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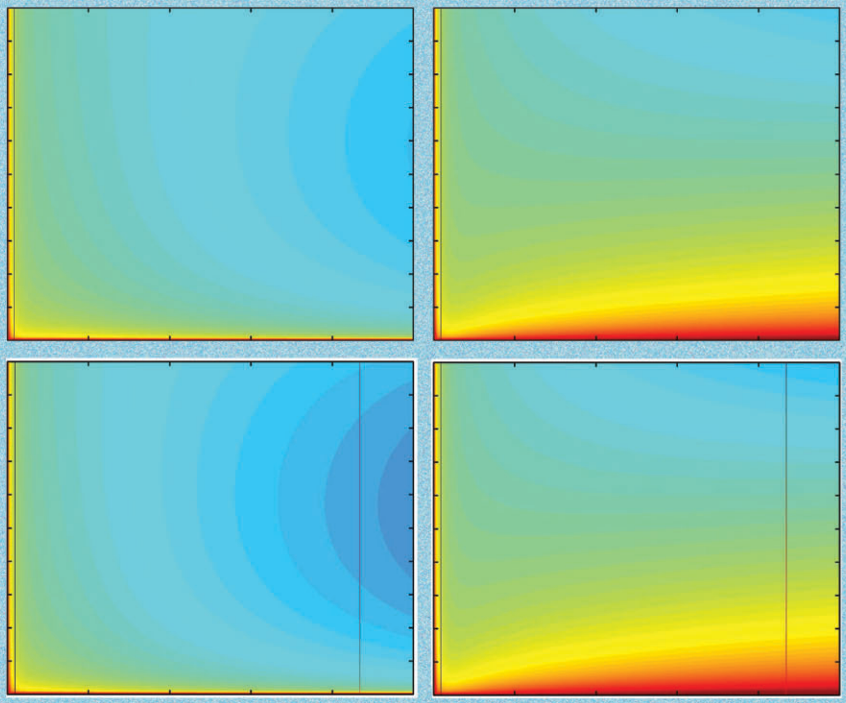
Signal and reverberation formulae including refraction



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NORTH ATLANTIC TREATY ORGANIZATION   ORGANISATION DU TRAITE DE L'ATLANTIQUE NORD

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Signal and Reverberation Formulae Including Refraction

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Dr Steven E. Ramberg
Director

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Signal and Reverberation Formulae including Refraction

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Executive Summary: An analytical solution, or a mathematical closed-form solution, is often worth more than a numerical sensitivity analysis because straight away, by inspection, it shows how a phenomenon behaves under many circumstances. The insight that it brings to such phenomena as range-dependent propagation and reverberation means that designers of sonars, military operational analysts and planners can cover more complicated scenarios, still retaining realistic trends and the basic physics. In addition closed-form solutions are valuable as benchmarks against which to test more computer-intensive methods.

Because reverberation is an incoherent process it is possible to base realistic solutions on very simple flux theory. This report extends the closed-form signal and reverberation formulae of an earlier report [SACLANTCEN SR-356] to include upward or downward refraction with potentially different scattering properties and boundary losses at each boundary. The inclusion of refraction is important for the credibility of these formulae. This report covers two-way propagation, target echoes and reverberation in a range-independent environment, giving explicit closed-form solutions for signal-to-background.

A by-product of these solutions is that they take negligible time to evaluate, so they can be incorporated in simple desk top programs. The approach is, of course, complementary to the standard computer-intensive techniques of under water acoustics.

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Signal and Reverberation Formulae including Refraction

C H Harrison

Abstract: Because reverberation is an incoherent process it is possible to base realistic solutions on very simple flux theory. This report extends the closed-form signal and reverberation formulae of an earlier report [SACLANTCEN SR-356] to include upward or downward refraction with potentially different scattering properties and boundary losses at each boundary. The inclusion of refraction is important for the credibility of these formulae although the additional effects are considered to be minor. This report covers two-way propagation, target echoes and reverberation in a range-independent environment, giving explicit closed-form solutions for signal-to-background. Typically these formulae can be evaluated in one line of computer code, and example plots are given. The solutions are given in two forms, one incorporating true focusing effects, the other simplified by merely adding one spherical spreading term for each eigenray. In all circumstances the simple solution performs very well, and in most it obviates the need for consideration of focusing.

Keywords: Reverberation, propagation, signal excess, refraction, analytical models, closed-form models.

Contents

1 Introduction

An earlier report [1] used simple flux concepts combined with Lambert's law to derive reverberation and signal excess in an isovelocity range-dependent environment. Subsequently this was extended to the bistatic case [2]. Here we concentrate on the effects of refraction on monostatic reverberation and target echo in a range-independent environment. Range-dependent effects are dealt with in a separate report [3].

Refraction has two effects: (i) selection and deselection of eigenrays, (ii) caustics and focusing. Although the effect of caustics in a refracting environment is well known (see, for instance [4]) there is considerable dependence on source and receiver depth through the simple selection of eigenrays alone. A moment's thought shows that a near surface source and receiver in a downward refracting environment have far fewer eigenrays than a bottom source and receiver. In truth there is only one flux mechanism but the two effects can be handled differently numerically.

Here we take account of reflection and scattering at both boundaries but restrict treatment to a linear profile. We present two solutions, one with, and one without focusing, for three reasons: (a) one solution is much simpler than the other, (b) for most source receiver positions the solutions are very close, (c) focusing effects are usually smudged by diffraction and therefore only properly seen at very high frequencies. The simple solution is based on eigenray addition as in [1]. The more detailed solution starts with an incoherent normal mode sum and uses WKB mode shapes.

Explicit formulae are given for signal-to-reverberation-ratio, and various new regimes are explored and compared with the isovelocity case. Dependence on normalised source depth and range is shown graphically

Note that in all the figures the quantities plotted are normalised received intensity in dBs. Thus the target or "propagation" curves are shown as a negative quantity. This is source level minus two-way propagation plus target strength, with source level and target strength set to zero. Similarly "reverberation" curves are shown as a negative quantity. This is source level minus two-way propagation plus scattering strength plus 10log(area), with source level and scattering strength set to zero. The linear area term ($r\Phi ct_p/2$) includes only the range dependence, and the product of the other terms (beam width, sound speed in water, half-pulse-length) is effectively set to unity. To turn these levels into received reverberation in the same band as the source transmission, add source level, scattering strength and 10×log($\Phi ct_p/2$) with ct_p in metres. In the earlier report [1] the plotted quantities are the same except for the sign. It adopted the increasing-loss-is-positive-dBs convention appropriate for propagation loss.

Note that the one-way propagation loss includes four eigenrays for each horizontal wavenumber (a full set of up- and down-going paths at source and receiver), but in the case of reverberation, with a notional receiver on the seabed this reduces to two. This reduction has been allowed for in this report. Reference [1] deliberately excluded this factor but the explanatory note was accidentally omitted. For compatibility divide the reverberation results in [1] by 4.

2
Refraction excluding focusing

2.1 Theory – eigenray sum

We start by taking a sum of eigenrays as in [1] in which we assume each eigenray still contributes an intensity $1/r^2$ despite its slight curvature.

$$I_P = \frac{4}{r^2}\sum_n \exp(-a_n r) = \frac{4}{r^2}\int \exp(-a_n r)\,dn \tag{2.1}$$

Instead of converting this to an angle integral we introduce the variable u which is the reciprocal of the cycle distance, $u = 1/r_c$. Now $n = r/r_c = r\,u$ so we can write Eq. (2.1) in a very simple form as

$$I_P = (4/r)\int \exp(-a_n r)\,du \tag{2.2}$$

The exponent is a function of cycle distance since it is loss per bounce $\alpha\theta$ (in the notation of [1]) times number of boundary interactions. If we were to write $u = \tan\theta/2H$ we would obtain exactly the same integrals as in [1]. In a refracting medium the cycle distance now depends not only on the sound speed gradient but also on whether there is refraction or reflection at the high sound speed side. In Section 2.2 we consider both cases and calculate propagation loss. Then in Section 2.3 we calculate reverberation in the corresponding two cases.

This report is applicable to upward and downward refraction with potentially different scattering parameters at each boundary. To reduce the number of permutations and simplify notation we use subscripts L, H (for all parameters such as α, μ, θ) to indicate the boundary with respectively low and high sound speed. The seabed and bottom can be either one. In this way we can cope concisely with scattering and losses at both boundaries.

2.2 One-way propagation

2.2.1 Propagation: rays interacting with only one boundary

We assume a linear relation between sound speed and depth so each ray is an arc of a circle as in Fig. 1.

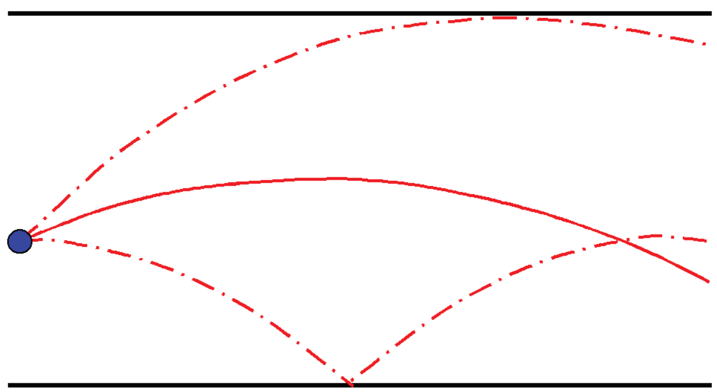

Figure 1 Diagram of downward refracting rays from the source

The ray curvature is given by

$$\rho = c_L / (c' \cos\theta_L) \tag{2.3}$$

Note that to be consistent with the L, H notation, c' is always positive. The cycle distance is given by

$$r_c = u^{-1} = 2\rho \sin\theta_L = (2c_L / c') \tan\theta_L \tag{2.4}$$

The boundary loss in nepers is

$$a_n r = \alpha_L \theta_L r u = \alpha_L \theta_L r c' / (2c_L \tan\theta_L) \cong \alpha_L r c' / (2c_L) \tag{2.5}$$

(note $\alpha = \alpha_{dB}/(10 \log(e))$; α is in nepers/radian whereas α_{dB} is in dB/radian)

so Eq. (2.2) becomes

$$I_{P1} = (4/r)\int_{u_1}^{u_0} du \exp\{-(\alpha_L c'/2c_L)r\} = (4/r)(u_0 - u_1)\exp\{-(\alpha_L c'/2c_L)r\} \qquad (2.6)$$

The integral limits u_1 and u_0 correspond[1] respectively to the longest and shortest cycle distances, determined by the ray that just grazes the high speed boundary and the ray whose phase speed (sound speed at turning point) is the maximum of the speeds at source and receiver. Thus

$$u_1 = \frac{1}{2}\sqrt{\frac{c'}{2H\,c_H}} \qquad (2.7)$$

$$u_0 = \frac{1}{2}\sqrt{\frac{c'}{2h_{sr}\,c_{sr}}} \qquad (2.8)$$

$$c_{sr} = MAX(c_s, c_r); \quad h_{sr} = MAX(h_s, h_r) \qquad (2.9)$$

where subscripts s, r refer to source and receiver, H is the water depth, and h_{sr} is the distance of the source-receiver speed maximum c_{sr} from the lower speed boundary.

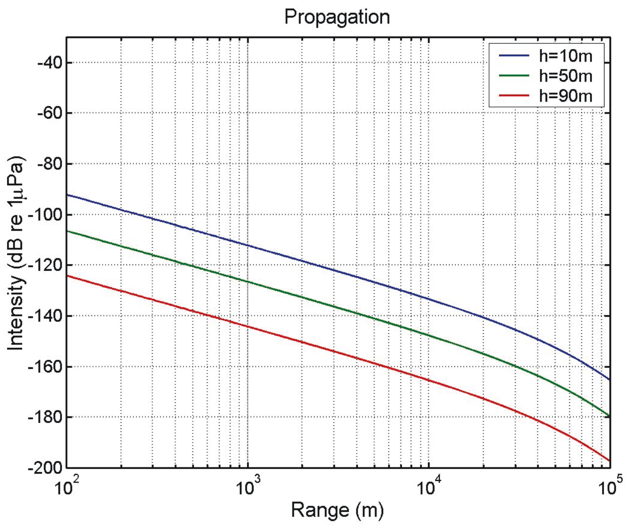

Figure 2 Two-way propagation contributions from rays interacting with only one boundary (I_{P1}^2) for h_{sr}=10,50,90m.

[1] The unusual choice of subscript ordering is chosen to agree with the natural angle ordering (low subscript; low angle). In this 1-boundary interaction regime u decreases with angle, but in the 2-boundary interaction regime u increases with angle. Thus the angle range θ_0 to θ_1 in the θ integral correspond to u_1 to u_0 in the u integral. Similarly θ_1 to θ_2 corresponds to u_1 to u_2.

2.2.2 Propagation: rays with interactions at two boundaries

The ray angles at the upper and lower boundaries can be written in terms of the angle to the straight line connecting the same reflecting points (Fig. 3)

$$\theta_L = \theta_o + \delta \tag{2.10}$$

$$\theta_H = \theta_o - \delta \tag{2.11}$$

where

$$r_c = 1/u = 2H \cot\theta_o ; \qquad \tan\theta_o = 2Hu \tag{2.12}$$

Since the ray is an arc of a circle the angle δ is given by

$$2\rho \sin\delta = H / \sin\theta_o \tag{2.13}$$

where again

$$\rho = c_L /(c' \cos\theta_L) \tag{2.14}$$

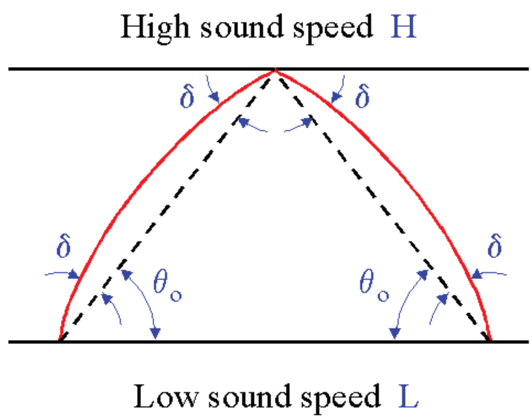

Figure 3 Diagram showing refracting ray (red) hitting both boundaries and the angles θ_o and δ. The ray's grazing angle at the high speed side (shown at top) is θ_H and at the low speed side (shown at bottom) is θ_L.

Substituting Eq. (2.10) into Eq. (2.13) and expanding $\sin(\theta_L - \theta_o)$ we find the exact relation

$$\tan\theta_L = 2Hu(1 + H c' / 2c_L) + c'/(4 c_L u) \cong 2Hu + c'/(4 c_L u) \tag{2.15}$$

and similarly

$$\tan\theta_H = 2Hu(1 + H\,c'/2c_H) - c'/(4c_L u) \cong 2Hu - c'/(4c_H u) \qquad (2.16)$$

Combining Eqs. (2.15) and (2.16) with (2.2), making the small angle approximation and writing $\rho \equiv c_H / c' \cong c_L / c$ we obtain

$$\begin{aligned} I_{P2} &= (4/r)\int_{u_1}^{u_2} \exp\{-(\alpha_L\theta_L + \alpha_H\theta_H)ru\}\,du \\ &= (4/r)\int_{u_1}^{u_2} \exp\{-2rHu^2(\alpha_L + \alpha_H)\}\,du\;\exp\{-(\alpha_L - \alpha_H)r/4\rho\} \end{aligned} \qquad (2.17)$$

The lower limit u_1 is defined as before and u_2 depends on the critical angle through

$$u_2 = \theta_c / 2H \qquad (2.18)$$

and the result is

$$\begin{aligned} I_{P2} &= \sqrt{\frac{2\pi}{(\alpha_L + \alpha_H)H r^3}}\;\exp\{-(\alpha_L - \alpha_H)r/4\rho\} \\ &\quad \times \left\{ \mathrm{erf}\left(\sqrt{\frac{(\alpha_L + \alpha_H)r}{2H}}\,\theta_c \right) - \mathrm{erf}\left(\sqrt{\frac{(\alpha_L + \alpha_H)r}{4\rho}} \right) \right\} \end{aligned} \qquad (2.19)$$

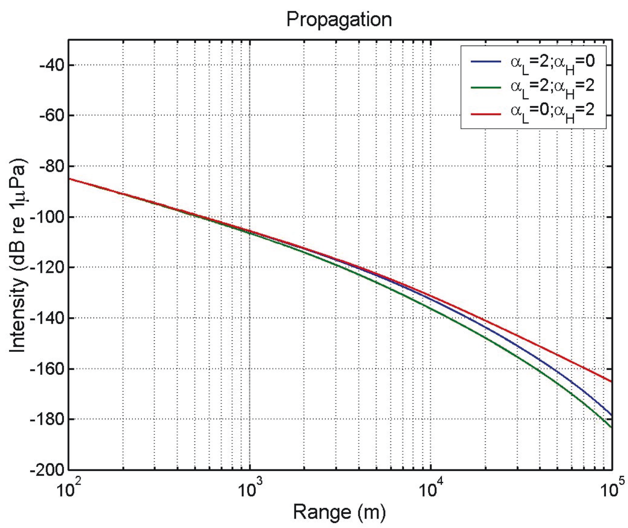

Figure 4 Two-way propagation contributions from rays interacting with both boundaries (I_{P2}^2) for labelled values of α_L and α_H.

2.2.3 Propagation: complete angle range

The complete solution for one-way propagation includes both angle ranges and is therefore the sum of the two integrals.

$$I_P = I_{P1} + I_{P2} \tag{2.20}$$

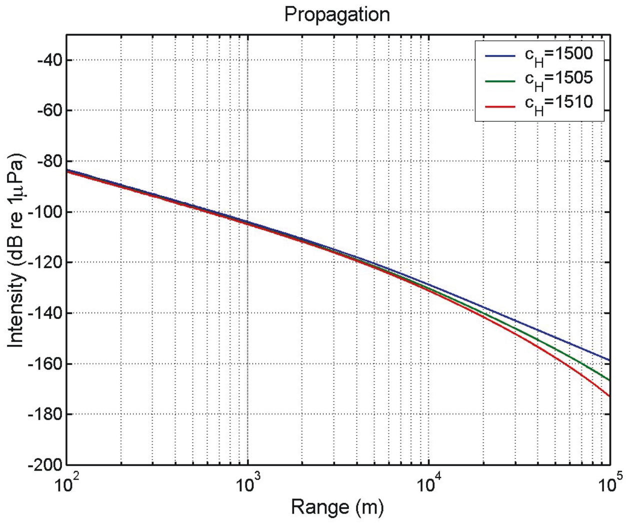

Figure 5 Two-way propagation for complete angle range (I_P^2) for three values of c_H.

2.3 Reverberation

If we consider Lambert's Law to be separable in angle ($S=\mu \sin\theta_{in}\sin\theta_{out}$) as in [1, 2, 3] then the reverberation can be written in terms of the product of ingoing and outgoing propagation where each angle integral contains an additional $\sin\theta$ term. These components are calculated here and the final joint solution will be described in Section 2.3.3. Each component is of the form

$$I_R = (2/r)\mu^{1/2} \int \theta \exp(-a_n r)\, du \tag{2.21}$$

where μ is the Lambert constant and θ is the (small) angle at the appropriate boundary. Here we have allowed for the fact that, of the upward and downward rays, only those going towards the boundary on the way in (and away from the boundary on the way out) contribute to the reverberation. This results in a factor of 2 reduction in Eq. (2.21).

2.3.1 Reverberation: rays interacting with only one boundary

Following Section 2.2.1, and remembering that in this regime there can only be a reverberation contribution from the L side of the duct, we have

$$\begin{aligned} I_{R1} &= (2/r)\,\mu_L^{\;1/2}\int_{u_1}^{u_0}\theta_L\ \exp\{-\alpha_L\theta_L ru\}\,du \\ &= (2/r)\,\mu_L^{\;1/2}\int_{u_1}^{u_0}\frac{1}{u}du\,(c'/2c_L)\ \exp\{-(\alpha_L c'/2c_L)r\} \\ &= (c'/rc_L)\,\mu_L^{\;1/2}\ \ln(u_0/u_1)\exp\{-(\alpha_L c'/2c_L)r\} \\ &= (c'/2rc_L)\,\mu_L^{\;1/2}\ \ln(H/h)\ \exp\{-(\alpha_L c'/2c_L)r\} \end{aligned} \tag{2.22}$$

This formula can be used for the outgoing or return path by setting h=h_s or h=h_r.

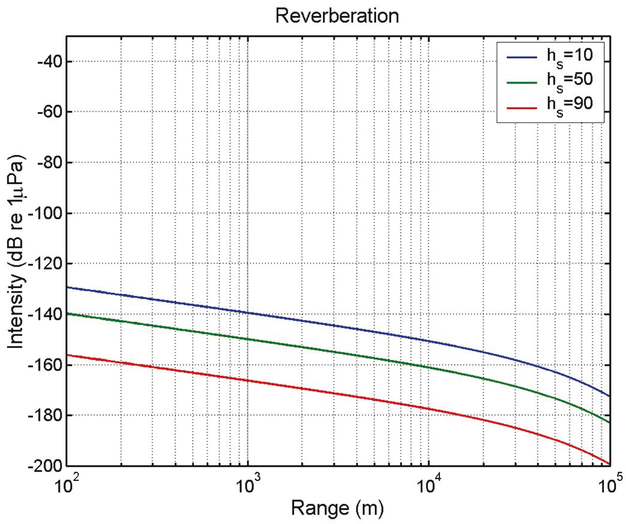

Figure 6 Reverberation contributions from rays interacting with only one boundary ($r\times(I_{R1})^2$) for h_{sr}=10,50,90m.

2.3.2 Reverberation: rays with interactions at two boundaries

Following Section 2.2.2 we can now calculate a reverberation contribution from each boundary. At the low speed boundary we have

$$I_{RL2} = (2/r)\,\mu_L^{\;1/2}\int_{u_1}^{u_2}\theta_L\ \exp\{-2rHu^2(\alpha_L+\alpha_H)\}\,du\ \exp\{-(\alpha_L-\alpha_H)r/4\rho\} \tag{2.23}$$

and at the high speed boundary we have

$$I_{RH2} = (2/r)\,\mu_H^{\;1/2} \int_{u_1}^{u_2} \theta_H \;\exp\{-2rHu^2(\alpha_L + \alpha_H)\}\,du\;\exp\{-(\alpha_L - \alpha_H)r/4\rho\} \quad (2.24)$$

For small angles Eqs (2.15) and (2.16) reduce to

$$\begin{aligned} \theta_L &= 2Hu + 1/(4\rho\, u) \\ \theta_H &= 2Hu - 1/(4\rho\, u) \end{aligned} \quad (2.25)$$

In both cases the *u*-integral splits into two terms *A*, *B* where

$$I_{RL2} = (2/r)\,\mu_L^{\;1/2}\;\exp\{-(\alpha_L - \alpha_H)r/4\rho\}\;(A + B) \quad (2.26)$$

$$I_{RH2} = (2/r)\,\mu_H^{\;1/2}\;\exp\{-(\alpha_L - \alpha_H)r/4\rho\}\,(A - B) \quad (2.27)$$

$$\begin{aligned} A &= 2H \int_{u_1}^{u_2} u\,\exp\{-2rHu^2(\alpha_L + \alpha_H)\}\,du \\ &= \{1/2r(\alpha_L + \alpha_H)\}\,[\exp\{-2rHu^2(\alpha_L + \alpha_H)\}]_{u_1}^{u_2} \\ &= \{1/2r(\alpha_L + \alpha_H)\}\,\{\exp\{-(\alpha_L + \alpha_H)r/4\rho\} - \exp\{-(\alpha_L + \alpha_H)r\theta_c^{\;2}/2H\}\} \end{aligned}$$

(2.28)

$$\begin{aligned} B &= (1/4\rho)\int_{u_1}^{u_2} (1/u)\exp\{-2rHu^2(\alpha_L + \alpha_H)\}\,du \\ &= (1/8\rho)\int_{X_1}^{X_2} (1/X)\exp\{-X\}\,dX\,; \quad X = 2rHu^2(\alpha_L + \alpha_H) \\ &= (1/8\rho)\{\mathrm{E}_1(X_1) - \mathrm{E}_1(X_2)\} \\ &= (1/8\rho)\{\mathrm{E}_1(-(\alpha_L + \alpha_H)r/4\rho) - \mathrm{E}_1(-(\alpha_L + \alpha_H)r\theta_c^{\;2}/2H)\} \end{aligned} \quad (2.29)$$

where $E_1(x)$ is the Exponential integral [5].

Of course there is no reverberation at all on the H side beyond the point where $\theta_H = 0$. However this is exactly the lower limit of the integral Eq. (2.24), where substituting u_1 from Eq. (2.7) into Eq. (2.25) we do indeed obtain $\theta_H = 0$. This ensures that I_{RH2} in Eq. (2.27) is always positive.

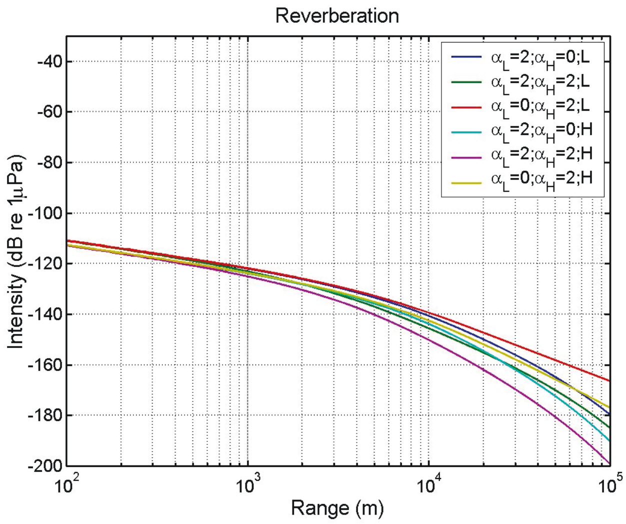

Figure 7 Reverberation contributions from rays interacting with both boundaries (lower boundary 'L': $r\times(I_{RL})^2$); upper boundary 'H': $r\times(I_{RH})^2$) for three combinations of α_L and α_H.

2.3.3 Reverberation: complete angle range

The joint reverberation for the L boundary is given by

$$I_{RL} = (I_{R1} + I_{RL2})^2 \, r \, \Phi \, p \tag{2.30}$$

and for the H boundary is

$$I_{RH} = I_{RL2}^{\;2} \, r \, \Phi \, p \tag{2.31}$$

Although we have not bothered here it would be possible to handle different parameters on the outward and return paths of Eq. (2.22), in particular, we can allow for different source and receiver depths in I_{R1}.

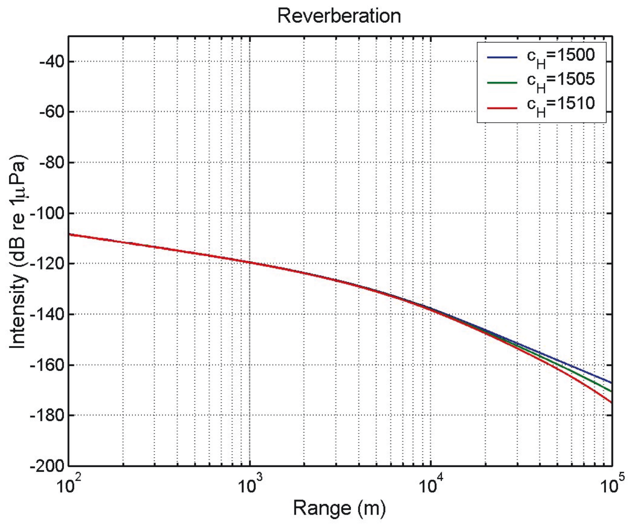

Figure 8 Reverberation from both boundaries, for complete angle range for three values of c_H..

3
Reverberation including focusing

3.1 Theory – normal modes with WKB

Here we derive the acoustic intensity in a stratified medium from the discrete normal mode sum formula assuming WKB modes and initially excluding boundary losses. The modulus of the wave function ψ is [4]

$$|\psi|^2 = 2\pi \sum_n \phi_n^{\,2}(z_s)\phi_n^{\,2}(z_r)/(K_n r) \tag{3.1}$$

Propagation loss is, by definition, the ratio of the acoustic pressure squared at the desired location and at one metre from the source. Since $p = \rho\, \partial\psi / \partial t = i\,\omega\,\rho\,\psi$ and $|\psi|^2$ is unity at unit distance from the source, Eq. (3.1) in dBs already represents propagation loss if the medium has uniform density. The ratio of intensities is slightly different since we also have $v = \nabla\psi = i\,k\,\psi$ so $p/v = c\,\rho$ and $I = pv = p^2/c\,\rho$.

The WKB formula for the mode shape is

$$\phi_n(z) = A_n \sin\left(\int_z \gamma_n\, dz + \varepsilon_n\right)/\gamma_n^{\,1/2}\;; \qquad \gamma_n = (k^2(z) - K_n^{\,2})^{1/2} \tag{3.2}$$

and the ‘phase integral’ for the nth WKB mode [6] is

$$(n + \tfrac{1}{2})\pi = \int \gamma_n\, dz \tag{3.3}$$

By substituting this mode shape, assuming that there are many modes ($\sum_n \to \int dn$), using orthogonality to calculate A_n, and noting that the cycle distance $r_c = 2\pi\, dn/dK_n$ (obtained from differentiating Eq. (3.3) with respect to K_n and comparing with the ray formula for r_c) we find

$$|\psi|^2 = (4/r)\int \frac{K\, dK}{r_c\,(k_r^{\,2} - K^2)^{1/2}(k_s^{\,2} - K^2)^{1/2}} \tag{3.4}$$

Putting

$$K = k_s \cos\theta_s \;; \quad dK = k_s \sin\theta_s \, d\theta_s \tag{3.5}$$

we obtain

$$|\psi|^2 = \frac{4k_s}{rk_r} \int \frac{\cos\theta_s \, d\theta_s}{r_c \sin\theta_r} = \frac{4c_r}{rc_s} \int \frac{\cos\theta_s \, d\theta_s}{r_c \sin\theta_r} \tag{3.6}$$

Now it is interesting to note that if we had started with a flux argument where we inspect the power travelling down a tube of rays (see, for example [7]) and we derive an intensity by comparing tube areas, we would have found for a single ray

$$I = I_o \frac{\cos\theta_s}{r \sin\theta_r \left| d\theta_s / dr \right|} \tag{3.7}$$

If we think of the receiving range element *dr* at a long range where it is large compared with the cycle distance (Fig. 9) then the received intensity is increased by a factor dr/r_c.

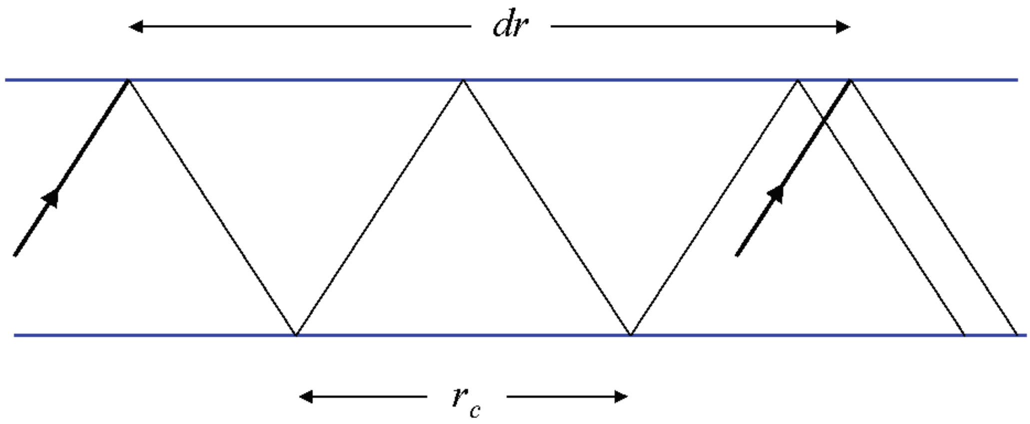

Figure 9 Geometric interpretation of the factor dr/r_c increase. The two thick rays originate as adjacent rays separated by $d\theta_s$., but have spread apart at long range.

We have to add the contributions for all angles, and in addition we find the usual four eigenrays for each angle. Thus we obtain

$$I = I_o \frac{4}{r} \int \frac{\cos\theta_s \, d\theta_s}{r_c \sin\theta_r} \tag{3.8}$$

Since, by definition I_o is unity, Eq. (3.8) differs from Eq. (3.6) only in missing the sound speed ratio. However, as pointed out in the first paragraph, this is because the former is a ratio of intensities whereas the latter is a ratio of pressures-squared. The difference stems from the sound speed difference at the source and receiver through $I = p^2 / c\rho$.

Making use of Snell's law

$$\frac{\cos\theta_s}{c_s} = \frac{\cos\theta}{c}; \quad \frac{\sin\theta_s \, d\theta_s}{c_s} = \frac{\sin\theta \, d\theta}{c}; \quad \tan\theta_s \, d\theta_s = \tan\theta \, d\theta \tag{3.9}$$

we can write Eq. (3.6) as

$$|\psi|^2 = \frac{4}{r} \int \frac{u \tan\theta \, d\theta}{\tan\theta_s \, \tan\theta_r} \tag{3.10}$$

where θ is anywhere on the ray. In particular we can write the integral either in terms of the ray angle at the low speed boundary

$$|\psi|^2 = \frac{4}{r} \int \frac{u \tan\theta_L \, d\theta_L}{\tan\theta_s \, \tan\theta_r} \tag{3.11}$$

or in terms of the reciprocal cycle distance u if we know the relation between θ_L and u. As before, this relationship depends on whether the ray reflects at one or two boundaries. From now on we will continue with the ratio of pressures-squared but we will rename it I for simplicity, despite the fact it is not really an intensity.

3.2 Propagation with focusing

3.2.1 Propagation: rays interacting with only one boundary

Following Section 2.2.1 we define

$$\tan\theta_L = (c'/2c_L)/u \tag{3.12}$$

So after including the same boundary loss term as before, Eq. (3.11) becomes either

$$I_{P1} = \frac{2c'}{rc_L} \int \frac{d\theta_L}{\tan\theta_s \, \tan\theta_r} \exp\{-(\alpha_L c'/2c_L) r\} \tag{3.13}$$

or

$$I_{P1} = \frac{4}{r} \int \frac{\sin^2 \theta_L \, du}{\tan\theta_s \, \tan\theta_r} \exp\{-(\alpha_L c'/2c_L)r\} \tag{3.14}$$

From Snell's law

$$\tan\theta_{s,r} = \frac{c_L}{c_{s,r}} \sqrt{\tan^2\theta_L - \tan^2\xi_{s,r}} \tag{3.15}$$

where ξ happens to be the (constant) angle at the low speed boundary of the ray that is horizontal at source or receiver

$$\tan^2\xi_{s,r} = (c_{s,r}{}^2 - c_L{}^2)/c_L{}^2 \tag{3.16}$$

This is not to be confused with the angle of the 'cycling' ray at the source or receiver $\theta_{s,r}$. However MAX(ξ_s ,ξ_r) is the same thing as the angle integral's lower limit θ_0.

Note that when source and receiver are at the L boundary Eq. (3.14) only deviates from the simpler version, Eq. (2.6), by $\cos^2\theta_L$.

Thus Eq. (3.13) becomes

$$I_{P1} = \frac{2c' c_s c_r}{r c_L{}^3} \int \frac{d\theta_L}{\sqrt{\tan^2\theta_L - \tan^2\xi_s}\sqrt{\tan^2\theta_L - \tan^2\xi_r}} \exp\{-(\alpha_L c'/2c_L)r\} \tag{3.17}$$

or in the low angle approximation, also ignoring sound speed ratios

$$I_{P1} = \frac{2c'}{r c_L} \int_{\theta_0}^{\theta_1} \frac{d\theta_L}{\sqrt{\theta_L{}^2 - \xi_s{}^2}\sqrt{\theta_L{}^2 - \xi_r{}^2}} \exp\{-(\alpha_L c'/2c_L)r\} \tag{3.18}$$

where θ_0 and θ_1 correspond to u_0 and u_1 , ie

$$\theta_0 = \mathrm{MAX}(\xi_s, \xi_r) = \left(\frac{2c' \mathrm{MAX}(h_s, h_r)}{c_L} \right)^{1/2} \tag{3.19}$$

$$\theta_1 = \left(\frac{2c' H}{c_L} \right)^{1/2} \tag{3.20}$$

Weston [8] showed that this could be solved in terms of incomplete elliptic integrals of the first kind F(μ,ν) [5].

$$I_{P1} = \frac{1}{r}\sqrt{\frac{2c'}{c_L \, \mathrm{MAX}(h_s, h_r)}} \; \mathrm{F}(\mu,\nu) \exp\{-(\alpha_L c'/2c_L)\}$$

$$\mu = \arcsin\left(\frac{\theta_1^{\,2} - \mathrm{MAX}(\xi_s,\xi_r)^2}{\theta_1^{\,2} - \mathrm{MIN}(\xi_s,\xi_r)^2}\right)^{1/2} \qquad (3.21)$$

$$\nu = \arcsin\left(\frac{\mathrm{MIN}(\xi_s,\xi_r)}{\mathrm{MAX}(\xi_s,\xi_r)}\right)$$

It is also possible to do a numerical calculation avoiding the singularity at the lower limit (see Annex).

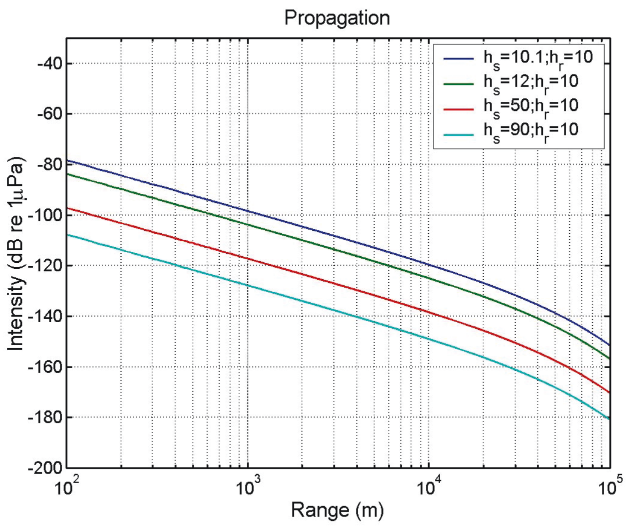

Figure 10 Two-way propagation contributions with focusing from rays interacting with only one boundary (I_{P1}^2) for h_{sr}=10,50,90m.

3.2.2 Propagation: rays with interactions at two boundaries

Starting again with Eq. (3.11) (with small angle approximation) and including boundary losses as in Eq. (2.17)

$$I_{P2} = (4/r)\int_{\theta_1}^{\theta_2} \frac{u\theta_L}{\theta_s\theta_r} \exp\{-2rHu^2(\alpha_L + \alpha_H)\}\, d\theta_L \;\exp\{-(\alpha_L - \alpha_H) r / 4\rho\} \qquad (3.22)$$

Again we have from Eqs. (2.15), (2.16) and (3.15)

$$\theta_L = 2Hu + c'/(4c_L u) = 2Hu + 1/(4\rho u) \qquad (3.23)$$

$$\theta_H = 2Hu - c'/(4c_H u) = 2Hu - 1/(4\rho u) \tag{3.24}$$

$$\theta_{s,r} = \frac{c_L}{c_{s,r}}\sqrt{\theta_L{}^2 - \xi_{s,r}{}^2} \tag{3.25}$$

and, noting that

$$u\,\theta_L\ d\theta_L = u\{2Hu + c'/(4c_L u)\}\{2H - c'/(4c_L u^2)\}\,du = \theta_L\,\theta_H\,du \tag{3.26}$$

Eq. (3.22) becomes

$$\begin{aligned} I_{P2} &= (4/r)\int_{\theta_1}^{\theta_2}\frac{u\theta_L}{\sqrt{\theta_L{}^2-\xi_s{}^2}\sqrt{\theta_L{}^2-\xi_r{}^2}}\exp\{-2rHu^2(\alpha_L+\alpha_H)\}\,d\theta_L \\ &\quad\times\exp\{-(\alpha_L-\alpha_H)r/4\rho\} \\ &= (4/r)\int_{u_1}^{u_2}\frac{\theta_L\ \theta_H}{\sqrt{\theta_L{}^2-\xi_s{}^2}\sqrt{\theta_L{}^2-\xi_r{}^2}}\exp\{-2rHu^2(\alpha_L+\alpha_H)\}\,du \\ &\quad\times\exp\{-(\alpha_L-\alpha_H)r/4\rho\} \\ &= (4/r)\int_{u_1}^{u_2}\frac{\{u^2+u_1{}^2\}\{u^2-u_1{}^2\}\exp\{-2rHu^2(\alpha_L+\alpha_H)\}}{\sqrt{\{u^2+u_1{}^2\}^2-4u^2u_1{}^2h_s/H}\sqrt{\{u^2+u_1{}^2\}^2-4u^2u_1{}^2h_r/H}}\,du \\ &\quad\times\exp\{-(\alpha_L-\alpha_H)r/4\rho\} \end{aligned} \tag{3.27}$$

Again the limits are

$$u_1 = \frac{1}{2}\sqrt{\frac{c'}{2H\,c_H}} \tag{3.28}$$

$$u_2 = \theta_c/2H \tag{3.29}$$

and θ_1, θ_2 are the corresonding angle limits. In Eq. (3.27) we have used the fact that $u_1{}^2 = 1/(8\rho H)$ and $\xi_{s,r}{}^2/(2H)^2 = 4u_1{}^2 h_{s,r}/H$.

The only ways of solving Eq. (3.27) are numerically, series solution or some kind of approximation. An approximation is discussed in the Annex. There are some special cases. When h_s and h_r are zero Eq. (3.27) differs from the simpler one, Eq. (2.17) in the additional factor θ_H/θ_L which is close to unity except when $u \sim u_1$. When $h_s = h_r = H$ this additional factor is θ_L/θ_H. If source and receiver are at opposite sides of the duct the factor is unity and we obtain the simple formula again.

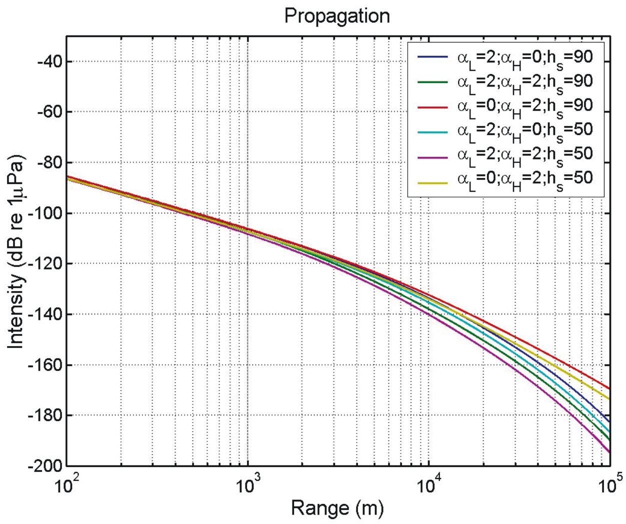

Figure 11 Two-way propagation contributions with focusing from rays interacting with both boundaries (I_{P2}^2) for labelled values of α_L and α_H and source depth..

3.2.3 Propagation: complete angle range

As before the complete solution for one-way propagation includes both angle ranges and is therefore the sum of the two integrals.

$$I_P = I_{P1} + I_{P2} \tag{3.30}$$

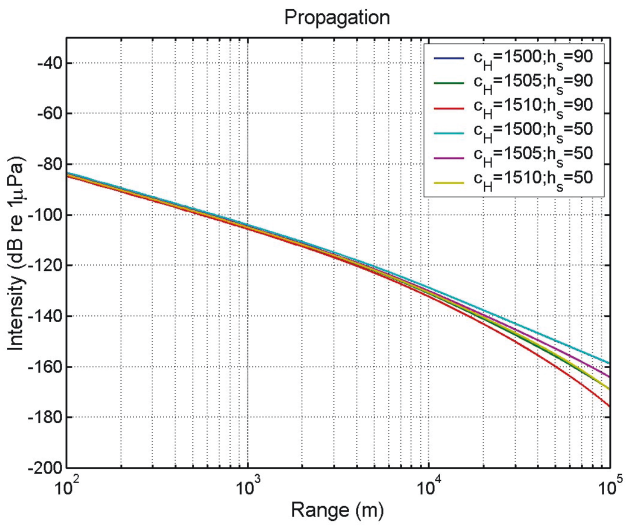

Figure 12 Two-way propagation with focusing for complete angle range (I_P^2) for three values of c_H. and two source depths.

3.3 Reverberation with focusing

3.3.1 Reverberation: rays interacting with only one boundary

To get the one-way angle integral from a source (or receiver) to a scatterer we use Eq. (3.13) with the arrival on the boundary $\theta_r = \theta_L$, but there is an additional Lambert θ_L in the numerator and these cancel. This is an important difference between the focusing and not-focusing case. At the H boundary we would have expected reduced reverberation because the angle is smaller, but focusing makes the intensity higher by the same amount.

$$I_{R1} = \frac{c'}{rc_L}\mu_L^{1/2}\int_{\theta_0}^{\theta_1}\frac{d\theta_L}{\sqrt{\theta_L^2-\xi_s^2}}\exp\{-(\alpha_L c'/2c_L)r\} \tag{3.31}$$

Remembering that $\xi_s = \theta_0$, we find

$$\begin{aligned} I_{R1} &= \frac{c'}{rc_L}\mu_L^{1/2}\exp\{-(\alpha_L c'/2c_L)r\}\ \cosh^{-1}(\theta_1/\theta_0) \\ &= \frac{c'}{rc_L}\mu_L^{1/2}\exp\{-(\alpha_L c'/2c_L)r\}\ \cosh^{-1}\{(H/h_s)^{1/2}\} \end{aligned} \tag{3.32}$$

In the limit of $h_s \to 0$ this is identical with Eq. (2.22).

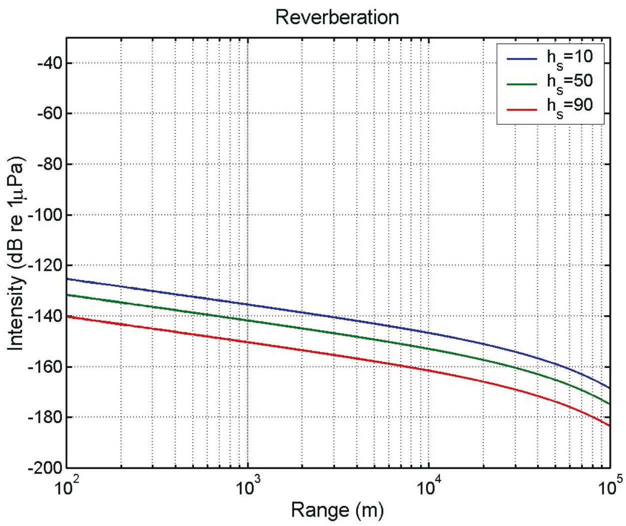

Figure 13 Reverberation contributions with focusing from rays interacting with only one boundary ($r\times(I_{R1})^2$) for h_{sr}=10,50,90m.

3.3.2 Reverberation: rays with interactions at two boundaries

Taking the second line of Eq. (3.27) and adding the Lambert θ term, which again cancels with the same angle in the denominator, regardless of which boundary is the scatterer

$$
\begin{aligned}
I_{R2} &= (2/r)\mu^{1/2}\int_{u_1}^{u_2}\frac{\theta_L\ \theta_H}{\sqrt{\theta_L{}^2-\xi_s{}^2}}\exp\{-2rHu^2(\alpha_L+\alpha_H)\}\,du\ \exp\{-(\alpha_L-\alpha_H)r/4\rho\}\\
&= (2/r)\mu^{1/2}\int_{u_1}^{u_2}\frac{\{u^2+u_1{}^2\}\exp\{-2rHu^2(\alpha_L+\alpha_H)\}}{\sqrt{\{u^2+u_1{}^2\}^2-4u^2u_1{}^2h_s/H}}2H\{u-u_1{}^2/u\}\,du\\
&\qquad\times\exp\{-(\alpha_L-\alpha_H)r/4\rho\}\\
&= (2H/r)\mu^{1/2}\int_{V_1}^{V_2}\frac{\{V+V_1\}\exp\{-2rHV(\alpha_L+\alpha_H)\}}{\sqrt{\{V+V_1\}^2-4VV_1h_s/H}}\{1-V_1/V\}\,dV\\
&\qquad\times\exp\{-(\alpha_L-\alpha_H)r/4\rho\}
\end{aligned}
\tag{3.33}
$$

where $V = u^2$, and μ applies to either boundary.

One possible approximation is small h_s/H, and it is clear that for $h_s = 0$ the second line of Eq. (3.33) is identical to Eq. (2.24), i.e. Eq. (2.27), for the H boundary. This is discussed in the Annex.

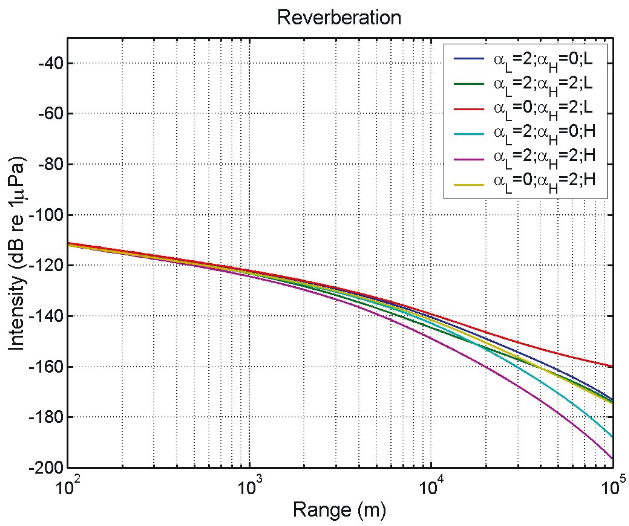

Figure 14 Reverberation contributions from rays interacting with both boundaries (lower boundary 'L': $r\times(I_{RL})^2$); upper boundary 'H': $r\times(I_{RH})^2$) for three combinations of α_L and α_H.

3.3.3 Reverberation: complete angle range

As before the joint reverberation for the L boundary is given by

$$I_{RL} = (I_{R1} + I_{RL2})^2\, r\,\Phi\, p \tag{3.34}$$

and for the H boundary is

$$I_{RH} = I_{RL2}{}^2\, r\,\Phi\, p \tag{3.35}$$

Although we have not bothered here it would be possible to handle different parameters on the outward and return paths of Eq. (3.34), in particular, we can allow for different source and receiver depths in I_{R1}.

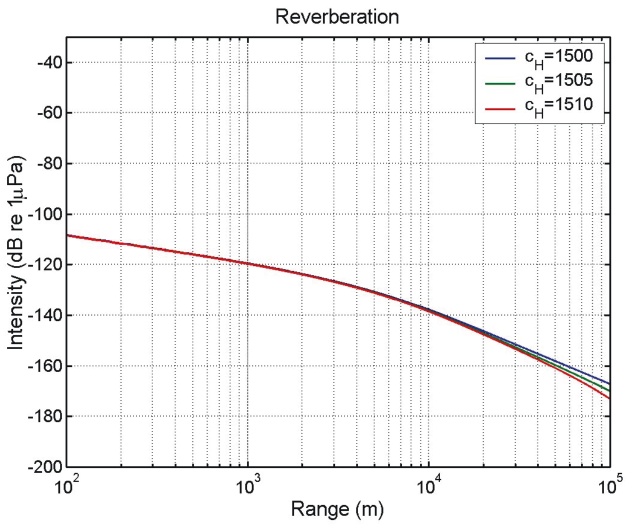

Figure 15 Reverberation with focusing from both boundaries, for complete angle range for three values of c_H..

3.4 Comparison: with and without focusing

It is instructive to compare the four focusing integrals in u (for one/two boundary reflection and propagation/reverberation) with the earlier cases where focusing was ignored. In each case the appropriate boundary loss term is represented by A and the rest of the integrand is written in terms of the angles at the L and H boundary and at the source and receiver. In both P cases the angle terms in the simplified formulas were unity. In the R cases the angle terms were simply θ_L except in the two-boundary case with scattering at the H boundary in which case the factor was θ_H.

Note that A is not a function of range in P1 and R1!! Therefore focusing or not makes no difference to the range dependence in these cases.

• *Propagation – one boundary*

$$P_1 = \int_{u_1}^{u_0} \frac{\sin^2\theta_L}{\tan\theta_s \tan\theta_r} A\, du \tag{3.36}$$

• *Propagation – two boundaries*

$$P_2 = \int_{u_1}^{u_2} \frac{\tan\theta_L \tan\theta_H}{\tan\theta_s \tan\theta_r} A\, du \tag{3.37}$$

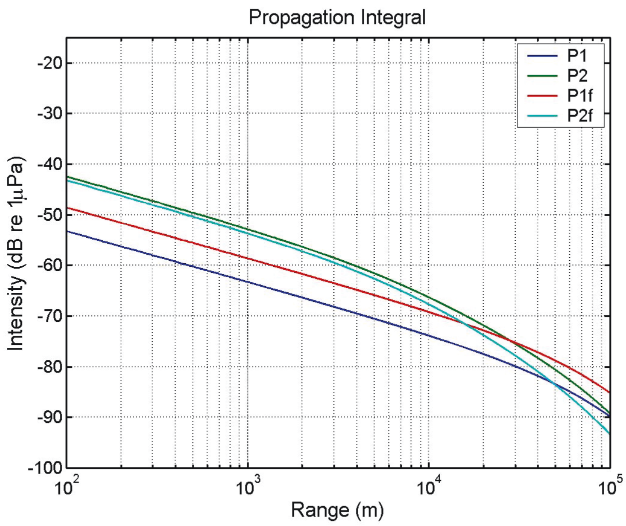

Figure16 Propagation integrals with (P1f,P2f) and without (P1,P2) focusing.

• *Reverberation – one boundary*

$$R_1 = \int_{u_1}^{u_0} \frac{\sin^2\theta_L}{\tan\theta_s} A\, du \tag{3.38}$$

• *Reverberation –two boundaries*

$$R_2 = \int_{u_1}^{u_2} \frac{\tan\theta_L \tan\theta_H}{\tan\theta_s} A\, du \tag{3.39}$$

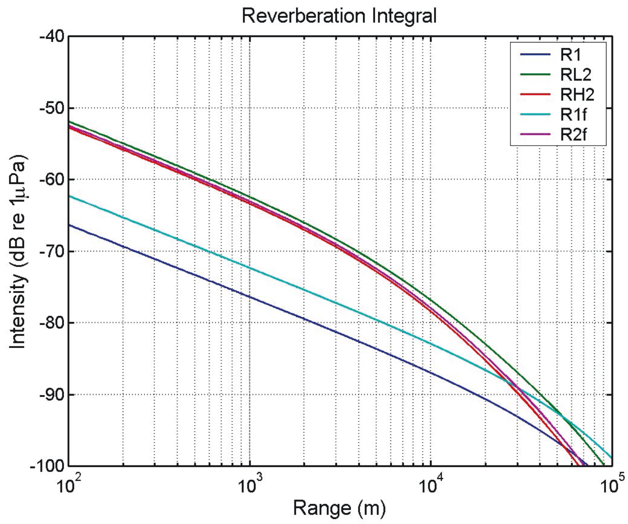

Figure 17 Reverberation integrals with (R1f,R2f) and without (R1,RL2,RH2) focusing.

In the first equation the minimum of θ_s, θ_r must go to zero at the integral limit. Since it is in the denominator approximations are difficult and the result is an elliptic function. There is therefore a fundamental difference in the focused and non-focused solution. In fact if MIN(ξ_s,ξ_r) = 0 (see Eqs. (3.16), (3.18)) and MAX(ξ_s,ξ_r) is finite but small it is shown in the Annex that there is a factor difference of $\pi/2$. However it is still not obvious that the differences are that important to underwater acoustics at around a kilohertz.

In the second equation θ_s, θ_r cannot have such a dramatic effect unless one of them is near the H boundary. But even then θ_H is always smaller so it would need both θ_s and θ_r to be near the H boundary to cause serious deviation from the non-focused solution.

In the first reverberation equation there is also only a problem if the source (or receiver) is near the H boundary.

In the second reverberation equation there can only be a slight underestimate since $\tan\theta_H < \tan\theta_s$. The focused and unfocused integrands only differ, and differ by a small amount, near the lower integration limit. So the net effect is small. In fact the focused solution must lie between the H and L solutions for no focusing because

$$\theta_L > \frac{\theta_L \theta_H}{\theta_s} > \theta_H .$$

4
Signal-to-Reverberation-Ratio

4.1 General formula

Taking equations for propagation and reverberation from Section 2 (Eqs. 2.6, 2.19, 2.22, 2.26-2.29) we can write an explicit formula for SRR. Here we use the shortahnd $\rho = c/c'$ (ignoring distinctions between c_H , c_L , etc.), $\alpha_0 = \alpha_L + \alpha_H$, $\alpha_1 = \alpha_L - \alpha_H$. We ignore the difference between $I_{RL\ 2}$ and $I_{RH\ 2}$ (i.e. $B \to 0$) except for the Lambert coefficient. We introduce a switch Δ so that we can write a single formula for a background of either reverberation from the L boundary (Δ = 1) or reverberation from the H boundary (Δ = 0). The Lambert coefficient is $\mu = \mu_L\ \Delta + \mu_H\ (1 - \Delta)$. We introduce V the linear equivalent of SRR where SRR = 10 log V . Target echo is determined by the maximum of source and target distance from L, h_{st} , whereas reverberation is determined by only h_s .

$$I_P = \mathrm{e}^{-\alpha_1 r/2\rho}\left[\frac{4}{r}(u_0 - u_1)\,\mathrm{e}^{-\alpha_0 r/4\rho} + \sqrt{\frac{2\pi}{\alpha_0 H r^3}}\left\{\mathrm{erf}\left(\sqrt{\frac{\alpha_0 r}{2H}}\theta_c\right) - \mathrm{erf}\left(\sqrt{\frac{\alpha_0 r}{4\rho}}\right)\right\}\right]^2 \quad (4.1)$$

$$I_R = \mathrm{e}^{-\alpha_1 r/2\rho}\, r\mu\Phi p\left[\frac{\Delta}{2r\rho}\ln\left(\frac{H}{h_s}\right)\mathrm{e}^{-\alpha_0 r/4\rho} + \frac{1}{\alpha_0 r^2}\left\{\exp\left(-\frac{\alpha_0 r}{4\rho}\right) - \exp\left(-\frac{\alpha_0 r}{2H}\theta_c^{\ 2}\right)\right\}\right]^2 \quad (4.2)$$

Note that the first reverberation term vanishes (Δ = 0) for scattering from the H boundary. Substituting for u_o , u_1 and rearranging we have

$$V = \frac{2\alpha_0}{\mu\Phi p H}\left[\frac{\sqrt{\frac{\alpha_0 r}{4\rho}}\left(\sqrt{\frac{H}{h_{st}}} - 1\right)\mathrm{e}^{-\alpha_0 r/4\rho} + \frac{\sqrt{\pi}}{2}\left\{\mathrm{erf}\left(\sqrt{\frac{\alpha_0 r}{2H}}\theta_c\right) - \mathrm{erf}\left(\sqrt{\frac{\alpha_0 r}{4\rho}}\right)\right\}}{\Delta\frac{\alpha_0 r}{4\rho}\ln(H/h_s)\mathrm{e}^{-\alpha_0 r/4\rho} + \frac{1}{2}\left\{\exp\left(-\frac{\alpha_0 r}{4\rho}\right) - \exp\left(-\frac{\alpha_0 r}{2H}\theta_c^{\ 2}\right)\right\}}\right]^2 \quad (4.3)$$

We now investigate three range limits. These are set by the relative sizes of the gaussian width $\theta_g = (2H/\alpha_o r)^{1/2}$, the bottom's critical angle θ_c and (half) the water column's critical angle $\theta_w = (2\delta c/c)^{1/2}/2 = (\delta c/2c)^{1/2}$ where δc is the velocity contrast in the water. Thus there are three possibilities: $\theta_g > \theta_c$ (short range, $r < 2H/\alpha_0\theta_c^{\ 2}$); $\theta_c > \theta_g > \theta_w$ (intermediate range, $2H/\alpha_0\theta_c^{\ 2} < r < (2H/\alpha_0)(2c/\delta c)$); and $\theta_g < \theta_w$ (long range, $r > (2H/\alpha_0)(2c/\delta c)$).

Typical magnitudes of the limits can be seen as follows. Suppose θ_c = 0.5 radians, so $1/\theta_c^2 = 4$. Suppose sound speed contrast is 1% (e.g. 15m/s in 1500m/s), so $2c/\delta c = 200$. Then the intermediate range limits can be expressed as $4 < \alpha_0 r/2H < 200$. For $\alpha_{dB} = 2$ dB/rad $\alpha_0 = 0.46$ rad^{-1} (joint boundary loss) and H = 100m the limits are 1.7km < r < 86km. These values are not particularly restrictive or well chosen; there will always be a middle range as long as the bottom has a critical angle significantly bigger than the water column's critical angle.

Equation 4.3 can be further simplified by introducing the notation

$$W \equiv \theta_w/\theta_g = \sqrt{\frac{r\alpha_0}{2H}}\sqrt{\frac{\delta c}{2c}} \tag{4.4}$$

$$C \equiv \theta_c/\theta_g = \sqrt{\frac{r\alpha_0}{2H}}\,\theta_c \tag{4.5}$$

$$\chi \equiv H/h \quad ; \quad h \equiv h_s = h_t \tag{4.6}$$

and then normalising V by the isovelocity long range value $2\pi\,\alpha_0/\mu\Phi pH$. Thus

$$V = \left[\frac{\dfrac{W}{\sqrt{\pi}}\left(\chi^{1/2}-1\right)\mathrm{e}^{-W^2} + \dfrac{1}{2}\left\{\mathrm{erf}(C) - \mathrm{erf}(W)\right\}}{\Delta\, W^2 \ln(\chi)\mathrm{e}^{-W^2} + \dfrac{1}{2}\left\{\exp\left(-W^2\right) - \exp\left(-C^2\right)\right\}}\right]^2 \tag{4.7}$$

4.2 Short range: $r < 2H/\alpha_0\theta_c^{\ 2}$; $\theta_g > \theta_c$; $1 > C > W$

At short range all exponentials can be approximated as $\exp(z) = 1 + z$, and erf functions as $\mathrm{erf}(z) = (2/\sqrt{\pi})\, z$. Thus

$$V = \frac{1}{\pi}\left[\frac{W\left(\chi^{1/2}-1\right)\mathrm{e}^{-W^2}+(C-W)}{\Delta\, W^2 \ln(\chi)\mathrm{e}^{-W^2}+\frac{1}{2}(C^2-W^2)}\right]^2$$
$$= \frac{1}{\pi W^2}\left[\frac{\left(\chi^{1/2}-1\right)\mathrm{e}^{-W^2}+(C/W-1)}{\Delta\, \ln(\chi)\mathrm{e}^{-W^2}+\frac{1}{2}(C^2/W^2-1)}\right]^2 \tag{4.8}$$

Now the order of magnitude of $C^2/W^2 = 2\rho\theta_c^{\ 2}/H = (2c/\delta c)/(1/\theta_c^{\ 2}) \approx 50$ for the typical environment considered. In the denominator of Eq. 4.8 the log term is always smaller than this for practical cases. Therefore there is no significant difference between L and H boundary reverberation. In the numerator the equivalent changeover is $\chi = H/h_{st} \approx 25$. Thus if h_{st} (i.e. both source/receiver and target) is more than a few metres from the L boundary in 100m of water then Eq. 4.8 reduces to

$$V = \frac{1}{\pi C^2} = \frac{2H}{\pi\,\alpha_0\theta_c^{\ 2} r} \tag{4.9}$$

for scattering from either boundary. This is precisely the same as the isovelocity short range behaviour and $V \propto 1/r$.

4.3 Intermediate range: $2H/\alpha_0\theta_c^{\ 2} < r < (2H/\alpha_0)(2c/\delta c)$; $\delta c/2c < \theta_g < \theta_c$; $C > 1 > W$

In this regime the exponential and erf functions containing *W* take the low argument form whereas the equivalents containing *C* have large argument. Thus Eq. 4.7 becomes

$$V = \left[\frac{\frac{W}{\sqrt{\pi}}\left(\chi^{1/2}-1\right)\mathrm{e}^{-W^2}+\frac{1}{2}\left\{1-2W/\sqrt{\pi}\right\}}{\Delta\, W^2 \ln(\chi)\mathrm{e}^{-W^2}+\frac{1}{2}\mathrm{e}^{-W^2}}\right]^2$$
$$= \frac{1}{\pi}\left[\frac{W\left(\chi^{1/2}-2\right)+\sqrt{\pi}/2}{\Delta\, W^2\ln(\chi)+1/2}\right]^2 \tag{4.10}$$

For scattering at the H boundary ($\Delta = 0$) this reduces to

$$V = \left(1 + 2W(\chi^{1/2} - 2)/\sqrt{\pi}\right)^2 \tag{4.11}$$

Thus for $\chi = 4$, i.e. $h = 0.25\ H$ we obtain the isovelocity solution at all ranges (in this regime). For larger h, V reduces with range; for smaller h, V increases with range.

For scattering at the L boundary ($\Delta = 1$) Eq. 4.10 is difficult to simplify. However for the case where $\chi = 4$ ($h = H/4$) we have

$$V = \left(1 + 2.77\ W^2\right)^{-2} \tag{4.12}$$

Clearly this results in reducing V as W^2 (i.e. r) increases.

4.4 Long range: $r > (2H/\alpha_0)(2c/\delta c)\,;\delta c/2c > \theta_g\,; C > W > 1$

Finally the arguments of both exponentials and erfs become large and we have 1 – erf(z) = ($1/\sqrt{\pi}\,z$) exp($-z^2$).

$$\begin{aligned} V &= \frac{1}{\pi}\left[\frac{\left(\chi^{1/2} - 1\right) + \dfrac{1}{2W^2}\left\{1 - \dfrac{W}{C}\mathrm{e}^{-(C^2 - W^2)}\right\}}{\Delta\, W \ln(\chi) + \dfrac{1}{2W}\left\{1 - \mathrm{e}^{-(C^2 - W^2)}\right\}}\right]^2 \\ &= \frac{1}{\pi W^2}\left[\frac{\left(\chi^{1/2} - 1\right) + 1/2W^2}{\Delta\ \ln(\chi) + 1/2W^2}\right]^2 \end{aligned} \tag{4.13}$$

For scattering from the H boundary we obtain

$$V = \frac{1}{\pi W^2}\left(1 + 2W^2(\chi^{1/2} - 1)\right)^2 \tag{4.14}$$

which for large W tends to

$$V = \frac{4}{\pi}\left(\chi^{1/2} - 1\right)^2 W^2 \propto r \tag{4.15}$$

For scattering from the L boundary the corresponding limit is

$$V = \frac{1}{\pi W^2}\left(\frac{\chi^{1/2} - 1}{\ln(\chi)}\right)^2 \propto 1/r \tag{4.16}$$

4.5 Summary

To summarise these results four numerical examples are given in Figs. 18-21, each for scattering from the L and H boundary. These plots are SRR in dBs against normalised depth ($h/H = 1/\chi$) and normalised range (r/H). Thus for a water depth of 100m a normalised range of 1000 means a range of 100 km. The limits $C = 1$ and $W = 1$ are indicated by a vertical line (whenever they are on scale). Two sets of sound speed contrasts are taken: 1500-1507 m/s and 1500-1515 m/s; and two sets of reflection properties are taken: $\alpha_{dB} = 2.00$ dB/rad, $\theta_c = 28°$ and $\alpha_{dB} = 1.35$ dB/rad, $\theta_c = 35°$.

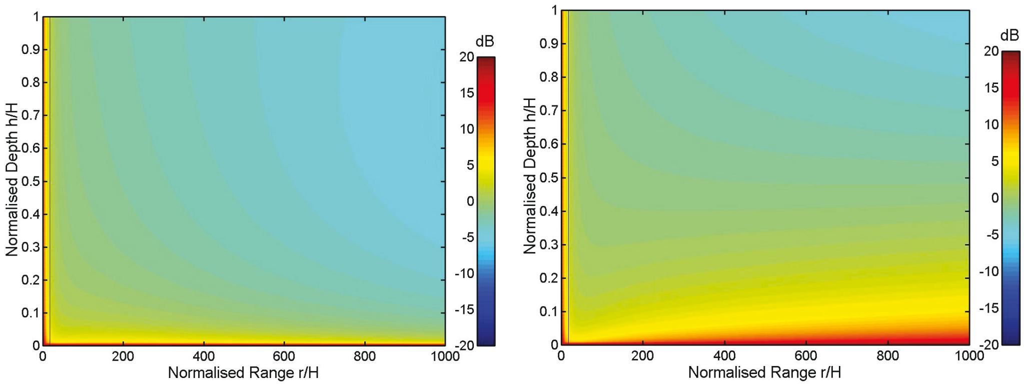

Figure 18 SRR for scattering from the L boundary (left) and the H boundary (right). Parameters are: $\alpha_{dB} = 1.35$ *dB/rad,* $\theta_c = 35°$*,* $c_L = 1500$*m/s,* $c_H = 1507$*m/s.*

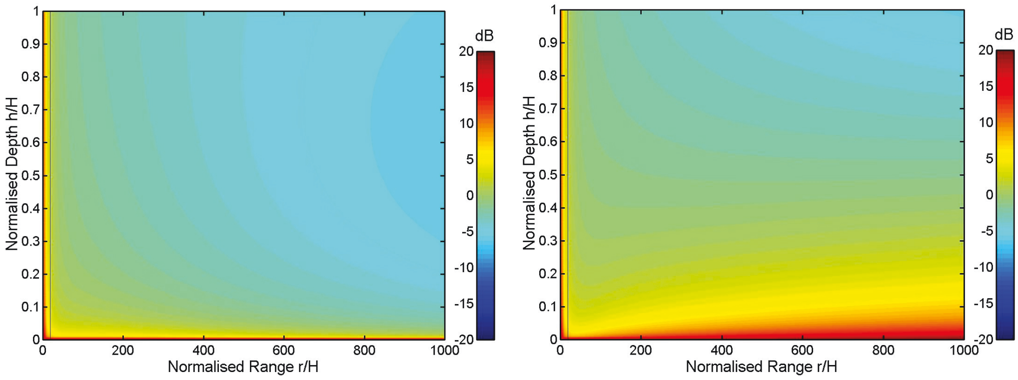

Figure 19 SRR for scattering from the L boundary (left) and the H boundary (right). Parameters are: $\alpha_{dB} = 2.00$ *dB/rad,* $\theta_c = 28°$*,* $c_L = 1500$*m/s,* $c_H = 1507$*m/s.*

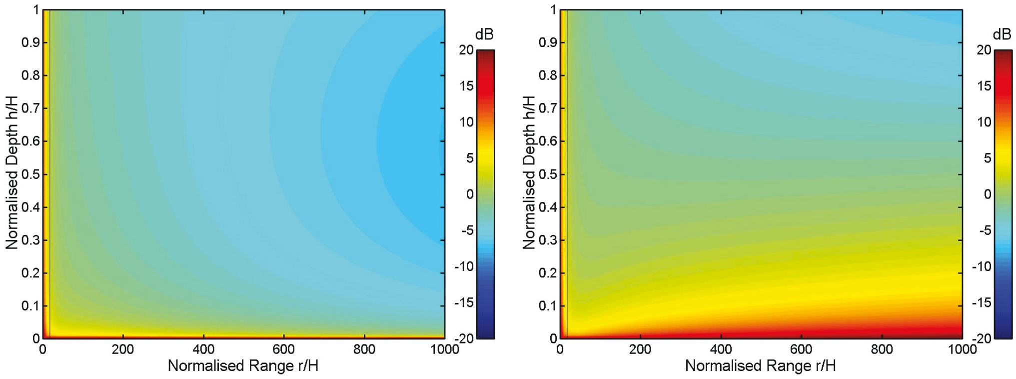

Figure 20 SRR for scattering from the L boundary (left) and the H boundary (right). Parameters are: α_{dB} = 1.35 dB/rad, θ_c = 35°, c_L = 1500m/s, c_H = 1515m/s.

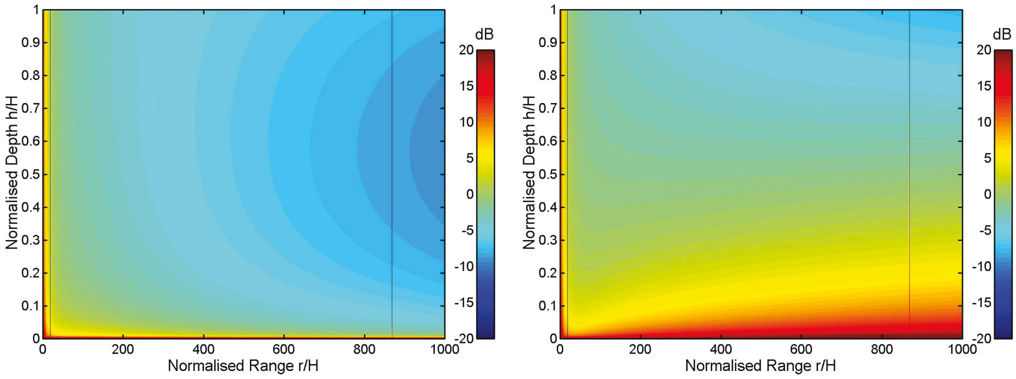

Figure 21 SRR for scattering from the L boundary (left) and the H boundary (right). Parameters are: α_{dB} = 2.00 dB/rad, θ_c = 28°, c_L = 1500m/s, c_H = 1515m/s.

5 Conclusions

Fomulae for propagation and reverberation have been extended to the case where there is refraction from a linear sound speed profile and scattering and losses at either or both boundaries. Because the rays are curved there is a low angle duct with single boundary interaction as well as the high angle duct with two-boundary interaction considered here. Thus the two-boundary contribution tends to behave as the mode-stripping regime except that there is a small gap in the middle of the gaussian angle distribution filled by the single-boundary contribution. At long range, therefore, the mode-stripping $r^{-3/2}$ range law reverts to the exponential decay of the single-boundary duct. Naturally the solution depends on the depth of the source and receiver, and there is a different result for scattering from the high and the low sound speed sides of the duct.

The formulae are based on two approaches. The first and simplest is to sum eigenrays and to allocate a simple spherical spreading contribution to each ignoring focusing and caustics except in as far as the ray shapes determine the number of received eigenrays. The second includes focusing effects as well. The latter approach was derived here from the incoherent mode sum although it can also be derived from Weston's flux approach. The propagation solution for single boundary interaction (surface or bottom duct) is, apart from a trivial decay term, identical to Weston's.

Explicit formulae were given for the signal-to-reverberation-ratio generally and under particular conditions. These were shown also in graphical form. Although there are some obvious differences from the isovelocity case many of the effects found with isovelocity are still seen, for instance there are still regimes where SRR increases with bottom loss, decreases with water depth, and is independent of range.

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Annex A: Approximations and numerical solutions

A.1 Numerical solution for elliptic integral and Eq. (3.18)

Elliptic integrals can be written in many forms (see [5], p. 596). The particular problem with incomplete elliptic integrals is that the upper limit is also a variable so there are two arguments and typically ready-made computer functions are hard to come by. The problem with a straightforward numerical integration of Eq. (3.18) as it stands

$$I_{P1} = \frac{2c'}{rc_L} \int_{\theta_0}^{\theta_1} \frac{d\theta_L}{\sqrt{\theta_L{}^2 - \xi_s{}^2}\sqrt{\theta_L{}^2 - \xi_r{}^2}} \exp\{-(\alpha_L c'/2c_L)r\} \tag{A.1}$$

is that there is a singularity at the lower limit. More generally the integral can be written as

$$\begin{aligned} F(\mu,\nu) &= a\int_a^u \frac{dt}{\sqrt{t^2 - a^2}\sqrt{t^2 - b^2}} \\ \mu &= \arcsin\left(\frac{u^2 - a^2}{u^2 - b^2}\right)^{1/2} \\ \nu &= \arcsin\left(\frac{b}{a}\right) \end{aligned} \tag{A.2}$$

where $u > a > b$. Weston [8] also wrote this integral in terms of depths ($h = \theta_L{}^2 c_L/2\,c'$) as

$$\begin{aligned} F(\mu,\nu) &= \frac{\sqrt{a'}}{2}\int_{a'}^{u'} \frac{dh}{\sqrt{h(h-a')(h-b')}} \\ \mu &= \arcsin\left(\frac{u'-a'}{u'-b'}\right)^{1/2} \\ \nu &= \arcsin\left(\frac{b'}{a'}\right) \end{aligned} \tag{A.3}$$

where $u' > a' > b'$. This also has a singularity at the lower limit, but if we change variable to $x^2 = h - a'$ we find

$$F(\mu,\nu) = \sqrt{a'} \int_0^{\sqrt{u'-a'}} \frac{dx}{\sqrt{x^2 + a'}\sqrt{x^2 + a' - b'}} \qquad \text{(A.4)}$$

This integral does not have any singularities and can be robustly and accurately evaluated using an integration subroutine such as Matlab's 'quad'. In fact it is a practical way of evaluating incomplete elliptic integrals.

A.2 Difference between solution with focusing (Eq. 3.18) and without focusing (Eq. 2.6).

Intuitively one might expect that in the limit of source and receiver being near the L boundary the focused and non-focused solutions would be the same. As noted in Eq. (3.36) there is a simple factor difference between Eqs. (2.6) and (3.14) which one would expect to tend to unity. However several of the plots (in particular, Figs.16 and 17) show that this is not so. To try and understand why, we evaluate both integrals for the special case where the receiver is on the boundary, i.e. $\xi_r = 0$, $\xi_s = \theta_0$ and $\theta_1 > \theta_2 > 0$.

In the θ integral form Eq. (2.6) is

$$\begin{aligned} I_{NF} &= \frac{2c'}{rc_L} \int_{\theta_0}^{\theta_1} \frac{d\theta_L}{\theta_L^2} \quad \exp\{-(\alpha_L c'/2c_L)r\} \\ &= \frac{1}{r}\sqrt{\frac{2c'}{c_L}} \left(\frac{1}{\sqrt{h_s}} - \frac{1}{\sqrt{H}} \right) \quad \exp\{-(\alpha_L c'/2c_L)r\} \end{aligned} \qquad \text{(A.5)}$$

but Eq. (3.18) is

$$I_F = \frac{2c'}{rc_L} \int_{\theta_0}^{\theta_1} \frac{d\theta_L}{\theta_L \sqrt{\theta_L^2 - \theta_0^2}} \quad \exp\{-(\alpha_L c'/2c_L)r\} \qquad \text{(A.6)}$$

and this can be solved by substitution (also [9], 2.266) to give

$$\begin{aligned} I_F &= \frac{2c'}{rc_L} \frac{1}{\theta_0} \arccos\left(\frac{\theta_0}{\theta_1} \right) \quad \exp\{-(\alpha_L c'/2c_L)r\} \\ &= \frac{1}{r}\sqrt{\frac{2c'}{c_L h_s}} \arccos\left(\sqrt{\frac{h_s}{H}} \right) \quad \exp\{-(\alpha_L c'/2c_L)r\} \end{aligned} \qquad \text{(A.7)}$$

In the limit of large H, i.e. $H >> h_s$ we have

$$I_{NF} = \frac{1}{r}\sqrt{\frac{2c'}{c_L h_s}} \quad \exp\{-(\alpha_L c'/2c_L)r\} \tag{A.8}$$

and

$$I_{NF} = \frac{1}{r}\sqrt{\frac{2c'}{c_L h_s}} \quad \exp\{-(\alpha_L c'/2c_L)r\} \quad \times \pi/2 \tag{A.9}$$

Therefore even in this limit there is still a factor of π/2 difference between the focused and non-focused case.

A.3 Approximations to integrals for two boundaries

A.3.1 Reverberation integral with focusing

Here we consider an approximation to the integral for two-boundary reverberation (Eq. (3.33)).

$$I_{R2} = (2H/r)\mu^{1/2}\int_{V_1}^{V_2}\frac{\{V+V_1\}\exp\{-2rHV(\alpha_L+\alpha_H)\}}{\sqrt{\{V+V_1\}^2-4VV_1h_s/H}}\{1-V_1/V\}\,dV \\ \times\exp\{-(\alpha_L-\alpha_H)r/4\rho\} \tag{A.10}$$

This can also be written as

$$I_{R2} = (2H/r)\mu^{1/2}\exp\{-(\alpha_L-\alpha_H)r/4\rho\} \\ \times\int_{V_1}^{V_2}\left(1-\frac{4VV_1h_s/H}{\{V+V_1\}^2}\right)^{-1/2}\{1-V_1/V\}\exp\{-2rHV(\alpha_L+\alpha_H)\}dV \tag{A.11}$$

If $h_s << H$ we can expand the square root term and the first term is identical to I_{RH2} (Eq. 2.24))

$$I_{R2} = (2H/r)\mu^{1/2}\exp\{-(\alpha_L-\alpha_H)r/4\rho\} \\ \times\left\{\begin{array}{l}\displaystyle\int_{V_1}^{V_2}\{1-V_1/V\}\exp\{-2rHV(\alpha_L+\alpha_H)\}dV \\ \displaystyle+(2V_1h_s/H)\int_{V_1}^{V_2}\frac{(V-V_1)}{\{V+V_1\}^2}\exp\{-2rHV(\alpha_L+\alpha_H)\}dV\end{array}\right\} \tag{A.12}$$

Changing variable in the second integral to $V' = V + V_1$ we can write it in terms of the exponential integrals E_1 and E_2 [5].

A.3.2 Propagation integral with focusing

The equivalent rearrangement and expansion of the square root can be done with the propagation integral (Eq. (3.27)), but now it is convenient to measure the source and receiver depths from opposite sides of the duct, i.e. either $h_1 = h_s$; $h_2 = H - h_r$ or $h_1 = h_r$; $h_2 = H - h_s$.

$$
\begin{aligned}
I_{P2} &= (4/r)\int_{u_1}^{u_2} \frac{\{u^2+u_1{}^2\}\{u^2-u_1{}^2\}\exp\{-2rHu^2(\alpha_L+\alpha_H)\}}{\sqrt{\{u^2+u_1{}^2\}^2-4u^2u_1{}^2h_s/H}\sqrt{\{u^2+u_1{}^2\}^2-4u^2u_1{}^2h_r/H}}\,du \\
&\quad\times\exp\{-(\alpha_L-\alpha_H)r/4\rho\} \\
&= (4/r)\int_{u_1}^{u_2} \frac{\{u^2+u_1{}^2\}\{u^2-u_1{}^2\}\exp\{-2rHu^2(\alpha_L+\alpha_H)\}}{\sqrt{\{u^2+u_1{}^2\}^2-4u^2u_1{}^2h_1/H}\sqrt{\{u^2-u_1{}^2\}^2+4u^2u_1{}^2h_2/H}}\,du \\
&\quad\times\exp\{-(\alpha_L-\alpha_H)r/4\rho\} \\
&= (4/r)\exp\{-(\alpha_L-\alpha_H)r/4\rho\} \\
&\times\int_{u_1}^{u_2}\left(1-\frac{4u^2u_1{}^2h_1/H}{\{u^2+u_1{}^2\}^2}\right)^{-1/2}\left(1+\frac{4u^2u_1{}^2h_2/H}{\{u^2-u_1{}^2\}^2}\right)^{-1/2}\exp\{-2rHu^2(\alpha_L+\alpha_H)\}\,du \\
&\cong (4/r)\exp\{-(\alpha_L-\alpha_H)r/4\rho\} \\
&\times\int_{V_1}^{V_2}\left(1+\frac{2VV_1h_1/H}{\{V+V_1\}^2}\right)\left(1-\frac{2VV_1h_2/H}{\{V-V_1\}^2}\right)\frac{\exp\{-2rHV(\alpha_L+\alpha_H)\}}{2\sqrt{V}}\,dV
\end{aligned}
\tag{A.13}
$$

Again the first term is identical to the simple no focusing formula and the second, being of the form $\int x^{-1/2}(x+a)^{-2}\exp(-x)\,dx$ can be expressed in terms of incomplete gamma functions ([9], e.g. 3.384). However this does not add much insight! If $V_2 \ll V_1$ one could ignore the a leaving $\int x^{-3/2}\exp(-x)\,dx$, but this is still a gamma function.

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Title

Signal and reverberation formulae including refraction

Abstract

Because reverberation is an incoherent process it is possible to base realistic solutions on very simple flux theory. This report extends the closed-form signal and reverberation formulae of an earlier report [SACLANTCEN SR-356] to include upward or downward refraction with potentially different scattering properties and boundary losses at each boundary. The inclusion of refraction is important for the credibility of these formulae although the additional effects are considered to be minor. This report covers two-way propagation, target echoes and reverberation in a range-independent environment, giving explicit closed-form solutions for signal-to-background. Typically these formulae can be evaluated in one line of computer code, and example plots are given. The solutions are given in two forms, one incorporating true focusing effects, the other simplified by merely adding one spherical spreading term for each eigenray. In all circumstances the simple solution performs very well, and in most it obviates the need for consideration of focusing.

Keywords

Reverberation – propagation – signal excess – refraction – analytical models – closed-form models

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