

*Equation 7. The controllability Gramian*

$$\mathcal{W}_{\mathcal{K}} = \int_0^t e^{-\mathcal{L}\tau} \mathcal{B}_{\mathcal{K}} \mathcal{B}_{\mathcal{K}}^T e^{-\mathcal{L}^T\tau} d\tau \quad (7)$$

is the controllability Gramian at time  $t$ , which is positive definite if and only if the system in (4) is leader-follower controllable. In this work, we focused on the infinite-horizon Gramian, i.e., the case when  $t \rightarrow \infty$  in (7), due to asymptotic or exponential convergence/stability of most dynamical systems.

Since the controllability Gramian  $\mathcal{W}_{\mathcal{K}}$  provides an energy-related measure of network controllability, various quantitative metrics of controllability were developed based on  $\mathcal{W}_{\mathcal{K}}$ . As discussed in [17], the trace  $tr(\mathcal{W}_{\mathcal{K}})$  provides an overall measure of network controllability in all directions. The average control energy required to move the system in (4) to a target state is obtained as the trace  $tr(\mathcal{W}_{\mathcal{K}}^{-1})$ . The volumetric control energy, given by  $\log(\det \mathcal{W}_{\mathcal{K}})$ , measures the volume of the ellipsoid containing the target states that can be reached with a unit control input. The objective of this work is to establish these metrics in the context of signed networks and develop topological characterizations on how these metrics are related to the graph Laplacian in quantifying the energy-related performance in the control of signed networks.

### 3.2 Energy-Related Controllability of Signed Networks

Based on the topological structures, signed graphs can be classified as either structurally balanced or structurally unbalanced. To develop controllability results on signed graphs, the subsequent sections give results developed on structurally unbalanced and balanced networks.

### 3.2.1 Structurally Unbalanced Signed Graphs

Controllability Gramian based metrics are widely used to characterize the energy required in network control. However, the eigen-properties of the Gramian are typically challenging to characterize analytically. In addition, topological characterizations of the energy-related controllability are hard to be extracted from the Gramian matrix. To overcome this issue, inspired by the Gramian based nodal centrality, a notion termed nodal metric [57] is defined to quantify how much each node contributes to the energy-related controllability of structurally unbalanced signed graphs, which is defined based on the signed graph Laplacian.

**Definition 1** (Nodal Metric). Consider a structurally unbalanced signed network  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  with graph Laplacian  $\mathcal{L}(\mathcal{G})$ . Let  $\mathcal{M}(\cdot): \mathcal{V} \rightarrow \mathbb{R}$  denotes a metric associated with node  $v_i$ , which is defined as the  $i$ th diagonal entry of the inverse of the graph Laplacian  $\mathcal{L}(\mathcal{G})$ , i.e.,  $\mathcal{M}(v_i) = \mathcal{L}_{ii}^{-1}(\mathcal{G})$ .

Based on the nodal metric defined in Definition 1, the following theorems, developed through our work [57], determine how various energy-related controllability measures, including the average controllability (Theorem 5), the average control energy (Theorem 6), and the volumetric control energy (Theorem 7), are related to the total nodal metric of the control nodes (i.e., leaders) via graph Laplacian.

**Theorem 5** (Average Controllability). Consider an undirected signed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  evolving according to the dynamics in (4) with the leader set  $\mathcal{K}$ . If  $\mathcal{G}$  is structurally unbalanced, the average controllability  $\text{tr}(\mathcal{W}_{\mathcal{K}})$  with the controllability Gramian  $\mathcal{W}_{\mathcal{K}}$  defined in (7) can be characterized by the sum of the total  $\mathcal{M}$  of the leaders in  $\mathcal{K}$  as

$$\text{tr}(\mathcal{W}_{\mathcal{K}}) = \frac{1}{2} \sum_{i \in \mathcal{V}_l} \mathcal{M}(v_i).$$

**Theorem 6** (Average Control Energy). Consider an undirected signed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  evolving according to the dynamics in (4). If  $\mathcal{G}$  is structurally unbalanced and the system (4) under the leader group  $\mathcal{V}_l$  is stable, the average control energy metric  $\text{tr}(\mathcal{W}_{\mathcal{K}}^{-1})$  can be lower bounded by the total  $\mathcal{M}$  of the leaders in  $\mathcal{V}_l$  as

$$\text{tr}(\mathcal{W}_{\mathcal{K}}^{-1}) \geq \frac{2n^2}{\sum_{i \in \mathcal{V}_l} \mathcal{M}(v_i)}$$

**Theorem 7** (Volumetric Control Energy). Consider an undirected signed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  evolving according to the dynamics in (4). If  $\mathcal{G}$  is structurally unbalanced, the volumetric control energy metric  $\log(\det \mathcal{W}_{\mathcal{K}})$  can be upper bounded by the total  $\mathcal{M}$  of the leaders in  $\mathcal{V}_l$  as

$$\log(\det \mathcal{W}_{\mathcal{K}}) \leq \sum_{i \in \mathcal{V}_l} n \log \left( \frac{\mathcal{M}(v_i)}{n} \right) + c'_{\psi},$$

where  $c'_{\psi} \in \mathbb{R}^+$  is a constant determined by the eigenvalues of the graph Laplacian.

Theorems 5-7 not only show how various energy measures are related to the nodal metric of the leader group in the network, but also reveal that these measures are closely related to the inverse

of the graph Laplacian. As such, these theorems offer a new perspective to design networks from energy considerations. For different controllability metrics, including the average controllability, the average control energy, and the volumetric control energy, it is shown that the respective metrics can be improved if the network topology is designed such that the diagonal entries of the inverse graph Laplacian corresponding to the leaders are maximized. In addition, as discussed in the works of [46,58,59], the inverse of the graph Laplacian can be interpreted as the graph resistance, which plays an important role in distributed network control and estimation. For instance, the graph resistance appears in some network control problems in which nodes are steered towards the desired formation [58], and also appears in least-squares estimation problems in which global information can be reconstructed from noisy measurements. In the recent work [46], the graph resistance was also used to quantify the information centrality. Based on Theorems 5-7, additional research will consider exploring how controllability metrics are related to the leader groups via graph resistance.

### 3.2.2 Structurally Balanced Signed Graphs

This subsection considers the case that the signed graph  $\mathcal{G}$  is structurally balanced. For a balanced graph, we find out that the controllability and the Gramian-based control energy metrics are equivalent to its corresponding unsigned case, i.e., the same network structure with edge weights being replaced by their absolute values [57]. The results are captured by the following Theorem and Corollaries.

**Theorem 8.** *Consider a signed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  and its corresponding unsigned graph  $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}})$  with their controllability Gramians  $\mathcal{W}_{\mathcal{K}}$  and  $\bar{\mathcal{W}}_{\mathcal{K}}$ , respectively. Let the nodes in  $\mathcal{G}$  and  $\bar{\mathcal{G}}$  evolve according to the dynamics in (4) but with their respective adjacency matrices  $\mathcal{A}$  and  $\bar{\mathcal{A}}$ . If  $\mathcal{G}$  is structurally balanced, then  $\mathcal{W}_{\mathcal{K}}$  has the same matrix spectrum as  $\bar{\mathcal{W}}_{\mathcal{K}}$ .*

**Corollary 1.** *Consider a signed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  and its unsigned correspondence graph  $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}})$ . If  $\mathcal{G}$  is structurally balanced, then the leader-follower controllability of  $\mathcal{G}$  is equivalent to that of  $\bar{\mathcal{G}}$  under the same leader set.*

**Corollary 2.** *If  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  is a signed acyclic graph, then there always exists an unsigned correspondence  $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}})$ , such that the controllability Gramian  $\mathcal{W}_{\mathcal{K}}$  of  $\mathcal{G}$  has the same matrix spectrum as that of  $\bar{\mathcal{W}}_{\mathcal{K}}$  of  $\bar{\mathcal{G}}$ .*

The energy-related controllability metrics analyzed in Section 3.2.1 (i.e., average controllability  $tr(\bar{\mathcal{W}}_{\mathcal{K}})$ , average control energy  $tr(\bar{\mathcal{W}}_{\mathcal{K}}^{-1})$ , and volumetric control energy  $\log(\det \bar{\mathcal{W}}_{\mathcal{K}})$ , along with other metrics, including the worst-case controllability  $\lambda_{min}(\bar{\mathcal{W}}_{\mathcal{K}})$  and the dimension of the controllable subspace  $rank(\bar{\mathcal{W}}_{\mathcal{K}})$ , can be directly obtained from the eigenvalues of  $\bar{\mathcal{W}}_{\mathcal{K}}$ . Since  $\mathcal{W}_{\mathcal{K}}$  and  $\bar{\mathcal{W}}_{\mathcal{K}}$  have the same set of eigenvalues, Theorem 8 indicates that the structurally balanced signed graph  $\mathcal{G}$  and its corresponding unsigned graph  $\bar{\mathcal{G}}$  are equivalent in terms of the above controllability metrics. Therefore, Theorem 8 provides a means to investigate the energy-related

controllability of signed networks by examining its corresponding unsigned graph  $\bar{\mathcal{G}}$  using the analysis and design methods developed for unsigned graphs. Further, it is well known that a system is controllable if and only if its controllability Gramian is positive definite [16,17]. Thus, Corollary 1 follows immediately from Theorem 8 by the fact that  $\mathcal{W}_{\mathcal{K}}$  and  $\bar{\mathcal{W}}_{\mathcal{K}}$  share the same set of eigenvalues when  $\mathcal{G}$  is structurally balanced. Since acyclic graphs, e.g., tree or path graphs, are inherently structurally balanced [49], Corollary 2 is an immediate result of Theorem 8 as well. Corollary 2 states that, instead of investigating the signed acyclic graph  $\mathcal{G}$ , the unsigned correspondence  $\bar{\mathcal{G}}$  of  $\mathcal{G}$  can be explored to enable leader group selection using the energy-related controllability metrics based on  $\mathcal{W}_{\mathcal{K}}$ .

The problem of leader group selection on structurally balanced signed graphs has been partially studied in [60], which requires the leaders to be selected from the same partitioned set to ensure network controllability. Corollary 1 relaxes this constraint by allowing the leaders to be selected from different partitioned sets, as long as the corresponding unsigned graph is controllable under the selected leader group. Additionally, the discovered equivalence of controllability between  $\mathcal{G}$  and  $\bar{\mathcal{G}}$  enables the use of existing leader group selection methods developed for unsigned graphs [19,41,61].

### 3.3 Summary

Energy-related controllability was investigated through the structure of signed Laplacian matrices on general signed weighted networks in this section. A nodal metric capturing the energy-related performance of a networked control system was defined. The average controllability, the average control energy, and the volumetric control energy were discovered to be related to the defined nodal metric. Therefore, the performance of networked control systems depends on the contribution of individual leaders. Further, the structural balance was found to play an important role in network controllability, that controllability and energy-related controllability of a signed network were equivalent to the corresponding unsigned cases if the network was structurally balanced. The developed theorems paved a way for creating a new leader selection algorithm to ensure better energy-related controllability of signed networks.

## 4. CONCLUSIONS

Controllability ensures a leader-follower networked control system to enjoy the property so that the system can be driven into any desired state through designed control input. However, a lot of controllable networked systems could potentially be unrealizable in practice. This comes from the difference between traditional control systems and large-scaled networked control systems in terms of energy-related performance. Consequently, energy-related controllability provides a networked control system with the capability to be controllable and to have a better system performance in terms of energy-related metrics. This report tries to bridge the gap between controllability/energy-related controllability and network topology. Topological characterizations of controllability and energy-related controllability of networked control systems were investigated in this report. Particularly, a leader-follower framework was adopted on networked

control systems. Controllability and energy-related controllability were explored through various graph characteristics. The specific contributions of each section are summarized as follows.

In Section 2, the controllability of networked control systems with Laplacian dynamics was considered. Leader selection to ensure the controllability of signed networks with unit edge weights was investigated through partitioning the network into several signed path and cycle networks. Based on the developed theorems and propositions, the network controllability was invariant under alterations of leader-to-leader connections. Sufficient conditions to ensure signed path and cycle networks to be controllable were developed via graph topology to facilitate the development of the leader selection algorithm on general signed graphs. The mathematical proofs of the theorems and propositions discussed in Section 2 can be found in [51].

Although the effectiveness of the developed leader selection rules [51] was demonstrated via examples, there might exist different leader sets that are capable of ensuring network controllability with additional constraints (e.g., minimal leader number). Future research will be conducted by extending the results in this work with the consideration of additional constraints. For instance, minimal controllability problems were considered in [31], where a greedy heuristic approach was developed to ensure network controllability while minimizing the number of selected leaders. Robust minimal controllability was investigated in [62], where additional constraints were included in the minimal controllability problem.

In Section 3, the presented work focused on characterizing the energy-related controllability in signed networks [57]. To this end, the controllability Gramian-based measures were exploited to quantify the difficulty of the control problem on signed networks in terms of the energy-related performance. Fundamental relationships between these measures and network topology were developed via graph Laplacian to characterize energy-related controllability. It was revealed that, for structurally unbalanced signed graphs, the energy-related controllability was closely related to the diagonal entries of the inverse of the graph Laplacian. It was also discovered that structurally balanced signed graphs and their corresponding unsigned graphs had the same energy-related controllability. Although graph Laplacian has direct implications on the network topological properties, the potential connections between the energy-related controllability and its network topology have not been fully explored in this work. Future research will explore from topological perspectives (e.g., the graph resistances based on the inverse graph Laplacian) to facilitate the selection of optimal leader groups for energy-efficient network control.

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